RADIATION AND SCATTERING OF ELECTROMAGNETIC
WAVES IN LAYERED MEDIA

by

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TAREK MOHAMED EL-HABASHY

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ABSTRACT

The stratified medium formulation using the Green's function approach is employed to study the high frequency radiation and scattering of electromagnetic waves in layered media as applied to three current applications problems.

The analysis of resonance, impedance parameters and radiation pattern calculations of two coupled circular disks microstrip structure are rigorously formulated using the Vector Hankel Transform together with the translational properties of the cylindrical harmonics. The resonance frequencies are calculated exactly using Galerkin's moment method and approximately using both the matched asymptotic expansion approach and a perturbational method. The results obtained are then compared. It is also shown that the perturbational formula can be derived from the exact solution in the limit of a thin substrate. The self and mutual impedances and the radiation patterns are derived both exactly and approximately using the single mode approximation.

In studying the resonance and radiation of the elliptic disk structure, a Scaler and a Vector Mathieu Transform are derived and their properties are studied. With the help of these transforms the resonance frequencies of the elliptic disk resonator can be derived exactly and approximately using a perturbational approach. Expressions for the input impedance and the radiation pattern are also obtained for the elliptic disk radiator.

The scattering and radiation from a dielectric coated perfectly conducting cylinder, as a two-dimensional analogue for the propagation over earth, is analyzed using two independent approaches, the modal approach, in which the fields are represented in terms of the natural modes of the structure and the ray expansion approach, in which the fields are represented as multiply reflected rays inside the dielectric. In the modal approach it is shown that there are two kinds of modes, the creeping wave modes and the surface wave modes. The attenuation of the
surface wave modes are calculated by two methods. In one, a perturbational approach is applied to the exact eigenvalue equation obtained by solving the boundary value problem in the limit of a large radius of curvature and in the second method, a perturbational formula is developed from Maxwell's equation to obtain an expression for the propagation constant.
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Dedicated to

Amr M. El-Habashy
Table of Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Title Page</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Abstract</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Acknowledgements</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Dedication</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Table of Contents</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>List of Principal Symbols</td>
<td>9</td>
</tr>
<tr>
<td>I</td>
<td>Introduction</td>
<td>11</td>
</tr>
<tr>
<td>II</td>
<td>Resonance in Two Coupled Circular Microstrip Disk Resonators</td>
<td>17</td>
</tr>
<tr>
<td>II.</td>
<td>Introduction</td>
<td>17</td>
</tr>
<tr>
<td>II.</td>
<td>Formulation of the Problem</td>
<td>19</td>
</tr>
<tr>
<td>III.</td>
<td>Galerkin's Method and the Eigenvalue Equation</td>
<td>27</td>
</tr>
<tr>
<td>IV.</td>
<td>A Perturbation Formula</td>
<td>33</td>
</tr>
<tr>
<td>V.</td>
<td>The Perturbation Formula as a Zeroth-Order Theory</td>
<td>42</td>
</tr>
<tr>
<td>VI.</td>
<td>Evaluation of the Resonant Frequencies Using the Matched Asymptotic Expansion Approach</td>
<td>46</td>
</tr>
<tr>
<td>VII.</td>
<td>Results and Conclusions</td>
<td>61</td>
</tr>
<tr>
<td></td>
<td>Impedance Parameters and Radiation Pattern of Two Coupled Circular Microstrip Disk Antennas</td>
<td>79</td>
</tr>
<tr>
<td>I.</td>
<td>Introduction</td>
<td>79</td>
</tr>
<tr>
<td>II.</td>
<td>Field Expressions Due to the Current on the Disks</td>
<td>80</td>
</tr>
<tr>
<td>III.</td>
<td>Field Expressions Due to Probe Excitation</td>
<td>87</td>
</tr>
<tr>
<td>IV.</td>
<td>The Basic Vector Integral Equations for the Current Distributions</td>
<td>89</td>
</tr>
<tr>
<td>V.</td>
<td>Calculations of the Self and Mutual Impedances</td>
<td>101</td>
</tr>
<tr>
<td>VI.</td>
<td>The Radiation Pattern</td>
<td>113</td>
</tr>
<tr>
<td>VII.</td>
<td>Results and Conclusions</td>
<td>116</td>
</tr>
<tr>
<td></td>
<td>Appendix. The Input Impedance of a Coax Probe</td>
<td>130</td>
</tr>
</tbody>
</table>
CHAPTER 4.  RESONANCE AND RADIATION OF THE ELLIPTIC DISK MICROSTRIP STRUCTURE................................. 138
I.  Introduction........................................... 138
II.  Field Expressions Due to the Current Distribution on the Disk................................................. 140
III.  The Vector Integral Equations Governing the Current Distribution on the Disk......................... 144
IV.  Perturbation Formula for the Resonant Frequencies................................................................. 150
V.  Calculation of the Input Impedance.......................................................................................... 153
VI.  Radiation Pattern........................................ 158
VII.  Conclusions.............................................. 162
APPENDIX 4.1  MATHIEU FUNCTIONS (REVIEW)............. 163
APPENDIX 4.2  ORTHOGONALITY RELATIONS OF THE ANGULAR HARMONICS.......................... 167
APPENDIX 4.3  ORTHOGONALITY RELATIONS OF THE ELLIPTIC HARMONICS............................. 171
APPENDIX 4.4  SCALER MATHIEU TRANSFORM (SMT)............................... 175
APPENDIX 4.5  VECTOR MATHIEU TRANSFORM (VMT)................................. 178
APPENDIX 4.6  FIELD EXPRESSIONS DUE TO PROBE EXCITATION................................. 184
APPENDIX 4.7  INTEGRAL EVALUATIONS................................. 190

CHAPTER 5.  HIGH FREQUENCY SCATTERING FROM A DIELECTRIC COATED PERFECTLY CONDUCTING CYLINDER...................................................................... 194
I.  Introduction................................................ 194
II.  Formulation................................................ 197
   A.  The E-polarized Incident Plane Wave.............. 197
   B.  The H-polarized Incident Plane Wave.............. 203
III.  Methods of Solution..................................... 205
   A.  The Modal Expansion.................................. 205
      i.  The Creeping Waves................................. 208
      ii.  Guided Modes (Surface Modes).................. 220
            1.  Exact Eigenvalue Equation..................... 225
            2.  Perturbational Formula......................... 231
   B.  The Ray Expansion.................................... 248
IV.  Results and Conclusions............................... 253
APPENDIX.  ASYMPTOTIC EXPANSIONS OF $H_v^{(1)}(x)$, $H_v^{(2)}(x)$ AND THEIR DERIVATIVES.............. 257
### LIST OF PRINCIPAL SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_i$</td>
<td>Wavenumber in the i-th medium</td>
</tr>
<tr>
<td>$k_\rho$</td>
<td>Lateral wavenumber</td>
</tr>
<tr>
<td>$k_z$</td>
<td>Vertical wavenumber</td>
</tr>
<tr>
<td>$a$</td>
<td>Radius of a circular disk or half the semi-major axis of an elliptic disk</td>
</tr>
<tr>
<td>$b$</td>
<td>Half the semi-minor axis of an elliptic disk</td>
</tr>
<tr>
<td>$c_0$</td>
<td>Half the focal length of an elliptic disk</td>
</tr>
<tr>
<td>$h$</td>
<td>Height of a disk above the substrate</td>
</tr>
<tr>
<td>$d$</td>
<td>Thickness of the dielectric substrate</td>
</tr>
<tr>
<td>$\varepsilon_i$</td>
<td>Dielectric permittivity of the i-th medium</td>
</tr>
<tr>
<td>$R^{TM}$</td>
<td>Generalized reflection coefficient for a TM wave incident on a stratified medium</td>
</tr>
<tr>
<td>$R^{TE}$</td>
<td>Generalized reflection coefficient for a TE wave incident on a stratified medium</td>
</tr>
<tr>
<td>$R_{ij}^{TM}$</td>
<td>Local Fresnel reflection coefficient of a TM wave incident from the i-th medium</td>
</tr>
<tr>
<td>$R_{ij}^{TE}$</td>
<td>Local Fresnel reflection coefficient of a TE plane wave incident from the i-th medium</td>
</tr>
<tr>
<td>$\tilde{H}<em>n(k</em>\rho)$</td>
<td>Kernel of Vector Hankel Transform (VHT)</td>
</tr>
<tr>
<td>$\tilde{M}<em>n(c_0,k</em>\rho,u,v)$</td>
<td>Kernel of Vector Mathieu Transform (VMT)</td>
</tr>
<tr>
<td>$\alpha, \gamma, \beta$</td>
<td>Parity of the elliptic harmonic functions, either even (e) or odd (o)</td>
</tr>
<tr>
<td>$e_n^{(j)}(k_\rho), h_n^{(j)}(k_\rho)$</td>
<td>The electric and magnetic spectral amplitudes of the electric and magnetic fields respectively for the n-th harmonic of the j-th disk, in the cylindrical coordinate system.</td>
</tr>
</tbody>
</table>
\( e_{\alpha_n}(k_\rho), h_{\alpha_n}(k_\rho) \)

The electric and magnetic spectral amplitudes of the electric and magnetic fields respectively for the \( n \)-th harmonic of parity \( \alpha \), in the elliptic coordinate system.

\( K_n^{(j)}(k_\rho) \)

The Vector Hankel Transform of the \( n \)-th harmonic \( K_n^{(j)}(k_\rho) \).

\( K_{\alpha_n}(k_\rho) \)

The Vector Mathieu Transform of the \( n \)-th harmonic \( K_{\alpha_n}(u,v) \) which has the parity \( \alpha \).

\( J_n(x) \)

Regular Bessel function of order \( n \) and argument \( x \).

\( H_n^{(1)}(x) \)

Hankel function of first kind, of order \( n \) and argument \( x \).

\( H_n^{(2)}(x) \)

Hankel function of second kind, of order \( n \) and argument \( x \).

\( J_n'(x) \)

Derivative of \( J_n(x) \).

\( H_n^{(1)'}(x) \)

Derivative of \( H_n^{(1)}(x) \).

\( \psi_{\alpha_n}(c_0 k_\rho, u, v) \)

Elliptic harmonic of \( n \)-th order and parity \( \alpha \).

\( J_{\alpha_n}(c_0 k_\rho, u) \)

Elliptic regular radial function of \( n \)-th order and parity \( \alpha \).

\( H_{\alpha_n}^{(1)}(c_0 k_\rho, u) \)

Elliptic radial function of \( n \)-th order and parity \( \alpha \), representing outgoing waves.

\( S_{\alpha_n}(c_0 k_\rho, v) \)

Elliptic angular function of \( n \)-th order and parity \( \alpha \).

\( Ai(x) \)

Airy function.

A superscript single prime denotes the real part of a quantity and a superscript double prime denotes the imaginary part.
CHAPTER 1
INTRODUCTION

Many radiation, diffraction and scattering problems in electromagnetic wave theory can often be formally solved in terms of harmonic series representations in the coordinate system appropriate to the geometry of the structure considered. This harmonic series representation is usually effective numerically especially when the characteristic dimension of the structure is small relative to the wavelength. On the other hand, the numerical handling of such solutions becomes almost practically impossible because of the slow convergence of such representations, if the dimension to wavelength ratio is relatively large. In such cases, these harmonic series representations have to be converted to other representations in which the fields can be computed efficiently.

In this thesis, both types of problems are considered, the microstrip antenna problem in which the dimensions to wavelength ratios are small and the scattering from the dielectric coated cylinder in which this ratio is very large. The stratified medium formulation using the Green's function approach is employed to rigorously study the high frequency radiation and scattering of electromagnetic waves in layered media as applied to three current applications problems.

The formulation of these problems are first rigorously carried out and for the sake of simplifying the analysis to give a deeper physical insight, approximate solutions are then obtained.
The resonance and radiation of single-element microstrip antennas have been extensively studied in the past few years due to their wide applications as resonators and radiators. Three approaches which have been often used are the magnetic wall cavity model [14], the transmission line model [56] and the moment method [3-5]. On the other hand, less efforts have been directed to the analysis of the coupling effects between several elements. The coupling between two circular microstrip disk resonators has been studied using an electrostatic approach [9,55] in which the coupling is modeled only by a gap capacitance. In reference [9], the coupling is assumed to be mainly due to the fringing of the electric field and thus the interaction between the disks is modeled by a gap capacitance, which is computed by solving the corresponding electrostatic problem. In [55], the coupling problem is formulated electrostatically by applying what is known as the Kobayashi potential which is the name given to the expression for the potential constructed by using the properties of the Weber-Schafheitlin integrals proposed by Kobayashi in 1931. Also, the mutual coupling in the cases of two rectangular and two circular microstrip antennas has been investigated experimentally by several measurements of the S-parameters [10]. In Chapters 2 and 3 of this thesis, a full wave analysis of the coupling between two circular disks is carried out rigorously using the Vector Hankel Transform together with the translational properties of the cylindrical harmonics. This Vector Hankel Transform [3] was originally developed to study the resonance of a single circular disk microstrip structure. The resonance in the two coupled resonators is then analyzed
using three independent approaches: (a) the Galerkin moment method in which the current is expanded in terms of the TE and TM modes of the magnetic wall cavity model which form a complete set of basis functions, (b) a perturbative approach and (c) matched asymptotic expansion approach in which the range, over which the problem is defined, is divided into a number of overlapping subranges. On each subinterval, perturbative methods are used, in the limit of small substrate thickness, to obtain an asymptotic approximation to the solution of the problem valid on that interval. Finally the matching is done by requiring that the asymptotic approximations have the same functional form on the overlap of every pair of subintervals. Exact expressions for the self, mutual and input impedances together with the radiation pattern have been derived in terms of the current distribution on the feeding probe. In the limit of thin substrate, these expressions are greatly simplified. The formulation is carried out for two disks of equal radii, however, the formulation can be easily generalized for the case of two disks of unequal radii.

Another problem of interest in the microstrip antenna field is to obtain circular polarization using the simplest feeding setup possible. It is known that circular or rectangular disk microstrip antennas usually radiate waves which are linearly polarized [23,53,54]. However, in such structures, circular polarization can be obtained using a complicated feed setup of more than one feed.

From experimental measurements [53] and recent theoretical work by Shen [23] it is shown that by using a slightly elliptical shaped disk, circular polarization can be achieved while retaining a single, simple
feed.

Also, for applications in harmonic multipliers and parametric amplifiers, the circular microstrip disk resonator is unsuitable due to a harmonious relationship of mode frequencies. However, suitable modes for both applications can be found in the resonances of an ellipse where a further degree of freedom, the eccentricity enables the desired frequency relationships to be achieved. Thus the eccentricity as a design parameter provides additional flexibility and enhances the usefulness of the elliptic disk structure [21-23].

In the past studies of the elliptic disk microstrip structure more research was devoted to the analysis of the structure as a resonator [21,22] than as an antenna [23,53,54]. In [21], the analysis of the elliptic resonator is carried out by using the magnetic wall model, whereas in [22], the analysis is carried out using a quasi-static approximation in which information about capacitance and associated end effects are presented. In both these analyses, the resonant frequencies obtained are pure real and therefore do not account for the radiation losses thus rendering the obtained results very limited.

In the analysis of the elliptic structure as a radiator and in calculating the radiation pattern [23], the fringing effects of the electric field are taken into account by superficially imposing an impedance boundary condition at the opening of the cavity. This surface impedance is approximated by the one obtained from solving the circular disk antenna, thus limiting the obtained results for small eccentricity.
In [54], a resonant circuit model for the input impedance of the elliptic disk antenna has been developed by proposing an equivalent circuit consisting of two parallel resonant circuits connected in series and obtaining the values of the circuit elements by comparing with experimental data.

In Chapter 4, a rigorous analysis of the elliptic microstrip structure is carried out by first developing Scaler and Vector Mathieu Transforms. These transforms allow an exact formulation of the elliptic disk structure which has long been thought almost impossible to formulate rigorously [22]. The properties of these transforms are also studied. It is shown that the current distribution on the disk is rigorously derived using these transforms and that it is governed by vector integral equations. These equations are then solved using the Galerkin's moment method. The resonance in the elliptic disk structure is analyzed using two independent methods: Galerkin's method and a perturbative approach. The input impedance together with the radiation pattern are derived both exactly and in the small substrate thickness limit.

In the previous considered problems, the ratio of dimension to wavelength is relatively small and hence the harmonic series representation of the fields are efficient for numerical computation. In Chapter 5, the scattering from a dielectric coated cylinder, as a two dimensional analogue of the problem of line of sight propagation over the earth, is considered. In this case, the circumference of the cylinder to the wavelength is very large and hence the harmonic series becomes numerically inefficient and thus other representations have to be
considered in order to avoid the very slow convergence of the harmonic series.

The interest in scattering from large perfectly conducting cylinders has been extensively studied in literature [27,44-48]. Also, the problem of radiation from slots on a perfectly conducting circular cylinders were considered by several authors [43,44]. Scattering from a circular cylinder with a surface impedance was recently studied by Wang [52] and the case of an abrupt change in the surface impedance was also considered by Wait [51].

The problem of scattering from a dielectric coated perfectly conducting cylinder, was considered by Adey [50] in which the fields are represented in the form of a harmonic series, i.e. infinite series of integral order Bessel and Hankel functions, thus limiting the results obtained to small values of circumference to wavelength ratio. In Chapter 5, the analysis of the dielectric coated cylinder is formulated using two independent methods. The first is the modal approach in which the fields are represented in terms of the natural modes of the structure and the second is the ray expansion method in which the fields are expanded in terms of rays which have undergone multiple reflections inside the dielectric. In the modal approach, the different modes excited are studied and their attenuation coefficients are derived.
Chapter 2
Resonance in Two Coupled Circular Microstrip Disk Resonators

I. Introduction

Due to the recent wide application of microstrip antenna arrays [1,2], usable analytical techniques for predicting mutual coupling effects between different elements need to be investigated. Different microstrip structures as isolated resonators and antenna elements have been analyzed to predict the resonant frequencies of these structures. The resonance in a microstrip disk has been studied extensively [3-5]. Also, triangular and rectangular microstrip resonators have been studied [6-8]. More recently, the resonance in an annular-ring microstrip resonator has been studied using different approaches [25,40].

The coupling between two microstrip disk resonators has been investigated using an electrostatic approximation [9], where the coupling is assumed to be mainly due to the fringing effects of the electric fields, and thus the coupling between the two disks was modeled only by a gap capacitance. Also, the mutual coupling in the cases of two rectangular and two circular microstrip antennas has been investigated experimentally by several measurements of the S-parameters [10].

This chapter provides a full-wave analysis of two coupled circular microstrip disk resonators. Using vector Hankel transform (VHT), the problem is rigorously formulated in terms of a set of vector integral equations. The resonance in the two coupled resonators is then analyzed
using three approaches: Galerkin's basis function expansion method (GM), a perturbative approach (PA), and by the matched asymptotic expansion approach (MA). It is shown that the structure formed by the two coupled circular microstrip disk resonators can support four different types of resonant modes. Using (VHT) and (GM), it is also shown that the perturbative formula can be derived as a special case of the rigorous equations when the dielectric substrate is thin. Several plots are presented for the four different resonant modes of the TM₁₁ when the radii of both disks are equal.
II. Formulation of the Problem

The two coupled circular microstrip disk resonators are shown in Fig. 1. The coordinates measured with respect to O₁ will be denoted by \((ρ₁, φ₁, z)\) and those measured with respect to O₂ will be \((ρ₂, φ₂, z)\).

Let the disk D₂ be carrying an arbitrary distribution of current and placed at a distance h from the surface of the dielectric substrate backed by the ground plane, and which is characterized by reflection coefficients \(R_{TM}\) and \(R_{TE}\) for TM and TE waves respectively. The z-component of the electric and magnetic fields referred to the cylindrical coordinates at O₂ can be written as [17]

\[
E_{Z}^{(2)}(ρ₂, φ₂, z) = \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} dk ρ m \, e_m^{(2)}(k) \left[ e^{\pm ikZ} \cdot R_{TM} \exp(ikZ + 2ik_h) \right] \cdot J_m(κ_0 ρ₂) e^{imφ₂} \tag{1a}
\]

\[
H_{Z}^{(2)}(ρ₂, φ₂, z) = \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} dk ρ m \, h_m^{(2)}(k) \left[ e^{\pm ikZ} \cdot R_{TE} \exp(ikZ + 2ik_h) \right] \cdot J_m(κ_0 ρ₂) e^{imφ₂} \tag{1b}
\]

where \(k_z = (k^2 - k_ρ^2)^{\frac{1}{2}}, k^2 = \omega^2 \mu \varepsilon, \) the upper sign in the above is chosen for \(z > 0,\) the lower sign for \(z < 0,\) and \(e_m(κ_0)\) and \(h_m(κ_0)\) are unknowns to be determined. Applying the addition theorem for cylindrical waves [19]

\[
J_m(κ_0 ρ₂) e^{imφ₂} = \sum_{k=-\infty}^{\infty} J_k(κ_0) J_{m+k}(κ_0 φ₁) e^{i(m+k)φ₁} \tag{2}
\]
Figure 1. Geometrical configuration of the two coupled circular microstrip disk resonators.
equations (1a,b) referred to the coordinates at $O_1$, reduce to

\[
E_z^{(2)}(\rho_1, \phi_1, z) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} dk \, k \cdot e_m^{(2)}(k) \left[ e^{-i k z} - R_{\text{TM}} \exp(i k z + 2ik z) \right] e^{i(m+k)\phi_1} \tag{3a}
\]

\[
H_z^{(2)}(\rho_1, \phi_1, z) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} dk \, k \cdot h_m^{(2)}(k) \left[ e^{i k z} + R_{\text{TE}} \exp(i k z + 2ik z) \right] e^{i(m+k)\phi_1} \tag{3b}
\]

The transverse electric field due to the arbitrary distribution of current on disk $D_2$, for $h = 0$ and $z = 0$, can be obtained from (3a) as

\[
E_z^{(2)}(\rho_1, \phi_1) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} dk \, k \cdot e_m^{(2)}(k) \left\{ e_k^{(2)}(k, \rho_1) \frac{i k z (1 - R_{\text{TM}})}{k_\rho^{\rho_1}} J_k(k, \rho_1) J_{m+k}(k_\rho^{\rho_1}) \right\} e^{i(m+k)\phi_1}
\]

\[+ \frac{i \omega}{k_\rho^{\rho_1}} h_m^{(2)}(k, \rho_1) (1 + R_{\text{TE}}) J_k(k, \rho_1) \frac{i(m+k)}{k_\rho^{\rho_1}} J_{m+k}(k_\rho^{\rho_1}) \tag{4a}
\]

\[
E_\phi^{(2)}(\rho_1, \phi_1) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} dk \, k \cdot e_m^{(2)}(k) \left\{ e_k^{(2)}(k, \rho_1) \frac{i k z (1 - R_{\text{TM}})}{k_\rho^{\rho_1}} J_k(k, \rho_1) \frac{i(m+k)}{k_\rho^{\rho_1}} \right\} e^{i(m+k)\phi_1}
\]

\[\times J_{m+k}(k_\rho^{\rho_1}) - \frac{i \omega}{k_\rho^{\rho_1}} h_m^{(2)}(k, \rho_1) (1 + R_{\text{TE}}) J_k(k, \rho_1) J_{m+k}(k_\rho^{\rho_1}) \tag{4b}
\]
From the discontinuity of the transverse magnetic field components at the interface $z = 0$, the electric surface current $K_2^{(2)}(\rho_1, \phi_1)$ on the disk $D_2$ is obtained as

$$K_2^{(2)}(\rho_1, \phi_1) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_0^{\infty} dk_\rho \, 2 \left\{ -h_m^{(2)}(k_\rho) i k_z J_k(k_\rho c) \frac{i(m+k)}{k_\rho \rho_1} J_{m+k}(k_\rho \rho_1) ight. $$

$$+ \left. \cdot i \omega e_m^{(2)}(k_\rho) J_k(k_\rho c) J_{m+k}^*(k_\rho \rho_1) \right\} e^{i(m+k)\phi_1} \quad (5a)$$

$$K_2^{(2)}(\rho_1, \phi_1) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_0^{\infty} dk_\rho \, 2 \left\{ h_m^{(2)}(k_\rho) i k_z J_k(k_\rho c) J_{m+k}^*(k_\rho \rho_1) ight. $$

$$+ \left. \cdot i \omega e_m^{(2)}(k_\rho) J_k(k_\rho c) \frac{i(m+k)}{k_\rho \rho_1} J_{m+k}(k_\rho \rho_1) \right\} e^{i(m+k)\phi_1} \quad (5b)$$

We define the Fourier coefficients

$$E_n^{(j)}(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} E_n^{(j)}(\rho, \phi, \theta) e^{-in\theta} d\theta \quad (6a)$$

$$K_n^{(j)}(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} K_n^{(j)}(\rho, \phi, \theta) e^{-in\theta} d\theta \quad (6b)$$

where $n$ denotes $\rho$- or $\phi$-component and $j = 1$ or 2. Multiplying (4) and (5) by $e^{-i\phi_1}$, performing the Fourier integration, and making use of
the Vector Hankel Transform [4], we obtain

\[
\vec{E}_n^{(2)}(\rho_1) = \begin{bmatrix} -E_\rho^{(2)}(\rho_1) \\ iE_\phi^{(2)}(\rho_1) \end{bmatrix} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk_\rho \vec{h}_n(k_\rho \rho_1) \cdot \vec{g}(k_\rho) \cdot \vec{J}_{n-m}(k_\rho c) \cdot \vec{K}_m^{(2)}(k_\rho)
\]

(7)

\[
\vec{K}_n^{(2)}(\rho_1) = \begin{bmatrix} k_\rho^{(2)}(\rho_1) \\ -iK_\phi^{(2)}(\rho_1) \end{bmatrix} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk_\rho \vec{h}_n(k_\rho \rho_1) \cdot \vec{J}_{n-m}(k_\rho c) \cdot \vec{K}_m^{(2)}(k_\rho)
\]

(8)

where \(\vec{J}_{n-m}(k_\rho c) = J_{n-m}(k_\rho c) \vec{I}\) and

\[
\vec{K}_m^{(2)}(k_\rho) = \begin{bmatrix} -2i\omega \frac{e_m^{(2)}(k_\rho)}{k_\rho} \\ 2k_\rho \frac{h_m^{(2)}(k_\rho)}{k_\rho} \end{bmatrix}
\]

(9)

\[
\vec{h}_n(k_\rho \rho_1) = \begin{bmatrix} J_n^{(2)}(k_\rho \rho_1) \\ \frac{n}{k_\rho \rho_1} J_n^{(2)}(k_\rho \rho_1) \\ \frac{n}{k_\rho \rho_1} J_n^{(2)}(k_\rho \rho_1) \\ J_n^{(2)}(k_\rho \rho_1) \end{bmatrix}
\]

(10)

and
\[
\mathbf{\bar{\sigma}}(k_\rho) = \begin{bmatrix}
\frac{k_z}{2\omega_0} (1 - R^{TM}) & 0 \\
0 & \frac{\omega_0}{2k_z} (1 + R^{TE})
\end{bmatrix}
\] (11)

The reflection coefficients \(R^{TM}\) and \(R^{TE}\) are given by

\[
R^{TM} = \frac{i\varepsilon_1 k_z \cos k_1 z d - \varepsilon k_1 z \sin k_1 z d}{i\varepsilon_1 k_z \cos k_1 z d + \varepsilon k_1 z \sin k_1 z d}
\] (12)

\[
R^{TE} = \frac{ik_z \sin k_1 z d + k_1 z \cos k_1 z d}{ik_z \sin k_1 z d - k_1 z \cos k_1 z d}
\] (13)

where

\[k_1 z = (k_1^2 - k_\rho^2)^{1/2}, \quad k_1^2 = \omega^2 \mu \varepsilon_1 .\]

Similarly, for arbitrary distribution of current on disk \(D_1\), we can get

\[
\mathbf{\bar{E}}^{(1)}(\rho_1) = \begin{bmatrix}
-E^{(1)}(\rho_1) \\
\int_0^\infty dk_\rho k_\rho \mathbf{\bar{n}}(k_\rho \rho_1) \cdot \mathbf{\bar{\sigma}}(k_\rho) \cdot R^{(1)}(k_\rho)
\end{bmatrix}
\] (14)
\[ \bar{K}_n^{(1)}(\rho_1) = \begin{bmatrix} \bar{k}_n^{(1)}(\rho_1) \\ \bar{k}_n^{(2)}(\rho_1) \end{bmatrix} = \int_0^\infty dk_\rho \bar{A}_n(k_\rho \rho_1) \cdot \bar{K}_n^{(1)}(k_\rho) \]

where \( \bar{K}_n^{(1)}(k_\rho) \) is related to \( e_m^{(1)}(k_\rho) \) and \( h_m^{(1)}(k_\rho) \) by an equation similar to (9).

If we express the fields and currents in terms of the coordinates at \( O_2 \) by making use of the translation relation

\[ J_m(k_\rho \rho_1) e^{i m \phi_1} = \sum_{k=-\infty}^{\infty} J_k(k_\rho c) J_{m+k}(k_\rho \rho_2) e^{i (m+k) \phi_2} (-1)^k \]

we can get

\[ \bar{E}_n^{(2)}(\rho_2) = \int_0^\infty dk_\rho \bar{A}_n(k_\rho \rho_2) \cdot \bar{G}(k_\rho) \cdot \bar{K}_n^{(2)}(k_\rho) \]

\[ \bar{K}_n^{(2)}(\rho_2) = \int_0^\infty dk_\rho \bar{A}_n(k_\rho \rho_2) \cdot \bar{K}_n^{(2)}(k_\rho) \]

and

\[ \bar{E}_n^{(1)}(\rho_2) = \sum_{m=-\infty}^{\infty} \int_0^\infty dk_\rho \bar{A}_n(k_\rho \rho_2) \cdot \bar{G}(k_\rho) \cdot \bar{J}_{n-m}(k_\rho c) \cdot \bar{K}_n^{(1)}(k_\rho) (-1)^{n-m} \]

\[ \bar{K}_n^{(1)}(\rho_2) = \sum_{m=-\infty}^{\infty} \int_0^\infty dk_\rho \bar{A}_n(k_\rho \rho_2) \cdot \bar{J}_{n-m}(k_\rho c) \cdot \bar{K}_n^{(1)}(k_\rho) (-1)^{n-m} \]
Imposing the boundary conditions on the tangential field components at the plane \( z = 0 \), we obtain a set of vector integral equations given by

\[
\int_0^\infty \frac{dk_p}{k_p} \bar{H}_n(k_{p\rho_1}) \cdot \bar{G}(k_p) \cdot \bar{K}_n^{(1)}(k_p) + \sum_{m=-\infty}^{\infty} \int_0^\infty \frac{dk_p}{k_p} \bar{H}_n(k_{p\rho_1}) \cdot \bar{G}(k_p) \\
\cdot \bar{J}_{n-m}(k_p) \cdot \bar{K}_m^{(2)}(k_p) = 0, \quad \rho_1 \in D_1 \tag{21}
\]

\[
\int_0^\infty \frac{dk_p}{k_p} \bar{H}_n(k_{p\rho_2}) \cdot \bar{G}(k_p) \cdot \bar{K}_n^{(2)}(k_p) + \sum_{m=-\infty}^{\infty} \int_0^\infty \frac{dk_p}{k_p} \bar{H}_n(k_{p\rho_2}) \cdot \bar{G}(k_p) \\
\cdot \bar{J}_{n-m}(k_p) \cdot \bar{K}_m^{(1)}(k_p)(-1)^{n-m} = 0, \quad \rho_2 \in D_2 \tag{22}
\]

\[
\int_0^\infty \frac{dk_p}{k_p} \bar{H}_n(k_{p\rho_1}) \cdot \bar{K}_n^{(1)}(k_p) = 0, \quad \rho_1 \notin D_1 \tag{23}
\]

\[
\int_0^\infty \frac{dk_p}{k_p} \bar{H}_n(k_{p\rho_2}) \cdot \bar{K}_n^{(2)}(k_p) = 0 \quad \rho_2 \notin D_2 \tag{24}
\]

The task is to solve this set of vector integral equations with the Galerkin's method and compare with the results which will be obtained using a perturbative approach.
III. Galerkin's Method and the Eigenvalue Equation

The formulation up to this stage is exact. We now solve the set of vector integral equations (21)-(24) by using Galerkin's method. First, the unknown current distributions on $D_1$ and $D_2$ will be expanded in terms of a complete set of vector basis functions. Since for $d \to 0$, the current distributions on $D_1$ and $D_2$ approach that of a magnetic-wall model for each disk independently, it is possible to find a complete set of vector basis functions to approximate these current distributions. By noting that the superposition of the currents due to TM and TE modes of a magnetic-wall cavity form a complete set, the current distribution of the $n$-th mode on each disk can be written as

$$
K_n^{(j)}(\rho_j) = \begin{cases} 
\sum_{r=1}^{8} a_{nr}^{(j)} \psi_{nr}(\rho_j) + \sum_{p=1}^{\infty} b_{np}^{(j)} \phi_{np}(\rho_j), & \rho_j \in D_j \\
0 & \rho_j \notin D_j
\end{cases} \quad (25)
$$

where $j = 1$ and 2. $\psi_{nr}(\rho)$ and $\phi_{np}(\rho)$ are, respectively, the TM and TE modes of a magnetic-wall cavity and are given by

$$
\psi_{nr}(\rho_j) = \begin{bmatrix} J_n^{(j)}(\beta_{nr} \rho_j/a_j) \\
\frac{n a_j}{\beta_{nr} \rho_j} J_n^{(j)}(\beta_{nr} \rho_j/a_j)
\end{bmatrix} \quad (26)
$$
\[
\frac{\phi_{np}(\rho_j)}{\alpha_{np}^{\rho_j}} J_n(\alpha_{np}^{\rho_j}/a_j)
\]

\[
\hat{j}(j) = \begin{bmatrix} \frac{na_j}{\alpha_{np}^{\rho_j}} J_n(\alpha_{np}^{\rho_j}/a_j) \\ 0 \end{bmatrix}
\]

and \(\beta_{nr}\) and \(\alpha_{np}\) are the \(r\)-th and \(p\)-th zeros of \(J'_n(x)\) and \(J_n(x)\), respectively.

In practice, we need to consider only finite number of terms of both series of (25). Thus the vector Hankel transforms of \(\overline{K}_n^{(1)}(\rho_1)\) and \(\overline{K}_n^{(2)}(\rho_2)\) are given by

\[
\overline{K}_n^{(1)}(k_\rho) = \sum_{r=1}^{R} a_{nr}^{(1)} \overline{\psi}_{nr}^{(1)}(k_\rho) + \sum_{p=1}^{P} b_{np}^{(1)} \overline{\phi}_{np}^{(1)}(k_\rho)
\]

\[
\overline{K}_n^{(2)}(k_\rho) = \sum_{q=1}^{Q} a_{nq}^{(2)} \overline{\psi}_{nq}^{(2)}(k_\rho) + \sum_{s=1}^{S} b_{ns}^{(2)} \overline{\phi}_{ns}^{(2)}(k_\rho)
\]

where \(\overline{\psi}_{nr}(k_\rho)\) and \(\overline{\phi}_{np}(k_\rho)\) are given by

\[
\overline{\psi}_{nr}(k_\rho) = \beta_{nr} J_n(\beta_{nr})
\]

\[
\overline{\phi}_{np}(k_\rho) = \frac{na_j}{\beta_{nr}^{\rho_j}} J_n(\beta_{nr}^{\rho_j}/a_j)
\]

\[
\overline{\psi}(j)(k_\rho) = \beta_{nr} J_n(\beta_{nr})
\]

\[
\overline{\phi}(j)(k_\rho) = \frac{na_j}{\beta_{nr}^{\rho_j}} J_n(\beta_{nr}^{\rho_j}/a_j)
\]
and

\[ \tilde{\phi}_{np}(k_\rho) = \frac{k_\rho a_j J_j'(\alpha_{np})}{k^2 - (\alpha_{np}/a_j)^2} \begin{bmatrix} 0 \\ J_n(k_\rho a_j) \end{bmatrix} \]  

(31)

The choice of \( \tilde{\phi}_{n}^{(j)}(\rho_j) \) as given by (25) satisfies (23) and (24) identically. Also, this choice is suitable in the limiting case of the magnetic-wall model when \( d \to 0 \).

Substituting (28) and (29) in (21) and (22), we get

\[
\begin{align*}
R \sum_{r=1}^{\infty} a_{nr}^{(1)} & \int_{0}^{\infty} dk_\rho \bar{A}_n(k_\rho a_1) \cdot \bar{G}(k_\rho) \cdot \psi_{nr}^{(1)}(k_\rho) + \sum_{p=1}^{P} b_{np}^{(1)} \int_{0}^{\infty} dk_\rho \bar{A}_n(k_\rho a_1) \\
& \cdot \bar{G}(k_\rho) \cdot \tilde{\phi}_{np}(k_\rho) \\
+ \sum_{m=-\infty}^{\infty} \sum_{q=1}^{Q} a_{mq}^{(2)} \int_{0}^{\infty} dk_\rho \bar{A}_n(k_\rho a_1) \cdot \bar{G}(k_\rho) \cdot \bar{J}_{n-m}(k_\rho c) \cdot \psi_{mq}^{(2)}(k_\rho) \\
+ \sum_{m=-\infty}^{\infty} \sum_{s=1}^{S} b_{ms}^{(2)} \int_{0}^{\infty} dk_\rho \bar{A}_n(k_\rho a_1) \cdot \bar{G}(k_\rho) \cdot \bar{J}_{n-m}(k_\rho c) \cdot \tilde{\phi}_{ms}^{(2)}(k_\rho) = 0,
\end{align*}
\]

(32)

and
\[
\sum_{q=1}^{Q} a^{(2)}_{nq} \int_{0}^{\infty} dk_{\rho} \frac{\bar{H}_{n}(k_{\rho}^{2}) \cdot \bar{g}(k_{\rho}) \cdot \psi(2)(k_{\rho})}{nq} \\
+ \sum_{s=1}^{S} b^{(2)}_{ns} \int_{0}^{\infty} dk_{\rho} \frac{\bar{H}_{n}(k_{\rho}^{2}) \cdot \bar{g}(k_{\rho}) \cdot \phi(2)(k_{\rho})}{ns} \\
+ \sum_{m=-\infty}^{\infty} \sum_{r=1}^{R} a^{(1)}_{mr} \int_{0}^{\infty} dk_{\rho} \frac{\bar{H}_{n}(k_{\rho}^{2}) \cdot \bar{g}(k_{\rho}) \cdot \tilde{J}_{n-m}(k_{\rho} c) \cdot \psi(1)(k_{\rho})}{mr} (-1)^{n-m} \\
+ \sum_{m=-\infty}^{\infty} \sum_{p=1}^{P} b^{(1)}_{mp} \int_{0}^{\infty} dk_{\rho} \frac{\bar{H}_{n}(k_{\rho}^{2}) \cdot \bar{g}(k_{\rho}) \cdot \tilde{J}_{n-m}(k_{\rho} c) \cdot \phi(1)(k_{\rho})}{mp} (-1)^{n-m=0} 
\]

(33)

Multiplying (32) by \(\rho_{1} \psi_{nf}(\rho_{1})\) and \(\rho_{1} \phi_{nk}(\rho_{1})\) and integrating from 0 to \(a_{1}\) for \(f = 1, 2, \ldots, R\), \(k = 1, 2, \ldots, P\), and using Parseval's relation for Hankel transform. Similarly, multiplying (33) by \(\rho_{2} \psi_{nt}(\rho_{2})\) and by \(\rho_{2} \phi_{nu}(\rho_{2})\) and integrating from 0 to \(a_{2}\) for \(t = 1, 2, \ldots, Q\), \(u = 1, 2, \ldots, S\) and using Parseval's relation for Hankel transform we get a system of

\((R + P + Q + S) \times (2M + 1)\) linear algebraic equations which can be written as

\[
\sum_{r=1}^{R} a^{(1)}_{nr} A^{(1)}_{n}(fr) + \sum_{p=1}^{P} b^{(1)}_{np} A^{(1)}_{n}(fp) + \sum_{m=-M}^{M} \sum_{q=1}^{Q} a^{(2)}_{mq} A^{(2)}_{nm}(fq) \\
+ \sum_{m=-M}^{M} \sum_{s=1}^{S} b^{(2)}_{ms} A^{(2)}_{nm}(fs) = 0 \text{, for } f = 1, \ldots, R \quad (34a)
\]
\[ 
\sum_{r=1}^{R} a_{nr} \phi^1_{1r} + \sum_{p=1}^{P} b_{np} \phi^1_{1p} + \sum_{m=-M}^{M} \sum_{q=1}^{Q} a_{mq} \phi^1_{mq} + \sum_{m=-M}^{M} \sum_{s=1}^{S} b_{ms} \psi^1_{ms} = 0, \quad \text{for } k = 1, \ldots, P 
\]

\[
\sum_{q=1}^{Q} a_{nq} \psi^2_{nq} + \sum_{s=1}^{S} b_{ns} \psi^2_{ns} + \sum_{m=-M}^{M} \sum_{r=1}^{R} a_{mr} \psi^2_{mr} + \sum_{m=-M}^{M} \sum_{p=1}^{P} b_{mp} \psi^2_{mp} = 0, \quad \text{for } t = 1, \ldots, Q 
\]

\[
\sum_{p=1}^{P} a_{np} \psi^2_{np} + \sum_{s=1}^{S} b_{ns} \psi^2_{ns} + \sum_{m=-M}^{M} \sum_{r=1}^{R} a_{mr} \psi^2_{mr} + \sum_{m=-M}^{M} \sum_{p=1}^{P} b_{mp} \psi^2_{mp} = 0, \quad \text{for } u = 1, \ldots, S 
\]

where

\[ A_{s(pq)}^{\psi_{ij}} = \int_0^{\infty} dk_{\rho} k_{\rho} \bar{\psi}_{sp}^{(i)}(k_{\rho}) \cdot \bar{\psi}_{sq}^{(j)}(k_{\rho}) 
\]

\[ A_{s(pq)}^{\phi_{ij}} = \int_0^{\infty} dk_{\rho} k_{\rho} \bar{\psi}_{sp}^{(i)}(k_{\rho}) \cdot \bar{\psi}_{sq}^{(j)}(k_{\rho}) 
\]

\[ A_{s(pq)}^{\psi_{ij}} = \int_0^{\infty} dk_{\rho} k_{\rho} \bar{\psi}_{sp}^{(i)}(k_{\rho}) \cdot \bar{\psi}_{sq}^{(j)}(k_{\rho}) 
\]

\[ A_{s(pq)}^{\phi_{ij}} = \int_0^{\infty} dk_{\rho} k_{\rho} \bar{\psi}_{sp}^{(i)}(k_{\rho}) \cdot \bar{\psi}_{sq}^{(j)}(k_{\rho}) 
\]
\begin{align}
A_{st(pq)}^{\psi \psi} &= \int_0^{\infty} dk_{\rho} \psi_{sp}^{(i)T}(k_{\rho}) \cdot \bar{g}(k_{\rho}) \cdot \bar{J}_{s-t}(k_{\rho} c) \cdot \bar{\psi}_{tq}^{(j)}(k_{\rho}) \cdot \zeta(i) \\
A_{st(pq)}^{\phi \phi} &= \int_0^{\infty} dk_{\rho} \phi_{sp}^{(i)T}(k_{\rho}) \cdot \bar{g}(k_{\rho}) \cdot \bar{J}_{s-t}(k_{\rho} c) \cdot \bar{\psi}_{tq}^{(j)}(k_{\rho}) \cdot \zeta(i) \\
A_{st(pq)}^{\psi \phi} &= \int_0^{\infty} dk_{\rho} \psi_{sp}^{(i)T}(k_{\rho}) \cdot \bar{g}(k_{\rho}) \cdot \bar{J}_{s-t}(k_{\rho} c) \cdot \bar{\psi}_{tq}^{(j)}(k_{\rho}) \cdot \zeta(i) \\
A_{st(pq)}^{\phi \psi} &= \int_0^{\infty} dk_{\rho} \phi_{sp}^{(i)T}(k_{\rho}) \cdot \bar{g}(k_{\rho}) \cdot \bar{J}_{s-t}(k_{\rho} c) \cdot \bar{\psi}_{tq}^{(j)}(k_{\rho}) \cdot \zeta(i)
\end{align}

where \( i = 1,2, j = 1,2, \) and \( \zeta(i) = \begin{cases} (-1)^{s-t}, & i = 2 \\ 1, & i = 1 \end{cases} \)

Nontrivial solutions can exist if the determinant of the system of equations (34a)-(34d) vanishes, that is

\[ \det|\bar{A}| = f(\omega) = 0. \]

This is the characteristic equation for eigenmodes for two coupled microstrip circular disk resonators. The accuracy of the obtained resonant frequencies can be improved by increasing \( R, P, Q, S \) and \( M \), arbitrarily.
IV. A Perturbation Formula

In the limit when \( d \to 0 \), the two microstrip cavities formed by the two disks \( D_1 \) and \( D_2 \) are uncoupled and their resonant frequencies approach that for the magnetic-wall cavities. Therefore for small \( d \), we can view the two coupled microstrip-disk resonators as a perturbation of the magnetic-wall model. Assuming both disks \( D_1 \) and \( D_2 \) to have the same radius \( a \), the resonant frequency shift of the magnetic-wall cavity with the magnetic wall can be derived as \([3]\)

\[
\Delta \omega = \omega_f - \omega_i = \frac{L(1) + L(2)}{4[<W_T^{(1)}>_i + <W_T^{(2)}>_i]}
\]  

(44)

where

\[
L(j) = -i \iint_{\Delta S_j} (E_i^{(j)}* \times H_f^*) \cdot \hat{n} \ dS_j, \quad j = 1 \text{ or } 2
\]

(45)

\[
<W_T^{(j)}>_i = \frac{1}{2} \epsilon_i \iint_{V_j} |E_i^{(j)}|^2 \ dV_j
\]

(46)

and the asterisk (*) denotes the complex conjugate. In the above, we denote the initial \( E \) and \( H \) fields and resonant frequency of the \( j \)-th cavity before perturbation by \( E_i^{(j)}, H_i^{(j)} \) and \( \omega_i \), respectively. The final \( E \) and \( H \) fields and resonant frequency after perturbation are denoted by \( E_f, H_f \) and \( \omega_f \), respectively. \(<W_T^{(j)}>_i\) is the time-average total energy stored in the \( j \)-th cavity before perturbation. \( \Delta S_j \) is the surface of the magnetic-wall of the \( j \)-th cavity.
Let us consider the perturbation of the TM₁₀-mode whose resonant frequency of the magnetic-wall cavity is ωᵥα. From the knowledge of the field in the magnetic-wall cavity, the unperturbed electric fields can be written in a general form as

\[
E_{1j}(ρ,φ, j) = \frac{8}{ω₁} \left[ E_{+j}(j) (β_{να} ρ_j/a) e^{iνφ_j} + E_{-j}(j) (β_{να} ρ_j/a) e^{-iνφ_j} \right]
\]

where j = 1,2.

The total time-average energy stored in the j-th cavity can be found as

\[
\langle W_j \rangle_i = \frac{1}{2} \varepsilon_1 \int_{-d}^{0} \int_{0}^{2π} \int \frac{|E_{1j}|^2 ρ_j dφ_j dz}{2ω₁ε₁} = \frac{πd}{2ω₁ε₁} (β_{να}^2 - ν^2) J_{ν}^2(β_{να})
\]

Noting that \(E_1^{(1)}\) has only \(z\)-component, we can write (45) for j = 1 as

\[
L^{(1)} = i \int_{-d}^{0} dz \int_{0}^{2π} dφ_1 E_1^{(1)*} (a, φ_1, z) H_{φ} (a, φ_1, z)
\]

The magnetic field \(H_{φ}\) in the dielectric region can be easily shown to be given by
\[ H_{F} (\rho_{1}, \phi_{1}, z) \big|_{\rho_{1}=a} = \sum_{p=-\infty}^{\infty} \frac{ip}{a} \int_{0}^{\infty} dk_{\rho} \frac{k_{1z}}{k_{o}} h_{p}^{(1)}(k_{\rho}) (1 + R^{TE}) \]

\[ \cdot \frac{\cos k_{1z}(z+d)}{\sin k_{1z}d} \cdot J_{p}(k_{\rho}a) + i\omega \int_{0}^{\infty} dk_{\rho} e_{p}^{(1)}(k_{\rho}) (1 + R^{TM}) \frac{\cos k_{1z}(z+d)}{\cos k_{1z}d} J'_{p}(k_{\rho}a) \]

\[ + \sum_{k=\infty}^{\infty} \frac{ik_{\rho}^{2}}{a} \int_{0}^{\infty} dk_{\rho} \frac{k_{1z}}{k_{o}} h_{p}^{(2)}(k_{\rho}) (1 + R^{TE}) \]

\[ - i\omega \int_{0}^{\infty} dk_{\rho} e_{p}^{(2)}(k_{\rho}) (1 + R^{TM}) \frac{\cos k_{1z}(z+d)}{\cos k_{1z}d} \cdot J_{k}(k_{\rho}c) J_{p+k}(k_{\rho}a) \]

(50)

For small \( d/a \), \( e_{p}^{(j)}(k_{\rho}) \) and \( h_{p}^{(j)}(k_{\rho}) \) can be obtained as

\[ e_{p}^{(j)}(k_{\rho}) = - \frac{k_{\rho}}{2i\omega \beta_{p} a} J_{p}(\beta_{p}a) \frac{J_{p}(k_{\rho}a)}{[(\beta_{p}a)^{2} - k_{p}^{2}]^{2}} E^{(j)}_{p} \quad (51) \]

\[ h_{p}^{(j)}(k_{\rho}) = \frac{p a}{2k_{z}^{2} \beta_{p} a} J_{p}(\beta_{p}a) J_{p}(k_{\rho}a) E^{(j)}_{p} \quad (52) \]

where \( p \) will be limited to the values \( \nu \) and \( -\nu \) and \( j = 1,2 \). Thus
\[
H_T(a, \phi, z) = \frac{1}{2} \sum_{p=-\nu, \nu} e^{i p \phi} \left[ E_p^{(1)} J_p(\beta_p a) \int_0^\infty dk_p \frac{k_{1z}}{k_{c} k_z} (1 + R^{TE}) J_{p+1}(k_p a) \right.
\]
\[
\cdot \left. \frac{\cos k_{1z}(z + d)}{\sin k_{1z} d} + \beta p \int_0^\infty dk_p k_{p} (1 + R^{TM}) \frac{J_p^{(2)}(k_p a)}{[(\beta_p a)^2 - k_p^2]} \frac{\cos k_{1z}(z + d)}{\cos k_{1z} d} \right]
\]
\[
+ \sum_{k=-\infty}^{\infty} E_p^{(2)} e^{i k \phi} \left[ E_p^{(2)} J_p(\beta_p a) \int_0^\infty dk_p \frac{k_{1z}}{k_{p} k_{c} k_z} (1 + R^{TE}) J_k(k_{p} c) \right.
\]
\[
\left. \cdot J_{p+k}(k_{p} a) J_{p+k+1}(k_{p} a) \frac{\cos k_{1z}(z + d)}{\sin k_{1z} d} + \beta p \int_0^\infty dk_p k_{p} (1 + R^{TM}) \right]
\]
\[
\cdot \left. \frac{J_p^{(2)}(k_p a)}{[(\beta_p a)^2 - k_p^2]} J_k(k_{p} c) J_{p+k}(k_{p} a) \frac{\cos k_{1z}(z + d)}{\cos k_{1z} d} \right]. \tag{53}
\]

From (45), (47), and (53) we can get

\[
L(1) = -\pi \frac{\beta^2_{\nu a}}{\omega_{\nu \nu}} J^2(\beta_{\nu a}) \left[ \eta(E^{(1)}_{\nu} E^{(1)}_{\nu}) + \eta_c(E^{(1)}_{\nu} E^{(2)}_{\nu} + E^{(1)}_{\nu} E^{(2)}_{\nu}) \right.
\]
\[
+ \eta_c(E^{(1)}_{\nu} E^{(2)}_{\nu} + E^{(1)}_{\nu} E^{(2)}_{\nu}) \right] \tag{54}
\]

where
\[ n = \frac{i \nu^2}{\beta_{\nu \alpha}^2} \int_0^\infty dk \frac{J_2(k a)}{k^2} \left(1 + R^{TE}\right) + \int_0^\infty dk \frac{J_0(k a)}{k^2} \left(1 + R^{TM}\right) \frac{\tan(k_{1z} d)}{k_{1z}} \]  \hspace{1cm} (55)

\[ n_c = \frac{i \nu^2}{\beta_{\nu \alpha}^2} \int_0^\infty dk \frac{J_2(k a)}{k^2} J_0(k c) \left(1 + R^{TE}\right) + \int_0^\infty dk \frac{J_0(k a)}{k^2} \left(1 + R^{TM}\right) \frac{\tan(k_{1z} d)}{k_{1z}} \]  \hspace{1cm} (56)

\[ \bar{n}_c = -\frac{i \nu^2}{\beta_{\nu \alpha}^2} \int_0^\infty dk \frac{J_2(k a)}{k^2} J_2(k c) \left(1 + R^{TE}\right) + \int_0^\infty dk \frac{J_0(k a)}{k^2} \left(1 + R^{TM}\right) \frac{\tan(k_{1z} d)}{k_{1z}} \]  \hspace{1cm} (57)

Similarly, \( L^{(2)} \) can be obtained as

\[ L^{(2)} = -\frac{\beta_{\nu \alpha}^2}{\omega_1 \nu} J_2(\beta_{\nu \alpha}) \left[ n(E_{\nu} E_{\nu}^*) + n_c(E_{\nu} E_{\nu}^*) + \bar{n}_c(E_{\nu} E_{\nu}^*) \right] \]

\[ \overset{(2)}{\bar{n}_c} \left( E_{\nu} E_{\nu}^* + E_{\nu} E_{\nu}^* \right) \]  \hspace{1cm} (58)

Thus from (44), (48), (54) and (58) we can get

\[ \frac{\Delta \omega}{\omega_{\nu \alpha}} = -\frac{\beta_{\nu \alpha}^2}{2d(\beta_{\nu \alpha}^2 - \nu^2)} \left( n + g_1 n_c + g_2 \bar{n}_c \right) \]  \hspace{1cm} (59)

where
where

\[ g_1 = 2 \frac{E_{1}^{(1)} E_{1}^{(2)} + E_{-1}^{(1)} E_{-1}^{(2)}}{E} \]  \hfill (60)

\[ g_2 = 2 \frac{E_{1}^{(1)} E_{-1}^{(2)} + E_{1}^{(2)} E_{-1}^{(1)}}{E} \]  \hfill (61)

\[ E = \sum_{j=1,2} \left( E_{j1}^{(2)} + E_{j-1}^{(2)} \right) \]  \hfill (62)

Since the single isolated disk resonator can support two orthogonal resonant modes with electric field varying as \( \sin(v\phi) \) or \( \cos(v\phi) \) and both have the same resonant frequency, thus placing two circular disk resonators beside each other as in Fig. 1, there can be 8 different possibilities for the variation of the unperturbed electric fields in the structure formed by the two disks as given in Table I. These 8 possibilities can be obtained by assuming the proper amplitudes of the unperturbed electric field \( E_{1}^{(1)}, E_{-1}^{(1)}, E_{1}^{(2)} \) and \( E_{-1}^{(2)} \). For \( v = 1 \), using (60) and (61) it is easy to show that the only possible nonzero values of both \( (g_1, g_2) \) are \((1,1), (1,-1), (-1,-1) \) and \((-1,1) \) and thus there exist four possible resonant modes for the coupled structure. The mode when \( (g_1, g_2) \equiv (1,1) \), corresponds to unperturbed electric fields varying as \( \sin(\phi_1) \) in cavity (1) and as \( \sin(\phi_2) \) in cavity (2). Such mode will be called odd symmetric mode (os). The mode when \( (g_1, g_2) \equiv (1,-1) \) corresponds to electric fields in cavity (1) and cavity (2) varying respectively as \( \cos(\phi_1) \) and \( \cos(\phi_2) \).
TABLE I

<table>
<thead>
<tr>
<th>φ-variation of $E_{z_1}$ in cavity 1</th>
<th>φ-variation of $E_{z_2}$ in cavity 2</th>
<th>The structure formed by the two cavities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \phi_1$</td>
<td>$\sin \phi_2$</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$-\sin \phi_1$</td>
<td>$\uparrow$</td>
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<tr>
<td>$\cos \phi_1$</td>
<td>$\cos \phi_2$</td>
<td>$\rightarrow$</td>
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<td>$-\cos \phi_1$</td>
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<td>$\cos \phi_1$</td>
<td>$\sin \phi_2$</td>
<td>$\rightarrow$</td>
</tr>
<tr>
<td>$-\sin \phi_1$</td>
<td>$\rightarrow$</td>
<td></td>
</tr>
</tbody>
</table>

Table I. The eight possibilities for the unperturbed electric fields in the structure formed by the two cavities. The arrows indicate the direction of the maximum of the unperturbed electric fields $E_z^{(1)}$ and $E_z^{(2)}$ for $\nu = 1$. 
and is called even symmetric mode (es). Similarly, \((g_1, g_2) \equiv (-1, -1)\) corresponds to electric fields varying as \(\sin(\phi_1)\) in cavity (1) and 
\(-\sin(\phi_2)\) in cavity (2) and is called odd antisymmetric (oa). Finally when \((g_1, g_2) \equiv (-1, 1)\) we have the case of even antisymmetric mode (ea) 
corresponding to unperturbed electric field varying as \(\cos(\phi_1)\) and 
\(-\cos(\phi_2)\) in cavity (1) and (2), respectively. It can be shown that these 
four resonant modes are the only modes which are orthogonal to each other. 
The other four possibilities of Table I, give \((g_1, g_2) \equiv (0, 0)\) and thus give the resonant frequency of a single isolated disk which means that 
fields varying as \(\sin(\phi_1)\) or \(\cos(\phi_1)\) in cavity (1) do not couple with 
fields varying as \(\pm \cos(\phi_2)\) or \(\pm \sin(\phi_2)\) in cavity (2), respectively.

Thus, the resonant frequency shift for the four modes of the 
coupled structure, can be written, in general, as

\[
\begin{align*}
\left\{ \frac{\Delta \omega}{\omega} \right\}_{es} &= \beta(n + \eta_c + (-1)^{\nu+1} \pi_c) \\
\left\{ \frac{\Delta \omega}{\omega} \right\}_{oa} &= \beta(n + \eta_c - (-1)^{\nu+1} \pi_c) \\
\left\{ \frac{\Delta \omega}{\omega} \right\}_{es} &= \beta(n - \eta_c + (-1)^{\nu+1} \pi_c) \\
\left\{ \frac{\Delta \omega}{\omega} \right\}_{oa} &= \beta(n - \eta_c - (-1)^{\nu+1} \pi_c)
\end{align*}
\]
where

\[ \beta = - \frac{\beta_{\nu\alpha}^2}{2d(\beta_{\nu\alpha}^2 - \nu^2)} \]  \hspace{1cm} (67)
V. The Perturbation Formula as a Zeroth-Order Theory

In this section, it is shown that the perturbation formulae (63)-(66) for the resonant frequency shift can also be derived from Galerkin's equations in the limiting case when $d \to 0$.

Assuming the currents $\overline{R}_n^{(1)}$ and $\overline{R}_n^{(2)}$ in (28) and (29) to be expressed in terms of TM-modes only, equations (34a)-(34d) reduce to

$$\sum_{r=1}^{R} a_n^{(1)} \psi_1 \psi_1 + \sum_{m=-M}^{M} \sum_{q=1}^{Q} a_{nq}^{(2)} A_{nm}(f_q) = 0, \quad f = 1, \ldots, R$$
$$n = -M, \ldots, M \quad (68a)$$

$$\sum_{q=1}^{Q} a_n^{(2)} \psi_2 \psi_2 + \sum_{m=-M}^{M} \sum_{r=1}^{R} a_{nr}^{(1)} A_{nm}(t_r) = 0, \quad t = 1, \ldots, Q$$
$$n = -M, \ldots, M \quad (68b)$$

Let us assume that we are interested in the resonant frequency shift as a perturbation from the resonant frequency $\omega_{\nu\alpha}$ of the magnetic-wall cavity. In such a case, the only coefficients having considerable contribution are $a_{\nu\alpha}^{(1)}$, $a_{-\nu\alpha}^{(1)}$, $a_{\nu\alpha}^{(2)}$ and $a_{-\nu\alpha}^{(2)}$. Thus in (68a) and (68b) both $n$ and $m$ will be limited to the values $-\nu$ and $\nu$ and $f$, $t$, $r$ and $q$ will equal $\alpha$. So, we get a system of four algebraic equations which can be put in the form

$$\begin{bmatrix}
A & 0 & A_c & \bar{A}_c \\
0 & A & \bar{A}_c & A_c \\
A_c & \bar{A}_c & A & 0 \\
\bar{A}_c & A_c & 0 & A
\end{bmatrix}
\begin{bmatrix}
a_{\nu\alpha}^{(1)} \\
a_{-\nu\alpha}^{(1)} \\
a_{\nu\alpha}^{(2)} \\
a_{-\nu\alpha}^{(2)}
\end{bmatrix}
= 0,$$  \hspace{1cm} (69)
where

\[ A = \frac{k^2v^2a^2}{\beta_{v\alpha}^4} \int_0^\infty dk_\rho \frac{J_0^2(k_\rho a)}{k_\rho^2 k_z} (1 + R_{TE}) + \int_0^\infty dk_\rho k_\rho k_z \frac{J_0^1(k_\rho a)}{[(\beta_{v\alpha}/a)^2 - k_\rho^2]^2} (1 - R_{TM}) \]  

(70)

\[ A_c = \frac{k^2v^2a^2}{\beta_{v\alpha}^4} \int_0^\infty dk_\rho \frac{J_0^2(k_\rho a)}{k_\rho^2 k_z} J_0(k_\rho c)(1 + R_{TE}) + \int_0^\infty dk_\rho k_\rho k_z \frac{J_0^1(k_\rho a)}{[(\beta_{v\alpha}/a)^2 - k_\rho^2]^2} \cdot J_0(k_\rho c)(1 - R_{TM}) \]  

(71)

\[ \bar{A}_c = \frac{-k^2v^2a^2}{\beta_{v\alpha}^4} \int_0^\infty dk_\rho \frac{J_0^2(k_\rho a)}{k_\rho^2 k_z} J_2(k_\rho c)(1 + R_{TE}) + \int_0^\infty dk_\rho k_\rho k_z \frac{J_0^1(k_\rho a)}{[(\beta_{v\alpha}/a)^2 - k_\rho^2]^2} \cdot J_2(k_\rho c)(1 - R_{TM}) . \]  

(72)

After some algebraic manipulations (70) can be written as

\[ A = -i \frac{\varepsilon}{\varepsilon_1} \left[ \eta + \frac{1}{a^2} (k^2a^2 - \beta_{v\alpha}^2) \int_0^\infty dk_\rho \frac{J_0^2(k_\rho a)}{k_\rho^2 k_z} (1 + R_{TE}) \right. \]

\[ \left. + \int_0^\infty dk_\rho k_\rho \frac{J_0^1(k_\rho a)}{[(\beta_{v\alpha}/a)^2 - k_\rho^2]^2} (1 + R_{TM}) \tan(k_{1z}d) \frac{\tan(k_{1z}d)}{k_{1z}} \right] \]  

(73)

where \( \eta \) is given by (55). When \( d \to 0 \), \( (1 + R_{TE}) = -2ik_zd \), \( (1 + R_{TM}) = 2 \), \( \tan(k_{1z}d) = k_{1z}d \) and the last two integrals in (73) can be evaluated exactly leading to
\[ A = -i \frac{e}{\epsilon_1} \left[ n + 2(k_1 a - \beta_{\nu \alpha}) \frac{d(\beta_{\nu \alpha}^2 - \nu^2)}{\beta_{\nu \alpha}^3} \right]. \] (74)

Similarly, (71) and (72) can be put in the form

\[ A_c = -i \frac{e}{\epsilon_1} \left[ n_c + \frac{1}{a^2} (k_1^2 a^2 - \beta_{\nu \alpha}^2) \right] \left\{ \int_0^\infty dk_\rho \frac{J^2(k_\rho a)}{k_\rho k_z} J_0(k_\rho c)(1 + RTE) \right. \]

\[ + \left. \int_0^\infty dk_\rho k_\rho \frac{J^2(k_\rho a)}{[(\beta_{\nu \alpha}/a)^2 - k_\rho^2]^2} J_0(k_\rho c)(1 + RTM) \frac{\tan(k_1 z d)}{k_1 z} \right\} \] (75)

\[ \bar{A}_c = -i \frac{e}{\epsilon_1} \left[ \bar{n}_c + \frac{1}{a^2} (k_1^2 a^2 - \beta_{\nu \alpha}^2) \right] \left\{ -\frac{iv^2 a^2}{\beta_{\nu \alpha}^4} \int_0^\infty dk_\rho \frac{J^2(k_\rho a)}{k_\rho k_z} J_2(k_\rho c)(1 + RTE) \right. \]

\[ + \left. \int_0^\infty dk_\rho k_\rho \frac{J^2(k_\rho a)}{[(\beta_{\nu \alpha}/a)^2 - k_\rho^2]^2} J_2(k_\rho c)(1 + RTM) \frac{\tan(k_1 z d)}{k_1 z} \right\} \] (76)

where \( n_c \) and \( \bar{n}_c \) are given by (56) and (57), respectively. As \( d \to 0 \), the last two integrals in (75) and (76) reduce to zero. Thus

\[ A_c = -i \frac{e}{\epsilon_1} n_c \] (77)

\[ \bar{A}_c = -i \frac{e}{\epsilon_1} \bar{n}_c. \] (78)
For nontrivial solution of $a_{-\nu\alpha}^{(1)}$, $a_{\nu\alpha}^{(1)}$, $a_{-\nu\alpha}^{(2)}$ and $a_{\nu\alpha}^{(2)}$, we require $\det|\bar{A}| = 0$, where $\bar{A}$ is the matrix in (69). In this case, the determinant can be factorized as

$$\det|\bar{A}| = 0 = (A + A_c + \bar{A}_c)(A + A_c - \bar{A}_c)(A - A_c + \bar{A}_c)(A - A_c - \bar{A}_c). \quad (79)$$

By equating each factor of (79) to zero, we get the four resonant modes of the two coupled microstrip disk resonators as given by the perturbative approach in (63)-(66). For example, using (74) the factor $A + n_c + \bar{n}_c = 0$ can be written as

$$\left(\frac{k_1 a - \beta_{\nu\alpha}}{\beta_{\nu\alpha}}\right) \frac{2d(\beta_{\nu\alpha}^2 - \nu^2)}{\beta_{\nu\alpha}^2} = -(n + n_c + \bar{n}_c). \quad (80)$$

Thus,

$$\left(\frac{\Delta \omega}{\omega_{\nu\alpha}}\right) = -\frac{\beta_{\nu\alpha}^2}{2d(\beta_{\nu\alpha}^2 - \nu^2)} (n + n_c + \bar{n}_c) \quad (81)$$

which is one of the expressions obtained in Section IV, for the resonant frequency shift.
VI. Evaluation of the Resonant Frequencies Using the Matched Asymptotic Expansion Approach

The use of the matched asymptotic expansion approach in solving boundary value problems is well explained in detail in [26]. In such an approach, the range over which a boundary value problem is defined, is divided into a number of overlapping subranges or subintervals. On each subinterval, perturbation methods are used to obtain an asymptotic approximation to the solution of the problem valid on that interval. Finally, the matching is done by requiring that the asymptotic approximations have the same functional form on the overlap of every pair of subintervals.

In the analysis of resonance of microstrip structure this method was used in [24] and [25] to develop an asymptotic formula for the resonant frequencies of a circular and an annular ring microstrip antenna. In these microstrip structures, the space around the open resonator is divided into essentially three regions, where the field varies differently in each of these regions. These regions are the edge, the interior and the exterior regions. Approximate solutions are then obtained in each of these regions. Leading order solutions are then obtained in these regions in the limit of small dielectric substrate thickness. Finally, these approximate solutions are then joined together by matching their leading order terms to, ultimately, obtain an asymptotic eigenequation for the resonant frequencies. This eigenequation is then solved approximately.
In carrying out the asymptotic expansions we will only keep track of terms of the order of \( \delta = d/a \).

In the case of the two coupled circular disks, the space is divided into five regions, two interior regions, two edge regions and an exterior region.

A. The Edge Regions

To emphasize the edge regions around the two circular disks (\( D_1 \) and \( D_2 \)), the following coordinate transformation is introduced

\[
\rho_j = a(1 + \delta x_j), \quad \phi_j = \phi_j, \quad z = a\delta Z, \quad j = 1, 2, \quad \delta = d/a
\]

where \( x_j, \phi_j \) and \( Z \) are the local edge coordinates of disk \( D_j \) and where both disks are assumed to have the same radius \( a \).

From the results of reference [24], and in the limit \( \delta \to 0 \), the leading order interior expansion of the edge solution is given by

\[
h_{1\phi}^{(j)} = n \sum_{n} e^{i n \phi \frac{n}{a} \left( k_0^2 a^2 - n^2 \right) \delta x_j + C_n^{(j)}} \quad \text{as} \quad x_j \to -\infty
\]

where \( h_{1\phi}^{(j)} \) is the \( \phi \)-component of the magnetic field of the \( j \)-th disk in the substrate region, \( B_n^{(j)} \) and \( C_n^{(j)} \) are unknowns that will be determined through asymptotic matching.

The leading exterior expansion of the edge solution is given by

\[
h_{0\phi}^{(j)} = n \sum_{n} e^{i n \phi \frac{n}{a} \left( k_0^2 a^2 - n^2 \right) \ln \left( x_j^2 + Z^2 \right) + k_0^2 a^2 \frac{A - n^2 \frac{1}{2} (n+1)}{Z} + C_n^{(j)}}
\]
where \( h^{(j)}_{\phi} \) is the \( \phi \)-component of the magnetic field of the \( j \)-th disk in the exterior region and

\[
A = -2\varepsilon_r \sum_{m=1}^{\infty} \left( \frac{1 - \varepsilon_r}{1 + \varepsilon_r} \right)^m \ln(m) + \varepsilon_r \ln(\pi) + (\varepsilon_r - 1) \ln(2) + 1
\]

and

\[
\varepsilon_r = \varepsilon_1/\varepsilon_0.
\]

### B. The Interior Regions

This region is emphasized by the coordinate transformation

\[
\rho_j = \rho_j \quad \quad \phi_j = \phi_j \quad \quad z = a\delta Z
\]

Since we are interested in the leading order solution, then in the limit of \( \delta \to 0 \), we observe the following.

1. In this limit, the circular disk structure approaches that of the magnetic wall cavity model and thus the coupling between the two disks disappear.

2. The \( z \)-variation of the fields is negligible.

The solution which reflects these two properties is given by the TM mode.

\[
e_{1Z}^{(j)} = \sum_n e^{in\phi_j} E_n^{(j)} J_n(k_{lj}\rho_j)
\]

and therefore
\[ h_{1\phi} = \frac{ik_1}{\omega \mu} \sum_n e^{in\phi} E_n(j) J_n(k_1\rho_j) \]

where \( E_n(j) \) is an unknown.

The edge expansion of this interior solution is obtained by setting \( \rho_j = a(1 + \delta x_j) \) and making \( \delta \to 0 \)

\[ h_{1\phi} = \frac{ik_1}{\omega \mu} \sum_n e^{in\phi} E_n(j) \left[ J_n(k_1a) + \delta x_j \frac{(n^2 - k_1^2) a^2}{k_1 a} J_n(k_1a) \right] \]

C. The Exterior Region

For \( z > 0 \), it can be easily shown that the \( \phi \)-component of the magnetic field is given, alternatively, by either one of the following two expressions:

\[ H_\phi = -\frac{1}{2} \sum_n e^{in\phi} \left\{ \int_0^\infty dk_\rho \frac{1}{R_{\rho}} (k_\rho) (1-R_{TM}) e^{ik_\rho z} J_n(k_\rho \rho_1) \right\} \left\{ \int_0^\infty dk_\rho \frac{1}{R_{\rho}} (k_\rho) (1+R_{TE}) e^{ik_\rho z} J_n(k_\rho \rho_1) \right\} \]

\[ + \sum_r \left\{ \int_0^\infty dk_\rho \frac{1}{R_{\rho}} J_{n-r}(k_\rho \rho_1) K_\rho^{(1)}(k_\rho) (1-R_{TM}) e^{ik_\rho z} J_n(k_\rho \rho_1) \right\} \left\{ \int_0^\infty dk_\rho \frac{1}{R_{\rho}} J_{n-r}(k_\rho \rho_1) K_\rho^{(2)}(k_\rho) (1+R_{TE}) e^{ik_\rho z} J_n(k_\rho \rho_1) \right\} \]

\[ + \sum_r \left\{ \int_0^\infty dk_\rho \frac{1}{R_{\rho}} J_{n-r}(k_\rho \rho_1) K_\rho^{(1)}(k_\rho) (1-R_{TM}) e^{ik_\rho z} J_n(k_\rho \rho_1) \right\} \left\{ \int_0^\infty dk_\rho \frac{1}{R_{\rho}} J_{n-r}(k_\rho \rho_1) K_\rho^{(2)}(k_\rho) (1+R_{TE}) e^{ik_\rho z} J_n(k_\rho \rho_1) \right\} \]
which is the expression for $H_\phi$ referred to $O_1$ whereas that referred to $O_2$ is given by

$$H_\phi = -\frac{1}{2} \sum_n e^{in\phi_2} \left\{ \int_0^\infty dk \kappa(k) (1-R_{TM}) e^{ikz} J_n(k) \right\}$$

$$+ \frac{n}{\rho_2} \int_0^\infty dk_\phi \kappa(k) (1+R_{TE}) e^{ikz} J_n(k)$$

$$+ \sum_r (-1)^{n-r} \left\{ \int_0^\infty dk_\rho \kappa(k) (1-R_{TM}) e^{ikz} J_n(k) \right\}$$

In obtaining the first expression, the following addition theorem has been used

$$J_m(k_\rho) e^{im\phi_2} = \sum_n J_{n-m}(k_\rho) J_n(k_\rho) e^{in\phi_1}$$

and in obtaining the second expression, the following addition theorem has been used

$$J_m(k_\rho) e^{im\phi_1} = \sum_n (-1)^{n-m} J_{n-m}(k_\rho) J_n(k_\rho) e^{in\phi_2}$$
In the small $\varepsilon$ limit, only modal fields with no $z$-variations inside the substrate will be excited and since all the TE modes have $z$-variation, none of them will be excited, therefore the currents on the disks are thus expanded in terms of the TM modes only.

Therefore

\[ k_{p}^{(j)}(k_{p}) = A_n^{(j)} \beta_{nm} J_n(\beta_{nm}) \frac{J'_n(k_{p}a)}{\left(\frac{\beta_{nm}}{a}\right)^2 - k_{p}^2} \]

and

\[ k_{p}^{(j)}(k_{p}) = A_n^{(j)} \frac{na}{\beta_{nm} k_{p}} J_n(\beta_{nm}) J_n(k_{p}a) \]

Also in this limit as $\varepsilon \to 0$, we have

\[ (1 - R^{TM}) = -i2d \frac{\varepsilon}{\varepsilon_1} \frac{1}{k_{z}} k_{z}^2 - i2d \frac{\varepsilon}{\varepsilon_1} \frac{1}{k_{z}} \left(\frac{\beta_{nm}}{a}\right)^2 - k_{p}^2 \]

and

\[ (1 + R^{TE}) = -i2d k_{z} \]

and since we are only interested in the $TM_{nm}$ mode.

Therefore $H_{\phi}$ can be shown to be given by

\[ H_{\phi} = i \delta a^2 \frac{J_n(\beta_{nm})}{\beta_{nm}} \left[ e^{i n_{1}f} \left\{ A_n^{(1)} n'(\rho_1) + A_n^{(2)} n_c'(\rho_1) + A_n^{(2)} n_c'(\rho_1) \right\} + (-1)^n e^{i n_{1}f} \left\{ A_{-n}^{(1)} n'(\rho_1) + A_{-n}^{(2)} n_c'(\rho_1) + A_{-n}^{(2)} n_c'(\rho_1) \right\} \right] \]
or alternatively

\[
H_\phi = i\delta a^2 \frac{J_n(\beta_{nm})}{\beta_{nm}} \left[ e^{i\phi_2} \left\{ A_n^{(2)} n'(\rho_2) + A_n^{(1)} n'_c(\rho_2) + A_{-n}^{(1)} \overline{n}'_c(\rho_2) \right\} 
+ (-1)^n e^{-i\phi_2} \left\{ A_{-n}^{(2)} n'(\rho_2) + A_n^{(1)} \overline{n}'_c(\rho_2) + A_{-n}^{(1)} n'_c(\rho_2) \right\} \right]
\]

where

\[
n'(\rho_j) = \int_0^\infty dk_p k_p \left[ \frac{n^2 k_z}{k_{p \rho_j}^2} J_n(k_p a) J_n(k_{p \rho_j}) + \frac{k_0^2 a}{k_z} J'_n(k_p a) J'_n(k_{p \rho_j}) \right] e^{ik_z z} \]

\[
n'_c(\rho_j) = \int_0^\infty dk_p k_p J_0(k_p c) \left[ \frac{n^2 k_z}{k_{p \rho_j}^2} J_n(k_p a) J_n(k_{p \rho_j}) + \gamma \frac{k_0^2 a}{k_z} J'_n(k_p a) J'_n(k_{p \rho_j}) \right] e^{ik_z z} \]

\[
\overline{n}'_c(\rho_j) = \int_0^\infty dk_p k_p J_2n(k_p c) \left[ -\alpha \frac{n^2 k_z}{k_{p \rho_j}^2} J_n(k_p a) J_n(k_{p \rho_j}) + \gamma \frac{k_0^2 a}{k_z} J'_n(k_p a) J'_n(k_{p \rho_j}) \right] e^{ik_z z} \]

where

\[
\alpha = \frac{(1 + R^{TE})}{(-i2d)} \quad \text{and} \quad \gamma = \frac{(1 - R^{TM})}{-i2d} \left( \frac{\beta_{nm}}{a} \right)^2 \frac{1}{k_z} \left[ \left( \frac{\beta_{nm}}{a} \right)^2 - k_{p \rho_j}^2 \right] \]

Notice that as \( \delta \to 0, \alpha \to 1 \) and \( \gamma \to 1 \). To get the edge expansion of the exterior solution, substitute \( \rho_j = a(1 + \delta x_j), z = a\delta Z \) and let \( \delta \to 0 \).
Therefore

\[
h^{(1)}_{\phi \phi} = i \delta a^2 \frac{J_n(\beta_{nm})}{\beta_{nm}} e^{i \phi_1} \left\{ A_n(1) \eta_1 + A_n(2) \eta_c + A_{-n}(2) \bar{\eta}_c \right\} \\
+ (-1)^n e^{-i \phi_1} \left\{ A_{-n}(1) \eta_1 + A_{-n}(2) \bar{\eta}_c + A_{-n}(2) \eta_c \right\}.
\]

or

\[
h^{(2)}_{\phi \phi} = i \delta a^2 \frac{J_n(\beta_{nm})}{\beta_{nm}} e^{i \phi_2} \left\{ A_n(2) \eta_2 + A_n(1) \eta_c + A_{-n}(1) \bar{\eta}_c \right\} \\
+ (-1)^n e^{-i \phi_2} \left\{ A_{-n}(2) \eta_2 + A_{-n}(1) \bar{\eta}_c + A_{-n}(1) \eta_c \right\}.
\]

where \( \eta_j \) is given in reference [24]

\[
\eta_j = \frac{i}{\pi a^2} \left[ (k_0^2 a^2 - n) \left\{ \ln \left( \frac{\delta}{\beta} \right) + \ln(x_j^2 + z^2)^{3/2} + 2 \sum_{k=1}^{n} \frac{1}{2k-1} \right\} - \frac{2k_0^2 a^2}{4n^2 - 1} \right] \\
+ (-1)^n \int_0^{\pi/2} d\psi \cos(2n\psi) \frac{\exp(ik_0 a \cos \psi) - 1}{\cos \psi} \left( n^2 + k_0^2 a^2 \cos 2\psi \right)
\]

and \( \eta_c \) and \( \bar{\eta}_c \) are given by
\[
\eta_c = \int_0^\infty dk \rho J_0(k, \rho) \left[ \alpha \frac{n^2 k_z}{k_\rho^2 a} J_2(k, \rho, a) + \gamma \frac{k_\rho^2 a}{k_z} J_1^2(k, \rho, a) \right]
\]

\[
\overline{\eta_c} = \int_0^\infty dk \rho J_2(k, \rho) \left[ -\alpha \frac{n^2 k_z}{k_\rho^2 a} J_2(k, \rho, a) + \gamma \frac{k_\rho^2 a}{k_z} J_1^2(k, \rho, a) \right]
\]

In the limit when \( \delta \to 0 \), the leading order terms in the expansions of \( \eta_c \) and \( \overline{\eta_c} \) are independent of \( \delta \) (i.e. nonzero quantities).

Therefore, in summary we get the following for the TM\(_{nm}\) mode.

A) The edge expansion of the interior solution

\[
h^{(j)}_{1\phi} = \frac{ik_1}{\omega} e^{i \phi} \left[ E_n \left\{ J_1'(k_1 a) + \delta x_j \frac{n^2 - k_1^2 a^2}{k_1 a} J_n(k_1 a) \right\} \right.
\]

\[
+ \left. e^{-i \phi} E_n \left\{ J_1'(k_1 a) + \delta x_j \frac{n^2 - k_1^2 a^2}{k_1 a} J_n(k_1 a) \right\} \right] \tag{82}
\]

B) The interior expansion of the edge solution

\[
h^{(j)}_{1\phi} = e^{i \phi} \left\{ \frac{B(j)}{n} (k_1^2 a^2 - n^2) \delta x_j + C_n^{(j)} \right\} + e^{-i \phi} \left\{ -i \frac{B(j)}{n} (k_1^2 a^2 - n^2) \delta x_j + C_n^{(j)} \right\} \tag{83}\]
C) The exterior expansion of the edge solution

\[ h_{\phi}(j) \sim e^{\pm i \phi_{\phi}} \left\{ \frac{B_{\phi}(j)}{n} \frac{\delta}{\pi} \left\{ \left( k_{0} a_{j}^{2} - n^{2} \right) \ln(x_{j}^{2} + Z_{\phi}^{2})^{3/2} + S_{n} \right\} + C_{n}(j) \right\} \]

where

\[ S_{n} = k_{0}^{2} a_{j}^{2} \frac{A}{2} - \frac{n^{2}}{2} (\ln(\pi) + 1) \]

D) The edge expansion of the exterior solution

\[ h_{\phi}(j) \sim i \delta a_{j} \frac{J_{n}(\beta_{nm})}{\beta_{nm}} \left[ e^{\pm i \phi_{\phi}} \left\{ A_{n}(j) \eta_{j} + A_{n}(j') \eta_{c} + A_{-n}(j') \eta_{c} \right\} \right. \]

\[ + \left. (-1)^{n} e^{\pm i \phi_{\phi}} \left\{ A_{-n}(j) \eta_{j} + A_{n}(j') \eta_{c} + A_{-n}(j') \eta_{c} \right\} \right] \]

where \( j' = 1 \) if \( j = 2 \) and \( j' = 2 \) if \( j = 1 \). Comparing (82) with (83) we get

\[ B_{\phi}(j) = - \frac{n}{\omega a_{j}} J_{n}(k_{1} a_{j}) E_{n}(j) \]

\[ C_{n}(j) = \frac{ik_{1}}{\omega a_{j}} J_{n}^{*}(k_{1} a_{j}) E_{n}(j) \]
Comparing the coefficients multiplying $\delta(k_o^2a^2 - n^2) \ln(x_j^2 + z^2)^{\mp}$ in equations (84) and (85) we get

$$A_n(j) = -i \frac{\beta_{nm}}{n} \frac{1}{J_n(\beta_{nm})} B_n(j)$$

and from (86) we get

$$A_n(j) = i \frac{\beta_{nm}}{J_n(\beta_{nm})} \frac{J_n(k_1a)}{\omega \mu a} E_n(j)$$

Equating the remaining terms in equations (84) and (85) we get the eigenequation for the resonant frequencies as follows

$$k_1a \ J_n(k_1a) = \frac{\delta}{n} J_n(k_1a) \left[ S_n + i \pi a^2 \left\{ n + \frac{E_n(j')}{E_n(j)} n_c + \frac{E_n(j')}{E_n(j)} \ n_c \right\} \right]$$

where

$$n = \frac{i}{\pi a^2} \left[ (k_o^2a^2 - n^2) \left\{ \ln \left( \frac{\delta}{8} \right) + 2 \sum_{k=1}^{n} \frac{1}{2k - 1} \right\} - \frac{2k_o^2a^2}{4n^2 - 1} \right]$$

$$+ (-1)^n \int_0^{\pi/2} d\psi \cos(2n\psi) \frac{\exp(i2k_0a \cos \psi) - 1}{\cos \psi} (n^2 + k_o^2a^2 \cos 2\psi)$$

It is clear from equation (87), that we have two eigenequations; one corresponds to $j = 1$ ($j' = 2$) which is the eigenequation resulting from carrying out the asymptotic matching process at the edge of disk $D_1$ and
the other eigenequation corresponds to \( j = 2 \) \((j' = 1)\) which is the equation resulting from carrying out the asymptotic matching process at the edge of disk \( D_2 \).

From the symmetry of the problem, these two eigenequations should be identical. This results in the following two conditions on the mode amplitudes

\[
\frac{E_n^{(2)}}{E_n^{(1)}} = \frac{E_{-n}^{(2)}}{E_{-n}^{(1)}} \\
\frac{E_n^{(2)}}{E_n^{(1)}} = \frac{E_{-n}^{(2)}}{E_{-n}^{(1)}}
\]

Recalling that the \( n \)-th mode on disk \( D_1 \) is given by

\[
E_z^{(1)} = [E_n^{(1)} e^{i \phi_1} + (-1)^n E_{-n}^{(1)} e^{-i \phi_1}] J_n(\beta nm \frac{\omega}{a})
\]

and on disk \( D_2 \) is given by

\[
E_z^{(2)} = [E_n^{(2)} e^{i \phi_2} + (-1)^n E_{-n}^{(2)} e^{-i \phi_2}] J_n(\beta nm \frac{\omega}{a})
\]

The only possible mode amplitude allowed by conditions (88) together with the corresponding \( \phi \)-variation of \( E_z \) on both disks are given in the following table.
\[ E_n^{(1)} \] is taken as a normalization factor, i.e. \[ E_n^{(1)} \] is set equal to unity.

These four cases give the four possible modes excited by the two coupled circular disk structures. The first mode is the odd-symmetric (OS) mode, the second is the odd-antisymmetric (OA), the third is the even-symmetric (ES) and the fourth is the even-antisymmetric (EA).

Thus the resonant frequencies of the four possible modes in the coupled structure are given by the following four equations.

\[ k_1 a \ J'_n(k_1 a) = \frac{\delta}{\pi} \ J_n(k_1 a) \left[ S_n + i\pi a^2 \left( n + \eta_c - (-1)^n \frac{\eta_c}{n_c} \right) \right] \quad (89a) \]

for the odd-symmetric modes

\[ k_1 a \ J'_n(k_1 a) = \frac{\delta}{\pi} \ J_n(k_1 a) \left[ S_n + i\pi a^2 \left( n - \eta_c + (-1)^n \frac{\eta_c}{n_c} \right) \right] \quad (89b) \]

for the odd-antisymmetric modes

<table>
<thead>
<tr>
<th>( E_n^{(1)} )</th>
<th>( E_{-n}^{(1)} )</th>
<th>( \phi)-variation of ( E_z ) in cavity (1)</th>
<th>( E_n^{(2)} )</th>
<th>( E_{-n}^{(2)} )</th>
<th>( \phi)-variation of ( E_z ) in cavity (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-1)^n\</td>
<td>(\sin (n\phi_1))</td>
<td>1</td>
<td>(-1)^n\</td>
<td>(\sin (n\phi_2))</td>
</tr>
<tr>
<td>1</td>
<td>(-1)^n\</td>
<td>(\sin (n\phi_1))</td>
<td>-1</td>
<td>(-1)^n\</td>
<td>(-\sin (n\phi_2))</td>
</tr>
<tr>
<td>1</td>
<td>(-1)^n\</td>
<td>(\cos (n\phi_1))</td>
<td>1</td>
<td>(-1)^n\</td>
<td>(\cos (n\phi_2))</td>
</tr>
<tr>
<td>1</td>
<td>(-1)^n\</td>
<td>(\cos (n\phi_1))</td>
<td>-1</td>
<td>(-1)^n\</td>
<td>(-\cos (n\phi_2))</td>
</tr>
</tbody>
</table>
\[ k_1 a \, J_n'(k_1 a) = \frac{\delta}{\pi} \, J_n(k_1 a) \left[ S_n + i \pi a^2 \left\{ n + \eta_c + (-1)^n \frac{n}{\eta_c} \right\} \right] \] (89c)

for the even-symmetric modes, and finally

\[ k_1 a \, J_n'(k_1 a) = \frac{\delta}{\pi} \, J_n(k_1 a) \left[ S_n + i \pi a^2 \left\{ n - \eta_c - (-1)^n \frac{n}{\eta_c} \right\} \right] \] (89d)

for the even-antisymmetric modes.

Notice that when \( \delta \rightarrow 0 \), the eigenequations reduce to

\[ J_n'(k_1 a) = 0 \quad \text{or} \quad k_1 a = \beta_{nm} \]

which are the resonant frequencies of the magnetic wall cavity. Also notice that when \( c \rightarrow \infty \), \( \eta_c \) and \( \eta_c \) vanish and the eigenequations reduce to that of the single disk [24].

The eigenequations given by (89) are nonlinear equations in \( \omega \). To simplify these equations we do the following [24]:

1. Substitute \( k_1 a \) by \( \beta_{nm} \) on the left hand side of equations (89).
2. Replace \( k_1 a \, J_n'(k_1 a) \) by \( -(k_1 a - \beta_{nm}) \left\{ \frac{\beta_{nm}^2 - n^2}{\beta_{nm}} \right\} J_n(\beta_{nm}) \).

Thus the eigenequations simplify to the following equations:

\[ k_1 a - \beta_{nm} \left[ 1 - \frac{\delta}{\pi(\beta_{nm}^2 - n^2)} \left\{ S_n + i \pi a^2 \left\{ n + \eta_c - (-1)^n \frac{n}{\eta_c} \right\} \right\} \right] \]
for the odd-symmetric modes

\[ k_{1}a = \beta_{nm} \left[ 1 - \frac{\delta}{\pi(\beta_{nm}^{2} - n^{2})} \left\{ S_{n} + i\pi a^{2}[n - n_{c} + (-1)^{n} \frac{n}{n_{c}}] \right\} \right] \]

for the odd-antisymmetric modes

\[ k_{1}a = \beta_{nm} \left[ 1 - \frac{\delta}{\pi(\beta_{nm}^{2} - n^{2})} \left\{ S_{n} + i\pi a^{2}[n + n_{c} + (-1)^{n} \frac{n}{n_{c}}] \right\} \right] \]

for the even-symmetric modes, and finally

\[ k_{1}a = \beta_{nm} \left[ 1 - \frac{\delta}{\pi(\beta_{nm}^{2} - n^{2})} \left\{ S_{n} + i\pi a^{2}[n - n_{c} + (-1)^{n} \frac{n}{n_{c}}] \right\} \right] \]

for the even-antisymmetric modes.
VII. Results and Conclusions

A rigorous full-wave analysis of the two coupled circular microstrip disk resonators is presented using Vector Hankel Transform. The resonance in the coupled structure is analyzed using three different approaches: Galerkin's moment method (GM), perturbative approach (PA) and the matched asymptotic expansion approach (MA). The formulae obtained for the resonance frequencies are shown to account for the coupling effects between the two disks. It is also shown that the structure can support four different resonant modes: odd-symmetric (os), even-symmetric (es), odd-antisymmetric (oa) and even-antisymmetric (ea).

For a structure having two disks of equal radii \( a_1 = a_2 = a \), Figs. 2a-9a show the real part of the resonant frequency of the four different resonant modes of the TM_{11} whereas Figs. 2b-9b show the imaginary part as a function of the substrate thickness \( d/a \) for different values of separation \( c/a \). Figures 2, 5, and 8 are for a separation of \( c = 2.05a \) between the two disks, Figs. 3 and 6 are for \( c = 2.1a \) and Figs. 4, 7 and 9 are for \( c = 2.25a \).

On the same figures, the resonant frequency of the single disk (sd) is plotted for the sake of comparison.

Figures 2-4 show the results obtained by using the perturbative approach (PA) for the resonant frequencies. It is shown that the four different resonant modes approach that of the single disk (sd) as the separation between the two disks increases.

Figures 5-7 show the results obtained by using the matched asymptotic expansion approach (MA) for the resonant frequencies of the different
Figure 2a. Real part of the resonant frequencies of the different modes of the TM11 as a function of d/a using the perturbative approach (PA), c = 2.05a, ε1 = 2.65ε and σ = 5.8 x 10^7 mho m⁻¹.
Figure 2b. Imaginary part of the resonant frequencies of the $TM_{11}$ mode using (PA). Same parameters as in fig. 2a.
Figure 3a. Real part of the resonant frequencies of the TM$_{11}$ mode using (PA). Same parameters as in fig. 2a except that $c = 2.1 \, a$. 
Figure 3b. Imaginary part of the resonant frequencies of the $TH_{11}$ mode using (PA). Same parameters as in fig. 3a.
Figure 4a. Real part of the resonant frequencies of the $\text{TM}_{11}$ mode using (PA). Same parameters as in fig. 2a except that $c = 2.25a$. 
Figure 4b. Imaginary part of the resonant frequencies of the TM₁₁ mode using (PA). Same parameters as in fig. 4b.
Figure 5a. Real part of the resonant frequencies of the different modes of the TM$_{11}$ as a function of d/a using the matched asymptotic expansion approach (MA), c = 2.05a, $\varepsilon_1 = 2.65\varepsilon$ and $\sigma = 5.8 \times 10^4$ mho m$^{-1}$.
Figure 5b. Imaginary part of the resonant frequencies of the TM$_{11}$ mode using (MA). Same parameters as in fig. 5a.
Figure 6a. Real part of the resonant frequencies of the TM$_{11}$ mode using (MA). Same parameters as in fig. 5a except that $c = 2.1a$. 
Figure 6b. Imaginary part of the resonant frequencies of the TM$_{11}$ mode using (MA). Same parameters as in fig. 6a.
Figure 7a. Real part of the resonant frequencies of the TM\textsubscript{11} mode using (MA). Same parameters as in fig. 5a except that $c = 2.25a$. 
Figure 7b. Imaginary part of the resonance frequencies of the TM$_{11}$ mode. Same parameters as in fig. 7a.
modes of the \( \text{TM}_{11} \). Comparing the results obtained by the matched asymptotic expansion approach to that obtained from the perturbation approach, the two methods agree quite satisfactorily for values of \( d/a \leq 0.1 \).

Figures 8 and 9 show the results obtained by using Galerkin's method (GM); with \( (R,P,Q,S,M) = (1,0,1,0,1) \); for the resonant frequencies of the \( \text{TM}_{11} \) mode. Comparing these results with those obtained from either the (PA) or the (MA), we see that all methods approach each other for \( d/a \leq 0.5 \) which is the value of \( d/a \) that sets the limit of validity of the matched asymptotic expansion approach and the perturbational method. It is also clear from the figures, that all three methods predict the same trend of how the resonant frequencies vary as a function of \( c/a \).

From the plots of the imaginary part of the resonant frequencies for the different cases presented, it can be seen that the odd-symmetric mode has the largest imaginary part and thus it is a good radiating mode while the odd-antisymmetric mode is suitable for resonators.
Figure 8a. Real part of the resonant frequencies of the different modes of the TM₁₁ using Galerkin's method (GM1), \( c/a = 2.05 \), \( c_1 = 2.65 \varepsilon \) and \( \sigma = 5.8 \times 10^7 \) mho m⁻¹.
Figure 8b. Imaginary part of the resonant frequencies of the TM$_{11}$ mode using (GM). Same parameters as in fig. 8a.
Figure 9a. Real part of the resonant frequencies of the TM$_{11}$ mode using (GM). Same parameters as in fig. 8a except that $c = 2.25a$. 
Figure 9b. Imaginary part of the resonant frequencies of the $TM_{11}$ mode using (GM). Same parameters as in fig. 9a.
Chapter 3

Impedance Parameters and Radiation Pattern of Two
Coupled Circular Microstrip Disk Antennas

I. Introduction

Microstrip antenna arrays have found wide application in recent years due to its inherent advantages [2]. The single isolated microstrip antenna elements of different shapes have been extensively analyzed to predict its input impedance, bandwidth and radiation pattern [11-14]. However, less efforts have been directed to the analysis of the coupling effects between several elements.

The coupling capacitor between microstrip disk resonators have been analyzed using an electrostatic approximation [9]. The mutual impedance between printed dipoles has also been studied [15]. Mutual coupling between L-band rectangular and circular microstrip antennas has been investigated experimentally [10].

In this chapter, the mutual coupling effects between two circular microstrip disk antennas excited by vertical probes, has been studied rigorously. Using the Hankel Transform [3], the problem is formulated in terms of a set of vector integral equations. Exact expressions for the self, mutual and input impedances together with the radiation pattern have been derived. In the thin substrate limit these expressions are greatly simplified. We show the excitation of both the odd symmetric and antisymmetric modes by properly adjusting the positions and currents of the feeding probes. Several plots for self, mutual and input impedances and radiation patterns will be illustrated.
II. Field Expressions Due to the Current on the Disks

Figure 1 shows the geometry of the problem. The two circular perfectly conducting disks are of equal radii \( a \), separated by a distance \( c \) and placed on top of a dielectric substrate backed by a perfectly conducting ground plane. The dielectric substrate has a permittivity of \( \varepsilon_1 \) and thickness \( d \). The cylindrical coordinates referred to the center \( O_1 \) of disk \( D_1 \) are \((\rho_1, \phi_1, z)\) and those referred to the center \( O_2 \) of disk \( D_2 \) are \((\rho_2, \phi_2, z)\).

Disk \( D_1 \) is excited by a probe \( P_1 \) situated at \((\rho_{01}, \phi_{01})\) referred to \( O_1 \) or \((\rho'_{01}, \phi'_{01})\) referred to \( O_2 \), similarly disk \( D_2 \) is fed by another probe \( P_2 \) located at \((\rho_{02}, \phi_{02})\) referred to \( O_2 \) or \((\rho'_{02}, \phi'_{02})\) referred to \( O_1 \). The time convention \( e^{-i\omega t} \) is used.

Using the stratified medium formulation [3,17], the \( z \)-component of the electric and magnetic fields in the free space region due to the current distribution on disk \( D_1 \) referred to \( O_1 \) can be easily shown to be given by

\[
E_z^{(1)}(\rho_1, \phi_1, z) = \sum_m e^{im\phi_1} \int_0^\infty dk \frac{e^{(1)(k_r)}}{k_r} \left[ e^{\pm i k_z z} - R^{TM} e^{(ik_z z + 2ik_h)} \right] J_m(k_r \rho_1) \tag{1a}
\]

\[
H_z^{(1)}(\rho_1, \phi_1, z) = \sum_m e^{im\phi_1} \int_0^\infty dk \frac{e^{(1)(k_r)}}{k_r} \left[ e^{\pm i k_z z} + R^{TE} e^{(ik_z z + 2ik_h)} \right] J_m(k_r \rho_1) \tag{1b}
\]

The positive sign is for \( z > 0 \) and the negative sign is for \( z < 0 \), \( k_z = \ldots \).
Figure 1. Geometrical configuration of the two coupled circular disk excited by two probes.
\[(k^2 - k_\rho^2)^{1/2}, \quad k^2 = \omega^2 \mu_\varepsilon, \quad R_{\text{TM}}^{\text{TM}} \text{ and } R_{\text{TE}}^{\text{TE}} \text{ are the TM and TE reflection coefficients at } z = 0 \text{ and are given by}
\]

\[
R_{\text{TM}}^{\text{TM}} = \frac{i \varepsilon_1 k_z \cos k_{1z} d - \varepsilon k_{1z} \sin k_{1z} d}{i \varepsilon_1 k_z \cos k_{1z} d + \varepsilon k_{1z} \sin k_{1z} d}, \quad (2a)
\]

\[
R_{\text{TE}}^{\text{TE}} = \frac{ik_z \sin k_{1z} d + k_{1z} \cos k_{1z} d}{ik_z \sin k_{1z} d - k_{1z} \cos k_{1z} d}, \quad (2b)
\]

\[
k_{1z} = (k_{1z}^2 - k_\rho^2)^{1/2}
\]

\[
k_{1z}^2 = \omega^2 \mu_\varepsilon_1.
\]

\(h\) is the height of the disks above the substrate which will be later set equal to zero, \(e_m^{(1)}(k_\rho)\) and \(h_m^{(1)}(k_\rho)\) are the spectral amplitudes of the electric and magnetic fields respectively which are determined by the current distribution on disk \(D_1\). These amplitudes are obtained by equating the discontinuity in the tangential magnetic field at the disk to the current distribution on the disk. When this is done we get

\[
R^{(1)}(\rho_1, \phi_1) = \begin{bmatrix} \rho \cr \phi \end{bmatrix}^{(1)}(\rho_1, \phi_1) = \hat{z} \times (\hat{R}^+ - \hat{R}^-) = \bar{M} \cdot \sum_{m} e^{im\phi_1} \int_{0}^{\infty} dk_\rho \rho_1
\]

\[
\bar{M}_m(k_\rho) R^{(1)}(k_\rho) \quad \text{for } \rho_1 \in D_1 \quad (3)
\]

where
\[
\tilde{\mathbf{r}}_m^{(1)}(k_{\rho}) = \begin{bmatrix}
-2i\omega & a_m^{(1)}(k_{\rho})/k_{\rho} \\
2k_z h_m^{(1)}(k_{\rho})/k_{\rho}
\end{bmatrix}
\] (4)

is the vector Hankel transform [12] of the \(m\)-th harmonic of the vector,

\[
\begin{bmatrix}
k_{\rho}^{(1)}(\rho_1, \phi_1) \\
-i k_{\phi}^{(1)}(\rho_1, \phi_1)
\end{bmatrix},
\]

\[
\tilde{\mathbf{h}}_m(k_{\rho\rho_1}) = \begin{bmatrix}
-j_m'(k_{\rho\rho_1}) & m J_m(k_{\rho\rho_1})/k_{\rho}\rho_1 \\
-m J_m(k_{\rho\rho_1})/k_{\rho}\rho_1 & J_m'(k_{\rho\rho_1})
\end{bmatrix}
\] (5)

and

\[
\bar{M} = \begin{bmatrix}
1 & 0 \\
0 & i
\end{bmatrix}.
\] (6)

The transverse electric field due to the current distribution on disk \(D_1\) is thus given, for \(z = 0\) and \(h = 0\), by [4,12]

\[
\tilde{E}_s^{(1)}(\rho_1, \phi_1) = \begin{bmatrix}
\tilde{E}_\rho^{(1)}(\rho_1, \phi_1) \\
\tilde{E}_\phi^{(1)}(\rho_1, \phi_1)
\end{bmatrix} = -\bar{M} \cdot \sum_n \int_0^{\infty} dk_{\rho} k_{\rho} \tilde{h}_m(k_{\rho\rho_1})
\]

\[
\cdot \tilde{a}(k_{\rho}) \cdot \tilde{r}_m^{(1)}(k_{\rho})
\] (7)
where
\[
\bar{G}(k_\rho) = \begin{bmatrix}
\frac{k_z}{2\omega_c} (1 - R_{TM}) & 0 \\
0 & \frac{\omega_m}{2k_z} (1 + R_{TE})
\end{bmatrix}.
\]

Similarly the expression for \( E^{(1)}_s \) referred to \( O_2 \) can be obtained by using the following addition theorem for cylindrical waves [19]
\[
J_m(k_\rho \rho_1) e^{i\phi_1} = \sum_k (-1)^k J_k(k_\rho c) J_{m+k}(k_\rho \rho_2) e^{i(m+k)\phi_2}.
\]

Thus \( E^{(1)}_z \) and \( H^{(1)}_z \) referred to \( O_2 \) becomes
\[
E^{(1)}_z(\rho_2, \phi_2, z) = \sum_k \sum_m (-1)^k e^{i(m+k)\phi_2} \int_0^\infty dk_\rho k_m e^{(1)}(k_\rho)
\begin{align*}
&\left[ e^{\pm ik_z z} + R_{TM} e^{ik_z z + 2ik_z h} \right] J_k(k_\rho c) J_{m+k}(k_\rho \rho_1) \\
&= \sum_k \sum_m (-1)^k e^{i(m+k)\phi_2} \int_0^\infty dk_\rho k_m h^{(1)}(k_\rho)
\end{align*}
\]

\[
H^{(1)}_z(\rho_2, \phi_2, z) = \sum_k \sum_m (-1)^k e^{i(m+k)\phi_2} \int_0^\infty dk_\rho k_m e^{(1)}(k_\rho)
\begin{align*}
&\left[ e^{\pm ik_z z} + R_{TE} e^{ik_z z + 2ik_z h} \right] J_k(k_\rho c) J_{m+k}(k_\rho \rho_2) \\
&= \sum_k \sum_m (-1)^m e^{i(m+k)\phi_2} \int_0^\infty dk_\rho k_m e^{(1)}(k_\rho)
\end{align*}
\]

Substituting \( k = n - m \) we get
\[
E^{(1)}_z(z_2, \rho_2, z) = \sum_n \sum_m (-1)^{n-m} \int_0^\infty dk_\rho k_m e^{(1)}(k_\rho)
\]
\[ e^{\pm ikz} [ - \mathcal{R}^{TM} e^{(ikz + 2ikzh)} ] J_{n-m}(k_\rho c) J_n(k_\rho \rho_2) \]  
(11a)

\[ H_z^{(1)}(\rho_2, \phi_2, z) = \sum_n e^{in\phi_2} \sum_m (-1)^{n-m} \int_0^\infty dk_\rho k_\rho h_m^{(1)}(k_\rho) \]
\[ [e^{\pm ikz} + \mathcal{R}^{TE} e^{(ikz + 2ikzh)} ] J_{n-m}(k_\rho c) J_n(k_\rho \rho_2). \]  
(11b)

Thus the transverse electric field \( E_s^{(1)}(\rho_2, \phi_2) \) at \( z = 0 \), referred to \( O_2 \) is given by

\[ E_s^{(1)}(\rho_2, \phi_2) = -\bar{M} \cdot \sum_n e^{in\phi_2} \sum_m (-1)^{n-m} \int_0^\infty dk_\rho k_\rho J_{n-m}(k_\rho c) \bar{h}_n(k_\rho \rho_2) \]
\[ \cdot h(k_\rho) \cdot R_m^{(1)}(k_\rho). \]  
(12)

Similarly, the transverse electric field due to the current distribution on disk \( D_2 \) referred to \( O_2 \), for \( z = 0 \) is given by

\[ E_s^{(2)}(\rho_2, \phi_2) = -\bar{M} \cdot \sum_n e^{in\phi_2} \int_0^\infty dk_\rho k_\rho \bar{h}_n(k_\rho \rho_2) \cdot h(k_\rho) \cdot R_n^{(2)}(k_\rho) \]  
(13)

and referred to \( O_1 \) is given by

\[ E_s^{(2)}(\rho_1, \phi_1) = -\bar{M} \cdot \sum_n e^{in\phi_1} \sum_m \int_0^\infty dk_\rho k_\rho J_{n-m}(k_\rho c) \bar{h}_n(k_\rho \rho_1) \cdot h(k_\rho) \cdot R_m^{(2)}(k_\rho) \]
(14)
where we have made use of the following addition theorem

\[ J_m(k_\rho \rho_2) e^{im\phi_2} = \sum_k J_k(k_\rho \rho_0) J_{m+k}(k_\rho \rho_1) e^{i(m+k)\phi_1}. \]  (15)
III. Field Expressions Due to Probe Excitation

To simplify the analysis, the current distribution on the probe can be approximated by a uniform current sheet on a cylindrical surface of radius $R$, the current distribution on the probe referred to its central line is thus represented by

$$\overline{J}(\rho) = z I \frac{\delta(\rho - R)}{2\pi \rho}$$

for $-d < z < 0$  \hspace{1cm} (16)

where we have assumed a vertical probe of length $d$, radius $R$ and with uniform current $I$. Using the dyadic Green's function formulation [12,17], the $z$-component of the electric field of the probe in the upper half space, referred to its central line is given by

$$E_z^p = -\frac{iI}{4\pi\omega e} \int_{0}^{\infty} \frac{k^3}{k^2_{1z}} \frac{k_{\rho}}{k_{\rho}^2} (1 - R_{TM}^2) J_0(k_{\rho} R) J_0(k_{\rho} \rho) e^{ik_{1z}z} \hspace{1cm} (17)$$

If the probe is located at $(\rho',\phi')$ referred to an origin $0$, the electric field referred to this origin will be given by

$$E_z^p = -\frac{iI}{4\pi\omega e} \sum_{m} e^{i\phi'} e^{-i\phi} \int_{0}^{\infty} \frac{k^3}{k^2_{1z}} \frac{k_{\rho}}{k_{\rho}^2} (1 - R_{TM}^2) J_0(k_{\rho} R) J_m(k_{\rho} \rho') J_m(k_{\rho} \rho) e^{ik_{1z}z} \hspace{1cm} (18)$$

where we have made use of the following addition theorem
\begin{align}
\mathcal{J}_0(k_\rho |\vec{z} - \vec{z}'|) = \sum_m J_m(k_\rho) J_m(k_\rho') e^{im(\phi - \phi')}
\end{align}

(19)

and thus the transverse component of the electric field at $z = 0$ is given by

\begin{align}
\mathcal{E}_s^p(\rho, \phi) = -\vec{M} \cdot \sum_m e^{im\phi} \int_0^\infty dk_\rho k_\rho \vec{H}_m(k_\rho) \cdot \vec{s}_m(k_\rho)
\end{align}

(20)

where

\begin{align}
\vec{s}_m(k_\rho) &= \begin{bmatrix} P_m(k_\rho) \\ 0 \end{bmatrix}, \\

P_m(k_\rho) &= -\frac{1}{4\pi\omega e k_z^2} k_\rho k_z(1 - R_{TM}^T) J_0(k_\rho R) J_m(k_\rho') e^{-im\phi'}.
\end{align}

(21a) (21b)
IV. The Basic Vector Integral Equations for the Current Distributions

These equations are obtained by applying the boundary conditions on the tangential electric field components on the two disks.

\[ E_s^{(1)} + E_s^{(2)} + E_s^p + E_s^{p2} = 0 \quad \text{on } D_1 \text{ and } D_2 \quad (22) \]

from which we get

\[ \int_0^\infty dk_\rho k_\rho \tilde{H}_n(k_\rho \rho_1) \cdot \tilde{G}(k_\rho) \cdot \tilde{R}_n^{(1)}(k_\rho) + \sum_m \int_0^\infty dk_\rho k_\rho J_{n-m}(k_\rho c) \tilde{H}_n(k_\rho \rho_1) \]

\[ \cdot \tilde{G}(k_\rho) \cdot \tilde{R}_m^{(2)}(k_\rho) \]

\[ = - \int_0^\infty dk_\rho k_\rho \tilde{H}_n(k_\rho \rho_1) \cdot \tilde{G}_n^{(1)}(k_\rho) - \int_0^\infty dk_\rho k_\rho \tilde{H}_n(k_\rho \rho_1) \cdot \tilde{G}_n^{(2)'}(k_\rho) \]

\[ 0 \leq \rho_1 < a \quad (23a) \]

\[ \int_0^\infty dk_\rho k_\rho \tilde{H}_n(k_\rho \rho_2) \cdot \tilde{G}(k_\rho) \cdot \tilde{R}_n^{(2)}(k_\rho) + \sum_m (-1)^{n-m} \int_0^\infty dk_\rho k_\rho J_{n-m}(k_\rho c) \]

\[ \cdot \tilde{H}_n(k_\rho \rho_2) \cdot \tilde{G}(k_\rho) \cdot \tilde{R}_m^{(1)}(k_\rho) \]

\[ = - \int_0^\infty dk_\rho k_\rho \tilde{H}_n(k_\rho \rho_2) \cdot \tilde{G}_n^{(1)'}(k_\rho) - \int_0^\infty dk_\rho k_\rho \tilde{H}_n(k_\rho \rho_2) \cdot \tilde{G}_n^{(2)}(k_\rho) \]

\[ 0 \leq \rho_2 < a \quad (23b) \]
where

\[ g(m)(k_\rho) = \begin{bmatrix} p_m(j)(k_\rho) \\ 0 \end{bmatrix}, \] (24a)

\[ g(m)'(k_\rho) = \begin{bmatrix} p_m(j)'(k_\rho) \\ 0 \end{bmatrix}, \quad j = 1, 2 \] (24b)

\[ p_m(j)(k_\rho) = -\frac{I_j}{4\pi\omega} \frac{k_\rho}{k_{1z}^2} k_z (1 - R^{TM}) J_0(k_\rho R) J_m(k_\rho R) e^{-im\phi_{0j}} \] (25a)

\[ p_m(j)'(k_\rho) = -\frac{I_j}{4\pi\omega} \frac{k_\rho}{k_{1z}^2} k_z (1 - R^{TM}) J_0(k_\rho R) J_m(k_\rho R) e^{-im\phi'_{0j}}. \] (25b)

From the condition on the currents being zero outside the disks we have

\[ \int_0^\infty dk_\rho k_\rho \vec{H}_n(k_\rho, \rho_1) \cdot \vec{R}_n^{(1)}(k_\rho) = 0 \quad \rho_1 > a \] (26a)

\[ \int_0^\infty dk_\rho k_\rho \vec{H}_n(k_\rho, \rho_2) \cdot \vec{R}_n^{(2)}(k_\rho) = 0, \quad \rho_2 > a. \] (26b)

The current distribution on the disks can be expanded in terms of the orthogonal set of the TM and TE modes of the magnetic wall cavity since they form a complete set of basis functions

\[ \vec{R}_n^{(j)}(k_\rho) = \sum_r a_r^{(j)} \tilde{\psi}_{nr}(k_\rho) + \sum_p b_p^{(j)} \tilde{\varphi}_{np}(k_\rho) \] (27)
where \( \tilde{\psi}_{nr}(k_\rho) \) and \( \tilde{\phi}_{np}(k_\rho) \) are the vector Hankel transforms of the TM and TE modes of the magnetic wall cavity and are given by [12]

\[
\tilde{\psi}_{nr}(k_\rho) = \tilde{\beta}_{nr} J_n(\tilde{\beta}_{nr}) \left[ \frac{J_n'(k_\rho a)}{(\beta_{nr}/a)^2 - k_\rho^2} \right. \\
\left. + \frac{na}{\beta_{nr}^2 k_\rho} J_n(k_\rho a) \right] \tag{28a}
\]

and

\[
\tilde{\phi}_{np}(k_\rho) = \frac{k_\rho a J_n'(\alpha_{np})}{k_\rho^2 - (\alpha_{np}/a)^2} \left[ \begin{array}{c} 0 \\ J_n(k_\rho a) \end{array} \right] \tag{28b}
\]

where

\[
J_n'(\beta_{nm}) = 0, \\
J_n(\alpha_{np}) = 0,
\]

and with this choice of current, (26a) and (26b) are automatically satisfied.

In the small \( d/a \) limit and because of the uniformity of the current on the probe along the \( z \)-direction, only modal fields with no \( z \)-variation will be excited, and since all the TE modes of the cavity have \( z \)-variation, none of them will be excited. Therefore, the currents on the disk are thus expanded in terms of the TM modes only as

\[
R_n^{(j)}(k_\rho) = \sum_{\rho} a_n^{(j)} \tilde{\psi}_{nr}(k_\rho). \tag{29}
\]
Substituting this expansion into (23a) and (23b) and multiplying the first obtained equation by \( p_1 \bar{\psi}_{nf}(\rho_1) \) and integrating from 0 to \( a \) and the second equation by \( p_2 \bar{\psi}_{nt}(\rho_2) \) and integrating from 0 to \( a \), we get

\[
\sum r a_{nr} \int_0^\infty dk \frac{k}{\rho} \bar{\psi}_{nf}(k) \cdot \bar{\psi}_{nr}(k) + \sum s q \int_0^\infty dk \frac{k}{\rho} J_{n-s}(k) c \bar{\psi}_{nf}(k) \cdot \bar{\psi}_{sq}(k) \cdot \bar{G}(k) \cdot \bar{\psi}_{nt}(k) = - \int_0^\infty dk \frac{k}{\rho} \bar{\psi}_{nf}(k) \cdot \bar{\psi}_n(1)(k) - \int_0^\infty dk \frac{k}{\rho} \bar{\psi}_{nt}(k) \cdot \bar{\psi}_n(2)''(k)
\]

\( f = 1, \ldots, \infty \), \hspace{1cm} (30a)

\[
\sum q a_{nr} \int_0^\infty dk \frac{k}{\rho} \bar{\psi}_{nt}(k) \cdot \bar{\psi}_{nr}(k) \cdot \bar{\psi}_{nt}(k) \cdot \bar{\psi}_{n}(k) + \sum s r \int_0^\infty \left( -a_{sr}(1) \int_0^\infty dk \frac{k}{\rho} J_{n-s}(k) c \right. \cdot \bar{\psi}_{nt}(k) \cdot \bar{\psi}_{n}(k) \cdot \bar{\psi}_{sr}(k)
\]

\( = - \int_0^\infty dk \frac{k}{\rho} \bar{\psi}_{nt}(k) \cdot \bar{\psi}_n(1)'(k) - \int_0^\infty dk \frac{k}{\rho} \bar{\psi}_{nt}(k) \cdot \bar{\psi}_n(2)(k) \)

\( t = 1, \ldots, \infty \). \hspace{1cm} (30b)

When \( d = 0 \), the resonant frequency of the structure approaches that of the magnetic wall cavity and the vector integral equations give a solution similar to the magnetic wall model, thus, for small \( d \) we can consider the structure as a perturbation of the magnetic wall model.
If we are interested in operating frequencies around $\omega_{nm}$, which is the unperturbed resonant frequency of the magnetic wall cavity, only modes which have resonant frequencies of $\omega_{nm}$ in the unperturbed state, will have dominant contribution to the currents on the disks and all other modes will have negligible contribution.

Therefore the current on the disks can be approximated by the two harmonics

$$R_n^{(j)}(k_\rho) = a_n^{(j)} \psi_{nm}^{(j)}(k_\rho) \quad (31a)$$

and

$$R_{-n}^{(j)}(k_\rho) = a_{-n}^{(j)} \psi_{-nm}^{(j)}(k_\rho). \quad (31b)$$

That is,

$$R_n^{(j)}(k_\rho) = e^{i\phi} R_n^{(j)}(k_\rho) + e^{-i\phi} R_{-n}^{(j)}(k_\rho). \quad (31c)$$

At this point we should mention that this is not a very good approximation to the currents if we are interested in the current itself, but is a good approximation for the calculation of the self, mutual and input impedances of the structure since we will be using a stationary formula for calculating these impedances.

Thus the equations governing $a_{nm}^{(1)}$, $a_{-nm}^{(1)}$, $a_{nm}^{(2)}$ and $a_{-nm}^{(2)}$ reduce to

$$A_n a_{nm}^{(1)} + A_c a_{-nm}^{(1)} + \bar{A}_c a_{-nm}^{(2)} = E_n^{(1)} \quad (32a)$$
\[ A a_{-nm}^{(1)} + A_{c} a_{nm}^{(2)} + A c a_{-nm}^{(2)} = E_{-n}^{(1)} \]  
\[ A a_{nm}^{(2)} + A_{c} a_{nm}^{(1)} + A c a_{-nm}^{(1)} = E_{n}^{(2)} \]  
\[ A a_{-nm}^{(2)} + A_{c} a_{nm}^{(1)} + A c a_{-nm}^{(1)} = E_{-n}^{(2)} \]

where

\[ E_{n}^{(1)} = - \int_{0}^{\infty} dk_{\rho} k_{\rho} \psi_{nm}^{T}(k_{\rho}) \cdot \xi_{n}^{(1)}(k_{\rho}) - \int_{0}^{\infty} dk_{\rho} k_{\rho} \psi_{nm}(k_{\rho}) \cdot \xi_{n}^{(2)}(k_{\rho}) \]

\[ = \frac{I_{1}}{4\pi\omega_{e}} \beta_{nm} J_{n}(\beta_{nm}) e^{-i\phi_{01}} \int_{0}^{\infty} dk_{\rho} \frac{k_{\rho}^{2}}{k_{1z}^{2}} k_{z}(1 - R^{TM}) \frac{J_{n}^{'}(k_{\rho} a)}{(\frac{\beta_{nm}}{a})^{2} - k_{\rho}^{2}} J_{n}(k_{\rho} a_{01}) J_{0}(k_{\rho} R) \]

\[ + \frac{I_{2}}{4\pi\omega_{e}} \beta_{nm} J_{n}(\beta_{nm}) e^{-i\phi_{02}} \int_{0}^{\infty} dk_{\rho} \frac{k_{\rho}^{2}}{k_{1z}^{2}} k_{z}(1 - R^{TM}) \frac{J_{n}^{'}(k_{\rho} a)}{(\frac{\beta_{nm}}{a})^{2} - k_{\rho}^{2}} J_{n}(k_{\rho} a_{02}) J_{0}(k_{\rho} R) \]

\[ E_{n}^{(2)} = - \int_{0}^{\infty} dk_{\rho} k_{\rho} \psi_{nm}^{T}(k_{\rho}) \cdot \xi_{n}^{(1)}(k_{\rho}) - \int_{0}^{\infty} dk_{\rho} k_{\rho} \psi_{nm}(k_{\rho}) \cdot \xi_{n}^{(2)}(k_{\rho}) \]

\[ = \frac{I_{1}}{4\pi\omega_{e}} \beta_{nm} J_{n}(\beta_{nm}) e^{-i\phi_{01}^{*}} \int_{0}^{\infty} dk_{\rho} \frac{k_{\rho}^{2}}{k_{1z}^{2}} k_{z}(1 - R^{TM}) \frac{J_{n}^{'}(k_{\rho} a)}{(\frac{\beta_{nm}}{a})^{2} - k_{\rho}^{2}} J_{n}(k_{\rho} a_{01}) J_{0}(k_{\rho} R) \]

\[ + \frac{I_{2}}{4\pi\omega_{e}} \beta_{nm} J_{n}(\beta_{nm}) e^{-i\phi_{02}^{*}} \int_{0}^{\infty} dk_{\rho} \frac{k_{\rho}^{2}}{k_{1z}^{2}} k_{z}(1 - R^{TM}) \frac{J_{n}^{'}(k_{\rho} a)}{(\frac{\beta_{nm}}{a})^{2} - k_{\rho}^{2}} J_{n}(k_{\rho} a_{02}) J_{0}(k_{\rho} R) \]
\[ J_n(k_{\rho}r_{02}) J_0(k_{\rho}R) \]  

(35) 

\[ A = \int_0^\infty dk_{\rho} k_{\rho} \frac{\tilde{T}_{nm}(k_{\rho})}{\bar{G}(k_{\rho})} \frac{\tilde{\psi}_{nm}(k_{\rho})}{\psi_{nm}(k_{\rho})} = \frac{1}{2\omega_e} \beta_{nm}^2 J_n^2(\beta_{nm}) \]  

\[ \int_0^\infty dk_{\rho} k_{\rho} k_z (1 - R_{TM}) \frac{J_n^2(k_{\rho}a)}{\left\{ \left( \frac{\beta_{nm}}{a} \right)^2 - k_{\rho}^2 \right\}^2} + \frac{k^2 a^2 n^2}{\beta_{nm}} \int_0^\infty dk_{\rho} (1 + R_{TE}) \frac{J_n^2(k_{\rho}a)}{k_{\rho} k_z} \]  

(36a) 

\[ A_c = \int_0^\infty dk_{\rho} k_{\rho} J_0(k_{\rho}a) \frac{\tilde{T}_{nm}(k_{\rho})}{\bar{G}(k_{\rho})} \frac{\tilde{\psi}_{nm}(k_{\rho})}{\psi_{nm}(k_{\rho})} = \frac{1}{2\omega_e} \beta_{nm}^2 J_n^2(\beta_{nm}) \]  

\[ \int_0^\infty dk_{\rho} k_{\rho} k_z (1 - R_{TM}) J_0(k_{\rho}a) \frac{J_n^2(k_{\rho}a)}{\left\{ \left( \frac{\beta_{nm}}{a} \right)^2 - k_{\rho}^2 \right\}^2} \]  

+ \frac{k^2 a^2 n^2}{\beta_{nm}} \int_0^\infty dk_{\rho} (1 + R_{TE}) J_0(k_{\rho}a) \frac{J_n^2(k_{\rho}a)}{k_{\rho} k_z} \]  

(36b) 

\[ A_c = \int_0^\infty dk_{\rho} k_{\rho} J_2n(k_{\rho}a) \frac{\tilde{T}_{nm}(k_{\rho})}{\bar{G}(k_{\rho})} \frac{\tilde{\psi}_{nm}(k_{\rho})}{\psi_{nm}(k_{\rho})} \]  

= \frac{1}{2\omega_e} \beta_{nm}^2 J_n^2(\beta_{nm}) \int_0^\infty dk_{\rho} k_{\rho} k_z (1 - R_{TM}) J_2n(k_{\rho}a) \frac{J_n^2(k_{\rho}a)}{\left\{ \left( \frac{\beta_{nm}}{a} \right)^2 - k_{\rho}^2 \right\}^2} \]  

- \frac{k^2 a^2 n^2}{\beta_{nm}} \int_0^\infty dk_{\rho} (1 + R_{TE}) J_2n(k_{\rho}a) \frac{J_n^2(k_{\rho}a)}{k_{\rho} k_z} \]  

(36c) 

In the small \( d/a \) limit, \( A, A_c, A', \) \( E_{n}^{(j)} \) and \( E_{-n}^{(j)} \) can be expanded in terms
of powers of $d/a$. It can be easily shown that the leading term in the expansion of $A, A_c$ and $\bar{A}_c$ is $(d/a)^2$ whereas that in the expansion of $E_n^{(j)}$ and $E_{-n}^{(j)}$ is $(d/a)$. Therefore the leading term in the expansion of $a_{nm}^{(j)}$, $a_{-nm}^{(j)}$ is $(d/a)^{-1}$.

Thus if we are interested in expressing $a_{nm}^{(j)}$ and $a_{-nm}^{(j)}$ in terms of its dominant term, the series expansion of $A, A_c$ and $\bar{A}_c$ should be correctly evaluated for terms up to $(d/a)^2$ whereas $E_n^{(j)}$ and $E_{-n}^{(j)}$ should be expanded correctly up to terms of the order $(d/a)$.

If this is done for $E_n^{(j)}$ and $E_{-n}^{(j)}$ we find that they are given as follows:

$$E_n^{(j)} = \frac{iI_j d}{2\pi \omega_1} e^{-i\phi_{oj}} \left( \frac{\beta_{nm}}{a} \right) J_n \left( \frac{\beta_{nm}}{a} \right) J_0 \left( \frac{\beta_{nm} R}{a} \right)$$  \hspace{1cm} (37a)

$$E_{-n}^{(j)} = \frac{iI_j d}{2\pi \omega_1} (-1)^n e^{i\phi_{oj}} \left( \frac{\beta_{nm}}{a} \right) J_n \left( \frac{\beta_{nm}}{a} \right) J_0 \left( \frac{\beta_{nm} R}{a} \right)$$  \hspace{1cm} (37b)

As for $A$, it can be easily shown that it can be written in the following form

$$A = -\frac{i}{2\omega_1} \beta_{nm}^2 J_n^2(\beta_{nm}) \left[ n + \frac{1}{a^2} (k_1 a^2 - \beta_{nm}^2) \int_0^\infty dk_\rho k_\rho \frac{J_n^2(k_\rho a)}{\left( \frac{\beta_{nm}}{a} \right)^2 - k_\rho^2} \right]$$

$$-\frac{\tan k_1 d}{k_1} \left( 1 + R^{TM} \right) + \frac{in^2 a^2}{\beta_{nm}^4} \int_0^\infty dk_\rho \frac{J_n^2(k_\rho a)}{k_\rho k_z} (1 + R^{TE}) \right]$$  \hspace{1cm} (38)

where
\[
\eta = \int_0^\infty dk_\rho k_\rho \frac{J_n^2(k_\rho a)}{\beta_{nm}^2} \frac{\tan k_{1z}d}{(1 + R_{TM})} + \frac{\ln^2}{\beta_{nm}^2} \int_0^\infty dk_\rho (1 + R_{TE}) \frac{J_n^2(k_\rho a)}{k_\rho k_{1z}}.
\]

(39)

Now since \((k_1a - \beta_{nm})\) is of the order \((d/a)\), we need to keep terms of the order up to \((d/a)\) in the expansion of the quantity between the curly brackets. It can be easily shown that this expansion is given by \((d/\beta_{nm}^4)(\beta_{nm}^2 - n^2)a^2\).

It can also be shown that the leading term in the expansion of \(\eta\) is of the order of \((d/a)^2\), however this term cannot be obtained in a closed form, so we might as well keep \(\eta\) as it is

\[
A = -\frac{i}{2\omega e_1} \beta_{nm}^2 J_n^2(\beta_{nm})a
\]

(40)

where

\[
\alpha = n + (k_1^2a^2 - \beta_{nm}^2)(\beta_{nm}^2 - n^2) \frac{d}{\beta_{nm}^4}.
\]

(41)

Similarly, \(A_c\) and \(\bar{A}_c\) can be put in the form
\[
+ \frac{\text{i} n^2 a^2}{\beta_{nm}} \int_0^\infty dk_\rho J_0(k_\rho c) \frac{J_n^2(k_\rho a)}{k_\rho k_z} \left(1 + R^\text{TE}\right) \bigg] \quad (42)
\]

and

\[
\bar{A}_c = -\frac{1}{2\omega_1} \beta_{nm}^2 \frac{2}{J_n^2(\beta_{nm})} \left[\eta_c + \frac{R^\text{TM}}{a^2} (k_1^2 a^2 - \beta_{nm}^2) \right]
\]

\[
\left\{ \int_0^\infty dk_\rho J_2(k_\rho c) \frac{J_n^2(k_\rho a)}{k_\rho k_z} \frac{\tan k_1 z^d}{k_1 z} (1 + R^\text{TM}) \right\}
\]

\[
- \frac{\text{i} n^2 a^2}{\beta_{nm}} \int_0^\infty dk_\rho J_2(k_\rho c) \frac{J_n^2(k_\rho a)}{k_\rho k_z} (1 + R^\text{TE}) \bigg) \quad (43)
\]

where

\[
\eta_c = \int_0^\infty dk_\rho J_0(k_\rho c) - \frac{J_n^2(k_\rho a)}{\left(\frac{\beta_{nm}}{a}\right)^2 - k_\rho^2} \frac{\tan k_1 z^d}{k_1 z} (1 + R^\text{TM})
\]

\[
\eta_c = \int_0^\infty dk_\rho J_2(k_\rho c)(1 + R^\text{TE}) \frac{J_n^2(k_\rho a)}{k_\rho k_z} \quad (44a)
\]

\[
\bar{\eta}_c = \int_0^\infty dk_\rho J_2(k_\rho c) \frac{J_n^2(k_\rho a)}{\left(\frac{\beta_{nm}}{a}\right)^2 - k_\rho^2} \frac{\tan k_1 z^d}{k_1 z} (1 + R^\text{TM})
\]

\[
- \frac{\text{i} n^2 a^2}{\beta_{nm}} \int_0^\infty dk_\rho J_2(k_\rho c)(1 + R^\text{TE}) \frac{J_n^2(k_\rho a)}{k_\rho k_z} \quad (44b)
\]

In the case of \( A_c \) and \( \bar{A}_c \), the term of order \((d/a)\) in the expansion
of the quantity between curly brackets is identically equal to zero. Thus $A_c$ and $\bar{A}_c$ can be approximated by

$$A_c = -\frac{i}{2\omega_e} B_{nm}^2 J_n^2(\beta_{nm}) n_c$$  \hspace{1cm} (45)$$

$$\bar{A}_c = -\frac{i}{2\omega_e} B_{nm}^2 J_n^2(\beta_{nm}) \bar{n}_c.$$  \hspace{1cm} (46)$$

When equations (32) and (33) are solved in $a_{nm}^{(j)}$, $a_{-nm}^{(j)}$ we get

$$a_{nm}^{(1)} = -\frac{1}{\pi} \left( \frac{d}{a} \right) \frac{J_0(\beta_{nm} R/a)}{B_{nm} J_n^2(\beta_{nm})} (\Delta_1/\Delta)$$  \hspace{1cm} (47a)$$

$$a_{-nm}^{(1)} = -\frac{1}{\pi} \left( \frac{d}{a} \right) \frac{J_0(\beta_{nm} R/a)}{B_{nm} J_n^2(\beta_{nm})} (\Delta_2/\Delta)$$  \hspace{1cm} (47b)$$

$$a_{nm}^{(2)} = -\frac{1}{\pi} \left( \frac{d}{a} \right) \frac{J_0(\beta_{nm} R/a)}{B_{nm} J_n^2(\beta_{nm})} (\Delta_3/\Delta)$$  \hspace{1cm} (47c)$$

$$a_{-nm}^{(2)} = -\frac{1}{\pi} \left( \frac{d}{a} \right) \frac{J_0(\beta_{nm} R/a)}{B_{nm} J_n^2(\beta_{nm})} (\Delta_4/\Delta)$$  \hspace{1cm} (47d)$$

where

$$\Delta_1 = D_1 X_1 + D_4 X_2 + D_2 Y_1 + D_3 Y_2$$  \hspace{1cm} (48a)$$

$$\Delta_2 = D_4 X_1 + D_1 X_2 + D_3 Y_1 + D_2 Y_2$$  \hspace{1cm} (48b)$$
\[ \Delta_3 = D_2 X_1 + D_3 X_2 + D_1 Y_1 + D_4 Y_2 \] (48c)

\[ \Delta_4 = D_3 X_1 + D_2 X_2 + D_4 Y_1 + D_1 Y_2 \] (48d)

\[ D_1 = \alpha (\alpha^2 - \eta_c^2 - \bar{\eta}_c^2), \] (49a)

\[ D_2 = \eta_c (\eta_c^2 - \alpha^2 - \bar{\eta}_c^2), \] (49b)

\[ D_3 = \bar{\eta}_c (\bar{\eta}_c^2 - \eta_c^2 - \alpha^2) \] (49c)

\[ D_4 = 2 \alpha n_c \bar{\eta}_c \] (49d)

\[ \Delta = (\alpha + \eta_c + \bar{\eta}_c)(\alpha + \eta_c - \bar{\eta}_c)(\alpha - \eta_c + \bar{\eta}_c)(\alpha - \eta_c - \bar{\eta}_c) \] (50)

\[ X_1 = I_1 J_n (\beta_{nm}^0 / a) \ e^{-\im \phi_{01}}, \] (51a)

\[ X_2 = (-1)^n I_1 J_n (\beta_{nm}^0 / a) \ e^{\im \phi_{01}}, \] (51b)

\[ Y_1 = I_2 J_n (\beta_{nm}^0 / a) \ e^{-\im \phi_{02}}, \] (51c)

\[ Y_2 = (-1)^n I_2 J_n (\beta_{nm}^0 / a) \ e^{\im \phi_{02}}. \] (51d)
V. Calculations of the Self and Mutual Impedances

From the appendix, the self impedance $Z_{11}$ across the terminals of probe $p_1$ is

$$Z_{11} = -\frac{1}{I_1^2} \int \bar{E}_1 \cdot \bar{J}_1 \, dV_1$$

(52a)

and the mutual impedance $Z_{12}$ is

$$Z_{12} = -\frac{1}{I_1 I_2} \int \bar{E}_1 \cdot \bar{J}_2 \, dV_2$$

(52b)

where $\bar{J}_1$ and $\bar{J}_2$ are the current distributions on probes $p_1$ and $p_2$ respectively and are given by

$$\bar{J}_1 = \hat{z} I_1 \frac{\delta (\rho_1' - R)}{2\pi \rho_1'} \quad -d < z < 0$$

(53a)

$$\bar{J}_2 = \hat{z} I_2 \frac{\delta (\rho_2' - R)}{2\pi \rho_2'} \quad -d < z < 0$$

(53b)

$(\rho_1', \gamma_1')$ is the local coordinate system of probe $p_1$ while $(\rho_2', \gamma_2')$ is that of probe $p_2$. $\bar{E}_1$ is the total electric field in the substrate due to the current on probe $p_1$ and the currents induced on disks $D_1$ and $D_2$ i.e. with $I_2$ set equal to zero. These impedance formulas are stationary with respect to errors in the assumed current distributions on the probe and on the two disks [18]. Thus useful approximate expressions can be obtained for $Z_{11}$ and $Z_{12}$ using these formulas.
Using the dyadic Green's function formulation, the $z$-component of the electric field due to the current on probe $p_1$ referred to the local coordinates at $p_1$, can be shown to be given by \[12\]

\[
E_z^{p_1} = - \frac{I_1}{4\pi \omega_1} \int_0^\infty dk_1 \frac{k_{\rho_1}}{k_{1z}} \mathbf{J}_0(k_{\rho_1} r_1') \mathbf{J}_0(k_{\rho} R) \left\{ \frac{2}{ik_{1z}} 
\right. \\
\left. [k_{\rho}^2 e^{ik_{1z}z/2} - k_{\rho_1}^2] + \frac{2k_{\rho}^2 \sin(k_{1z}z/2) e^{ik_{1z}z/2}}{k_{1z}(1 + R_{01}^{TM} e^{ik_{1z}z/2})} \right. \\
\left. [(1 + R_{01}^{TM} e^{ik_{1z}z/2}) e^{ik_{1z}z/2} - (1 + e^{ik_{1z}z/2})R_{01}^{TM} e^{-ik_{1z}z/2}] \right\}
\]

where

\[
R_{01}^{TM} = \frac{\varepsilon_1 k_{1z} - \varepsilon k_{1z}}{\varepsilon_1 k_{1z} + \varepsilon k_{1z}}.
\]

The electric field due to the induced currents on the disks, referred to the local coordinates at $0_1$, can be shown to be given by

\[
E_z^i = \frac{1}{2i\omega_1} \sum_n e^{in\phi} \int_0^\infty dk \frac{k^2}{k_{1z}} \{K^{(1)}(k_{\rho}) + \sum_r J_n(k_{\rho}) K^{(2)}(k_{\rho}) \}
\]

\[
(1 + R_{01}^{TM}) \frac{\cos k_{1z}(z + d)}{\cos k_{1z}d} J_n(k_{\rho}).
\]

Using the following addition theorem
\[ J_n(k_\rho, 0) e^{i\phi} = \sum_k J_k(k_\rho^0, 0) J_{n+k}(k_\rho^0, 0) e^{i(n+k)\phi} e^{-ik(\pi + \phi_1')}. \] (57)

The self and mutual impedances can be easily obtained, to give

\[ Z_{11} = -\frac{1}{I_1} \int_0^R \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} d\phi_1' d\phi_1'' \, E_{1z} J_{1z} \]

\[ = \frac{1}{4\pi \omega_1 I_1} \int_0^\infty dk_\rho \frac{k_\rho}{k_1 z} J_0^2(k_\rho R) \left\{ \frac{2e^{i k_1 d/2}}{ik_1 z} \frac{k_\rho^2 \sin(k_1 z d/2)}{k_1 z'} - k_1 z' \right\} \]

\[ + \frac{i k_\rho^2}{k_1^2} \frac{(1 - R_{TM}) \tan(k_1 z d/2)}{k_1 z} e^{-ik_1 d/2} \left\{ 1 + \frac{2R_{TM}}{X_{10} \omega_1} e^{ik_1 d/2} \right\} \]

\[ - \frac{1}{2i \omega_1 I_1} \sum_n e^{i\phi_1} \int_0^\infty dk_\rho \frac{k_\rho^2}{k_1 z} \left\{ J_n(k_\rho, 0) + \sum_{r} J_{n-r}(k_\rho, c) k_r^2(k_\rho, r) \right\} \]

\[ (1 + R_{TM}) \tan k_1 z d \int J_n(k_\rho^0, 0) J_0(k_\rho R) \] (58)

where

\[ X_{10} = 1 + R_{TM}. \]

Using the addition theorem

\[ J_r(k_\rho^0, 0) e^{i\phi_1'} \sum_n e^{i\phi_1} J_{n-r}(k_\rho, c) J_n(k_\rho^0, 0). \] (59)
The expression for $Z_{11}$ simplifies to

$$Z_{11} = \frac{1}{4\pi \omega} \int_0^\infty dk_1 \frac{k^2}{k_1^2} \text{J}_0^2(k_1 R) \left\{ \frac{2}{ik_1} \left[ \frac{2e^{ik_1d/2}}{k_1^2 \sin(k_1d/2)} \right] - k_1^{-2d} \right\}$$

$$+ \frac{k}{k_1^2} (1 - R_{TM}) \tan(k_1d/2) e^{-ik_1d/2} \left[ 1 + \frac{2e^{R_{TM}/10}}{e^{ik_1d/2}} \right]$$

$$- \frac{1}{2i \omega I_1} \int_0^\infty dk_1 \frac{k^2}{k_1^2} \left[ \sum_n k_{n1}^2 \text{J}_n(k_1 R) e^{i\phi_{n1}} \right] (1 + R_{TM}) \tan k_1d \text{J}_0(k_1 R). \quad (60a)$$

Similarly,

$$Z_{12} = \frac{1}{4\pi \omega} \int_0^\infty dk_1 \frac{k^2}{k_1^2} \text{J}_0(k_1 t) \text{J}_0^2(k_1 R) \left\{ \frac{2}{ik_1} \left[ \frac{2e^{ik_1d/2}}{k_1^2 \sin(k_1d/2)} \right] - k_1^{-2d} \right\}$$

$$+ \frac{k}{k_1^2} (1 - R_{TM}) \tan(k_1d/2) e^{-ik_1d/2} \left[ 1 + \frac{2e^{R_{TM}/10}}{e^{ik_1d/2}} \right]$$

$$- \frac{1}{2i \omega I_1} \int_0^\infty dk_1 \frac{k^2}{k_1^2} \left[ \sum_n k_{n2}^2 \text{J}_n(k_1 R) e^{i\phi_{n2}} \right] (1 + R_{TM}) \tan k_1d \text{J}_0(k_1 R). \quad (60b)$$

where $t$ is the distance between $p_1$ and $p_2$. Similar expressions can be obtained for $Z_{22}$ and $Z_{21}$ with $Z_{21} = Z_{12}$.
In the limit of small \( d/a \), the currents can be substituted by their approximate expressions obtained in Section IV, after setting \( I_2 = 0 \) to obtain \( Z_{11} \) and \( Z_{12} \). When this is done we get the following simple expressions for \( Z_{11} \) and \( Z_{12} \):

\[
Z_{11} = \frac{i2}{\pi\omega\epsilon_1} (d/a)^2 J_n^2 \left( \frac{\beta_{nm}}{a} \right) \frac{J_o^2(\beta_{nm} R/a)}{J_n^2(\beta_{nm})} \alpha \\
\frac{a^2 - \eta_c^2 - \bar{\eta}_c^2 + 2(-1)^n \eta_c \bar{\eta}_c \cos(2n\phi_{01})}{(\alpha + \eta_c + \bar{\eta}_c)(\alpha + \eta_c - \bar{\eta}_c)(\alpha - \eta_c + \bar{\eta}_c)(\alpha - \eta_c - \bar{\eta}_c)}
\]

(61a)

and

\[
Z_{12} = \frac{i2}{\pi\omega\epsilon_1} (d/a)^2 J_n \left( \frac{\beta_{nm}}{a} \right) J_n \left( \frac{\beta_{nm}}{a} \right) \frac{J_o^2(\beta_{nm} R/a)}{J_n^2(\beta_{nm})} \\
\frac{\eta_c(\eta_c^2 - a^2 - \bar{\eta}_c^2) \cos(n(\phi_{01} - \phi_{02})) + (-1)^n \bar{\eta}_c(\bar{\eta}_c^2 - a^2 - \eta_c^2) \cos(n(\phi_{01} + \phi_{02}))}{(\alpha + \eta_c + \bar{\eta}_c)(\alpha + \eta_c - \bar{\eta}_c)(\alpha - \eta_c + \bar{\eta}_c)(\alpha - \eta_c - \bar{\eta}_c)}
\]

(61b)

When \( \epsilon_c \to \infty \) that is in the case when the two disks become uncoupled we have \( \eta_c = \bar{\eta}_c \to 0 \) and

\[
Z_{11} = -\frac{i2d(\beta_{nm}/a)^4 J_n^2(\beta_{nm}^2 01/a) J_o^2(\beta_{nm} R/a)}{\pi\omega\epsilon_1(\beta_{nm}^2 - n^2) J_n^2(\beta_{nm}) \left( \frac{\beta_{nm}}{a} \right)^2 \omega^2 \mu_1 \epsilon_1 (1 + \alpha)}
\]

(62)
where

$$\ddot{a} = \left( \frac{\beta_{nm}}{ak_1} \right)^2 \frac{\beta^2_{nm}}{d(\beta^2_{nm} - n^2)} a$$  (63)

and

$$Z_{12} \to 0.$$  (64)

This expression for $Z_{11}$ agrees with the expression obtained in Reference [13] for the input impedance of a single disk and which was obtained by an entirely different approach. Thus the expression obtained for $Z_{11}$ agrees with previously obtained expressions in the limit $c \to \infty$.

The expressions for $Z_{11}$ and $Z_{12}$ can be put in a more convenient form as follows:

$$Z_{11} = \frac{i}{\pi \omega \epsilon_1} (d/a)^2 J_n^2(\beta_{nm} a) \frac{J_o^2(\beta_{nm} R/a)}{J_n^2(\beta_{nm})} \left[ \left\{ \frac{1}{\alpha + \eta_c + \bar{\eta}_c} + \frac{1}{\alpha - \eta_c - \bar{\eta}_c} \right\} \cos^2\left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} + \left\{ \frac{1}{\alpha + \eta_c - \bar{\eta}_c} + \frac{1}{\alpha - \eta_c + \bar{\eta}_c} \right\} \sin^2\left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} \right]$$  (65a)

$$Z_{12} = \frac{i}{\pi \omega \epsilon_1} (d/a)^2 J_n(\beta_{nm} a) \frac{J_n(\beta_{nm})}{J_o^2(\beta_{nm})} \left[ \left\{ \frac{1}{\alpha + \eta_c + \bar{\eta}_c} - \frac{1}{\alpha - \eta_c - \bar{\eta}_c} \right\} \cos\left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} \cos\left\{ n \left( \frac{\pi}{2} - \phi_{02} \right) \right\} \right]$$
\[
+ \left\{ \frac{1}{\alpha + \eta_c - \bar{\eta}_c} - \frac{1}{\alpha - \eta_c + \bar{\eta}_c} \right\} \sin\left(\eta_c \left(\frac{\pi}{2} - \phi_{01}\right)\right) \sin\left(\eta_c \left(\frac{\pi}{2} - \phi_{02}\right)\right) \right].
\] (65b)

The factors \((\alpha \pm \eta_c \pm \bar{\eta}_c)\) can be approximated by the following expressions:

\[
\alpha + \eta_c + \bar{\eta}_c = \frac{2d(\beta_{nm}^2 - n^2)}{\beta_{nm}^2} \left( \frac{\omega - \omega_{ys}}{\omega_{nm}} \right) \tag{66a}
\]

\[
\alpha + \eta_c - \bar{\eta}_c = \frac{2d(\beta_{nm}^2 - n^2)}{\beta_{nm}^2} \left( \frac{\omega - \omega_{ys}}{\beta_{nm}^2} \right) \tag{66b}
\]

\[
\alpha - \eta_c + \bar{\eta}_c = \frac{2d(\beta_{nm}^2 - n^2)}{\beta_{nm}^2} \left( \frac{\omega - \omega_{ya}}{\omega_{nm}} \right) \tag{66c}
\]

\[
\alpha - \eta_c - \bar{\eta}_c = \frac{2d(\beta_{nm}^2 - n^2)}{\beta_{nm}^2} \left( \frac{\omega - \omega_{ya}}{\omega_{nm}} \right) \tag{66d}
\]

where \(\gamma\) stands for 0 (odd) in case of odd values of \(n\) and for \(e\) (even) for even values of \(n\) and vice versa for \(\bar{\gamma}\), whereas \(s\) and \(a\) stand for symmetrical or antisymmetrical.

Thus \(\omega_{os}\) is the resonant frequency of the odd symmetrical mode, that is a mode which varies as \(\sin n\phi_1\) on disk (1) and \(\sin n\phi_2\) on disk (2), \(\omega_{oa}\) is that of the odd antisymmetrical mode, that is a mode varying as \(\sin n\phi_1\) on \(D_1\) and \(-\sin n\phi_2\) on \(D_2\). Likewise \(\omega_{es}\) is the resonant frequency of the even symmetric mode, that is varying as \(\cos n\phi_1\) on \(D_1\) and \(\cos n\phi_2\) on \(D_2\) and finally \(\omega_{ea}\) is the resonant frequency of the even antisymmetric mode,
that is varying as \(\cos n\phi_1\) on \(D_1\) and \(-\cos n\phi_2\) on \(D_2\). These resonant frequencies are given by

\[
\omega_{\gamma S} = \omega_{nm} \left[ 1 - \frac{\beta_{nm}^2}{2d(\beta_{nm}^2 - n^2)} \left( \eta' + \eta'_c + \bar{\eta}'_c \right) \right]
\]
(67a)

\[
\omega_{\gamma a} = \omega_{nm} \left[ 1 - \frac{\beta_{nm}^2}{2d(\beta_{nm}^2 - n^2)} \left( \eta' - \eta'_c - \bar{\eta}'_c \right) \right]
\]
(67b)

\[
\omega_{\gamma S} = \omega_{nm} \left[ 1 - \frac{\beta_{nm}^2}{2d(\beta_{nm}^2 - n^2)} \left( \eta' + \eta'_c - \bar{\eta}'_c \right) \right]
\]
(67c)

\[
\omega_{\gamma a} = \omega_{nm} \left[ 1 - \frac{\beta_{nm}^2}{2d(\beta_{nm}^2 - n^2)} \left( \eta' - \eta'_c + \bar{\eta}'_c \right) \right]
\]
(67d)

\(\eta', \eta'_c\) and \(\bar{\eta}'_c\) are the values of \(\eta, \eta_c\) and \(\bar{\eta}_c\) evaluated at \(\omega = \omega_{nm}\).

Using these expressions we get the following simple forms for \(Z_{11}\) and \(Z_{12}\)

\[
Z_{11} = \frac{i}{2\pi} \sqrt{\frac{\omega}{\varepsilon_1}} \frac{(d/a)}{\beta_{nm}^2 - n^2} J_n^2 \left( \frac{\beta_{nm} \phi_{01}}{a} \right) J_\phi^2 \left( \frac{\beta_{nm} R/a}{\omega} \right) \left[ \frac{\omega_{nm}}{\omega - \omega_{\gamma S}} + \frac{\omega_{nm}}{\omega - \omega_{\gamma a}} \right] \cos^2 \left( \eta \left( \frac{\pi}{2} - \phi_{01} \right) \right) + \left\{ \frac{\omega_{nm}}{\omega - \omega_{\gamma S}} + \frac{\omega_{nm}}{\omega - \omega_{\gamma a}} \right\} \sin^2 \left( \eta \left( \frac{\pi}{2} - \phi_{01} \right) \right)
\]
(68a)

\(Z_{22}\) has exactly the same form as \(Z_{11}\) except that \(\phi_{01}\) and \(\phi_{01}\) are replaced by \(\phi_{02}\) and \(\phi_{02}\) and
\[ Z_{12} = Z_{21} = \frac{i}{2\pi} \sqrt{\mu/\varepsilon_1} \left( \frac{d}{a} \right) \frac{\beta_{nm}}{\beta_{nm}^2 - n^2} \sum_{p_{01}} J_n \left( \frac{\beta_{nm} p_{01}}{a} \right) J_n \left( \frac{\beta_{nm} p_{02}}{a} \right) \frac{J_0^2 \left( \beta_{nm} R/a \right)}{J_n^2 \left( \beta_{nm} \right)} \]

\[
\begin{bmatrix}
\frac{\omega_{nm}}{\omega - \omega_{gs}} - \frac{\omega_{nm}}{\omega - \omega_{ga}} \\
\frac{\omega_{nm}}{\omega - \omega_{gs}} - \frac{\omega_{nm}}{\omega - \omega_{ga}}
\end{bmatrix}
\cos \left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} \cos \left\{ n \left( \frac{\pi}{2} - \phi_{02} \right) \right\} 
\]

\[
+ \begin{bmatrix}
\frac{\omega_{nm}}{\omega - \omega_{gs}} - \frac{\omega_{nm}}{\omega - \omega_{ga}} \\
\frac{\omega_{nm}}{\omega - \omega_{gs}} - \frac{\omega_{nm}}{\omega - \omega_{ga}}
\end{bmatrix}
\sin \left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} \sin \left\{ n \left( \frac{\pi}{2} - \phi_{02} \right) \right\}. \tag{68b}
\]

Thus it is clear from the expression of \( Z_{11} \), that in the case of odd values of \( n \) and for \( \phi_{01} = \pi/2 \), the odd modes, that is the modes which are varying as \( \sin n\phi \) are the only modes excited whereas for \( \phi_{01} = 0 \), the even modes, that is the modes which are varying as \( \cos n\phi \) are the only modes excited and vice versa in the case of even values of \( n \).

The input impedance \( Z_{in} \) which is defined as

\[ Z_{in} = Z_{11} + \frac{I_2}{I_1} Z_{12} \]

is given by

\[
Z_{in} = \frac{i}{2\pi} \sqrt{\mu/\varepsilon_1} \left( \frac{d}{a} \right) \frac{\beta_{nm}}{\beta_{nm}^2 - n^2} \sum_{p_{01}} J_n \left( \frac{\beta_{nm} p_{01}}{a} \right) J_n \left( \frac{\beta_{nm} p_{02}}{a} \right) \frac{J_0^2 \left( \beta_{nm} R/a \right)}{J_n^2 \left( \beta_{nm} \right)} \left[ \cos \left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} \right]
\]

\[
\begin{bmatrix}
\frac{\omega_{nm}}{\omega - \omega_{gs}} \left[ J_n \left( \frac{\beta_{nm} p_{01}}{a} \right) \right] \cos \left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} + \frac{I_2}{I_1} J_n \left( \frac{\beta_{nm} p_{02}}{a} \right) \cos \left\{ n \left( \frac{\pi}{2} - \phi_{02} \right) \right\} \\
\frac{\omega_{nm}}{\omega - \omega_{gs}} \left[ J_n \left( \frac{\beta_{nm} p_{01}}{a} \right) \right] \cos \left\{ n \left( \frac{\pi}{2} - \phi_{01} \right) \right\} - \frac{I_2}{I_1} J_n \left( \frac{\beta_{nm} p_{02}}{a} \right) \cos \left\{ n \left( \frac{\pi}{2} - \phi_{02} \right) \right\}
\end{bmatrix}
\]
\[ + \sin\left\{n\left(\frac{\pi}{2} - \phi_{01}\right)\right\}\left\{\frac{\omega_{nm}}{\omega - \omega_{\gamma_s}} \left[J_n\left(\beta_{nm} \frac{\phi_{01}}{a}\right) \sin\left\{n\left(\frac{\pi}{2} - \phi_{01}\right)\right\} + \frac{I_2}{I_1} J_n\left(\beta_{nm} \frac{\phi_{02}}{a}\right) \sin\left\{n\left(\frac{\pi}{2} - \phi_{02}\right)\right\}\right]\right\} + \frac{\omega_{nm}}{\omega - \omega_{\gamma_s}} \left[J_n\left(\beta_{nm} \frac{\phi_{01}}{a}\right) \sin\left\{n\left(\frac{\pi}{2} - \phi_{01}\right)\right\}\right] - \frac{I_2}{I_1} J_n\left(\beta_{nm} \frac{\phi_{02}}{a}\right) \sin\left\{n\left(\frac{\pi}{2} - \phi_{02}\right)\right\}\right]\right\}. \quad (69) \]

Thus it is clear from the expression of \( Z_{in} \), that by proper choice of \( \phi_{01}, \phi_{02}, \phi_{01}, \phi_{02} \), and \( I_2/I_1 \), only one of the four modes can be excited separately. Thus in the case of odd values of \( n \) and in order to excite the odd symmetric mode, for example, we choose

\[ \phi_{02} = \phi_{01} = \frac{\pi}{2} \]

and

\[ \frac{I_2}{I_1} = \frac{J_n(\beta_{nm} \phi_{01}/a)}{J_n(\beta_{nm} \phi_{02}/a)}. \quad (70) \]

In such a case, \( Z_{in} \) is given by

\[ Z_{in} = \frac{i}{\pi} \frac{\nu/\varepsilon_1}{(d/a)} \frac{\beta_{nm}}{\beta_{nm}^2 - n^2} J_n^2\left(\beta_{nm} \frac{\phi_{01}}{a}\right) \frac{J_0^2(\beta_{nm} R/a)}{J_n^2(\beta_{nm})} \frac{\omega_{nm}}{\omega - \omega_{os}} \]

\[ = Z_r \frac{\omega_{nm}}{\omega_{os} - i(\omega - \omega_{os})} \quad (71) \]
where $\omega_{os} = \omega'_{os} - i\omega''_{os}$, that is $\omega'_{os}$ and $\omega''_{os}$ are the real and imaginary parts of $\omega_{os}$ and

$$Z_r = \frac{1}{\pi} \frac{\sqrt{\omega/c_1}}{(d/a)} \frac{s_{nm}}{s_{nm}^2 - n^2} J_n^2 \left( \frac{p_{01}}{a} \right) \frac{J_o^2(s_{nm}R/a)}{J_n^2(s_{nm})}. \quad (72)$$

If we use the time convention $e^{j\omega t}$, which is the one used in circuit theory, instead of $e^{-i\omega t}$, we get

$$Z_{in} = Z_r \frac{\omega_{nm}}{\omega_{os} + j(\omega - \omega'_{os})}. \quad (73)$$

Thus the input into the terminals of probe $P_1$ is acting like a parallel resonant circuit which has the following element factors

$$R = Z_r \frac{\omega_{nm}}{\omega_{os}}, \quad C = \frac{1}{2Z_r \omega_{nm}}, \quad L = \frac{2Z_r \omega_{nm}}{\omega_{os}^2}. \quad (74)$$

Thus

$$Q = \frac{\omega'_{os}}{2 \omega''_{os}}, \quad \text{B.W.} = \frac{\omega_{os}}{\pi} \quad (75)$$

where $R$, $C$, $L$, $Q$ and B.W. are the resistance, capacitance, inductance, quality factor and bandwidth, respectively, of the parallel resonant circuit.

Other modes will have similar equivalent circuits. Thus the mode which has the higher $\omega''$, that is offering the highest radiation, would give a wider
bandwidth and will thus cause the antenna to act as a good radiator, whereas the mode which has the lowest $\omega''$, that is offering the lowest radiation, would have the highest $Q$ and will cause the antenna to act as a good resonator.

Thus, in the $TM_{11}$ mode, the two disk antenna is a better radiator than the single disk antenna if the odd symmetric mode is excited since this mode has the highest $\omega''$, whereas the two disk antenna acts as a better resonator than the single disk antenna if the odd antisymmetric mode is excited since this mode has the lowest $\omega''$. 
VI. The Radiation Pattern

In the small d/a limit, the contribution of the current on the probe to the radiation field is negligible to that of the currents induced on the disks. Using the stratified medium formulation, the $z$-component of the electric field produced by the currents induced on the disks in the upper half space can be shown to be given by

$$E_z = -\frac{1}{2i\omega} \left[ \sum_{p} e^{ip\phi_1} \int_{0}^{\infty} dk_p k_p^2 k_p^{(1)}(k_p)(1 - R^{TM}) e^{ikz} J_p(k_p\rho_1) + \sum_{q} e^{iq\phi_2} \int_{0}^{\infty} dk_q k_q^2 k_q^{(2)}(k_q)(1 - R^{TM}) e^{ikz} J_q(k_q\rho_2) \right]. \quad (76)$$

Now, in the small d/a limit, the currents on the disks will be replaced by the approximate expressions previously discussed in Section IV. If the integral along the positive real axis in the expression of $E_z$ is extended to an integration on the whole real axis we get the following expression for $E_z$

$$E_z = -\frac{1}{4i\omega} \left[ \sum_{p=-n,n} e^{ip\phi_1} a_p^{(1)} \beta_{pm} J_p(\beta_{pm}) \int_{-\infty}^{\infty} dk_p k_p^2 \frac{J_p'(k_p\rho)}{(\beta_{pm})^2 - k_p^2} (1 - R^{TM}) e^{ikz} H_p^{(1)}(k_p\rho_1) + \sum_{q=-n,n} e^{iq\phi_2} a_q^{(2)} \beta_{qm} J_q(\beta_{qm}) \int_{-\infty}^{\infty} dk_q k_q^2 \frac{J_q'(k_q\rho)}{(\beta_{qm})^2 - k_q^2} \right].$$
\[
(1 - R^{TM}) e^{ikz} H_q^{(1)}(k \rho_2) \]

(77)

In the far field where \( r_1, r_2 \to \infty \), the leading order approximation of these integrals can be obtained using the saddle point method \([13, 17]\) and since in the far field \( \theta_1 \sim \theta_2 = \theta \) and \( \phi_1 \sim \phi_2 = \phi \) and \( r_2 \sim r - c \sin \theta \cos \phi \) where \( r = r_1 \), we get the following expression for \( E_z \):

\[
E_z = \frac{(-i)^n}{2\omega c} k^2 \sin \theta \cos \theta \{1 - R^{TM}(k_\rho = k \sin \theta)\} \frac{\gamma_{nm}}{\left(\alpha_{nm}^2 - k^2 \sin^2 \theta\right)} J_n(\beta_{nm}) J_n(ka \sin \theta) e^{ikr \sin \theta \cos \phi} \]

\[
[\{a_{nm}^{(1)} e^{i\phi} + (-1)^n a_{-nm}^{(1)} e^{-i\phi}\} + e^{-ikc \sin \theta \cos \phi} \{a_{nm}^{(2)} e^{i\phi} + (-1)^n a_{-nm}^{(2)} e^{-i\phi}\}] \]

(78)

Similarly the expression for \( H_z \) in the far field is given by

\[
H_z = \frac{i}{2} a(-1)^n \frac{nJ_n(\beta_{nm})}{\beta_{nm}} e^{ikr \sin \theta \cos \phi} J_n(ka \sin \theta) \{1 + R^{TE}(k_\rho = k \sin \theta)\} \]

\[
[\{a_{nm}^{(1)} e^{i\phi} + (-1)^n a_{-nm}^{(1)} e^{-i\phi}\} + e^{-ikc \sin \theta \cos \phi} \{a_{nm}^{(2)} e^{i\phi} + (-1)^n a_{-nm}^{(2)} e^{-i\phi}\}] \]

(79)
Substituting for the current amplitudes by the expressions obtained in Section IV and since in the far field \( E_\theta = -E_z \sin \theta \), \( E_\phi = -\nu H_z \sin \theta \), we get the following expressions for \( E_\theta \) and \( E_\phi \)

\[
E_\theta = -\frac{1}{\Delta} (-i)^n \frac{1}{2\pi \omega e a^2} \frac{(d/a)(ka)^3}{J_0(\beta nm R/a)} \frac{\text{e}^{ikr}}{J_n(\beta nm)} R_\theta(\theta, \phi) \tag{80}
\]

where

\[
R_\theta(\theta, \phi) = \cos \theta \frac{J_n'(ka \sin \theta)}{\beta^2_{nm} - (ka \sin \theta)^2} \{1 - R_{\text{TM}}(k_\rho = k \sin \theta)\}
\]

\[
[(\Delta_1 \text{e}^{i \phi} + (-1)^n \Delta_2 \text{e}^{-i \phi}) + \text{e}^{-ikc \sin \theta \cos \phi} \text{e}^{i \phi}]
\]

\[
(\Delta_3 \text{e}^{i \phi} + (-1)^n \Delta_4 \text{e}^{-i \phi})] \tag{81}
\]

\[
E_\phi = -\frac{1}{\Delta} (-i)^{n+1} \frac{1}{2\pi \omega e a^2} \frac{(d/a)(ka)^3}{J_0(\beta nm R/a)} \frac{\text{e}^{ikr}}{J_n(\beta nm)} R_\phi(\theta, \phi) \tag{82}
\]

where

\[
R_\phi(\theta, \phi) = \frac{n}{\beta^2_{nm}} \frac{1}{ka \sin \theta} \frac{J_n(ka \sin \theta)}{\sin \theta} \{1 + R_{\text{TE}}(k_\rho = k \sin \theta)\}[(\Delta_1 \text{e}^{i \phi} - (-1)^n \Delta_2 \text{e}^{-i \phi}) + \text{e}^{-ikc \sin \theta \cos \phi} \text{e}^{i \phi} \text{e}^{i \phi} \text{e}^{i \phi}]
\]

\[
-(-1)^n \Delta_2 \text{e}^{-i \phi} + \text{e}^{-ikc \sin \theta \cos \phi} \text{e}^{i \phi} \text{e}^{i \phi} \text{e}^{i \phi}]
\]

\[
(\Delta_3 \text{e}^{i \phi} - (-1)^n \Delta_4 \text{e}^{-i \phi})] \tag{83}
\]
Results and Conclusions

Vector integral equations governing the current amplitudes on the two disks are rigorously derived using the Vector Hankel Transform. The limit of small \( d/a \) is applied to these equations to get simple approximate expressions for the current amplitudes.

The impedance parameters, namely the self, mutual and input impedances, together with the radiation pattern are derived both exactly and in the small \( d/a \) limit. In Fig. 2 we show the self and mutual impedances \( Z_{11} \) and \( Z_{12} \) for the antenna in the \( \text{TM}_{11} \) case, for \( \varepsilon_r = \varepsilon_1/\varepsilon = 2.65 \) and \( \phi_{01} = \phi_{02} = 0^\circ \) and \( 90^\circ \). The curves are the superposition of two resonant curves corresponding either to those of the odd symmetric and odd antisymmetric modes, that is modes varying as \( \sin n\phi \), in the case of \( \phi_{01} = \phi_{02} = 90^\circ \) or to the even symmetric and even antisymmetric modes, that is varying as \( \cos n\phi \), in the case of \( \phi_{01} = \phi_{02} = 0^\circ \). One of the resonant curves is seen to be predominant over the other and because of the small spacing between the resonant frequencies, the overall resonant curve appears as a single resonant behavior. In Fig. 3 we illustrate the case of \( \varepsilon_r = 9.6 \). Because of the increasing coupling effect, the suppressed mode starts to give a pronounced resonance behavior and thus the overall resonance curve of \( R_{11} \) shown in Fig. 3a exhibits two peaks corresponding to the odd symmetric and odd antisymmetric modes which have the largest resonant frequency spacing among the four modes.

In Fig. 4 we illustrate the input impedance for the \( \text{TM}_{11} \) case, in which only one of the four modes is excited. The input impedance of the single disk is also plotted to compare with the results obtained for the two
Figure 2a. Normalized self impedance $z_{11}$ for the TM$_{11}$ mode with $r_{01} = r_{02} = 0.9a$, $d = 0.1a$, $c = 2.05a$, $r_{11} = 2.65$, $a = 1.88$ cm. Solid lines are for $r_{01} = r_{02} = 0$; dashed lines are for $r_{01} = r_{02} = 90^\circ$. 
Figure 2b. Normalized mutual impedance $z_{12}$. Same parameters as in fig. 2a.
Figure 3a. Normalized self impedance $z_{11}$ for the TM$_{11}$ mode with $\phi_0 = \omega_0 = 0.9a$, $d = 0.1a$, $c = 2.05a$, $l/l_1 = 9.6$, $a = 1.88$ cm.
Figure 3b. Normalized mutual impedance $z_{12}$. Same parameters as in fig. 3a.
Figure 4a. Normalized input impedance for the odd-symmetric mode of the TM_{11} (solid line).

\( \phi_1 = \phi_2 = 0.9a, d = 0.1a, c = 2.05a, \epsilon_{11} = 2.65, a = 1.88 \text{ cm, } \gamma_0 = \theta_2 = 90^\circ. \)

\( l_2 = l_1. \) The dashed lines are for a single disk.
Figure 4b: Normalized input impedance for the odd-antisymmetric mode of the \( T_{11} \) (solid line). Same parameters as in Fig. 4a except that \( I_2 = -1 \).
Figure 4c. Normalized input impedance for the even-symmetric mode of the TM_{11} (solid line). Same parameters as in fig. 4a except that \( \phi_0 = \phi_2 = 0^\circ \).
Figure 4d. Normalized input impedance for the even-antisymmetric mode of the TM_{11} (solid line). Same parameters as in fig. 4a except that \( \phi_{01} = \phi_{02} = 0^\circ \) and \( I_2 = -I_1 \).
coupled disks. In all previous figures, the impedance parameters are normalized to the free space characteristic impedance. It is shown, from the results obtained, that the two disk structure can act as a good radiator when the odd symmetric mode is excited and acts as a good resonator in the case of the odd antisymmetric mode. Figs. 5 are the radiation patterns, $|R_\theta(\theta, \phi)|$ for $E_\theta$ and $|R_\phi(\theta, \phi)|$ for $E_\phi$, around resonance whereas Figs. 6 are those off resonance.
Figure 5a. Radiation pattern around resonance for the TM_{11} mode, 
\( k_1a = 1.782, \phi_01 = \phi_02 = 0.9a, d = 0.1a, c = 2.05a, \)
\( \varepsilon_1/\varepsilon = 2.65, I_2 = I_1, a = 1.88 \text{ cm}. \) Solid line for 
\( \phi_01 = \phi_02 = 0^\circ, \phi = 90^\circ; \) dashed lines for \( \phi_01 = \phi_02 = 90^\circ, \)
\( \phi = 0^\circ. \)
Figure 5b. Same parameters as in fig. 5a except that solid line is for \( \theta_1 = \theta_2 = 0^\circ \), \( z = 0^\circ \) and dashed line is for \( \theta_1 = \theta_2 = 90^\circ \), \( z = 90^\circ \).
Figure 6a. Radiation pattern off resonance for the TM_{11} mode, 
\( k_1 a = 1.6, \phi_0 = \phi_2 = 0.9a, d = 0.1a, c = 2.05a, \)
\( \varepsilon_1 / \varepsilon = 2.65, I_2 = I_1, \) \( a = 1.88 \) \( \) cm. Solid line for
\( \phi_1 = \phi_2 = 0^\circ, \) \( \psi = 90^\circ; \) dashed line for \( \phi_1 = \phi_2 = 90^\circ, \)
\( \phi_1 = \phi_2 = 0^\circ, \) \( \psi = 90^\circ; \) dashed line for \( \phi_1 = \phi_2 = 90^\circ, \)
Figure 6b. Same parameters as in fig. 6a except that solid line is for \( \gamma_1 = \gamma_2 = 0^\circ, \gamma = 0^\circ \) and dashed line is for \( \gamma_1 = \gamma_2 = 90^\circ, \gamma = 90^\circ \).
Appendix

The Input Impedance of a Coax Probe

In this appendix, the input impedance of the coax probe opening onto a ground plane is derived using the induced emf method [18].

Consider a metallic probe of length \( \ell \) which is the center conductor of a coax, opening into a conducting ground plane. Let \( a \) and \( b \) be the radii of the center and outer conductors respectively. \( V \) is the line voltage (i.e. the voltage difference between the inner and outer conductors).

If \( \mathbf{J}_i(z) \) is the induced surface current distribution on the probe, then the field in the region outside the probe produced by the coax aperture in the presence of the metallic probe is the same as that produced by the coax aperture plus the induced surface current \( \mathbf{J}_i(z) \) in the absence of the metallic probe.

Thus as far as the field, in the region outside the probe, is concerned, the solution to the problem in Fig. (A1) is exactly the same as that in Fig. (A2), where in Fig. (A2), \( \mathbf{J}_i(z) \) is a cylindrical current sheet.

Next we will set-up the problem equivalent to that in Fig. (A2) in which the coax aperture is replaced by an equivalent magnetic surface current.

In setting up the equivalent problem, we will use the uniqueness theorem which states that [17,35]:

"A field in a region is uniquely specified by the sources within the region plus the tangential components of \( \vec{E} \) over the boundary, or the tangential components of \( \vec{H} \) over the boundary, or the former over part of the boundary and the latter over the rest of the boundary."

The problem equivalent to that in Fig. (A2) is thus given in Fig. (A3) which consists of the cylindrical current sheet \( \vec{J}_i(z) \) plus the infinite perfectly electric conductor plane at \( z = 0 \). On top of the part of the infinite plane surface extending from \( \rho = a \) to \( \rho = b \) (originally the coax aperture), we place a magnetic current sheet \( \vec{J}_{ma} \) given by

\[
\vec{J}_{ma} = \vec{E}_a \times \hat{z} = -\hat{\phi} \vec{E}_a
\]

where \( \vec{E}_a \) is the electric field at the aperture (whose only component is the \( \phi \)-component).

From the uniqueness theorem, it is clear that the problem given in Fig. (A3) is equivalent to that of Fig. (A2), since they have the same source distribution \( (\vec{J}_i(z)) \) and the tangential components of the electric field at the boundary surfaces are the same in both problems (for the part of the infinite plane extending from \( \rho = a \) to \( \rho = b \), in the equivalent problem of A3, the tangential components of \( \vec{E} \) are zero on the conductor, just below \( \vec{J}_{ma} \), and equal to the original field components just above \( \vec{J}_{ma} \).

Let \( \vec{E}_i \) and \( \vec{H}_i \) be the electric and magnetic fields produced by \( \vec{J}_i \) in the absence of \( \vec{J}_{ma} \), whereas \( \vec{E}_a \) and \( \vec{H}_a \) be those produced by \( \vec{J}_{ma} \) in the absence of \( \vec{J}_i \).
Figure (A3) Equivalent problem
Thus, by the method of superposition, the total field at any point (in the region outside the probe) in the original problem of (A1) is given by

\[ \bar{E} = \bar{E}_i + \bar{E}_a \quad \text{and} \quad \bar{H} = \bar{H}_i + \bar{H}_a \]

From the boundary condition at the metallic probe, \( \bar{E}_{\tan} = 0 \)

i.e.

\[ (\bar{E}_a)_{\tan} = -(\bar{E}_i)_{\tan} \]

or

\[ \bar{J}_i \cdot \bar{E}_a = -\bar{J}_i \cdot \bar{E}_i \quad (2) \]

Applying the reciprocity theorem \([17,35]\) between sources c and d which states that

\[ <c,d> = <d,c> \]

where

\[ <c,d> \equiv \int_v dv(\bar{J}_c \cdot \bar{E}_d - \bar{J}_{mc} \cdot \bar{H}_d) \]

which is the interaction of source c, denoted by \( \bar{J}_c \) and \( \bar{J}_{mc} \), with the field \( \bar{E}_d \) and \( \bar{H}_d \) produced by source d (denoted by \( \bar{J}_d \) and \( \bar{J}_{md} \)).

In the problem of (A3) we have \( \bar{J}_c = 0, \bar{J}_{mc} = \bar{J}_{ma}, \bar{J}_d = \bar{J}_i \) and \( \bar{J}_{md} = 0 \), thus we get

\[ -\int_v \bar{J}_{ma} \cdot \bar{H}_i = \int_v \bar{J}_i \cdot \bar{E}_a \quad (3) \]
The integral on the left of this equation can be expressed in terms of the input current and voltage as follows.

\[
\int dv \mathbf{J}_m \cdot \mathbf{H}_i = \int_a^b d\phi \int_0^{2\pi} d\phi \rho (-\hat{\phi} \mathbf{E}_a) \cdot \mathbf{H}_i(z=0)
\]

\[
= -\int_a^b d\phi \mathbf{E}_a \int_0^{2\pi} d\phi \rho \mathbf{H}_i(z=0)
\]

\[
= -VI
\]  \hspace{1cm} (4)

where \( I = \int_0^{2\pi} d\phi \rho \mathbf{H}_i(z=0) \) is the input current to the probe and \( V = \int_a^b d\phi \mathbf{E}_a \) is the line voltage.

Thus from eqs. (3) and (4) we get

\[
IV = \int dv \mathbf{J}_i \cdot \mathbf{E}_a
\]

and from eq. (2) we get

\[
IV = -\int dv \mathbf{J}_i \cdot \mathbf{E}_i
\]

From the circuit definition of the input impedance, we finally get

\[
z_{in} = \frac{V}{I} = -\frac{1}{I^2} \int dv \mathbf{J}_i \cdot \mathbf{E}_i
\]  \hspace{1cm} (5)
Thus the input impedance is given in terms of a volume integral over the current induced \( \mathbf{J}_i(z) \) on the metallic probe.

This \( \mathbf{J}_i(z) \) is a cylindrical current sheet replacing the metallic probe which produces the field \( \mathbf{E}_i \) whose tangential component doesn't vanish at the surface of the probe instead it is the sum of \( \mathbf{E}_i \) and \( \mathbf{E}_a \) (the field produced by the coax aperture) whose tangential component vanishes at the probe surface.

Formula (5) for the input impedance is an exact one, provided that both \( \mathbf{J}_i \) and \( \mathbf{E}_i \) are exact. However because \( \mathbf{J}_i \) is usually unknown and has to be approximated, this formula thus gives approximate values for the impedance. Fortunately, this impedance formula is stationary [36, p. 53] with respect to small errors in the assumed current distribution and therefore can give satisfactory results for the input impedance.

In the previous analysis, we assumed that the probe is open circuited at its other end which complicates the current distribution on the probe, thus making it difficult to assume an approximate current distribution on the probe close to the exact one.

However, in the case of the circular microstrip structure, the other end of the probe touches the circular conducting disk and thus for \( d \ll a \) and from the image theory, the metallic probe will appear as a very small part of a much longer probe (formed by the images of the metallic probe), thus allowing us to approximate the current on the probe (when its physical length is much smaller than the wavelength) by a uniform current distribution (fig. (A4)).
Chapter 4

Resonance and Radiation of the Elliptic Disk Microstrip Structure

I. Introduction

It is known that circular or rectangular disk microstrip antennas usually radiate waves which are linearly polarized. However in such structures, circular polarization can be obtained using a complicated feed setup of more than one feed.

From experimental measurement [23], it is known that the elliptic disk antenna can be made to radiate circularly polarized waves using a simple feeding setup consisting of one feed.

Also the elliptic microstrip disk resonator, in various practical applications is much more preferable than the circular one because of the fact that eccentricity as a design parameter provides additional flexibility and enhances the usefulness of the structure [20].

In the past, studies of the elliptic disk microstrip structure more work was devoted to the analysis of the structure as a resonator [21,22] than as an antenna [23]. In the analysis of the structure as a resonator, the analysis is carried out either by using an electrostatic approximation or by using the magnetic wall model. In both analyses the resonant frequencies obtained are pure real and therefore the results are unsatisfactory since the actual resonant frequency is complex because of the radiation loss. Hence all the available theoretical results are very limited.

In the calculation of the radiation pattern [23], the fringing effects of the electric field are taken into account by superficially imposing an
impedance boundary condition at the opening of the cavity. This surface impedance is obtained from the circular disk antenna, thus limiting the results obtained for small eccentricity.

To carry out an analysis which is more rigorous than what has been done in the past, a Vector Mathieu Transform is developed in the appendices of this chapter which allows the rigorous formulation of the problem. It is shown that the current on the disk is governed by a set of vector integral equations. These equations are then solved using Galerkin's moment method, in which the current is expended in terms of the orthogonal set of the TM and TE modes of the magnetic wall cavity since they form a complete set of basis functions to obtain linear algebraic equations in the expansion coefficients.

The eigenvalue equation is then obtained by setting the determinant of the coefficients of these algebraic equations to zero. In the limit of small d (the substrate thickness) a perturbative approach is used to get a much simpler formula for the resonant frequencies.

Finally, the input impedance and the radiation pattern are derived both exactly and in the small d limit.
II. Field Expressions Due to the Current Distribution on the Disk

Figure 1 shows the geometrical configuration of the problem. A perfectly conducting elliptical disk is placed on top of a dielectric substrate backed by a perfectly conducting ground plane. The disk has a semi-major axis a and a semi-minor axis b and of a focal length $2c_0$. The dielectric substrate has a permittivity of $\varepsilon_1$ and thickness d. Elliptical coordinates $(u,v,z)$ are used to express points in space. These are related to the cartesian coordinates as follows:

$$x = c_0 \cosh u \cos v, \quad y = c_0 \sinh u \sin v$$

The disk is fed by a probe $P$ situated at $(u_0,v_0)$. Using the stratified medium formulation [3,17], the $z$-component of the electric and magnetic fields in the free space region due to the current distribution on the disk can be easily shown to be given by

$$E_z = \sum_{n,\alpha} \int_0^\infty dk \rho_n (\rho)\left[ e^{\pm ik_0 z} - e^{-R \cdot z} \right] \psi_n (c_0 k_0, u, v)$$

$$H_z = \sum_{n,\alpha} \int_0^\infty dk \rho_n (\rho)\left[ e^{\pm ik_0 z} + e^{R \cdot z} \right] \psi_n (c_0 k_0, u, v)$$

where, as before, the positive sign is for $z > 0$ and the negative sign is for $z < 0$. $\alpha_n (c_0 k_0, u, v)$ is an elliptic harmonic and is defined in Appendix (4.3). $h$ is the height of the disk above the substrate which will be later set equal to zero and is introduced to avoid the ambiguity in applying the boundary conditions at the disk. The transverse components of the $E$ and $H$ fields can be readily obtained from $E_z$ and $H_z$ using the following formulae [17]
Figure 1. Geometrical configuration of the elliptic disk excited by a probe.
\[
E_s(k_\rho) = \frac{1}{k_\rho^2 \hbar} \left[ \hat{u} \left( \frac{\partial^2}{\partial u \partial z} E_z(k_\rho) + i \omega u \frac{\partial}{\partial v} H_z(k_\rho) \right) + \hat{v} \left( \frac{\partial^2}{\partial v \partial z} E_z(k_\rho) - i \omega u \frac{\partial}{\partial u} H_z(k_\rho) \right) \right]
\]

\[
H_s(k_\rho) = \frac{1}{k_\rho^2 \hbar} \left[ \hat{u} \left( \frac{\partial^2}{\partial u \partial z} H_z(k_\rho) - i \omega e \frac{\partial}{\partial v} E_z(k_\rho) \right) + \hat{v} \left( \frac{\partial^2}{\partial v \partial z} H_z(k_\rho) + i \omega e \frac{\partial}{\partial u} E_z(k_\rho) \right) \right]
\]

where \(E_z(k_\rho), H_z(k_\rho), E_s(k_\rho)\) and \(H_s(k_\rho)\) are the integrands of \(E_z, H_z, E_s\) and \(H_s\), respectively.

\(e_\alpha_n(k_\rho)\) and \(h_\alpha_n(k_\rho)\) are the spectral amplitudes of the electric and magnetic fields respectively and are obtained by equating the current on the disk to the discontinuity in the tangential magnetic field at the disk.

Thus, we get

\[
\mathcal{K}(u,v) = \begin{bmatrix} K_u(u,v) \\ K_v(u,v) \end{bmatrix} = - \sum_{\alpha, n} \int_0^\infty dk_\rho \bar{\alpha}_n(c_0 k_\rho, u, v) \cdot \bar{K}_\alpha_n(k_\rho) \quad \text{for} \quad (u, v) \in \mathcal{D}
\]

(3)

where

\[
\bar{K}_\alpha_n(k_\rho) = \begin{bmatrix} 2i\omega e e_\alpha_n(k_\rho)/k_\rho \\ 2i k_z h_\alpha_n(k_\rho)/k_\rho \end{bmatrix}
\]

(4)

and \(\bar{\alpha}_n(c_0 k_\rho, u, v)\) is defined in Appendix (4.5).
The transverse electric field due to the current distribution on the disk \( D \) is thus given, for \( z = 0 \) and \( h = 0 \), by

\[
\mathbf{E}_s(u,v) = \begin{bmatrix} E_u(u,v) \\ E_v(u,v) \end{bmatrix} = \sum_{\alpha, n} \int_0^\infty dk_\rho \overline{K}_n(c_0 k_\rho, u, v) \cdot \mathcal{G}(k_\rho) \cdot \overline{K}_n(k_\rho)
\]

(5)

where \( \mathcal{G}(k_\rho) \) is given by equation (11) of Chapter 2.
III. The Vector Integral Equations Governing the Current Distribution on the Disk

The current distribution on the disk is governed by the condition that the current has to vanish outside the disk, together with the boundary condition on the tangential component of the electric field on the disk. These two conditions give rise to two dual vector integral equations which can then be solved using Galerkin's method.

From the condition on the current we have

\[ \mathcal{K}(u,v) = 0 \quad \text{for} \quad u > u_s \]  \hfill (6)

From the condition on the tangential electric field, we have

\[ \vec{E}_S(u,v) + \vec{E}_S^D(u,v) = 0 \quad \text{for} \quad u < u_s \]  \hfill (7)

where \( \vec{E}_S(u,v) \) is given by Eq. (5) and \( \vec{E}_S^D(u,v) \) is given in Appendix (4.6) and \( u = u_s \) are the points on the outer edge of the elliptic disk.
Expressing the two conditions explicitly in terms of the current spectral amplitudes \( \bar{\kappa}_n(k_\rho) \), we get

\[
- \sum_{n, \alpha} \int_0^\infty dk_\rho \bar{\bar{\alpha}}_n(c_0 k_\rho, u, v) \cdot \bar{\kappa}_n(k_\rho) = 0 \quad \text{for } u > u_s \tag{8}
\]

\[
\sum_{n, \alpha} \int_0^\infty dk_\rho \bar{\bar{\alpha}}_n(c_0 k_\rho, u, v) \cdot \bar{\varepsilon}(k_\rho) \cdot \bar{\kappa}_n(k_\rho) = - \sum_{n, \alpha} \int_0^\infty dk_\rho \bar{\bar{\alpha}}_n(c_0 k_\rho, u, v) \cdot \bar{\varepsilon}_n(k_\rho) \quad \text{for } u < u_s \tag{9}
\]

which are two dual vector integral equations governing the unknown current spectral amplitudes \( \bar{\kappa}_n(k_\rho) \).

To solve these equations using Galerkin's approach, the current distribution on the disk is expanded in terms of the orthogonal set of the TM and TE modes of the magnetic wall cavity since they form a complete set of basis functions.

\[
\bar{\kappa}(u, v) = \sum_{n, \alpha, m} a_{\alpha nm} \bar{\varphi}_{\alpha nm}(u, v) + \sum_{p, q} b_{\gamma pq} \bar{\varphi}_{\gamma pq}(u, v) \tag{10}
\]

where \( a_{\alpha nm} \), \( b_{\gamma pq} \) are the unknown coefficients of expansion, \( \bar{\varphi}_{\alpha nm}(u, v) \) and \( \bar{\varphi}_{\gamma pq}(u, v) \) are the TM and TE current modes of the magnetic wall cavity and are given by

\[
\bar{\varphi}_{\alpha nm}(u, v) = \frac{1}{\hbar} \begin{bmatrix} \varphi/\varphi u \, \psi_\alpha(c_0 \, k_{\alpha nm} u, v) \\ \varphi/\varphi v \, \psi_\alpha(c_0 \, k_{\alpha nm} u, v) \end{bmatrix} \tag{11}
\]
\[
\bar{\gamma}_{pq}(u,v) = \frac{1}{n} \left[ e^{i\beta \nu} \psi_p(c_0 k_{pq}^m u, v) - e^{-i\beta \nu} \psi_p(c_0 k_{pq}^m u, v) \right]
\]

where

\[
J_{\alpha n}(c_0 k_{nm}^e u_s) = 0
\]

\[
J_{\gamma p}(c_0 k_{pq}^m u_s) = 0
\]

Substituting Eq. (10) into Eq. (6), we get

\[
\sum_{\gamma, \beta} \int_0^{\infty} dk' \bar{\Phi}_{\gamma}(c_0 k', u, v) \cdot \bar{K}_{\delta}(k') = -\sum_{\gamma, \beta} \alpha_{nm} \phi_{\alpha nm}(u, v) - \sum_{\gamma, \beta} b_{\gamma pq} \bar{\gamma}_{pq}(u, v)
\]

\[
= 0
\]

Applying the orthogonality relation (4.5.1) developed in Appendix (4.5) we get

\[
\bar{K}_{\delta}(k_o) = -c_{\delta}(k_o) \left[ \sum_{\gamma, \beta} \alpha_{nm} F_{nm, r}(\alpha, \beta, c_0 k_o) + \sum_{\gamma, \beta} b_{\gamma pq} T_{pq, r}(\gamma, \beta, c_0 k_o) \right]
\]

where

\[
F_{nm, r}(\alpha, \beta, c_0 k_o) = \int_0^{u_s} du \int_0^{2\pi} dv \ h \bar{\Phi}_{\gamma}(c_0 k_o u, v) \cdot \bar{\phi}_{nm}(u, v)
\]

(16)
and
\[
\mathcal{T}_{pq, r}(\gamma, \beta, c_0 k_\rho) = \int_0^{u_s} du \int_0^{2\pi} dv \ h^2 \mathcal{R}_{r}(c_0 k_\rho, u, v) \overline{\gamma}_{pq}(u, v)
\]  
(17)

This equation supplies a relation between the current spectral amplitude \( \mathcal{R}_{r}(k_\rho) \) and the coefficients of modal expansion \( a_{nm} \) and \( b_{pq} \). The factors \( F_{nm, s}(\alpha, \beta, c_0 k_\rho) \) and \( T_{pq, s}(\gamma, \beta, c_0 k_\rho) \) are given by integrals \( I_2 \) and \( I_3 \) evaluated in Appendix (4.7).

The next step is to find the values of the modal expansion coefficients \( a_{nm} \) and \( b_{pq} \). This is done by substituting the expression for \( \mathcal{R}_{r}(k_\rho) \) given by eq. (15) into the second boundary condition given by eq. (9), multiplying the resulting equation once by \( h^2 \bar{\xi}_{T_j}(u, v) \) and a second time by \( h^2 \bar{\xi}_{T_j}(u, v) \) and in each time integrating over the area of the ellipse we get the following two sets of equations:

\[
\sum_{n, \alpha, m} a_{nm} A_{nm, js}^f(\alpha, \xi) + \sum_{p, \gamma, q} b_{pq} A_{pq, js}^t(\gamma, \xi) = B_{js}^f(\xi)
\]  
(18)

\[
\sum_{n, \alpha, m} a_{nm} A_{nm, js}^t(\alpha, \xi) + \sum_{p, \gamma, q} b_{pq} A_{pq, js}^f(\gamma, \xi) = B_{js}^t(\xi)
\]  
(19)

where

\[
A_{nm, js}^f(\alpha, \xi) = \sum_{r, \beta} \int_0^\infty dk_\rho \ c_\beta r(k_\rho) \ F_{js, r}(\epsilon, \beta, c_0 k_\rho) \cdot \tilde{g}(k_\rho) \cdot F_{nm, r}(\alpha, \beta, c_0 k_\rho)
\]

\[
A_{js, pq}^t(\xi, \gamma) = A_{pq, js}^t(\gamma, \xi) = \sum_{r, \beta} \int_0^\infty dk_\rho \ c_\beta r(k_\rho) \ F_{js, r}(\epsilon, \beta, c_0 k_\rho) \cdot \tilde{g}(k_\rho) \cdot T_{pq, r}(\gamma, \beta, c_0 k_\rho)
\]
\[ A_{p,q,j,s}^{t}(\gamma,\xi) = \sum_{r,B} \int_{0}^{\infty} dk_{\rho} \, c_{r}(k_{\rho}) \, F_{j,s,r}(\xi,\beta,c_{0}k_{\rho}) \cdot g(k_{\rho}) \cdot F_{p,q,r}(\gamma,\beta,c_{0}k_{\rho}) \]

\[ B_{j,s}^{f}(\xi) = \sum_{k,n} \int_{0}^{\infty} dk_{\rho} \, F_{j,s,k}(\xi,n,c_{0}k_{\rho}) \cdot S_{n}(k_{\rho}) \]

\[ B_{j,s}^{t}(\xi) = \sum_{k,n} \int_{0}^{\infty} dk_{\rho} \, F_{j,s,k}(\xi,n,c_{0}k_{\rho}) \cdot S_{n}(k_{\rho}) \]

where the superscript $T$ denotes transposing the vector.

For these equations to be of practical use, the series expansion has to be truncated and these equations can then be recast in the matrix form

\[ \bar{A}^{FF} \cdot \bar{a} + \bar{A}^{FT} \cdot \bar{b} = \bar{B}^{F} \]  \hspace{1cm} (20)

\[ \bar{A}^{TF} \cdot \bar{a} + \bar{A}^{TT} \cdot \bar{b} = \bar{B}^{T} \]  \hspace{1cm} (21)

where \( \bar{A}^{FF}, \bar{A}^{FT}, \bar{A}^{TF} \) and \( \bar{A}^{TT} \) are the matrices whose elements are \( A_{nm,j,s}^{ff}(\alpha,\xi), A_{p,q,j,s}^{ft}(\gamma,\xi), A_{nm,j,s}^{tf}(\alpha,\xi) \) and \( A_{p,q,j,s}^{tt}(\gamma,\xi) \), respectively, whereas \( \bar{a}, \bar{b}, \bar{B}^{F} \) and \( \bar{B}^{T} \) are the vectors whose elements are \( a_{nm}, b_{\gamma pq}, B_{j,s}^{f}(\xi) \) and \( B_{j,s}^{t}(\xi) \), respectively. The two matrix equations (20) and (21) can be thus easily solved for the unknown coefficients \( \bar{a} \) and \( \bar{b} \).

In the case of a thin substrate, the structure approaches that of the magnetic wall cavity model and in the limit \( d \to 0 \), the vector integral equations give a solution similar to the magnetic wall model. Thus in the case of a thin substrate structure, the dominant modes are the TM modes.
Furthermore, if we are interested in frequencies around $\omega_{anm}^e$, which is the unperturbed resonant frequency of the magnetic wall cavity, we will find that only modes which have resonant frequency of $\omega_{anm}$ in the unperturbed state, will have dominant contribution to the currents on the disk and all other modes will be negligible.

Thus in the case of a thin substrate and for frequencies around $\omega_{anm}^e$, a single mode approximation can be employed in which the TM$_{anm}$ is the only mode considered. Thus, in this case, equations (18) and (19) simplify to

$$a_{anm} = B_{nm}^f(a)/A_{nm,nm}^{ff}(a,a)$$

(22)

and therefore

$$K_a (k_\rho) = -c a_n (k_\rho) \cdot a_{anm} \cdot F_{nm,nm}(a,a,c_0 k_\rho)$$

(23)
IV. Perturbation Formula for the Resonant Frequencies

The exact resonant frequency of the structure, acting as a resonator, can be obtained by setting the determinant of the coefficients of equations (20) and (21) to zero, to get
\[ \text{det}[\begin{bmatrix} \mathbf{A}^F & \mathbf{A}^T \end{bmatrix}^{-1} \mathbf{A}^T - 1 \mathbf{A}^F - 1 \mathbf{A}^T] = 0. \]

As it is clear from this equation, the calculation of the resonant frequencies using this equation is not a simple task. However, in the limit of a thin substrate, a perturbational approach can be used to calculate the resonant frequencies in a much simpler way.

In this limit, the elliptic microstrip structure can be viewed as a perturbation of an elliptic resonator with a perfectly magnetic wall. The resonant frequency shift of this perturbed magnetic wall cavity is given by [3]:

\[ \Delta \omega = \omega_f - \omega_i - \frac{L}{4<\omega_T>_i} \]  \hspace{1cm} (24)

where
\[ L = -i \int_{\Delta S} \int ds \, \hat{n} \cdot (\mathbf{E}^* \times \mathbf{H}_f) \]
\[ <\omega_T>_i = \frac{1}{2} \epsilon_1 \int_{V} \int \int dV |\mathbf{E}_i|^2 \]

Where \( \mathbf{E}_i \) and \( \omega_i \) are the electric field and the resonant frequency of the unperturbed cavity, \( \mathbf{H}_f \) and \( \omega_f \) are the magnetic field and the resonant frequency of the perturbed cavity, \( <\omega_T>_i \) is the unperturbed time average total energy stored in the cavity and \( \Delta S \) is the surface area of the magnetic wall.

In the unperturbed case and because the substrate thickness is assumed to be small, the field components are independent of \( z \). Thus, the only existing modes are the \( \text{TM}_{anm} \) modes for which \( \mathbf{E}_z \) is the only nonvanishing electric field component.
Let us consider the perturbation of the $TM_{\alpha nm}$ mode whose resonant frequency of the magnetic wall cavity is $\omega_{\alpha nm}$ given as the solution of the equation

$$J_{\alpha n}(c_0 k_{\alpha nm}^e, u_s) = 0$$

where $k_{\alpha nm}^e = \omega_{\alpha nm} \sqrt{\mu_0}$

$$E_i = \frac{\dot{z}}{A} \psi_{\alpha n}(c_0 k_i^e, u, v), \quad k_i = k_{\alpha nm}^e, \quad \omega_i = \omega_{\alpha nm} \quad (25)$$

and therefore

$$L = i \int_{-d}^{0} dz \int_{0}^{2\pi} dv \ h E_{iz}^* H_{fv}$$

evaluated at $u = u_s$. Applying the boundary conditions, it can be easily shown that

$$E_{fz} = -\frac{e}{\varepsilon_1} \sum_{\gamma, r} k_0 \sum_{k_0} e_{\gamma r}(k_0) (1 + R_{TM}) \frac{\cos k_{1z}(z+d)}{\cos k_{1z}d} \psi_{\gamma r}(c_0 k_0^e, u, v)$$

$$H_{fz} = \sum_{\gamma, r} \int_{0}^{\infty} dk_0 k_0 h_{\gamma r}(k_0) (1 + R_{TE}) \frac{\sin k_{1z}(z+d)}{\sin k_{1z}d} \psi_{\gamma r}(c_0 k_0^e, u, v)$$

and therefore

$$H_{fv} = \frac{i}{h} \sum_{\gamma, r} \int_{0}^{\infty} dk_0 \frac{k_{1z}}{k_0} h_{\gamma r}(k_0) (1 + R_{TE}) \frac{\cos k_{1z}(z+d)}{\sin k_{1z}d} \frac{\partial}{\partial u} \psi_{\gamma r}(c_0 k_0^e, u, v)$$

$$- i\omega \int_{0}^{\infty} dk_0 \frac{k_{1z}}{k_0} e_{\gamma r}(k_0) (1 + R_{TM}) \frac{\cos k_{1z}(z+d)}{\cos k_{1z}d} \frac{\partial}{\partial u} \psi_{\gamma r}(c_0 k_0^e, u, v)$$
\[ L = \pi A^* \int_{c_0 k_i, u_s}^{c_0 k_i, u_s} dk \rho \left( \frac{1}{k^2} \right) h_{\bar{\alpha}}(k_\rho) (1 + \mathbf{R}_{TE}) \int_{c_0 k_\rho, u_s}^{c_0 k_\rho, u_s} Q_{\bar{\alpha}}(c_0 k_\rho, c_0 k_i) + \omega \mathcal{E} \int_{c_0 k_\rho, u_s}^{c_0 k_\rho, u_s} \frac{1}{k^2} e_{\alpha}(k_\rho) (1 + \mathbf{R}_{TM}) \tan \frac{k_1 z_d}{k_1 z} J_{\alpha}(c_0 k_\rho, u_s) N_{\alpha}(c_0 k_i, c_0 k_\rho) \right] \]

where \( Q_{\alpha}(c_0 k_\rho, c_0 k_i) \) and \( N_{\alpha}(c_0 k_i, c_0 k_\rho) \) are defined and given in Appendix (4.7) and by \( \bar{\alpha} \) we mean the parity opposite to \( \alpha \).

In the limit when \( d/a \to 0 \), and from (4), (15), (10) and (25), \( h_{\bar{\alpha}}(k_\rho) \) and \( e_{\alpha}(k_\rho) \) can be approximated as

\[
e_{\alpha}(k_\rho) = -c_{\alpha}(k_\rho) \frac{k_\rho}{2 \omega \mathcal{E}} a_{\alpha}(\alpha, \alpha, c_0 k_\rho) \]

\[
h_{\bar{\alpha}}(k_\rho) = -c_{\bar{\alpha}}(k_\rho) \frac{k_\rho}{2 k \nu} a_{\alpha}(\alpha, \bar{\alpha}, c_0 k_\rho) \]

and

\[
a_{\alpha} = \frac{i \omega \nu \epsilon}{k_1} A
\]

and from Appendix (4.7) we get

\[
e_{\alpha}(k_\rho) = \frac{A}{2} \pi k_\rho c_{\alpha}(k_\rho) \frac{\omega \nu \epsilon}{\omega \epsilon} \frac{1}{k^2} \int_{c_0 k_\rho, u_s}^{c_0 k_\rho, u_s} J_{\alpha}(c_0 k_\rho, u_s) J_{\alpha}(c_0 k_\rho, u_s) N_{\alpha}(c_0 k_i, c_0 k_\rho)
\]

\[
h_{\bar{\alpha}}(k_\rho) = \frac{A}{2} \pi c_{\bar{\alpha}}(k_\rho) \frac{k_z}{k_1} \frac{\omega \nu \epsilon}{\omega \nu \epsilon} \int_{c_0 k_\rho, u_s}^{c_0 k_\rho, u_s} J_{\bar{\alpha}}(c_0 k_\rho, u_s) J_{\alpha}(c_0 k_i, u_s) Q_{\alpha}(c_0 k_i, c_0 k_\rho)
\]
and finally
\[
\langle \omega_i \rangle = \frac{1}{2} \varepsilon_1 |A|^2 d \int_0^{2\pi} dv \int_0^u du \ h^2 [\psi \alpha_n(c_0k_i, u, v)]^2
\]

and from Appendix (4.7), we have

\[
\langle \omega_i \rangle = - \frac{1}{4} \pi \varepsilon_1 \frac{|A|^2}{k_i} d J_{\alpha_n}(c_0k_i, u_s) \frac{3}{\delta k_o} J_{\alpha'_n}(c_0k_o, u_s) \bigg|_{k_o=k_i} R_{\alpha_n}(c_0k_i)
\]

(29)

Substituting (26), (27), (28) and (29) into (24) we get a perturbational formula for the resonant frequency shift of the $TM_{\alpha m}$ mode.

V. Calculation of the Input Impedance

The input impedance across the terminals of the probe $P$ is given by [18] (see appendix of Chapter 3):

\[
Z_{in} = - \frac{1}{I^2} \int dV \ \overline{E}_1 \cdot \overline{J}
\]

where $\overline{J}$ is the current distribution on probe $P$ and is given by

\[
\overline{J} = \hat{z} I \frac{1}{h^2} \delta(z-z_0) \delta(v-v_0) \quad -d < z < 0
\]

and $\overline{E}_1$ is the total electric field in the substrate region due to the current on the disk $D$ as well as the current on the probe $P$. The volume integration in the expression of the input impedance is carried out over
the volume of the probe.

\[ Z_{in} = - \frac{1}{I^2} \int d\mathbf{v} \mathbf{E}_{1z} \mathbf{J}_z = - \frac{1}{I^2} \int_{-d}^{0} dz \int_{0}^{2\pi} dv \int_{0}^{u_s} du^2 \mathbf{h}^2 \mathbf{E}_{1z} \mathbf{J}_z \]

with

\[ E_{1z} = E_{1z}^p + E_{1z}^i \]

where \( E_{1z}^p \) is the electric field due to the current on the probe and is given by equation (4.6.3) of Appendix (4.6).

\( E_{1z}^i \) is the electric field due to the induced currents on the disk and can be shown to be given by

\[ E_{1z}^i = - \frac{1}{2i\omega e_1} \sum_{p,\gamma} \int_0^\infty dk_p \ k_p^2 \left[ K_p(k_p) \right]_u \ \frac{\cos k_{1z}(z+d)}{\cos k_{1z}d} \ \psi_p(c_o k_p, u, v)(1+\eta) \]

Thus the input impedance is given by

\[ Z_{in} = \frac{1}{2\omega e_1} \sum_{n,\alpha} \int_0^\infty dk_p \ \frac{\omega_n(k_p)\{\psi_{\alpha_n}(c_o k_p, u_o, v_o)\}}{k_{1z}^2} \]

\[ \left[ \frac{2k_p^2 \sin(k_{1z}d/2)}{k_{1z}} - k_{1z}^2 \right] + \frac{ik_{1z}^2}{\eta} \left( 1+\eta \right) \tan(k_{1z}d/2) e^{-ik_{1z}d/2} \]

\[ \left[ 1 + 2 \eta \frac{R_{10}^{TM} e^{i k_{1z}d}}{\chi_{10}^{TM}} \right] \]

\[ + \frac{1}{2i\omega e_1} \sum_{p,\gamma} \int_0^\infty dk_p \ \frac{k_p^2}{k_{1z}^2} \left[ K_p(k_p) \right]_u \left( 1+\eta \right) \tan(k_{1z}d) \]

\[ \psi_p(c_o k_p, u_o, v_o) \]
where \([K_{\gamma p}(k_\rho)]_u\) is the \(u\)-component of \(K_{\gamma p}(k_\rho)\) and \(K_{\gamma p}(k_\rho)\) is given by equations (15), (20) and (21). Alternatively, the current distribution on the probe \(P\) is given by

\[
\mathbf{J} = z \mathbf{I} \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \quad -d < z < 0
\]

where \(\rho_0\) and \(\phi_0\) are related to \(u_0\) and \(v_0\) by

\[
\rho_0 \cos \phi_0 = c_0 \cosh u_0 \cos v_0 \quad \rho_0 \sin \phi_0 = c_0 \sinh u_0 \sin v_0
\]

and the electric field due to the current distribution on the probe is given by equation (54) of Chapter 3. Thus, an alternative expression for the input impedance is

\[
z_{in} = \frac{1}{4\pi \omega e_1} \int_0^\infty dk_\rho \frac{k_\rho}{k_{1z}} \left\{ 2 \left[ \frac{e^{ik_{1z}d/2}}{ik_{1z}} \right] \frac{\sin(k_{1z}d/2)}{k_{1z}} - k_{1z}^2 \right\}
\]

\[
+ \frac{i k_{1z}^2}{k_{1z}^2} (1 - R_{TM}^*) \tan(k_{1z}d/2) e^{-ik_{1z}d/2} \left[ 1 + 2 \frac{R_{TM}^*}{e^{ik_{1z}d}} \right]
\]

\[
+ \frac{1}{2i \omega e_1} \sum_{p, \gamma} \int_0^\infty dk_\rho \frac{k_\rho^2}{k_{1z}} [K_{\gamma p}(k_\rho)]_u (1 + R_{TM}^*) \tan(k_{1z}d) \psi_p(c_0 k_\rho, u_0, v_0)
\]
In the case of thin substrate and since the dominant modes are the TM modes, the current is given approximately by

\[
[K_{\gamma_p}(k_\rho)]_u = -c_{\gamma_p}(k_\rho) \sum_{r,s} a_{\gamma rs} \{F_{rs,p}(\gamma,\gamma,c_0 k_\rho)\}_u
\]

\[
= \pi c_{\gamma p}(k_\rho) J_{\gamma r}(c_0 k_\rho, u_s)
\]

\[
\cdot \sum_{r,s} a_{\gamma rs} \frac{(k_{rs}^e)^2}{k_\rho^2 - (k_{rs}^e)^2} J_{\gamma r}(c_0 k_{rs}^e, u_s) N_{\gamma p}(c_0 k_{rs}^e, c_0 k_\rho)
\]

(30)

where \(a_{\alpha rs}\) are given by the following equation

\[
\sum_{r,\gamma,s} a_{\gamma rs}^{ff} A_{rs, \gamma j k}^{ff}(\gamma, \xi) = B_{jk}^{f}(\xi)
\]

(31)

and under the single mode approximation we get

\[
[K_{\alpha_p}(k_\rho)]_u = \pi c_{\alpha_p}(k_\rho) J_{\alpha p}(c_0 k_\rho, u_s) a_{\alpha nm} \frac{(k_{\alpha nm}^e)^2}{k_\rho^2 - (k_{\alpha nm}^e)^2} J_{\alpha n}(c_0 k_{\alpha nm}^e, u_s) N_{\alpha np}(c_0 k_{\alpha nm}, c_0 k_\rho)
\]

(32)

and thus

\[
z_{in} = \frac{1}{4\pi \omega \epsilon_1} \int_0^\infty dk_\rho \frac{k_\rho}{k_{1z}} \left[ \frac{2}{ik_{1z}} \frac{2k_{1z}^2 d/2}{k_{1z}^2} \frac{\sin(k_{1z} d/2)}{k_{1z}} - k_{1z}^2 \right]
\]

\[
+ \frac{ik_{1z}^2}{k_{1z}} (1 - R_{TM}) \tan(k_{1z} d/2) e^{-ik_{1z} d/2} \left[ i + 2 \frac{R_{TM}}{e} \frac{ik_{1z}^d}{10} \right]
\]

\[
+ \frac{\gamma}{2i \omega \epsilon_1} a_{\alpha nm} J_{\alpha n}(c_0 k_{\alpha nm}^e, u_s) \int_0^\infty dk_\rho \frac{k_\rho^2}{k_{1z}} \frac{(k_{\alpha nm}^e)^2}{k_\rho^2 - (k_{\alpha nm}^e)^2} (1 + R_{TM}) \tan(k_{1z} d)
\]

\[
\sum_{p} c_{\alpha_p}(k_\rho) J_{\alpha p}(c_0 k_\rho, u_s) N_{\alpha np}(c_0 k_{\alpha nm}, c_0 k_\rho) \nu_{\alpha p}(c_0 k_\rho, u_0, v_0)
\]
where $a_{nm}$ is given by equation (22).
VI. Radiation Pattern

In this section, we will be only interested in developing a far field expression for the electric field in the thin substrate case. In this case and because the structure approaches the magnetic wall model, and as we mentioned before (section (VI) of Chapter 3), the contribution of the current on the probe to the radiation field is negligible to that of the currents induced on the disk.

From equations (1) and (4), the z-component of the electric field due to the current on the disk in the upper half space is given by

\[ E_z = \frac{1}{2i\omega} \sum_{p,\gamma} \int_0^{\infty} dk_p \, k_p^2 [\mathcal{K}_p(k_p)]_u \psi_p(c_0 k_p, u, v) (1-R^TM) e^{ik_z z} \]

Under the single mode approximation, \([\mathcal{K}_p(k_p)]_u\) is given by equations (32) and (22) and thus we have

\[ E_z = \frac{\pi}{2i\omega} \sum_{p} \alpha_{nm} J_\alpha_n(c_0 k_{nm} u_s) \int_0^{\infty} dk_p \, k_p^2 \frac{(k_{nm}^e)^2}{k_p^2 - (k_{nm}^e)^2} (1-R^TM) e^{ik_z z} \]

\[ \sum_p c_{\alpha p}(k_p) J_{\alpha p}(c_0 k_p, u_s) N_{\alpha p}(c_0 k_{nm} e, c_0 k_p) S_{\alpha p}(c_0 k_p, v) J_{\alpha p}(c_0 k_p, u) \]

It can be easily shown that the characteristic value \(b\) together with the expansion coefficients \(b^m_{n}(c_0 k_p)\), \(F^m_{n}(c_0 k_p)\) are even functions of \(k_p\).

Therefore

\[ J_{\alpha p}(c_0 k_p, u_s) = (-1)^p J_{\alpha p}(c_0 k_p, u_s) \]
\[ H_{\alpha p}(1)(c_0 k_p, u) = (-1)^p H_{\alpha p}(1)(c_0 k_p, u) \]

and \(N_{\alpha p}(c_0 k_{nm} e, c_0 k_p)\), \(S_{\alpha p}(c_0 k_p, v)\) are even in \(k_p\), whereas \(c_{\alpha p}(k_p)\) is odd in \(k_p\).
Thus the integral along the positive real axis in the expression of $E_z$ can be extended to an integral over the whole real axis, to get

$$E_z = \frac{\pi}{4iw\varepsilon} a_{\alpha nm} J_\alpha(c_0k^e_{nm}, u_s) \sum_p \int_{-\infty}^{\infty} dk_\rho \frac{(k^e_{nm})^2}{k_\rho^2 - (k^e_{nm})^2} (1-R^{TM}) e^{ik_\rho z}$$

$$\cdot c_\alpha(k_\rho) J_\alpha'(c_0k^e_{\rho}, u_s) N_{\alpha p}(c_0k^e_{nm}, c_0k_\rho) S_{\alpha p}(c_0k^e_{\rho}, v) H_\alpha^{(1)}(c_0k^e_{\rho}, u)$$

In the far field, it can be easily shown that $c_0 \cosh u \rightarrow \rho$, and from Appendix (4.1), the asymptotic expansion of $H_\alpha^{(1)}(c_0k^e_{\rho}, u)$ is thus given by

$$H_\alpha^{(1)}(c_0k^e_{\rho}, u) - L_\alpha'(c_0k^e_{\rho}) \frac{1}{\sqrt{k^e_{\rho}}} \exp\left\{i\left(k^e_{\rho} - 2p + \frac{1}{2}\right)\right\}$$

$$- \frac{\pi}{2} (-1)^P L_\alpha'(c_0k^e_{\rho}) H_0^{(1)}(k^e_{\rho})$$

where

$$L_\alpha(p_{c_0k^e_{\rho}}) = S_{\alpha p}(c_0k^e_{\rho}, 0)$$

and

$$L_\alpha'(p_{c_0k^e_{\rho}}) = S_{\alpha p}'(c_0k^e_{\rho}, 0)$$

Thus in the far field, i.e. for large values of $\rho$ and $z$, the integrand is rapidly oscillating and the integral can be thus evaluated using the method of stationary phase. A stationary point exists at $k^e_{\rho} = k \sin \theta$ where $\theta = \tan^{-1}(\rho/z)$. Thus we get
\[ E_z = -\frac{\pi}{4i\omega_e} a_{nm} k^2 \sin \theta \cos \theta \frac{(k_{nm}^e)^2}{(k_{nm}^e)^2 - k^2 \sin^2 \theta} J_\alpha n(c_0 k_{nm}^e, u_s) \]

\[ \cdot [1 - R^{TM}(k_{p} = k \sin \theta)] \int \frac{\pi}{Z} (-i)^p c_\alpha p(k \sin \theta) L_\alpha p(c_0 k \sin \theta) \]

\[ \cdot J_\alpha'(c_0 k \sin \theta, u_s) N_\alpha np(c_0 k_{nm}^e, c_0 k \sin \theta) S_\alpha p(c_0 k \sin \theta, \phi) \]

\[ \cdot \int_{-\infty}^{\infty} dk_\phi \frac{k_{\phi}}{k_z e^{ik_z z}} H_1(1)(k_{\phi}) \]

Using Sommerfeld's identity [17] we finally get:

\[ E_z = \frac{\pi}{2\omega_e} \sqrt{\frac{\pi}{Z}} a_{nm} k^2 \sin \theta \cos \theta \frac{(k_{nm}^e)^2}{(k_{nm}^e)^2 - k^2 \sin^2 \theta} J_\alpha n(c_0 k_{nm}^e, u_s) \]

\[ \cdot [1 - R^{TM}(k_{p} = k \sin \theta)] \sum_p (-i)^p c_\alpha p(k \sin \theta) L_\alpha p(c_0 k \sin \theta) \]

\[ \cdot N_\alpha np(c_0 k_{nm}^e, c_0 k \sin \theta) J_\alpha'(c_0 k \sin \theta, u_s) S_\alpha p(c_0 k \sin \theta, \phi) \frac{e^{ikr}}{r} \]

where

\[ r = \sqrt{\rho^2 + z^2} \]

and in the far field \( v \to 0 \). Similarly the expression for \( H_z \) in the far field is given by
\[ H_z = - \frac{\pi}{2} \sqrt{\frac{\pi}{2}} a_{\alpha nm} k \sin \theta J_{\alpha n}(c_0 k a_{\alpha nm}^e u_s) \left[ 1 + R^{TE}(k_\phi = k \sin \theta) \right] \]

\[ \sum \frac{(-i)^p c_{\alpha p}(k \sin \theta) L_{\alpha p}(c_0 k \sin \theta) Q_{\alpha np}(c_0 k a_{\alpha nm}^e, c_0 k \sin \theta)}{J_{\alpha p}(c_0 k \sin \theta, u_s) S_{\alpha p}(c_0 k \sin \theta, \phi) \frac{e^{ikr}}{r}} \]

where \( \bar{\alpha} = e \) if \( \alpha = 0 \) and \( \bar{\alpha} = 0 \) if \( \alpha = e \) and \( a_{\alpha nm} \) is given by equation (22) and finally in the far field

\[ E_\phi = -E_z / \sin \theta \]

\[ E_\phi = -\sqrt{u/\epsilon} \frac{H_z}{\sin \theta} \]
VII. **Conclusions**

Scaler and Vector Mathieu Transforms have been developed, allowing the rigorous formulation of the elliptic disk microstrip antenna which has long been thought almost impossible to formulate exactly [22]. The properties of these transforms are also studied.

It is shown that the current distribution on the disk is rigorously derived using these transforms and that it is governed by vector integral equations. These vector integral equations are then solved using Galerkin's moment method in which the current distribution on the disk is expanded in terms of the TM and TE current modes of the magnetic wall cavity model which form a complete set of basis functions. The limit of small substrate thickness is then applied to these equations to get simple approximate expressions for the current amplitudes.

The resonance in the elliptic disk structure is then analyzed using two different approaches: Galerkin's method and a perturbative approach.

The input impedance together with the radiation pattern are derived both exactly and in the small substrate thickness limit.
Appendix 4.1

Mathieu Functions

\( \psi_{\alpha_n}(c_0 k_\rho, u, v) \) is an elliptic wave function which is the solution to the wave equation in the elliptic cylinder coordinate system. It satisfies the differential equation

\[
\left( \frac{1}{h^2} \frac{\partial^2}{\partial u^2} + \frac{1}{h^2} \frac{\partial^2}{\partial v^2} + k_\rho^2 \right) \psi_{\alpha_n}(c_0 k_\rho, u, v) = 0
\]

It is given by the product of an angular function \( \alpha_n(c_0 k_\rho, v) \) and a radial function \( \alpha_n(c_0 k_\rho, u) \) which satisfy the following differential equations

\[
\frac{d^2}{dv^2} \alpha_n(c_0 k_\rho, v) + (b_n - c_0^2 k_\rho^2 \cos^2 v) \alpha_n(c_0 k_\rho, v) = 0
\]

\[
\frac{d^2}{du^2} \alpha_n(c_0 k_\rho, u) - (a_n - c_0^2 k_\rho^2 \cosh^2 u) \alpha_n(c_0 k_\rho, u) = 0
\]

and are given by the following expansions

for \( \alpha = \text{e} \) (stands for even)

\[
\text{Se}_n(c_0 k_\rho, v) = \sum_r r^n \rho_r(c_0 k_\rho) \cos rv
\]

\[
\text{Je}_n(c_0 k_\rho, u) = \sqrt{\frac{\pi}{2}} \sum_r i^{n-r} \rho_r(c_0 k_\rho) J_r(c_0 k_\rho \cosh u)
\]

\[
\psi_{\text{e}}_n(c_0 k_\rho, u, v) = \sqrt{\frac{\pi}{2}} \sum_r i^{r-n} \rho_r(c_0 k_\rho) \cos r \phi J_r(k_\rho \phi)
\]
for \( \alpha = 0 \) (stands for odd)

\[
S_{\alpha n}(c_0k_\rho, v) = \sum_r' F_r^n(c_0k_\rho) \sin rv
\]

\[
J_{\alpha n}(c_0k_\rho, u) = \sqrt{\frac{\pi}{2}} \tanh u \sum_r' r^{n-r} F_r^n(c_0k_\rho) J_r(c_0k_\rho \cosh u)
\]

\[
\psi_{\alpha n}(c_0k_\rho, u, v) = \sqrt{\frac{\pi}{2}} \sum_r' i^{r-n} F_r^n(c_0k_\rho) \sin r\phi J_r(k_\rho)
\]

Another radial harmonic which is a solution to the wave equation and which is linearly independent of \( J_{\alpha n}(c_0k_\rho, u) \), representing outgoing waves, is given by

\[
H_{\alpha n}^{(1)}(c_0k_\rho, u) = \sqrt{\frac{\pi}{2}} \sum_r' i^{n-r} D_r^n(c_0k_\rho) H_r^{(1)}(c_0k_\rho \cosh u)
\]

\[
H_{\alpha n}^{(1)}(c_0k_\rho, u) = \sqrt{\frac{\pi}{2}} \tanh u \sum_r' r^{n-r} F_r^n(c_0k_\rho) H_r^{(1)}(c_0k_\rho \cosh u)
\]

It can be easily shown, using the asymptotic expansion of the Bessel and Hankel functions, that the asymptotic expansions of the radial functions are given by

\[
J_{\alpha n}(c_0k_\rho, u) \sim \frac{1}{\sqrt{c_0k_\rho \cosh u}} S_{\alpha n}(c_0k_\rho, 0) \cos \left( c_0k_\rho \cosh u - \frac{2n+1}{4} \pi \right)
\]

\[
J_{\alpha n}(c_0k_\rho, u) \sim \frac{1}{\sqrt{c_0k_\rho \cosh u}} S_{\alpha n}(c_0k_\rho, 0) \cos \left( c_0k_\rho \cosh u - \frac{2n+1}{4} \pi \right)
\]
\[ H_{\nu}^{(1)}(c_0 k_{\rho}, u) = -\frac{1}{\sqrt{c_0 k_{\rho} \cosh u}} S_{\nu}(c_0 k_{\rho}, o) \exp\left\{ i \left( c_0 k_{\rho} \cosh u - \frac{2n+1}{4} \pi \right) \right\} \]

\[ H_{\nu}^{(1)}(c_0 k_{\rho}, u) = -\frac{1}{\sqrt{c_0 k_{\rho} \cosh u}} S_{\nu}'(c_0 k_{\rho}, o) \exp\left\{ i \left( c_0 k_{\rho} \cosh u - \frac{2n+1}{4} \pi \right) \right\} \]

where these asymptotic expansions are for \( u \to \infty \), and

\[ S_{\nu}(c_0 k_{\rho}, o) = \sum_{r} p_{r}(c_0 k_{\rho}) \quad \quad \quad S_{\nu}'(c_0 k_{\rho}, o) = \sum_{r} r f_{r}(c_0 k_{\rho}) \]

From the expansions of the angular and radial functions, it can be easily shown that

\[ S_{\nu}^{(1)}(c_0 k_{\rho}, o) = 0, \quad J_{\nu}^{(1)}(c_0 k_{\rho}, o) = 0 \]

\[ S_{\nu}(c_0 k_{\rho}, o) = 0, \quad J_{\nu}(c_0 k_{\rho}, o) = 0 \]

\[ J_{\nu}(c_0 k_{\rho}, u) \big|_{u=\infty} = J_{\nu}(c_0 k_{\rho}, u) \big|_{u=\infty} = 0 \]

For further details on properties of Mathieu functions, see Straton [19], pp. 52-55, pp. 375-387, Morse and Feshback [16], pp. 562-568, pp. 1406-1422, pp. 1568-1573 and McLachlan [27].
Normalization of the Mathieu Functions

The coefficients of expansions $D_n^m(c_o k_\rho)$ and $F_n^m(c_o k_\rho)$ are determined by a recursion formula in which all the coefficients of the series are referred to an initial one which is arbitrary.

There are two ways, popular in literature, to normalize the Mathieu functions:

The first is to choose the initial coefficients in such a way that $S_{e_m}$ and the slope of $S_{o_m}$ are unity at $v = 0$. Thus, we get $\sum_n D_n^m(c_o k_\rho) = 1$ and $\sum_n F_n^m(c_o k_\rho) = 1$. This would be done in order to make the angular functions correspond as closely as possible to the trigonometric functions, to which they reduce as $k_\rho \to 0$.

Another method of normalization is to set $\int_0^{2\pi} dv [S_{e_m}]^2 = \pi (1 + \delta_{m0})$ and $\int_0^{2\pi} dv [S_{o_m}]^2 = \pi$ as it is the case for the limiting form of $S_{e_m}$ and $S_{o_m}$ as $k_\rho \to 0$. In this case, we get $R_{e_m}(c_o k_\rho) = \sum_n (1 + \delta_{n0}) [D_n^m(c_o k_\rho)]^2 = (1 + \delta_{m0})$ and $R_{o_m}(c_o k_\rho) = \sum_n [F_n^m(c_o k_\rho)]^2 = 1$.

It seems that the convenient normalization method to our application is the second one, since this will simplify the form of the Sclar and Vector Mathieu Transforms that will be developed in the next appendices.

*See Morse and Feshback [16], p. 1409.
Appendix 4.2

Orthogonality Relations of the Angular Harmonics

Let \( b_n \) and \( b_m \) be two characteristic values and \( \text{Se}_n(c_0 k_\rho, \nu) \), \( \text{Se}_m(c_0 k_\rho, \nu) \) the associated angular harmonics. These satisfy the following differential equations:

\[
\frac{d^2 \text{Se}_n}{dv^2} + (b_n - c_0^2 k_\rho^2 \cos^2 \nu) \text{Se}_n = 0 \quad (4.2.1)
\]

\[
\frac{d^2 \text{Se}_m}{dv^2} + (b_m - c_0^2 k_\rho^2 \cos^2 \nu) \text{Se}_m = 0 \quad (4.2.2)
\]

Multiplying (4.2.1) by \( \text{Se}_m \) and (4.2.2) by \( \text{Se}_n \) and subtracting we get

\[
\frac{d}{dv}\left(\frac{\text{Se}_n}{\text{Se}_n} \frac{d}{dv} \text{Se}_m - \frac{\text{Se}_m}{\text{Se}_m} \frac{d}{dv} \text{Se}_n\right) + (b_m - b_n) \text{Se}_n \text{Se}_m = 0
\]

Integrating over \( \nu \) from 0 to \( 2\pi \) we get

\[
(b_m - b_n) \int_0^{2\pi} dv \text{Se}_m(c_0 k_\rho, \nu) \text{Se}_n(c_0 k_\rho, \nu) = - \left[ \text{Se}_n \frac{d}{dv} \text{Se}_m - \text{Se}_m \frac{d}{dv} \text{Se}_n \right]_0^{2\pi} \quad (4.2.3)
\]

From the periodicity of the functions \( \text{Se}_n \), \( \text{Se}_m \) and their derivatives \( \frac{d}{dv} \text{Se}_m \), \( \frac{d}{dv} \text{Se}_n \), the RHS of (4.2.3) vanishes. Therefore,
\[ \int_0^{2\pi} \text{d}v \, S_n(c_0 k_\rho, v) \, S_m(c_0 k_\rho, v) = 0 \quad \text{for } n \neq m \quad (4.2.4) \]

Alternatively, we have

\[ S_n(c_0 k_\rho, v) = \sum_r D^n_r(c_0 k_\rho) \cos rv \]

\[ S_m(c_0 k_\rho, v) = \sum_k D^m_k(c_0 k_\rho) \cos kv \]

Therefore

\[ \int_0^{2\pi} \text{d}v \, S_n(c_0 k_\rho, v) \, S_m(c_0 k_\rho, v) = \sum_r \sum_k D^n_r(c_0 k_\rho) D^m_k(c_0 k_\rho) \int_0^{2\pi} \text{d}v \cos rv \cos kv \]

but

\[ \int_0^{2\pi} \text{d}v \cos rv \cos kv = \pi (1 + \delta_{r0}) \delta_{kr} \]

Therefore

\[ \int_0^{2\pi} \text{d}v \, S_n(c_0 k_\rho, v) \, S_m(c_0 k_\rho, v) = \pi \sum_r (1 + \delta_{r0}) D^n_r(c_0 k_\rho) D^m_r(c_0 k_\rho) \quad (4.2.5) \]

From (4.2.4) and (4.2.5) we get
\[ \int_{0}^{2\pi} \text{dv } S_n(c_0 k_\rho, v) S_m(c_0 k_\rho, v) = 0 \quad n \neq m \]

\[ = \pi \sum' \left(1 + \delta_{r_0}\right) [D^n_r(c_0 k_\rho)]^2 \quad n = m \]

Or in a more concise form

\[ \int_{0}^{2\pi} \text{dv } S_n(c_0 k_\rho, v) S_m(c_0 k_\rho, v) = \delta_{nm} \pi \text{Re}_n(c_0 k_\rho) \]

where

\[ \text{Re}_n(c_0 k_\rho) = \sum' \left(1 + \delta_{r_0}\right) [D^n_r(c_0 k_\rho)]^2 \]

Also, we have

\[ \sum' \left(1 + \delta_{r_0}\right) D^n_r(c_0 k_\rho) D^m_r(c_0 k_\rho) = \delta_{nm} \sum' \left(1 + \delta_{r_0}\right) [D^n_r(c_0 k_\rho)]^2 \quad (4.2.6) \]

Similarly we get

\[ \int_{0}^{2\pi} \text{dv } S_n(c_0 k_\rho, v) S_m(c_0 k_\rho, v) = \delta_{nm} \pi \text{Ro}_n(c_0 k_\rho) \]

where

\[ \text{Ro}_n(c_0 k_\rho) = \sum' [F^n_r(c_0 k_\rho)]^2 \]
and

\[ \sum_r' F_r^n(c_o k_\rho) F_r^m(c_o k_\rho) = \delta_{nm} \sum_r' [F_r^n(c_o k_\rho)]^2 \]  

(4.2.7)

whereas

\[ \int_0^{2\pi} dv \, S_{n}(c_o k_\rho, v) \, S_{m}(c_o k_\rho, v) = 0 \]

\[ \therefore \text{In general,} \]

\[ \int_0^{2\pi} dv \, S_{\alpha n}(c_o k_\rho, v) \, S_{\gamma m}(c_o k_\rho, v) = \delta_{\alpha \gamma} \delta_{nm} \pi R_{n}(c_o k_\rho) \]  

(4.2.8)
Appendix 4.3

Orthogonality Relations of the Elliptic Harmonics

Define an elliptic harmonic by

\[ \psi_\alpha_n(c_0k_\rho, u, v) = J_\alpha_n(c_0k_\rho, u) S_\alpha_n(c_0k_\rho, v) \]

In this appendix, we will develop the orthogonality relationship of these elliptic harmonics. To do so, let us define

\[ I_{nm}(\alpha, \gamma, c_0k_\rho, c_0k'_\rho) = \int_0^\infty \int_0^{2\pi} h^2 \psi_\alpha_n(c_0k_\rho, u, v) \psi_\gamma_m(c_0k'_\rho, u, v) \, \text{d}u \, \text{d}v \]

where \( h^2 = c_0^2 (\cosh^2 u - \cos^2 v) \) and is the Jacobian of the transformation from the \( x, y \) coordinates to the \( u, v \) coordinates.

In what follows, we will find the value of \( I_{nm} \). \( \psi_\alpha_n \) and \( \psi_\gamma_m \) can be represented in terms of cylindrical harmonics [19] as follows

\[ \psi_\alpha_n(c_0k_\rho, u, v) = \sqrt{\frac{\pi}{2}} \sum_r r^{n-r} F_r^n(c_0k_\rho) \sin(r\phi) J_r(k_\rho) \]

\[ \psi_\gamma_m(c_0k_\rho, u, v) = \sqrt{\frac{\pi}{2}} \sum_r r^{n-r} G_r^n(c_0k_\rho) \cos(r\phi) J_r(k_\rho) \]
Case (i): \( \alpha = \gamma = \epsilon \)

since \( h^2 dudv = \rho d\phi d\rho \)

\[
I_{nm}(e,e,c_o k_\rho, c_o k'_\rho) = \frac{\pi}{2} \sum_r \sum_{k'} i^{r-n} i^{k-m} D^n_r(c_o k_\rho) D^m_r(c_o k'_\rho) \\
\quad \cdot \int_0^{2\pi} d\phi \cos r\phi \cos k\phi \cdot \int_0^\infty d\rho \ J_r(k_\rho) J_r(k'_{\rho'})
\]

since \( \int_0^{2\pi} d\phi \cos k\phi \cos r\phi = \pi(1 + \delta_{r0}) \delta_{kr} \)

\[
I_{nm}(e,e,c_o k_\rho, c_o k'_\rho) = \frac{\pi^2}{2} \sum_r i^{2r-n-m}(1 + \delta_{r0}) D^n_r(c_o k_\rho) D^m_r(c_o k'_\rho) \int_0^\infty d\rho \ J_r(k_\rho) J_r(k'_{\rho'})
\]

with both \( n \) and \( m \) should be either even or odd, and since

\[
\int_0^\infty d\rho \ J_r(k_\rho) J_r(k'_{\rho'}) = \frac{1}{k_\rho} \delta(k_\rho - k'_{\rho'})
\]

and

\[
i^{2r-n-m} = i^{2r-2n}, \quad i^{n-m} = i^{n-m}
\]

Therefore

\[
I_{nm}(e,e,c_o k_\rho, c_o k'_\rho) = \frac{\pi^2}{2} \frac{\delta(k_\rho - k'_{\rho'})}{k_\rho} i^{n-m} \sum_r (1 + \delta_{r0}) D^n_r(c_o k_\rho) D^m_r(c_o k'_\rho)
\]

and from equation (4.1.6) of Appendix (4.1), we get
\[ I_{nm}(e,e,c_0k'_\rho,c_0k'_\rho) = \delta_{nm} \frac{\pi^2}{2} \frac{\delta(k'_\rho - k'_\rho)}{k'_\rho} \text{Re}_n(c_0k'_\rho) \]

Case (ii): \( \alpha = \gamma = 0 \) (odd)

In exactly the same way as in case (i) and using equation (4.2.7) of Appendix (4.2), it can be easily shown that

\[ I_{nm}(o,o,c_0k'_\rho,c_0k'_\rho) = \delta_{nm} \frac{\pi^2}{2} \frac{\delta(k'_\rho - k'_\rho)}{k'_\rho} \text{Re}_n(c_0k'_\rho) \]

Case (iii): \( \alpha \neq \gamma \)

In this case and because of the orthogonality of the sine and cosine functions, it can be easily shown that

\[ I_{nm}(\alpha,\gamma,c_0k'_\rho,c_0k'_\rho) = 0 \quad \text{for} \ \alpha \neq \gamma \]

Therefore in a more concise form, we have the following orthogonality relationship for the elliptic harmonics

\[
\int_0^{2\pi} du \int_0^\infty dv \ h^2 \psi_\alpha_n(c_0k'_\rho,u,v) \psi_\gamma_m(c_0k'_\rho,u,v) = \delta_{\alpha\gamma} \delta_{nm} \frac{\pi^2}{2} \frac{\delta(k'_\rho - k'_\rho)}{k'_\rho} \text{Re}_n(c_0k'_\rho)
\]

(4.3.1)

Thus the elliptic harmonics \( \{\psi_\alpha_n\} \) form a complete set of basis functions in the \( u-v \) plane.

Thus any function \( K(u,v) \) can be represented as a sum of these elliptic harmonics in the following form
\[ K(u, v) = \sum_n \sum_\alpha \int_0^\infty dk_\rho \psi_\alpha_n(c_{o\rho}k_\rho, u, v) K_\alpha_n(k_\rho) \] (4.3.2)

To get the coefficient of expansion \( K_\alpha_n(k_\rho) \), multiply (4.3.2) by \( h^2\psi_m(c_{o\rho}k'_\rho, u, v) \) and integrate over \( u \) and \( v \), then applying the orthogonality relation (4.3.1), we get

\[ K_\alpha_n(k_\rho) = \frac{2}{\pi^2} \frac{k_\rho}{R_\alpha_n(c_{o\rho}k_\rho)} \int_0^\infty du \int_0^{2\pi} dv \ h^2\psi_\alpha_n(c_{o\rho}k_\rho, u, v) K(u, v) \] (4.3.3)

In particular, when \( K(u, v) = \frac{1}{h^2} \delta(u - u') \delta(v - v') \)

\[ K_\alpha_n(k_\rho) = \frac{2}{\pi^2} \frac{k_\rho}{R_\alpha_n(c_{o\rho}k_\rho)} \psi_\alpha_n(c_{o\rho}k_\rho, u', v') \]

and therefore we get the following orthogonality relation in the \( k_\rho \) domain

\[ \frac{1}{h^2} \delta(u - u') \delta(v - v') = \frac{2}{\pi^2} \sum_n \sum_\alpha \int_0^\infty dk_\rho \frac{k_\rho}{R_\alpha_n(c_{o\rho}k_\rho)} \psi_\alpha_n(c_{o\rho}k_\rho, u, v) \psi_\alpha_n(c_{o\rho}k_\rho, u', v') \] (4.3.4)
Appendix 4.4

Scaler Mathieu Transform (SMT)

From Appendix (4.3), it was shown that a function \( K(u,v) \) can be represented as follows:

\[
K(u,v) = \frac{2}{\pi^2} \sum_n \sum_\alpha \int_0^\infty dk_\rho \frac{k_\rho}{R_n(c_o k_\rho, u, v)} \psi_n(c_o k_\rho, u, v) K_n(k_\rho)
\]

where

\[
K_n(k_\rho) = \int_0^\infty du \int_0^{2\pi} dv \ h^2 \psi_n(c_o k_\rho, u, v) K(u,v)
\]  \hspace{1cm} (4.4.1)

If furthermore we define \( K_n(u,v) \) to be the \( n \)-th and \( \alpha \)-th harmonic of \( K(u,v) \), i.e. \( K(u,v) = \sum_n \sum_\alpha K_n(u,v) \), \( \therefore K_n(u,v) \) and \( K_n(k_\rho) \) are related by

\[
K_n(u,v) = \frac{2}{\pi^2} \int_0^\infty dk_\rho \frac{k_\rho}{R_n(c_o k_\rho)} \psi_n(c_o k_\rho, u, v) K_n(k_\rho)
\]

and by using the orthogonality relation (4.3.1) of Appendix (4.3) we get

\[
\int_0^\infty du \int_0^{2\pi} dv \ h^2 \psi_m(c_o k_\rho, u, v) K_n(u,v) = \delta_{\alpha \gamma} \delta_{nm} K_n(k_\rho)
\]  \hspace{1cm} (4.4.2)

from (4.4.1) and (4.4.2) we get

\[
K_n(k_\rho) = \int_0^\infty du \int_0^{2\pi} dv \ h^2 \psi_n(c_o k_\rho, u, v) K_n(u,v) \quad \therefore K(u,v) = \sum_m \sum_\gamma K_m(u,v)
\]
In summary, a function \( K(u,v) \) is defined by the following pair of (SMT)

\[
K(u,v) = \sum_n \sum_\alpha K_n(\alpha, u, v) \tag{4.4.3}
\]

\[
K(k_\rho) = \sum_n \sum_\alpha K_n(k_\rho) \tag{4.4.4}
\]

where

\[
K_n(\alpha, u, v) = \int_0^\infty dk_\rho \omega_n(k_\rho) \psi_n(c_0 k_\rho, u, v) K_\alpha(k_\rho) \tag{4.4.5}
\]

and

\[
\int_0^\infty \frac{2\pi}{0} dv h^2 \psi_m(c_0 k_\rho, u, v) K_n(u, v) = \delta_{\alpha \gamma} \delta_{nm} K_n(k_\rho) \tag{4.4.6}
\]

and

\[
\omega_n(k_\rho) = \frac{2}{\pi^2} \frac{k_\alpha}{R_n(c_0 k_\rho)}
\]

\( K_n(u,v) \) and \( K_n(k_\rho) \) are also related by Parseval's Theorem as follows.

Assume that \( K_n(k_\rho) \) and \( F_n(k_\rho) \) are the SMT of \( K_n(u,v) \) and \( F_\alpha(u,v) \) respectively. Each function and its SMT is related by an expression similar to (4.4.5) and (4.4.6).
Therefore

\[
\int_0^{\infty} du \int_0^{2\pi} dv \ h^2 K_\alpha_n(u,v) F_\alpha_n(u,v) = \int_0^{\infty} du \int_0^{2\pi} dv h^2 \left[ \int_0^{\infty} dk_\rho \omega_n(k_\rho) \psi_n(c_0k_\rho, u, v) K_\alpha_n(k_\rho) \right]
\]

\[
\cdot \left[ \int_0^{\infty} dk'_\rho \omega_n(k'_\rho) \psi'_n(c_0k'_\rho, u, v) F_\alpha_n(k'_\rho) \right]
\]

\[
= \int_0^{\infty} dk_\rho \int_0^{\infty} dk'_\rho \omega_n(k_\rho) \omega_n(k'_\rho) K_\alpha_n(k_\rho) F_\alpha_n(k'_\rho)
\]

\[
\cdot \int_0^{\infty} du \int_0^{2\pi} dv h^2 \psi_n(c_0k_\rho, u, v) \psi'_n(c_0k'_\rho, u, v)
\]

and from the orthogonality relation of the elliptic harmonics we finally get

\[
\int_0^{\infty} du \int_0^{2\pi} dv \ h^2 K_\alpha_n(u,v) F_\alpha_n(u,v) = \int_0^{\infty} dk_\rho \omega_n(k_\rho) K_\alpha_n(k_\rho) F_\alpha_n(k_\rho)
\]

(4.4.7)
Appendix 4.5

Vector Mathieu Transform

In this appendix, we will develop a Vector Mathieu Transform (VMT) which will facilitate the analysis of the elliptic disk antenna.

Let us first define $\tilde{M}_n(c_0 k_\rho, u, v)$ to be the following matrix

$$
\tilde{M}_n(c_0 k_\rho, u, v) = \frac{1}{h} \begin{bmatrix}
\frac{\partial}{\partial u} \psi_n(c_0 k_\rho, u, v) & \frac{\partial}{\partial v} \psi_n(c_0 k_\rho, u, v) \\
\frac{\partial}{\partial v} \psi_n(c_0 k_\rho, u, v) & -\frac{\partial}{\partial u} \psi_n(c_0 k_\rho, u, v)
\end{bmatrix}
$$

Next we will derive the orthogonality relations of $\tilde{M}_n(c_0 k_\rho, u, v)$ and to do so, we will define

$$
\tilde{N}_{nm}(\alpha, \gamma, c_0 k_\rho, c_0 k'_\rho) = \int_0^{2\pi} \int_0^\infty dv \ du \ h^2 \tilde{M}_n(c_0 k_\rho, u, v) \tilde{M}_m(c_0 k'_\rho, u, v) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
$$

where it is obvious that $a_{11} = a_{22}$ and $a_{12} = -a_{21}$

$$
a_{11} = \int_0^{2\pi} \int_0^\infty dv \ S_{\gamma n}(c_0 k_\rho, v) S_{\gamma m}(c_0 k'_\rho, v) \int_0^\infty dv \ J_{\alpha n}(c_0 k_\rho, u) J_{\gamma m}(c_0 k'_\rho, u)
$$

$$
+ \int_0^{2\pi} \int_0^\infty dv \ S_{\alpha n}(c_0 k_\rho, v) S_{\gamma m}(c_0 k'_\rho, v) \int_0^\infty du \ J_{\alpha n}(c_0 k_\rho, u) J_{\gamma m}(c_0 k'_\rho, u)
$$

Using integration by parts together with the D.E. for the radial and the angular harmonics
The following integrals can be recast in the form below

\[ \int_{0}^{\infty} du \ J_{\alpha'}(c_0 k, u) \ J_{\gamma'}(c_0 k', u) = -\int_{0}^{\infty} du (b_n - c_0 k^2 \cosh^2 u) J_{\alpha}(c_0 k, u) J_{\gamma}(c_0 k', u) \]

\[ \int_{0}^{2\pi} dv \ S_{\alpha}(c_0 k, v) \ S_{\gamma}(c_0 k', v) = \int_{0}^{\infty} dv (b_n - c_0 k^2 \cos^2 v) S_{\alpha}(c_0 k, v) S_{\gamma}(c_0 k', v) \]

Where we have used the fact that the radial harmonic functions vanish as \( u \to \infty \) and that \( J_{\alpha}(c_0 k, u)|_{u=0} = 0 \), we also used the periodic properties of the angular harmonic functions.

\[ \therefore a_{11} \text{ reduces to the following form} \]

\[ a_{11} = k_0^2 \int_{0}^{2\pi} dv \int_{0}^{\infty} du \ h_\alpha(c_0 k, u, v) \ h_\gamma(c_0 k', u, v) \]

\[ = k_0^2 \delta_\alpha \delta_\gamma \delta_{nm} \frac{\pi^2}{2} \frac{\delta(k - k')}{k_0^2} R_{\alpha}(c_0 k) \]

where we have used the orthogonality properties of \( \psi_\alpha \) as given by (4.3.1) in Appendix (4.3).
As for \( a_{12} \) we have

\[
a_{12} = \int_0^{2\pi} dv \ S_n(c_0 k_p, v) \ S'_m(c_0 k'_p, v) \int_0^\infty du \ J_n(c_0 k_p, u) J_m(c_0 k'_p, u)
- \int_0^{2\pi} dv \ S'_n(c_0 k_p, v) S'_m(c_0 k'_p, v) \int_0^\infty du \ J_n(c_0 k_p, u) J'_m(c_0 k'_p, u)
\]

Case (i): \( \alpha = \gamma \)

In this case and because of the orthogonality of the cosine and sine functions

\[
\int_0^{2\pi} dv \ S_n(c_0 k_p, v) S'_m(c_0 k'_p, v) = 0
\]

\[
\therefore \text{in this case} \ a_{12} = a_{21} = 0.
\]

Case (ii): \( \gamma \neq \alpha \)

In this case we have

\[
\int_0^{2\pi} dv \ S_n(c_0 k_p, v) S'_m(c_0 k'_p, v) = -\int_0^{2\pi} dv \ S'_n(c_0 k_p, v) S'_m(c_0 k'_p, v)
\]

and

\[
\int_0^\infty du \ J_n(c_0 k_p, u) J_m(c_0 k'_p, u) = \left[ J_n(c_0 k_p, u) J_m(c_0 k'_p, u) \right]_0^\infty
- \int_0^\infty du \ J_n(c_0 k_p, u) J'_m(c_0 k'_p, u)
\]
and for \( \gamma \neq \alpha \),

\[
J_\alpha_n(c_0\kappa, u) J_\gamma_m(c_0\kappa', u) = 0 \quad \text{at } u = 0
\]

therefore in this case

\[
a_{12} = a_{21} = 0
\]

and thus in general

\[
a_{12} = a_{21} = 0
\]

Thus, the orthogonality relation has the following form

\[
\int_0^{2\pi} \int_0^\infty du \, h^2 \tilde{\alpha}_n(c_0\kappa, u, v) \cdot \tilde{\alpha}_m(c_0\kappa', u, v) = \frac{\pi^2}{2} k \delta(k-k') R_{\alpha n}(c_0\kappa)
\]

Thus in the u-v plane, \( \tilde{\alpha}_n(c_0\kappa, u, v) \) form an orthogonal set.

The orthogonality relation in the \( k_\rho \)-domain can be derived by obtaining the expansion coefficient in the following

\[
\int \frac{1}{h^2} \delta(u-u') \delta(v-v') = \sum \sum dk_\rho \tilde{\alpha}_n(c_0\kappa, u, v) \cdot \tilde{\alpha}_n(c_0\kappa', u', v')
\]

Using the orthogonality relation (4.5.1) we can easily find

\( \tilde{\alpha}_n(c_0\kappa, u', v') \) and therefore we get the following orthogonality relation in the \( k_\rho \)-domain
\begin{equation}
\frac{1}{\hbar^2} \delta(u-u')\delta(v-v') = \sum_n \sum_\alpha \int_0^\infty dk_\rho \, c_\alpha_n(k_\rho) \bar{\alpha}_n(c_0k_\rho, u, v) \cdot \bar{\alpha}_n(c_0k_\rho, u', v')
\end{equation}

where

\begin{equation}
c_\alpha_n(k_\rho) = \frac{2}{\pi^2} \frac{1}{k_\rho R_\alpha_n(c_0k_\rho)}
\end{equation}

At this stage, we can introduce the Vector Mathieu Transform (VMT) of a vector \( \mathbf{K}(u, v) \) in a way similar to the Scalar Mathieu Transform (SMT) of a function \( K(u, v) \).

A vector \( \mathbf{K}(u, v) \) is defined by the following pair of (VMT)

\begin{equation}
\mathbf{K}(u, v) = \sum_n \sum_\alpha \mathbf{K}_\alpha_n(u, v)
\end{equation}

\begin{equation}
\mathbf{K}(k_\rho) = \sum_n \sum_\alpha \mathbf{K}_\alpha_n(k_\rho)
\end{equation}

where

\begin{equation}
\mathbf{K}_\alpha_n(u, v) = \int_0^\infty dk_\rho \, c_\alpha_n(k_\rho) \bar{\alpha}_n(c_0k_\rho, u, v) \cdot \bar{\alpha}_n(k_\rho)
\end{equation}

and

\begin{equation}
\int_0^\infty dv \, h^2 \bar{\alpha}_m(c_0k_\rho, u, v) \cdot \bar{\alpha}_n(u, v) = \delta_{\alpha m} \delta_{nm} \bar{\alpha}_n(k_\rho)
\end{equation}
\[ c_n^\alpha(k_\rho) = \frac{2}{\pi^2} \frac{1}{k_\rho} \mathcal{K}_n^\alpha(c_0, k_\rho) \]

Using the symmetry property \( \mathcal{K}_n^\alpha(c_0, k_\rho, u, v) = \mathcal{K}_n^\alpha(c_0, k_\rho, u, v) \) we have the following Parseval's theorem

\[
\int_0^\infty \int_0^{2\pi} \mathcal{K}_n^\alpha(u, v) F_n(u, v) = \int_0^\infty \frac{dk_\rho}{k_\rho} c_n^\alpha(k_\rho) \mathcal{K}_n^\alpha(k_\rho) F_n(k_\rho)
\]

(4.5.7)

where \( \mathcal{K}_n^\alpha(k_\rho) \) and \( F_n^\alpha(k_\rho) \) are the VMT of \( \mathcal{K}_n^\alpha(u, v) \) and \( F_n(u, v) \), respectively.
Appendix 4.6

Field Expressions Due to Probe Excitation

First we will develop field expressions due to the probe excitation in unbounded medium. The electric field \( \vec{E} \) is governed by the following equation

\[

\nabla \times \nabla \times \vec{E}(u,v,z) - k^2 \vec{E}(u,v,z) = i \omega \mu \vec{J}(u,v,z) \tag{4.6.1}

\]

where \( \vec{J}(u,v,z) \) is the current distribution on the probe.

To solve this D.E. we will first develop the dyadic Green's function in the elliptic coordinate system.

Let

\[

\vec{E}(u,v,z) = \int d\vec{r}' \ \vec{g}(\vec{r},\vec{r}') \cdot \vec{J}(u',v',z')

\]

and

\[

\vec{J}(u,v,z) = \int d\vec{r}' \ \vec{J}(u',v',z') \frac{1}{h^2} \delta(u - u') \delta(v - v') \delta(z - z')

\]

Substituting into (4.6.1) we get

\[

\nabla \times \nabla \times \vec{g}(\vec{r},\vec{r}') - k^2 \vec{g}(\vec{r},\vec{r}') = i \omega \mu \frac{1}{h^2} \delta(u - u') \delta(v - v') \delta(z - z')

\]
which has the solution

\[ \bar{G}(\bar{r}, \bar{r}') = \imath \omega \left\{ \mathbb{1} + \frac{1}{k^2} \mathbb{V} \mathbb{V} \right\} g(\bar{r}, \bar{r}') \]

where

\[ (\nabla^2 + k^2) g(\bar{r}, \bar{r}') = -\frac{1}{\hbar^2} \delta(u-u') \delta(v-v') \delta(z-z') \]

Now \( g(\bar{r}, \bar{r}') \) can be expanded in terms of the scaler elliptic harmonics (see Appendices (4.2) and (4.3)) as follows

\[ g(\bar{r}, \bar{r}') = \sum_{n} \sum_{\alpha} \int_{0}^{\infty} dk_{\rho} f(k_{\rho}, z; \bar{r}') \psi_{\alpha n}(c_{0k_{\rho}}, u, v) \]

and therefore

\[ (\nabla^2 + k^2) g(\bar{r}, \bar{r}') = \sum_{n} \sum_{\alpha} \int_{0}^{\infty} dk_{\rho} (\nabla^2 + k^2) \{ f(k_{\rho}, z; \bar{r}') \psi_{\alpha n}(c_{0k_{\rho}}, u, v) \} \]

Using the differential equation governing \( \psi_{\alpha n}(c_{0k_{\rho}}, u, v) \)

\[ \left\{ \frac{1}{\hbar^2} \frac{\partial^2}{\partial u^2} + \frac{1}{\hbar^2} \frac{\partial^2}{\partial v^2} \right\} \psi_{\alpha n}(c_{0k_{\rho}}, u, v) = -k_{\rho}^2 \psi_{\alpha n}(c_{0k_{\rho}}, u, v) \]

It can be easily shown that

\[ (\nabla^2 + k^2) \{ f(k_{\rho}, z; \bar{r}') \psi_{\alpha n}(c_{0k_{\rho}}, u, v) \} = \psi_{\alpha n}(c_{0k_{\rho}}, u, v) \left\{ \frac{\partial^2}{\partial z^2} + k_{\rho}^2 \right\} f(k_{\rho}, z; \bar{r}') \]
where
\[ k_z^2 = k^2 - k_\rho^2 \]

therefore
\[
(\gamma^2 + k^2) \, g(\vec{r}, \vec{r}') = \sum \sum \int_0^{\infty} \frac{dk_\rho}{2} \psi_n(c_0 k_\rho, u, v) \left\{ \frac{3^2}{2z^2} + k_z^2 \right\} f(k_\rho, z; \vec{r}')
\]

and therefore
\[
\sum \sum \int_0^{\infty} \frac{dk_\rho}{2} \psi_n(c_0 k_\rho, u, v) \left\{ \frac{3^2}{2z^2} + k_z^2 \right\} f(k_\rho, z; \vec{r}') = -\frac{1}{h^2} \delta(u-u') \, \delta(v-v') \, \delta(z-z')
\]

Multiplying by \( h^2 \psi_m(c_0 k'_\rho, u, v) \) and integrating over \( u \) and \( v \) and applying

the orthogonality relation developed in Appendix (4.4) we get
\[
\left\{ \frac{3^2}{2z^2} + k_z^2 \right\} f(k_\rho, z; \vec{r}') = -\omega_n(k_\rho) \psi_n(c_0 k_\rho, u', v') \, \delta(z-z')
\]

Let
\[
f(k_\rho, z; \vec{r}') = -\omega_n(k_\rho) \psi_n(c_0 k_\rho, u', v') \, q(z, z')
\]

therefore
\[
\left\{ \frac{3^2}{2z^2} + k_z^2 \right\} q(z, z') = \delta(z-z')
\]

which has the solution
\[-187\]

\[ q(z,z') = c \ e^{\frac{ik_z}{|z-z'|}} \]

and since

\[
\lim_{\varepsilon \to 0} \left\{ \frac{\partial}{\partial z} q(z,z') \bigg|_{z=z'+\varepsilon} - \frac{\partial}{\partial z} q(z,z') \bigg|_{z=z'-\varepsilon} \right\} = 1
\]

we get

\[ c = \frac{1}{2ik_z} \]

therefore

\[
g(\vec{r},\vec{r}') = \frac{i}{2} \sum_n \sum_{\alpha} \int_0^\infty dk_\rho \frac{1}{k_z} \omega_n(k_\rho) \psi_\alpha_n(c_0 k_\rho, u', v') \psi_\alpha_n(c_0 k_\rho, u, v) e^{\frac{ik_z}{|z-z'|}}
\]

and the electric field is thus given by

\[
\vec{E} = -\frac{1}{2\omega_e} \sum_n \sum_{\alpha} (\vec{\mathbf{k}}_z^2 + \omega^2) \int d\vec{r}' J(\vec{r}') \int_0^\infty dk_\rho \frac{1}{k_z} \omega_n(k_\rho) \psi_\alpha_n(c_0 k_\rho, u', v') \psi_\alpha_n(c_0 k_\rho, u, v) e^{\frac{ik_z}{|z-z'|}}
\]

To simplify the analysis, the current distribution on the probe is approximated by a uniform line current situated at \( u = u_0 \) and \( v = v_0 \) and extends from \( z = -d \) to \( z = 0 \), pointing in the \( z \)-direction. The current distribution on the probe is thus represented by

\[
\vec{J}(\vec{r}) = \hat{z} I \ \frac{1}{h^2} \delta(u - u_0) \ \delta(v - v_0) \quad -d < z < 0
\]
Therefore

\[ E = - \frac{1}{2 \omega \varepsilon} \sum \sum \left( \hat{z} k^2 + v \frac{\partial}{\partial z} \right) \int_0^\infty dk_\rho \frac{1}{k_z} \omega_n(k_\rho) \psi_n(c_0k_\rho, u_0, v_0) \psi_n(c_0k_\rho, u, v) f(k_\rho, z) \]

where \( f(k_\rho, z) \) is given by [37]

\[
f(k_\rho, z) = \int_{-d}^0 dz' e^{ik_z|z-z'|} = \begin{cases} 2 \frac{ik_z d/2}{ik_\rho} [e^{ik_z(z-d/2)} - 1] & -d < z < 0 \\ \frac{2}{k_z} e^{ik_z(z+d/2)} \sin (k_z d/2) & 0 < z < -d \end{cases}
\]

Thus in the case of the structure considered in Fig. (1) and applying the stratified medium formulation [17], the \( z \)-component of the electric field in the air region and the substrate is given as follows [12,37]:

\[ E^P_z = - \frac{i I}{2 \omega \varepsilon} \sum \sum \int_0^\infty dk_\rho \frac{k_\rho^2}{k_{1z}^2} \omega_n(k_\rho) (1 - R_{01}^{TM}) \psi_n(c_0k_\rho, u_0, v_0) \psi_n(c_0k_\rho, u, v) e^{ik_z z} \]

for \( z > 0 \) (4.6.2)

\[ E^P_{1z} = - \frac{i I}{2 \omega \varepsilon} \sum \sum \int_0^\infty dk_\rho \frac{1}{k_{1z}} \omega_n(k_\rho) \psi_n(c_0k_\rho, u_0, v_0) \psi_n(c_0k_\rho, u, v) \]

\[
\left\{ \begin{aligned}
\frac{2}{ik_{1z}} & \left[ k_\rho^2 e^{ik_{1z} d/2} \cos(k_{1z}(z+d/2)) - k_{1z}^2 \right] + \frac{2k_\rho^2 \sin(k_{1z} d/2) e^{ik_{1z} d/2}}{k_{1z}(1 + R_{01}^{TM} e^{i2k_{1z} d})} \\
(1 - R_{01}^{TM} e^{ik_{1z} d}) e^{ik_{1z} (z+d/2)} & - (1 + e^{ik_{1z} d}) R_{01}^{TM} e^{-ik_{1z} (z+d/2)}
\end{aligned} \right\}
\]

for \( -d < z < 0 \) (4.6.3)
$R_{\text{TM}}$ and $R_{01}$ are given by equations (2a) and (55) of Chapter 3. At $z = 0$, the transverse component of the electric field is given by

$$E^p_{s}(u,v) = \sum \sum n \alpha \int_{0}^{\infty} dk_{\rho} \tilde{\alpha}_{n}(c_{0}k_{\rho},u,v) \tilde{S}_{n}(k_{\rho})$$

where

$$\tilde{S}_{n}(k_{\rho}) = \begin{bmatrix} p_{n}(k_{\rho}) \\ 0 \end{bmatrix}$$

$$p_{n}(k_{\rho}) = \frac{1}{2\omega e} \frac{k z}{k_{\rho}^{2}} \omega_{n}(k_{\rho}) (1 - R_{\text{TM}}) \psi_{n}(c_{0}k_{\rho},u_{0},v_{0})$$
Appendix 4.7

I. Evaluation of \( I_1 = \int_{0}^{2\pi} \int_{0}^{u'} dv \int_{0}^{u} du \ h^2 \psi_{\beta r}(c_0 k_{\rho}, u, v) \psi_{\alpha n}(c_0 k_{\rho}', u, v) \)

\( \psi_{\beta r}(c_0 k_{\rho}, u, v) \) and \( \psi_{\alpha n}(c_0 k_{\rho}', u, v) \) are governed by the following equations

\[
\frac{3}{u^2} \psi_{\beta r} + \frac{3}{v^2} \psi_{\beta r} + h^2 k_{\rho}^2 \psi_{\beta r} = 0 \tag{4.7.1}
\]

\[
\frac{3}{u^2} \psi_{\alpha n} + \frac{3}{v^2} \psi_{\alpha n} + h^2 k_{\rho}'^2 \psi_{\alpha n} = 0 \tag{4.7.2}
\]

Multiplying (4.7.1) by \( \psi_{\alpha n} \) and (4.7.2) by \( \psi_{\beta r} \) and subtracting we get

\[
\frac{3}{u} \left[ \frac{\partial}{\partial u} \psi_{\alpha n} - \frac{3}{u} \psi_{\beta r} \right] + \frac{3}{v} \left[ \frac{\partial}{\partial v} \psi_{\alpha n} - \frac{3}{v} \psi_{\beta r} \right] + h^2 (k_{\rho}^2 - k_{\rho}'^2) \psi_{\alpha n} \cdot \psi_{\beta r} = 0
\]

Integrating over \( u \) from 0 to \( u' \) and over \( v \) from 0 to \( 2\pi \). Using the periodicity property of the angular functions together with the fact that \( J_{\gamma r}(c_0 k_{\rho}, 0) = 0 \) and finally employing the orthogonality relation of the sine and cosine functions, we get

\[
I_1 = - \frac{\pi}{k_{\rho}^2 - k_{\rho}'^2} \left[ J_{\alpha n}(c_0 k_{\rho}', u') J_{\alpha r}(c_0 k_{\rho}', u') - J_{\alpha n}(c_0 k_{\rho}', u') J_{\alpha r}(c_0 k_{\rho}, u') \right]
\]

\[ \cdot N_{\alpha r}(c_0 k_{\rho}', c_0 k_{\rho}) \delta_{\alpha \beta} \]
provided that \( k_\rho \neq k'_\rho \) and both \( n \) and \( r \) have to be even or odd (\( I_1 \) vanishes if \( n \) and \( r \) are of different parity), and where

\[
N_{\alpha\,nr}(c_o k'_\rho, c_o k_\rho) = \frac{1}{\pi} \int_0^{2\pi} dv \, S_{\alpha\,n}(c_o k'_\rho, v) \, S_{\alpha\,r}(c_o k_\rho, v)
\]

\[
N_{\alpha\,nr}(c_o k'_\rho, c_o k_\rho) = \sum_m (1 + \delta_{m0}) \, D_m^n(c_o k'_\rho) \, D_m^n(c_o k_\rho)
\]

\[
N_{\alpha\,nr}(c_o k'_\rho, c_o k_\rho) = \sum_m F_m^n(c_o k'_\rho) \, F_m^n(c_o k_\rho)
\]

If \( k_\rho = k'_\rho \), we have to consider the following two cases.

Case (i): \( r \neq n \)

In this case and as \( k_\rho \to k'_\rho \), the quantity in the square brackets has a nonvanishing value whereas \( N_{\alpha\,nr}(c_o k'_\rho, c_o k_\rho) \) vanishes. Applying L'Hôpital's rule we get

\[
I_1 = -\frac{\pi}{2k'_\rho} \left[ J_{\alpha\,n}(c_o k'_\rho, u') J_{\alpha\,r}(c_o k_\rho, u') - J_{\alpha\,n}(c_o k'_\rho, u') J_{\alpha\,r}(c_o k_\rho, u') \right] \frac{\partial}{\partial k'_\rho} N_{\alpha\,nr}(c_o k'_\rho, c_o k_\rho) \bigg|_{k'_\rho = k_\rho} \delta_{\alpha\beta}
\]

Case (ii): \( r = n \)

In this case and in the limit \( k_\rho \to k'_\rho \), \( N_{\alpha\,nn}(c_o k'_\rho, c_o k'_\rho) \to R_{\alpha\,n}(c_o k'_\rho) \)

whereas the quantity between the square brackets vanishes and thus we get

\[
I_1 = -\frac{\pi}{2k'_\rho} \left[ J_{\alpha\,n}(c_o k'_\rho, u') \left. \frac{\partial}{\partial k'_\rho} \right|_{k'_\rho = k_\rho} J_{\alpha\,n}(c_o k'_\rho, u') - J_{\alpha\,n}(c_o k'_\rho, u') \left. \frac{\partial}{\partial k'_\rho} \right|_{k'_\rho = k_\rho} J_{\alpha\,n}(c_o k'_\rho, u') \right] \right. \delta_{\alpha\beta}
\]

\[
- R_{\alpha\,n}(c_o k'_\rho) \delta_{\alpha\beta}
\]
II. Evaluation of \( I_2 = \left[ \int_0^{u'} du \int_0^{v'} dv \right] \left\{ \frac{\partial}{\partial u} \psi_{r,c_0 k', u,v} \frac{\partial}{\partial u} \psi_{n,c_0 k', u,v} \right. \\
+ \left. \frac{\partial}{\partial v} \psi_{r,c_0 k', u,v} \frac{\partial}{\partial v} \psi_{n,c_0 k', u,v} \right\} \)

Using the integration by parts, two alternative forms for \( I_2 \) can be obtained. The first is

\[
I_2 = \pi J_{\alpha r,c_0 k', u'} J_{\alpha n,c_0 k', u'} N_{\alpha n r,c_0 k', c_0 k'} \delta_{\alpha \beta} + k^2 I_1
\]

The second form is

\[
I_2 = \pi J_{\alpha r,c_0 k', u'} J_{\alpha n,c_0 k', u'} N_{\alpha n r,c_0 k', c_0 k'} \delta_{\alpha \beta} + k^2 I_1
\]

\( n \) and \( r \) have to be of the same parity (\( I_2 \) vanishes if they are of opposite parity).
III. Evaluation of $I_3 = \int_0^\infty du \int_0^{2\pi} dv \left[ \frac{3}{3v} \psi_\alpha (c_o k_p, u, v) \frac{\partial}{\partial u} \psi_\alpha (c_o k_p, u, v) - \frac{3}{3u} \psi_\beta (c_o k_p, u, v) \frac{\partial}{\partial v} \psi_\alpha (c_o k_p, u, v) \right]

Using integration by parts, it can be easily shown that $I_3$ is given by

$$I_3 = -\pi J_\rho (c_o k_p, u') J_\alpha (c_o k_p, u') Q_\alpha (c_o k_p, c_o k_p) \delta_\beta$$

where $\alpha = e$ if $\alpha = 0$ and $\alpha = 0$ if $\alpha = e$

and

$$Q_{\alpha} (c_o k_p, c_o k_p) = \frac{1}{\pi} \int_0^{2\pi} dv S_{\alpha} (c_o k_p, v) S' (c_o k_p, v)$$

$$Q_{\alpha} (c_o k_p, c_o k_p) = \sum_m^n F_m (c_o k_p) D_m (c_o k_p)$$

$$Q_e (c_o k_p, c_o k_p) = \sum_m^r D_m (c_o k_p) F_m (c_o k_p)$$

with $n$ and $r$ of the same parity (i.e. both $n$ and $r$ have to be either even or odd).
Chapter 5
High Frequency Scattering from a Dielectric Coated
Perfectly Conducting Cylinder

I. Introduction
The scattering from a perfectly conducting cylinder has been extensively studied in literature \([27,44-48]\). This study goes back to half a century ago, where such a study was stimulated originally by the phenomenon of radio propagation over the earth and later by that of scattering by radar targets.

The very analogous case of slots on perfectly conducting circular cylinders were also considered by Sensiper \([43]\) and Wait \([44]\). Scattering from a circular cylinder with a surface impedance was recently studied by Wang \([52]\) and the case of an abrupt change in the surface impedance was also considered by Wait \([51]\).

Following Watson's approach for calculating the fields in the case of a source near a large sphere, Franz and Beckmann \([43]\) derived the Green's function for a circular cylinder with a finite conductivity.

The problem of scattering from a dielectric coated perfectly conducting cylinder, was considered by Adey \([50]\) in which the fields were represented in the form of a harmonic series, i.e. infinite series of integer order Bessel and Hankel functions and numerical values were supplied for small values of \(ka\) (for values of \(ka\) up to 4).

In this chapter, the high frequency scattering from a dielectric coated perfectly conducting cylinder is studied. The interest in this
problem arises from the line of sight propagation between a transmitter and a receiver on the earth's surface. To simplify the analysis of the actual problem, namely the line of sight propagation over the earth, the earth will be modelled by a perfectly conducting cylinder with the earth's radius of curvature on top of which there is a homogeneous layer of dielectric with an effective permittivity to model the vegetation or snow cover. Thus the scattering from the dielectric covered cylinder is essentially the two dimensional analogue of the propagation over a spherical earth.

The analysis of the scattering from the dielectric coated cylinder will be carried out using two methods. The first is the modal approach and the second is the ray expansion method.

In the modal approach, the total field is first represented as a sum of cylindrical harmonics and by matching the boundary conditions, the total field can be obtained exactly. This series expansion is not suitable for numerical computation at high frequency (ka can get as large as $10^8$) since it is a slowly convergent series. Using the well known Watson transformation this series can be converted to an integral which can be carried out by residue calculus, thus enabling us to represent the field in terms of the natural modes of the structure.

There appear two kinds of modes. The first kind are modes which correspond to the usual creeping waves for the perfectly conducting cylinder. These modes arise from poles in the vicinity of the zeros of either $H^{(1)}_\nu(kR_0)$ or $H^{(1)'}_\nu(kR_0)$ where $R_0$ is the outermost
radius of the cylinder and $k$ is the free space wavenumber. The second kind of modes are the surface wave modes which are a small perturbation to those of the flat slab waveguide except that they are radiating. These modes arise from poles which are the zeros of an expression similar to the guidance condition of a slab waveguide. The attenuation coefficient for the second kind of modes are obtained using two methods; the first method is by using the exact modal equation in the limit of very large radius of curvature. The second method is a perturbative method.

In the image approach, the field is represented in terms of multiply reflected rays which have undergone multiple reflections within the dielectric.
II. **Formulation**

Consider a perfectly conducting cylinder of radius $R$, coated with a dielectric of permittivity $\varepsilon_1$ and thickness $d$, extending from $r = R$ to $r = R_0$ (see Fig. (1)).

The incident wave is a plane wave with a $\vec{k}$-vector assumed to be perpendicular to the axis of the cylinder.

Two types of polarization are considered, the E-polarized (TE) incident wave and the H-polarized (TM) incident wave.

The TE case will be studied in detail and the analysis of the TM case will be almost the same.

A. **The E-polarized Incident Plane Wave**

In this case, the electric field vector $\vec{E}$ is assumed to be parallel to the axis of the cylinder ($\hat{z}$-axis).

Therefore the incident electric field is given by

$$\vec{E}_i = \hat{z} E_0 \exp(-ikr) = \hat{z}E_0 \exp(-ik\rho \cos \phi)$$

and for the purpose of matching the boundary conditions at the surface of the cylinder, this electric field is expanded in terms of the cylindrical harmonics as follows $[17, 19, 27]$:

$$\vec{E}_i = \hat{z} E_0 \sum_{n=-\infty}^{\infty} J_n(k\rho) \exp(in(\phi - \pi/2))$$
Figure (1). A perfectly conducting cylinder of radius $R$ coated with a dielectric of thickness $d$. Incident wave is either TE or TM.
The scattered field is then expanded in terms of the cylindrical outgoing waves as

$$\mathcal{E}_s = \hat{z} E_0 \sum_n D_n H_n^{(1)}(k_\rho) \exp\{in(\phi - \pi/2)\}$$

Thus the total field in the outermost region is given by

$$\mathcal{E}_o = \hat{z} E_0 \sum_n [J_n(k_\rho) + D_n H_n^{(1)}(k_\rho)] \exp\{in(\phi - \pi/2)\} \quad (1)$$

Whereas, in the dielectric region, the field is given as a superposition of outgoing and ingoing cylindrical waves as

$$\mathcal{E}_1 = \hat{z} E_0 \sum_n [B_n H_n^{(1)}(k_1\rho) + C_n H_n^{(2)}(k_1\rho)] \exp\{in(\phi - \pi/2)\}$$

and the magnetic field in the respective regions is thus obtained from

$$\mathcal{H}_j = -\frac{i\omega \epsilon_j}{k_j^2} \nabla \times \hat{z} \mathcal{E}_{jz} = \frac{1}{i\omega} \left[ -\frac{\partial}{\partial \phi} E_{jz} + \frac{\partial}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \phi} E_{jz} \right]$$

Applying the boundary conditions, $\frac{\partial E_z}{\partial \rho}$ and $E_z$ continuous at $\rho = R_0$, $E_z = 0$ at $\rho = R$, we get

$$D_n = -\frac{J_n'(kR_0) - S_n J_n(kR_0)}{H_n^{(1)'}(kR_0) - S_n H_n^{(1)}(kR_0)}$$

with
\[
S_n = \frac{k}{\pi} \frac{H^{(1)'}_n(k_1 R_0) H^{(2)}_n(k_1 R) - H^{(2)'}_n(k_1 R_0) H^{(1)}_n(k_1 R)}{H^{(1)}_n(k_1 R) H^{(2)}_n(k_1 R) - H^{(2)}_n(k_1 R_0) H^{(1)}_n(k_1 R)}
\]

And since \( J_n(x) = \frac{1}{2} [H^{(1)}_n(x) + H^{(2)}_n(x)] \), we get the following expression for the field in the outer region

\[
E_o = \hat{z} \frac{1}{2} E_0 \sum_n \left[ H^{(2)}_n(k_\rho) - \frac{N^e_n}{D^e_n} H^{(1)}_n(k_\rho) \right] \exp\{i(n - \pi/2)\}
\]

where

\[
N^e_\nu = k H^{(2)'}_\nu(k R_0)[H^{(1)}_\nu(k_1 R_0) H^{(2)}_\nu(k_1 R) - H^{(2)'}_\nu(k_1 R_0) H^{(1)}_\nu(k_1 R)]
\]

\[
- k_1 H^{(2)}_\nu(k R_0)[H^{(1)'}_\nu(k_1 R_0) H^{(2)}_\nu(k_1 R) - H^{(2)'}_\nu(k_1 R_0) H^{(1)}_\nu(k_1 R)]
\]

(2)

\[
D^e_\nu = k H^{(1)'}_\nu(k R_0)[H^{(1)}_\nu(k_1 R_0) H^{(2)}_\nu(k_1 R) - H^{(2)}_\nu(k_1 R_0) H^{(1)}_\nu(k_1 R)]
\]

\[
- k_1 H^{(1)}_\nu(k R_0)[H^{(1)'}_\nu(k_1 R_0) H^{(2)}_\nu(k_1 R) - H^{(2)'}_\nu(k_1 R_0) H^{(1)}_\nu(k_1 R)]
\]

(3)

It can be shown that the series expansion given in (1) becomes slowly convergent as \( k R_0 \) increases and to be of impractical value for numerical computation when \( k R_0 \) exceeds 10 [27,28].

Using the well known Watson's transformation, this slowly convergent series is converted to a fastly convergent one, in which the field is represented in terms of the natural modes of the structure or in terms of a ray series.
The Watson transformation [17,27] expresses a slowly convergent series in terms of a contour integral in the complex plane which then can be deformed to evaluate the sum of residues of the poles of the integrand to obtain a residue series which is a fastly convergent series than the original one.

The Watson transformation is essentially based on the following equality

$$\oint_{c} \frac{A(\nu)}{\sin(\nu \pi)} \, d\nu = -2i \sum_{n=-\infty}^{\infty} e^{-in\pi} A(n) \quad (4)$$

provided that $A(\nu)$ has no singularities within the contour $c$ where $c$ is a closed contour around the real axis as shown in Fig. (2) enclosing the zeros of $\sin(\nu \pi)$ which occur at real positive and negative values of $\nu$. These zeros of the $\sin(\nu \pi)$ are all first order poles of the integrand $A(\nu)/\sin(\nu \pi)$ and thus is is clear that equation (4) readily follows.

Thus the electric field is given by

$$\mathbf{E} = \hat{z} E_0 \sum_n e^{in\phi} \mathbf{E}_n = \hat{z} E_0 \frac{i}{2} \oint_{c} \frac{e^{i\nu(\phi - \pi)}}{\sin \nu \pi} \mathbf{E}_\nu \, d\nu$$

where

$$E_\nu = \frac{1}{2} \left[ \frac{H_{(2)}^{(2)}(k\rho)}{D_{\nu}} - \frac{N_e}{D_{\nu}} \frac{H_{(1)}^{(1)}(k\rho)}{D_{\nu}} \right] \exp(-i\nu \pi/2)$$
Figure 2. Integration Path of the Watson Transformation.
It can be easily shown that

\[ N_{-\nu}^{e} = N_{\nu}^{e} \exp(-i\nu\pi), \quad D_{-\nu}^{e} = D_{\nu}^{e} \exp(i\nu\pi) \]

and thus \( E_{-\nu} = E_{\nu} \).

Thus the integral over the closed contour \( c \) which is equivalent to two line integrals, one above and one below the real axis, can be converted into a line integral from \(-\infty\) to \(+\infty\) above the real axis given by

\[
\bar{E}_0 = \frac{\hat{z} E_0}{2} \int_{-\infty+i\delta}^{\infty+i\delta} d\nu \frac{\cos(\nu(\phi-\pi))}{\sin(\nu\pi)} \left\{ H_{\nu}^{(2)}(k_\rho) - \frac{N_{\nu}^{e}}{D_{\nu}^{e}} H_{\nu}^{(1)}(k_\rho) \right\} \exp(-i\nu\pi/2) \tag{5}
\]

where \( \delta \) is a very small real number, to indicate that the path is slightly above the real axis.

**B. The H-polarized Incident Plane Wave**

In this case, the magnetic field vector \( \overline{H} \) is parallel to the axis of the cylinder.

In region (0), the magnetic field is given by

\[
\overline{H}_0 = \frac{\hat{z}}{H_0} \sum_n \left[ J_n(k_\rho) + D_n H_n^{(1)}(k_\rho) \right] \exp\{in(\phi-\pi/2)\}
\]

while in region (1),

\[
\overline{H}_1 = \frac{\hat{z}}{H_0} \sum_n \left[ B_n H_n^{(1)}(k_1\rho) + C_n H_n^{(2)}(k_1\rho) \right] \exp\{in(\phi-\pi/2)\}
\]
and the electric field in the respective regions is thus obtained from

\[ \vec{E}_j = -\frac{1}{i\omega \varepsilon_j} \nabla \times \hat{z} H_j z = -\frac{1}{i\omega \varepsilon_j} \left[ -\hat{\phi} \frac{\partial}{\partial \rho} H_j z + \frac{1}{\rho} \frac{\partial}{\partial \phi} H_j z \right] \]

Applying the boundary conditions, \( \frac{1}{k_i^2} \frac{\partial}{\partial \rho} H_z \) and \( H_z \) continuous at \( \rho = R_0 \), \( \frac{\partial H_z}{\partial \rho} = 0 \) at \( \rho = R \), we get

\[ H_0 = \hat{z} \frac{1}{2} H_0 \sum_n \left[ H_n^{(2)}(k_\rho) - \frac{N_n^m}{D_n^m} H_n^{(1)}(k_\rho) \right] \exp\{i(n - \pi/2)\} \]

where

\[ N_n^m = k_1 H_n^{(2)}(kR_0)[H_n^{(1)}(kR_0) H_n^{(2)}(k_1 R_0) - H_n^{(2)}(k_1 R_0) H_n^{(1)}(k_1 R_0)] \]

\[ - k H_n^{(2)}(kR_0)[H_n^{(1)}(k_1 R_0) H_n^{(2)}(k_1 R_0) - H_n^{(2)}(k_1 R_0) H_n^{(1)}(k_1 R_0)] \] (6)

\[ D_n^m = k_1 H_n^{(1)}(kR_0)[H_n^{(1)}(k_1 R_0) H_n^{(2)}(k_1 R_0) - H_n^{(2)}(k_1 R_0) H_n^{(1)}(k_1 R_0)] \]

\[ - k H_n^{(1)}(kR_0)[H_n^{(1)}(k_1 R_0) H_n^{(2)}(k_1 R_0) - H_n^{(2)}(k_1 R_0) H_n^{(1)}(k_1 R_0)] \] (7)

and using the Watson's transformation we get

\[ H_0 = \hat{z} H_0 \int_{-\infty + i\delta}^{\infty + i\delta} \frac{d\nu}{\sin(\nu \pi)} \left\{ H_n^{(2)}(k_\rho) - \frac{N_n^m}{D_n^m} H_n^{(1)}(k_\rho) \right\} \exp(-i\nu \pi/2) \] (8)
III. **Methods of Solution**

The integrals in (5) and (8) can be evaluated using two methods. The first method is to deform the path of integration, thus picking up the contributions from the poles of the integrand. In this case, the series obtained represents the natural modes of the structure.

The second method is to expand the integrand in terms of the cylindrical Fresnel's reflection coefficients. In this case, the series obtained represents rays which are multiply reflected inside the dielectric.

A. **The Modal Expansion**

Examining the integrands of both (5) and (8), it can be easily shown that the integrands vanish on the circle at infinity in the upper half of the \( v \)-plane.

Thus the electric and magnetic fields are evaluated by deforming the path of integration upwards, thereby picking up the residues of the poles of the integrand.

Moreover the first term in the integrands [containing \( H^{(2)}_v(k_0) \)] vanishes since it doesn't have any poles. Thus we have

\[
\vec{F} = \hat{\mathbf{z}} \pi \frac{F_0}{2} \sum_{n=1}^{\infty} \frac{N_v}{n} \frac{\cos \frac{\nu_n(\phi-\pi)}{\sin \left( \frac{\nu_n \pi}{2} \right)} \exp(-i\nu_n \pi/2) H^{(1)}_{\nu_n}(k_0)}{\frac{\partial}{\partial \nu} D_{\nu_n}}
\]

where \( \vec{F} \) is either the electric or magnetic field and \( \nu_n \) are the roots of the equation \( D_{\nu_n} = 0 \) which is the modal equation.
There appear two kinds of modes. The first kind are modes which correspond to the usual creeping waves for a perfectly conducting cylinder. These modes arise from poles in the vicinity of the zeros of either \( H^{(1)}_\nu (k R_0) \) or \( H^{(1)'}_\nu (k R_0) \) depending on the thickness of the dielectric and the type of polarization. Tracing their propagation path to an observation point in the shadow region, these modes correspond to waves travelling along the outermost cylindrical surface at the air-dielectric interface (Fig. (3)). Thus these modes appear as if they don't penetrate the dielectric, however their attenuation is determined by the electrical properties of the dielectric as well as the dielectric thickness.

The second kind of modes correspond to waves penetrating the dielectric and undergoing multiple reflection within the dielectric. These modes arise from poles which are the zeros of an expression similar to the guidance condition of a slab waveguide and are therefore slight perturbations of the surface wave modes of the flat slab waveguide. These surface waves modes are different from the surface modes on a Goubau line \([19,41,42]\) in that, the latter are modes which are propagating along the axis of the cylinder and are nonradiating whereas the former are waves which propagate parallel to the circumference and are slightly radiating modes. Furthermore, the surface wave modes considered in this chapter are either pure TM or pure TE waves whereas each Goubau surface mode is a combination of a TM and TE mode except for the lowest order mode (called the Sommerfeld-Goubau wave) which is a TM wave and is axially symmetrical (i.e. independent of the angular coordinate \( \phi \)).
Figure 3. Ray Path of Creeping Waves.
(i) **The Creeping Waves**

The location of the poles giving rise to the creeping waves are given by the solution of the modal equation \( \nu = 0 \) around \( kR_0 \).

For the TE-case, the modal equation is given by

\[
D_{\nu}^e = kH_{\nu}^{(1)'}(kR_0)[H_{\nu}^{(1)}(kR_0) H_{\nu}^{(2)}(k_1R) - H_{\nu}^{(2)}(k_1R_0) H_{\nu}^{(1)}(k_1R)]
- k_1H_{\nu}^{(1)}(kR_0)[H_{\nu}^{(1)'}(k_1R_0) H_{\nu}^{(2)}(k_1R) - H_{\nu}^{(2)'}(k_1R_0) H_{\nu}^{(1)}(k_1R)] = 0
\]

To solve this equation, we first assume a solution for \( \nu \) and later verify the assumption. Thus let us assume that \( \nu \) is very close to \( kR_0 \) and since we are interested in the high frequency limit, then both \( \nu \) and \( kR_0 \) are very large and so are \( k_1R_0 \) and \( k_1R \).

Thus for \( H_{\nu}^{(1)'}(kR_0) \) and \( H_{\nu}^{(1)}(kR_0) \) both the argument and the order are large and almost equal, in this case, these Hankel functions are represented by the uniform asymptotic expansion (ii) given in terms of the Airy function supplied in the appendix.

As for \( H_{\nu}^{(1)}(k_1R_0) \), \( H_{\nu}^{(1)'}(k_1R_0) \), \( H_{\nu}^{(2)}(k_1R_0) \), \( H_{\nu}^{(2)'}(k_1R_0) \), \( H_{\nu}^{(1)}(k_1R) \) and \( H_{\nu}^{(2)}(k_1R) \), the order and argument are large and unequal and since \( |\nu| = |kR_0| < |k_1R_0|, |k_1R| \), thus we have to use the asymptotic expansion (i) of the appendix in region (1) for \( H_{\nu}^{(1)} \) and in region (4) for \( H_{\nu}^{(2)} \).

Using these asymptotic expansions and after some algebraic manipulations we get
\[ D_\nu^e = - \frac{8i}{\pi R_0} \left( \frac{2}{kR_0} \right)^{1/3} \exp(-i\pi/3) \frac{S_0}{S_0} \cos (\phi_0 - \psi) \]

\[ \cdot \left[ \text{Ai}(-a) - \left( \frac{2}{kR_0} \right)^{1/3} \frac{kR_0}{S_0} \exp(-i\pi/3) \text{Ai}'(-a) \tan (\phi_0 - \psi) \right] = 0 \]

\[ N_\nu^e = \frac{8i}{\pi R_0} \left( \frac{2}{kR_0} \right)^{1/3} \exp(-i\pi/3) \frac{S_0}{S_0} \cos (\phi_0 - \psi) \]

\[ \cdot \left[ \exp(-i\pi/3) \text{Ai}(a \exp(-i\pi/3)) - \left( \frac{2}{kR_0} \right)^{1/3} \frac{kR_0}{S_0} \text{Ai}'(a \exp(-i\pi/3)) \tan (\phi_0 - \psi) \right] \]

where

\[ a = (\nu - kR_0) \left( \frac{2}{kR_0} \right)^{1/3} \exp(-i\pi/3) \]

\[ S_0 = \sqrt{k_1^2 R_0^2 - \nu^2}, \quad S = \sqrt{k_1^2 R^2 - \nu^2} \]

\[ \phi_0 - \phi = c + \nu b \]

\[ c = S_0 - S \quad \text{(10)} \]

\[ b = \cos^{-1} \left( \frac{\nu}{k_1 R} \right) - \cos^{-1} \left( \frac{\nu}{k_1 R_0} \right) \quad \text{(11)} \]

From equations (10) and (11), it is clear that both \( c \) and \( b \) are the difference between two large numbers (large compared to the difference). Thus an appropriate way to express \( c \) and \( b \), is to expand the quantities on the right hand side of equations (10) and (11) as a series expansion.
in terms of \( \delta = d/R \). Keeping terms of the order of \( \delta^3 \) we get

\[
c = \delta k_1 R \sqrt{1 + \eta} \left[ 1 - \frac{\delta}{2} n + \frac{\delta^2}{2} n (1 + n) \right]
\]

(12)

\[
b = -\frac{\nu}{k_1 R} \frac{\delta}{\sqrt{1 + \eta}} \left( (1 + n) - \delta [1 + \frac{\nu}{k_1^2 R^2} (3 + \nu n)] + \frac{\nu^2}{k_1 R} [1 + \frac{\nu}{k_1 R} (4 + n + n^2)] \right) - \frac{1}{6} n^{3/2} \delta
\]

(13)

where

\[
n = \frac{\nu^2}{k_1^2 R^2 - \nu^2}
\]

(14)

For convenience, let us redefine \( D^e_\nu \) and \( N^e_\nu \) as follows:

\[
D^e_\nu = \text{Ai}(-a) - \left( \frac{2}{kR_0} \right)^{1/3} \frac{kR_0}{\mu_0} \exp(-i\pi/3) \text{Ai}'(-a) \tan(\psi - \psi)
\]

(15)

\[
N^e_\nu = -\exp(-i\pi/3) \text{Ai}(a \exp(-i\pi/3) + \left( \frac{2}{kR_0} \right)^{1/3} \frac{kR_0}{\mu_0} \text{Ai}'(a \exp(-i\pi/3)) \tan(\psi - \psi)
\]

(16)

and thus

\[
\frac{\partial D^e_\nu}{\partial \nu} = \frac{\partial D^e_\nu}{\partial a} \cdot \frac{\partial a}{\partial \nu}
\]

(17)

where

\[
\frac{\partial a}{\partial \nu} = \left( \frac{2}{kR_0} \right)^{1/3} \exp(-i\pi/3)
\]

(18)
\[
\frac{\partial D^0}{\partial a} = -Ai'(-a) \left[ 1 + \frac{v k R_0}{S_0^3} \tan(\psi_0 - \psi) + \frac{b k R_0}{S_0} \sec^2(\psi_0 - \psi) \right] \\
- \left( \frac{2}{k R_0} \right)^{1/3} \frac{k R_0}{S_0} \exp(-i\pi/3) a \ Ai(-a) \tan(\psi_0 - \psi)
\]  

(19)

where we have used the differential equation \(Ai''(-a) = -a Ai(-a)\) and the fact that \(\frac{\partial c}{\partial \nu} + \nu \frac{\partial b}{\partial \nu} = 0\).

For the TM-case, it can be similarly shown that

\[
D^m_\nu = Ai'(-a) + \left( \frac{k R_0}{2} \right)^{1/3} \frac{k}{k_1} \frac{S_0}{k_1 R_0} \exp(i\pi/3) Ai(-a) \tan(\psi_0 - \psi)
\]

(20)

\[
N^m_\nu = -\exp(i\pi/3) Ai'(a \exp(-i\pi/3)) - \left( \frac{k R_0}{2} \right)^{1/3} \frac{k}{k_1} \frac{S_0}{k_1 R_0} Ai(a \exp(-i\pi/3)) \tan(\psi_0 - \psi)
\]

(21)

and

\[
\frac{\partial D^m}{\partial a} = Ai(-a) \left[ a + \left( \frac{k R_0}{2} \right)^{2/3} \frac{k}{k_1} \frac{1}{k_1 R_0} \exp(-i\pi/3) \left( \frac{v}{S_0} \tan(\psi_0 - \psi) - b S_0 \sec^2(\psi_0 - \psi) \right) \right] \\
- \left( \frac{k R_0}{2} \right)^{1/3} \frac{k}{k_1} \frac{S_0}{k_1 R_0} \exp(i\pi/3) Ai'(-a) \tan(\psi_0 - \psi)
\]

(22)

If \(a\) is the root to either the TM or TE modal equation \(D_\nu = 0\), \(\nu\) will thus be given by

\[
\nu = k R_0 + a \left( \frac{k R_0}{2} \right)^{1/3} \exp(i\pi/3)
\]

(23)
The phase factor is thus given by

$$\nu' = kR_0 + \frac{1}{2} \left( \frac{kR_0}{2} \right)^{1/3} \left( a' - \sqrt{3} a'' \right)$$  \hspace{1cm} (24)$$

and the attenuation factor is

$$\nu'' = \frac{1}{2} \left( \frac{kR_0}{2} \right)^{1/3} \left( a'' + \sqrt{3} a' \right)$$  \hspace{1cm} (25)$$

where $\nu = \nu' + i\nu''$ and $a = a' + ia''$ and the dielectric is assumed to be lossless. Examining the TM and TE modal equations, it is clear that $a'$ varies between $a_m = 1.019$ and $a_e = 2.338$ for the dominant mode, where $Ai'(-a_m) = 0$ and $Ai'(-a_e) = 0$.

At this point, we can see that the initial assumption, that $\nu$ is almost equal to $kR_0$, is a valid assumption. This is because, since $kR_0 > 1$, then $kR_0 >> \left( \frac{kR_0}{2} \right)^{1/3}$ and therefore $\nu' - kR_0$ and $\nu'' << \nu'$.

Thus the assumption that $\nu - kR_0$ is a self consistent assumption.

In summary, we have the following:

The TE modal equation for the coated cylinder is

$$D_{\nu}^E = \text{Ai}(-a) - \left( \frac{2}{kR_0} \right)^{1/3} \frac{kR_0}{\sqrt{k_1 R_0^2 - \nu^2}} \exp(-i\pi/3) \text{Ai}'(-a) \tan(\psi_0 - \psi) = 0$$

where $(\psi_0 - \psi)$, for the first order term in $\delta = d/R$, is given by

$$\psi_0 - \psi = k_1 d \sqrt{1 - \left( \frac{kR_0}{k_1 R} \right)^2} - k_1 d \sqrt{1 - 1/\varepsilon_r} \hspace{1cm} \varepsilon_r = \varepsilon_1/\varepsilon$$
whereas for the uncoated cylinder, the TE modal equation is

\[ D^e_v = \text{Ai}(-a) = 0 \quad a = 2.338, 4.088, 5.521, \ldots \]

As for the TM wave and for the coated cylinder

\[ D^m_v = \text{Ai}'(-a) + \frac{(\frac{kR_o}{2})^{1/3}}{\sqrt{k_1^2R_o^2 - \nu^2}} \cdot \frac{k}{k_1} \cdot \frac{\text{Ai}(-a) \exp(i\pi/3)}{\tan(\psi_o - \psi)} = 0 \]

and for the uncoated cylinder

\[ D^m_v = \text{Ai}'(-a) = 0 \quad a = 1.019, 3.248, 4.82, \ldots \]

From these modal equations and since \((kR_o/2)^{1/3} \gg 1\), it is clear that:

- For values of d such that \(\tan(\psi_o - \psi)\) is of the order of unity or higher [i.e. \(\tan(\psi_o - \psi) \gg (2/kR_o)^{1/3}\)], the modal equation of the TM wave for the coated cylinder is nearly the same as that of the TE wave for the uncoated one and thus the TM wave is more attenuated in the case of the coated cylinder compared to the uncoated one.

- For values of \(d = \frac{r\lambda}{2\sqrt{E_r-1}}\), where \(r\) is an integer, for which \(\tan(\psi_o - \psi) \sim 0\), the modal equation of the TM wave for the coated cylinder is nearly the same as that of the TM wave for the uncoated cylinder.

Thus, for an incident TM wave in the case of a coated cylinder, the lowest
attenuation that one can hope for, is the same as that of the uncoated one.

- For values of $d$ for which $\tan(\psi_0 - \psi)$ is of the order of unity or smaller [i.e. $\tan(\psi_0 - \psi) \ll (kR_0/2)^{1/3}$], the modal equation of the TE wave for the coated cylinder is nearly the same as that of the TE wave for the uncoated one. Thus in this case there is no change in attenuation between the coated and the uncoated cylinders.

- For values of $d = \frac{(2r+1)\lambda}{4\sqrt{\mu_\perp-1}}$, where $r$ is an integer, for which $\tan(\psi_0 - \psi) \to \infty$, the modal equation of the TE wave for the coated cylinder approaches that of the TM wave for the uncoated cylinder and therefore in this case, the TE wave for the coated cylinder is less attenuated than the TE wave for the uncoated cylinder.

Thus in summary, the attenuation coefficient of both the TM and TE waves, as a function of the coating thickness $d$, will oscillate between the TM and TE attenuation factors of the uncoated cylinder.

This can be physically interpreted by recalling that from the boundary conditions at the surface of the inner perfectly conducting cylinder, the incident and reflected tangential electric fields are out of phase whereas those of the magnetic fields are in phase. If the coating thickness is a multiple of $\lambda_d/2$ ($\lambda_d$ being the wavelength in the dielectric), this phase relation between the incident and reflected tangential electric and magnetic fields and hence the boundary condition ($E_{\text{tang}} = 0$) will still hold at the outer surface of the dielectric, making the dielectric look like a perfectly electric conductor. Hence in this case, the modal
equation for the coated cylinder will be the same as that of the uncoated cylinder for the same type of polarization. On the other hand, if the coating thickness is an odd multiple of $\lambda_d/4$, an additional phase of $\pi$ will result between the incident and reflected fields at the outer surface of the dielectric thus causing the tangential incident and reflected electric fields to be in phase whereas the magnetic fields to be out of phase. Therefore the total tangential magnetic field will vanish at the outer surface of the dielectric causing the dielectric to act like a perfectly magnetic conductor. From the duality principle, the modal equation in the case of the coated cylinder (acting like a perfectly magnetic conductor) will thus be the same as that of the opposite polarization for the uncoated cylinder (the perfectly electric conductor).

In summary, and for both types of polarization, the attenuation coefficient, as a function of the coating thickness $d$, will oscillate between the TM and TE attenuation coefficients of the uncoated cylinder. The period of oscillation is roughly equal to $\lambda_d/2$.

The above argument holds only for the case of a lossless dielectric. In the case of a lossy dielectric, the modal equation will not exhibit the same oscillatory behavior because in addition to the phase introduced to the fields propagating in the dielectric, the amplitudes will be also attenuated inside the dielectric. This can be also seen mathematically by noticing that the factor $\tan(\psi_0 - \psi)$ which oscillates as a function of $d$ in the case of a lossless dielectric, is no more oscillatory once we assume the dielectric to be lossy.
To calculate the field intensity in the shadow region we have to consider two cases, the first is for observation points at the outer surface of the dielectric \((\rho - R_0)\) and the second case is for points elevated above the dielectric's outer surface \((\rho > R_0)\).

Case (i): \(\rho - R_0\)

In this case, \(H_0^1(k\rho)\) in the expression of the field given by equation (9), has to be replaced by its uniform asymptotic expansion (ii) given in terms of the Airy function supplied in the appendix.

\[
F = -i\varepsilon 2\pi F_0 \left(\frac{2}{k\rho}\right)^{1/3} \exp(-i\pi/3) \sum_{n=1}^{\infty} C_{\nu_n} \left[ \exp\left\{i\nu_n(\phi - \pi/2)\right\} + \exp\left\{i\nu_n\left(\frac{3\pi}{2} - \phi\right)\right\} \right]
\]

\[
\cdot \text{Ai}
\left[\left(\nu_n - k\rho\right)\left(\frac{2}{k\rho}\right)^{1/3} \exp(-i\pi/3)\right]
\]

(26)

Furthermore if \(k(\rho - R_0) \ll \left(\frac{kR_0}{2}\right)^{1/3}\), this expression simplifies to

\[
F = -i\varepsilon 2\pi F_0 \left(\frac{2}{k\rho}\right)^{1/3} \exp(-i\pi/3) \sum_{n=1}^{\infty} C_{\nu_n} \left[ \exp\left\{i\nu_n(\phi - \pi/2)\right\} + \exp\left\{i\nu_n\left(\frac{3\pi}{2} - \phi\right)\right\} \right]
\]

\[
\cdot \left[\text{Ai}(-a) + k(\rho - R_0)\left(\frac{2}{kR_0}\right)^{1/3} \exp(-i\pi/3) \text{Ai}'(-a)\right]
\]

where

\[
C_{\nu_n} = \frac{N_{\nu_n}}{\sqrt{\left(\frac{3D_{\nu_n}}{3\nu_n}\right)_{\nu = \nu_n}}} \cdot \frac{1}{1 - \exp\left(i\nu_n 2\pi\right)}
\]
It is clear from this expression that for observation points near the outer surface of the dielectric, the field varies linearly with the radial distance \((\rho - R_0)\) from the dielectric's surface.

Case (ii): \(\rho > R_0\)

Expression (26) is still a valid representation for the field, however when \(\rho\) becomes significantly larger than \(R_0\) this expression becomes numerically unsuitable and in this case the Airy function is more suitably replaced by its asymptotic expansion.

Alternatively, \(H^{(1)}_\nu(k\rho)\) in equation (9) can be replaced by its asymptotic expansion (i) given in the appendix to get an expression which provides an interesting geometrical interpretation of the way the creeping waves propagate to an observation point in the shadow region.

\[
\bar{F} = -iz\pi F_0 \exp(-i\pi/4) \sum_{n=1}^{\infty} B_{\nu_n} \left[ \exp \left\{ i \nu_n \left( \phi - \frac{\pi}{2} \right) \right\} + \exp \left\{ i \nu_n \left( \frac{3\pi}{2} - \phi \right) \right\} \right] \\
\cdot \exp \left[ i \left( \sqrt{k^2 \rho^2 - \nu_n^2} - \nu_n \cos^{-1} \left( \frac{\nu_n}{k\rho} \right) \right) \right]
\]

where

\[
B_{\nu_n} = C_{\nu_n} \frac{\sqrt{2}}{\pi \left( k^2 \rho^2 - \nu_n^2 \right)^{1/4}}
\]

Recalling that \(\nu'_n \approx kR_0\) and \(\nu''_n \ll \nu'_n\), therefore expanding both \(\sqrt{k^2 \rho^2 - \nu_n^2}\) and \(\nu_n \cos^{-1} \left( \frac{\nu_n}{k\rho} \right)\) in terms of \(\nu''_n/\nu'_n\) and keeping terms of zeroth order for the real part and first order for the imaginary part we get
\[
\sqrt{k^2\rho^2 - \nu^2} - k\sqrt{\rho^2 - R_0^2} - \frac{i}{\sqrt{k^2\rho^2 - \nu^2}}\frac{\nu n}{n n}
\]

\[
\nu_n \cos^{-1}\left(\frac{\nu n}{k\rho}\right) - \nu_n' \cos^{-1}\left(R_0 \frac{R_0}{\rho}\right) + i \left(\nu_n'' \cos^{-1}\left(R_0 \frac{R_0}{\rho}\right) - \frac{\nu_n' \nu_n''}{\sqrt{k^2\rho^2 - \nu^2}}\right)
\]

and thus the field is given by

\[
\mathbb{F} = -i\hat{z} F_0 \exp(-i\pi/4) \sum_{n=1}^{\infty} \nu_n \left[ \exp\left(ik\sqrt{\rho^2 - R_0^2} + y_c\right) \exp\left(-\alpha_n y_c\right)
\right.
\]

\[
+ \exp\left(ik\sqrt{\rho^2 - R_0^2} + y_a\right) \exp\left(-\alpha_n y_a\right)
\]

where

\[
y_c = R_0 \left\{ \hat{\phi} - \frac{\pi}{2} - \cos^{-1}\left(R_0 \frac{R_0}{\rho}\right) \right\}, \quad y_a = R_0 \left\{ \frac{3\pi}{2} - \hat{\phi} - \cos^{-1}\left(R_0 \frac{R_0}{\rho}\right) \right\}
\]

and

\[
\alpha_n = \frac{\nu_n''}{kR_0} k = \frac{a_n'' + \sqrt{3} a_n'}{(4kR_0)^{2/3}} k
\]

This expression provides a geometrical interpretation [17,27] of how these creeping waves travel to observation points in the shadow region. From the above expression, it is clear that the n-th mode consists of two rays both of which creep along the outer cylindrical surface of the dielectric for distances \( y_c \) and \( y_a \) during which they are attenuated by the factor \( \alpha_n \) which is determined by the coating thickness and the
electrical properties of the dielectric. Eventually these rays leave
the surface and radiate into free space for a distance $\sqrt{D^2 - R_0^2}$ before
they join together at the observation point. Thus these modes appear as
if they don't penetrate the dielectric (see Fig. (3)).
(ii) **Guided Modes (Surface Modes):**

In order to have a better insight of the pole distribution of this class of modes in the case of the coated cylinder, the cylinder is better visualized as a slightly bent dielectric slab waveguide.

For the straight slab waveguide with a lossless dielectric it is well known that the natural modes of such a structure consist of a finite number of surface wave modes whose lateral wavenumbers are real and lie between \( k \) and \( k_1 \). Thus these modes are evanescent and don't contribute to radiation.

In the case of the coated cylinder one would expect a similar class of modes which are essentially a small perturbation of these surface wave modes of the straight slab waveguide.

However these perturbed surface modes are no more nonradiating, because of the fact that dielectric waveguides can't perfectly guide electromagnetic energy around bends without losing power by radiation [30, 31].

The process by which a surface wave mode is radiated from a flat slab waveguide that has been slightly bent into a cylindrical surface can be physically explained qualitatively by noticing that while the phase velocity of the mode inside the dielectric is essentially the same as that in the case of the flat waveguide, which is smaller than the velocity of plane waves in the outer region, this phase velocity increases proportionally to the radial distance from the curvature center. At some radius, it exceeds the speed of light in the outer medium. At this point, the field detaches itself from the guided mode field and radiates
into the outer medium [30-33].

This can also be visualized by the following crude quantitative analysis: the $\phi$-variation of a mode in the bent waveguide is $\exp(i\nu\phi)$. By comparison with the modes of the straight guide which vary as $\exp(ik_yy)$ we obtain the relation

$$\nu\phi = k_yy$$

where the $y$-axis is now the arc length along the bent surface of the waveguide $y = \rho\phi$. Thus we get

$$k_y(\rho) = \nu/\rho$$

Since the coated cylinder is assumed to be a slightly bent slab waveguide, the propagation constant of the wave inside the dielectric is very nearly identical for both the slightly bent waveguide and the corresponding flat guide, i.e. $k_y(R) = k_{1y}$, therefore $\nu = k_{1y}R$ and $k_y = k_{1y}R/\rho$, where $k_{1y}$ is the solution to the guidance condition of the flat guide.

Thus the vertical wavenumber (radial wavenumber) in the outer medium is

$$k_{ox} = \sqrt{k_0^2 - k_{1y}^2 \left(\frac{R}{\rho}\right)^2}$$

and the field varies as $\exp(ik_{ox}x)$ in the x-direction (radial direction).
Thus it is clear that when \( \rho = R \), and since \( k_1 y > k_0 \), the wave is evanescent however as \( \rho \) increases and start to exceed \( \rho_0 = \frac{k_1 y}{k_0} \) \( R \), the wave starts to propagate along the \( x \)-axis and is no more attached to the surface, thus causing radiation losses. The point at which this occurs (i.e. \( \rho = \rho_0 \)) is known as the turning point.

Because of this radiation loss, the wave will be attenuated as it propagates along the \( \phi \)-direction. However, in practice, this bending loss is negligibly small if the radius of curvature of the bend is sufficiently large. Hence, in the case of a lossless dielectric, this class of modes, namely the surface wave modes, are far less attenuated than the creeping waves.

The calculation of the bending loss has been of interest since a long time in the field of fiber communication because of the great practical importance of the effect of the bending loss in setting a limit on how sharp bends can be made in fiber optics without resulting in unreasonable loss.

A variety of techniques has been used to treat the problem of calculating the bending loss. The difficulty and sophistication of these methods vary according to the waveguide configuration. Most of these approaches use either a variational expression or a perturbational approach to approximate the attenuation coefficient.

The simplest of these methods to calculate the bending loss of a slightly bent slab waveguide was performed by Marcuse [33]. His approach is based on a power balance concept. The power lost by the guide must be supplied by the power travelling down the guide, thus by
conservation of power the decrease in power flow per unit length in the
y-direction (ϕ-direction) must be equal to the power radiated per unit
length along the guide.

So if the field is assumed to decay as e^{-a_y} and the power as e^{-2\alpha y}
we get

\[ \alpha = \frac{P_d}{2P_f} \]

where \( P_f \) is the unperturbed power flowing along the guide, i.e. the
power flowing in the flat slab waveguide, and \( P_d \) is the power lost per
unit guide length which is the power radiated in the radial direction.

The result obtained by Marcuse is given, in terms of the notation
used in this chapter, by

\[ \nu'' = \frac{\gamma^2}{k_y(1+\gamma d)} \frac{k_{1x}^2}{k_1^2 - k_y^2} R \exp(2\gamma d) \cdot \exp \left[ -2R \left\{ k_y \tanh^{-1} \left( \frac{\gamma}{k_y} \right) - \gamma \right\} \right] \]

where

\[ \gamma = \sqrt{k_y^2 - k_x^2}, \quad k_{1x} = \sqrt{k_1^2 - k_y^2} \]

and \( k_y \) is the solution to the modal equation of the flat slab waveguide.
This is the attenuation constant for the TE wave.

In this section, two perturbational methods are used to obtain
the attenuation constant caused by the bending loss. In the first method
a perturbational approach is applied to the exact eigenvalue equation
obtained by solving the boundary value problem which has been carried out in the previous section. In the second method, a perturbational formula is developed starting from Maxwell's equations to obtain an expression for the propagation constant \( \nu \).
1) Exact Eigenvalue Equation

In this approach, we will use the exact model equation of the dielectric coated cylinder which was derived in the previous section. The modal equation is first recasted in a convenient form and by using appropriate approximations, the attenuation coefficients of the surface wave modes (which are a small perturbation to those of the flat slab waveguide) are obtained.

In what follows, two forms for the asymptotic expansion of the Hankel functions, are used depending on the relation between the order and argument.

The first case is for \( \nu < x \), for which the appropriate form is the asymptotic expansion (i) in regions (1) and (4) supplied in the appendix.

The second case is for \( \nu > x \). In this case, the expansions given in the appendix are not appropriate for this specific problem, namely the calculation of the bending loss (this will be discussed in detail in the next section). The appropriate expansions are given by [34, p. 963]

\[
H^{(1)}_\nu(x) = J_\nu(x) + iN_\nu(x) - i \sqrt{\frac{2}{\pi}} \frac{1}{(\nu^2 - x^2)^{1/4}} e^{ip} \left[ 1 + \frac{i}{2} e^{-2p} \right] \quad (27a)
\]

\[
H^{(1)'}_\nu(x) = \frac{i}{x} \sqrt{\frac{2}{\pi}} (\nu^2 - x^2)^{1/4} e^{ip} \left[ 1 - \frac{i}{2} e^{-2p} \right] \quad (27b)
\]

and

\[
\frac{H^{(1)'}_\nu(x)}{H^{(1)}_\nu(x)} = - \frac{\sqrt{\nu^2 - x^2}}{x} (1 - ie^{-2p})
\]
where
\[ p = \sqrt[2]{\tanh^{-1}\left(\frac{\sqrt{\gamma^2 - x^2}}{\gamma}\right)} - \sqrt{\gamma^2 - x^2} \]

To calculate the attenuation coefficients for the surface wave modes, the eigenvalue equation has to be recasted first in a convenient way similar to that of the flat slab waveguide. It can be easily shown after some algebraic manipulations, that the modal equation for the TE wave is rewritten as

\[ D_e^e = 1 + R_e^e e_e^e = 0 \]  \hspace{1cm} (28)

whereas that for the TM wave is given by

\[ D_m^m = 1 - R_m^m e_m^m = 0 \]  \hspace{1cm} (29)

where
\[ R_e^e = \frac{b k_{1z} - k_z}{k_{1z} + k_z}, \quad R_m^m = \frac{b \varepsilon k_{1z} - \varepsilon_1 k_z}{\varepsilon k_{1z} + \varepsilon_1 k_z} \]  \hspace{1cm} (30)

\[ b = - \frac{H_\nu^{(1)'}(k_{1r_0})}{H_\nu^{(2)'}(k_{1r_0})} \cdot \frac{H_\nu^{(2)}(k_{1r_0})}{H_\nu^{(1)}(k_{1r_0})} - 1 \]

\[ k_z = -i k \frac{H_\nu^{(1)'}(k_{r_0})}{H_\nu^{(1)}(k_{r_0})} - i \gamma(1 - i e^{-2s}) \]

\[ \gamma = \sqrt{\left(\frac{\nu}{R}\right)^2 - k^2}, \quad S = R \tanh^{-1}\left(\frac{\gamma}{\nu R}\right) = \gamma d \]
\[ k_{1z} = i k_1 \frac{H_v^2(k_1R_0)}{H_v^2(k_1R_0)} \sqrt{k_1^2 - \left( \frac{\nu}{R} \right)^2} \]

\[ e^{e} = \frac{H_v^1(k_1R_0)}{H_v^2(k_1R_0)} \cdot \frac{H_v^2(k_1R)}{H_v^1(k_1R)} \exp(i2k_{1z}d) \]

\[ e^{m} = -\frac{H_v^1(k_1R_0)}{H_v^2(k_1R_0)} \cdot \frac{H_v^2(k_1R)}{H_v^1(k_1R)} \exp(i2k_{1z}d) \]

From the way the eigenvalue equation is recasted, we can see the close resemblance between the modal equation of both the coated cylinder and the flat slab waveguide in the limit \( R \to \infty \)

Since \( e^{-2s} \ll 1 \), it can be shown, for the TE case, that

\[
R_v^e - r_v^e + \frac{i}{2} \{1 - (r_v^e)^2\} e^{-2s} \quad \text{where} \quad r_v^e = \frac{k_{1z} - i\gamma}{k_{1z} + i\gamma}
\]

and

\[
D_v^e - 1 + r_v^e e_v^e + \frac{i}{2} \{1 - (r_v^e)^2\} e_v^e e^{-2s} = 0 \quad (31)
\]

In this equation it is clear that the second term is an exponentially small term.

Now assume

\[ \nu = k_y R + \nu_s \]

where \( k_y \) is the solution to the guidance equation of the flat slab waveguide and thus \( k_y R \) is the zeroth order term in \( \nu \) and \( \nu_s \) is the
higher order term.

The next step is to expand \((1 - \Rev_{\nu} \Rev_{\nu})\) into a zeroth order term corresponding to \(k_y R\) (zeroth order term of \(\nu\)) and a higher order term corresponding to \(\nu_s\) and by substituting these expansions and balancing terms of the same order in equation (31), we get the required expression for \(\nu_s\) and consequently for the attenuation coefficient.

It can be easily shown that

\[
1 + \Rev_{\nu} \Rev_{\nu} = [1 + \Rev_{\nu} \Rev_{\nu}]_{\nu = k_y R} - \nu_s \left[\frac{12}{\gamma R} \frac{k_y}{k_{1z}} (1 + \gamma d) \Rev_{\nu} \Rev_{\nu} \right]_{\nu = k_y R}
\]

and from the process of balancing terms of the same order in equation (31) we get

\[
[1 + \Rev_{\nu} \Rev_{\nu}]_{\nu = k_y R} = 0 \quad \text{for the zeroth order term}
\]

\[
\nu_s = \left[\frac{1}{4} \frac{\gamma R (1 - (\Rev_{\nu})^2)}{\Rev_{\nu} (1 + \gamma d)} \frac{k_{1z}}{k_y} e^{-2s}\right]_{\nu = k_y R} \quad \text{for the higher order term}
\]

The first equation is the modal equation of the flat slab waveguide and therefore \(k_y\) is indeed the solution to the flat slab waveguide modal equation, and from the second equation, we get the attenuation coefficient as

\[
\nu_s = \frac{i \gamma^2}{k_y (1 + \gamma d)} \frac{k_{1z}^2}{k_1^2 - k^2} R e^{-2s}
\]
where
\[ \gamma = \sqrt{k_y^2 - k^2}, \quad k_{1z} = \sqrt{k_1^2 - k_y^2}, \quad S = R \left( k_y \tanh^{-1} \left( \frac{\gamma}{k_y} \right) \right) - \gamma d \]

which agrees perfectly with the expression obtained by Marcuse.

As for the TM case, we have

\[ R^m_v - r^m_v + \frac{i}{2} (1 - (r^m_v)^2) e^{-2s} \quad \text{where} \quad r^m_v = \frac{ek_{1z} - i\varepsilon_1 \gamma}{ek_{1z} + i\varepsilon_1 \gamma} \]

and

\[ D^m_v - 1 - r^m_v e^m_v - \frac{i}{2} (1 - (r^m_v)^2) e^m_v e^{-2s} = 0 \]

Assuming
\[ v = k_y R + v_s \]

where \( k_y \), in this case, is the solution to the guidance equation of the flat slab waveguide in the TM case.

It can be thus shown that

\[ 1 - r^m_ve^m_v = [1 - r^m_v e^m_v]_{v=k_y R} - v_s \left[ \frac{i2}{k_y} \frac{\varepsilon_1 (k_1^2 - k^2)}{k_{1z} \left( \varepsilon^2 k_{1z}^2 + \varepsilon_1^2 \gamma^2 \right)} + \frac{\gamma d}{k_y} \right] r^m_v e^m_v \]

Substituting into the expression for \( D^m_v \) and balancing terms of the same order we get
\[ v_s = i \epsilon \epsilon_1 \frac{\gamma^2}{k_y \left( \frac{\epsilon \epsilon_1 (k_1^2 - k^2)}{\epsilon^2 k_1^2 + \epsilon_1^2 \gamma^2} + \gamma d \right)} \cdot \frac{k_1^2}{\epsilon^2 k_1^2 + \epsilon_1^2 \gamma^2} R e^{-2s} \]

and \( k_y \) is the solution to the equation

\[ [1 - e^m e^{m_1}]_u = k_y R = 0 \]
2) Perturbational Formula

In this approach, the coated cylinder is assumed to be a slightly bent dielectric slab waveguide. Thus the very large coated cylinder will be treated as a small perturbation of the flat dielectric slab waveguide.

For the flat dielectric slab waveguide [Fig. (4)] we have

\[ \mathbf{E}(x,y) = \mathbf{E}(x) \exp(ik_y y), \quad \mathbf{H}(x,y) = \mathbf{H}(x) \exp(ik_y y) \]

Thus

\[ \nabla \times \mathbf{E}(x,y) = [\nabla \times \mathbf{E}(x) + ik_y \hat{y} \times \mathbf{E}(x)] \exp(ik_y y) \]

and since

\[ \nabla \times \mathbf{E}(x,y) = i\omega \mu \mathbf{H}(x,y) \]

we get

\[ \nabla \times \mathbf{E}(x) = i\omega \mu \mathbf{H}(x) - ik_y \hat{y} \times \mathbf{E}(x) \]

where the curl operator in this equation is the one-dimensional curl operator. Similarly,

\[ \nabla \times \mathbf{H}(x) = -i\omega \varepsilon \mathbf{E}(x) - ik_y \hat{y} \times \mathbf{H}(x) \quad 0 < x < d \]
Figure 4. The Flat Slab Waveguide.

Figure 5. The Slightly Bent Slab Waveguide.
For the slightly bent dielectric slab waveguide [Fig. (5)] we have

\[ \mathbf{E}(\rho, \phi) = \mathbf{E}(\rho) \exp(i \nu \phi), \quad \mathbf{H}(\rho, \phi) = \mathbf{H}(\rho) \exp(i \nu \phi) \]

The \( \phi \)-dependence is singled out, since we are interested in obtaining a perturbational formula in \( \nu \).

\[ \nabla \times \mathbf{E}(\rho, \phi) = \left[ \nabla \times \mathbf{E}(\rho) + \frac{i \nu}{\rho} \hat{\phi} \times \mathbf{E}(\rho) \right] \exp(i \nu \phi) \]

and since

\[ \nabla \times \mathbf{E}(\rho, \phi) = i \omega \mu \mathbf{H}(\rho, \phi) \]

we get

\[ \nabla \times \mathbf{E}(\rho) = i \omega \mu \mathbf{H}(\rho) - \frac{i \nu}{\rho} \hat{\phi} \times \mathbf{E}(\rho) \quad (32) \]

Similarly

\[ \nabla \times \mathbf{H}(\rho) = -i \omega \epsilon \mathbf{E}(\rho) - \frac{i \nu}{\rho} \hat{\phi} \times \mathbf{H}(\rho) \quad R < \rho < R_0 \quad (33) \]

For the region inside the dielectric, i.e. \( R + d < \rho < R \) and in the narrow wedge around \( \phi = 0 \), we introduce the following change of variables
\[ \rho = R + x \quad \quad \quad \quad y = \rho \phi - R\phi \]

Therefore

\[ \frac{\partial}{\partial \rho} \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \rho} \frac{\partial}{\partial y}, \quad \rho \to \hat{x}, \quad \phi \to \hat{y} \]

Under this change of variables, equations (32) and (33) reduce to

\[ \nabla \times \vec{E}(x) = i \omega \mu \vec{H}(x) - \frac{i}{R} \hat{y} \times \vec{E}(x) \]

\[ \nabla \times \vec{H}(x) = -i \omega \varepsilon \vec{E}(x) - \frac{i}{R} \hat{y} \times \vec{H}(x) \]

It should be pointed out that \( \vec{E}(x) \) and \( \vec{H}(x) \) in the above two equations are now expressed in terms of the localized cartesian coordinate system introduced by the change of variables. It should also be pointed out that \( \nabla \times \vec{E}(\rho) \) and \( \nabla \times \vec{H}(\rho) \) give the terms \( \frac{1}{R} E_y \hat{z} \) and \( \frac{1}{R} H_y \hat{y} \) respectively in addition to \( \nabla \times \vec{E}(x) \) and \( \nabla \times \vec{H}(x) \), however these terms are negligibly small (of an order \( 1/R \) less than the other terms involved).

Applying the divergence theorem, where we define \((\vec{E}_1, \vec{H}_1)\) to be the field components for the flat slab waveguide and \((\vec{E}_2, \vec{H}_2)\) to be those for the bent waveguide.

\[ \nabla \times \vec{E}_1 \times \vec{H}_2 + \vec{E}_2 \times \vec{H}_1 = -2 \omega \varepsilon_1 \vec{E}_1 \cdot \vec{E}_2 - i \left( \frac{\nu}{R} - k_y \right) \hat{y} \cdot [\vec{E}_1 \times \vec{H}_2 + \vec{E}_2 \times \vec{H}_1] \]
where $\varepsilon''$ is the imaginary part of $\varepsilon_1$.

Integrating over a volume $V$ enclosed by the surface $S$ and applying Gauss' theorem we get

$$\int_S dS \cdot (E_1 \times \hat{H}_2 + E_2 \times \hat{H}_1) = -i \left( \frac{\nu}{R} - k_y \right) \int_V dv \cdot \hat{y} \cdot (E_1 \times \hat{H}_2 + E_2 \times \hat{H}_1) - 2\omega \varepsilon'' \int_V dv \cdot E_1 \cdot E_2$$

Thus we get the following perturbational formula for $\nu$

$$\frac{\nu}{R} = k_y + i \frac{\int_S dS \cdot (E_1 \times \hat{H}_2 + E_2 \times \hat{H}_1) + 2\omega \varepsilon'' \int_V dv \cdot E_1 \cdot E_2}{\int_V dv \cdot \hat{y} \cdot (E_1 \times \hat{H}_2 + E_2 \times \hat{H}_1)}$$

(34)

where

$$E_1 \times \hat{H}_2 + E_2 \times \hat{H}_1 = (E_1 H_{x_2} + E_2 H_{x_1})\hat{y} - (E_1 H_{y_2} + E_2 H_{y_1})\hat{x}$$

for the TE case

and

$$E_1 \times \hat{H}_2 + E_2 \times \hat{H}_1 = -(E_1 H_{x_2} + E_2 H_{x_1})\hat{y} + (E_1 H_{y_2} + E_2 H_{y_1})\hat{x}$$

for the TM case

Taking the volume $V$ to be the volume of the infinitesimal wedge around $\phi = 0$, which has the dimensions $dy$, $dz$ and extends over the thickness of the coating (see Fig. (6)).

With this choice of the volume $V$, we find that the part of the surface integral in (34) at $x = 0$ vanishes from the boundary condition on the tangential component of the electric field. The front and back surface integrals are also equal to zero since the normal to these surfaces (which is along the $\hat{z}$-direction) is normal to $(E_1 \times \hat{H}_2 + E_2 \times \hat{H}_1)$ (which
lies exclusively on the x-y plane). Finally the side surface integrals cancel each other in the limit when dy → 0. Thus the surface integral reduces to that on the top surface at x = d (ρ = R + d) which has the dimensions dy and dz.

\[ \int_S dS \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) = dzdy \hat{x} \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) \]  

at x = d

As for the volume integral in the numerator of (34), it reduces to

\[ \int_v dv \, \mathbf{E}_1^* \cdot \mathbf{E}_2 = dz \, dy \int_0^d dx \, \mathbf{E}_1^* \cdot \mathbf{E}_2 \]

and that in the denominator reduces to

\[ \int_v dv \, \hat{y} \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) = dz \, dy \int_0^d dx \, \hat{y} \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) \]

Thus the perturbational formula reduces to

\[ \frac{2\omega e_1''}{R} \int_0^d dx \, \mathbf{E}_1^* \cdot \mathbf{E}_2 + \hat{x} \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) \]

at x = d

(35)

and \( k_y \) is the solution to the guidance condition of the flat waveguide with a lossless dielectric.
In the case of the TE polarization

\[ \frac{\nu}{R} = k_y + i \frac{2\omega \varepsilon_1''}{d} \int_0^d dx \left( E_1^* E_2 - (E_1^* H_2 + E_2^* H_1) \right) \]
\[ at \ x = d \]
\[ \int_0^d dx (E_1^* H_2 + E_2^* H_1) \]

and \( k_y \) is the solution of the equation

\[ 1 - R_{01} \exp(i2k_1 x d) = 0, \]
\[ R_{01} = \frac{i \gamma - k_1 x}{\gamma + k_1 x} \]

and

\[ k_i^2 - k_1^2 = k^2 + \gamma^2 = k_y^2 \]

and since

\[ H_x = \frac{k_y}{\omega \mu} E \quad \text{and} \quad H_y = -\frac{1}{\omega \mu} \frac{dE}{dx} \]

These relations are valid exactly for the flat slab waveguide and asymptotically for the slightly bent waveguide.

Using these relations, the perturbational formula simplifies to

\[ \frac{\nu}{R} = k_y + i \frac{\omega^2 \varepsilon_1''}{k_y} \left( \frac{1}{2k_y} \int_0^d dx E_1^* E_2^* \right) \int_0^d dx \left( \frac{dE_1^*}{dx} - E_1^* \frac{dE_2}{dx} \right) \text{at} \ x = d + 0 \]
where \( x = d + 0 \) means that the quantity between the square brackets is evaluated at \( x = d \) on the side of the outer region (from the boundary conditions this quantity is continuous at \( x = d \)). The evaluation of the quantity between the square brackets at \( x = d + 0 \) rather than \( x = d - 0 \) will allow us to account for the radiation loss. This is because the information about the radiation loss lies in the field expression of the outer region.

For the flat slab waveguide

\[
E_{1}^{-} = A_{1} \sin k_{1x}x, \quad E_{1}^{+} = A_{1} \sin(k_{1x}d)e^{yd}e^{-\gamma x}
\]

where \( E^{-} \) and \( E^{+} \) are the z-component of the electric field for \( x < d \) and \( x > d \) respectively and \( \gamma^2 = k_y^2 - k_1^2 \). As for the slightly bent waveguide

\[
E_{2}^{-} = A_{2}[H_{\nu}^{(1)}(k_{1\rho}) H_{\nu}^{(2)}(k_{1R}) - H_{\nu}^{(1)}(k_{1R}) H_{\nu}^{(2)}(k_{1\rho})] \quad \text{where} \quad \rho = R + x
\]

\[
E_{2}^{+} = B H_{\nu}^{(1)}(k_{\rho})
\]

In the expression of \( E_{2}^{-} \) and since \( \nu - k_y R < k_1 \rho \), the Hankel functions are replaced by the asymptotic expansion (i) of regions (1) and (4) given in the appendix. Thus we get

\[
E_{2}^{-} \approx A_{2} \frac{i4}{k_y R} \sin(k_{1x}x)
\]

(36)
As for $E^+_2$ at the surface i.e. at $\rho = R + d$ and since $\nu > k_y R > k R$, in this case the Hankel function $H^{(1)}_\nu(k R)$ is replaced by the expansion given by equation (27a) to give

$$E^+_2(\rho = R_0) = B H^{(1)}_\nu(k R_0) - i B \sqrt{\frac{2}{\pi R}} e^a \left[ 1 + \frac{i}{2} e^{-2a} \right]$$

where we have kept the real and imaginary leading order terms and neglected higher order terms in $(d/R)$ and

$$a = R \left( k_y \tanh^{-1} \left( \frac{\gamma}{k_y} \right) - \gamma \right) - \gamma d$$

The expansion given in equation (27a) gives the Hankel function as the sum of a growing (but very small) wave and an evanescent wave. As will be shown, this growing wave component of the field outside the surface is the one which results in the attenuation of the wave caused by radiation loss. In the absence of this growing wave component, the perturbation solution will give no attenuating contribution to the propagation constant due to radiation loss.

The presence of this growing wave component can be easily explained by recalling that for values of $\rho > \rho_0$ ($\rho_0$ is the turning point defined at the beginning of section (ii)) the field is an outgoing unattenuated cylindrical wave whereas for $\rho < \rho_0$ the field must be evanescent, thus the character of the solution at $\rho = \rho_0$ must change causing the field to be partially transmitted and partially reflected at $\rho = \rho_0 - 0$, returning
toward the slab as an incoming evanescent wave. Thus near the slab surface, the field must have a finite although small, exponentially growing component [29]. This growing component is the local form of the radiation condition that has to be applied at the surface in order to ensure the satisfaction of the radiation condition at infinity [35].

Comparing with Marcuse's approach, he uses only the evanescent part of the local field (without the growing component) to relate it to the field in the case of a flat slab waveguide. He then calculates the Poynting vector radiation using the far field approximation, from which he finally calculates the bending less attenuation coefficient. Thus, in his approach, he does not require the growing component of the local field, in other words, he employs the radiation condition at infinity instead of using its localized form.

From the boundary conditions at $\rho = R_0$ $(x = d)$, and comparing (36) and (37) we get

$$B = -A_2 \sqrt{\frac{\pi \gamma R}{2}} \frac{4}{\pi k_y R} \sin(k_{1x}d) e^{-a\left(1 + \frac{i}{2} e^{-2a}\right)}^{-1}$$

It can also be shown that

$$\frac{dE^+}{d\rho} (\rho = R_0) = B \frac{\gamma}{\sqrt{2\pi}} \frac{2k_y}{k_y} e^{a\left(1 - \frac{i}{2} e^{-2a}\right)} - i A_2 \frac{4\gamma}{\pi k_y R} \sin(k_{1x}d)(1-ie^{-2a})$$
Substituting into the perturbational formula we get

\[ \frac{\nu}{R} = k_y + \frac{i \omega \varepsilon^\prime_1}{k_y} + \frac{i 2k_1 x \gamma \sin^2(k_1 x d)}{k_y (2k_1 x d - \sin(2k_1 x d))} e^{-2a} \]

Therefore

\[ \frac{\nu}{R} = \frac{\omega^2 \varepsilon_1^\prime}{k_y} + \frac{2k_1 x \gamma \sin^2(k_1 x d)}{k_y (2k_1 x d - \sin(2k_1 x d))} e^{-2a} \]

Where the first part is the attenuation caused by the dielectric losses (where we assumed that \( \varepsilon_1^\prime << \varepsilon_1^\prime \)) and the second term is the attenuation caused by the radiation loss.

Using \( \sin^2 k_1 x d = \frac{k_1^2}{k_1^2 - k^2} \) and \( \sin(2k_1 x d) = -2 \frac{k_1 x \gamma}{k_1^2 - k^2} \), it can be shown that this expression agrees satisfactorily with that of Marcuse.

For the case of the TM polarization

\[ \frac{\nu}{R} = k_y + \frac{i \omega^2 \varepsilon^\prime_1}{k_y} \left[ 1 + \frac{1}{k_1^2} \int_0^d \frac{dH^*_1}{dx} \frac{dH^*_2}{dx} \right] \left[ 1 + \frac{1}{2k_y} \frac{H_2}{\varepsilon} \left( \int_0^d \frac{dH^*_1}{dx} - \frac{H^*_1}{\varepsilon} \int_0^d \frac{dH^*_2}{dx} \right) \right] \]

where \( k_y \) is the solution to the equation

\[ 1 + R_{01} \exp(i2k_1 x d) = 0, \quad R_{01} = \frac{i \varepsilon_1^\prime \gamma - \varepsilon k_1 x}{i \varepsilon_1^\prime \gamma + \varepsilon k_1 x} \]

and where we have used the following relations between the electric and magnetic field components.
\[ E_x = - \frac{k_y}{\omega \varepsilon_j} H, \quad E_y = \frac{1}{i \omega \varepsilon_j} \frac{dH}{dx} \]

and

\[ \frac{d^2H}{dx^2} = -k_1^2 x H \]

for \( 0 < x < d \)

where

\[ \varepsilon_j = \varepsilon \]

\[ = \varepsilon \]

\[ \text{for } 0 < x < d \]

\[ d < x \]

For the flat slab waveguide

\[ H_1^- = A_1 \cos(k_1 x) \cos(k_1 d) e^{\gamma d} e^{-\gamma x} \]

and

\[ H_1^+ = A_1 \cos(k_1 x) \cos(k_1 d) e^{\gamma d} e^{-\gamma x} \]

and for the slightly bent waveguide

\[ H_2^- = A_2 [H_\nu^{(1)}(k_1 \rho) H_\nu^{(2)}(k_1 R) - H_\nu^{(1)'}(k_1 R) H_\nu^{(2)'}(k_1 \rho)] \]

\[ - \frac{i4}{\pi k_1 R} A_2 \cos(k_1 x) \]

and

\[ H_2^+ = B H_\nu^{(1)}(k \rho) \]

\[ - iB \sqrt{\frac{2}{\pi \gamma R}} e^{a} \left[ 1 + \frac{i2}{e^{-2a}} \right] \text{ at } \rho = R_0 \]

where \( a \) is given by Eq. (38). Thus
\[ \frac{dH_2^+}{dx} (\text{at } x=d) = A_2 \frac{i4\gamma}{\pi k_1 R} \cos(k_1 x d) (1 - ie^{-2a}) \]

and finally we get

\[ \frac{\nu}{R} = k_y + \frac{i\omega \varepsilon_0}{k_y} \left[ 1 - \frac{k_1 x}{k_1} \frac{\sin(2k_1 x d)}{d + \frac{1}{2k_1^2} \sin(2k_1 x d)} \right] + i \frac{\varepsilon_0}{k_y} \frac{\gamma}{\varepsilon} \frac{\cos^2(k_1 x d)}{d + \frac{1}{2k_1^2} \sin(2k_1 x d)} e^{-2a} \]

where, from the guidance condition, we have

\[ \sin(2k_1 x d) = \frac{2\varepsilon_1 k_1 x \gamma}{\varepsilon^2 k_1^2 + \varepsilon_1^2 \gamma^2} \quad \text{and} \quad \cos^2(k_1 x d) = \frac{\varepsilon^2 k_1^2}{\varepsilon^2 k_1^2 + \varepsilon_1^2 \gamma^2} \]

Comparing between the attenuation coefficients of the creeping wave modes and the surface wave modes we find that whether the dielectric is lossless or lossy, the attenuation coefficient of the creeping wave modes is of the order \((kr)^{1/3}\) whereas that of the surface wave modes depends on the loss in the dielectric. In the case of a lossless dielectric, the attenuation constant is an exponentially small quantity of the order \(kr e^{-kR}\) and therefore the surface modes are far less attenuated than the creeping wave modes. Whereas, if the dielectric is lossy, the attenuation coefficient of the surface modes is of the order \(k_1^2 R\) and therefore are more attenuated than the creeping wave modes.
The electric field of the TE polarization, recast in a form suitable for calculating the contributions from the surface wave modes, is given by

\[
E = \frac{1}{2} \int_{-\infty + i\delta}^{+\infty + i\delta} d\nu \frac{\cos(\nu(\phi-\pi))}{\sin(\nu\pi)} \frac{B_n^e}{1 + R_n^e} \frac{H_n^1(kR_o) \exp(-i\nu\pi/2)}{H_n^1(kR_o) \exp(-i\nu\pi/2)}
\]

(39)

\[= \pi E_0 \sum_{n=1}^{\infty} \frac{1}{\partial D_n^e / \partial \nu} \frac{\cos(\nu(\phi-\pi))}{\sin(\nu\pi)} \frac{B_n^e}{H_n^1(kR_o) \exp(-i\nu\pi/2)}
\]

where \(D_n^e = 1 + R_n^e\) given by equation (28)

\[B_n^e = S_n^e \left( \frac{k_{1z} + c_n^e}{k_{1z} + c_n^e} \right) \]

where

\[S_n^e = \frac{H_n^2(kR_o)}{H_n^1(kR_o)}, \quad c_n^e = -ik \frac{H_n^2(kR_o)}{H_n^1(kR_o)}, \quad c_n^e = -ik \frac{H_n^1(kR_o)}{H_n^1(kR_o)}
\]

\(B_n^e\) given in this form is not appropriate for numerical computation as it results in an underflow.

After simple algebraic manipulation, \(B_n^e\) can be put in the following form

\[B_n^e = S_n^e \left[ 1 + R_n^e \frac{(1 - e_n^e)}{k_{1z} + c_n^e} (c_n^e - c_n^e) \right]
\]
at a surface wave mode \((1 + R^e_{\nu} e^e_{\nu})\) vanishes and therefore \(B^e_{\nu}\) reduces to

\[
B^e_{\nu} = -\frac{4}{\pi R_o} \frac{(1 - e^e_{\nu})}{k_{1z} + k_z} \frac{1}{[H^{(1)}_{\nu}(kR_o)]^2}
\]

where we have used the Wronskian of the Hankel function in simplifying the expression of \(B^e_{\nu}\).

Therefore for the surface wave modes, in the case of the TE wave

\[
E = -\hat{z} 2E_o \sum_{n=1}^{\infty} E^\nu_n \frac{[\exp(i\nu_n(\phi-\pi/2)) + \exp(i\nu_n(3\pi/2-\phi))]}{H^{(1)}_{\nu_n}(kR_o)}
\]

where

\[
E^\nu_n = \frac{\gamma}{k_{1z}} \frac{1 - e^e_{\nu}}{1 + \gamma d} \frac{1}{1 - \exp(i2\pi R_k)}
\]

As for the TM polarization, the magnetic field is given by

\[
\bar{H} = \hat{z} \pi H_o \sum_{n=1}^{\infty} \frac{1}{2D^m_{\nu_n}} \frac{\cos(\nu_n(\phi-\pi))}{\sin(\nu_n\pi)} \frac{H^{(1)}_{\nu_n}(k\rho)}{H^{(1)}_{\nu_n}(kR_o)} \exp(-i\nu_n\pi/2) B^m_{\nu_n}
\]

where \(D^m_{\nu_n} = 1 - R^m_{\nu_n} e^m_{\nu_n}\), given by equation (29), and \(\nu_n\) is the solution to the modal equation \(D^m_{\nu_n} = 0\) corresponding to the surface modes.

\[
B^m_{\nu} = S_{\nu} \frac{\varepsilon k_{1z} (1 - b e^m_{\nu}) + \varepsilon_1 (1 + e^m_{\nu}) c^e_{\nu}}{\varepsilon k_{1z} + \varepsilon_1 c^e_{\nu}}
\]

This expression for \(B^m_{\nu}\) reduces to the following form

\[
B^m_{\nu} = -\frac{4}{\pi R_o} \frac{\varepsilon_1 (1 + e^m_{\nu})}{\varepsilon k_{1z} + \varepsilon_1 k_z} \frac{1}{[H^{(1)}_{\nu}(kR_o)]^2}
\]
where we have used the fact that at a surface mode \((1 - R_{\nu} e^{i\nu})\) vanishes and we have also used the Wronskian of the Hankel function in simplifying \((c_{\nu}' - c_{\nu})\).

Thus for the TM surface wave modes

\[
\overline{H} = \frac{\hat{z}}{2} \sum_{n=1}^{\infty} H_{\nu} \left[ \exp(i\nu(\phi - \pi/2)) + \exp(i\nu(3\pi/2 - \phi)) \right] \frac{H^{(1)}(kr)}{[H^{(1)}(kR_0)]^2}
\]

where

\[
H_{\nu} = \frac{\gamma}{k y} \frac{\varepsilon_1 k_{1z}}{\varepsilon k_{1z} - i \varepsilon_1 \gamma} \left( \frac{1}{\varepsilon_1 (k_{1z}^2 - k^2)} - \frac{1}{\varepsilon_1 (k_{1z}^2 + \varepsilon_1 \gamma^2)} \right) \frac{1 + \varepsilon^m}{\varepsilon^m} \frac{1}{1 - e^{i2\pi B y}}
\]
B. The Ray Expansion

In the previous sections, the field was represented in the form of a modal series expansion, in which each term in the series is identified as a contribution from one of the natural modes of the structure.

In this section a ray series expansion is developed to represent the fields. In this case, the series obtained represents rays which are multiply reflected inside the dielectric. This ray series expansion is obtained by expanding the integrand in terms of the cylindrical Fresnel's reflection coefficient.

In equation (30), it can be easily shown that $R_{\nu}^e$ and $R_{\nu}^m$ are the Fresnel's reflection coefficients of TE and TM cylindrical waves incident from the dielectric side onto the air-dielectric interface. It can also be shown that $|R_{\nu}^e e|_v$ and $|R_{\nu}^m e|_v$ $< 1$ for real values of $\nu$. Thus the factors $1/(1 + R_{\nu}^e e)$ and $1/(1 - R_{\nu}^m e)$ can be expanded in a Taylor series to get the following ray expansion series for both the TE and TM waves

$$ F = -\hat{z} F_0 \frac{1}{2} \sum_{m=0}^{\infty} \frac{d \nu}{\sin(\nu \pi)} \frac{A_\nu(Q_\nu)^m}{(M_\nu)^{m+1}} (e_\nu)^m H_{\nu}^{(1)}(k \rho) \exp(-i \nu \pi/2) $$

(40)

where for the TE wave

$$ A_\nu = k_{1z} (1 + be_\nu^e) H_{\nu}^{(2)}(k R_0) - i k (1 - e_\nu^e) H_{\nu}^{(2)}(k R_0) $$

$$ Q_\nu = -[b k_{1z} H_{\nu}^{(1)}(k R_0) + i k H_{\nu}^{(1)}(k R_0)] $$

$$ M_\nu = k_{1z} H_{\nu}^{(1)}(k R_0) - i k H_{\nu}^{(1)}(k R_0) $$
\[ \bar{F} = \bar{E} , \quad F_0 = E_0 , \quad e_v = e \]

and for the TM wave

\[ A_v = \varepsilon k_1 z (1 - \beta \rho_v) H^{(2)}(kR_0) - i \varepsilon_1 k (1 + \rho_v) H^{(2)'}(kR_0) \]

\[ Q_v = b k_1 H^{(1)}(kR_0) + i \varepsilon_1 k H^{(1)'}(kR_0) \]

\[ M_v = \varepsilon k_1 z H^{(1)}(kR_0) - i \varepsilon_1 k H^{(1)'}(kR_0) \]

\[ \bar{F} = \bar{H} , \quad F_0 = H_0 , \quad e_v = e \]

It can be easily shown that the term corresponding to \( m = 0 \), is the field scattered from a dielectric cylinder of radius \( R_0 \) (i.e. in the absence of the inner conducting cylinder) whereas the term corresponding to \( m = 1 \) is a singly reflected ray and the \( m \)-th term is a ray which has suffered \( m \) reflections inside the dielectric coating.

The integral in equation (40) may now be evaluated by a residue series. The poles are located by solving for the equation \( M_v = 0 \) which for the TE case has the form

\[ k_1 z H_v^{(1)}(kR_0) - ik H_v^{(1)'}(kR_0) = 0 \]

and for the TM case has the form
\[ \varepsilon k_{1z} H^{(1)}_{\nu}(k R_0) - i \varepsilon \alpha k H^{(1)'}_{\nu}(k R_0) = 0 \]

In terms of the Airy function, this eigenvalue equation is given by

\[ \text{Ai}(-a) = i \alpha \left( \frac{2}{k R_0} \right)^{1/3} \exp(-i \pi/3) \text{Ai}'(-a) \]

where

\[ \nu = k R_0 + a \left( \frac{k R_0}{2} \right)^{1/3} \exp(i \pi/3) \]

and

\[ \alpha = 1 \quad \text{for TE case} \]
\[ = \varepsilon_1 / \varepsilon \quad \text{for TM case} \]

In calculating the residue series of the integral in (40), notice that the m-th term has a pole of multiplicity \(m + 1\).

To calculate the residues, let us consider the m-th term in the series which can be represented as

\[ I_m = \int \frac{p^{(m)}_{\nu}}{(M_{\nu})^{m+1}} \]

where

\[ p^{(m)}_{\nu} = \frac{\cos(\nu(\phi - \pi))}{\sin(\nu \pi)} A_{\nu}(Q_{\nu})^m (e_{\nu})^m H^{(1)}_{\nu}(k R_0) \exp(-i \nu \pi/2) \]
For the \( n \)-th pole which occurs at \( \nu = \nu_n \)

\[
M_{\nu} = (\nu - \nu_n) J_{\nu}
\]

where

\[
J_{\nu} = \sum_{r=1}^{\infty} \frac{(\nu - \nu_n)^{r-1}}{r!} \left[ \frac{\partial^r M_{\nu}}{\partial \nu^r} (\nu = \nu_n) \right]
\]

and thus

\[
I_m = \frac{2\pi i}{m!} \sum_{n=1}^{\infty} \left[ \frac{\partial^m}{\partial \nu^m} \left\{ \frac{p(m)}{(J_{\nu})^{m+1}} \right\} \right]_{\nu = \nu_n}
\]

and finally the field is given by

\[
F = \pi F_0 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \left[ \frac{\partial^m}{\partial \nu^m} \left\{ \frac{p(m)}{(J_{\nu})^{m+1}} \right\} \right]_{\nu = \nu_n}
\]

with

\[
\frac{\partial^s J_{\nu}}{\partial \nu^s} (\nu = \nu_n) = \frac{1}{(s+1)} \left[ \frac{\partial^{(s+1)} M_{\nu}}{\partial \nu^{(s+1)}} (\nu = \nu_n) \right]
\]

and

\[
J_{\nu_n} = M'_{\nu_n}
\]
From these above expressions, it is clear that the computation of the field using the ray expansion approach is much more complicated than the modal expansion approach. This method, thus becomes impractical if a large number of ray contributions has to be included in the field computation and hence it is only of practical use, if the dielectric is lossy and thick.
IV. Results and Conclusions

In this chapter, the scattering from a dielectric coated cylinder is formulated using two approaches. The first method of solution is the modal approach, in which the field is represented in terms of the natural modes of the structure. The second method is the ray expansion approach, in which the field is represented in terms of multiply reflected rays.

In the modal approach, there appear two kinds of modes, the creeping wave modes and the surface wave modes, with the first contributing dominantly to the field. The first kind of modes, are modes which correspond to the usual creeping waves for a perfectly conducting cylinder. These modes arise from poles in the vicinity of the zeros of either $H_{\nu}^{(1)}(kR)$ or $H_{\nu}^{(1)'}(kR)$ depending on the thickness of the dielectric and the type of polarization. For both types of polarizations, the attenuation coefficient, as a function of the coating thickness $d$, will oscillate between values limited by the TM and TE attenuation coefficients of the uncoated cylinder. This is only true if the dielectric is lossless. In the case of a lossy dielectric, the attenuation coefficient will not exhibit the same oscillatory behavior as in the case of the lossless dielectric. Figures (7) and (8) show the attenuation coefficients for the TE and TM polarizations, respectively as a function of the coating thickness $d$ for a lossless dielectric of permittivity $\varepsilon_1 = 3\varepsilon_0$ at a frequency of 0.1 GHz and $R_0 = 6700$ km. Figure (9) shows the electric field for the TE and TM polarizations for a thickness of $d = 8.5$ m., $R_0 = 6700$ km and at a frequency of 0.1 GHz, for $\varepsilon_1 = 3\varepsilon_0$. The electric field for both polarization is normalized to the same value at $\rho = R_0$. From this
figure it is clear that the TE polarization is more attenuated than the TM one.

The second class of modes are the surface waves modes which are a small perturbation of those of the flat slab waveguide except that, unlike the nonradiating surface modes of the flat slab waveguide, these modes radiate and therefore attenuate as they propagate because of this bending loss. The attenuation of these surface modes are calculated by two methods. In the first method a perturbational approach is applied to the exact eigenvalue equation obtained by solving the boundary value problem. In the second method a perturbational formula is developed starting from Maxwell's equations to obtain an expression for the propagation constant \( \nu \). It is shown that both methods agree satisfactorily, it is also shown that this attenuation coefficient is practically negligible because of the large radius of curvature and therefore in the case of a lossy dielectric, the attenuation coefficient is dominated by the absorption effect inside the dielectric.

The field computation using the ray expansion approach is much more complicated than the modal expansion approach. The method of ray expansion, thus becomes impractical if a large number of ray contributions has to be included in the field computation and hence it is only of practical use, if the dielectric is lossy and thick.
Figure 7: Attenuation Coefficient of TE Polarized Wave

Figure 8: Attenuation Coefficient of TM Polarized Waves
Figure 9

Azimuthal distance in km

Normalized Electric Field

TM polarized wave

TE polarized wave
Appendix

Asymptotic Expansions of $H^{(1)}_\nu(x)$, $H^{(2)}_\nu(x)$ and Their Derivatives

Two types of asymptotic expansions are used for the Hankel functions, the first is for large unequal order and argument and the second is for large and almost equal order and argument.

(i) Large and Unequal Order and Argument \([34,38,39]\):

There are three different asymptotic expansions for $H^{(1)}_\nu(x)$ and $H^{(2)}_\nu(x)$ depending on the location of \(\nu\) in the complex \(\nu\)-plane.

For $H^{(1)}_\nu(x)$, the complex \(\nu\)-plane is divided into three different regions (Fig. (A1)), separated by the curves $C_1$, $C_2$ and the solid part of the line $\nu'' = \nu' \tan \phi$, whereas for $H^{(2)}_\nu(x)$, the \(\nu\)-plane is divided into regions (4), (5) and (6) separated by the curves $C_3$, $C_4$ and the solid part of the line $\nu'' = \nu' \tan \phi$ (Fig. (A2)), where $C_1$, $C_2$ are the curves on which the zeros of $H^{(1)}_\nu(x)$ are located and $C_3$, $C_4$ are those on which the zeros of $H^{(2)}_\nu(x)$ are situated and $\phi = \arg(x)$.

\[
H^{(1)}_\nu(x) - \sqrt{\frac{2}{\pi x^2}} e^{i\psi}, \quad H^{(1)'}_\nu(x) - i \sqrt{\frac{2}{\pi x^2}} e^{i\psi} \quad \text{in region (1)}
\]

\[
H^{(1)}_\nu(x) - -\sqrt{\frac{2}{\pi x^2}} e^{-i\psi}, \quad H^{(1)'}_\nu(x) - i \sqrt{\frac{2}{\pi x^2}} e^{-i\psi} \quad \text{in region (2)}
\]

\[
H^{(1)}_\nu(x) - -\sqrt{\frac{2}{\pi x^2}} e^{-i\psi} e^{-i2\nu\pi}, \quad H^{(1)'}_\nu(x) - i \sqrt{\frac{2}{\pi x^2}} e^{-i\psi} e^{-i2\nu\pi} \quad \text{in region (3)}
\]
\[ H_{\nu}^{(2)}(x) = \sqrt{\frac{2}{\pi x^2}} e^{-i\psi}, \quad H_{\nu}^{(2)'}(x) = -i \sqrt{\frac{2z}{\pi x^2}} e^{-i\psi} \text{ in region (4)} \]

\[ H_{\nu}^{(2)}(x) = -i \sqrt{\frac{2}{\pi z}} e^{i\psi}, \quad H_{\nu}^{(2)'}(x) = -i \sqrt{\frac{2z}{\pi x^2}} e^{i\psi} \text{ in region (5)} \]

and finally

\[ H_{\nu}^{(2)}(x) = -i \sqrt{\frac{2z}{\pi x^2}} e^{i\psi} e^{i2\nu\pi}, \quad H_{\nu}^{(2)'}(x) = -i \sqrt{\frac{2z}{\pi x^2}} e^{i\psi} e^{i2\nu\pi} \text{ in region (6)} \]

where

\[ z = \sqrt{x^2 - \nu^2} \quad \text{with} \quad \Re\left(\frac{z}{x}\right) > 0 \]

and

\[ \psi = z - \nu \cos^{-1}\left(\frac{\nu}{x}\right) - \frac{\pi}{4}, \]

and the prime in \( H_{\nu}^{(1)'}(x) \) and \( H_{\nu}^{(2)'}(x) \) denotes differentiation with respect to the argument \( x \).

(ii) Large and Almost Equal Order and Argument (Uniform Asymptotic Expansion) \([34,38,39]\):

From the expressions given in section (i), it is clear that when the order approaches the argument, the formulae which determine \( H_{\nu}^{(1)}(x) \) diverge whereas the ones which determine \( H_{\nu}^{(2)}(x) \) approach zero, which suggests that the formulae in section (i) are not valid when the order approaches the argument.

In this section a uniform asymptotic expansion is given for \( H_{\nu}^{(1)}(x) \) and \( H_{\nu}^{(2)}(x) \). These uniform asymptotic expansions are more powerful than the expansions given in section (i) in the sense that they
reduce to the previous expansions when the argument is no more equal
to the order.

These expansions are valid when \(-\pi/2 < \arg(v) < 3\pi/2\) for \(H_{\nu}^{(1)}(x)\)
and \(-3\pi/2 < \arg(v) < \pi/2\) for \(H_{\nu}^{(2)}(x)\). Outside these regions one can
use the relations \(H_{\nu}^{(1)}(x) = \exp(i\nu\pi) H_{\nu}^{(1)}(x)\), \(H_{\nu}^{(2)}(x) = \exp(-i\nu\pi) H_{\nu}^{(2)}(x)\).

These uniform asymptotic expansions are given by

\[
H_{\nu}^{(1)}(x) = \frac{2}{\nu^3} \exp(-i\nu/3) \text{Ai}'(-\nu - x) \left\{ \frac{2}{\nu x} \right\}^{1/3} \exp(-i\nu/3)
\]

\[
H_{\nu}^{(2)}(x) = \frac{2}{\nu^3} \exp(i\nu/3) \text{Ai}'(-\nu - x) \left\{ \frac{2}{\nu x} \right\}^{1/3} \exp(i\nu/3)
\]

\[
H_{\nu}^{(1)}(x) = \frac{2}{\nu^3} \exp(i\nu/3) \text{Ai}'(-\nu - x) \left\{ \frac{2}{\nu x} \right\}^{1/3} \exp(i\nu/3)
\]

\[
H_{\nu}^{(2)}(x) = \frac{2}{\nu^3} \exp(-i\nu/3) \text{Ai}'(-\nu - x) \left\{ \frac{2}{\nu x} \right\}^{1/3} \exp(i\nu/3)
\]

where \(\text{Ai}(y)\) and \(\text{Ai}'(y)\) is the Airy function and its derivative which has
the following properties.

(i) The Airy function is given in terms of the \(\pm 1/3\) and \(\pm 2/3\) order
Bessel function by the following expressions

\[
\text{Ai}(-z) = \frac{1}{3} \sqrt{z} \left[ J_{1/3}\left(\frac{2}{3} z^{3/2}\right) + J_{-1/3}\left(\frac{2}{3} z^{3/2}\right) \right]
\]
\[ A_i'(z) = \frac{1}{3} z \left[ J_{2/3}\left(\frac{2}{3} z^{3/2}\right) - J_{-2/3}\left(\frac{2}{3} z^{3/2}\right) \right] \]

(ii) The Airy function satisfies the following differential equation

\[ A_i''(z) = z A_i(z) \]

(iii) The Wronskian of the Airy function is given by

\[ A_i(z) \frac{d}{dz} \left\{ A_i \left[ z \exp \left( \pm \frac{i2\pi}{3} \right) \right] \right\} - A_i'(z) A_i \left[ z \exp \left( \pm \frac{i2\pi}{3} \right) \right] = \frac{1}{2\pi} \exp \left( \pm \frac{i\pi}{6} \right) \]

\[ A_i \left[ z \exp \left( \frac{i2\pi}{3} \right) \right] \frac{d}{dz} \left\{ A_i \left[ z \exp \left( - \frac{i2\pi}{3} \right) \right] \right\} - A_i \left[ z \exp \left( - \frac{i2\pi}{3} \right) \right] \frac{d}{dz} \left\{ A_i \left[ z \exp \left( \frac{i2\pi}{3} \right) \right] \right\} \]

\[ = \frac{i}{2\pi} \]

(iv) The Airy function satisfies the recurrence relation

\[ A_i(z) = \exp \left( \frac{i\pi}{3} \right) A_i \left[ -z \exp \left( \frac{i\pi}{3} \right) \right] + \exp \left( - \frac{i\pi}{3} \right) A_i \left[ -z \exp \left( - \frac{i\pi}{3} \right) \right] \]
References


BIOGRAPHICAL NOTE

Tarek M. El-Habashy was born in Cairo, Egypt on May 7, 1953. He received the B.S. degree from Cairo University, Cairo, Egypt in July 1976 and the M.S. degree from Massachusetts Institute of Technology in June 1980.

During 1976-1977 he was a teaching instructor at the Military Technical College, Cairo, Egypt. During 1977-1978 he was both a teaching instructor and a research assistant at Cairo University, Cairo, Egypt. In 1978 he joined MIT as a graduate student. In the years as a graduate student, he served both as a research assistant and a teaching assistant. In 1982, he received the Carlton E. Tucker Award for excellence in teaching.

Mr. Habashy is a member of Sigma Xi.