MIT Sloan School of Management

Working Paper 4248-02
April 2002

ISOTONE EQUILIBRIUM IN GAMES OF INCOMPLETE INFORMATION

David McAdams

Isotone Equilibrium in Games of Incomplete Information

David McAdams*
MIT Sloan School of Management

This version: April 26, 2002
First draft: April 2001

Abstract

An isotone pure strategy equilibrium exists in any game of incomplete information in which (1) each player i's action set is a finite sublattice of multi-dimensional Euclidean space, (2) types are multi-dimensional and atomless, and each player's interim expected payoff function satisfies two "non-primitive conditions" whenever others adopt isotone pure strategies: (3) single-crossing in own action and type and (4) quasisupermodularity in own action. Similarly, given that (134) and (2') types are multi-dimensional (with atoms) an isotone mixed strategy equilibrium exists. Conditions (34) are satisfied in supermodular and log-supermodular games given affiliated types, and in games with independent types in which each player's ex post payoff satisfies (a) supermodularity in own action and (b) non-decreasing differences in own action and type. These results also extend to games with a continuum action space when each player's ex post payoff is also continuous in his and others' actions.

*I am grateful to Susan Athey, Eddie Dekel, John McMillan, Barry Nalebuff, Robert Wilson, and two anonymous referees for helpful suggestions and most especially to Paul Milgrom for introducing me to lattice theory. This research has been supported by the John Olin Foundation through a grant to the Stanford Institute for Economic Policy Research, as well as by the State Farm Companies Foundation. E-mail: mcadams@mit.edu. Post: MIT Sloan School of Management, E52-448, 50 Memorial Drive, Cambridge MA 02142

1 Introduction

Monotone methods have proven to be powerful in the study of games with strategic complementarity. For example, Milgrom and Roberts (1990) and Vives (1990) show that supermodular games possess several useful properties, including existence of pure strategy equilibrium, monotone comparative statics on equilibrium sets, and coincidence of the predictions of various solution concepts such as Nash equilibrium, correlated equilibrium, and rationalizability. Milgrom and Shannon (1994) generalize these results to games with strategic complementarity including, as Athey (1998) shows, log-supermodular games with affiliated types. This paper adds to this literature by providing sufficient conditions for existence of a *monotone* pure (mixed) strategy equilibrium in settings with multidimensional actions and multi-dimensional atomless (atom) types. A player's pure strategy is monotone, technically "isotone", iff his action is non-decreasing along every dimension of his action space as his type increases along any dimension of his type space. A mixed strategy is isotone, similarly, if *all* actions played by a higher type are greater than or equal to *all* actions played by a lower type. The sufficient conditions for these existence results are satisfied in the two most widely studied sorts of games with strategic complementarity, supermodular games and log-supermodular games, given affiliated types. Isotonicity is important since it often provides testable empirical implications. For instance, in the Cournot-with-advertising example discussed in Section 1.1, lower production and advertising costs are each associated with (weakly) higher sales and advertising levels.

This paper departs from the usual strategic complements framework, however, and considers a broad class of games in which only some of the requirements of strategic complementarity are satisfied. Strategic complementarity requires two sorts of monotone relationships between strategic variables: "strategic complementarity between actions" and "complementarity within own action". This paper extends a new approach pioneered by Athey (2001) to develop monotone methods that apply to games of incomplete information which may fail to exhibit strategic complementarity between actions but in which payoffs satisfy "monotone incremental returns in own type".[1] (Mil-

[1]Formally, strategic complementarity between actions = single-crossing in own action and others' actions, complementarity within own action = quasisupermodularity in own action, and monotone incremental returns in own type = single-crossing in own action and own type. Athey (2001) refers to monotone incremental returns in own type given any

grom and Shannon (1994) do not require monotone incremental returns in own type to prove existence of a pure strategy equilibrium but, naturally, they can not guarantee existence of an isotone equilibrium.)

In a setting with one-dimensional actions and one-dimensional types, Athey (2001) shows that a non-decreasing pure strategy equilibrium exists in any finite game in which each player's interim expected payoff satisfies monotone incremental returns in own type given any non-decreasing strategies by others. This paper generalizes this result in a setting with multi-dimensional actions and multi-dimensional types, showing that an isotone pure strategy equilibrium exists in any finite game in which each player's interim expected payoff satisfies complementarity in own action and monotone incremental returns in own type given any isotone strategies by others. This result generalizes to games with a continuum action space whenever each player's ex post payoff is also continuous in his and others' actions, just as Athey (2001)'s results generalize in this case.

The rest of the paper is organized as follows. Section 1.1 continues the introduction, showing how a corollary of the main existence result can be applied in an example with a continuum action space and differentiable payoffs. Section 2 lays out the basic model of incomplete information games with finite action spaces and atomless types and provides definitions of important terms. Section 3 states the main theorem as well as interesting corollaries and three sets of sufficient primitive conditions. Section 4 provides the proof of the main Theorem and related discussion, followed by some concluding remarks and an appendix.

1.1 Illustration in games with differentiable payoffs

Consider an incomplete information game in which n players each receive a signal $t_i = (t_i^1, ..., t_i^h) \in [0,1]^h$ and choose an action $a_i = (a_i^1, ..., a_i^k) \in [0,1]^k$. Define each player's interim expected payoff function π_i^{int} given others' pure strategies $a_{-i}(\cdot)$ as follows:

$$\pi_i^{int}(a_i, t_i; a_{-i}(\cdot)) \equiv \int_{[0,1]^{h(n-1)}} \pi_i^{post}(a_i, a_{-i}(t_{-i}), t_i, t_{-i}) f(t_{-i}|t_i) dt_{-i}$$

where π_i^{post} is his ex post payoff and $f(\cdot|t_i)$ is the conditional p.d.f. of others' types given that player i's type is t_i. Suppose also that $\pi_i^{post}(\mathbf{a}, \mathbf{t})$, $f(t_{-i}|t_i)$ are

non-decreasing strategies by others as "the single-crossing condition".

smooth functions (of $\mathbf{a}, \mathbf{t}$ and of t_i, respectively) so that π_i^{int} is differentiable in a_i, t_i. A specialized version of Corollary 2 of the main theorem applies to this class of games:

Corollary. *Suppose that, for each bidder $i = 1, ..., n$ and all isotone strategy profiles $a_{-i}(\cdot)$ of others,*

(1) $\dfrac{\partial^2 \pi_i^{int}}{\partial a_i^{j_1} \partial a_i^{j_2}}(\cdot, a_i^{-j_1,j_2}, t_i; a_{-i}(\cdot)) \geq 0$ *for all* $a_i^{-j_1,j_2}, t_i, 1 \leq j_1 < j_2 \leq k$

(2) $\dfrac{\partial^2 \pi_i^{int}}{\partial a_i^{j_1} \partial t_i^{j^2}}(\cdot, a_i^{-j_1}, t_i^{-j_2}; a_{-i}(\cdot)) \geq 0$ *for all* $a_i^{-j_1}, t_i^{-j_2}, 1 \leq j_1 \leq k, 1 \leq j_2 \leq h.$

Then an isotone pure strategy equilibrium exists.

((1) implies assumption (A4) of the Theorem, (2) implies (A5).) For illustration purposes it is simplest to consider examples in which player types are independent, since then the cross-partial inequalities (1),(2) on expected payoffs are implied directly by the corresponding cross-partial inequalities on ex post payoffs.

Example (Cournot with 2 advertising channels, n firms). Consider an undifferentiated product Cournot competition game in which n risk-neutral firms each choose a quantity q_i and levels of two sorts of advertising e_i^1, e_i^2 to expand the size of the total market. In the pharmaceutical context, for example, drug companies advertise to patients through media advertising and to doctors through detailing (such as office visits from company reps). Firms also receive possibly multi-dimensional independent private information t_i, where higher own type implies (weakly) lower own advertising and production costs. In particular, suppose that (i) $D(p; \mathbf{e}) = D(p) + \gamma(\mathbf{e})$ is total demand, (ii) $\phi_i(\mathbf{e}, \mathbf{t})$ is firm i's advertising cost function, and (iii) $c_i(q_i; \mathbf{t})$ is firm i's production cost function,[2] where $\mathbf{q}, \mathbf{e}, \mathbf{t}$ refer to vectors of all firms' quantities, advertising levels, and types. Firm i's ex post payoff is

$$\pi_i^{post}(\mathbf{q}, \mathbf{e}, \mathbf{t}) = \pi_i(\mathbf{q}, \mathbf{e}, \mathbf{t}) \equiv q_i p(\mathbf{q}, \mathbf{e}) - c_i(q_i; \mathbf{t}) - \phi_i(\mathbf{e}, \mathbf{t})$$

[2]For simplicity, suppose further that all functions are smooth and that $\frac{\partial D}{\partial p} \leq 0$, $\frac{\partial D}{\partial \mathbf{e}} \geq 0$, $\frac{\partial c_i}{\partial t_i} \leq 0$, and $\frac{\partial \phi_i}{\partial t_i} \leq 0$. In particular, this implies that $\frac{\partial p}{\partial \mathbf{q}} \leq 0$ and $\frac{\partial p}{\partial \mathbf{e}} \geq 0$, hence that $\frac{\partial^2 \pi_i}{\partial q_i \partial t_i} \geq 0$ and $\frac{\partial^2 \pi_i}{\partial e_i \partial t_i} \geq 0$. Also, $\frac{\partial^2 \pi_i}{\partial e_i \partial q_i} \geq 0$ since advertising increases marginal revenue.

where $p(\mathbf{q}, \mathbf{e})$ is the market-clearing price. If there were just one advertising channel, an isotone pure strategy equilibrium would always exist in this example[3] since

$$\frac{\partial^2 \pi_i}{\partial q_i \partial e_i} \geq 0, \frac{\partial^2 \pi_i}{\partial q_i \partial t_i} \geq 0, \frac{\partial^2 \pi_i}{\partial e_i \partial t_i} \geq 0.$$

Given two advertising channels, however, an isotone equilibrium may fail to exist if $\frac{\partial^2 \pi_i}{\partial e_i^1 \partial e_i^2} \ngeq 0$. In this way, the results of this paper clarify the effect of omitted strategic variables on monotone equilibrium relationships between own action and own type. If the econometrician observes only one of the channels (say detailing) in this augmented Cournot game, then she will conclude that lower costs are associated with more detailing and more production. This relationship may not hold in the actual underlying game, however, if the marginal returns to detailing are not monotone in media advertising levels. On the other hand, this is the *only* unobserved interaction that matters to this monotone conclusion.

Example (Cournot with 1 advertising channel, 2 firms). This special case of the previous example is a supermodular game in the sense of Milgrom and Roberts (1990) if $\frac{\partial^2 \pi_i}{\partial e_i \partial e_j} \geq 0$. By providing existence of an equilibrium in the alternative that ex post payoff does not satisfy this condition, the results of this paper help the econometrician to test for supermodularity in this game.

2 Model: Incomplete Information Games

This Section lays out the model of incomplete information games with atomless types and finite action spaces. Assumptions are numbered by (A#). Corollaries to the main theorem apply to related models having atom types and/or a continuum action space. These models are presented in the Appendix.

[3] Note that existence of an isotone equilibrium does not provide the basis for monotone comparative statics. For example, suppose that a change in the tax code lowers all firms' costs. In the new isotone equilibrium, some firms may produce and/or advertise less than they did in the original equilibrium.

2.1 Actions and Lattices

(A1) Each player $i = 1, ..., n$ has a common *action set* $L \subset \mathcal{R}^k$ that is a finite sublattice of k-dimensional Euclidean space with respect to the product order on $\mathcal{R}^k$.[4]

Definition ($\vee$, $\wedge$). Let $(L, \geq)$ be a partially ordered set and let $S \subset L$. The *least upper bound of* S, $\vee S$, is the unique element of L – if it exists! – satisfying $\vee S \leq c \Leftrightarrow a \leq c$ for all $a \in S$ and all $c \in L$. The *greatest lower bound of* S, $\wedge S$, is the unique element of L satisfying $\wedge S \geq c \Leftrightarrow a \geq c$ for all $a \in S$ and all $c \in L$. When $S = \{a, b\}$, I use the notation $a \vee b$ and $a \wedge b$.

Definition (Lattice, Sublattice, Complete). A *lattice* $(L, \geq, \vee, \wedge)$ is a partially ordered set $(L, \geq)$ such that $a \vee b, a \wedge b \in L$ for all $a, b \in L$. $L_1 \subset L$ is a *sublattice* of L iff $(L_1, \geq, \vee, \wedge)$ is a lattice with respect to the same order and operators as on L. L is *complete* if $\vee S, \wedge S \in L$ for every subset $S \subset L$.

Every finite lattice is complete (Birkhoff (1967)).

Definition (Product order). Let $x' = (x'_1, ..., x'_k), x = (x_1, ..., x_k)$ be elements of $\mathcal{R}^k$. $x' \geq x$ in the *product order* iff $x'_m \geq x_m$ for all $m = 1, ..., k$. $\mathcal{R}^k$ forms a lattice with respect to the product order:

$$(x'_1, ..., x'_k) \vee (x_1, ..., x_k) = (\max\{x'_1, x_1\}, ..., \max\{x'_k, x_k\})$$
$$(x'_1, ..., x'_k) \wedge (x_1, ..., x_k) = (\min\{x'_1, x_1\}, ..., \min\{x'_k, x_k\})$$

A typical action is $a_i \equiv \left(a_i^1, ..., a_i^k\right) \equiv \left(a_i^m, a_i^{-m}\right)$, for $m = 1, ..., k$. A typical action profile is $\mathbf{a} \equiv (a_1, ..., a_n) \equiv (a_i; a_{-i}) \in \Pi_{i=1}^n L$. Similar subscript, superscript, and bold notation will be used consistently throughout the paper to refer to types and strategies as well as actions. For each $m = 1, ..., k$, define

$$L_m \equiv \left\{a_i^m \in \mathcal{R} : \left(a_i^m, a_i^{-m}\right) \in L \text{ for some } a_i^{-m} \in \mathcal{R}^{k-1}\right\}$$

By definition, $L \subset \Pi_{m=1}^k L_m$. (I do not assume that $L = \Pi_{m=1}^k L_m$.) For easy reference, let $L_m = \{0, 1, ..., |L_m| - 1\}$.

[4] All results are easily generalizable to settings in which players have different action sets that may be of different dimensionality. Similarly, the assumption of a common type space is purely for expositional simplicity.

2.2 Types and Strategies

(A2) Player i's *type* t_i is drawn from common support $T = [0,1]^h$ and $\mathbf{t}$ has an atomless joint distribution with a continuous density function $f(\mathbf{t}) : [0,1]^{nh} \to \mathcal{R}_+$.

The type space is endowed with the product order.

Definition (Pure strategy, Isotone pure strategy). A *pure strategy* $a_i(\cdot)$ specifies an action $a_i(t_i) \in L$ for each type $t_i \in T$. A pure strategy $a_i(\cdot)$ is *isotone* iff $t_i' > t_i$ implies $a_i(t_i') \geq a_i(t_i)$.

$\mathcal{I}_i$ denotes the space of all of player i's isotone pure strategies, $\mathcal{I}_{-i} = \Pi_{j \neq i} \mathcal{I}_j$ the space of others' isotone pure strategy profiles, and $\mathcal{I} = \Pi_{i=1}^n \mathcal{I}_j$ the space of isotone pure strategy profiles.

2.3 Payoffs

Player i's *ex post payoff* (or utility) $\Pi_i^{post}(\mathbf{a}, \mathbf{t})$ depends on the vector of types and the vector of actions.

(A3) Π_i^{post} is bounded.

His *interim expected payoff* given his own type, similarly, depends on his action and others' strategies:

$$\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot)) = E_{t_{-i} \big| t_i} \left[\Pi_i^{post}(a_i, t_i; a_{-i}(t_{-i}), t_{-i}) | t_i \right]$$

Definition (Quasisupermodular). Let $(L, \geq, \vee, \wedge)$ be a lattice and Θ an index set. $g : L \times \Theta \to \mathcal{R}$ is quasisupermodular in x iff

$$g(x'; \theta) \geq (>) g(x' \wedge x; \theta) \Rightarrow g(x' \vee x; \theta) \geq (>) g(x; \theta)$$

for all $x', x \in L$ and all $\theta \in \Theta$. (Weak inequality implies weak inequality and strict inequality implies strict inequality.)

(A4) $\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot))$ is quasisupermodular in a_i for all $t_i \in T$ and all $a_{-i}(\cdot) \in \mathcal{I}_{-i}$. (The relevant index set is $\Theta = T \times \mathcal{I}_{-i}$.)

Definition (Single-crossing property). Let $(L, \geq, \vee, \wedge)$ be a lattice, $(T, \geq)$ a partially ordered set, and Θ an index set. $g : L \times T \times \Theta \to \mathcal{R}$ satisfies the *Milgrom-Shannon single-crossing property in* $(x; t)$ or, simply, *single-crossing in* $(x; t)$ iff

$$g(x', t; \theta) \geq (>) g(x, t; \theta) \Rightarrow g(x', t'; \theta) \geq (>) g(x, t'; \theta)$$

for all $x' > x \in L$, all $t' > t \in T$, and all $\theta \in \Theta$.

(A5) $\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot))$ satisfies single-crossing in $(a_i; t_i)$ for all $a_{-i}(\cdot) \in \mathcal{I}_{-i}$. (The relevant index set is $\Theta = \mathcal{I}_{-i}$.)

2.4 Best Response and Equilibrium

Let

$$BR_i\left(t_i, a_{-i}(\cdot)\right) \equiv \arg\max_{a \in L} \Pi_i^{int}\left(a, t_i; a_{-i}(\cdot)\right)$$

denote player i's best response action set when others follow pure strategies $a_{-i}(\cdot)$. Define bidder i's *isotone-restricted best response correspondence*

$$\begin{aligned} & BR_i^{\geq} : \mathcal{I}_{-i} \to \mathcal{P}\left(\mathcal{I}_i\right) \\ & \qquad a_{-i}(\cdot) \mapsto \{a_i(\cdot) \in \mathcal{I}_i : a_i(t_i) \in BR_i\left(t_i, a_{-i}(\cdot)\right) \text{ for all } t_i \in T\}. \end{aligned}$$

($\mathcal{P}(X)$ denotes the set of all subsets of X.) $\mathbf{a}^*(\cdot) \in \mathcal{I}$ is an *isotone pure strategy equilibrium* iff

$$a_i^*(\cdot) \in BR_i^{\geq}\left(a_{-i}^*(\cdot)\right) \text{ for all } i.$$

3 Existence of Isotone Equilibrium

Theorem 1. *Under assumptions A1-A5, an isotone pure strategy equilibrium exists.*

Corollary 1. *Under assumptions A1, A2',A3-A5, an isotone mixed strategy equilibrium exists.*

See page 16 for A2', which replaces the atomless type assumption with one of finitely many atom types. That is to say, each player's type has finite support. An isotone mixed strategy is a sort of mixed strategy equilibrium that has a lot of monotone structure. If $t_i' > t_i$, then *all* actions played with positive probability by type t_i' are greater than or equal to *all* actions played by type t_i. Isotone mixed strategy equilibrium is defined in the Appendix.

Corollary 2. *Under assumptions A1', A2-A5, and A6, an isotone pure strategy equilibrium exists.*

Corollary 3. *Under assumptions A1', A2', A3-A5, and A6, an isotone mixed strategy equilibrium exists.*

See page 16 for A1' and A6. A1' replaces the finite lattice action set assumption with an assumption that each player's action set is the unit cube $[0,1]^k$. A6 adds the requirement that each player's ex post payoff is continuous in own and others' actions.

3.1 Sufficient primitive conditions for A4,A5

I gather here three sets of primitive conditions that others' work proves are sufficient for interim expected payoff to satisfy quasisupermodularity in own action and single-crossing in own action and type. I refer the reader to this other work for the formal definitions of such standard terms as affiliated, supermodular, log-supermodular, and non-decreasing differences.

> Types are affiliated and $\Pi_i^{post}(a_i, t_i; a_{-i}, t_{-i})$ is supermodular in $(\mathbf{a}, t_j)$ for all j. In this case, Athey (2002) proves that $\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot))$ is supermodular in $(\mathbf{a}, t_i)$.
>
> Types are affiliated and $\Pi_i^{post}(a_i, t_i; a_{-i}, t_{-i})$ is log-supermodular in $(\mathbf{a}, \mathbf{t})$. In this case, Athey (2001) proves that $\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot))$ is log-supermodular in $(\mathbf{a}, t_i)$.
>
> Types are independent and $\Pi_i^{post}(a_i, t_i; a_{-i}, t_{-i})$ is supermodular in a_i with non-decreasing differences in (a_i, t_i). In this case, expected payoff $\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot))$ is supermodular in a_i and has non-decreasing differences in (a_i, t_i). See Topkis (1979).

In Milgrom and Roberts (1990) and Vives (1990), a supermodular game is one in which $\Pi_i^{post}(a_i, t_i; a_{-i}, t_{-i})$ is supermodular in $\mathbf{a}$, with no conditions placed on the distribution of types. Thus, these sufficient primitive conditions are only satisfied in a subclass of supermodular (and log-supermodular) games. This stands to reason, of course, since I prove that an *isotone* pure strategy equilibrium exists.

4 Proof of Theorem 1

4.1 Monotonicity Theorem

Theorem 1 is essentially a corollary of Milgrom and Shannon (1994)'s powerful Monotonicity Theorem. Indeed, in my view, the main contribution of this paper is to uncover an amazing amount of structure possessed by $\arg\max_x g(x,t)$ when x is multidimensional and g satisfies its conditions. This structure in turn happens to be exactly what is required to extend Athey (2001b)'s ingenious approach to proving existence of monotone pure strategy equilibrium in the multi-dimensional action case. It seems worthwhile, then, to discuss these prior contributions and thereby indicate what I feel is at the heart of my contribution. First, I state a specialized (and weakened) version of the Monotonicity Theorem.[5]

Theorem. *(Milgrom and Shannon (1994)) Let* $g : L \times T \to \mathcal{R}$*, where* $(L, \geq, \vee, \wedge)$ *is a complete lattice and* $(T, \geq)$ *a partially ordered set. Then* $\arg\max_{x \in L} g(x,t)$ *is a complete sublattice for all* t *and increasing in the strong set order if* g *is quasisupermodular in* x *and satisfies single-crossing in* $(x;t)$*.*

Definition (Strong set order). Let $(L, \geq, \vee, \wedge)$ be a lattice. The *strong set order* $\geq_L$ is a partial ordering on $\mathcal{P}(L)$, the space of subsets of L. For $A, A' \subset L$, $A' \geq_L A$ iff $a' \in A', a \in A$ implies that $a' \vee a \in A', a' \wedge a \in A$.

Definition (Increasing in the strong set order). Let $(L, \geq, \vee, \wedge)$ be a lattice and $(T, \geq)$ a partially ordered set. A correspondence $g : T \to \mathcal{P}(L)$ is *increasing in the strong set order* iff $g(t') \geq_L g(t)$ whenever $t' > t$.

Given assumptions A4,A5, for any fixed profile $a_{-i}(\cdot)$ of others' isotone pure strategies, one may apply the Monotonicity Theorem to the interim expected payoff function $\pi_i^{int}(\cdot, \cdot; a_{-i}(\cdot)) : L \times T \to \mathcal{R}$. Thus, $BR_i\left(t_i, a_{-i}(\cdot)\right)$ (shorthand $BR_i(t_i)$) is a complete sublattice for all t_i and increasing in the strong set order. Since L is finite, this set of best response actions is always non-empty. Theorem 4.1 also implies directly that an *isotone* best response strategy always exists. *Proof:* $\vee BR_i(\cdot)$ is an isotone best response pure strategy. (i) $\vee BR_i(t_i)$ is a best response action for each t_i since $BR_i(t_i)$ is

[5] Milgrom and Shannon (1994) have a stronger "if and only if" formulation that also accounts for how the $\arg\max_x g(x,t)$ set varies with a constraint $S \subset L$. I do not leverage this aspect of their result, since each player's action set is fixed.

a sublattice. (ii) $\vee BR_i(\cdot)$ is isotone since $BR_i(\cdot)$ is increasing in the strong set order. For any $t'_i > t_i$, $(\vee BR_i(t'_i)) \vee (\vee BR_i(t_i)) = z \in BR_i(t'_i)$. But $z > \vee BR_i(t'_i)$ unless $\vee BR_i(t'_i) \geq \vee BR_i(t_i)$. $\square$

4.2 Order-interval inclusivity

In the context of one-dimensional actions, the meaning and strength of this structure on the best response set is relatively easy to grasp. In this case, every subset of actions is a complete sublattice and increasingness in the strong set order has a simple, equivalent formulation.

Fact. Suppose that X is a finite totally ordered set and that $B(t) \subset X \neq \emptyset$ for all t. Then $B(t)$ is increasing in the strong set order iff (1) $\min B(t)$ and $\max B(t)$ are non-decreasing in t and (2) $x \in B(t) \cap B(t'')$ implies $x \in B(t')$ for all $t < t' < t''$.

Thus, the set of types t_i for whom $a_i \in BR_i(t_i)$ is order-interval inclusive, for all $a_i \in L$.

Definition (Order-interval inclusive). [6] Let $(T, \geq)$ be a partially ordered set. $S \subset T$ is order-interval inclusive if $t'' > t' > t$ and $t'', t \in S$ implies that $t' \in S$.

The labelled regions in Figures 1, 2 on page 13 provide several examples of order-interval inclusive subsets of $[0,1]^2$. (Note that an order-interval inclusive set need not be connected.) Order-interval inclusivity is also intimately linked to (1) the definition of an isotone strategy and (2) the convexity of the set of isotone strategies with respect to certain convex combination operators as well as (3) the convexity of the set of isotone best response strategies in the one-dimensional case.

For each dimension $m = 1, ..., k$ of the action space and each $j = 0, 1, ..., |L_m|$, define

$$X_i^m(j, a_i(\cdot)) \equiv \{t_i \in [0,1]^h : a_i^m(t_i) < j\}$$
$$A_i^m(j, a_i(\cdot)) \equiv \cup_{t_i^{-1} \in [0,1]^{h-1}} cl\left(\{t_i \in C(t_i^{-1}) : a_i^m(t_i) < j\}\right).$$

where $C(t_i^{-1}) \equiv \{(x, t_i^{-1}) : x \in [0,1]\}$ and $cl(X)$ is the closure of X. When there can be no confusion, I drop notational reference to the strategy $a_i(\cdot)$. By

[6] I have not found this concept defined in previous work. I chose "order-interval inclusive" since $t'' > t \in S$ implies that the order-interval $[t, t''] \subset S$.

definition, let $X_i^m(|L_m| = A_i^m(|L_m|) = T$. (All types play an action less than $|L_m|$.) $X_i^m(j)$ is the less-than set of types that play an action whose m-th coordinate is strictly less than j. $A_i^m(j)$ is a closely related set that happens to be convenient for technical purposes. Note that $X_i^m(j) \subseteq A_i^m(j) \subseteq cl(X_i^m(j))$ and any pair of these sets has zero measure symmetric difference (union minus intersection).

Lemma 1. *$a_i(\cdot)$ is an isotone strategy iff (i) $A_i^m(j, a_i(\cdot))$ is order-interval inclusive for all $m, j \in L_m$ and (ii) $0 \in A_i^m(0, a_i(\cdot))$.*

Proof. $a_i(\cdot)$ is isotone iff $a_i^m(\cdot)$ is non-decreasing for $m = 1, ..., k$. Suppose that $t' > t$ but $j = a_i^m(t') < a_i^m(t) = j'$ where $j' > j$. Since $0 \in A_i^m(0)$ and $A_i^m(j+1)$ is order-interval inclusive, every type t such that $0 < t < t'$ must play an action in the set $\{0, ..., j\}$, contradicting $j' > j$. □

4.3 Athey Map and Convexity of Isotone Strategies

Define *equivalence classes* of isotone strategies as follows. $a_i'(\cdot), a_i(\cdot)$ are equivalent iff the symmetric difference of $A_i^m(j, a_i'(\cdot)), A_i^m(j, a_i(\cdot))$ has zero measure for all $m = 1, ..., k$ and all $j \in L_m$. (In all equivalent strategies, the boundaries of the less-than sets in the type space coincide and the same action is played by types who are not on any of these boundaries.)

This representation of equivalence classes of strategies as less-than sets in the type space is an extension of Athey's representation of strategies as vectors. (Each vector component in Athey's representation corresponds to the degenerate boundary between the regions in $[0, 1]$ playing two consecutive actions.) Indeed, when invoking Glicksberg's Fixed Point Theorem, I leverage the fact that this "Athey map" A_i is a homeomorphism between the space of strategy classes and a convex, compact subset of an (infinite-dimensional) vector space, where the domain and range are both endowed with the topology of pointwise convergence.[7] Note that interim expected payoffs are continuous in the players' strategies when endowed with the topology of pointwise convergence. Consequently, the "projection" of the isotone-restricted best response correspondence via the Athey map,

$$A^{-1} \circ BR^{\geq} \circ A \equiv \left(A_1^{-1} \circ BR_1^{\geq} \circ A_{-1}, ..., A_{-n}^{-1} \circ BR_n^{\geq} \circ A_{-n}\right)$$

[7]Each less-than set $A_i^m(j)$ is identified with its upper boundary which in turn may be represented by the vector $\left(\max\left(A_i^m(j) \cap C(t_i^{-1})\right)\right)^{t_i^{-1} \in [0,1]^{h-1}}$.

has a **closed graph**.[8] Since an isotone best response strategy always exists, $A^{-1} \circ BR^{\geq} \circ A$ is **non-empty valued**. Also note that properties (i) and (ii) of Lemma 1 are preserved under the pointwise limit, so the range of $A^{-1} \circ BR^{\geq} \circ A$ is closed in the product space $\Pi_{k=1,\ldots,m,j\in L_m,\theta\in\Theta}[0,1]$. Since this product space is itself compact in the topology of pointwise convergence by Tychonoff's Theorem, $A^{-1} \circ BR^{\geq} \circ A$ has **compact range**.

Now, with respect to the partition $[0,1]^h = \cup_{t_i^{-1}\in[0,1]^{h-1}} C(t_i^{-1})$, one may define a convex combination operation on the space of isotone strategies as follows. For any two given isotone strategies $a_i'(\cdot)$,$a_i(\cdot)$, define the "convex combination strategy" $a_i(\cdot;\alpha) \equiv \alpha a_i'(\cdot) + (1-\alpha)a_i(\cdot)$ by taking the "line-by-line" convex combination of the less than sets corresponding to strategies $a_i'(\cdot), a_i(\cdot)$:

$$A_i^m(j;a_i(\cdot;\alpha)) \equiv \cup_{t_i^{-1}} \left(\alpha\left(A_i^m\left(j;a_i'(\cdot)\right)\cap C(t_i^{-1})\right) + (1-\alpha)\left(A_i^m\left(j;a_i(\cdot)\right)\cap C(t_i^{-1})\right)\right)$$

where for $X, Y \subset C(t_i^{-1})$, $\alpha X + (1-a)Y$ represents the usual convex combination of sets. (When $X = [0, \max X], Y = [0, \max Y]$, $\alpha X + (1-\alpha)Y = [0, \alpha \max X + (1-\alpha)\max Y]$.) For example, Figures 1 and 2 illustrate $\alpha = .5$ convex combination of two strategies when $T = [0,1]^2$ and $L = \{0,1,2\}$. The number 0,1,2 in each region of the type-space is the action played in that region.

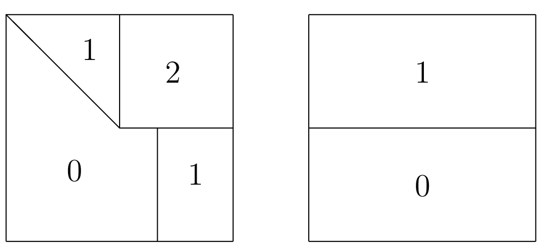

Figure 1: Two isotone strategies

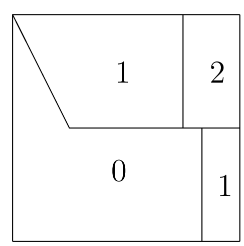

Figure 2: Convex combination

Lemma 2. *$\{A_i^m(j,a(\cdot;\alpha))\}^{m,j\in L_m}$ satisfies conditions (i,ii) of Lemma 1 whenever $\{A_i^m(j,a'(\cdot))\}^{m,j\in L_m}$ and $\{A_i^m(j,a(\cdot))\}^{m,j\in L_m}$ satisfy these conditions.*

Proof. In the Appendix. □

[8] I thank an anonymous referee for suggesting this space-saving approach for proving the closed graph property.

Thus, the map $A^{-1} \circ BR^{\geq} \circ A$ has **convex range**.

More important than the convexity of the set of isotone strategies is the convexity of the set of isotone *best response* strategies, i.e. the fact that $A^{-1} \circ BR^{\geq} \circ A$ is **convex-valued**. In the case of one-dimensional actions, this fact also follows immediately from order-interval inclusivity: The set of types who play a given action in the strategy $a_i(\cdot;\alpha)$ is a subset of the smallest order-interval inclusive set containing the union of the sets of types who play that action in strategies $a_i'(\cdot)$ and $a_i(\cdot)$. In my view, this is the fundamental structure at the heart of Athey (2001)'s lovely proof in the one-dimensional action case.

When actions are multi-dimensional, however, order-interval inclusivity of the set of types who find each given action to be a best response is not nearly sufficient to conclude that the set of isotone best response strategies is convex.

Example. $L = \{0,1,2\} \times \{0,1\} \times \{0,1,2\}$, $T = [0,1]$. Let $a_{-i}(\cdot)$ be a given profile of others' isotone pure strategies and let $BR_i(t_i) \equiv BR_i(t_i; a_{-i}(\cdot))$, the set of best response actions for type t_i. Consider two isotone pure strategies:

$$\begin{aligned} a_i'(t_i) &= (0,0,1) \text{ for all } t_i \in [0,1/2) \\ &= (1,1,2) \text{ for all } t_i \in [1/2,1] \\ a_i(t_i) &= (2,0,0) \text{ for all } t_i \in [0,1/2) \\ &= (2,1,0) \text{ for all } t_i \in [1/2,3/4) \\ &= (2,1,1) \text{ for all } t_i \in [3/4,1] \end{aligned}$$

The Athey map suggests the following (self-explanatory) tabular representation of these strategies:

$a_i'(\cdot)$	$j=0$	$j=1$	$j=2$
$m=1$	$[0,1/2]$	$[1/2,1]$	$[1,1]$
$m=2$	$[0,1/2]$	$[1/2,1]$	N/A
$m=3$	$[0,0]$	$[0,1/2]$	$[1/2,1]$

$a_i(\cdot)$	$j=0$	$j=1$	$j=2$
$m=1$	$[0,0]$	$[0,0]$	$[0,1]$
$m=2$	$[0,1/2]$	$[1/2,1]$	N/A
$m=3$	$[0,3/4]$	$[3/4,1]$	$[1,1]$

Taking a convex combination ($\alpha = .5$), one gets

$a_i(\cdot;.5)$	$j=0$	$j=1$	$j=2$
$m=1$	$[0,1/4]$	$[1/4,1/2]$	$[1/2,1]$
$m=2$	$[0,1/2]$	$[1/2,1]$	N/A
$m=3$	$[0,3/8]$	$[3/8,3/4]$	$[3/4,1]$

In strategy $a_i(\cdot;.5)$, all types in $(0,1/4)$ play $(0,0,0)$, types in $(1/4,3/8)$ play $(1,0,0)$, types in $(3/8,1/2)$ play $(1,0,1)$, types in $(1/2,3/4)$ play $(2,1,1)$, and types in $(3/4,1)$ play $(2,1,2)$. Note that several actions played in the convex combination strategy are not played in either of the two original strategies. Indeed, revealed preference and the order-interval inclusivity structure provide no useful leverage by themselves in proving that the new strategy is a best response.

Rather, I leverage an even more powerful structure that is based (in part) on the observation that, for each $m=1,...,k$ and $j \in L_m$, the set of types who have *some* best response action whose m-th coordinate equals j is order-interval inclusive. The specific idea of the proof of Theorem 2 is to consider first the strategies derived from $a_i'(\cdot)$, $a_i(\cdot)$ by taking the g.l.b. and the l.u.b. of the actions played in both strategies. Then leverage the increasing in the strong set order property to enlarge the set of actions that are best responses for type t_i. For example, consider a type $t_i \in (3/8,1/2)$. By the lattice property, type t_i finds $(2,0,1)=(0,0,1)\vee(2,0,1)$ to be a best response whereas types greater than t_i find $(1,1,0)=(1,1,2)\wedge(2,1,0)$ to be a best response. Now we can use increasingness in the strong set order to conclude that $(2,0,1)\wedge(1,1,0)=(1,0,0)\in BR_i(t_i)$. Repeating the process, with attention focused only on those actions with initial coordinates $(1,0)$, my proof demonstrates that there is a best response action whose initial coordinates are $(1,0,1)$, as desired. (An induction argument applies to any number of action dimensions.)

Convexity of the set of isotone best response strategies is at the heart of the paper, so I label it as a Theorem:

Theorem 2. *$A^{-1}\circ BR^{\geq}\circ A$ is convex-valued.*

Proof. In the Appendix. □

Once all preceding Lemmas and Theorems have been proven, I may apply Glicksberg (1952)'s Fixed Point Theorem to $A^{-1}\circ BR^{\geq}\circ A$. A fixed point of this correspondence corresponds to a profile of *equivalence classes* of isotone strategies. As a final step, Part 4 of the proof of Theorem 3 suffices to imply that some selection from these classes is an isotone equilibrium.

5 Concluding Remarks

This paper shows how two non-primitive conditions, (i) quasisupermodularity in own action and (ii) single-crossing in own action and type of interim expected payoff whenever others follow isotone strategies, are sufficient for existence of an isotone pure strategy equilibrium in a very general setting with finitely many multi-dimensional actions and a continuum of multi-dimensional types. Furthermore, these conditions are satisfied in a variety of important classes of games such as supermodular and log-supermodular games with affiliated types. This includes games such as my Cournot-with-advertising example in which there is a lot of monotone structure but not all of the requirements of strategic complementarity hold. Another interesting application studied by McAdams (2001) is to multi-unit auctions of identical objects with multi-unit demand. Although strategic complementarity fails in wholesale fashion in the uniform-price and pay-as-bid auctions, for instance, each player's bid turns out to be additively separable in his bid for a first unit, for a second unit, and so on. This additive separability is a very strong form of complementarity in own action, implying condition (i). Given independent types and risk-neutral bidders, McAdams (2001) proves that bidders' expected payoffs satisfy (ii) as well.

Appendix

Atom Types and/or Continuum Action Space

Alternative / Additional Model Assumptions

(A1') Player i's action set is $[0,1]^k$.

(A2') Player i's *type* t_i is drawn from finite support in $(0,1)^h$.

(A6) $\Pi_i^{post}(\mathbf{a},\mathbf{t})$ is continuous in $\mathbf{a}$ for all $\mathbf{t}$.

Definition (Mixed strategy, isotone mixed strategy). A mixed strategy $\alpha_i(\cdot,\cdot)$ specifies a probability distribution over actions $\alpha_i(\cdot,t_i)$ for each type t_i. An isotone mixed strategy is one in which $t_i' > t_i$, $\alpha_i(a_i', t_i') > 0$, and $\alpha_i(a_i, t_i) > 0$ imply that $a_i' \geq a_i$.

A profile of isotone mixed strategies $\alpha(\cdot, \cdot)$ is an equilibrium iff $\alpha_i(a_i, t_i) > 0$ implies that $a_i \in BR_i(t_i; \alpha_{-i}(\cdot, \cdot))$.

Proof of Corollary 2

Using a well understood technique[9], each type t_i in the model with atoms can be associated with a *region* of types in an equivalent model without atoms. In this equivalent model, an isotone pure strategy equilibrium exists by Theorem 1. This implies that in the original model a mixed strategy equilibrium exists in which (i) the set of actions played with positive probability by any given type is a totally ordered set and (ii) $t_i' > t_i$ implies that *all* actions played by type t_i' are greater than or equal to all actions played by t_i. But (ii) is precisely the definition of an isotone mixed strategy equilibrium. (So I have proven that the mixed strategy equilibrium is isotone *and* all actions played by each type are comparable.) □

Proof of Corollaries 3,4

These results follow directly from Theorem 2 in Athey (2002), so I omit details. The only difference is that each player's action is multi-dimensional, so the only step that does not obviously immediately carry through is the conclusion that any sequence of isotone pure strategy profiles $\mathbf{a}_j(\cdot)$ has a subsequence that converges to an isotone pure strategy profile $\mathbf{a}_*(\cdot)$. But it is straightforward to apply Helly's Selection Theorem to the sequences $\mathbf{a}_j^m(\cdot)$ separately, each of which has a subsequence converging to $\mathbf{a}_*^m(\cdot)$. □

Proof of Lemma 2

Define $\overline{A}_i^m(j, a_i(\cdot); t_i^{-1}) \equiv \max\left(A_i^m(j, a_i(\cdot)) \cap C(t_i^{-1})\right)$. $\{A_i^m(j, a_i(\cdot))\}^{m,j \in L_m}$ satisfy conditions (i,ii) iff conditions (a,b) are satisfied

(a) $A_i^m(j, a(\cdot)) \cap C(t_i^{-1}) = [0, \overline{A}_i^m(j, a_i(\cdot); t_i^{-1})] \times t_i^{-1}$ for all t_i^{-1}.

(b) $\tilde{t}_i^{-1} > t_i^{-1}$ implies that $\overline{A}_i^m(j, a_i(\cdot); \tilde{t}_i^{-1}) \leq \overline{A}_i^m(j, a_i(\cdot); t_i^{-1})$.

[9] See for instance footnote 8 in Milgrom (1981).

"$\Rightarrow$": (a) follows immediately from order-interval inclusivity of $A_i^m(j, a_i(\cdot)) \ni 0$. (b) follows from isotonicity. Suppose to the contrary that $\overline{A}_i^m(j, a_i(\cdot); t_i^{-1}) < x < \overline{A}_i^m(j, a_i(\cdot); \tilde{t}_i^{-1})$. Then $a_i^m\left(x, t_i^{-1}\right) \geq j$ whereas $a_i^m\left(x, \tilde{t}_i^{-1}\right) < j$, violating the fact that $a_i^m(\cdot)$ is non-decreasing. "$\Leftarrow$": (ii) follows immediately from (a) for $t_i^{-1} = 0$. Now let $\tilde{t}_i \geq t_i$. For order-interval inclusivity of $A_i^m(j, a_i(\cdot))$ it suffices to show that $a_i^m(\tilde{t}_i) < j$ implies $a_i^m(t_i) < j$ for all $m, j \in L_m$. But then

$$t_i^{-1} \leq \tilde{t}_i^{-1} \leq \overline{A}_i^m(j, a_i(\cdot); \tilde{t}_i^{-1}) \leq \overline{A}_i^m(j, a_i(\cdot); t_i^{-1})$$

implying that $a_i^m(t_i) < j$. (The second inequality follows from $a_i^m(\tilde{t}_i) < j$, the third from (b).) Finally, properties (a,b) are clearly preserved by convex combination: whenever (a,b) are satisfied by $\{A_i^m(j, a_i'(\cdot))\}$ and $\{A_i^m(j, a_i(\cdot))\}$, then they are satisfied by $\{A_i^m(j, a_i'(\cdot; \alpha))\}$ as defined in Section 4. □

Proof of Theorem 2

Preliminaries: The type space has partition $T = \{C(t_i^{-1})\}^{t_i^{-1} \in [0,1]^{h-1}}$, where $C(t_i^{-1}) \equiv \{(x, t_i^{-1}) : x \in [0, 1]\}$. This partition induces a convex combination operation of the equivalence classes of isotone strategies, as defined in Section 4. Let $a_i(\cdot; \alpha)$ be any isotone strategy in the class gotten by taking a convex combination with weights α, $1 - \alpha$ on the classes containing $a_i'(\cdot)$, $a_i(\cdot)$. All equivalent strategies specify the same action for all types in the interior of the regions $A_i^m(j, a_i(\cdot; \alpha)$ for $k = 1, ..., m, j \in L_m$. In Parts 1-3, I will prove that the action $a_i(t_i; \alpha)$ is a best response action for such types. Then in Part 4 I will construct an equivalent isotone strategy $\hat{a}_i(\cdot; \alpha)$ that specifies a best response action for all types.

In the proof of Lemma 2 I used the notation $\overline{A}_{i,t_i^{-1}}^m(j; a_i(\cdot))$ to denote the l.u.b. of the types in the one-dimensional subset $C(t_i^{-1})$ that play an action whose m-th dimension is strictly less than j. (As type increases through such a so-called "switching point", the m-th coordinate of the action played switches from one strictly less than j to one that is weakly greater than j.) Let $\mathcal{D}_i$ denote the set of all such switching points. For all types $t_i \notin \mathcal{D}_i$, I will prove that the action played by that type is a best response *given only that* the actions played by types in $C(t_i^{-1})$ in strategies $a_i'(\cdot), a_i(\cdot)$ are all best response actions. **Without loss, then, I may focus entirely on the one-dimensional set of types** $C(t_i^{-1})$ and, indeed, drop all reference to t_i^{-1}. Thus, for Parts 1-3, I will treat the notationally simpler case in which $T = [0, 1]$. (All superscripts are dropped and any reference to the full set of types refers instead to the subset $C(t_i^{-1})$.)

For a given type $\hat{t}_i \notin \mathcal{D}_i$, I need to show that $a_i(\hat{t}_i;\alpha) \equiv \left(a^1(\alpha), ..., a^k(\alpha)\right)$ belongs to the set $BR_i(\hat{t}_i)$ of best response actions. Define

$$BR_i^m(t_i) \equiv \left\{a^m \in L_m : \left(a^m, a^{-m}\right) \in BR_i(t_i) \text{ for some } a^{-m} \in L_{-m}\right\}$$

Part 1: For the duration of this part, fix $m \in \{1, ..., k\}$. I make five points which will be referred to throughout the proof as "the first point", "the second point", etc... *First, define notation related to revealed preference.* Given that $a_i(\cdot)$, $a_i'(\cdot)$ are best response strategies, revealed preference implies that $a^m(\alpha) \in BR_i^m(t_i)$ for all types t_i who play an action with $a^m(\alpha)$ as its m-th coordinate in either strategy. By the definition of switching points, this includes all types $t_i \in int\left(S^{a^m(\alpha)}(a_i(\cdot))\right) \cup int\left(S^{a^m(\alpha)}(a_i'(\cdot))\right)$ where

$$S^{a^m(\alpha)}(a_i(\cdot)) \equiv \left[\overline{A}_i^m(a^m(\alpha), a_i(\cdot)), \overline{A}_i^m(a^m(\alpha)+1, a_i(\cdot))\right]$$
$$S^{a^m(\alpha)}(a_i'(\cdot)) \equiv \left[\overline{A}_i^m(a^m(\alpha), a_i'(\cdot)), \overline{A}_i^m(a^m(\alpha)+1, a_i'(\cdot))\right]$$

$S^{a^m(\alpha)}(a_i(\cdot))$ is the closure of the order-interval of types who play an action with m-th coordinate $a^m(\alpha)$ in the strategy $a_i(\cdot)$. Similarly, $S^{a^m(\alpha)}(a_i'(\cdot))$ contains types who play an action with m-th coordinate $a^m(\alpha)$ in the strategy $a_i'(\cdot)$. Define the following shorthand:

$$H^m \equiv \left[\hat{t}_i, 1\right] \cap \left(S^{a^m(\alpha)}(a_i'(\cdot)) \cup S^{a^m(\alpha)}(a_i(\cdot))\right)$$
$$L^m \equiv \left[0, \hat{t}_i\right] \cap \left(S^{a^m(\alpha)}(a_i'(\cdot)) \cup S^{a^m(\alpha)}(a_i(\cdot))\right)$$

H^m (L^m) is mnemonic for "types that are *H*igher (*L*ower) than $\hat{t}_i$ that play an action equal to $a^m(\alpha)$ on the m-th dimension in either strategy $a_i'(\cdot)$ or $a_i(\cdot)$". (L^m should not be confused with the action lattice $L = \Pi_{m=1}^k L_m$.)

Note that these sets are closed and that all types t_i in the interior of $H^m \cup L^m$ have a best response action whose m-th coordinate equals $a^m(\alpha)$. (By construction, the action played by such a type must have m-th coordinate equal to $a^m(\alpha)$ in either strategy $a_i'(\cdot)$ or $a_i(\cdot)$. Types not in the interior – i.e. at a switching point – may not have such a best response action.)

Second, reduce the problem to 1/2-1/2 *convex combinations.* The set of types $t_i \notin \mathcal{D}_i$ such that $a_i(t_i, \alpha) = a^m(\alpha)$ is the interior of the interval

$$S^{a^m(\alpha)}(\hat{a}_i(\cdot;\alpha)) \equiv \alpha S^{a^m(\alpha)}(a_i(\cdot)) + (1-\alpha) S^{a^m(\alpha)}(a_i'(\cdot))$$

where this is the usual convex combination of sets. (Endpoints of this interval are switching points at which the player's action jumps along the m-th dimension.) In particular, for any such type t_i, the action $a_i(t_i;\alpha) = a_i(t_i;\tilde{\alpha})$ for all

$\tilde{\alpha}$ in a neighborhood of α. Thus, I only need to prove that $a_i(t_i;\alpha) \in BR_i(t_i)$ for α belonging to a dense subset of $[0,1]$. By an induction argument, therefore, it suffices to prove that $a_i(t_i;1/2) \in BR_i(t_i)$ (i.e. for $\alpha = 1/2$). *Base step:* $a_i(t_i;\alpha) \in BR_i(t_i)$ for all t_i when $\alpha \in \{0,1\}$ since $a_i'(\cdot)$, $a_i(\cdot)$ are best response strategies. *Induction step* which I will address in the rest of the proof: $a_i^*(t_i;\alpha) \in BR_i(t_i)$ for all t_i whenever $a_i^*(t_i;\alpha+\partial), a_i^*(t_i;\alpha-\partial) \in BR_i(t_i)$ for all t_i. ($a_i^*(\cdot;\alpha)$ is the well-defined isotone best response pure strategy constructed in Part 4 of the proof.) Note that the induction step is equivalent to proving that $a_i^*(t_i;1/2) \in BR_i(t_i)$ for all t_i whenever $a_i'(t_i), a_i(t_i) \in BR_i(t_i)$ for all t_i (by shifting and rescaling $\alpha-\partial \to 0$ and $\alpha+\partial \to 1$).

Third, some type has a best response action whose m-th coordinate equals $a^m(1/2)$. Since $\hat{t}_i \notin \mathcal{D}_i$, one of the intervals $S^{a^m(1/2)}(a_i'(\cdot))$, $S^{a^m(1/2)}(a_i(\cdot))$ must have non-empty interior. Thus, there must be some type t_i so that either $a_i^m(t_i) = a^m(1/2)$ or $a_i'^m(t_i) = a^m(1/2)$, implying that $a^m(1/2) \in BR_i^m(t_i)$.

Fourth, define $\triangle_m$ and derive some of its important properties. Since $\hat{t}_i \notin \mathcal{D}_i$,

$$\hat{t}_i \in int\left(S^{a^m(\alpha)}(a_i(\cdot;1/2))\right)$$

where $S^{a^m(1/2)}(a_i(\cdot;1/2))$ was defined in the first point. Thus,

$$W \equiv H^m \cap \left(2\hat{t}_i - L^m\right)$$

also has non-empty interior. Define

$$\triangle_m \equiv \max W - \hat{t}_i$$

In words, $\triangle_m$ is the maximum length y such that $\hat{t}_i - y \in L^m$ and $\hat{t}_i + y \in H^m$. Key properties of $\triangle_m$ include:

- $\triangle_m > 0$*:* Follows from the fact that W has non-empty interior and $\min W \geq \hat{t}_i$. (This fact will be used in Part 2 when I argue that types $\hat{t}_i - \triangle_m + \varepsilon$ and $\hat{t}_i + \triangle_m - \varepsilon$ have a best response action with m-th coordinate equal to $a^m(1/2)$.)

- *It can not be that both $\hat{t}_i - \triangle_m = \max L^m$ and $\hat{t}_i + \triangle_m = \min H^m$:* Otherwise, by definition of $\triangle_m$, one of the sets L^m, H^m must be a singleton and $\hat{t}_i \notin int\left(S^{a^m(\alpha)}(a_i(\cdot;1/2))\right)$, a contradiction. (For example, if $|L^m| = 1$ and $\hat{t}_i + \triangle_m = \min H^m$, then $\hat{t}_i = \min S^{a^m(\alpha)}(a_i(\cdot;1/2))$.)

- $\max\left\{a_i^{m\prime}(t_i), a_i^m(t_i)\right\} \leq a^m(1/2)$ *for all* $t_i < \hat{t}_i - \triangle_m$*:* This and the next facts follow immediately from the definition of $\triangle_m$.

- $\min\left\{a_i^{m'}(t_i), a_i^m(t_i)\right\} \leq a^m(1/2)$ *for all* $t_i < \hat{t}_i + \triangle_m$.
- $\max\left\{a_i^{m'}(t_i), a_i^m(t_i)\right\} \geq a^m(1/2)$ *for all* $t_i > \hat{t}_i - \triangle_m$.
- $\min\left\{a_i^{m'}(t_i), a_i^m(t_i)\right\} \geq a^m(1/2)$ *for all* $t_i > \hat{t}_i + \triangle_m$.

By the second of these bulleted observations, *either* $\hat{t}_i - \triangle_m + \varepsilon \in L^m$ *or* $\hat{t}_i + \triangle_m - \varepsilon \in H^m$ *for small enough* ε. This implies that *either*

$$\max\{a_i^m\left(\hat{t}_i - \triangle_m + \varepsilon\right), a_i'^m\left(\hat{t}_i - \triangle_m + \varepsilon\right)\} = a^m(1/2) \text{ or}$$
$$\min\{a_i^m\left(\hat{t}_i + \triangle_m - \varepsilon\right), a_i'^m\left(\hat{t}_i + \triangle_m - \varepsilon\right)\} = a^m(1/2)$$

and hence that

$$a^m(1/2) \in BR_i^m\left(\hat{t}_i - \triangle_m + \varepsilon\right) \cup BR_i^m\left(\hat{t}_i + \triangle_m - \varepsilon\right)$$

Fifth, define the meet and join strategies $a_i^\wedge(\cdot)$, $a_i^\vee(\cdot)$ *and derive some of their important properties.* Reorder the dimensions of player i actions with a permutation ρ on $\{1, ..., k\}$ so that

$$\rho(m_1) \geq \rho(m_2) \Leftrightarrow \triangle_{m_1} \geq \triangle_{m_2}$$

where the positive constants $\triangle_m$ were defined in the fourth point. To simplify notation, relabel dimensions so that $\rho(m)$ becomes m. Under this new reordering, then, $m_1 \geq m_2 \Leftrightarrow \triangle_{m_1} \geq \triangle_{m_2}$. Now, for each non-negative $z \notin \{\triangle_1, ..., \triangle_m\}$, note that there exists $m(z) \in \{1, ..., k\}$ such that

$$\triangle_m < z \text{ for all } m \leq m(z), \triangle_m > z \text{ for all } m > m(z)$$

For each such z, consider the actions

$$a_i^\vee\left(\hat{t}_i - z\right) \equiv a_i'\left(\hat{t}_i - z\right) \vee a_i\left(\hat{t}_i - z\right)$$
$$a_i^\wedge\left(\hat{t}_i + z\right) \equiv a_i'\left(\hat{t}_i + z\right) \wedge a_i\left(\hat{t}_i + z\right)$$

Note that $a_i^\vee(\cdot)$ is defined over $[0, t_i]\backslash\left\{\hat{t}_i - \triangle_1, ..., \hat{t}_i - \triangle_k\right\}$ whereas $a_i^\wedge(\cdot)$ is defined over $[\hat{t}_i, 1]\backslash\left\{\hat{t}_i + \triangle_1, ..., \hat{t}_i + \triangle_k\right\}$. $a_i^\vee\left(\hat{t}_i - z\right) \in BR_i\left(\hat{t}_i - z\right)$ and $a_i^\wedge\left(\hat{t}_i + z\right) \in BR_i\left(\hat{t}_i + z\right)$ since the set of best response actions is a sublattice. Furthermore, by the bulleted observations in the fourth point,

$$a_i^\vee\left(\hat{t}_i - \triangle_m - \varepsilon\right) \leq a^m(1/2), a_i^\vee\left(\hat{t}_i - \triangle_m + \varepsilon\right) \geq a^m(1/2)$$
$$a_i^\wedge\left(\hat{t}_i + \triangle_m - \varepsilon\right) \leq a^m(1/2), a_i^\wedge\left(\hat{t}_i + \triangle_m + \varepsilon\right) \geq a^m(1/2)$$

for all $\varepsilon > 0$. Thus, since $m > m(z)$ implies that $\triangle_m > z$, $\hat{t}_i - \triangle_m < \hat{t}_i - z$ and $\hat{t}_i + \triangle_m > \hat{t}_i - z$ and therefore that

$$a_i^{m\vee}\left(\hat{t}_i - z\right) \geq a^m(1/2) \text{ for all } m > m(z)$$
$$a_i^{m\wedge}\left(\hat{t}_i + z\right) \leq a^m(1/2) \text{ for all } m > m(z)$$

Similarly, since $m \leq m(z)$ implies $\triangle_m < z$,

$$a_i^{m\vee}\left(\hat{t}_i - z\right) \leq a^m(1/2) \text{ for all } m \leq m(z)$$
$$a_i^{m\wedge}\left(\hat{t}_i + z\right) \geq a^m(1/2) \text{ for all } m \leq m(z)$$

Part 2: For a given $m \in \{1, ..., k\}$, define

$$\underline{m} \equiv \lim_{\varepsilon \to 0} m\left(\triangle_m - \varepsilon\right)$$
$$\overline{m} \equiv \lim_{\varepsilon \to 0} m\left(\triangle_m + \varepsilon\right)$$

where $m(\cdot)$ is defined in the fifth point above. Thus, $\underline{m} \leq m \leq \overline{m}$, $\triangle_{\underline{m}} = ... = \triangle_{\overline{m}}$, $\triangle_j < \triangle_m$ for all $j < \underline{m}$, and $\triangle_j > \triangle_m$ for all $j > \overline{m}$. Define the shorthand

$$\hat{a}_{m>} \equiv a_i^{\wedge}\left(\hat{t}_i + \triangle_m + \varepsilon\right) \wedge a_i^{\vee}\left(\hat{t}_i - \triangle_m + \varepsilon\right)$$
$$\tilde{a}_{m>} \equiv \hat{a}_{m>} \vee a_i^{\wedge}\left(\hat{t}_i + \triangle_m - \varepsilon\right)$$
$$\hat{a}_{m<} \equiv a_i^{\vee}\left(\hat{t}_i - \triangle_m - \varepsilon\right) \vee a_i^{\wedge}\left(\hat{t}_i + \triangle_m - \varepsilon\right)$$
$$\tilde{a}_{m<} \equiv \hat{a}_{m<} \wedge a_i^{\vee}\left(\hat{t}_i - \triangle_m + \varepsilon\right)$$

($a_i^{\wedge}(\cdot)$, $a_i^{\vee}(\cdot)$ are defined in the fifth point above, $\triangle_m$ in the fourth.) I use an m subscript here not to confuse the fact that $\tilde{a}_{m>}$, etc.. as actions are *vectors*: $\tilde{a}_{m>}^j$ is the j-th coordinate of the action $\tilde{a}_{m>}$.

The key step is to show that

$$\tilde{a}_{m>} \in BR_i\left(\hat{t}_i + \triangle_m - \varepsilon\right), \tilde{a}_{m<} \in BR_i\left(\hat{t}_i - \triangle_m + \varepsilon\right)$$

for some small $\varepsilon > 0$. First,

$$\hat{a}_{m>} \in BR_i\left(\hat{t}_i - \triangle_m + \varepsilon\right)$$

since $BR_i(\cdot)$ is increasing in the strong set order, $a_i^{\wedge}\left(\hat{t}_i + \triangle_m + \varepsilon\right) \in BR_i\left(\hat{t}_i + \triangle_m + \varepsilon\right)$, and $a_i^{\vee}\left(\hat{t}_i - \triangle_m + \varepsilon\right) \in BR_i\left(\hat{t}_i - \triangle_m + \varepsilon\right)$. And again, since $BR_i(\cdot)$ is increasing in the strong set order and $a_i^{\vee}\left(\hat{t}_i + \triangle_m - \varepsilon\right) \in BR_i\left(\hat{t}_i + \triangle_m - \varepsilon\right)$, I conclude that

$$\tilde{a}_{m>} \in BR_i\left(\hat{t}_i + \triangle_m - \varepsilon\right)$$

By similar logic,

$$\hat{a}_{m<} \in BR_i\left(\hat{t}_i + \triangle_m - \varepsilon\right),\ \tilde{a}_{m<} \in BR_i\left(\hat{t}_i - \triangle_m + \varepsilon\right)$$

Now, I complete this part of the proof by showing that

$$\begin{aligned} \tilde{a}^j_{m<} \geq a^j(1/2) \geq \tilde{a}^j_{m>} &\text{ for all } j < \underline{m} \\ \tilde{a}^j_{m<} = a^j(1/2) = \tilde{a}^j_{m>} &\text{ for all } \underline{m} \leq j \leq \overline{m} \\ \tilde{a}^j_{m<} \leq a^j(1/2) \leq \tilde{a}^j_{m>} &\text{ for all } j > \underline{m} \end{aligned}$$

These facts all follow from properties of the meet and join strategies $a_i^{\wedge}(\cdot)$, $a_i^{\vee}(\cdot)$. For $j < \underline{m}$ and small enough ε, the following types are ranked:

$$\begin{aligned} &\hat{t}_i - \triangle_{\underline{m}} - \varepsilon < \hat{t}_i - \triangle_{\underline{m}} + \varepsilon < \hat{t}_i - \triangle_j < \\ &\qquad < \hat{t}_i + \triangle_j < \hat{t}_i + \triangle_{\underline{m}} - \varepsilon < \hat{t}_i + \triangle_{\underline{m}} + \varepsilon \end{aligned}$$

inducing the following rankings among the j-th dimension of actions played by those types:

$$\begin{aligned} &\text{(a1): } a_i^{j\vee}\left(\hat{t}_i - \triangle_{\underline{m}} + \varepsilon\right) \leq a^j(1/2) \\ &\text{(a2): } a_i^{j\wedge}\left(\hat{t}_i + \triangle_{\underline{m}} - \varepsilon\right) \geq a^j(1/2) \\ &\text{(a3): } a_i^{j\wedge}\left(\hat{t}_i + \triangle_{\underline{m}} + \varepsilon\right) \geq a^j(1/2) \\ &\text{(a4): } a_i^{j\vee}\left(\hat{t}_i - \triangle_{\underline{m}} - \varepsilon\right) \leq a^j(1/2) \end{aligned}$$

(a13) imply that $\hat{a}_{m>} \leq a^j(1/2)$; (a123) that $\tilde{a}_{m>} \geq a^j(1/2)$. Similarly, (a24) imply that $\hat{a}_{m<} \geq a^j(1/2)$; (a124) that $\tilde{a}_{m<} \leq a^j(1/2)$.

Now take $j > \overline{m}$. In this case, the type rankings are

$$\begin{aligned} &\hat{t}_i - \triangle_j < \hat{t}_i - \triangle_{\underline{m}} - \varepsilon < \hat{t}_i - \triangle_{\underline{m}} + \varepsilon < \\ &\qquad < \hat{t}_i + \triangle_{\underline{m}} - \varepsilon < \hat{t}_i + \triangle_{\underline{m}} + \varepsilon < \hat{t}_i + \triangle_j \end{aligned}$$

inducing the following rankings among the j-th dimension of actions:

$$\begin{aligned} &\text{(b1): } a_i^{j\vee}\left(\hat{t}_i - \triangle_{\underline{m}} + \varepsilon\right) \geq a^j(1/2) \\ &\text{(b2): } a_i^{j\wedge}\left(\hat{t}_i + \triangle_{\underline{m}} - \varepsilon\right) \leq a^j(1/2) \\ &\text{(b3): } a_i^{j\wedge}\left(\hat{t}_i + \triangle_{\underline{m}} + \varepsilon\right) \leq a^j(1/2) \\ &\text{(b4): } a_i^{j\vee}\left(\hat{t}_i - \triangle_{\underline{m}} - \varepsilon\right) \geq a^j(1/2) \end{aligned}$$

(b13) imply that $\hat{a}_{m>} \leq a^j(1/2)$; (b123) that $\tilde{a}_{m>} \leq a^j(1/2)$. Similarly, (b24) imply that $\hat{a}_{m<} \geq a^j(1/2)$; (b124) that $\tilde{a}_{m<} \geq a^j(1/2)$.

Finally, consider $\underline{m} \leq j \leq \overline{m}$. Here,

$$\hat{t}_i - \triangle_{\underline{m}} - \varepsilon < \min\{\hat{t}_i - \triangle_j, \hat{t}_i + \triangle_j\} < \hat{t}_i - \triangle_{\underline{m}} + \varepsilon < \\ < \hat{t}_i + \triangle_{\underline{m}} - \varepsilon < \max\{\hat{t}_i - \triangle_j, \hat{t}_i + \triangle_j\} < \hat{t}_i + \triangle_{\underline{m}} + \varepsilon$$

inducing the relationships

$$\begin{aligned}
&\text{(c1): } a_i^{j\vee}\left(\hat{t}_i - \triangle_{\underline{m}} + \varepsilon\right) \geq (=) a^j(1/2) \\
&\text{(c2): } a_i^{j\wedge}\left(\hat{t}_i + \triangle_{\underline{m}} - \varepsilon\right) = (\leq) a^j(1/2) \\
&\text{(c3): } a_i^{j\wedge}\left(\hat{t}_i + \triangle_{\underline{m}} + \varepsilon\right) \geq a^j(1/2) \\
&\text{(c4): } a_i^{j\vee}\left(\hat{t}_i - \triangle_{\underline{m}} - \varepsilon\right) \leq a^j(1/2)
\end{aligned}$$

In the fourth point above, I proved that *either*

$$\begin{aligned}
&a_i^{\vee}\left(\hat{t}_i - \triangle_j + \varepsilon\right) = a^j(1/2) \text{ or} \\
&a_i^{\wedge}\left(\hat{t}_i + \triangle_j - \varepsilon\right) = a^j(1/2)
\end{aligned}$$

In the first case, (c13) imply that $\hat{a}^j_{m>} = a^j(1/2)$; (c123) that $\hat{a}^j_{m>} = a^j(1/2)$; (c24) that $\hat{a}_{m<} \leq a^j(1/2)$; (c124) that $\tilde{a}_{m<} = a^j(1/2)$. In the second case, (c13) imply that $\hat{a}^j_{m>} \geq a^j(1/2)$; (c123) that $\hat{a}^j_{m>} = a^j(1/2)$; (c24) that $\hat{a}_{m<} = a^j(1/2)$; (c124) that $\tilde{a}_{m<} = a^j(1/2)$.

Part 3: Now I am ready to prove that $a(1/2) \in BR_i\left(\hat{t}_i\right)$. First of all, $BR_i\left(\hat{t}_i\right) \neq \emptyset$ since the action space is finite. Let $\dot{a} \in BR_i\left(\hat{t}_i\right)$. Next, note from previous results that

$$\overline{a} \equiv a_i^{\vee}(\underline{z}) \geq a(1/2) \geq a_i^{\wedge}(\overline{z}) \equiv \underline{a}$$

where $\underline{z} \in \left(\hat{t}_i - \triangle_1, \hat{t}_i\right)$ and $\overline{z} \in \left(\hat{t}_i, \hat{t}_i + \triangle_1\right)$. As I argued earlier, $\overline{a} \in BR_i(\underline{z})$ and $\underline{a} \in BR_i(\overline{z})$. Thus,

$$\overline{a} \leq \overline{a} \vee \dot{a} \in BR_i\left(\hat{t}_i\right), \underline{a} \geq \underline{a} \wedge \dot{a} \in BR_i\left(\hat{t}_i\right)$$

This fact establishes the base step of the induction argument that completes the proof:

> *Base step ($j = 0$):* $\overline{\mathbf{a}}, \underline{\mathbf{a}} \in BR_i\left(\hat{t}_i\right)$, where $\underline{a}^m \leq a^m(1/2) \leq \overline{a}^m$ for $m = 1, .., k$.

Induction step: Suppose that $\left(a(1/2),...,a^j(1/2),\overline{a}^{j+1},...,\overline{a}^k\right)$, $\left(a(1/2),...,a^j(1/2),\underline{a}^{j+1},...,\underline{a}^k\right) \in BR_i\left(\hat{t}_i\right)$, where $\underline{a}^m \leq a^m(1/2) \leq \overline{a}^m$ for $m = j+1,..,k$. Then $\left(a(1/2),...,a^{j+1}(1/2),\overline{a}^{j+2},...,\overline{a}^k\right)$, $\left(a(1/2),..,a^{j+1}(1/2),\underline{a}^{j+2},..,\underline{a}^k\right) \in BR_i\left(\hat{t}_i\right)$, where $\underline{a}^m \leq a^m(1/2) \leq \overline{a}^m$ for $m = j+2,..,k$. (The new $\overline{a}^{j+2},...,\overline{a}^k$ and $\underline{a}^{j+2},...,\underline{a}^k$ may be different than before.)

Under the presumption of the induction step, since $\hat{t}_i + \triangle_{j+1} - \varepsilon > \hat{t}_i$ and $a_i^{\vee}\left(\hat{t}_i + \triangle_{j+1} - \varepsilon\right) \in BR_i\left(\hat{t}_i + \triangle_{j+1} - \varepsilon\right)$ I have that

$$\breve{a} \in BR_i\left(\hat{t}_i + \triangle_{j+1} - \varepsilon\right)$$

where $\breve{a} \equiv a_i^{\wedge}\left(\hat{t}_i + \triangle_{j+1} - \varepsilon\right) \wedge \left(a(1/2),..,a^j(1/2),\overline{a}^{j+1},..,\overline{a}^k\right)$. By its construction, $\breve{a}$ has the required form of a lower bound on $a(1/2)$ that coincides on the first $j+1$ dimensions:

$$\breve{a} = \left(a(1/2),..,a^{j+1}(1/2),\underline{a}^{j+1},..,\underline{a}^k\right)$$

For observe that

$\breve{a}^m = a^m(1/2)$ for all $m = 1,...,j$ since $a_i^{m\wedge}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) \geq a^m(1/2)$;

$\breve{a}^{j+1} = a^{j+1}(1/2)$ since $a_i^{j+1\wedge}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) = a^{j+1}(1/2)$; and

$\breve{a}^m \leq a^{j+1}(1/2)$ since $a_i^{m\wedge}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) \leq a^{j+1}(1/2)$ for $m = j+2,..,k$.

Similarly, $\hat{t}_i - \triangle_{j+1} + \varepsilon < \hat{t}_i$ and $a_i^{\vee}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) \in BR_i\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right)$ imply that

$$\grave{a} \in BR_i\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right)$$

where $\grave{a} \equiv a_i^{\vee}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) \vee \left(a(1/2),..,a^j(1/2),\underline{a}^{j+1},..,\underline{a}^k\right)$ has the required form of a upper bound on $a(1/2)$ that coincides on the first $j+1$ dimensions:

$$\grave{a} = \left(a(1/2),..,a^{j+1}(1/2),\overline{a}^{j+1},..,\overline{a}^k\right)$$

For observe that

$\grave{a}^m = a^m(1/2)$ for all $m = 1,...,j$ since $a_i^{m\vee}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) \leq a^m(1/2)$;

$\grave{a}^{j+1} = a^{j+1}(1/2)$ since $a_i^{j+1\vee}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) = a^{j+1}(1/2)$; and

$\breve{a}^m \geq a^{j+1}(1/2)$ since $a_i^{m\vee}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) \geq a^{j+1}(1/2)$ for $m = j + 2, ..., k$.

Part 4: From Parts 1-3, $a_i(t_i;\alpha) \in BR_i(t_i)$ for all $t_i \notin \mathcal{D}_i$. Now, for each t_i define

$$\underline{a}(t_i;\alpha) \equiv \lim_{\varepsilon\to 0} a_i\left(t_i - \varepsilon;\alpha\right)$$
$$\overline{a}(t_i;\alpha) \equiv \lim_{\varepsilon\to 0} a_i\left(t_i + \varepsilon;\alpha\right)$$

When ε is small enough, the action $a_i(t_i;\alpha)$ is well-defined and, in fact, constant over the intervals $[t_i - \varepsilon, t_i)$ and $(t_i, t_i + \varepsilon]$. Thus, these limits exist. Furthermore, the order interval

$$[\underline{a}(t_i;\alpha), \overline{a}(t_i;\alpha)] \equiv \{\mathbf{a} : \underline{a}(t_i;\alpha) \leq \mathbf{a} \leq \overline{a}(t_i;\alpha)\}$$

is non-empty and increasing in the strong set order. Why? $a_i(\cdot;\alpha)$ is isotone over the types $t_i \notin \mathcal{D}_i$ at which it is defined and, for any two given types $\tilde{t}_i$ and t_i, there exists $\varepsilon > 0$ such that

$$\tilde{t}_i - \varepsilon, \tilde{t}_i + \varepsilon, t_i - \varepsilon, t_i + \varepsilon \notin \mathcal{D}_i$$
$$a_i\left(\tilde{t}_i - \varepsilon;\alpha\right) = \underline{a}(\tilde{t}_i;\alpha), a_i\left(\tilde{t}_i + \varepsilon;\alpha\right) = \overline{a}(\tilde{t}_i;\alpha)$$
$$a_i\left(t_i - \varepsilon;\alpha\right) = \underline{a}(t_i;\alpha), a_i\left(t_i + \varepsilon;\alpha\right) = \overline{a}(t_i;\alpha)$$

Since $a_i(\cdot;\alpha)$ is isotone when restricted to types not in $\mathcal{D}_i$, I conclude that $\underline{a}(\tilde{t}_i;\alpha) \geq \underline{a}(t_i;\alpha)$, $\overline{a}(\tilde{t}_i;\alpha) \geq \overline{a}(t_i;\alpha)$ and hence that the order interval $\left[\underline{a}(\tilde{t}_i;\alpha), \overline{a}(t_i;\alpha)\right]$ is increasing in the strong set order. Furthermore, for the same reason, $\underline{a}(t_i;\alpha) \leq \overline{a}(t_i;\alpha)$ and I conclude that this order interval is non-empty.

Now, define the strategy $a_i^*(\cdot;\alpha)$ as follows:

$$a_i^*(t_i;\alpha) \equiv \max\left(BR_i(t_i) \cap [\underline{a}(t_i;\alpha), \overline{a}(t_i;\alpha)]\right)$$

Since $BR_i(\cdot)$ and $[\underline{a}(\cdot;\alpha), \overline{a}(\cdot;\alpha)]$ are both sublattices and increasing in the strong set order, part (2) of Lemma 3 implies that there intersection is a sublattice and increasing in the strong set order. Furthermore, part (1) of Lemma 3 implies that their intersection is non-empty. Thus, the maximum of this intersection is well-defined and isotone. Finally, note that the strategy $a_i^*(\cdot;\alpha)$, so defined, coincides with $a_i(\cdot;\alpha)$ on all types $t_i \notin \mathcal{D}_i$. Hence, $a_i^*(\cdot;\alpha)$ is an isotone best response pure strategy.

Lemma 3. *Suppose that $BR_i(t_i)$ is a sublattice, increasing in the strong set order, and non-empty for all t_i. Then:*

1. *Suppose that $a_1 \in BR_i(t_i^1)$, $a_3 \in BR_i(t_i^3)$, $a_1 \leq a_3$, and $t_i^1 < t_i^2 < t_i^3$. Then there exists $a_2 \in BR_i(t_i^2)$ such that $a_1 \leq a_2 \leq a_3$.*

2. *Suppose that $X(t_i)$ is a sublattice and increasing in the strong set order. Then $BR_i(t_i) \cap X(t_i)$ is a sublattice (possibly empty) and increasing in the strong set order.*

Proof. (1) $z \in BR_i(t_i^2)$ by non-emptiness. Since $t_i^2 > t_i^1$, $z \vee a_1 \in BR_i(t_i^2)$ since $BR_i(\cdot)$ is increasing in the strong set order. Since $t_i^2 < t_i^3$, similarly, $(z \vee a_1) \wedge a_3 \in BR_i(t_i^2)$. $a_1 \leq (z \vee a_1) \wedge a_3 \leq a_3$, so one may set $a_2 = (z \vee a_1) \wedge a_3$.

(2) Suppose that $a_1, a_2 \in BR_i(t_i) \cap X(t_i)$. Then $a_1 \vee a_2$, $a_1 \wedge a_2 \in BR_i(t_i) \cap X(t_i)$ since $BR_i(t_i)$, $X(t_i)$ are sublattices. Similarly, suppose that $a_1 \in BR_i(t_i^1) \cap X(t_i^1)$, $a_2 \in BR_i(t_i^2) \cap X(t_i^2)$. Then $a_1 \wedge a_2 \in BR_i(t_i^1) \cap X(t_i^1)$, $a_1 \vee a_2 \in BR_i(t_i^2) \cap X(t_i^2)$ since $BR_i(t_i)$, $X(t_i)$ are increasing in the strong set order. □

References

Athey, S. (1998): "Comparative Statics Under Uncertainty: Single Crossing Properties and Log-Supermodularity," MIT Working Paper Number 96-22R, Available at www.stanford.edu/˜athey.

——— (2001): "Single Crossing Properties and the Existence of Pure-Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69(4), 861–889.

——— (2002): "Monotone Comparative Statics under Uncertainty," *Quarterly Journal of Economics*, 117(1), 187–223.

Birkhoff, G. (1967): *Lattice Theory.* American Mathematical Society Colloquium Publications, 3rd edn.

Glicksberg, I. (1952): "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points," *Proceedings of the American Mathematical Society*, 38, 170–174.

McAdams, D. (2001): "Isotone Equilibrium in Multi-Unit Auctions," Manuscript, MIT Sloan, Available at http://www.mit.edu/~mcadams/papers/multiunit.pdf.

Milgrom, P. (1981): "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica*, 49(4), 921–943.

Milgrom, P., and J. Roberts (1990): "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 51, 981–994.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180.

Shannon, C. (1995): "Weak and Strong Monotone Comparative Statics," *Economic Theory*, 5(2), 209–227.

Topkis, D. (1979): "Equilibrium Points in Nonzero-Sum n-Person Submodular Games," *SIAM Journal of Control and Optimization*, 17, 773–787.

Vives, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19, 305–321.