THIN SHELL AND NEW INVARIANT ELEMENTS

BY HYBRID STRESS METHOD

by

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Thin Shell and New Invariant Elements
by Hybrid Stress Method

by

Kiyohide Sumihara

Submitted to the Department of Aeronautics and Astronautics on
March 14, 1983 in partial fulfillment of the requirements for the
degree of Doctor of Philosophy.

ABSTRACT

This research work attempts to resolve some of the fundamental
difficulties existing at the present time for the construction of
finite elements by hybrid stress method. The study is based upon
new versions of Hu-Washizu and Hellinger-Reissner principles for
solid mechanics. These versions of variational principles are
derived from the introduction of internal displacement parameters
in addition to the displacements expressed in terms of physical
nodal quantities inside a finite element. The internal displace-
ment parameters essentially play the role of the constraint on the
stress equilibrium equations. The inversion of the flexibility matrix
in hybrid stress elements can be simplified by employing these varia-
tional principles and by the use of natural coordinates for the inter-
polation functions of physical quantities.

Through the semiLoof element and the new version of hybrid stress
element with partial satisfaction of stress equilibrium equations
shell elements can be formed that can represent the fundamental
characteristic modes, i.e. the rigid body motion, the momentless mem-
brane state and the inextensional bending state. This can be inter-
preted as the passing of the patch tests for thin shells. Further-
more the capability of the representation of inextensional bending
behavior of plate under large deflection in hybrid semiLoof element
is also demonstrated.

The similarity and equivalence between Wilson's incompatible
elements and the corresponding hybrid stress elements are shown by
means of the new versions of variational principles. Also the
relationship between quasi-conforming elements initiated by Tang
and hybrid stress elements is clarified.

Finally a new invariant hybrid stress element for plane stress
problem is created by applying a perturbation of the element geometry
and new version of Hellinger-Reissner principle. The finite element is essentially invariant with respect to reference coordinate systems, less sensitive to distortion and free from zero energy deformation modes. The new method can be further developed and applied to other hybrid stress elements.

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To the originator of Hybrid
To my parents and my sister
To my semi-fiancée
To people of Kobayashi laboratory in Tokyo University
To the founders and the president of
    Murata Overseas Scholarship Foundation
To people who supported me
To beauty in nature
To Wittgenstein
To great future in Hybrid
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1. Introduction

The formulation of hybrid stress model [1] involves more than one field variable, and the corresponding background is sophisticated. The model has many advantageous features but there are also difficulties in its usage.

The most fundamental and important problem in hybrid stress model is to establish the systematic procedure to select the pattern of assumed stresses. For the purpose, the suppression of kinematic deformation modes must be taken into consideration because the kinematic deformation modes might cause unreasonable behaviours of finite elements. The number of stress terms should be limited in order to maintain the computing efficiency and to yield not overly rigid elements. But the key point for choosing the stress assumptions is the consideration on the stress equilibrium conditions. So far the complete satisfaction of the stress equilibrium conditions in hybrid stress method has been taken as the necessary requirement for the construction of hybrid stress elements. However in the thin shell element formulation, because of the presence of the coupling between membrane and bending actions the complete satisfaction of the equilibrium conditions becomes a very difficult requirement to be satisfied.

Another fundamental issue is the extension of the patch test proposed by Irons [2,3] to thin shells. The validation of the patch test for thin shells would require the examination whether the characteristic solution modes for thin shells can be represented in a finite element.

As pointed out by Morley [4,5,6], the important characteristics modes for thin shells correspond to rigid body motions, momentless membrane states and inextensional bending. Therefore it follows that those three modes should be satisfactorily represented by a finite element for thin shells.
For isoparametric displacement model, it is known that the isoparametric representation admits exact recovery of rigid body movements while it does not provide an acceptable description of inextensional bending of curved surfaces because of the presence of coupling between bending strains and middle surface strains. One remedy that Morley [6] has suggested for a dramatic improvement in the approximation to inextensional bending for a triangular shell element is to introduce in the formulation, additional higher order internal displacements.

The concept of the patch test can be also extended to the case of large displacement bending of plates, although the attainment of independant constant curvatures is complicated by the nonlinear coupling between bending and stretching. Allman [7,8,9] investigated the way of introducing the inextensional bending behavior for finite elements by using an equilibrium finite element solution for the plate stretching behavior or by using hybrid stress model and introducing higher order modes for interelement boundary membrane displacements to recover the inextensional bending behavior.

In finite element analyses of plates and shells, one difficult task is to construct the equilibrium equations at a corner node of which the reference planes of the joining elements are not co-planar. At such nodes all six degrees of freedom should be taken into consideration, but, in general, only five degrees of freedom are used at a node for plate and shell elements. Irons [10] has suggested the use of the so-called SemiLoof element for which normal rotations along each edge are defined at nodes which are not located at corners of the element. It has been observed by Martins and Owen [11,12] that the results obtained by SemiLoof elements with coarse finite element meshes compare well with those given by using much larger number of elements.
But it has also been pointed out that, under a point loading situation, fine mesh should be used in order to obtain satisfactory solutions. Also the numerical formulation of the element is quite complex.

It has been shown by Pian [13] that SemiLoof elements for plates and shells can be easily constructed by hybrid stress method and, in fact, for thin plates the equilibrium model ofFraeijs de Veubeke [14] is equivalent to a SemiLoof element and can be derived by the same procedure used to derive the hybrid stress element. The theoretical basis of hybrid semiLoof element is, however, quite different from that of the SemiLoof element by Irons.

The more basic problems related to the construction of a finite element based upon hybrid stress method are the development of systematic methods to maintain the invariant property with respect to the reference coordinate system and to minimize the deterioration of the quality due to distortion. The conditions for invariance of the element stiffness matrix obtained by hybrid stress method have been given for two-dimensional problems by Spilker [15]. The invariance of the element stiffness matrix under general rotation and translation requires that complete polynomials be used for the stress interpolations. But such condition of invariance can lead to a relatively large number of stress parameters compared with the minimum required although it has the advantage of the search for plausible stress interpolations. The enforcement of appropriate stress compatibility conditions are then used as a rational procedure for reducing the number of independent stress parameters (β-stress used in ref. 1). It should be noted that under such procedure a four node plane stress element with complete linear distributions for stresses will have 7β-stress parameters while the eight node plane stress element with complete cubic distributions will have stress parameters after the imposition of compatibility condition.
They are not the minimum numbers to avoid the kinematic deformation modes.

So far as the effect of distortion in hybrid stress elements is concerned, there has been no systematic investigation.

In the construction of the stiffness matrix especially for higher order elements by hybrid stress method, there exists another problem, i.e., the time-consuming inversion of the flexibility matrix, i.e. the $H$ matrix [1]. In order to resolve the difficulty, so called uncoupled version was introduced by Pian and Chen [16] and Pian, Chen and Kang [17]. The new formulation is essentially based upon the Hellinger-Reissner principle and the Hu-Washizu principle with the introduction of additional internal displacement modes.

The present study is an attempt to resolve many difficulties in the hybrid stress method existing at the present time. These studies are all based upon the new versions of the variational principles in solid mechanics. Three main topics considered here are the construction of thin shell elements, a discussion of the similarity and equivalence of different finite element models and the formulation of new invariant hybrid stress element. Section 2 of this thesis is to present Hu-Washizu and Hellinger-Reissner principles with additional displacement modes and the next three sections are devoted to the three main topics mentioned above.
2. **Hu-Washizu and Hellinger-Reissner principles and**
   **new version of hybrid stress elements**

2.1 **Modified Hu-Washizu principle**

A summary of conventional variational principles in solid mechanics is indicated in figure 1. The generalized variational principle is the Hu-Washizu principle [18] given by

\[
\Pi_{\pi}(\mathcal{U}, \varepsilon, \sigma, \lambda) = \int_{\mathcal{V}} \left[ \frac{1}{2} \varepsilon : \mathcal{C} e - \sigma : \varepsilon + \frac{1}{2} (\tilde{\varepsilon} : \tilde{\mathcal{C}} \varepsilon + \mathcal{C} \tilde{\varepsilon}) - \mathcal{F} \cdot \mathcal{U} \right] d\mathcal{V} - \int_{\mathcal{S}_u} \mathcal{I} \cdot (\mathcal{U} - \mathcal{U}) dS - \int_{\mathcal{S}_f} \mathcal{F} \cdot \mathcal{U} dS = \text{stationary} \tag{2-1-1}
\]

where

- \( \mathcal{U} \) = displacement tensor
- \( \varepsilon \) = strain tensor
- \( \sigma \) = stress tensor
- \( \mathcal{I} \) = gradient vector
- \( \mathcal{F} \) = body forces
- \( \mathcal{I} \) = \( \mathcal{F} \cdot \lambda \)
- \( \lambda \) = normal vector along boundary
- \( (\text{--}) \) = prescribed quantities
- \( \mathcal{S}_u \) = surface with prescribed displacements
- \( \mathcal{S}_f \) = surface with prescribed tractions

We now suppose that the domain \( \mathcal{V} \) is subdivided fictitiously into finite number of elements, \( \mathcal{V}_i \) (\( i = 1, 2, \ldots, n \)) . We denote two arbitrary adjacent elements by \( \mathcal{V}_a \) and \( \mathcal{V}_b \) , and the interelement boundary between \( \mathcal{V}_a \) and \( \mathcal{V}_b \) by \( \text{Sab} \). We shall denote displacements \( \mathcal{U}_i \) in each element by

\[
\mathcal{U}_i^{(a)}, \mathcal{U}_i^{(b)}, \ldots, \mathcal{U}_i^{(a)}, \mathcal{U}_i^{(b)}, \ldots, \mathcal{U}_i^{(a)}; i = 1, 2, 3
\]

each of which will be called displacement functions. Then the assembly of these displacement functions may be taken as admissible functions for the functional of the generalized variational principle.
if they satisfy the following requirements;

(1) They are continuous and single-valued in each element
(2) There are conforming on interelement boundaries:
\[ U_i^{(a)} = U_i^{(b)} \quad \text{on} \quad S_{ab} \]

Consequently, if the displacement functions are so chosen as to satisfy the requirements (1) & (2), the functional becomes
\[ \mathcal{J}_G = \sum_{m \in \mathcal{M}} \int_{\Omega_m} \left[ \frac{1}{2} \varepsilon : \varepsilon - \varepsilon' : \varepsilon + \varepsilon' : (\bar{\varepsilon} \gamma + \bar{\varepsilon} \bar{\gamma}) - \varepsilon : \varepsilon' \right] \, dV \\
- \int_{S_{um}} \tau \cdot (\gamma - \bar{\gamma}) \, dS - \int_{S_{om}} \tau \cdot \dot{\gamma} \, dS \] (2-1-2)

where the independent quantities subject to variation in \( \mathcal{J}_G \) are \( \gamma^{(i)}, \varepsilon^{(i)} \) and \( \varepsilon'^{(i)} \) \( (i = 1, 2, \ldots, n) \).

Next, we formulate a variational principle in which the subsidiary conditions are introduced into the framework of the variational expression. When we introduce two kinds of Lagrangian multipliers to relax the traction reciprocity and interelement compatibility, the physical meaning of the Lagrangian multipliers can be identified as interelement boundary displacements and tractions on interelement boundary respectively. The following modified Hu-Washizu principle results;
\[ \mathcal{J}_{HG} = \sum_{n} \left[ \int_{\Omega_n} \left[ \frac{1}{2} \varepsilon : \varepsilon - \varepsilon' : \varepsilon + \varepsilon' : (\bar{\varepsilon} \gamma + \bar{\varepsilon} \bar{\gamma}) - \varepsilon : \varepsilon' \right] \, dV \\
- \int_{\partial \Omega_n} \tau \cdot (\gamma - \bar{\gamma}) \, dS - \int_{S_{on}} \tau \cdot \dot{\gamma} \, dS \right] \] (2-1-3)

where
\[ \partial \Omega_n = \text{interelement boundary surface} \]
\[ S_{on} = \text{interelement boundary surface with prescribed tractions.} \]
\[ \bar{\gamma} = \text{interelement boundary displacements} \]
2.2 New version of modified Hu-Washizu principle

An element stiffness matrix by assumed compatible displacements can be derived not only by the conventional potential energy principle but also, indirectly, by generalized variational principles such as the Hu-Washizu principle and the Hellinger-Reissner principle. Fraeijs de Veubeke [19] had cited his limitation principle and indicated that if no restrictions are applied to the assumed stress distribution the Hellinger-Reissner principle will yield the same element stiffness matrix as that by the assumed displacement method. Compatible elements are often found to be too rigid for finite element analyses and incompatible elements have been suggested. There is, however, the lack of a rational procedure for constructing shape functions that will guarantee the resulting element to pass the patch test.

Various kinds of variational principles and modified versions of the variational principles for finite element method are indicated in figure 2 which includes recent developments in assumed stress hybrid elements. We shall derive the new version of modified Hu-Washizu principle.

In the finite element formulation, we separate element displacements \( \mathbf{U} \) into two parts - the displacement \( \mathbf{U}_q \) which is expressed in terms of nodal parameters \( \mathbf{q} \) and the additional part \( \mathbf{U}_\lambda \) which is expressed in terms of internal displacement parameters \( \mathbf{\lambda} \) that can be statically condensed in the element level. Here \( \mathbf{U}_\lambda \) may be incompatible along the boundary or it may be bubble functions which are zero along the boundary;

\[
\mathbf{U} = \mathbf{U}_q + \mathbf{U}_\lambda \tag{2-2-1}
\]

The generalized principle given by eq.(2-1-3) can be rewritten in the form,

\[
\mathcal{L}_{eq}^* = \sum_n \left[ \int_{V_n} \left[ \frac{1}{2} \mathbf{E} : \mathbf{ \varepsilon} - \mathbf{ \varepsilon} : \mathbf{ \varepsilon} + \mathbf{ \varepsilon} : \frac{1}{2} (\nabla \mathbf{U}_q + \nabla^\top \mathbf{U}_q) + \mathbf{ \varepsilon} : \frac{1}{2} (\nabla \mathbf{U}_\lambda + \nabla^\top \mathbf{U}_\lambda) \right] dV - \int_{\partial V_n} \mathbf{ \varepsilon}(\mathbf{U}_q + \mathbf{U}_\lambda - \mathbf{U}) \cdot \mathbf{n} dS - \int_{\partial V_n} \mathbf{ \varepsilon} \cdot \mathbf{U}_\lambda dS \right] \tag{2-2-2}
\]
Since
\[
\sum_n \left[ \int_{V_n} \frac{1}{2} \left( \nabla \cdot \mathbf{e} - \mathbf{e} \right) \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{e} - \frac{1}{2} \left( \mathbf{V} \cdot \mathbf{u}_q + \mathbf{u}_q \cdot \mathbf{V} \right) - \mathbf{F} \cdot \mathbf{u}_q 
right]
- \left[ \int_{\partial V_n} \mathbf{t} \cdot \mathbf{n} \, dS - \int_{S_{\Omega_n}} \mathbf{t} \cdot \mathbf{n} \, dS \right]
\]
(2-2-3)
the equation (2-2-2) may be modified as
\[
\mathcal{T}_{HG} = \sum_n \left[ \int_{V_n} \left[ \frac{1}{2} \mathbf{e} : \mathbf{e} - \mathbf{e} : \mathbf{e} + \frac{1}{2} \left( \nabla \cdot \mathbf{u}_q + \mathbf{u}_q \cdot \nabla \right) - \mathbf{F} \cdot \mathbf{u}_q 
right]
- \left[ \int_{\partial V_n} \mathbf{t} \cdot \mathbf{n} \, dS - \int_{S_{\Omega_n}} \mathbf{t} \cdot \mathbf{n} \, dS \right]
\]
(2-2-4)
and modifying this equation furthermore, we get
\[
\mathcal{T}_{HG} = \sum_n \left[ \int_{V_n} \left[ \frac{1}{2} \mathbf{e} : \mathbf{e} - \mathbf{e} : \mathbf{e} - \left( \nabla \cdot \mathbf{u}_q + \mathbf{u}_q \cdot \nabla \right) \cdot \mathbf{u}_q - \left( \nabla \cdot \mathbf{u}_q + \mathbf{u}_q \cdot \nabla \right) \cdot \mathbf{u}_q \right] dV 
+ \left[ \int_{\partial V_n} \mathbf{t} \cdot \mathbf{n} \, dS - \int_{S_{\Omega_n}} \mathbf{t} \cdot \mathbf{n} \, dS \right] \right]
\]
(2-2-5)
When we choose the appropriate form for \( \mathbf{u}_q \) in order to satisfy the stress equilibrium equations, the expression for the functional can be simplified further in the form:
\[
\mathcal{T}_{HG} = \sum_n \left[ \int_{V_n} \left[ \frac{1}{2} \mathbf{e} : \mathbf{e} - \mathbf{e} : \mathbf{e} - \left( \nabla \cdot \mathbf{u}_q + \mathbf{u}_q \cdot \nabla \right) \cdot \mathbf{u}_q \right] dV 
+ \left[ \int_{\partial V_n} \mathbf{t} \cdot \mathbf{n} \, dS - \int_{S_{\Omega_n}} \mathbf{t} \cdot \mathbf{n} \, dS \right] \right]
\]
(2-2-6)
It should be noted the internal displacements \( \mathbf{u}_q \) may also be considered as purely mathematical Lagrangian multipliers to be used to bring in the stress equilibrium equations approximately or exactly. So long as we constrain the stress equilibrium equations in the appropriate manner, we can use any function for \( \mathbf{u}_q \). On the other hand when \( \mathbf{u}_q \) are to be used to derive the consistent nodal forces and mass matrices, they should have realistic physical meanings. It is noted that the version by Tong [20] shown in figure 2 implies that the pointwise satisfaction of the stress equilibrium equations can be performed in the variational procedure by the use of Lagrangian multiplier method. It is obvious that the version is a special case of the use of eq.(2-2-6).

When we consider the possibility of partial satisfaction or relaxation of equilibrium equations and we have explicit expressions for constraint
equations, another version of modified Hu-Washizu principle can be written as

\[
\mathcal{P}_{HG}^{**} = \sum_{n} \left\{ \int_{V_n} \left[ \frac{1}{2} \varepsilon : \varepsilon - \sigma : \sigma - (\tilde{\varepsilon}, \varepsilon - (\tilde{\sigma}, \varepsilon - (\tilde{\varepsilon}, \varepsilon + \tilde{\varepsilon}) \cdot \tau \right] dV
+ \int_{\partial V_n} T \cdot \tilde{\varepsilon} dS - \int_{S_{\partial n}} \tau \cdot \tilde{\varepsilon} dS + \int_{V_n} \zeta^\mu \cdot \mu^\mu dV + \int_{\partial V_n} \tilde{\zeta}^\mu \cdot \tilde{\mu}^\mu dV \right\}
\] (2-2-7)

where \( \zeta^\mu \) and \( \tilde{\zeta}^\mu \) indicate any physically meaningful constraint condition in finite element;

\[
\zeta^\mu = \mathcal{Q} \quad \text{in} \quad V_n \quad \text{and} \quad \tilde{\zeta}^\mu = \mathcal{Q} \quad \text{on} \quad \partial V_n
\] (2-2-8)

and \( \mu^\mu \) and \( \tilde{\mu}^\mu \) are simply mathematical Lagrangian multipliers.

For example, when the imposition of compatibility is preferred in a finite element, the constraint condition would be

\[
\begin{align*}
\tilde{\varepsilon} & = \varepsilon \quad \text{in} \quad V_n \\
\tilde{\varepsilon} & = \varepsilon \quad \text{on} \quad \partial V_n
\end{align*}
\] (2-2-9)

Also when the additional physical requirement such as the quality of bending behavior in plane stress element is needed, the constraint condition can be incorporated in the functional of eq. (2-2-7). The example will be given in chapter 4. As the example of the constraint condition on the element boundary, the traction free condition;

\[
\Gamma = 0 \quad \text{on} \quad S_{\sigma_{\partial n}} \subset \partial V_n
\] (2-2-10)

can be applied.

Thus any physically desirable requirements in addition to stress equilibrium equations can be put into a finite element based upon the new version of the variational principle.

It is noted that, when the constitutive relation is satisfied, the modified Hu-Washizu principle of eq. (2-1-6) reduces to the modified Hellinger-Reissner principle;

\[
\mathcal{P}_{HR}^* = \sum_{n} \left\{ \int_{V_n} \left[ -\frac{1}{2} \varepsilon : \varepsilon - (\tilde{\varepsilon}, \varepsilon - (\tilde{\sigma}, \varepsilon + \tilde{\varepsilon}) \cdot \tau \right] dV
+ \int_{\partial V_n} T \cdot \tilde{\varepsilon} dS - \int_{S_{\partial n}} \tau \cdot \tilde{\varepsilon} dS \right\}
\] (2-2-11)

where
\[ \Pi_{HR}^* = \sum \left[ \int_{V_n} \left[ -\frac{1}{2} \bar{Q} : \bar{S} - (\bar{v} \cdot \bar{S} + \bar{E}) \cdot \bar{y} - (\bar{v} \cdot \bar{S} + \bar{E}) \cdot \bar{y}_2 \right] dV \\
+ \int_{\partial V_n} \bar{J} \cdot \bar{z} dS - \int_{S_{\partial n}} \bar{J} \cdot \bar{y} dS \right] \]

(2-2-12)

Similar derivation can be performed for eq. (2-2-7). In the development of new versions of variational principles, the most critical point is the treatment of stress equilibrium equations. The versions of \( \Pi_C, \Pi_{HC} \), and \( \Pi_{R(\bar{Q}, \bar{y})} \) indicated in figure 2 conceptually requires the complete satisfaction of stress equilibrium equations. But the present new versions do not require the complete satisfaction of equilibrium equations in a explicit manner.
2.3 "Invariance" in Finite Element Method

The field theory of continuum mechanics requires coordinate invariance. This implies that any governing equations should be invariant with respect to coordinate system. Then we can write those equations in tensor form;

\[ \mathbf{\nabla} \cdot \mathbf{\sigma} = 0 \quad \text{equilibrium equations} \quad (2-3-1) \]
\[ \mathbf{\nabla} \times \mathbf{\varepsilon} \times \mathbf{\nabla} = \mathbf{\sigma} \quad \text{compatibility conditions} \quad \text{etc.} \quad (2-3-2) \]

In the construction of finite elements, it is convenient for us to employ Cartesian coordinate system for the definitions of various physical quantities because every governing equation has the simplest form under this coordinate system. Then we can apply one variational principle which is also invariant with respect to coordinate system. The key part of the preservance of invariance in finite element method is the choice of coordinate system; global Cartesian coordinate system or natural coordinate system. It is obvious that the intrinsic natural coordinate system should be employed to preserve invariant properties for any physical quantities.

When the principle of minimum potential energy is applied for the finite element construction, one operation needed in the construction of the stiffness matrix is

\[ \mathbf{\varepsilon} = \frac{1}{2} (\mathbf{\nabla} \mathbf{u} + \mathbf{\nabla} \mathbf{u}^T) \quad (2-3-3) \]

where \[ \mathbf{\varepsilon} = \varepsilon_{ij} \mathbf{e}^i \mathbf{e}^j \], \[ \mathbf{\nabla} = \frac{\partial}{\partial x^i} \mathbf{e}^i \], \[ \mathbf{u} = u_i \mathbf{e}^i \].

As shown in figure 3, we suppose that one Cartesian coordinate system is transformed into another Cartesian coordinate system \( \vec{\mathbf{X}}^i \). The relation can be given by

\[ \mathbf{X}^i = \vec{\mathbf{X}}^i + \mathbf{\xi} \], \[ \mathbf{X}^i = \mathbf{X}^i \mathbf{e}^i \], \[ \vec{\mathbf{X}}^i = \vec{\mathbf{X}}^i \vec{\mathbf{e}}^i \]
\[ \vec{\mathbf{e}}^i = \mathbf{R}_i \mathbf{e}^i \], \[ \mathbf{R}_i = \mathbf{R}_{ij} \mathbf{e}^i \mathbf{e}^j \quad (2-3-4) \]

where \( \mathbf{R}_i \) is the orthogonal rotation tensor and \( \mathbf{\xi} \) is the translation vector.
Then the equation (2-3-3) is transformed into

$$\vec{\varepsilon} = \frac{1}{2} \left( \vec{\nabla} \vec{\xi} + \vec{\xi} \vec{\nabla} \right)$$

where

$$\vec{\varepsilon} = \varepsilon_j \vec{\xi} \varepsilon_j$$

$$\vec{\nabla} = \frac{\partial}{\partial x_i} \vec{\xi}$$

(Eq. 2-3-5)

The relationship between natural coordinate system and global Cartesian coordinate system for isoparametric displacement model can be represented by

$$\xi_i = \xi_j \varepsilon_j = J_{ij} \xi_j$$

or

$$\xi_i = G_{ij} \xi_j$$

Thus,

$$G_{ij} = R_{ik} G_{kj}$$

(Eq. 2-3-6)

The displacements are represented by

$$\vec{\xi} = u_i \xi_i = L_j q_j \xi_i = L_j q_j$$

$$= \xi_j \xi_i = L_j q_j \xi_i = L_j R_{ij} q_j$$

Thus,

$$q_j = R_{ij} q_j$$

(Eq. 2-3-7)

where $L_j$ are interpolation functions and $q_j$ and $\bar{q}_j$ are nodal displacement vectors.

The relation of the gradients is given as

$$\vec{\nabla} = \frac{\partial}{\partial x_i} \vec{\xi}_i = \frac{\partial}{\partial \xi_j} \vec{\xi}_i = G_{ik} \frac{\partial}{\partial \xi_i} R_{ij} \vec{\xi}_j$$

(Eq. 2-3-8)

Then the equation (2-3-5) takes the form of

$$\vec{\varepsilon} = \frac{1}{2} \left\{ \left( G_{ik} \frac{\partial}{\partial \xi_j} R_{ij} \vec{\varepsilon}_j \right) \left( L_m \frac{\partial}{\partial \xi_i} \vec{\varepsilon}_i \right) + \left( G_{ik} \frac{\partial}{\partial \xi_j} R_{ij} \vec{\varepsilon}_i \right) \left( L_m \frac{\partial}{\partial \xi_j} \vec{\varepsilon}_j \right) \right\}$$

$$= \frac{1}{2} \left\{ G_{ik} \frac{\partial}{\partial \xi_j} \left( L_m \frac{\partial}{\partial \xi_i} \vec{\varepsilon}_i \vec{\varepsilon}_j \right) + G_{ik} \frac{\partial}{\partial \xi_j} \left( L_m \frac{\partial}{\partial \xi_j} \vec{\varepsilon}_i \vec{\varepsilon}_j \right) \right\}$$

(Eq. 2-3-9)

Therefore the components of strains based upon $\xi^d$ coordinates can be expressed by the quantities described by $\xi^d$ coordinates.

This implies that the element stiffness matrix is invariant with respect to rotation.

It is noted that the invariance with respect to translational change in coordinates, $\xi^d$, immediately follows from the use of natural coordinate system to interpolate various physical quantities.
We now consider that the following versions of Hellinger-Reissner principle are employed;

\[ \mathit{\mathfrak{C}}_{\mathfrak{R}}^{\mathfrak{S}} = \int_V \left[ -\frac{1}{2} \mathfrak{S} : \mathfrak{S} + \mathfrak{C} : \frac{1}{2} (\mathfrak{V} \mathfrak{Y} + \mathfrak{Y} \mathfrak{V}) - (\mathfrak{V} \cdot \mathfrak{S} \cdot \mathfrak{V} \cdot \mathfrak{S}) \cdot \mathfrak{Y} \right] dV \]  
(2-3-10)

for \( \mathfrak{S} \) coordinates

\[ \mathit{\mathfrak{C}}_{\mathfrak{R}}^{\mathfrak{S}} = \int_V \left[ -\frac{1}{2} \mathfrak{S} : \mathfrak{S} + \mathfrak{C} : \frac{1}{2} (\mathfrak{V} \mathfrak{Y} + \mathfrak{Y} \mathfrak{V}) - (\mathfrak{V} \cdot \mathfrak{S} \cdot \mathfrak{V} \cdot \mathfrak{S}) \cdot \mathfrak{Y} \right] dV \]  
(2-3-11)

for \( \mathfrak{S} \) coordinates

and complete polynomials are used for stress distributions. We then examine the condition for invariance with respect to rotation.

First of all, since we use complete polynomials for stresses, it follows that

\[ \mathfrak{S} = \mathfrak{S}^{ij} \xi_i \xi_j \]
\[ = \mathfrak{S}^{ij} \pi_i \pi_j \xi_i \xi_j \]
\[ = \mathfrak{S}^{ij} \beta_i \beta_j \xi_i \xi_j \]
\[ \mathfrak{S}^{ij} = \mathfrak{S}^{ij} \beta_i \beta_j \]  
(2-3-12)

Therefore, as the consequence of eq. (2-3-9) and eq. (2-3-12) the first two terms in eqs. (2-3-10) and (2-3-11) inherently yields the same matrix formulation. As a final check, the third term of eq. (2-3-11) will be examined.

Using eqs. (2-3-8), (2-3-12), we can obtain

\[ \mathfrak{S} \cdot \mathfrak{S} = (G_{\mathfrak{S}} \frac{2}{\mathfrak{S}^{ij}} \pi_i \pi_j \mathfrak{S}^{ij} \frac{2}{\mathfrak{S}^{ij}} \beta_i \beta_j) \]
\[ = G_{\mathfrak{S}} \frac{2}{\mathfrak{S}^{ij}} (\pi_i \pi_j \beta_i \beta_j) \mathfrak{S}^{ij} \]  
(2-3-13)

For \( \mathfrak{S} \), we suppose that the interpolation function is given by

\[ \mathfrak{S}^{ij} = \xi_i \xi_j \]
\[ = \xi_i \xi_j \lambda_{ij} \xi_i \]
\[ = \xi_i \xi_j \lambda_{ij} \xi_i \]
\[ \lambda_{ij} = \rho_{kl} = \lambda_{kl} \]  
(2-3-14)

Then it is seen that
\[ (\mathbf{\varepsilon} \cdot \mathbf{\varepsilon}) \cdot \ddot{\mathbf{\omega}} = \varepsilon_{ik} \frac{\partial^2}{\partial \xi^2} \left( \rho \dot{p} \beta_r \beta_r \right) \ddot{\mathbf{\lambda}} \cdot \left( L_j^2 \rho \varepsilon \beta_r \dot{\mathbf{\lambda}} \right) \ddot{\mathbf{\lambda}}_i \]

\[ = \varepsilon_{ik} \frac{\partial^2}{\partial \xi^2} \rho \dot{p} \beta_r \beta_r L_j^2 \dot{\mathbf{\lambda}}_j \]

\[ = \varepsilon_{ik} \frac{\partial^2}{\partial \xi^2} \rho \dot{p} \beta_r \beta_r L_j^2 \dot{\mathbf{\lambda}}_j \]  

(2-3-15)

Since \( \beta_r \) are arbitrary stress parameters, eq. (2-3-15) becomes independent of rotation parameter \( \theta \).

Therefore, so long as we use natural coordinates system and complete polynomials for stresses, the stiffness matrix becomes invariant with respect to both translation and rotation.

The problem of the preservance of quality of a finite element under distortion concerns the part of stretch tensor components in the mapping representation, i.e., the Jacobian. This issue is discussed in chapter 5.
2.4 Rational way of obtaining the constraint equations

The introduction of $\mathbf{Y}_2$ leads to the constraint on the stress equilibrium equations. Physically $\mathbf{Y}_2$ are the internal displacement modes. Since the displacements $\mathbf{Y}_2$ are connected with physical nodal quantities, we can take any deformation modes for $\mathbf{Y}_2$. But when we choose lower deformation modes for $\mathbf{Y}_2$, the parameter $\lambda^2$ should become very small numbers. Therefore, $\mathbf{Y}_2$ should be higher order modes to have the physical significance in the finite element formulation.

Also such higher modes are preferred to give zero values at nodes of the finite element from the physical view point. At the same time, the choice of stress distributions should be consistent with $\mathbf{Y}_1$ and $\mathbf{Y}_2$. The consistency can be achieved by looking at the critical term in eq. (2-2-7);

$$I^* = \int_V (\mathbf{u}, \mathbf{q}) \cdot \mathbf{u}_2 dV = \int_V \mathbf{e}_2 : \dot{\mathbf{u}}_2^2 dV \tag{2-4-1}$$

As the result of the variation of $I^*$ with respect to the internal parameters, the constraint conditions can be obtained. But it is sometimes observed that redundant constraint conditions are obtained. In such cases, a geometric perturbation must be applied to the finite element.

When the geometry of one element is perturbed by the use of perturbation parameter $\Delta$, eq. (2-4-1) can be rewritten, in general, as

$$I^* = I_0^* + \Delta I_1^* + \Delta^2 I_2^* + \cdots + \Delta^n I_n^* \tag{2-4-2}$$

The variation of $I^*$ with respect to $\lambda_i$ ($i = 1, 2, \ldots, m$) should, then, be set to zero. We obtain

$$\delta I^* = \frac{\partial I^*}{\partial \lambda_i} \delta \lambda_i = 0 \quad ; \quad \frac{\partial I_1^*}{\partial \lambda_i} + \Delta \frac{\partial I_1^*}{\partial \lambda_i} + \Delta^2 \frac{\partial I_2^*}{\partial \lambda_i} + \cdots + \Delta^n \frac{\partial I_n^*}{\partial \lambda_i} = 0 \tag{2-4-3}$$

Since $\Delta$ is an arbitrarily small perturbation parameter, eq. (2-4-3) equivalently states that

$$\frac{\partial I_i^*}{\partial \lambda_i} = 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n) \tag{2-4-4}$$
At the present time, the use of MACSYMA enables us to obtain the explicit analytic forms of integrations which might include complicated expressions. Since the technique is available, the explicit expressions for the constraint equations which have the dependence of the element geometry specified by nodal coordinate values can be obtained. Then the constraint equations can be introduced by the use of the functional such as eq. (2-2-7). Or we can impose the constraint equations directly to the stress assumptions and use the original version of either Hellinger-Reissner principle or modified Hellinger-Reissner principle. The choice of the functional is totally dependent on the efficiency in the formulation and actual computation.

It should be emphasized that the complete satisfaction of equilibrium equation is not considered in general. What is carried out in this new formulation is the consistent introduction of physical deformation modes through the term given in eq. (2-4-1). The example will be given in the following chapters.
2.5 General procedure for the construction of hybrid stress elements

The general feature of hybrid stress elements is indicated in figure 4. As discussed in the previous sections, we should employ complete polynomials for any stress components so as to have the quality of invariance. The polynomials should be expressed in terms of natural coordinates. Because of the invariance, the specification of the geometry of a finite element can be done in terms of nodal coordinate values based upon any fixed global Cartesian coordinate system.

In short, we have the following expressions;

\[ \mathcal{L} = \mathcal{L}(\xi^i) = P_i(\xi^i) \mathcal{B} \]
\[ \mathcal{L} = \mathcal{N}(\xi^i) \mathcal{L}^0 \]

where

\[ \xi^i = \text{natural coordinate system} \]
\[ P_i(\xi^i) = \text{interpolation matrix for stresses in terms of complete polynomials} \]
\[ \mathcal{N}(\xi^i) = \text{shape functions} \]
\[ \mathcal{L}^0 = \text{position vectors at nodes in terms of Cartesian coordinate system} \]

Since in general the constitutive relation has the simplest form under Cartesian coordinate system, the use of stresses based upon such coordinate system is preferred. Also, the satisfaction of the patch test can be carried out most easily by using the stress expressions. Then the stress tensor has the form of

\[ \mathcal{L} = \sigma_{ij} \xi^i \xi^j = \sigma_{ij} \xi^i \xi^j \]

The constitutive relation may be described as

\[ \mathcal{L} = \mathcal{C} : \mathcal{E} \]
\[ \sigma_{ij} = C_{ijk\ell} \varepsilon_{k\ell} \]

where the strain tensor is given by

\[ \varepsilon = \varepsilon_{ij} \xi^i \xi^j \]
Since the expression for displacements is already determined by the geometry of a finite element, all we need is the expressions for \( \mathcal{U}_\lambda \) and additional physical constraint equations in general.

The choice of \( \mathcal{U}_\lambda \) may be determined by making the form of polynomials for \( \mathcal{U} \) complete because the completeness in polynomial expansion is desirable from the mathematical point of view. On the other hand, the additional physical constraint equations should be determined by the physical information that should be put into a finite element.

It should be stressed that the following equation which is necessary for the prevention of kinematic modes should be satisfied apriori;

\[
\mathcal{N}_g - \mathcal{N}_R \leq \mathcal{N}_\beta - \mathcal{N}_\lambda - \mathcal{N}_\mu \tag{2-5-5}
\]

where

\( \mathcal{N}_g \) = the number of nodal parameters
\( \mathcal{N}_R \) = the number of rigid body degrees of freedom
\( \mathcal{N}_\beta \) = the number of stress parameters
\( \mathcal{N}_\lambda \) = the number of internal displacement parameters
\( \mathcal{N}_\mu \) = the number of additional physical constraint equations on stresses.

In general, for the satisfaction of eq. (2-5-5) or for the prevention of kinematic modes we must add some higher order terms which would invalidate the completeness in stress assumptions. Then the invariant property might not be preserved. But we can expect that the invariant property may not be affected by the higher order terms.

Especially for higher order elements the inversion of \( \mathcal{H} \) matrix is time-consuming.
If the polynomial expansion for stresses is incomplete, the use of
Hu-Washizu principle or modified Hu-Washizu principle will reduce the
computation time for the inversion to a great extent because the inversion
is independent of the constitutive relation.

Anyway the formulation should be performed by use of the functional
which gives the most efficient computational procedure in individual
problem.
3. Thin shell elements

The basic idea of patch test is that any finite element should be consistent with the fundamental physical solutions as the element size becomes infinitesimal. Therefore the concept of patch test is based upon the physical characteristic solutions of a physical problem.

For thin shells, the fundamental physical characteristic solutions include

(i) rigid body movements
(ii) momentless membrane states
(iii) inextensional bending
(iv) boundary effects

In a displacement model, the displacement fields are usually described in terms of surface polynomial functions. In order to accommodate rigid body movements the displacements must often be properly expressed as transcendental functions in the shell surface coordinates. On the other hand, the inextensional bending modes should include polynomial expansion. Therefore in order to reach better approximations to both requirements, higher order interpolation functions must be employed. But in such a case large number of geometric parameters are required to describe each element. Also the use of higher derivatives in describing the displacement field makes the element unsatisfactory for problems where the shell surface is discontinuous, unless special treatments are adopted.

In the present investigation, more versatile methods to construct thin shell elements are sought based upon new versions of variational principles discussed in the previous chapter. For a well-balanced finite element for thin shells, the three displacement components should have the same order of polynomial functions from the physical viewpoint and to maintain the condition number of the stiffness
matrix as small as possible, [21]. On the other hand, one of the major
difficulties in shell analysis is the treatment of junction between two
surfaces. Those requirements can be satisfied by introducing the so called
"semiLoof" element initiated by Irons. The use in the formulation of
hybrid semiLoof elements for plate and shell was pointed out by Pian [12].
In this chapter, the construction of hybrid semiLoof shell elements is
investigated. It is noted that the tensor components are extensively
employed instead of the dyadic notations because they are more suitable
for systematic finite element formulations.

In this chapter, the formulations for plates and shallow shells will
first be discussed. A study of general shell elements is then followed.
Since some desirable qualities of shell elements can be demonstrated only
in large deflection solutions, it is decided to include a short investigation
of large deflection analysis by stationary Lagrangian formulation only.
Example solutions are given at the end of the chapter.
3.1 Hybrid semi-loof elements for plates and shallow shells

3.1.1 Modified Hu-Washizu principle for shallow shells

In finite element formulations of 3-dimensional elastic solids, when the displacements $\mathbf{u}_i$ in an element is not compatible with the boundary displacements $\mathbf{U}_i$, the modified Hu-Washizu principle for one element takes the form

$$\mathcal{W}_{HG} = \int_{\Omega} \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} \sigma_{ij} (u_{ij} + u_{ji}) - \mathbf{F}_i \cdot \mathbf{u}_i \right] dv$$
$$- \int_{\partial\Omega} T_i (u_i - U_i) ds - \int_{S_{\sigma}} T_i \mathbf{U}_i \cdot ds$$

(3-1-1)

where $i,j,k,l = 1,2,3$

$\sigma_{ij}$ = stresses
$\varepsilon_{ij}$ = strains
$C_{ijkl}$ = elastic coefficients
$\mathbf{F}_i$ = body forces
$T_i$ = boundary traction related to $\sigma_{ij}$
$\Omega$ = volume of the element
$\partial\Omega$ = entire boundary of the element
$S_{\sigma}$ = boundary with prescribed traction

and the independent quantities are $\sigma_{ij}, \varepsilon_{ij}, u_i$ and $\mathbf{U}_i$.

In the present formulation we separate element displacements into two parts, the incompatible displacements $\hat{\mathbf{u}}_i$ which are expressed in terms of the nodal displacements $\mathbf{d}$ and the additional part $\mathbf{u}_i^2$ which are expressed in terms of internal displacement parameters $\lambda^2$ that can be statically condensed in the element level.
Here \( U_i^2 \) may be incompatible along the boundary or it may be bubble functions which are zero along the boundary.

By realizing that
\[
\int_V \frac{1}{2} \sigma_{ij} (u_{ij} + u_{ji}) dV = -\int_V \sigma_{ij} u_i dV + \int_{\partial V} T_i u_i dS
\]

where \( T_i = \lambda \sigma_{ij}, \lambda, \lambda \) being the directional cosine, Eq. (3-1-1) becomes
\[
\tau_{HG}^* = \int_V \left[ \frac{1}{2} C_{ijkl} E_{ij} E_{kl} - \sigma_{ij} E_{ij} - (\sigma_{ij} u_i + F_i) \tilde{u}_i - (\sigma_{ij} u_i + F_i) \tilde{u}_i + \lambda \right] dV
\]
\[
+ \int_{\partial V} T_i \tilde{u}_i dS - \int_{S_o} T_i \tilde{u}_i dS
\]

Since the equation
\[
\sigma_{ij}, u_i + F_i = 0
\]
represents the stress equilibrium condition, the last term in the volume integral in Eq. (3-1-2) actually plays the role of the conditions of constraint and corresponding Lagrange multipliers. Here by proper choice of \( U_i^2 \), the equilibrium equations can be identically satisfied by the variational process, and we may write
\[
\tau_{HG}^{**} = \int_V \left[ \frac{1}{2} C_{ijkl} E_{ij} E_{kl} - \sigma_{ij} E_{ij} - \sigma_{ij} \tilde{u}_i - F_i \tilde{u}_i \right] dV
\]
\[
+ \int_{\partial V} T_i \tilde{u}_i dS - \int_{S_o} T_i \tilde{u}_i dS
\]

For Marguerre's shallow shell theory the functional which corresponds to Eq. (3-1-2) is of the following form:
\[
\tau_{HG}^* = \int_A \left[ \phi (E_{ij}, K_{ij}) - (N_{ij} E_{ij} + M_{ij} K_{ij})
- (N_{ij} E_{ij} + F_{ij}) \tilde{u}_j - (S_{ij} \tilde{w} + F_3) \tilde{w} - (N_{ij} E_{ij} + F_{ij}) u^2_i - (S_{ij} \tilde{w} + F_3) w^2 \right] dA
+ \int_{S_A} \left[ N_{ij} \tilde{u}_i E_{ij} + S_{ij} \tilde{w} \right] dS
- \int_{S_o} \left[ N_{ij} \tilde{u}_i E_{ij} + S_{ij} \tilde{w} \right] dS
where $\beta, \beta' = 1, 2$

$$\frac{\partial \Phi}{\partial \varepsilon_{\alpha\beta}} = N_{\alpha\beta}, \quad \frac{\partial \Phi}{\partial \kappa_{\alpha\beta}} = M_{\alpha\beta}$$

$$N_{\alpha\beta} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} dS = \text{membrane stress resultants}$$

$$M_{\alpha\beta} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} S dS = \text{stress couples}$$

$$Q_{\alpha} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} dS = M_{\alpha\beta, \beta}$$

$$S_{\alpha} \equiv Q_{\alpha} + N_{\alpha\beta} \gamma_{\alpha\beta}$$

$Z = \text{coordinate of shell mid-surface with respect to the base plane}$

$\varepsilon_{\alpha\beta} = \text{strains in mid-surface}$

$\kappa_{\alpha\beta} = \text{curvatures}$

$\gamma_{\alpha} = \text{outward normals}$

$A = \text{mid-surface of the element}$

$\partial A = \text{entire boundary of mid-surface of the element}$

$s_{\alpha} = \text{boundary with prescribed traction}$

$\hat{u}_{\alpha} = \text{inplane displacements expressed in terms of } \varphi$

$\lambda_{\alpha} = \text{inplane displacements expressed in terms of } \lambda$

$\hat{w} = \text{lateral deflection expressed in terms of } \varphi$

$\lambda_{\alpha} = \text{lateral deflection expressed in terms of } \lambda$

$\tilde{u}_{\alpha} = \text{inplane boundary displacements}$

$\tilde{w} = \text{lateral deflection at boundary}$
and it is noted that the independent quantities in $\mathcal{H}_H$ of Eq. (3-1-5) are $N_{\alpha\beta}, M_{\alpha\beta}, E_{\alpha\beta}, K_{\alpha\beta}, u^\alpha, \tilde{u}_\alpha$ and $w$.

With an appropriate choice of $\gamma^\lambda$ and $w^\lambda$ in Eq. (3-1-5), the equilibrium equations

$$N_{\alpha\beta, \beta} + F_{\alpha} = 0 \quad (3-1-6)$$

$$M_{\alpha\beta, \alpha} + (N_{\alpha\beta} Z_{\alpha\beta})_{,\alpha} + F_{\beta} = 0 \quad (3-1-7)$$

can be satisfied. But, in general, the exact satisfaction of Eq. (3-1-7) is difficult owing to the presence of coupling terms. Therefore, for the finite element formulation it is better to employ Eq. (3-1-2) directly.

It is noted that the introduction of a state function defined by

$$B(N_{\alpha\beta}, M_{\alpha\beta}) = (N_{\alpha\beta} E_{\alpha\beta} + M_{\alpha\beta} K_{\alpha\beta}) - \bar{\Phi}(E_{\alpha\beta}, K_{\alpha\beta}) \quad (3-1-8)$$

generates the functional for a modified Hellinger-Reissner principle in which the independent quantities are $N_{\alpha\beta}, M_{\alpha\beta}, u^\alpha, \tilde{u}_\alpha, w^\lambda$ and $\tilde{w}$.

If the stress equilibrium equations are satisfied completely, the functional of the modified Hellinger-Reissner principle reduces to the functional of modified complementary energy principle.

3-1-2, **Finite element formulation of modified Hu-Washizu principle for shallow shells**

For convenience matrix notation is used to replace the indicial notation. In the new formulation indicated in the previous section, the stress components are assumed to be uncoupled. They are represented by unknown stress parameters in the following manner:
\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \begin{bmatrix}
P_{Nxx} & P_{Nxy} & 0 \\
0 & P_{Nyy} & 0 \\
0 & 0 & P_{Nzz}
\end{bmatrix}
\begin{bmatrix}
\beta_x \\
\beta_y \\
\beta_{xy}
\end{bmatrix}
\]

(3-1-9)

The corresponding boundary tractions are expressed as

\[
\begin{bmatrix}
N_x \\
N_y \\
S_x \\
M_x \\
M_y
\end{bmatrix}
= \begin{bmatrix}
P_{Nxx} & P_{Nxy} & 0 & 0 & 0 \\
0 & P_{Nyy} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\beta_x \\
\beta_y \\
\beta_z \\
\beta_{xy} \\
\beta_{xz}
\end{bmatrix}
\]

(3-1-10)

The key step in the finite element formulation of the modified Hu-Washizu principle is that both stresses and strains are approximated by the same interpolation functions, i.e.
\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} = \begin{bmatrix}
P_{xx} & 0 & 0 \\
0 & P_{yy} & 0 \\
0 & 0 & P_{xy}
\end{bmatrix} \alpha
\] (3-1-11)

where \( P_{x_{i\beta}} = P_{N_{i\beta}} \) and \( P_{y_{i\beta}} = P_{M_{i\beta}} \).

The displacements on the interior of an element, \( \hat{u}, \hat{w}, \hat{v}, \hat{u}^\lambda \) and \( \hat{w}^\lambda \), are interpolated in terms of the nodal displacements \( \xi \) and inner parameters \( \lambda \);\n
\[
\hat{u} = \begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{w}
\end{bmatrix} = \begin{bmatrix}
xu \\
vu \\
wv
\end{bmatrix}\xi = L\xi
\] (3-1-12)

and

\[
\hat{u}^\lambda = \begin{bmatrix}
\hat{u}^\lambda \\
\hat{v}^\lambda \\
\hat{w}^\lambda
\end{bmatrix} = \begin{bmatrix}
Nu \\
Lv \\
0
\end{bmatrix}\lambda = N\lambda
\] (3-1-13)

The boundary displacement \( \tilde{u} \) are interpolated with respect to the same set of nodal displacements in \( \hat{u} \),
\[
\begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w} \\
-\tilde{w}_x \\
-\tilde{w}_y
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{u}_x \\
\tilde{v}_y \\
\tilde{w}_x \\
-\tilde{w}_y \\
-\tilde{w}_x
\end{bmatrix}
\quad \mathbf{Q} = \mathbf{\tilde{u}} \mathbf{g}
\]  

(3-1-14)

Using Eq. (3-1-9), the homogeneous part of equilibrium equations given by Eqs. (3-1-6) and (3-1-7) may be written as

\[
\mathbf{F} \beta = 0
\]

(3-1-15)

In terms of matrix notations, the functional \( \mathcal{N}_{HG}^* \) can be written in the form:

\[
\mathcal{N}_{HG}^* = \frac{1}{2} \mathbf{q}^T \mathbf{J} \mathbf{q} - \mathbf{\beta}^T \mathbf{H} \mathbf{\beta} - \mathbf{\beta}^T \mathbf{G} \mathbf{\tilde{x}} - \mathbf{\beta}^T \mathbf{H} \mathbf{\tilde{x}} - \mathbf{\beta}^T \mathbf{G} \mathbf{\tilde{Q}} - \mathbf{\beta}^T \mathbf{G} \mathbf{\tilde{Q}}
\]

(3-1-16)

where

\[
\mathbf{J} = \int_A \mathbf{P}^T \mathbf{C} \mathbf{P} \ dA
\]

\( \mathbf{C} = \) elastic coefficient matrix

\[
\mathbf{H} = \int_A \mathbf{P}^T \mathbf{H} \mathbf{P} \ dA
\]

\[
\mathbf{G} = \int_A \mathbf{E}^T \mathbf{G} \ dA
\]

(3-1-17)
\[ \tilde{R}^2 = \int_A \tilde{E}^T \tilde{N} dA \]

\[ \tilde{\varrho} = \int_{\partial A} \tilde{R}^T \tilde{\gamma} ds \]

\[ \hat{\varrho} = \text{load vector due to body forces and resulting from } \hat{\gamma} \]

\[ \hat{\varrho}^2 = \text{load vector due to body forces and resulting from } \hat{\gamma}^2 \]

\[ \overline{\varrho} = \text{load vector due to prescribed tractions from } \overline{\gamma} \]

From the first variation of \( \mathcal{N}_{HG}^* \) with respect to \( \varrho, \beta \) and \( \lambda \), we obtain

\[ \beta = \mathcal{H}^{-1} \mathcal{J} \varrho \]  
(3-1-18)

\[ \varrho = \mathcal{H}^{-1} \{ (G-\hat{G}) \beta - \tilde{R}^2 \lambda \} \]  
(3-1-19)

and

\[ \tilde{R}^2 \beta + \hat{\varrho}^2 = \overline{\varrho} \]  
(3-1-20)

By eliminating \( \lambda \) and \( \beta \) and substituting \( \beta \) into \( \mathcal{N}_{HG}^* \) it follows that

\[ \mathcal{N}_{HG}^* = \frac{1}{2} \int \mathcal{K} \mathcal{Q}^T - \mathcal{Q}^T \mathcal{Q} \]  
(3-1-21)

where an element stiffness matrix \( \mathcal{K} \) and an element external load vector \( \mathcal{Q} \) are given by

\[ \mathcal{K} = (G-\hat{G})^T \mathcal{M} (G-\hat{G}) - (G-\hat{G})^T \mathcal{M} \mathcal{B} (\tilde{R}^T \mathcal{M} \mathcal{B})^{-1} \mathcal{M} (G-\hat{G}) \]  
(3-1-22)

and
\[ Q = \tilde{Q} + \hat{Q} + (G - \hat{G})^T M R (R^T M R)^{-1} \tilde{Q} \]

respectively and

\[ \tilde{M} = H^T J H^{-1} \]

It is well known that the inversion of the \( \tilde{H} \) matrix takes a considerable amount of computation time in the finite element construction by the conventional assumed stress hybrid model. But, by using the modified Hu-Washizu principle \( \tilde{\pi}_{ HW} \) with uncoupled stress assumption, the inversion of the \( \tilde{H} \) matrix can be done more efficiently, because the \( \tilde{H} \) matrix has the form of

\[
\tilde{H} = \int_A T^T \tilde{T} dA = 
\begin{bmatrix}
\tilde{H}_{Nx} & \tilde{H}_{Ny} & \tilde{H}_{Nxy} \\
\tilde{H}_{Ny} & \tilde{H}_{Mx} & \tilde{H}_{My} \\
\tilde{H}_{Nxy} & \tilde{H}_{My} & \tilde{H}_{Mxy}
\end{bmatrix}
\]

(3-1-25)

where

\[
\tilde{H}_{N\alpha} = \int_A T^T P_{N\alpha} P_{N\alpha} dA \quad \text{and} \quad \tilde{H}_{M\alpha\beta} = \int_A P^T_{M\alpha\beta} P_{M\alpha\beta} dA.
\]

(\( \alpha, \beta \) not summed)

It should be remarked that in the new formulation it is also required to invert the matrix \( R^T M R \). Thus, such inversion should be considered as a primary governing factor of the computational time for the generation of the element stiffness matrix.
It should be pointed out, furthermore, that the use of a natural coordinate system for stress assumptions leads to more efficient method for constructing the element stiffness matrix. Natural coordinate system means area coordinate system \((\xi_1, \xi_2, \xi_3)\) for triangular element and \(\xi-\eta\) coordinate system for quadrilateral element. For convenience, \(\xi_1\) and \(\xi_2\) are employed as the representation of natural coordinate system in place of \((\xi_1, \xi_2, \xi_3)\) or \((\xi, \eta)\). In using a natural coordinate system, the surface area \(dA\) should be changed to

\[
dA = |J| d\xi_1 d\xi_2
\]  
(3-1-26)

where \(|J|\) indicates the Jacobian determinant.

In this case the strains and stresses may be expressed as

\[
\varepsilon = \mathbf{P} (\xi_1, \xi_2) \varepsilon
\]
(3-1-27)

\[
|J| \varepsilon = \mathbf{P} (\xi_1, \xi_2) \varepsilon
\]

Here the interpolation functions are no longer the same because \(|J|\) is generally the function of \(\xi_1\) and \(\xi_2\). With this formulation, it can be seen that \(\mathcal{J}\) and \(\mathcal{H}\) in Eq. (3-1-17) becomes

\[
\mathcal{J} = \int_A \frac{1}{|J|} \mathbf{P}^T (\xi_1, \xi_2) \mathcal{E} \mathbf{P}(\xi_1, \xi_2) d\xi_1 d\xi_2
\]  
(3-1-28)

and

\[
\mathcal{H} = \int_A \mathcal{E} \mathbf{P}^T (\xi_1, \xi_2) \mathbf{P}(\xi_1, \xi_2) d\xi_1 d\xi_2
\]  
(3-1-29)

Since the natural coordinate system is inherent in an element itself, the \(\mathcal{H}\) matrix of Eq. (3-1-29) is totally independent of the shape of an element. Hence, the \(\mathcal{H}\) matrix for \(\mathbf{\mathcal{T}}_{\text{HA}}^{*}\) is required to be inverted only once for all elements. But, as can be seen in Eq. (3-1-28), \(|J|\) appears in the denominator in the integrand in \(\mathcal{J}\). This gives rise to a certain amount of numerical errors. Also, the constitutive equations cannot be completely
satisfied from the variational procedure on account of the presence of \(|J|\). Therefore, it is clear that the formulation by Eqs. (3-1-27) should be limited to element shapes which are only slightly distorted from a rectangular shape.

Another problem which emerges in using the natural coordinate system is related to the satisfaction of equilibrium equations. Since the equilibrium equations of Eqs. (3-1-6) and (3-1-7) are expressed in terms of the x-y coordinate system, the derivatives such as \(\frac{\partial}{\partial x}\) and \(\frac{\partial^2}{\partial x \partial y}\) should be converted to the derivatives with respect to \(\xi_1\) and \(\xi_2\). It is obvious that it is again impossible, in general, to fulfill the satisfaction of the equilibrium equations. Nevertheless, in the variational formulation the stress equilibrium condition is not required to be satisfied exactly.

3-1-3, **Assumed functions for stresses and displacements for hybrid semiLoof shallow shell elements**

Choices of assumed stresses for a 24-DOF triangular shell element and a 32-DOF quadrilateral element shown in Figures 5 are arrived at through some trial process. Basically, from the following stress vs. stress-function relations for shallow shells which are obtained through the so-called static geometric analogy

\[
N_x = -\frac{\partial^2 w}{\partial y^2}, \quad N_y = -\frac{\partial^2 w}{\partial x^2}, \quad N_{xy} = \frac{\partial^2 w}{\partial x \partial y}
\]

\[
M_x = -\frac{\partial V}{\partial y} + W \frac{\partial^2 z}{\partial y^2}, \quad M_y = -\frac{\partial U}{\partial x} + W \frac{\partial^2 z}{\partial x^2}, \quad M_{xy} = \frac{1}{2} \left[ \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial V}{\partial x} \right] - W \frac{\partial^2 z}{\partial x \partial y}
\]
where \( U, V \) and \( W \) are stress functions, the required terms of membrane and moment stresses for equilibrium conditions can be determined. The stresses included are adequate to prevent any kinematic deformation modes and the symmetry conditions are satisfied. However, the completeness in all orders of polynomial expansions is not satisfied. The resulting choices are a 37\( \beta \) triangular element and a 59\( \beta \) quadrilateral element.

The coordinate \( z \) of the mid-surface of the shallow shell is interpolated in terms of the nodal values of \( z \) in terms of natural coordinates. For these two elements the interpolation functions are:

**24 DOF Triangular Element**

\[
Z(\xi_1, \xi_2, \xi_3) = \mathbf{L}_Z \mathbf{Z}^0
\]  
(3-1-31)

where

\[
\mathbf{Z}^0 = \left\{ \begin{array}{c} z_1 z_2 z_3 z_4 z_5 z_6 \end{array} \right\}^T
\]

and

\[
\mathbf{L}_Z = \left\{ \begin{array}{c} \xi_1(2\xi_1-1) \xi_2(2\xi_2-1) \xi_3(2\xi_3-1) 4\xi_1\xi_2 4\xi_2\xi_3 4\xi_3\xi_1 \end{array} \right\}
\]  
(3-1-32)

**32 DOF Quadrilateral Element**

\[
Z(\xi, \eta) = \mathbf{L}_Z \mathbf{Z}^0
\]  
(3-1-33)

where

\[
\mathbf{Z}^0 = \left\{ \begin{array}{c} z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 \end{array} \right\}^T
\]

and

\[
\left( \mathbf{L}_Z \right)_i = \frac{1}{4} (1+\xi_i \xi)(1+\eta_i \eta)(\xi \xi_i + \eta \eta_i - 1) \quad ; \quad i = 1, 4
\]

\[
= \frac{1}{2} (1-\xi^2)(1+\eta \eta) \quad ; \quad i = 5, 7
\]

\[
= \frac{1}{2} (1-\eta^2)(1+\xi \xi) \quad ; \quad i = 6, 8
\]  
(3-1-34)
The interpolation functions for displacements \( \tilde{u} \), \( \tilde{v} \), and \( \tilde{w} \) are represented as follows:

### 24 DOF Triangular Element

\[
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{w}
\end{bmatrix} =
\begin{bmatrix}
L_{0} & 0 \\
L_{0} & 0 \\
L_{0} & 0
\end{bmatrix}
\begin{bmatrix}
\gamma \\
\xi
\end{bmatrix}
\tag{3-1-35}
\]

where

\( L_{0} = L_{z} \) given by Eq. (3-1-32)

and

\( \gamma = [ u_{1}, \ldots, u_{6}, v_{1}, \ldots, v_{6}, w_{1}, \ldots, w_{6}, (w_{3}n)_{1}, \ldots, (w_{3}n)_{6}]^{T} \)

Considering only one edge for \( \tilde{u} \), we can write

\[
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{w}
\end{bmatrix} =
\begin{bmatrix}
\tilde{L}_{0} \\
\tilde{L}_{1}
\end{bmatrix}
\begin{bmatrix}
\tilde{\gamma} \\
\tilde{\xi}
\end{bmatrix}
\tag{3-1-36}
\]

where

\( \tilde{\gamma} = [ u_{1}, u_{2}, u_{3}, v_{1}, v_{2}, v_{3}, w_{1}, w_{2}, w_{3}, (w_{3}n)_{a}, (w_{3}n)_{b}] \)

and

\[
\tilde{L}_{0} = \begin{bmatrix}
2s'^2 - 3s' + 1, -4s'^2 + 4s', 2s'^2 - s'
\end{bmatrix}
\]

\[
\tilde{L}_{1} = \begin{bmatrix}
2 - 3s', -1 + 3s'
\end{bmatrix}
\]

\( s' = \frac{s}{l} \) and \( l = \) length of the edge

It is noted that the locations of Loof nodes in hybrid semiLoof elements presented in this paper are \( s = \frac{L}{3} \) unlike the SemiLoof element of Irons in which Loof nodes are located at Gaussian integration points.

For \( \tilde{u}^{\lambda} \), we have

\[
\begin{bmatrix}
\tilde{u}^{\lambda} \\
\tilde{v}^{\lambda} \\
\tilde{w}^{\lambda}
\end{bmatrix} =
\begin{bmatrix}
L_{0}^{\lambda} & 0 \\
L_{0}^{\lambda} & 0 \\
L_{0}^{\lambda} & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\xi
\end{bmatrix}
\tag{3-1-38}
\]
where \[ \lambda = [ \lambda_1 \lambda_2 \ldots \lambda_{12}]^T \]

\[ L_0^\lambda = \xi_1 \xi_2 \xi_3 \cdot [ \xi_1 \xi_2 \xi_3] \]  \hspace{1cm} (3-1-39)

and

\[ L_1^\lambda = \xi_1 \xi_2 \xi_3 \cdot [ \xi_1 \xi_2 \xi_3 \xi_1^2 \xi_2^2 \xi_3^2] \]

32 DOF Quadrilateral Element

\[
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{w}
\end{bmatrix}
= \begin{bmatrix}
L_0^\lambda & 0 \\
0 & L_0^\lambda \\
0 & L_0^\lambda
\end{bmatrix}
\varphi
\]  \hspace{1cm} (3-1-40)

where \[ L_0^\lambda = \frac{1}{2} \] given by Eq. (3-1-34)

and

\[ \varphi = [ u_1 \ldots u_8 , v_1 \ldots v_8 , w_1 \ldots w_8 , (w_5 n)_1 \ldots (w_5 n)_8 ]^T \]

As for \( \gamma \), the interpolation is performed with the same manner as described before.

For \( u^\lambda \), we have

\[
\begin{bmatrix}
u^\lambda \\
v^\lambda \\
w^\lambda
\end{bmatrix}
= \begin{bmatrix}
L_0^\lambda & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\lambda
\]  \hspace{1cm} (3-1-41)

where \[ \lambda = [ \lambda_1 \lambda_2 \ldots \lambda_{22}]^T \]

\[ L_0^\lambda = (1 - \xi^2)(1 - \eta^2) \cdot [ 1 \xi \eta \xi^2 \xi \eta \eta^2 ] \]  \hspace{1cm} (3-1-42)

and

\[ L_1^\lambda = (1 - \xi^2)(1 - \eta^2) \cdot [ 1 \xi \eta \xi^2 \xi \eta \eta^2 \eta^3 \xi^2 \eta^2 \eta^3 ] \]
As shown in Eqs. (3-1-39) and (3-1-42), bubble functions are employed for $Y^\lambda$. Since the Lagrange multipliers method simply plays a role of introducing the condition of constraint on the equilibrium equations, the choice of the assumed functions for $Y^\lambda$ may be quite arbitrary so far as the construction of the element stiffness matrix is concerned. But when we consider the consistent load vector, geometric nonlinear stiffness matrix, etc., the physical implication of $Y^\lambda$ should be taken into consideration. Then the use of bubble functions for $Y^\lambda$ is preferred.

The stress assumptions are given as follows:

24-DOF Triangular Element with 37 $\beta$'s

by $x,y$ coordinate system

$$
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix} P_0 \\ P_0 \\ P_0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{18} \end{bmatrix} ;
\text{ } \text{ } P_0 = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}
$$

$$
M_x = \beta_{19} + \beta_{20} x + \beta_{21} y + \beta_{22} xy + \beta_{23} x^2 + \beta_{24} x^3 + \beta_{25} x^2 y^2
$$

$$
M_y = \beta_{26} + \beta_{27} x + \beta_{28} y + \beta_{29} xy + \beta_{30} y^2 + \beta_{31} y^3 + \beta_{32} x^2 y^2
$$

$$
M_{xy} = \beta_{33} + \beta_{34} x + \beta_{35} y + \beta_{36} xy + \beta_{37} x^2 y^2
$$

or by area coordinate system

$$
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix} \tilde{P}_0 \\ \tilde{P}_0 \\ \tilde{P}_0 \end{bmatrix} \begin{bmatrix} \tilde{\beta}_1 \\ \vdots \\ \tilde{\beta}_{18} \end{bmatrix} ;
\text{ } \text{ } \tilde{P}_0 = \begin{bmatrix} \tilde{\xi}_1 & \tilde{\xi}_2 & \tilde{\xi}_3 & \tilde{\xi}_1^2 & \tilde{\xi}_2 & \tilde{\xi}_3^2 \end{bmatrix}
$$
\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
\sim \\
\sim \\
\sim 
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
\beta_{37}
\end{bmatrix}
\tag{3-1-46}
\]

where
\[
P_1 = \begin{bmatrix}
\xi_1 & \xi_2 & \xi_3 & \xi_2^2 & \xi_1 \xi_2 & \xi_2^2 \xi_3 & \xi_2 \xi_3^2
\end{bmatrix}
\]

and
\[
P_2 = \begin{bmatrix}
\xi_1 & \xi_2 & \xi_3 & \xi_2^2 & \xi_2 \xi_3^2
\end{bmatrix}
\]

32 DOF Quadrilateral Element with 59 \(\beta\)'s

By \(\xi-\eta\) coordinate system

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix}
P_0 \\
\sim \\
P_0
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_{30}
\end{bmatrix} \quad ; \quad P_0 = \begin{bmatrix}
1 & \xi & \xi^2 & \xi \eta & \eta & \eta^2 & \eta^3 & \xi \xi^2 & \xi \eta^2 & \xi^2 \eta & \xi \eta^3
\end{bmatrix}
\tag{3-1-47}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
P_1 \\
\sim \\
P_2
\end{bmatrix}
\begin{bmatrix}
\beta_{31} \\
\vdots \\
\beta_{59}
\end{bmatrix}
\tag{3-1-48}
\]

where
\[
P_1 = \begin{bmatrix}
1 & \xi & \xi^2 & \xi \eta & \eta & \eta^2 & \eta^3 & \xi \eta^2 & \xi \xi^2 & \xi \eta^3 & \xi^2 \eta & \xi^2 \eta^2 & \xi^2 \eta^3
\end{bmatrix}
\]

and
\[
P_2 = \begin{bmatrix}
1 & \xi & \xi^2 & \xi \eta & \eta \xi^2 & \xi \eta^2
\end{bmatrix}
\]
3.2 Hybrid semi-loof general shell element formulation

The general shell element formulation is established by the use of natural coordinate system. The formulation is different from the standard shell theory. By the use of natural coordinates system, the order estimation of each term in one equation can be facilitated.

3.2.1 Basic equations for thin shells based upon natural coordinate system

Representation of position vector

In the present formulation, "natural coordinate system" is used to describe the shell structure. In order to reduce the general three dimensional theory of elasticity to shell theory in terms of natural coordinate system, one should employ three different coordinate systems (figure 11);

1. global coordinate system
2. shell coordinate system
3. natural coordinate system

The relationship between global coordinate system and shell coordinate system is given by

\[ \mathbf{\gamma} = \mathbf{\gamma}_0 + \mathbf{\delta} \mathbf{\gamma} = \mathbf{X}_i \mathbf{\gamma}_i + \mathbf{\delta} \mathbf{\gamma} \quad (i = 1, 2, 3) \]  
(3.2-1)

\[ X_i = f_i^\alpha (\mathbf{\gamma}_\alpha) \quad (\alpha = 1, 2) \]  
(3.2-2)

where

- \( \mathbf{\gamma} \) = position vector of arbitrary point of shell
- \( \mathbf{\gamma}_0 \) = position vector of mid-surface of shell
- \( X_i \) = global coordinate system

\[ \xi_i = \text{orthonormal basis vector of global coordinate system} \]
\[ \xi^d \text{, } \xi = \text{shell coordinate system} \]
\[ \mathcal{F}^d \text{, } \mathcal{F} = \text{given functions to specify shell geometry} \]
and \[ \mathcal{N} = \text{unit normal vector of mid-surface} \]

On the other hand, the relationship between shell coordinate system and natural coordinate system can be written as
\[ \xi^d = \mathcal{L}_i (\xi^d) \xi_i \quad (\alpha, \beta = 1, 2; i = 1, 2, \ldots, n) \]
\[ \xi = \frac{1}{2} \mathcal{N} (\xi^d) \xi^3 \]  
(3-2-3)

where
- \[ \xi_i = \text{natural coordinate system} \]
- \[ \xi_{d+} = \text{shell coordinate values of mid-surface corresponding to node} \]
- \[ \mathcal{L}_i = \text{interpolation functions} \]
- \[ n = \text{number of nodes of one element} \]
- \[ \mathcal{N} = \text{thickness of shell} \]

and \[ \mathcal{N} \] will be assumed to be constant for the sake of simplicity. Substituting eq. (3-2-3) into eqs. (3-2-1) and (3-2-2), the relationship between global coordinate system and natural coordinate system can be obtained in the following form;
\[ x^i = \mathcal{F}_i (\xi^d) \xi^i \equiv \mathcal{F}_i (\xi^d) \]  
(3-2-4)
\[ \mathcal{L} = \mathcal{F}_i (\xi^d) \xi_i + \frac{1}{2} \mathcal{N} \xi^3 \mathcal{N} (\xi^d) \]  
(3-2-5)

**Transformation of various tensor components**

The transformation between covariant base vectors \[ \mathcal{Q}_d \] and \[ \mathcal{N} \] for shell coordinate system and the base vectors \[ \xi_i \] for natural coordinate system can be written as
\[ \xi^d = \frac{\partial \xi^d}{\partial \xi^d} \xi^d = \left[ \frac{\partial \xi^d}{\partial \xi^d} \mathcal{L}_i (\xi^d) \xi^i \right] \xi^d \equiv \mathcal{J}_d \xi^d \]
\[ \xi^3 = \frac{1}{2} \mathcal{N} \]  
(3-2-6)
On the other hand, the transformation between contravariant base vectors $\xi^\alpha$ and $\xi^\gamma$ for shell coordinate system and the base vectors $\tilde{\xi}^i$, 

$$\xi^\alpha = \frac{\partial \xi^\gamma}{\partial \tilde{\xi}^i} \tilde{\xi}^i \equiv J_{\alpha}^r \tilde{\xi}^r$$

$$\xi^3 = \frac{2}{R} \xi^r$$

The fundamental metric tensor components $\xi_{ij}$, $\xi_{ij}^r$, $\xi_{ij}^r$ and $\xi_{ij}^r$, can be obtained as

$$\xi_{ij} = \xi_{ij} = (J_{\alpha}^r J_{\beta}^s) = J_{\alpha}^r J_{\beta}^s a_{rs}$$

$$\xi_{33} = \xi_{33} = \xi_{3d} = \xi_{d3} = 0$$

$$\xi_{ij} = \frac{\xi_{ij}}{\xi^{\alpha}_{ij}} = J_{\alpha}^r J_{\beta}^s a_{rs}$$

$$\xi_{33} = \xi_{33} = \xi_{3d} = \xi_{d3} = 0$$

$$\xi_{ij} = \frac{\xi_{ij}}{\xi^{\alpha}_{ij}} = J_{\alpha}^r J_{\beta}^s a_{rs}$$

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$$\xi_{ij} = \frac{\xi_{ij}}{\xi^{\alpha}_{ij}} = J_{\alpha}^r J_{\beta}^s a_{rs}$$

It is noted that the tensor components do not, in general, all have the same dimensions. In applications, it is often preferable to work with the so-called physical components. As can be observed in eq. (3-2-8), the metrics have the same dimensions because the natural coordinate system is non-dimensional one.

The transformations of various tensor components follow immediately from eqs. (3-2-6) and (3-2-7);

(displacements)

$$\mathcal{U}_{\alpha} = J_{\alpha}^r \tilde{\mathcal{U}}^r$$

$$\mathcal{W} = \frac{1}{2} \xi^{r} \tilde{w}$$

(membrane strain tensor and bending tensor)

$$\mathcal{S}_{\alpha} = J_{\alpha}^r J_{\beta}^s \tilde{s}_{\alpha s}$$

$$\mathcal{S}_{\beta} = J_{\alpha}^r J_{\beta}^s \tilde{s}_{\alpha s}$$

(effective contact quantities)

$$\mathcal{L}_{\alpha} = J_{\alpha}^r J_{\beta}^s \tilde{l}_{\alpha s}$$

$$\mathcal{M}_{\alpha} = J_{\alpha}^r J_{\beta}^s \tilde{m}_{\alpha s}$$

$$\mathcal{Q}_{\alpha} = J_{\alpha}^r J_{\beta}^s \tilde{q}_{\alpha s}$$

(3-2-10)
(constitutive equations)

\[
\begin{align*}
\sigma^\alpha = \kappa \sigma^\alpha \\
\kappa^\alpha = \frac{\kappa^3}{3} \sigma^\alpha
\end{align*}
\]

Since

\[
\begin{align*}
\sigma^\alpha = \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \kappa^\alpha \\
= \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \kappa^\alpha
\end{align*}
\]

\[
\begin{align*}
\sigma^\alpha = \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \frac{\partial \xi^\alpha}{\partial s} \kappa^\alpha
\end{align*}
\]

where \((-\)) indicates the quantities based upon shell coordinate system.

\(\xi^\alpha = \) inplane displacements of mid-surface of shell
\(\psi = \) lateral displacement of mid-surface of shell
\(\kappa^\alpha = \) inplane strains
\(\kappa^\alpha = \) curvatures
\(\sigma^\alpha = \) stress resultants
\(\kappa^\alpha = \) resultant moments
\(\tau^\alpha = \) lateral shear stress

and for isotropic material, the elastic coefficients are

\[
\begin{align*}
\sigma^\alpha = \mu \left( \alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha + \frac{2\nu}{1-2\nu} \alpha^\alpha \beta^\alpha \right) \\
\mu = \frac{E}{2(1+\nu)}, \; \nu = \text{Poisson's ratio}, \; E = \text{Young's modulus}
\end{align*}
\]

and

\[
\begin{align*}
\sigma^\alpha = \mu \left( \alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha + \frac{2\nu}{1-2\nu} \alpha^\alpha \beta^\alpha \right)
\end{align*}
\]
Equations of linear shell theory based upon shell coordinate system

In the linear shell theory of Koiter-Sanders, there are twenty unknowns and twenty equations. There are six components related to displacements, three of which are linear displacements and three of which are angular displacements. These are the linear displacement tensor \( \overrightarrow{\mathbf{u}}_d \), which lies in the tangent plane and \( \overrightarrow{\mathbf{w}} \), the displacement perpendicular to the tangent plane; the rotation tensor \( \overrightarrow{\theta}^3_\alpha \), which are rotations about the base vectors, \( \overrightarrow{\mathbf{x}}_\alpha \), and the rotation tensor, \( \overrightarrow{\omega}_{12} \), about the normal \( \overrightarrow{\mathbf{n}} \). There are six components related to strain, three of which are extensional strains and three of which are changes in curvature. These are represented by \( \overrightarrow{\epsilon}^\beta_\alpha \) and \( \overrightarrow{\kappa}^\beta_\alpha \), respectively. Finally there are eight components related to stress; three of these lead to membrane-type forces, three to bending and twisting moments and two to transverse shears. These are denoted by the tensors \( \overrightarrow{D}^\beta_\alpha, \overrightarrow{m} \) and \( \overrightarrow{t}^\alpha \) respectively.

The twenty equations are apportioned as follows:

Nine kinematic relations

\[
\overrightarrow{\theta}^\beta_\alpha = \frac{1}{2} ( \overrightarrow{\mathbf{u}}_{\alpha \beta} + \overrightarrow{\mathbf{u}}_{\beta \alpha} - 2 \overrightarrow{\mathbf{w}} \overrightarrow{b} \overrightarrow{b}_{\alpha \beta} )
\]

\[
\overrightarrow{\kappa}^\beta_\alpha = -\frac{1}{2} ( \overrightarrow{\theta}^3_\alpha + \overrightarrow{\theta}^3_\beta + \overrightarrow{b} \overrightarrow{\omega}_{\beta \alpha} + \overrightarrow{b} \overrightarrow{\omega}_{\alpha \beta} )
\]

\[
\overrightarrow{\omega}_{\alpha \beta} = \frac{1}{2} ( \overrightarrow{\mathbf{u}}_{\alpha \beta} - \overrightarrow{\mathbf{u}}_{\beta \alpha} )
\]

where

\[
\overrightarrow{b} \overrightarrow{b}_{\alpha \beta} = \overrightarrow{n} \cdot \frac{\partial \overrightarrow{b}_\alpha}{\partial \zeta^3} \overrightarrow{n}^\beta
\]

Two equilibrium equations along the tangent plane:

\[
\overrightarrow{D}^\beta_\alpha + \frac{1}{2} ( \overrightarrow{m}^\beta \overrightarrow{b}_{\alpha \beta} - \overrightarrow{m}^\alpha \overrightarrow{b}_{\beta \beta} ) \overrightarrow{b}_{\alpha \alpha} - \overrightarrow{b} \overrightarrow{b}_{\alpha \beta} \overrightarrow{\theta}^\beta_\alpha + \overrightarrow{t}_\beta = 0
\]
One equilibrium equation which is for the direction perpendicular to the tangent plane

\[ \bar{\sigma}_{\alpha}^{\alpha} + b_{d\beta} \bar{I}^{\alpha\beta} + F^3 = 0 \]  

(3-2-15)

Two moment equilibrium equations

\[ \bar{m}^{d\beta}_{\alpha\beta} - \bar{\sigma}_{\alpha}^{\alpha} = 0 \]  

(3-2-16)

Six constitutive relations

\[ \bar{I}^{\alpha\beta} = \kappa \sum_{\sigma\theta} a_{\alpha\beta}^{\sigma\theta} \bar{h}_{\sigma\theta} \]
\[ \bar{m}^{d\beta}_{\alpha\beta} = \frac{\kappa^3}{12} \sum_{\sigma\theta} a_{\alpha\beta}^{\sigma\theta} \bar{h}_{\sigma\theta} \]  

(3-2-17)

where it is noted that \( \bar{I}^{\alpha\beta} \) and \( \bar{m}^{d\beta}_{\alpha\beta} \) are symmetric tensors.

The preceding set of equations are in tensor form. They are applicable to any shell which can be described in terms of a position vector to its undeformed position. The deformation behavior of the shell is restricted to deflections specified by the Kirchhoff hypothesis;

\[ \bar{u}^* (\xi, \zeta) = \bar{u}^\sigma (\xi^d) \bar{a}^\sigma + \bar{w} \bar{w} - \frac{3}{2} \bar{e}^\sigma \bar{e}^\sigma \]  

(3-2-18)

**Equilibrium equations**

The following equilibrium equations can be obtained from eqs. (3-2-14), (3-2-15) and (3-2-16);

\[ \bar{I}^{\alpha\beta} + \frac{1}{2} (\bar{m}^{\alpha\beta}_{\alpha\beta} - \bar{m}^{\alpha\beta}_{\alpha\beta}) b_{\alpha\beta} = \bar{b}_{\alpha\beta} \bar{m}^{\beta\alpha}_{\beta\alpha} F^3 = 0 \]  

(3-2-19)

Since

\[ \bar{I}^{\sigma\alpha} = \frac{1}{\sqrt{a}} \frac{\partial (\bar{I}^{\sigma\alpha})}{\partial \sqrt{a}} + \bar{I}^{\sigma\lambda} \bar{I}^{\lambda\alpha} \]

\[ a = \det (a_{\alpha\beta}) \]  

(3-2-20)

and

\[ \bar{m}^{\beta\sigma} = \frac{1}{\sqrt{a}} \frac{\partial (\bar{I}^{\sigma\alpha})}{\partial \sqrt{a}} + \bar{m}^{\beta\lambda} \bar{I}^{\lambda\sigma} \]
\[ \bar{m}^{\beta\sigma}_{\beta\sigma} = \frac{1}{\sqrt{a}} \frac{\partial^2}{\partial x^\sigma \partial x^\sigma} (\bar{w} \bar{m}^{\beta\sigma} + \frac{1}{\sqrt{a}} \frac{\partial}{\partial \sqrt{a}} (\sqrt{a} \bar{m}^{\beta\lambda} \bar{I}^{\lambda\sigma} \bar{I}^{\lambda\sigma} ) \]
the stress equilibrium equations can be expressed by

\[
\frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \xi} (\sqrt{\alpha} \bar{u}^a) + \frac{1}{\sqrt{\alpha}} \frac{\partial \bar{u}^a}{\partial \xi} + \frac{1}{\beta} \left( \bar{m}^a \bar{b}^a - \bar{m}^a \bar{b}^a \right) \beta - \bar{m}^a \bar{m}^a \beta + \bar{f}^a = 0
\]

\[
\frac{1}{\sqrt{\alpha}} \frac{\partial^2}{\partial \xi^a \partial \xi^b} (\sqrt{\alpha} \bar{m}^a) + \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \xi^a} (\sqrt{\alpha} \bar{m}^a \bar{b}^a) + \bar{b}^a \bar{b}^a \beta + \bar{f}^3 = 0
\]  

(3-2-21)

Since the equilibrium equations are written in tensor form, the form of the equations doesn't change under transformation between shell coordinate system and natural coordinate system. The equilibrium equations based upon natural coordinate system can be obtained simply by removing (\(\bar{\cdot}\)) and changing \(\xi\) to \(\xi = \det \xi^a \xi^b\) in eq. (3-2-21); i.e.

\[
\frac{1}{\sqrt{\xi}} \frac{\partial}{\partial \xi^a} (\sqrt{\xi} \bar{u}^a) + \frac{1}{\sqrt{\xi}} \frac{\partial \bar{u}^a}{\partial \xi^a} + \frac{1}{2} \left( \bar{m}^a \bar{b}^a - \bar{m}^a \bar{b}^a \right) \beta - \bar{m}^a \bar{m}^a \beta + \bar{f}^a = 0
\]

\[
\frac{1}{\sqrt{\xi}} \frac{\partial^2}{\partial \xi^a \partial \xi^b} (\sqrt{\xi} \bar{m}^a) + \frac{1}{\sqrt{\xi}} \frac{\partial}{\partial \xi^a} (\sqrt{\xi} \bar{m}^a \bar{b}^a) + \bar{b}^a \bar{b}^a \beta + \bar{f}^3 = 0
\]  

(3-2-22)

**Physical components**

The contravariant or covariant tensor components of a vector do not have the same kind of physical significance in a curvilinear coordinate system as they have in a rectangular Cartesian system. Therefore, the following physical components are introduced in linear shell theory;

**displacements**

\[ \bar{u}^a = \sqrt{\alpha} \bar{u}^a \]

\[ \bar{w}^3 = \bar{w}^3 \]

\[ \bar{\varphi}^a = \sqrt{\alpha} \bar{\varphi}^a \]

\[ \bar{\omega}^a = \frac{\bar{\omega}^a \bar{b}^a}{\sqrt{\alpha_1 \alpha_2}} \]  

(3-2-23)

**strains**
\[ \gamma_{d\beta}^* = \bar{\gamma}_{d\beta} + S \bar{K}_{d\beta} \]
\[ e_{d\beta} = \frac{\bar{d}_{d\beta}^*}{\sqrt{a_{dd} a_{\beta\beta}}} = e_{d\beta}^* + S \bar{K}_{d\beta} \]
\[ e_{d\beta}^* = \frac{\bar{d}_{d\beta}}{\sqrt{a_{dd} a_{\beta\beta}}} \]
\[ \bar{K}_{d\beta} = \frac{\bar{K}_{d\beta}}{\sqrt{a_{dd} a_{\beta\beta}}} \]

These quantities can be rewritten by the use of physical displacements as follows:
\[ e_{d\beta} = \frac{1}{2 \sqrt{a_{dd} a_{\beta\beta}}} \left[ \frac{\partial}{\partial x^i} \left( \frac{u_{d\beta}^o}{\sqrt{a_{dd}}} \right) + \frac{\partial}{\partial x^j} \left( \frac{u_{d\beta}^o}{\sqrt{a_{\beta\beta}}} \right) - \frac{2 u_{d\beta}^o}{\sqrt{a_{ii}}} \hat{d}_{i\beta} - \frac{2 u_{d\beta}^o}{\sqrt{a_{jj}}} \hat{d}_{j\beta} - 2 w^o \overline{b_{d\beta}} \right] \]
\[ \bar{K}_{d\beta} = -\frac{1}{2 \sqrt{a_{dd} a_{\beta\beta}}} \left[ \frac{\partial}{\partial x^i} \left( \frac{\Phi_d}{\sqrt{a_{dd}}} \right) + \frac{\partial}{\partial x^j} \left( \frac{\Phi_d}{\sqrt{a_{\beta\beta}}} \right) - \frac{2 \Phi_d}{\sqrt{a_{ii}}} \hat{d}_{i\beta} - \frac{2 \Phi_d}{\sqrt{a_{jj}}} \hat{d}_{j\beta} \right. \\
+ \sqrt{a_{dd} a_{\beta\beta}} \left( \frac{\partial}{\partial x^i} \left( \frac{\Omega_{d\beta}^o}{\sqrt{a_{dd}}} \right) + \frac{\partial}{\partial x^j} \left( \frac{\Omega_{d\beta}^o}{\sqrt{a_{\beta\beta}}} \right) \right) \right] \]
\[ \Phi_d = \sqrt{a_{dd}} \left[ \frac{\partial}{\partial x^i} \left( \frac{w^o}{\sqrt{a_{dd}}} \right) + \frac{\partial}{\partial x^j} \left( \frac{w^o}{\sqrt{a_{\beta\beta}}} \right) \right] \]
\[ \Omega_{d\beta} = \frac{1}{2 \sqrt{a_{dd} a_{\beta\beta}}} \left[ \frac{\partial}{\partial x^i} \left( \frac{u_{d\beta}^o}{\sqrt{a_{dd}}} \right) - \frac{\partial}{\partial x^j} \left( \frac{u_{d\beta}^o}{\sqrt{a_{\beta\beta}}} \right) \right] \]

Since the equilibrium equations are described in terms of natural coordinate system, it is necessary to convert tensor quantities to physical quantities in order to apply the fundamental functional to shell problems.

The relationship between physical quantities and tensor quantities based upon shell coordinate system is given by
\[ N^\alpha_{d\beta} = \sqrt{a_{dd}} a_{\alpha\beta} \bar{N}^\alpha_{d\beta} \]
\[ M^\alpha_{d\beta} = \sqrt{a_{dd}} a_{\alpha\beta} \bar{M}^\alpha_{d\beta} \]
and
\[ Q^\alpha = \sqrt{a_{dd}} \bar{Q}^\alpha \]

The use of eq. (3-2-10) yields the relationship between physical quantities based upon shell coordinate system and tensor quantities based upon natural coordinate system in the following form;
\[ N^{d\beta} = \sqrt{\frac{\alpha_{dd}}{\alpha_{\beta\beta}}} \int_{\gamma} \int_{\beta} \mathcal{I} \, d\gamma \, d\beta = \frac{\beta_{d\beta}}{\sqrt{\gamma}} \, d\beta \]

\[ M^{d\beta} = \sqrt{\frac{\alpha_{dd}}{\alpha_{\beta\beta}}} \int_{\gamma} \int_{\beta} \mathcal{M} \, d\gamma \, d\beta = \frac{\beta_{d\beta}}{\sqrt{\gamma}} \, d\beta \]

\[ \mathcal{Q} = \frac{1}{\gamma} \int_{\gamma} \int_{\beta} \mathcal{Q} \, d\gamma \, d\beta = \frac{\beta_{d\beta}}{\sqrt{\gamma}} \]

3.2.2 New version of modified Hu-Washizu principle for general shells

The new version of modified Hu-Washizu principle given by eq. (2-2-5) can be written in terms of tensor components based upon general curvilinear coordinate systems in the following form;

\[ \begin{align*}
\Pi_{HG}^* & = \Pi_{HG}^* (\tau^{ij}, \gamma^{ij}, \nu_{ij}, \nu_{ij}^2, \nu_{ij}^3) \\
& = \sum_{\tau} \left[ \int_{\gamma} \left[ \left( \Phi^{(\gamma)} \right)_{ij} - \gamma^{ij} \gamma^{ij}_{\gamma} - (\gamma^{ij}_{\gamma} + \gamma^{ij}_{\nu}) u_{ij} - (\gamma^{ij}_{\gamma} + \gamma^{ij}_{\nu}) u_{ij}^2 \right] d\gamma \\
& + \int_{\gamma} \left( \gamma^{ij}_{\gamma} \gamma^{ij}_{\nu} - \gamma^{ij}_{\gamma} \gamma^{ij}_{\nu} \right) ds \right] 
\end{align*} \tag{3-2-28} \]

where \((-\)) indicates the given quantities. The representation of displacements based upon natural coordinate system is given by

\[ U_i (\xi^i) = U^r (\xi^r) \xi^r + \omega_i (\xi^i) \xi^3 \xi^3 + \xi^3 \xi^3 \xi^3 \xi^3 \tag{3-2-29} \]

The strains are expressed by

\[ \gamma_{ij}^* = \gamma_{ij} + \xi^3 \kappa_{ij} \]

\[ \gamma_{3i}^* = \gamma_{3i} = 0 \quad [\text{Kirchhoff hypothesis}] \tag{3-2-30} \]

where \( \gamma_{ij} \) and \( \kappa_{ij} \) are defined with respect to the shell mid-surface.

and

\[ \gamma_{ij} = \frac{1}{2} \left\{ U_{ij} + U_{ij,\sigma} + 2 b_{ij\sigma} \right\} \tag{3-2-31} \]

\[ \kappa_{ij} = - \frac{1}{2} \left\{ \sigma_{ij}^3 + \sigma_{ij}^3 + b_{ij\sigma} \right\} \tag{3-2-32} \]

\[ \sigma_{ij}^3 \equiv \frac{2}{3} \frac{\partial w}{\partial \xi^3} + U_{ij} b^\sigma \tag{3-2-33} \]
\[ \omega_{\alpha \beta} = \frac{1}{2} ( \mathcal{V}_{\alpha \beta} - \mathcal{V}_{\beta \alpha} ) \]  \hspace{1cm} (3-2-34)

In order to obtain the corresponding functional to eq. \((3-2-28)\) for general shell, each term is examined;

\[
\int_{V} \Phi^* \left( \mathcal{V}_{\alpha \beta}^* \right) dV = \int_{S} \Phi^* \left( \mathcal{V}_{\alpha \beta}^*, \mathcal{K}_{\alpha \beta}^* \right) \sqrt{\mathcal{F}_{33}} \, d\mathcal{S} \, d\mathcal{S} \\
= \int_{S} \Phi \left( \mathcal{V}_{\alpha \beta}, \mathcal{K}_{\alpha \beta} \right) dS
\]

\[
\int_{V} \gamma^i \mathcal{V}_{ij}^* dV = \int_{S} \int_{S} \gamma^{ij} \left( \mathcal{V}_{ij} + \mathcal{K}_{ij} \right) \sqrt{\mathcal{F}_{33}} \, d\mathcal{S} \, d\mathcal{S} \\
= \int_{S} \left( \mathcal{L}^{ij} \mathcal{V}_{ij} + \mathcal{M}^{ij} \mathcal{K}_{ij} \right) dS
\]  \hspace{1cm} (3-2-35)

where the effective contact quantities \( \mathcal{L}^{ij} \) and \( \mathcal{M}^{ij} \) are defined by

\[ \mathcal{L}^{ij} \equiv \int_{S} \gamma^{ij} \sqrt{\mathcal{F}_{33}} \, d\mathcal{S} \]

and

\[ \mathcal{M}^{ij} \equiv \int_{S} \gamma^{ij} \sqrt{\mathcal{F}_{33}} \, d\mathcal{S} \]

Since eq. \((3-2-29)\) can be rewritten as

\[ \gamma^{ij} = \mathcal{U}_{ij}^{\alpha \beta} = \mathcal{U}_{ij}^{\alpha \beta} + \mathcal{W} \mathcal{S}^{\alpha \beta} - \mathcal{S}^{\alpha \beta} \mathcal{S}^{\alpha \beta} \mathcal{S}^{\alpha \beta} \mathcal{S}^{\alpha \beta} \]

\[ = ( \mathcal{U}_{ij} - \mathcal{S}^{\alpha \beta} \mathcal{S}^{\alpha \beta} ) \mathcal{S}^{\alpha \beta} + \mathcal{W} \mathcal{S}^{\alpha \beta} \]  \hspace{1cm} (3-2-37)

we obtain

\[ \mathcal{U}_{\alpha} = \mathcal{U}_{\alpha} - \mathcal{S}^{\alpha \beta} \mathcal{S}^{\alpha \beta} \]

\[ \mathcal{U}_{3} = \mathcal{W} \]  \hspace{1cm} (3-2-38)

Then it follows that

\[
\int_{V} \gamma^{ij} \mathcal{U}_{ij} \, dV = \int_{V} \left[ \gamma^{ij} \mathcal{U}_{ij} + \mathcal{S}^{ij} \mathcal{U}_{ij} + \mathcal{W} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \right] dV \\
= \int_{V} \left[ \mathcal{L}^{ij} \mathcal{U}_{ij} - \mathcal{M}^{ij} \mathcal{K}_{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \right] dV \\
= \int_{S} \left[ \mathcal{L}^{ij} \mathcal{U}_{ij} - \mathcal{M}^{ij} \mathcal{K}_{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \right] dV + I
\]  \hspace{1cm} (3-2-39)

where

\[ I = \int_{S} \left[ \mathcal{L}^{ij} \mathcal{U}_{ij} - \mathcal{M}^{ij} \mathcal{K}_{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \mathcal{S}^{ij} \right] \sqrt{\mathcal{F}_{33}} \, d\mathcal{S} \, d\mathcal{S} \]
Since
\[
\gamma^{3d}_{\alpha} = \frac{\partial \gamma^{3d}}{\partial \xi^3} + \gamma^{3d}_{\alpha} = \frac{\partial \gamma^{3d}}{\partial \xi^3} + \gamma^{3d}_{\beta} \gamma^{d}_{\beta} \\
= \frac{\partial \gamma^{3d}}{\partial \xi^3} - b^d_{\beta} \gamma^{3d}_{\beta}
\]
we have
\[
I = \int_{S} \varphi^{d} \Phi^{3}_{\alpha} dS - \int_{S} \varphi^{b}_{\alpha} b^d_{\beta} \varphi^{3}_{\alpha} dS \\
+ \int_{S} \gamma^{d}_{\beta} \gamma^{3d}_{\beta} \gamma^{3}_{\alpha} \gamma^{3}_{\alpha} dS \tag{3-2-40}
\]

Here, introducing the assumption \(O(s^{3d}_{\beta}) < 1\) and introducing effective contact forces and couples in order to satisfy the sixth equilibrium equation,
\[
\int_{V} \varphi^{d}_{ij} \varphi^{a}_{ij} dV = \int_{S} \left\{ \left[ l^{d}_{\beta} \gamma^{3}_{\beta} + \frac{1}{2} \left( m^{\alpha \beta} b^d_{\alpha} - m^{d \beta} b^d_{\alpha} \right) \right]_{\beta} - b^d_{\alpha} \varphi^{d}_{ij} \varphi^{3}_{\alpha} \right\} dS \\
+ \left\{ \varphi^{d}_{ij} + b^d_{\beta} \varphi^{d}_{ij} \right\} \varphi^{3}_{ij} + \varphi^{3}_{ij} dS \tag{3-2-42}
\]

When the boundary displacements are written by
\[
\tilde{\gamma}^d_{\alpha} = \tilde{\gamma}^d_{\alpha} - s^{3d}_{\beta} \gamma^{3}_{\beta}, \quad \tilde{\gamma}^3_{\alpha} = \tilde{\omega}_{\alpha}
\tag{3-2-43}
\]
the boundary integral in eq. (3-2-28) can be expressed as
\[
\int_{S} \varphi^{d}_{ij} \tilde{\gamma}^d_{ij} dS = \int_{S} \varphi^{d}_{ij} \gamma^{3}_{\beta} \tilde{\gamma}^d_{\alpha} dS + \int_{S} \varphi^{d}_{ij} \gamma^{3}_{\beta} \tilde{\gamma}^3_{\alpha} dS + \int_{S} \varphi^{d}_{ij} \gamma^{3}_{\beta} \tilde{\gamma}^3_{\alpha} dS \\
= \int_{S} \left\{ l^{d}_{\beta} \gamma^{3}_{\beta} \tilde{\gamma}^d_{\alpha} + \varphi^{d}_{ij} \tilde{\omega} - m^{d \beta} \gamma^{3}_{\beta} \tilde{\gamma}^3_{\alpha} \right\} dS
\tag{3-2-44}
\]

where \(\tilde{\gamma}^3 = \varphi^{3}_{ij} \) is used.

In a similar manner to eq. (3-2-32), we obtain
\[
\int_{V} \varphi^{d}_{ij} \tilde{\gamma}^d_{ij} dV = \int_{S} \left\{ \left[ l^{d}_{\beta} \gamma^{3}_{\beta} + \frac{1}{2} \left( m^{\alpha \beta} b^d_{\alpha} - m^{d \beta} b^d_{\alpha} \right) \right]_{\beta} - b^d_{\alpha} \varphi^{d}_{ij} \varphi^{3}_{\alpha} \right\} dS \\
+ \left\{ \varphi^{d}_{ij} + b^d_{\beta} \varphi^{d}_{ij} \right\} \varphi^{3}_{ij} + \varphi^{3}_{ij} dS \tag{3-2-45}
\]

The contribution of body forces can be written as
\[
\int_{V} \varphi^{d}_{ij} \tilde{\gamma}^d_{ij} dV = \int_{S} \left\{ \bar{F}^{d} \varphi^{3}_{ij} + \bar{F}^{3} \varphi^{3}_{ij} \right\} dS \tag{3-2-46}
\]
Thus the corresponding functional to eq. (3-2-28) for general shell can be expressed by

\[ \tau_{H}^{*} \left( \ell \partial_{\beta}, m_{g}^{\beta}; \tau_{g}, k_{g} \beta; \underline{u}, \underline{w}; \underline{u}_{\beta}, \underline{w}_{\beta}; \tilde{u}, \tilde{w} \right) \]

\[ = \sum_{n} \left[ \int_{S_{n}} \left[ \Phi(k_{g} \beta, k_{g} \beta) - (\ell \partial_{\beta} \tau_{g} \beta + m_{g} \partial_{\beta} k_{g} \beta) \right. \right. \]

\[ - \left\{ \frac{1}{\sqrt{\lambda}} \partial \left( \frac{\partial^{2}}{\partial z \partial \lambda} m_{g}^{\lambda} \right) \lambda \beta, \lambda \beta, \lambda \beta \lambda \beta + \frac{1}{2} \left( m_{g}^{\lambda} \lambda \beta \lambda \beta - m_{g} \partial_{\beta} \lambda \beta \lambda \beta \right) \right. \]

\[ - \left\{ \frac{1}{\sqrt{\lambda}} \partial \left( \frac{\partial^{2}}{\partial z \partial \lambda} m_{g}^{\lambda} \lambda \beta \lambda \beta \lambda \beta \right) \right. \]

\[ - \left. \left\{ \frac{1}{\sqrt{\lambda}} \partial \left( \frac{\partial^{2}}{\partial z \partial \lambda} m_{g}^{\lambda} \lambda \beta \lambda \beta \lambda \beta \lambda \beta \right) \right. \right. \]

\[ + \int_{S_{n}} \left[ \ell \partial_{\beta} \theta \beta + 2 \partial_{\beta} \theta \beta - m_{g} \partial_{\beta} \theta \beta \theta \beta \right] \right. \]

\[ - \sum_{S_{n}} \left[ \ell \partial_{\beta} \theta \beta \theta \beta - m_{g} \partial_{\beta} \theta \beta \theta \beta \right] \right. \]

\[ \right. \]  \hspace{1cm} (3-2-47)

3.2.3 Finite element formulation

The functional indicated by eq. (3-2-47) can be rewritten for one finite element as

\[ \tau_{H}^{*} = \sum_{S_{n}} \left[ \Phi(k_{g} \beta, k_{g} \beta) - (\ell \partial_{\beta} \tau_{g} \beta + m_{g} \partial_{\beta} k_{g} \beta) \right. \]

\[ - \left\{ \frac{1}{\sqrt{\lambda}} \partial \left( \frac{\partial^{2}}{\partial z \partial \lambda} m_{g}^{\lambda} \lambda \beta \lambda \beta \right) \right. \]

\[ - \left\{ \frac{1}{\sqrt{\lambda}} \partial \left( \frac{\partial^{2}}{\partial z \partial \lambda} m_{g}^{\lambda} \lambda \beta \lambda \beta \lambda \beta \right) \right. \]

\[ - \left\{ \frac{1}{\sqrt{\lambda}} \partial \left( \frac{\partial^{2}}{\partial z \partial \lambda} m_{g}^{\lambda} \lambda \beta \lambda \beta \lambda \beta \lambda \beta \right) \right. \]

\[ + \int_{S} \left[ \ell \partial_{\beta} \theta \beta + 2 \partial_{\beta} \theta \beta - m_{g} \partial_{\beta} \theta \beta \theta \beta \right] \right. \]

\[ - \sum_{S} \left[ \ell \partial_{\beta} \theta \beta \theta \beta - m_{g} \partial_{\beta} \theta \beta \theta \beta \right] \right. \]

\[ \right. \]  \hspace{1cm} (3-2-48)

In the finite element formulation, each independent quantity is interpolated by

\[ \begin{bmatrix} \ell \partial_{\beta} \\ m_{g}^{\beta} \end{bmatrix} = \tilde{P}_{\beta} \]

\[ \begin{bmatrix} \tau_{g} \beta \\ \kappa_{g} \beta \end{bmatrix} = \tilde{P}_{\beta} \]

\[ \left(3-2-49 \right) \]

\[ \left(3-2-50 \right) \]
\[
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} = \nu \begin{bmatrix} \nu \\ \nu \end{bmatrix} \\
\frac{\partial \lambda}{\partial t} = \lambda
\end{cases} \\
\begin{cases}
\frac{\partial^2 \lambda}{\partial \xi^2} = \lambda \\
\frac{\partial^2 \lambda}{\partial \eta^2} = \lambda
\end{cases} \\
\begin{cases}
\frac{\partial \lambda}{\partial \xi} = \gamma \\
\frac{\partial \lambda}{\partial \eta} = \gamma
\end{cases}
\end{aligned}
\] (3-2-51) (3-2-52) (3-2-53)

The homogeneous part of equilibrium equations can be expressed as
\[
\begin{aligned}
\begin{cases}
\frac{1}{\sqrt{s}} \frac{\partial (\frac{\partial d_{\lambda}}{\partial \xi^2})}{\partial \xi^2} + \frac{\partial \lambda}{\partial \xi} = \frac{1}{\nu} \left( \frac{m_{\lambda \lambda}^d b_{\lambda}^d - m_{\lambda \lambda}^d b_{\lambda}^d}_d - b_{\lambda}^d m_{\lambda \lambda}^d \right) \\
\frac{1}{\nu} \frac{\partial^2}{\partial \eta^2} (\sqrt{s} m_{\lambda \lambda}^d) + \frac{1}{\nu} \frac{\partial}{\partial \eta} (\sqrt{s} m_{\lambda \lambda}^d) + b_{\lambda \beta} \lambda_{\lambda \beta}
\end{cases}
\end{aligned}
\] (3-2-54)

When the portion or the entire part of homogeneous equilibrium equations are constrained by \( \lambda_{\lambda}^2 \) and \( \lambda_{\eta}^2 \), we may express it by
\[
\bar{E} \lambda_{\lambda}^2 \quad (3-2-55)
\]

The boundary tractions are expressed as
\[
\begin{aligned}
\begin{cases}
\lambda_{\lambda}^d \\
\lambda_{\eta}^d \\
m_{\lambda \lambda}^d
\end{cases}
\end{aligned} = \begin{bmatrix} \lambda_{\lambda}^d \\
\lambda_{\eta}^d \\
m_{\lambda \lambda}^d
\end{bmatrix}
\] (3-2-56)

The use of these notations enables us to compute each integrand of eq. (3-2-48) as
\[
\begin{aligned}
\int_S \bar{E} \begin{bmatrix} \lambda_{\lambda}^2 \\
\lambda_{\eta}^2 \\
m_{\lambda \lambda}^2
\end{bmatrix} \sqrt{s} d\xi d\eta d\tau = \frac{1}{2} \bar{E} \begin{bmatrix} \lambda_{\lambda}^2 \\
\lambda_{\eta}^2 \\
m_{\lambda \lambda}^2
\end{bmatrix} \\
\int_S (\lambda_{\lambda}^d \lambda_{\eta}^d + m_{\lambda \lambda}^d \lambda_{\lambda}^d) \sqrt{s} d\xi d\eta d\tau = \bar{E} \begin{bmatrix} \lambda_{\lambda}^2 \\
\lambda_{\eta}^2 \\
m_{\lambda \lambda}^2
\end{bmatrix} \\
\int_S \left\{ \frac{1}{\sqrt{s}} \frac{\partial (\frac{\partial \lambda_{\lambda}^d}{\partial \eta^2})}{\partial \eta^2} + \frac{\partial \lambda_{\lambda}^d}{\partial \eta} + \frac{1}{\nu} \left( \frac{m_{\lambda \lambda}^d b_{\lambda}^d - m_{\lambda \lambda}^d b_{\lambda}^d}_d - b_{\lambda}^d m_{\lambda \lambda}^d \right) \right. \\
\left. + \frac{1}{\nu} \frac{\partial^2}{\partial \eta^2} (\sqrt{s} m_{\lambda \lambda}^d) + \frac{1}{\nu} \frac{\partial}{\partial \eta} (\sqrt{s} m_{\lambda \lambda}^d) + b_{\lambda \beta} \lambda_{\lambda \beta} \right\} \begin{bmatrix} w + w_{\lambda}^2 \\
w_{\eta} + w_{\eta}^2 \\
m_{\lambda \lambda}^2
\end{bmatrix} \sqrt{s} d\xi d\eta d\tau
\end{aligned}
\]
where \( \overrightarrow{q} = \) load vector due to body forces related to internal displacements and when the part of homogeneous equilibrium equations is constrained, we have \( \overrightarrow{q} = \mathbf{q} \), \( \mathbf{R}^{\lambda} = \int_{S} \mathbf{E}^{*T} \mathbf{N} \frac{d\mathbf{S}^{\alpha}}{dS^{\alpha}} \cdot \mathbf{S}^{\beta} \),

\[
\int_{\partial S} \left[ \beta^{\alpha} d\mathbf{S}^{\alpha} \cdot \mathbf{w} - m^{\alpha} \mathbf{b}^{\beta} \mathbf{\tilde{w}} \right] \cdot \left[ \delta_{\alpha\beta} d\mathbf{S}^{\beta} \cdot \mathbf{S}^{\beta} \right]^{\lambda} = \beta^{\lambda} \mathbf{G} \mathbf{q} \quad ; \quad \mathbf{G} = \int_{\partial S} \mathbf{R}^{\lambda} \mathbf{N} \left[ \delta_{\alpha\beta} d\mathbf{S}^{\beta} \cdot \mathbf{S}^{\beta} \right]^{\lambda} = \mathbf{q}^{T} \mathbf{Q}
\]

and

\[
\int_{\partial S} \left[ \beta^{\alpha} d\mathbf{w} + \mathbf{F}^{\alpha} \mathbf{W} \right] \cdot \left[ \delta_{\alpha\beta} d\mathbf{S}^{\beta} \cdot \mathbf{S}^{\beta} \right]^{\lambda} = \mathbf{q}^{T} \mathbf{Q} \quad (3-2-57)
\]

Now the functional of \( \mathbf{J}_{\mathbf{HG}}^{*} \) can be represented in terms of matrix notations as

\[
\mathbf{J}_{\mathbf{HG}}^{*} = \frac{1}{2} \mathbf{q}^{T} \mathbf{G} \mathbf{q} - \mathbf{b}^{T} \mathbf{G} \mathbf{q} - \mathbf{b}^{T} \mathbf{G} \mathbf{q} - \mathbf{b}^{T} \mathbf{G} \mathbf{q} - \mathbf{q}^{T} \mathbf{Q} - \mathbf{q}^{T} \mathbf{Q} - \mathbf{q}^{T} \mathbf{Q} \quad (3-2-58)
\]

By static condensation, the final form of \( \mathbf{J}_{\mathbf{HG}}^{*} \) becomes

\[
\mathbf{J}_{\mathbf{HG}}^{*} = \frac{1}{2} \mathbf{q}^{T} \mathbf{G} \mathbf{q} - \mathbf{q}^{T} \mathbf{Q} \quad (3-2-59)
\]

where

\[
\mathbf{q} = (\mathbf{G} - \mathbf{\tilde{G}}) \mathbf{T} \mathbf{M} (\mathbf{G} - \mathbf{\tilde{G}}) - (\mathbf{G} - \mathbf{\tilde{G}}) \mathbf{T} \mathbf{M} (\mathbf{G} - \mathbf{\tilde{G}}) - \mathbf{b}^{T} \mathbf{G} \mathbf{q} \quad (3-2-60)
\]

and

\[
\mathbf{Q} = \mathbf{Q} + \mathbf{Q} + (\mathbf{G} - \mathbf{\tilde{G}}) \mathbf{T} \mathbf{M} \mathbf{b}^{T} (\mathbf{G} - \mathbf{\tilde{G}}) \mathbf{G} - \mathbf{G} \mathbf{q} - \mathbf{G} \mathbf{q} \quad (3-2-61)
\]

It is noted that the analytic expression for the boundary normal \( \nu_{\alpha} \) and the boundary tangent \( t^{\alpha} \) is given by

\[
\nu_{\alpha} = \frac{\varepsilon_{\alpha\beta} d\mathbf{S}^{\beta}}{ds} \quad , \quad t^{\beta} = \varepsilon_{\alpha\beta} \nu_{\alpha} \quad (3-2-62)
\]

where

\[
\varepsilon_{\alpha\beta} = \text{surface permutation tensor}
\]

and

\[
\varepsilon_{\alpha\beta} = \left[ \begin{array}{cc} 0 & 1 \sqrt{s} \\ -1 \sqrt{s} & 0 \end{array} \right] , \quad \varepsilon_{\beta\alpha} = \left[ \begin{array}{c} 0 \\ \frac{1}{\sqrt{s}} \end{array} \right]
\]
In the practical computation, the formation of a matrix $J$ should be carried out by using physical components because, in general, the constitutive relation is described by physical components of stresses and strains. But the use of tensor components for any other matrices enables a more systematic formulation.

3.2.4 Stress assumptions in hybrid semiLoof general shell element

The interpolation functions used for displacements of a quadrilateral hybrid semiLoof general shell element are the same functions as those for shallow shell element.

In order to obtain the appropriate stress assumption, we should consider the static-geometric analogy and stress functions. Then the effective contact forces and couples can be represented in terms of three stress functions which are, in fact, identical to the components of displacements:

$$
\ell^{\alpha \beta}_{(\omega)} = \epsilon^{\alpha \lambda} \epsilon^{\beta \mu} \left[ \frac{\partial}{\partial \lambda} \varphi_{3} \lambda + b_{\lambda \mu} \varphi_{3} \lambda + b_{\mu \lambda} \varphi_{3} \lambda + \frac{1}{2} b_{\lambda} (\varphi_{3} \lambda - \varphi_{3} \mu) \right] \\
m^{\alpha \beta} = \epsilon^{\alpha \lambda} \epsilon^{\beta \mu} \frac{1}{2} \left( \varphi_{\mu \lambda} \lambda + \varphi_{\lambda \mu} \lambda - 2 b_{\lambda \mu} \varphi_{3} \right) \\
q^{\alpha} = \epsilon^{\alpha \lambda} \epsilon^{\beta \mu} \frac{1}{2} \left( \varphi_{\mu \beta} + \varphi_{\beta \mu} - 2 (b_{\beta \mu} \varphi_{3}) \right) \beta \right] \tag{3-2-63}
$$

where

$$\ell^{\alpha \beta}_{(\omega)} = \ell^{\alpha \beta} + \frac{1}{2} \left( m^{\beta \lambda} b_{\lambda}^{\beta} - m^{\alpha \lambda} b_{\lambda}^{\beta} \right)$$

and $\varphi_{3}$ and $\varphi_{3}$ correspond to the displacements $u$ and $w$.

In hybrid semiLoof element, incomplete cubic interpolation functions are employed for internal displacement distributions.

Since $\ell^{\alpha \beta}_{(\omega)}$, $m^{\alpha \beta}$ and $q^{\alpha}$ should have the distributions represented by complete polynomials, the following assumptions are used for effective contact quantities:

$$\begin{bmatrix}
\ell^{(\beta)} \\
m^{(\beta)}
\end{bmatrix} = \begin{bmatrix}
P_{0} & P_{0} & P_{0} \\
0 & P_{0} & P_{0} \\
0 & 0 & P_{0}
\end{bmatrix} \begin{bmatrix}
P_{0} \\
P_{0} \\
P_{0}
\end{bmatrix} \begin{bmatrix}
J_{1} \\
\vdots \\
\beta_{36}
\end{bmatrix} \tag{3-2-64}
$$

where

$$P_{0} = \begin{bmatrix}
1 & \xi & \xi^{2} & \xi \eta & \eta^{2}
\end{bmatrix}.$$
As will be seen in example solutions, this stress distribution yields two kinematic modes. Therefore, the alternative stress distribution to prevent kinematic modes is considered;

\[
\begin{pmatrix}
q^{\alpha \beta} \\
\rho^{\alpha}
\end{pmatrix} =
\begin{bmatrix}
\mathbf{P}_1 & \mathbf{P}_2 \\
\mathbf{P}_2 & \mathbf{P}_2
\end{bmatrix}
\begin{pmatrix}
\beta_1 \\
\ldots
\end{pmatrix}
\] (3-2-65)

where

\[
\mathbf{P}_1 = \begin{bmatrix}
1 & n & \tilde{s} & \tilde{s}^2 & \tilde{n} & \tilde{n}^2
\end{bmatrix},
\mathbf{P}_2 = \begin{bmatrix}
1 & n & \tilde{s} & \tilde{s}^2 & \tilde{n} & \tilde{n}^2
\end{bmatrix}
\].

Also the following internal displacements are employed;

\[
\begin{pmatrix}
u^\alpha \\
\omega^\alpha
\end{pmatrix} =
\begin{bmatrix}
\mathbf{N}_0 & \mathbf{N}_1
\end{bmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_7
\end{pmatrix}
\] (3-2-66)

where

\[
\mathbf{N}_0 = \begin{bmatrix}
1 & n & \tilde{n}
\end{bmatrix},
\mathbf{N}_1 = \begin{bmatrix}
1
\end{bmatrix}
\]

3.2.5 The satisfaction of zeroth order patch test (rigid body motion)

In \(C^0\) class problems and plate bending problem, the inclusion of rigid body motion in displacement assumptions is easy to carry out, hence the zeroth order patch test can be fulfilled.

For thin shell the representation of rigid body motion is not straightforward. But in hybrid stress method the zeroth order patch test can be satisfied by the different method from the use of higher order displacement modes.

In the expression of \(\mathbf{N}^*_H\), we have \(u_\alpha, w\) and \(\bar{u}_\alpha, \bar{w}\) for displacements. Then it follows that when we satisfy either

\[
\mathcal{L}^{\alpha \beta}_{\alpha \alpha} + \frac{1}{2} \left( m^{\beta \alpha} b_\alpha - m^{\alpha \beta} b_\alpha \right) \alpha - b^{\beta}_{\alpha} q^\alpha = 0
\]

or

\[
q^{\alpha}_{\alpha \alpha} + b^{\beta}_{\alpha \beta} \mathcal{L}^{\alpha \beta} = 0
\].
we can resolve the problem of the coupling between $\mathcal{U}_d$ and $\mathcal{W}$. For cylindrical shells trigonometric functions are needed for the displacements to represent rigid body motions because of the coupling between membrane and bending actions. In the hybrid stress method the displacements $\mathcal{U}_d$ and $\mathcal{W}$ may be expressed in polynomial functions but by the satisfaction of either equation above, the rigid body motion can be represented satisfactorily. It should be stressed again that there is no difficulty of interelement compatibility in hybrid stress method because of the introduction of $\mathcal{U}_d$ and $\mathcal{W}$. 
3.3 Large Deflection Analysis by Stationary Lagrangian Formulation

For finite element analysis of plates and shells in large deflection it is also desirable to use elements for which the physically fundamental modes such as rigid body motion, momentless membrane state and inextensional bending can be accommodated. Thus, included in the example solutions at the end of this chapter are tests of the hybrid semiLoof elements for plates under large deflection and for buckling of unsymmetric laminates under thermal loads. This section is to present the formulation of stationary Lagrangian analysis of shells in large deflections. It follows basically the work of Boland [22]. The free energy function for the thermal loading problem is given in Section 3.3.1.

3.3.1 Free Energy Function for Thermal Loading Problem

From thermodynamics, we have

\[ d\bar{U} = \varepsilon \sigma \varepsilon + d\bar{Q} \]

(3-3-1)

where \( d\bar{U} \) and \( d\bar{Q} \) are the increments of the internal energy and heat energy supplied to the element, respectively. It is noted that \( \bar{U} \) is a state function while \( d\bar{Q} \) represents the infinitesimal amount of heat energy supplied.

\[ d\bar{Q} = TdS \]

(3-3-2)

where \( S \) indicates a state function "entropy".
Then we get

\[ d\Phi = T dS + \sigma_{ij}^d d\varepsilon_{ij} \]  

(3-3-3)

Introducing the Helmholtz free energy function leads to

\[ d\Phi = d\bar{U} - T dS - S dT = \sigma_{ij}^d d\varepsilon_{ij} - S dT \]  

(3-3-4)

where

\[ \sigma_{ij}^d = \frac{\partial \Phi}{\partial \varepsilon_{ij}} \quad \text{and} \quad S = -\frac{\partial \Phi}{\partial T} \]

For thermoelasticity, we have

\[ \Phi(\varepsilon_{ij}; T) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - C_{ijkl} \varepsilon_{kl} \varepsilon_{ij} \]  

(3-3-5)

and

\[ \sigma_{ij} = S_{ijkl} \varepsilon_{kl} + \sigma_{ij}^\theta \]  

(3-3-6)

where

\[ \sigma_{ij} \equiv \sigma_{ij}^T(\varepsilon_{kl}; T) \quad \sigma_{ij}^T(0,0) = 0 \]

\[ \varepsilon_{ij} = \alpha_{ij} \Theta \delta_{ij} = \alpha_{ij} \Delta T \delta_{ij} \quad \Delta T = T - T_0 \]

and

\[ \alpha_{ij} = \text{thermal coefficients} \]

### 3.3.2 New Version of Modified Hu-Washizu Principle for Large Deflections Based Upon a Stationary Lagrangian Coordinate System

The total Green strain displacement relations for the large deformation analysis may be written as

\[ \varepsilon_{ij} + \Delta \varepsilon_{ij} = \frac{1}{2} \left[ (\nabla_i + \nabla_i^T) \delta_{ij} + (\nabla_j + \nabla_j^T) i + (\nabla_k + \nabla_k^T) j \right] \]  

(3-3-7)
where all the quantities are defined based upon the Cartesian coordinate system and \( \Delta \) represents the increment of a quantity.

The kinematic boundary condition on \( Su \) can be written as

\[
\mathbf{u}_i + \Delta \mathbf{u}_i = \mathbf{u}_i + \Delta \mathbf{u}_i \tag{3-3-8}
\]

The compatibility condition of displacements on the discretized segment boundary of a continuum is given by

\[
\mathbf{u}_i + \Delta \mathbf{u}_i = \mathbf{\hat{u}}_i + \Delta \mathbf{\hat{u}}_i \tag{3-3-9}
\]

The resulting functional of the modified Hu-Washizu principle can be expressed by

\[
\mathcal{T}_{\text{HG}} = \sum_n \left[ \int_{V_n} \left[ \Phi(\mathbf{E}^f; \mathbf{\Theta}) - \Delta \mathbf{N}_f^T \mathbf{N}_f - \mathbf{N}_f^T \mathbf{K}_f \mathbf{N}_f + \frac{1}{2} \mathbf{N}_f^T \mathbf{S}_f \mathbf{N}_f \right] dV \right. \\
- \int_{\partial V_n} (\mathbf{T}_i + \Delta \mathbf{T}_i) \left[ (\mathbf{u}_i + \Delta \mathbf{u}_i) - (\mathbf{\hat{u}}_i + \Delta \mathbf{\hat{u}}_i) \right] dS \\
- \int_{S_{on}} (\mathbf{T}_i + \Delta \mathbf{T}_i) dS - \int_{S_{un}} (\mathbf{T}_i + \Delta \mathbf{T}_i) (\Delta \mathbf{\hat{u}}_i - \Delta \mathbf{\hat{u}}_i) dS \right] \tag{3-3-10}
\]

where

\[
\Phi(\mathbf{E}^f; \mathbf{\Theta}) = \text{Helmholtz free energy function is given by Eq. (3-3-5)}
\]

\[
\frac{\partial \Phi(\mathbf{E}^f; \mathbf{\Theta})}{\partial \mathbf{E}^f} = \Delta \mathbf{S}_f
\]

\( \mathbf{S}_f \) = Kirchhoff stresses

\( \mathbf{T}_i \) = Tractions on the interelement boundary

\( \mathbf{F}_i \) = body forces
and the independent quantities are $\Delta E_{ij}$, $\Delta \sigma_{ij}$, $\Delta u_i$ and $\Delta \bar{u}_i$.

Since

$$
\int_{V_n} \frac{1}{2} (\sigma_{ij} + \Delta \sigma_{ij}) (\Delta u_{ij} + \Delta u_i) dV
= \int_{V_n} (\sigma_{ij} + \Delta \sigma_{ij}) \Delta u_i dS - \int_{V_n} (\sigma_{ij} + \Delta \sigma_{ij}) \Delta \bar{u}_i dS
$$

(3-3-11)

the functional of Eq. (3-3-10) can be rewritten in the form;

$$
\mathcal{T}_{HG} = \sum_n \left[ \int_{V_n} \left[ \Phi(\Delta E_{ij}; \theta) - \Delta \sigma_{ij} \Delta E_{ij} - \frac{1}{2} (\Delta u_{ij} + \Delta u_i) \right] dV
+ \int_{S_n} \left[ (\sigma_{ij} + \Delta \sigma_{ij}) \Delta u_i + \frac{1}{2} (\Delta u_{ij} + \Delta u_i) \right] dS
- \int_{S_n} (\sigma_{ij} + \Delta \sigma_{ij}) \Delta \bar{u}_i dS
- \int_{S_n} (\sigma_{ij} + \Delta \sigma_{ij}) \Delta \bar{u}_i dS
\right] + (F_i - \Delta \bar{F}_i) \Delta \bar{u}_i dS
$$

(3-3-12)

The use of a relationship for tractions on the interelement boundary

$$
T_i + \Delta T_i = \left[ (\sigma_{ij} + \Delta \sigma_{ij}) + (\sigma_{ij} + \Delta \sigma_{ij}) (\Delta u_{ij} + \Delta u_i) \right] j
$$

(3-3-13)

and a equation

$$
\int_{V_n} \Delta \sigma_{ij} \frac{1}{2} (\Delta u_{ij} + \Delta u_i) dV = \int_{S_n} \Delta \sigma_{ij} \Delta u_i dS - \int_{V_n} \Delta \sigma_{ij} \Delta \bar{u}_i dV
$$

leads to the following functional:

$$
\mathcal{T}_{HG} = \sum_n \left[ \int_{V_n} \left[ \Phi(\Delta E_{ij}; \theta) - \Delta \sigma_{ij} \Delta E_{ij} + \frac{1}{2} (\Delta u_{ij} + \Delta u_i) \Delta u_{ij} + \Delta u_i \right] dV
+ \int_{S_n} \left[ \frac{1}{2} (\sigma_{ij} + \Delta \sigma_{ij}) (\Delta u_{ij} + \Delta u_i) \Delta u_{ij} + \Delta u_i \right] dS
- \int_{V_n} \Delta \sigma_{ij} \Delta u_i dV
- \int_{S_n} \Delta \sigma_{ij} \Delta \bar{u}_i dV
+ \int_{S_n} \left[ (\sigma_{ij} + \Delta \sigma_{ij}) \Delta u_i + \Delta \sigma_{ij} \Delta \bar{u}_i \right] dS
- \int_{S_n} (\sigma_{ij} + \Delta \sigma_{ij}) \Delta \bar{u}_i dS
\right] + (F_i - \Delta \bar{F}_i) \Delta \bar{u}_i dS
$$

(3-3-14)
When the portion of the stress equilibrium equations is introduced as the constraint equation, an alternative version of the modified Hu-Washizu principle can be written as

$$
\mathcal{T}_{\text{HG}}^{**} = \mathcal{T}_{\text{HG}} + \sum_n \int_{\Gamma_n} \mu_i C_i^\mu (\Delta \delta_{ij}, \Delta U_f) d\Gamma
$$

(3-3-15)

where $C_i^\mu (\Delta \delta_{ij}, \Delta U_f)$ represents the portion of stress equilibrium equations and $\mu_i$ are Lagrangian multipliers to constrain the portion of equilibrium equations.

When the functional given by Eq. (3-3-15) is applied to shallow shells based upon Marguerre's shallow shell theory, we have the following form:

$$
\mathcal{T}_{\text{HG}}^{**} = \sum_n \int_{\Gamma_n} \left[ \Phi (\Delta E_{ij}, \Delta K_{ij}; \Theta) - (\Delta N_{ij} \Delta E_{ij} + \Delta M_{ij} \Delta K_{ij}) \\
+ \frac{1}{2} (\Delta N_{ij} + \Delta N_{ji})(\omega_{ij} \omega_{ij} + \Delta \omega_{ij} \Delta \omega_{ij}) + \frac{1}{2} \Delta N_{ij} \Delta \omega_{ij} \Delta \omega_{ij} \\
- (\Delta N_{ij} \Delta E_{ij} + \Delta M_{ij} \Delta K_{ij}) + \Delta N_{ij} \frac{1}{2} \Delta \omega_{ij} \Delta \omega_{ij} \right] dA
\\
- \int_{\Gamma_n} \left[ \left\{ (\Delta N_{ij} + \Delta N_{ji} \omega_{ij}) \beta \right\} dA \\
+ \left\{ (\Delta S_{ij} + \Delta S_{ji} \omega_{ij}) + (\Delta F_i + \Delta F_j) \right\} dA \\
- \int_{\Gamma_n} \left[ (\Delta N_{ij} + \Delta N_{ji}) \beta \omega_{ij} + (\Delta S_{ij} + \Delta S_{ji}) \omega_{ij} \omega_{ij} \right] dA \\
+ \int_{\Gamma_n} \left[ (\Delta N_{ij} + \Delta N_{ji}) \beta \omega_{ij} + (\Delta S_{ij} + \Delta S_{ji}) \omega_{ij} \omega_{ij} \right] dA \\
- (\Delta M_{ij} + \Delta M_{ji}) \beta \omega_{ij} \omega_{ij} \right] dA \\
+ \int_{\Gamma_n} \left[ \left\{ (\Delta N_{ij} + \Delta N_{ji}) \beta \omega_{ij} + (\Delta S_{ij} + \Delta S_{ji}) \omega_{ij} \omega_{ij} \right] dA \\
+ \int_{\Gamma_n} \left[ \left\{ (\Delta N_{ij} + \Delta N_{ji}) \beta \omega_{ij} + (\Delta S_{ij} + \Delta S_{ji}) \omega_{ij} \omega_{ij} \right] dA \\
+ \sum_n \int_{\Gamma_n} \left[ (\Delta N_{ij} + \Delta N_{ji}) \beta \omega_{ij} + (\Delta S_{ij} + \Delta S_{ji}) \omega_{ij} \omega_{ij} \right] dA
$$

(3-3-16)

where the higher order terms than $O(\Delta^2)$ are omitted by assuming that the increments are sufficiently small and the magnitude of rotation is moderately
small, and the following assumptions are made

\[ w = \tilde{w} \quad \text{on } \partial A_n \]

\[ \Delta \tilde{u}_d = \Delta \tilde{u}, \Delta \tilde{w} = \Delta \bar{w}, \Delta \tilde{w}_d = \Delta \bar{w}_d \quad \text{on } A u_n \]

It can be seen in Eq. (3-3-16) that the following constraint equations are introduced by the use of the Lagrangian multipliers;

\[
\begin{align*}
(N \alpha \beta + \Delta N \alpha \beta),_{\beta} & = 0 \\
M \alpha \beta,_{\beta} & = 0
\end{align*}
\]

(3-3-17)

3.3.3 Matrix Formulation and Computation Procedure

The stresses and strains may be interpolated in terms of stress parameters and strain parameters respectively in the form,

\[
\begin{align*}
\sigma_{ij} + \Delta \sigma_{ij} & = \Pi \beta \\
\varepsilon_{ij} + \Delta \varepsilon_{ij} & = \Pi \alpha
\end{align*}
\]

(3-3-18)

The displacements may be represented by

\[
\tilde{u}_i = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}, \quad \Delta \tilde{u}_i = \begin{bmatrix} \Delta \hat{u} \\ \Delta \hat{v} \end{bmatrix}
\]

(3-3-19)

The Lagrangian multipliers are interpolated as

\[
\begin{bmatrix} M_d \\ M_3 \end{bmatrix} = \begin{bmatrix} N_d \\ N_3 \end{bmatrix} \mu = [N \mu]
\]

(3-3-20)

The various terms of the functional given by Eq. (3-3-16) can be written in matrix notations as follows:
\[ (1) \quad \int_{A_n} \Phi(\Delta e_{\rho}, \Delta K_{\rho} \theta) dA = \frac{1}{2} A^{T} J_1 A - \frac{1}{2} J_2 A - \frac{1}{2} A^{T} J_3 A^{T} - r A^{T} J^{\Theta} A^{T} \]

\[ (3-3-21) \]

where

\[ \theta = \theta_0 + \tau \Delta \theta \quad (\theta_0 \text{ at } n \text{ step, } \Delta \theta \text{ fixed}) \]

\[ J_1 = \int_{A_n} P^T C P dA \]

\[ J_2 = \int_{A_n} (e_{ij}^0 + e_{ij}^0) C P dA \]

\[ C = \text{elastic coefficient matrix} \]

\[ J^\theta = \int_{A_n} A \Theta [A^T_{ij} \theta, D^\theta_{ij}] P dS \]

\[ A^T_{ij} = \int_{\frac{h}{2}} \int_{\frac{h}{2}} C_{ijkl} \partial \epsilon_{kl} \partial \epsilon_{ij} d \zeta \]

\[ D^\theta_{ij} = \int_{\frac{h}{2}} \int_{\frac{h}{2}} C_{ijkl} \partial \epsilon_{kl} \partial \epsilon_{ij} \partial \theta d \zeta \]

\[ \tau = \text{load increment parameter} \]

\[ (2) \quad \int_{A_n} \left[ \frac{1}{2} (N_{\rho} + \Delta N_{\rho})(\Delta w_{\rho} + \Delta \delta_{\rho} + \Delta \delta_{\rho} + \Delta \delta_{\rho} + \Delta \delta_{\rho}) \right] dA \]

\[ = \beta^T C \Delta \theta \]

\[ (3-3-22) \]

where

\[ C = \int_{A_n} P^T N_{\rho} p^T \delta \partial w \partial w dA \]

\[ w_{ni} = b \partial w_{n} \partial x = \partial w_{n} \partial x \]

\[ (3) \quad \int_{A_n} \left[ \Delta N_{\rho} (E_{\rho} + \Delta E_{\rho}) + \Delta M_{\rho} (K_{\rho} + \Delta K_{\rho}) \right] dA = \beta^T H_1 + H_2 \]

\[ (3-3-23) \]

where

\[ H_1 = \int_{A_n} P^T P dA \]

\[ H_2 = \int_{A_n} C^T P dA \]

\[ (4) \quad \int_{A_n} \frac{1}{2} N_{\rho}(\Delta \delta_{\rho} + \Delta \delta_{\rho} + \Delta \delta_{\rho} + \Delta \delta_{\rho} + \Delta \delta_{\rho}) dA = \frac{1}{2} \gamma^T K_{\rho} \Delta \theta \]

\[ (3-3-24) \]
where

\[ K_a = \int_A (C_{\alpha \beta}) \frac{\partial}{\partial w} \frac{\partial}{\partial w} dA \]

(5)

\[ \int_A \Delta N_{\alpha \beta} \cdot \frac{1}{2} \mathbf{w}_{\alpha \beta} \mathbf{w}_{\alpha \beta} dA = \frac{1}{2} \mathbf{p}^T \hat{G} \mathbf{q} \]

(3-3-25)

(6)

\[ \int_A \left[ (N_{\alpha \beta} + \Delta N_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{u}_d + (S_{\alpha \beta} + \Delta S_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{w} \right] dA = \mathbf{q}^T \hat{G} \mathbf{q} \]

(3-3-26)

where

\[ \hat{G} = \int_A \mathbf{E}^T \mathbf{b} dA \]

(7)

\[ \int_A \left[ (F_{\alpha \beta} + \Delta F_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{u}_d + (F_{\alpha \beta} + \Delta F_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{w} \right] dA = \mathbf{q}^T \hat{G} \mathbf{q} \]

(3-3-27)

where

\( \mathbf{q} = \) load vector due to body forces.

(8)

\[ \int_A \left[ (N_{\alpha \beta} + \Delta N_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{u}_d + (S_{\alpha \beta} + \Delta S_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{w} \right] dA = \mathbf{q}^T \hat{G} \mathbf{q} \]

(3-3-28)

(9)

\[ \int_{\partial A} \left[ (N_{\alpha \beta} + \Delta N_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{u}_d + (S_{\alpha \beta} + \Delta S_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{w} - (M_{\alpha \beta} + \Delta M_{\alpha \beta})_{\alpha \beta} \Delta \mathbf{w}_s \right] d\mathbf{s} = \mathbf{q}^T \hat{G} \mathbf{q} \]

(3-3-29)

where

\[ \hat{G} = \int_{\partial A} (\mathbf{P} \cdot \mathbf{N})^T \mathbf{N} d\mathbf{s} \]

\( \mathbf{N} = \) direction cosine vector along interelement boundary.

(10)

\[ \int_{\partial A} \left[ \Delta N_{\alpha \beta} \mathbf{v}_{\alpha \beta} \Delta \mathbf{u}_d + \Delta S_{\alpha \beta} \mathbf{v}_{\alpha \beta} \Delta \mathbf{w} - \Delta M_{\alpha \beta} \mathbf{v}_{\alpha \beta} \Delta \mathbf{w}_s \right] d\mathbf{s} = \mathbf{q}^T \hat{G} \mathbf{q} \]

(3-3-30)
(11) \[ \int_{\partial \Omega} \left[ (\tilde{N}_{\alpha \beta} + \delta N_{\alpha \beta}) N_{\alpha \beta} \right] \partial \w + (\tilde{S}_{\alpha \beta} + \delta S_{\alpha \beta}) J_{\alpha} \partial \w - (\tilde{M}_{\alpha \beta} + \delta M_{\alpha \beta}) J_{\alpha} \partial \w_m \right] ds \\
\quad = - \tilde{Q} \Delta \eta \\
\text{where} \\
\tilde{Q} = \text{load vector due to prescribed boundary tractions.} \\

(12) \int_{A_n} \left( N_{\alpha \beta} J_{\alpha} + M_{\alpha \beta} J_{\alpha} \right) \partial \Omega + M_{\alpha \beta} J_{\alpha} \right] dA = \beta^T \zeta \mu \\
\text{where} \\
\left\{ \begin{array}{l}
( N_{\alpha \beta} + \delta N_{\alpha \beta}) J_{\alpha} \\
M_{\alpha \beta} J_{\alpha}
\end{array} \right\} \sim \beta^T \mu \\

and \\
\zeta = \int_{A_n} \zeta^T N dA \\

Hence the functional for one element can be expressed as \\
\[ \tilde{\tau}_{HZ}^{***} = \frac{1}{2} \beta^T J \Sigma \beta - \frac{1}{2} \beta^T J \Sigma - \frac{1}{2} \beta^T J \Sigma - \tau \beta^T \Sigma \beta^T \]
\[ - \beta^T H \Sigma + H \Sigma + \beta^T \Sigma \beta \beta + \frac{1}{2} \Delta \beta^T \kappa \beta \Delta \beta \]
\[ + \frac{1}{2} \beta^T \Sigma \beta \beta - \beta^T \Sigma \beta \beta - \frac{1}{2} \beta^T \Sigma \beta \beta + \beta^T \Sigma \beta \beta \]
\[ - (\tilde{Q} + \tilde{Q}^T) \Delta \eta + \beta^T \zeta \mu \] \\
\[ = \frac{1}{2} \beta^T J \Sigma \beta - \frac{1}{2} \beta^T J \Sigma - \frac{1}{2} \beta^T J \Sigma - \tau \beta^T \Sigma \beta^T \]
\[ - \beta^T H \Sigma + H \Sigma + \frac{1}{2} \Delta \beta^T \kappa \beta \Delta \beta \]
\[ + \frac{1}{2} \beta^T \Sigma \beta \beta - \beta^T \Sigma \beta \beta - \beta^T \Sigma \beta \beta + \beta^T \Sigma \beta \beta \]
\[ + \beta^T \Sigma \beta \beta \]
\[ - (\tilde{Q} + \tilde{Q}^T) \Delta \eta + \beta^T \zeta \mu \] \\
\[ \text{(3-3-33)} \]
where
\[
\mathcal{G}^{*0} = \frac{1}{2} \hat{\mathcal{G}} - \hat{\mathcal{G}} + \mathcal{G}
\]
\[
\mathcal{G}^{*} = \hat{\mathcal{G}} - \hat{\mathcal{G}} + \mathcal{G}
\]

Taking the variations with respect to \( \hat{\mathcal{G}}, \beta^T, \mu \) and \( \Delta \mathcal{G} \), we obtain
\[
\begin{align*}
\mathcal{G}^T \mathcal{J}_1 - \mathcal{J}_2 - \mathcal{G}^\theta \beta^T \mathcal{H}_1 + \mathcal{H}_2 &= 0 \\
- \mathcal{H}_1 \mathcal{G}^* + \mathcal{G}^{*0} \mathcal{G} + \mathcal{G}^* \Delta \mu + \mathcal{G}^\mu \mu &= 0 \\
\beta^T \mathcal{G}^\mu &= 0 \\
\Delta \mathcal{G}^T \mathcal{K} + \beta^T \mathcal{G}^* - (\bar{\mathcal{G}} + \mathcal{G}^\theta) &= 0
\end{align*}
\]
(3-3-34)

Then we get
\[
\begin{align*}
\beta &= \mathcal{H}_1^\dagger \left( \mathcal{J}_1 \mathcal{G} + \mathcal{H}_2^T - \mathcal{J}_2 - \mathcal{G}^\theta \mathcal{H}_1 \right) \\
\mathcal{G} &= \mathcal{H}_1^\dagger \left[ \mathcal{G}^{*0} \mathcal{G} + \mathcal{G}^* \mathcal{G} + \mathcal{G}^\mu \mu \right] \\
\mu &= \mathcal{M} \left[ \mathcal{G}^{*0} \mathcal{G} + \mathcal{G}^\mu \mu \right] + \mathcal{M} \mathcal{G}^{*0} \mathcal{G} + \mathcal{H}_1^{-1} (\mathcal{H}_2^T \mathcal{G}^\theta) - \mathcal{H}_1^{-1} \mathcal{G}^\theta \mathcal{H}_1^{-1} \mathcal{J}^T
\end{align*}
\]
(3-3-35)

where
\[
\mathcal{M} = \mathcal{H}_1^\dagger \mathcal{J}_1 \mathcal{H}_1^{-1}
\]

Substituting Eq. (3-3-35) into Eqs. (3-3-34), it follows that
\[
\begin{align*}
(\mathcal{G}^* \mathcal{M} \mathcal{G}^* + \mathcal{K} \mathcal{G}) \Delta \mathcal{G} + \mathcal{G}^* \mathcal{M} \mathcal{G} \mathcal{G}^\mu &= \bar{\mathcal{G}}^T + \bar{\mathcal{G}}^T - \mathcal{G}^* \mathcal{M} \mathcal{G}^* \mathcal{G} \\
+ \mathcal{H}_1^{-1} (\mathcal{H}_2^T - \mathcal{J}_2) - \mathcal{H}_1^{-1} \mathcal{J}^\theta \mathcal{H}_1^{-1} \mathcal{J}^T \} \\
\mathcal{G}^\mu \mathcal{M} \mathcal{G}^{*0} \mathcal{G} + \mathcal{G}^\mu \mathcal{M} \mathcal{G}^{*0} \mathcal{G} = \mathcal{G}^\mu \left[ \mathcal{M} \mathcal{G}^{*0} \mathcal{G} + \mathcal{H}_1^{-1} (\mathcal{H}_2^T - \mathcal{J}_2^T) \\
- \mathcal{H}_1^{-1} \mathcal{J}^\theta \mathcal{H}_1^{-1} \mathcal{J}^T \right]
\end{align*}
\]
(3-3-36)
or
\[
\begin{align*}
\mathcal{Y} \Delta \xi + \mathcal{U}^T \mu &= \mathcal{Z} - \gamma H_{1}^{-1} J_{\theta}^T \mathcal{G}^* T \\
\mathcal{W} \Delta \xi + \mathcal{W}^T \mu &= \mathcal{Z} - \gamma H_{1}^{-1} J_{\theta}^T \mathcal{G}^* T
\end{align*}
\]

(3-3-37)

where
\[
\begin{align*}
\mathcal{Y} &= \mathcal{G}^* T \mathcal{M} \mathcal{G}^* + \beta \xi \\
\mathcal{U} &= \mathcal{Z} - \gamma H_{1}^{-1} J_{\theta}^T \mathcal{G}^* \\
\mathcal{W} &= \mathcal{Z} - \gamma H_{1}^{-1} J_{\theta}^T \mathcal{G}^* \\
\end{align*}
\]

and
\[
\xi = \mathcal{M} \mathcal{G}^* \xi + H_{1}^{-1} (H_{2}^T - J_{\theta}^T)
\]

By carrying out static condensation, we finally obtain
\[
(\mathcal{Y} - \mathcal{U}^T \mathcal{W}^{-1} \mathcal{U}) \Delta \xi = \gamma B^T H_{1}^{-1} J_{\theta} T + Q^T + Q_{\theta}^T - B^T \xi
\]

(3-3-38)

or
\[
\kappa \Delta \xi = \gamma Q^T + Q^T + Q_{\theta}^T - B^T \xi
\]

where
\[
\mathcal{A} = \mathcal{G}^* - \mathcal{G}^* \mathcal{W}^{-1} \mathcal{U}
\]

In practical computation, the assurance of the satisfaction of the constitutive equation in integral sense can be performed by checking the value of
\[
\| \int_{\Omega} (\mathcal{A} - \mathcal{A} \xi) T_{P} dA \|
\]

When we rewrite Eq. (3-3-38), we have
\[
\kappa \Delta \xi = \gamma Q_{\theta} + \mathcal{P}
\]

(3-3-39)
where \( \Delta q = \text{fixed load vector} \)
\[ R = \text{unbalanced load vector} \]

In the present investigation, \( \Delta q \) is separated into two parts,
\[
\begin{align*}
\kappa \Delta q_1 &= \Delta q_0 \\
\kappa \Delta q_2 &= R
\end{align*}
\]

(3-3-40)

Then the displacements can be obtained from
\[
\Delta q = \tau \Delta q_1 + \Delta q_2
\]

(3-3-41)

When the unbalance load becomes too big at a certain step, the incremental solution might drift away from the exact solution. In order to avoid the situation, the value of the following norm is controlled so that
\[
d = ||\Delta q_1||^2 + ||\Delta q_2||^2 = a \text{ given constant}
\]

(3-3-42)

Plugging Eq. (3-3-41) into Eq. (3-3-42) leads to
\[
\begin{align*}
d &= \tau^2 ||\Delta q_1||^2 + 2\tau (\Delta q_1, \Delta q_2) + ||\Delta q_2||^2 \\
&= \tau^2 a + 2\tau b + C
\end{align*}
\]

(3-3-43)

where
\[
\begin{align*}
a &= ||\Delta q_1||^2 \\
b &= (\Delta q_1, \Delta q_2) \\
c &= ||\Delta q_2||^2
\end{align*}
\]

Now we can determine the magnitude of \( \gamma \) by solving Eq. (3-3-44) as
\[
\gamma = \frac{-b \pm (\Delta q^2 cd)^{1/2}}{a}
\]

(3-3-44)
It is noted that

(1) when \( a^2 - cd \leq 0 \), \( \gamma = -\frac{b}{a} \)

(2) when \( a^2 - cd > 0 \) for loading process, \( \gamma = \frac{-b + (a^2 cd)^{1/2}}{a} \)

(3) when \( a^2 - cd > 0 \) for unloading process, \( \gamma = \frac{-b - (a^2 cd)^{1/2}}{a} \)

### 3.4 Example Solutions

#### 3.4.1 Example Solutions by the Hybrid SemiLoof Elements for Plates and Shallow Shells

The following example problems are solved to examine the behavior of hybrid semiLoof elements proposed in the present paper:

(i) bending of square plate with clamped edges under a concentrated load

(ii) pinched cylindrical shell

(iii) cylindrical roof under uniform load.

To start with, bending of a square plate with clamped edges under a concentrated load is examined to check the characteristics of hybrid semiLoof plate bending elements.

As shown in Fig. 6, a triangular element with 9\( \theta \) yields the upper bound solution for the deflection under the applied load because this element is the same as the equilibrium model of Fraeijs de Veubeke [14]. The increase of the number of \( \theta \) for triangular elements gives a stiffer solution.
It is shown that a 13β triangular element performs quite well. Unlike the behavior of the triangular element, the increase of the number of β from 13 to 17 makes the difference in the solutions for rectangular elements. In fact, both solutions are almost identical to that by 9β triangular elements. A rectangular element with 19β does yield a little stiffer solution.

It should be pointed out that the semiLoof element developed by Iron doesn't behave very well under point loading conditions, while hybrid semiLoof elements respond quite reasonably in this problem.

In order to test hybrid semiLoof shallow shell elements under severe loading conditions, the elements are applied to the pinched cylindrical shell problem. The finite element solutions obtained by 4x4 mesh pattern for both the 37β triangular element and the 59β quadrilateral element are given in the form of distributions of displacement components in Fig. 7a and of stress components in Figs. 7b and 7c.

Figure 7a shows that by using triangular elements with only 4 x 4 mesh the solutions for both displacement components have more than 10% error at the vicinity of the concentrated load. While the solutions by using quadrilateral elements are considerably better. On the other hand, Figs. 7b and 7c indicate that both elements yield fairly good results for the stress components. Since the present problem involves a deep cylindrical shell, it is expected that a sufficiently large number of shallow shell elements should be used.

The final example application is a cylindrical shell roof problem. The shell is loaded by its own dead weight and is supported by diaphragms
at the ends but is free along the side edges (Fig. 8).

Under the present method of formulation considerable computational effort is spent on the inversion of the matrix \((R^{-1}M^{-1}R)^T\), the order of which is equal to the number of \(\lambda\)'s used. It is seen that when the satisfaction of equilibrium equations can be relaxed, the number of \(\lambda\)'s can be reduced. A numerical experiment was made for the cylindrical shell roof problem using quadrilateral elements derived by using 12, 15 and 22 \(\lambda\)'s. Only for the element obtained by using 22 \(\lambda\)'s, are the equilibrium equations satisfied competely, and as shown in the results given in Figs. 9a and 9b, when the number of \(\lambda\)'s is reduced furthermore the results for moments become gradually deteriorated.

The convergence characteristics of the hybrid semiLoof shallow shell elements is illustrated for the vertical deflection at the center of the straight edge in Fig. 10, in which the result of the triangular shallow shell element developed by Cowper, Lindberg and Olson [23] is also included. The exact solution is obtained based upon the shallow shell theory. It can be seen that the performance of the quadrilateral element compares well with that of the triangular element of Cowper et al., while the present triangular element performs quite poorly.

It turns out that the poor behavior of the 37-\(\beta\) triangular shallow shell element is due to the fact that rigid body motions of the element are not very well represented. An eigenvalue analysis indicates that the six lowest eigenvalues of the stiffness matrix of this 37-\(\beta\) element are only of the 4th order smaller than the next larger one and the corresponding eigenmodes are clearly not rigid body modes. Because the assumed boundary
distribution is only quadratic, it is not possible to represent the rigid body motion exactly. In the case of the quadrilateral element, the resulting eigenmodes for the six lowest eigenvalues are much closer to rigid body motions. Thus, it appears that for triangular elements further considerations should be made to improve the representation of rigid body motions.

The possibility of improving the element by partial satisfaction of these equilibrium equations were investigated. First of all, an estimation of the orders of the various terms of the stress equilibrium equations can be carried out, since the assumed stresses are represented in natural coordinates. For a shallow shell the orders are as follows:

\[
\begin{align*}
N_{\alpha\beta,\beta} &= O(\Delta) \\
F_{\alpha} &= O(1) \\
M_{\alpha\beta,\alpha\beta} &= O(\Delta^2) \\
N_{\alpha\beta z,\alpha\beta} &= O(z,_{\alpha\beta}) \\
N_{\alpha\beta z,\alpha} &= O(\Delta z,_{\alpha})
\end{align*}
\]

where \(1/\Delta\) is the size of the element. For shallow shells the terms involving geometric factors should be dropped and the following equilibrium equations are resulted:

\[
\begin{align*}
N_{\alpha\beta,\beta} &\simeq 0 \\
M_{\alpha\beta,\alpha} &\simeq 0
\end{align*}
\]
Through these considerations, the attempt of partial satisfaction of stress equilibrium equations was made such that the coupling effect can be avoided.

As an illustration, a 24-β triangular element with 5λ's was employed for the pinched cylindrical shell problem. It is shown in Fig. 7 that the solution for displacements became more flexible. Furthermore, the computation cost was reduced by a great extent. Thus, when the equilibrium equations for shallow shells are satisfied only partially, the two terms with the Lagrange multipliers $u_{\alpha}^{\lambda}$ and $w_{\lambda}$ in the variational functional $\pi_{HG}^{*}$ [Eq. (3-1-5)] can be modified. For example, in order to satisfy $N_{\alpha\beta,\beta} = 0$ and $M_{\alpha\beta,\beta} = 0$, these terms become simply $\mu N_{\alpha\beta,\beta}$ and $\mu M_{\alpha\beta,\alpha\beta}$, respectively. Apparently for the 24-β triangular element, such modification means not only simplicity in computation, but also improvement in solution accuracy.

3.4.2 Example Solutions by Hybrid SemiLoof General Shell Element

The following example problems were solved by the use of a hybrid semiLoof general shell quadrilateral element as shown in Fig. 12:

1. Cylindrical shell roof given in Fig. 10.

2. Cylindrical shell under internal pressure as the first order patch test for thin shells (momentless membrane).

3. Slit cylinder as the second order patch test for thin shells (inextensional bending).

First of all, the order estimation of each term in the stress equilibrium equations for the cylindrical shell was carried out by assuming that
0 [(membrane resultant stress) x (thickness h)] = 0 [moment resultant],
and the element size is given by $2\varepsilon x 2\theta_0$ whose element area becomes
\[ R\varepsilon \theta_0 = O(\varepsilon^2) \]
where $\Delta$ represents a control parameter for the finite element mesh size.

Since the homogeneous equilibrium equations are expressed as
\[
\begin{align*}
\frac{\partial \bar{m}_{11}}{\partial \bar{s}} - \frac{1}{2R} \frac{\partial \bar{m}_{12}}{\partial \bar{s}} &= 0 \\
\frac{\partial \bar{m}_{12}}{\partial \bar{s}} + \frac{3}{2R} \frac{\partial \bar{m}_{12}}{\partial \bar{s}} + \frac{1}{R} \frac{\partial \bar{m}_{22}}{\partial \bar{s}} &= 0 \\
\frac{\partial^2 \bar{m}_{ab}}{\partial \bar{s} \partial \bar{s}} - R\bar{L}_{ab} &= 0
\end{align*}
\]
the order estimation of each term is obtained in the form;
\[
\begin{align*}
\frac{\partial \bar{m}_{11}}{\partial \bar{s}} &= O\left(\frac{1}{\varepsilon^4}\right) \quad (d; \text{no sum}) \\
\frac{1}{2R} \frac{\partial \bar{m}_{12}}{\partial \bar{s}} &= O\left(\frac{1}{\varepsilon^4 \cdot R}\right) \\
\frac{\partial \bar{m}_{12}}{\partial \bar{s}} &= O\left(\frac{1}{\varepsilon^4 R}\right) \\
\frac{3}{2R} \frac{\partial \bar{m}_{12}}{\partial \bar{s}} &= O\left(\frac{1}{\varepsilon^4 \cdot R}\right) \\
\frac{1}{R} \frac{\partial \bar{m}_{22}}{\partial \bar{s}} &= O\left(\frac{1}{\varepsilon^4 \cdot R}\right) \\
\frac{\partial^2 \bar{m}_{ab}}{\partial \bar{s} \partial \bar{s}} &= O(h) \quad (\beta; \text{no sum}) \\
R\bar{L}_{ab} &= O\left(\frac{\varepsilon^2}{R}\right)
\end{align*}
\]
Note that the relative order of each term in each equilibrium equation is indicated.
In the present investigation, the partial satisfaction of the equilibrium equations is considered. In the first example of the cylindrical shell roof problem, the effect of some terms in the equilibrium equations is studied. The results are presented in Figures 13, 14 and 15. The three cases included in Figure 13 are,

(1) The terms \( \frac{1}{2R} \frac{\partial m_{12}^1}{\partial z}, \frac{3}{2R} \frac{\partial m_{12}^2}{\partial z} \), \( \frac{1}{R} \frac{\partial m_{33}^2}{\partial z} \) and \( R \frac{\partial \text{a}_{22}}{\partial z} \) are omitted from the stress equilibrium equations. Then by using 3\( \lambda \)'s the simplified equilibrium equations are satisfied with 36\( \beta \)s in integral sense.

(2) Only one term \( R \frac{\partial \text{a}_{22}}{\partial z} \) is omitted because of its dependence on the mesh size. Then by using 7\( \lambda \)s the simplified equilibrium equations are satisfied with 36\( \beta \)s.

(3) In this case, the same term as the case (2) is omitted. But the element has 38\( \beta \)s and 7\( \lambda \)s.

It is shown in Figure 13 that case (1) yields poor solutions and convergence characteristics while case (2) and (3) converges nicely. But the computation results indicated that there exists eight almost zero eigenvalues for case (1) and case (2). On the other hand, case (3) showed six almost zero eigenvalues because of two additional terms in \( \varepsilon_{11} \) and \( \varepsilon_{22} \). In Figure 14, the result for the convergence of the same displacement as that in Figure 10 is indicated with the results of a degenerated shell with reduced integration and 54 DOF triangular elements by Dawe which is known as an excellent shell element although it is not widely used because of
the nodal configuration with the higher order derivatives and its difficulty for applications to shell with junctions. It can be observed that the solution obtained by the degenerated shell element does not converge to the exact solution but to that by the shallow shell theory. On the other hand, the solutions obtained by the hybrid semiLoof elements of 36βs or 38βs with 7λs converge to the exact solution and are comparable to the solution of Dawe's shell elements. Also, the stress distributions obtained by the hybrid semiLoof element with 38βs are shown in Fig. 15. The results are satisfactory for both 3 x 3 and 4 x 4 mesh patterns.

The second example problem is a very severe test for degenerated shell elements and shallow shell elements because the element geometry cannot be exactly represented in those elements. The results given in Fig. 16 are obtained for two cases (\( \frac{h}{R} = 10^{-2}, 10^{-3} \)) by use of two different patch patterns (A and B). Obviously, the hybrid semiLoof element with 38βs passes the first order patch test regardless of the ratio \( \frac{h}{R} \) and the mesh size. In other words, the membrane stress state can be represented exactly and there is no locking phenomenon which is often observed in degenerated shell elements.

The final test example (Figure 17) is the so-called slit cylinder, i.e., a circular cylinder with reciprocal shear stresses applied along a longitudinal slit [24,25]. The solution is characterized by the fact that

(a) there is only constant torsional coupling

and

(b) the displacement field has no component normal to the surface but the other two components have a linear variation in the surface coordinates.
The method of attack is to solve the problem indicated in Figure 17 with the rigid-body modes secured so that normal displacement occurs. This test can be carried out by using only one element with the rigid-body modes secured [25]. Considering the equilibrium of the whole shell and assigning the equivalent nodal forces which are \( F_2 \) and \( F_3 = F_2 \alpha t ^{1/2} \phi \), it follows that with this loading condition the classical shell theory solution has only one significant stress field component \( M_{12} \).

The results are shown in Figure 18. First of all, it should be noted that when the rigid-body modes are secured so that no displacements normal to the shell are possible, the computation results gave the exact solution because the displacement distribution has linear one. As can be seen in Figure 18, the solutions for \( M_{12} \) at the center of the elements and \( w \) at the point C are presented.

In Figure 18, the relationship between the angle \( \phi \) and non-dimensional values of \( M_{12} \) and \( w \) divided by the corresponding solution to \( \phi = 0.01 \) is indicated. We can observe that the solution obtained by 368s yields a very satisfactory solution for this problem because the complete uncoupling of \( u_\alpha \) and \( w \) is carried out. On the other hand, the solution by 388s reveals that the element needs the use of a much finer mesh pattern so as to represent the rigid body motion satisfactorily. This consequence is primarily caused by the fact that the complete uncoupling of \( u_\alpha \) and \( w \) is prohibited by the presence of higher order terms in the assumption of \( \xi ^{22} \).

From this example, it can be concluded that, if the complete uncoupling of \( u_\alpha \) and \( w \) is possible by using the partial satisfaction of the equilibrium equation, rigid body motion (zero order patch test) and inextensional bending
(second order patch test) can all be satisfied without the introduction of special functions such as trigonometric functions.

It should be stressed here that the momentless membrane state (first order patch test) was satisfactorily obtained in the second example. Thus, the use of the hybrid stress method and the hybrid semiLoof element based upon new versions of the variational principles yields the most versatile and efficient method to construct a good thin shell element.

3.4.3 Example Solutions for Large Deflection Analysis by Stationary Lagrangian Formulation

In order to investigate the capability of the hybrid semiLoof element even for the large deflection analysis, the following examples were used:

(1) Cantilever plate under edge shear load as inextensional bending behavior

(2) Buckling of unsymmetric laminates under thermal loads.

The first example is a very simple test case. For the finite element analysis, in general, the inextensional bending behavior cannot be represented exactly because of the coupling between the membrane displacement and the lateral deflection. This situation is quite similar to thin shells. Figure 19 illustrates the test example. The exact solution of the tip deflection with a small magnitude is essentially linear except for the contribution of the boundary layer effect. The results were obtained by use of the 20 DOF element with \( u, v, w, w_x, w_y \) as four corner nodal
parameters and the hybrid semiLoof element with 38βs and 32 DOF. The solutions of the tip deflection and the stress distributions are presented in Figure 20.

In Figure 20a, we can see that the solution by the 20 DOF element yields quite poor solutions for the tip deflection even by using a 3 x 3 mesh pattern while the hybrid semiLoof element can accommodate the inextensional bending state nicely even by a 1 x 1 mesh pattern. The main reason for this consequence is that the hybrid semiLoof element has the same order of interpolation functions for $u_\alpha$ and $w$. On the other hand, the 20 DOF element has the linear distribution for $u_\alpha$ and the cubic distribution for $w$. Therefore, we can state that the hybrid semiLoof element is well balanced.

As shown in Figure 20b, the solution for the stress distributions obtained by the 20 DOF element indicates that a very large stress $N_x$ is induced; $N_x$ should be zero under an extensional bending. The result of the hybrid semiLoof element shows much better stress distributions.

As an illustration of the difference in the performance of those two elements, the second example was solved. The example consists of a thermal distortion of an unsymmetric laminated plate as shown in Figure 21. Under a decrease of temperature, the unsymmetric laminate deforms due to the presence of coupling between bending and membrane actions. For a cross-ply laminate, Reference 26 has shown that this is a buckling problem. The deformation pattern is a saddle-shaped one before buckling. But this deformation pattern becomes unstable at a certain bifurcation point as illustrated in Figure 22.
The computation results are given in Figures 23 and 24. In Figure 23, the relationship between temperature decrease and the deflection of the point A in Figure 22 is described for the case of a perfect structure. First of all, it should be pointed out that a 1 x 1 mesh pattern for the 20 DOF element did not yield any instability point.

As the mesh size becomes smaller and smaller, the temperature difference required for the bifurcation phenomena obtained by the 20 DOF element becomes lower and lower. On the other hand, the hybrid semiLoof element gives the bifurcation point even by a 1 x 1 mesh pattern. It should be noted that the bifurcation point is determined by getting the determinants of the tangent stiffness matrix at the Nth step and (N+1)th step \((\det \mathbf{K}_n > 0, \det \mathbf{K}_{n+1} < 0)\) and using the extrapolation to get \(\det \mathbf{K} = 0\).

In Figure 24, the behavior of the unsymmetric laminate with imperfection whose form is

\[ \omega^0(x, y) = \frac{h}{100} \left( \frac{x}{L} \right)^2 \]

is given. The solutions were obtained by the use of the 20 DOF elements with a 4 x 4 mesh pattern and the hybrid semiLoof element with a 2 x 2 mesh pattern. Both solutions show the drastic change in the deformation pattern of the unsymmetric laminate near the bifurcation point.

Through two test examples, it can be concluded that even for the large deflection analysis the hybrid semiLoof element showed the superior performance primarily because of the balanced choice of the displacement assumptions.
4. Use of new versions of $\mathcal{H}_g$ and $\mathcal{H}_k$ to show similarity and equivalence of different finite element models.

The present chapter is a discussion of similarity and equivalence among various finite element models. To be considered are the relationship between the quasi-conforming element originated by Tang (27) and the new version of modified Hu-Washizu principle and the relations of the displacement model with reduced, selective integration, the incompatible model and the hybrid stress model. Also since the modified Hu-Washizu principle leads to modified Hellinger-Reissner principle and modified complementary energy principle under some conditions, the hybrid stress model can be connected to the quasi-conforming element. Several topics related to this matter will be discussed. It is noted that the false shear phenomenon in four node axisymmetric solid element will be considered in Appendix 1 as a further discussion of the topics related to this chapter.
4.1 Quasi-conforming element and new version of modified Hu-Washizu principle

4.1.1 The relationship between quasi-conforming element and the modified Hu-Washizu principle and the patch test

In the construction of a stiffness matrix in finite element method applied to linear elastostatics, the strain energy can be written as

\[ U = \int_V \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} dV = \frac{1}{2} \mathbf{E}^T \mathbf{C} \mathbf{E} \]  

(4-1-1)

where \( \mathbf{E} \) = stiffness matrix
and \( \mathbf{C} \) = nodal quantities

In the quasi-conforming element, the strain-displacement relation is not satisfied in the pointwise manner. Instead, the strain is expressed as

\[ \varepsilon_{ij} = \mathbf{N} \alpha \]  

(4-1-2)

and, to satisfy the strain-displacement relation in integral sense, the weighting function \( W_{ij} \) is used such that

\[ \int_V W_{ij} \varepsilon_{ij} dV = \int_V W_{ij} \frac{1}{2} (u_{ij} + u_j i) dV \]

\[ = \int_{\partial V} W_{ij} \frac{1}{2} \hat{\mathbf{U}}_i dS - \int_V W_{ij} \frac{1}{2} \hat{\mathbf{U}}_idV \]  

(4-1-3)

where \( \hat{\mathbf{U}}_i \) and \( \hat{\mathbf{U}}_i \) are displacements on \( \partial V \) and \( V \) respectively. Here, in
general, $\hat{u}_i$ on $\partial V$ and $\hat{u}_i$ in $V$ may not be compatible and they can be expressed in terms of nodal displacements $\xi$ independently. The physical meaning of $W_{ij}$ needs not be given in eq. (4-1-3).

We can recognize that if $W_{ij}$ is replaced by $\sigma_{ij}$, this process is basically the same as the use of modified Hu-Washizu principle, i.e.

$$
\tau_{\text{HG}} = \sum_n \left[ \int_V \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} - \frac{1}{2} (\hat{u}_{ij} + \hat{u}_{ji}) \right] dV - \int_{\partial V} \sigma_{ij} \nu^i \left( \hat{u}_i - \hat{u}_i \right) dS \right] 
$$

(4-1-4)

Therefore, it can be seen that the finite element derived from eq. (4-1-4) is a kind of incompatible model in general.

For a distorted finite element, the following assumption can be employed;

$$
\sqrt{g} \sigma_{ij} = P \xi
$$

$$
\varepsilon_{ij} = P \xi
$$

(4-1-5)

where $dV = \sqrt{g} \, ds^i \, ds^j \, ds^3$ and $s^i$ defines the coordinate system.

As can be seen, the constant strain condition can be introduced in $\varepsilon_{ij}$. But the constitutive relation cannot be satisfied exactly because of the presence of $\sqrt{g}$ unless $\sqrt{g}$ is a constant. On the other hand, hybrid stress model can be derived from the use of the modified Hu-Washizu principle:

$$
\tau_{\text{HG}}^* = \sum_n \left[ \int_V \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} - \sigma_{ij} \hat{u}_i \right] dV 
+ \int_{\partial V} \sigma_{ij} \nu^i \hat{u}_i dS - \int_{\partial V} \sigma_{ij} \hat{u}_i dV \right] 
$$

(4-1-6)

For the distorted case, the following assumption is employed;
\[ \sigma_{ij} = \mathcal{P} \mathcal{B}, \quad \sqrt{\mathcal{J}} \varepsilon_{ij} = \mathcal{P} \mathcal{\Phi} \]  

(4-1-7)

Then, constant stress condition can be satisfied, but also in this case the constitutive relation can't be satisfied. Comparing \( \tau_{HG}^* \) with \( \tau_{HG}^* \), we can see that \( \tau_{HG}^* \) is more general than \( \tau_{HG} \) because the use of incompatible part of displacements \( u_i^2 \) enables us to constrain the stress equilibrium equations as discussed in chapter 2. The mathematical description of patch test for \( \tau_{HG}^* \) should be considered here. Originally, the displacement \( u_i \) was divided into two parts:

\[ u_i = \hat{u}_i + u_i^2 \quad \text{in } \mathcal{V}_n \]  

(4-1-8)

Alternative expression of \( \tau_{HG}^* \) given by eq. (4-1-6) can be written as

\[
\tau_{HG}^* = \sum_n \left[ \int_{\mathcal{V}_n} \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} \Omega_{ij} (u_{ij} + u_j^2) \right] d\mathcal{V} 
- \int_{\partial \mathcal{V}_n} \sigma_{ij} \mathcal{I}_j (\hat{u}_i + u_i^2 - \tilde{u}_i) dS \right] 
\]  

(4-1-9)

Therefore, the contribution of the incompatible part to the functional can be identified as

\[ I = \int_{\partial \mathcal{V}_n} \sigma_{ij} \mathcal{I}_j (\hat{u}_i + u_i^2 - \tilde{u}_i) dS \]  

(4-1-10)

This value should be zero under a given stress distribution \( \sigma_{ij}^* \) corresponding to the patch test;

\[ I^* = \int_{\partial \mathcal{V}_n} \sigma_{ij}^* \mathcal{I}_j (\hat{u}_i + u_i^2 - \tilde{u}_i) dS = 0 \]  

(4-1-11)
When the construction of a finite element is based upon a legitimate variational principle, the variational procedure ensures the satisfaction of the patch test if the constant stress or the constant strain state is appropriately introduced in the assumptions. To make this point clear, we shall consider $C^0$ continuity problem where $\hat{u}_i = \tilde{u}_i$ can be achieved easily. Now the equation (4-1-9) takes the form of

$$\mathcal{A}_{iH} = \sum_n \left[ \int_{\Gamma_n} \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} \sigma_{ij} (\hat{u}_{ij} + \tilde{u}_{ij}) \right. \right. \left. \left. - \sigma_{ij, k} u_i^k \right] \, d\Gamma \right]$$  \hspace{1cm} (4-1-12)

Therefore the finite element based upon eq. (4-1-12) should pass the patch test if the interpolation functions are chosen adequately.

On the other hand, the formulation of quasi-conforming element requires the vanishment of the term representing the effect of incompatibility; i.e.

$$I^* = \int_{\partial \Gamma_n} \sigma_{ij}^* \nu_i^* u_i^* \, dS = 0$$  \hspace{1cm} (4-1-13)

In the reference (28) by Chen and Tang, a four-node plane stress element named $Q_4e$ is developed. In the formulation, the strain assumptions are made by

$$\varepsilon_i = \left\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\}^T = \mathcal{P} \tilde{\varepsilon}$$  \hspace{1cm} (4-1-14)

Then the Cartesian components of strains are obtained as

$$\varepsilon = \left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\}^T = \mathcal{Z} \tilde{\varepsilon}$$  \hspace{1cm} (4-1-15)
where
\[
Z = \begin{pmatrix}
J_{11} & J_{12} & 0 & 0 \\
0 & 0 & J_{21} & J_{22} \\
J_{21} & J_{22} & J_{11} & J_{12}
\end{pmatrix}
\]

and
\[
J_{11}^* = \frac{1}{J} \frac{\partial y}{\partial n}, \quad J_{12}^* = -\frac{1}{J} \frac{\partial y}{\partial s} \\
J_{21}^* = -\frac{1}{J} \frac{\partial x}{\partial n}, \quad J_{22}^* = \frac{1}{J} \frac{\partial x}{\partial s}
\]

In the quasi-conforming element, the internal displacement \( \mathbf{u}_h \) was introduced in order to pass the patch test. The form of \( \mathbf{u}_h \) was determined by applying
\[
\iint B \, dxdy = 0 \quad (4-1-16)
\]

where
\[
B^h = \frac{1}{2} \left( \mathbf{V} \mathbf{u}_h + \mathbf{u}_h \mathbf{V} \right), \quad \mathbf{u}_h = B^h \lambda
\]

and this condition is equivalent to eq. (4-1-11). Also, the stress results are obtained from
\[
\mathbf{C} = \mathbf{C} \mathbf{E} = \mathbf{C} \mathbf{Z} \mathbf{E} = \mathbf{C} \mathbf{Z} \mathbf{E}^t \quad (4-1-17)
\]

where \( \mathbf{C} \) is an elastic coefficient matrix and the constitutive relation is not satisfied exactly. It should be noted that the introduction of \( \mathbf{u}_h \) in the formulation implies the constraint on the stress equilibrium equations.

Again we should emphasize that the satisfaction of the patch test can be performed easily for hybrid stress model because the model is
based upon a legitimate variational principle. Thus, even though the quasi-conforming element is based upon the functional given by eq. (4-1-9) as the consequence of the imposition of the patch test, the physical quantity \( \sigma_{ij} \) is treated as mathematical weighting factors or Lagrangian multipliers. Therefore, the considerations on equilibrium equations are not explicitly carried out.

### 4.1.2 Basis of quasi-conforming element

A given function is dependent upon two variables \( x^d \);

\[
f \equiv f(x^d) ; \quad d = 1, 2 \tag{4-1-18}
\]

Then, if this function is differentiable, the Taylor's expression can be written as

\[
f(x^d) = \sum_{m=0}^{n-1} \frac{1}{m!} \left( \frac{\partial^n f}{\partial x^d} \right)_0 f(x^d) + \frac{1}{n!} \left( \frac{\partial^n f}{\partial x^d} \right)_0 f(0) \quad (0 < \theta < 1) \tag{4-1-19}
\]

In the actual finite element formulation, the function may be approximated by using polynomials;

\[
|f(x^d) - f_i \xi_i (x^d)| < M h^n ; \quad i = 1, 2, \ldots, m \quad |x^d| < \frac{h}{2} \tag{4-1-20}
\]

where \( \xi_i \) is a polynomial function and \( M \) is a finite constant multiplying another polynomial function \( \xi_i^* \) and integrating over the domain leads to
\[ \left| \int_S f(x^d) \mathcal{G}_f^*(x^d) dS - \int_S f_i \mathcal{G}_i^*(x^d) dS \right| < M^* h^{n+m+2} \quad (4-1-21) \]

\[ \Rightarrow \left| \int_S f_i \mathcal{G}_i^* dS - \int_S \mathcal{G}_i^* dS \right| < M^* h^{n+m+2} \quad (4-1-22) \]

where \( \| \mathcal{G}_f^*(x^d) \| < M^{**} h^m \), \( \| \mathcal{G}_i^* \| = c h^2 \) and \( M^* = c M M^{**} \).

When \( f = g_y \), we have

\[ \int_S f \mathcal{G}_f^* dS = \int_S g_y \mathcal{G}_f^* dS = \int_S g_y \mathcal{G}_f^* dS - \int_S g \mathcal{G}_f^* dS \quad (4-1-23) \]

Also, when \( f = g_{y_d} \), we have

\[ \int_S f \mathcal{G}_f^* dS = \int_S g_{y_d} \mathcal{G}_f^* dS = \int_S g_{y_d} \mathcal{G}_f^* dS - \int_S g \mathcal{G}_f^* dS \]

\[ + \int_S g \mathcal{G}_f^* dS \quad (4-1-24) \]

The coefficients \( f_i \) can be obtained from eq. (4-1-22) approximately.

In order to approximate a function \( f \) by using polynomials, eq. (4-1-20) should be identical to the Taylor's expansion except error terms of higher order. This implies that \( f_i \) should converge to the coefficients of Taylor's expansion as \( h \to 0 \). But, in the limit of discretized finite element mesh, \( f \) should become constant. Therefore, so long as we have

\[
\left| f_i - f(x^d=0) \right| = O(h^m) \quad ; \quad m \geq 1
\]

\[
\| f \mathcal{G}_i^* (x^d) \| < |f_i| \| \mathcal{G}_i^* \| < M^{**} h^n \quad ; \quad i \geq 2, \ n \geq 1
\]

(4-1-25)

the convergence can be assured.
In a quasi-conforming element, the accuracy of the evaluation of \( \hat{f} \) is relying upon eq. (4-1-23) or eq. (4-1-24). In general, the construction of a compatible function \( g(\mathbf{x}) \) is difficult for \( C^1 \) continuity problem. Therefore, different functions are employed for \( g \) on \( \partial S \) and \( g \) on \( S \);

\[
\begin{align*}
    g &= \hat{g} \quad \text{on} \ S \\
    g &= \tilde{g} \quad \text{on} \ \partial S
\end{align*}
\] (4-1-26)

Because of this assumption, the convergence of the quasi-conforming element is not obvious.

For \( C^0 \) continuity problem, when \( \hat{g} - \tilde{g} \) stands for the incompatible functions, what is called "patch test" requires the following mathematical statement;

\[
\int_{\partial S} (\hat{g} - \tilde{g}) \cdot \mathbf{n} \, ds = 0
\] (4-1-27)

and, for the first order patch test,

\[
\int_{S} (\hat{g} - \tilde{g}) \, dS = 0
\] (4-1-28)

If eq. (4-1-27) holds true, the convergence of the finite element can be assured because the incompatible part doesn't contribute to the evaluation of eq. (4-1-23). Here, depending upon the degree of polynomial of \( \mathbf{f}^* \), the nth order patch test can be established.

If one can assume intuitively that \( g_{\partial S} \) should become constant as \( h \to 0 \), the intuitive requirement for convergence can be written simply
as eq. (4-1-28). But the justification is based upon only "intuition."

For \( C^1 \) continuity problem, the same argument can be made and the "patch test" can be represented by

\[
\int_{\partial S} \left[ (\tilde{\mathbf{g}} - \tilde{\mathbf{g}}) \nu_{d} \mathbf{G}_{d}^{*} - (\tilde{\mathbf{g}} - \tilde{\mathbf{g}}) \nu_{\beta} \mathbf{G}_{\beta}^{*} \right] ds = 0 \quad (4-1-29)
\]

and, for the first order patch test,

\[
\int_{S} (\tilde{\mathbf{g}} - \tilde{\mathbf{g}}) \nu_{\beta} ds = 0 \quad (4-1-30)
\]

So far as the convergence of solutions obtained by finite element method is concerned, the satisfaction of eqs. (4-1-28) and (4-1-29) is not necessary for the finite size of element. It has to be assured only in the limit of \( h \to 0 \) from mathematical point of view. The significance of the satisfaction of the first order patch test for the element of finite size which might be distorted is obviously depending upon how important the state of constant \( \mathbf{G}_{d}^{*} \) is in a given problem.

In the actual construction of a finite element, both \( \tilde{\mathbf{g}} \) and \( \tilde{\mathbf{g}} \) can be interpolated by use of nodal parameters \( \mathbf{q}_{i} \). Because of the assumption made in eq. (4-1-26), \( \tilde{\mathbf{g}} \) and \( \tilde{\mathbf{g}} \) are independent in general.

When \( \mathbf{q}_{i}^{*} \) is orthogonal to \( \mathbf{G}_{d}^{*} \) for the sake of simplicity,

\[
\langle \mathbf{q}_{i}^{*}, \mathbf{G}_{d}^{*} \rangle = \int_{S} \mathbf{q}_{i}^{*} \mathbf{G}_{d}^{*} ds = c^{*} h^{2(\nu + 1)} S_{d}^{*} \quad (4-1-31)
\]

The evaluation of \( \mathbf{f}_{i} \) can be done from eqs. (4-1-22), (4-1-23) and (4-1-31)
\[ | f_i - \frac{<\Phi_i, \Phi_i^*>}{<\Phi_i, \Phi_i^*>} | < \frac{M^*}{C^*} n^{-m} \]  \hspace{1cm} (4-1-32)

where

\[ <f, \Phi_i^*> = \begin{cases} \int_{\Sigma_i} \Phi_i^* \Phi_i d\Sigma - \int_{\Sigma} \Phi_i^* \Phi_i d\Sigma & \text{for } C^0 \text{ class} \hspace{1cm} (4-1-33) \\
\int_{\Sigma_i} \left[ \frac{\partial}{\partial \beta} \Phi_i^* d\Sigma - \frac{\partial}{\partial \alpha} \Phi_i^* d\Sigma \right] + \int_{\Sigma} \Phi_i^* \Phi_i d\Sigma & \text{for } C^1 \text{ class} \end{cases} \]

Therefore, the primary requirement of eq. (41-125) can be fulfilled if the proper choice of nodal displacements \( \Phi_i \) and the corresponding interpolation functions \( \hat{g} \) and \( \tilde{g} \). Here the kind of \( \Phi_i \) to be used is dependent upon the numerical integration scheme employed for the evaluation of eq. (4-1-33).

It should be pointed out that, unless

\[ | f_i - f^{(i-1)}(x^d=0) | = O(h^n) \quad ; \quad n \geq 1 \]  \hspace{1cm} (4-1-34)

where \( f^{(i-1)} \) represents the \((i-1)\)th order partial derivative of \( f \), the corresponding term \( \Phi_i^* \) does not contribute to the convergence rate.

Therefore, the inclusion of \( \Phi_i^* \) (higher order terms) does not improve the convergence characteristics in such cases.

4.2. The relationship between a model based upon Hu-Washizu principle and a model based upon the principle of minimum potential energy with reduced, selective integration or smoothed shape function.

First of all, the essential part of each variational principle for the construction of a stiffness matrix will be summarized for the sake of the following discussion.
The principle of minimum potential energy can be written as

$$\Pi_p = \sum_{i} \int_{\Omega} \frac{1}{2} C_{ijkl} \varepsilon_{ij}^*(u_m) \varepsilon_{kl}^*(u_n) dV + \int_{\partial\Omega} \tilde{u} i dS$$

$$\equiv \sum_{i} a_i^*(\varepsilon_{ij}^*) + \int_{\partial\Omega} \tilde{u} i dS = \text{minimum} \quad (4-2-1)$$

subject to

$$\varepsilon_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right) \quad \text{in} \; \Omega \quad (4-2-2)$$

The Euler equation in $\Omega$ is given by

equilibrium equations in integral sense

$$\left\{ C_{ijkl} \varepsilon_{ij}^*(u_m) \varepsilon_{kl}^*(u_n) \right\}_{i,j} = 0 \quad (4-2-3)$$

The corresponding expression to eq. (4-2-1) for incompatible model becomes

$$\Pi_p^* = \sum_{i} \int_{\Omega} \frac{1}{2} C_{ijkl} \varepsilon_{ij}^{**(u_m)} \varepsilon_{kl}^{*(u_n)} dV + \int_{\partial\Omega} \tilde{u} i dS$$

$$\equiv \sum_{i} a_i^{*(\varepsilon_{ij}^*)} + \int_{\partial\Omega} \tilde{u} i dS \quad (4-2-4)$$

subject to

$$\varepsilon_{ij}^{*} = \frac{1}{2} \left( u_{ij}^* + u_{ji}^* \right) \quad \text{in} \; \Omega \quad (4-2-5)$$

where $U_i^*$ represents incompatible displacements. To assure the convergence of the incompatible model, we should have

$$\left| a_i(\varepsilon_{ij}) - \sum_{i} a_i^{*(\varepsilon_{ij}^*)} \right| \leq M h^n \quad ; \; n \geq 1 \quad (4-2-6)$$

where $M$ is a constant with finite value.

It is noted that many investigators claimed that "patch test" is necessary or sufficient condition for the convergence of incompatible
model. But Stummel (30) proved that the patch test is neither necessary or sufficient for the convergence. Then he proposed generalized patch test which provides necessary and sufficient condition for the convergence.

The Hu-Washizu principle can be written in the form;

$$\mathcal{W}_G = \int_V \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} \sigma_{ij} \hat{E}_{ij} \right] dV + \int_{S_0} \bar{T}_i u_i dS = \text{stationary}$$  \hspace{1cm} (4-2-7)

where

$$\hat{E}_{ij} = \hat{E}_{ij}(u_i)$$

The limitation principle by Fraeijts de Veubeke requires that the order of assumed $\varepsilon_{ij}$ or $\sigma_{ij}$ (the constitutive eqs. are assumed to be satisfied) should be lower than that of $\hat{E}_{ij}$. Otherwise, the functional becomes identical to the principle of minimum potential energy and

$$\sigma_{ij} = C_{ijkl} \hat{E}_{kl}$$  \hspace{1cm} (4-2-8)

The modified Hu-Washizu principle indicated in Chapter 2 can be expressed as

$$\mathcal{W}_G^* = \sum_n \int_{V_n} \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} \sigma_{ij} (\hat{u}_{ij} + \hat{u}_{ji}) \right] dV + \int_{S_n} \bar{T}_i u_i dS + \int_{S_0} \bar{T}_i u_i dS = \text{stationary}$$ \hspace{1cm} (4-2-9)

where

$$\hat{u}_i = \text{displacements interpolated in terms of nodal quantities} \ \varphi_i$$

$$u^\lambda_i = \text{internal displacements in terms of internal parameters}$$

The Euler equations in $\mathcal{W}_n$ are given by

strain-displacement relation in integral sense
\[ \varepsilon_{ij} = \frac{1}{2} (\hat{u}_{ij} \delta_{ij} + \hat{u}_{ij} \delta_{ij}) + \frac{1}{2} (\hat{u}_{ij} \delta_{ij} + \hat{u}_{ij} \delta_{ij}) \]  \hspace{1cm} (4-2-10)

equilibrium equations in pointwise sense
\[ \sigma_{ij}^* = 0 \]
with appropriate choice of \( \sigma_{ij} \)
or \( \sigma_{ij} = 0 \) in integral sense
and constitutive relations
\[ \sigma_{ij}^* = C_{ijkl} \varepsilon_{kl} \]  \hspace{1cm} (4-2-11)

In the finite element formulation, the use of Guassian quadrature produces the following quality;
\[ \Phi = \frac{1}{2} \int_{\Gamma} \sigma_{ij}^* \varepsilon_{ij}^* d\Gamma \text{ (compatible model)} \]
\[ \tilde{\Phi} = \sum \frac{1}{2} \sigma_{ij}^* \varepsilon_{ij}^* w_n \]
\[ \Rightarrow \quad 0 \leq \tilde{\Phi} \leq \Phi \]  \hspace{1cm} (4-2-12)

Since every polynomial function \( f(x) \) can be expressed by
\[ \varepsilon(x) \equiv f(x) = d_i \varphi_i = d_i \varphi_i^{* \cdot P_i-1} \quad (i=1, 2, \ldots, n) \]  \hspace{1cm} (4-2-13)

where \( \varphi_i = x^{i-1} \)

and \( P_i \) represents Legendre polynomials.

Then, one has
\[ \Phi = \frac{1}{2} \int_{\Gamma} \varepsilon^2 dx = \frac{1}{2} \int_{\Gamma} (\varphi_i^{* \cdot P_i})^2 dx \]  \hspace{1cm} (4-2-14)
The use of orthogonality condition

\[ \int_{-1}^{1} P_m P_n \, dx = \frac{2}{2n+1} \delta_{mn} \quad (4-2-15) \]

leads to

\[ \Phi = \sum_{i=1}^{n} \frac{(\alpha_i^*)^2}{2i-1} \quad (4-2-16) \]

Since the definition of Gaussian points is given by

\[ P_n(\infty) = 0 \quad , \]
\[ \Phi = \sum_{i=1}^{n-1} \frac{(\alpha_i^*)^2}{2i-1} \quad . \]
\[ \therefore \quad \Phi - \tilde{\Phi} = \frac{(\alpha_i^*)^2}{2n-1} \quad (4-2-17) \]

Therefore, for a reduced integration of order \( n-p \), one should get

\[ \tilde{\Phi}_R = \sum_{i=1}^{n-p-1} \frac{(\alpha_i^*)^2}{2i-1} + \sum_{i=n-p+1}^{n} I_i \]
\[ \therefore \quad \Phi - \tilde{\Phi}_R = \sum_{i=n-p+1}^{n} \left[ I_i - \frac{(\alpha_i^*)^2}{2i-1} \right] \quad (4-2-18) \]

The numerical error due to reduced integration becomes bigger obviously. Sometimes this causes linear-dependence in a stiffness matrix; i.e. rank deficiency.

In the isoparametric formulation of two dimensional or three dimensional problems, the order of integration scheme can be determined by following the consequence of one dimensional ease. When the strain function can be represented by
\[ L \mathcal{U}_i(x) = \mathcal{E}(x) = 0_i^{\ast} \mathcal{G}_i^{\ast} + d_i^{\ast} \mathcal{G}_i^{\ast} \]  
\( (i = 1, 2, \ldots, m; \quad i = m+1, m+2, \ldots, n) \)  
(4-2-19)

where \( \mathcal{G}_i^{\ast} \) = a set of complete polynomials [order P-1]
\( \mathcal{G}_i^{\ast} \) = a set of incomplete polynomials of one order higher than \( \mathcal{G}_i^{\ast} \) [order p]

The strain energy can be expressed as
\[
\Phi = \frac{1}{2} \int_1^1 \left[ 0_i^{\ast} \mathcal{G}_i^{\ast} + d_i^{\ast} \mathcal{G}_i^{\ast} \right]^2 dx
\]
\[ = \Phi^{\ast}_{[2p-2]} + \Phi^{\ast}_{[2p-1]} + \Phi^{\ast}_{[2p]} \]  
(4-2-20)

When Gaussian quadrature of order p is used, the error due to numerical integration is confined to the last term;
\[
\Phi - \left\{ \Phi^{\ast}_{[2p-2]} + \Phi^{\ast}_{[2p-1]} + \Phi^{\ast}_{[2p]} \right\} = O(h^{2p}) \quad ; \quad h = \max |x|
\]

The error diminishes rapidly with the order of \( h^{2p} \) as \( h \to 0 \). When the convergence characteristics of \( \mathcal{E}(x) \) can be written in the form;
\[
\| L \mathcal{U} - L \mathcal{U}_h \|^2 \leq C h^{2p-1} 
\]
(4-2-21)

the higher order terms with incompleteness have no essential physical meanings. Therefore, so far as \( \Phi^{\ast}_{[2p-2]} \) can be evaluated in the exact manner, the convergence can be assured. Since the reduced integration yields the lower estimation of strain energy, this procedure is preferred especially to the compatible model which gives the over-estimation of strain energy.
Through the use of the numerical integration scheme, \( \mathcal{R}_P \) can be obtained as

\[
\mathcal{R}_P = \sum_n \frac{1}{2} C_{ijkl} \hat{E}^n_{ij} \hat{E}^n_{kl} w_n \quad (n=1, 2, \ldots, m) \quad (4-22)
\]

where \( w_n \) = weighting factors

\( m = \) number of integration points

Also, \( \mathcal{R}_G \) can be obtained by

\[
\mathcal{R}_G = \sum_n \left[ \frac{1}{2} C_{ijkl} E^n_{ij} E^n_{kl} - \sigma^n_{ij} E^n_{ij} + \sigma^n_{ij} E^n_{ij} \right] \quad (4-23)
\]

where

\[
\sigma^n_{ij} = C_{ijkl} E^n_{kl}
\]

is assumed to be satisfied.

Then, if we can identify the integration points where

\[
E^n_{ij} = \hat{E}^n_{ij} \quad (4-24)
\]

we can obtain

\[
\mathcal{R}_G = \mathcal{R}_P \quad (4-25)
\]

Since \( \mathcal{R}_G \) has three independent quantities \( \mathcal{Q}, \mathcal{E}, \mathcal{U} \), in general, there exists no difficulty for problems with strain constraints (nearly incompressible materials, inextensional deformation and thin plates under Reissner plate theory).

On the other hand, the formulation based upon \( \mathcal{R}_P \) has the difficulties in such cases. Of appropriate integration is used so that \( \mathcal{R}_P = \mathcal{R}_G \), the difficulties by \( \mathcal{R}_P \) can be resolved. The same kind of statement can be established between \( \mathcal{R}_G^* \) and \( \mathcal{R}_P \) with reduced/selective
integration.

Then, what is the difference between two models? The major
differences can be listed as follows;

(i) Since the number of integration points is that of constraint
equations, for a reduced integration scheme there is always
an equivalence of \( \mathcal{K} \) with the complete satisfaction of con-
stitutive equation or a near equivalence of \( \mathcal{K} \). But the opposite
statement is not always true.

(ii) The development of mixed models is simpler and more straight-
forward. This is particularly true for the cases where the
decomposition of energy into constrained and unconstrained
components cannot be easily made: An example is the shell
problem and the geometric nonlinear problem with the presence
of bending-extensional coupling.

Therefore we can claim that the formulation based upon the
functionals given by eqs. (4-2-7) and (4-2-9) is more versatile.

4.3. Equivalence between modified Hu-Washizu prinicple and modified
Hellinger-Reissner principle.

Since \( \mathcal{K}^{*}_{HR} \) can be given by

\[
\mathcal{K}^{*}_{HR} = \sum_{k} \left[ \int_{V} \left[ -\frac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl} + \frac{1}{2} \sigma_{ij} \left( \hat{u}_{ij} + \hat{u}_{ij} \right) \right] \right]
\]

comparing eq. (4-2-9) to eq. (4-3-1) yields the condition for the
equivalence;

\[
\int_{\Omega} \left[ \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \frac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl} \right] dV = \int_{\Omega} \left[ -\frac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl} \right] dV \quad \text{(4-3-2)}
\]
For quadrilateral element with regular shape, one can show that the grouping of polynomials is possible and each polynomial space is orthogonal to each other; i.e.

\[(2-D)\]
\[
\mathcal{G}_1 = C_{mn}^1 \xi^{2m} \eta^{2n} \\
\mathcal{G}_2 = C_{mn}^2 \xi^{2m+1} \eta^{2n} \\
\mathcal{G}_3 = C_{mn}^3 \xi^{2m} \eta^{2n+1} \\
\mathcal{G}_4 = C_{mn}^4 \xi^{2m+1} \eta^{2n+1}
\]
\[i \in \mathcal{P} \quad (i=1, \ldots, 4)\]
\[m, n = \text{integers} \]
\[-1 \leq |\xi|, |\eta| \leq 1\]
\[<\mathcal{G}_i, \mathcal{G}_j> = 0, \quad i \neq j\] \[\text{(4-3-3)}\]

\[(3-D)\]
\[
\mathcal{G}_1 = C_{mnp}^1 \xi^{2m} \eta^{2n} \zeta^{2p} \\
\mathcal{G}_2 = C_{mnp}^2 \xi^{2m+1} \eta^{2n} \zeta^{2p} \\
\mathcal{G}_3 = C_{mnp}^3 \xi^{2m} \eta^{2n+1} \zeta^{2p} \\
\mathcal{G}_4 = C_{mnp}^4 \xi^{2m+1} \eta^{2n+1} \zeta^{2p+1} \\
\mathcal{G}_5 = C_{mnp}^5 \xi^{2m} \eta^{2n+1} \zeta^{2p+1} \\
\mathcal{G}_6 = C_{mnp}^6 \xi^{2m+1} \eta^{2n} \zeta^{2p+1} \\
\mathcal{G}_7 = C_{mnp}^7 \xi^{2m+1} \eta^{2n+1} \zeta^{2p+1} \\
\mathcal{G}_8 = C_{mnp}^8 \xi^{2m} \eta^{2n} \zeta^{2p}
\]
\[i \in \mathcal{P} \quad (i=1, \ldots, 8)\]
\[m, n, p = \text{integers} \]
\[-1 \leq |\xi|, |\eta|, |\zeta| \leq 1\] \[\text{(4-3-4)}\]
\[<\mathcal{G}_i, \mathcal{G}_j> = 0, \quad i \neq j\]

where \(\mathcal{P}\) defines a polynomial space.

When we assume
\[\sigma_{ij} = \mathcal{P}_\beta, \quad \varepsilon_{ij} = \mathcal{P}_\gamma\]
and after grouping
\[\sigma_{ij} = \mathcal{P}^R_\beta \mathcal{P}^K, \quad \varepsilon_{ij} = \mathcal{P}^K_\gamma \mathcal{P}^R\]
we have
\[
S_\xi \left[ \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \varepsilon_{kl} - \sigma_{ij} \varepsilon_{ij} \right] d\xi = \frac{1}{2} \varepsilon' m \sum_{\xi} P_m^T P_m d\xi \varepsilon' m - \beta' m \sum_{\xi} P_m^T \varepsilon' m d\xi \beta' m
\]
\[
S_\xi \left[ -\frac{1}{2} S_{ijkl} \sigma_{ij} \varepsilon_{kl} \right] d\xi = -\frac{1}{2} \beta' m \sum_{\xi} P_m^T \varepsilon' m d\xi \beta' m
\]
where
\[
\text{it is assumed that } \xi \text{ and } \xi \text{ consist of constants and } \xi' = \xi. \text{ Therefore}
\]
the condition for equivalence can be given by
\[
\begin{align*}
\overline{H}_m &= \sum_{\xi} P_m^T \varepsilon' m d\xi \\
\overline{L}_m &= \sum_{\xi} P_m^T \overline{S}_m d\xi \\
\overline{H}_m &= \sum_{\xi} P_m^T \varepsilon' m d\xi
\end{align*}
\]
\[
\Rightarrow \quad \overline{H}_m^{-1} = \overline{H}_m^{-1} \overline{S}_m^{-1} \overline{H}_m^{-1} \quad (m; \text{no sum})
\]

4.4 The relationship between incompatible model and the model based upon new version of modified Hellinger-Reissner principle.

For the matrix formulation of the functionals given by eqs. (4.3-1) and (4.2-1), we assume
\[
\begin{align*}
\sigma_{ij} &= R_{ij} \beta', \quad \sigma_{ij}' = R_{ij}' \beta' \\
\eta_i &= G_i \varepsilon', \quad \eta_i = N_i \beta', \quad \varepsilon_i = D (G_i + N_i \beta')
\end{align*}
\]
\[
; \tilde{\eta}_i = \hat{\eta}_i
\]
The matrix formulation of \( \tau_{HR}^* \) becomes
\[
\tau_{HR}^* = \sum_n \left[ -\frac{1}{2} \beta' \varepsilon' \beta' + \beta' \varepsilon' \varepsilon' - \beta' R_{ij} \beta' \right]
\]
where
\[
H = \sum_n P_m^T \varepsilon' m dV, \quad G = \sum_n P_m^T (G_i + N_i \beta') dV
\]
and
\[
R = \sum_n (D_i + N_i \beta') (D_i + N_i \beta') dV.
\]
The matrix formulation of $\pi^*$ becomes

$$\nabla_{\pi}^* = \sum_n \left[ \frac{1}{2} \bar{z}^T \mathcal{K}^0 \bar{z} + \frac{1}{2} (\bar{z}^T \bar{A}^T \bar{A} + \bar{z}^T \bar{B} \bar{A}) + \frac{1}{2} \bar{z}^T \bar{B} \bar{B} \bar{z} \right]$$

(4-4-3)

where

$$\mathcal{K}^0 = \int_{V_n} (D \hat{G})^T \mathcal{C} (D \hat{G}) dV$$

$$A = \int_{V_n} (D N)^T \mathcal{C} (D \hat{G}) dV$$

and

$$B = \int_{V_n} (D N)^T \mathcal{C} (D N) dV$$

Taking the variations leads to

$$\frac{\partial \nabla_{\pi}^*}{\partial \bar{z}} = \bar{z}^T \bar{A} \bar{A}^T + \bar{z}^T \bar{B} = 0 \quad \Rightarrow \quad \bar{A} = -\bar{B}^T \bar{A} \bar{B}$$

$$\therefore \quad \nabla_{\pi}^* = \frac{1}{2} \bar{z}^T \left[ \mathcal{K}^0 - \bar{B}^T \bar{B} \bar{A} \bar{B} \bar{A} \right] \bar{z}$$

(4-4-4)

$$\frac{\partial \nabla_{\pi}^*}{\partial \beta} = -\beta^T \bar{A} + \bar{z}^T \bar{B} \bar{G}^T - \bar{A}^T \bar{B} = 0 \quad \Rightarrow \quad \beta = \bar{G}^T (\bar{G} \bar{G} - \bar{R} \bar{A})$$

$$\frac{\partial \nabla_{\pi}^*}{\partial \lambda} = -\beta^T \bar{B} = 0 \quad \Rightarrow \quad \lambda = [\bar{B}^T \bar{H}^{-1} \bar{B}]^{-1} [\bar{B}^T \bar{H}^{-1} \bar{G} \bar{z}]$$

$$\therefore \quad \nabla_{\pi}^* \frac{1}{2} \bar{z}^T \left[ \bar{G}^T \bar{H}^{-1} \bar{G} - (\bar{G}^T \bar{B}) (\bar{R} \bar{H} \bar{B})^{-1} (\bar{R}^T \bar{H}^{-1} \bar{G}) \bar{z} \right]$$

(4-4-5)

Eqs. (4-4-4) and (4-4-5) show the correspondence;

$$\mathcal{K}^0 \leftrightarrow \bar{G}^T \bar{H}^{-1} \bar{G}$$

$$A \leftrightarrow -\bar{R}^T \bar{H}^{-1} \bar{G}$$

$$B \leftrightarrow \bar{R}^T \bar{H}^{-1} \bar{R}$$

(4-4-6)

When we put $\mathcal{C} = \mathcal{S} = I$ (identity), we have

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} = \varepsilon_{ij} = \frac{1}{2} (u_{ij}^1 + u_{ji}^2 + u_{ij}^3 + u_{ji}^3)$$

(4-4-7)

Then, eq. (4-4-3) takes the form of
\[ \mathcal{T}_p^* = \sum \int_{\Omega_k} \left[ -\frac{1}{2} \sigma_{ij} \sigma_{ij}^* + \frac{1}{2} \sigma_{ij}^* (\hat{u}_{ij} + \hat{u}_{ji}) - (\sigma_{ij}^* \delta_{ij}) u^i \right] dV \] (4-4-8)

where the boundary terms are ignored because it is the basis of oncompatible model. Also, eq. (4-4-2) can be rewritten as

\[ \mathcal{T}_{HR} = \sum \int_{\Omega_k} \left[ -\frac{1}{2} \sigma_{ij} \sigma_{ij}^* + \frac{1}{2} \sigma_{ij} (\hat{u}_{ij} + \hat{u}_{ji}) - (\sigma_{ij} \delta_{ij}) u^i \right] dV \] (4-4-9)

It is obvious that the form of \( \mathcal{T}_p^* \) is identical to that of \( \mathcal{T}_{HR} \). But \( \sigma_{ij}^* \) in eq. (4-4-8) is given by eq. (4-4-7) and satisfies the compatibility condition. Because of this compatibility condition, the equivalence between incompatible model and the model based upon new version of modified Hellinger-Reissner principle can't be established except special cases where the compatibility condition is automatically satisfied. An example will be given in the following section.

4.5 Related topics to similar and equivalent models in finite element method.

In ref. 31, Froier et al., have pointed out the equivalence between Wilson's incompatible model and Pian's 5-\( \beta \) assumed stress hybrid model for the case of four-node rectangular plane stress element. A proof was given by Pian [32] using the present new version of Hellinger-Reissner principle. This section is to present a proof in a similar procedure as that given in ref. 32 and to give further discussions of this problem.
4.5.1 Wilson's incompatible model and Pian's 5-β hybrid stress model for four node rectangular plane stress element.

The isoparametric interpolations for two-dimensional four-node element are given by

\[ \mathcal{U}^q = \mathcal{U}^{qi} \xi_i(\xi^q) \quad (i=1,\ldots,4; q,\beta=1,2) \]
\[ \mathcal{X}^q = \mathcal{X}^{qi} \xi_i(\xi^q) \]  

(4-5-1)

where

\[ \xi_i(\xi^q) = \frac{1}{4} \left( 1 + \xi_i^1 \xi^1 + 1 + \xi_i^2 \xi^2 \right) \]

\[ \mathcal{U}^q = \text{displacements} \]
\[ \mathcal{X}^q = \text{coordinates} \]
\[ \xi^q = \text{natural coordinates} \]

In Wilson's element [29], we have

\[ \mathcal{U}^q = \mathcal{U}^{qi} \xi_i(\xi^q) + \mathcal{A}^{qi} \xi^q N_{\beta}^j(\xi^q) \quad (j=1,2) \]  

(4-5-2)

where

\[ \mathcal{U}^{qi} = A^{qi} N_{\beta}^j(\xi^q) \]
\[ N_{\beta}^j(\xi^q) = \{ 1 - (\xi^j)^2 \} \]

It is well known that this element is incompatible and doesn't pass the patch test under distorted shape except parallelograms. Through the strain-displacement relations

\[ \mathcal{E}_{\beta\beta} = \frac{1}{2} \left[ \frac{\partial \mathcal{U}^q}{\partial \xi^q} + \frac{\partial \mathcal{U}^q}{\partial \xi^q} \right] \]  

(4-5-3)

and the constitutive equation
\[
\sigma^{\alpha \beta} = C^{\alpha \beta \gamma} \varepsilon_{\gamma} \varepsilon_{\delta} \delta_{\delta}\delta_{\delta} \quad (4-5-4)
\]

One can obtain the corresponding stress patterns in the form

\[
\sigma^{\alpha \beta} = \begin{bmatrix}
\tilde{P}_0 \\
\tilde{P}_0 \\
\tilde{P}_0
\end{bmatrix} \beta
\quad (4-5-5)
\]

where

\[
\tilde{P}_0 = \begin{bmatrix} 1 & \xi & \eta \end{bmatrix}
\]

and

\[
\beta^T = [\beta_1, \beta_2, \ldots, \beta_q]
\]

Eq. (4-5-5) satisfies the compatibility condition

\[
\nabla^2 \sigma^{\alpha \beta} = 0 \quad (4-5-6)
\]

Therefore we must examine the consequence of the constraint equations resulted by the introduction of the additional incompatible modes.

In order to obtain the appropriate constraint equations, we shall consider a slightly distorted element as depicted in figure 25a.

The geometric representation should be modified to

\[
\varepsilon^{\alpha \beta} = \varepsilon^{\alpha \beta} I_l (\xi \beta) + \Delta^{\alpha \beta} \xi \left\{ 1 - (\xi^2)^2 \right\}
\]

\[
\varepsilon^\alpha = \varepsilon^\alpha , \Delta^\alpha = \Delta , \Delta^2 = 0
\]

The Jacobian has the form of

\[
J^{\beta \gamma} = \frac{\partial X^{\gamma}}{\partial \xi^\gamma} = \varepsilon^{\gamma \delta} \frac{\partial L_i}{\partial \xi^\delta} + \Delta^{\gamma \delta} \xi \left\{ \xi^l \xi^l (\xi^2)^2 \right\}
\]

\[
\Rightarrow \quad J^1_1 = 1 + \Delta \left\{ 1 - (\xi^2)^2 \right\} , \quad J^1_2 = -2 \xi \xi \xi \xi \Delta
\]

\[
J^2_1 = 0 , \quad J^2_2 = 1 \quad ; \quad J = 1 + \Delta \left\{ 1 - (\xi^2)^2 \right\}
\]

The computation of the term
\[ \mathbf{I}^* = \int_1^3 \int_1^2 \mathbf{O}^{\beta \beta} \mathbf{u}^{\alpha \alpha} J d \xi d \eta \]  \hspace{1cm} (4-5-9) \\

results in
\[ \mathbf{I}^* = \int_1^3 \int_1^2 \left[ \left\{ \beta_2 + 2 \xi^2 \xi^2 \alpha_2 \beta_8 + (1 + 4 \xi \xi - \xi^2) \beta_9 \right\} \right. \\
\cdot \left\{ (1 - \xi^2) \lambda_1 + (1 - \xi^2) \lambda_2 \right\} \\
+ \left\{ \beta_8 + 2 \xi^2 \xi^2 \alpha_2 \beta_5 + (1 + 4 \xi \xi - \xi^2) \beta_6 \right\} \\
\cdot \left\{ (1 - \xi^2) \lambda_3 + (1 - \xi^2) \lambda_3 \right\} d \xi d \eta \\
\]

\[ = \left[ \frac{8}{3} \beta_2 + \frac{8}{3} \beta_5 + \frac{16}{9} \beta_6 \right] d \lambda_1 \\
+ \left[ \frac{8}{3} \beta_2 + \frac{8}{3} \beta_5 + \frac{32}{15} \beta_6 \right] d \lambda_2 \\
+ \left[ \frac{8}{3} \beta_8 + \frac{8}{3} \beta_6 + \frac{16}{9} \beta_5 \right] d \lambda_3 \\
+ \left[ \frac{8}{3} \beta_8 + \frac{8}{3} \beta_6 + \frac{32}{15} \beta_5 \right] d \lambda_4 \]

\hspace{1cm} (4-5-10) \\

Carrying out the variational procedure
\[ \delta \mathbf{I}^* = \frac{\partial \mathbf{I}^*}{\partial \lambda_i} \delta \lambda_i = 0 \hspace{1cm} (i = 1, \ldots, 4) \]  \hspace{1cm} (4-5-11) \\

it follows that
\[ \beta_2 + \beta_5 = \beta_6 + \beta_8 = 0 \hspace{0.5cm} \beta_8 = \beta_6 = 0 \]

\[ \therefore \beta_2 = \beta_6 = \beta_8 = \beta_9 = 0 \]  \hspace{1cm} (4-5-12) \\

Therefore we obtained the following hybrid stress model with 5\( \beta \).
\[
\sigma_{xx} = \beta_1 + \beta_4 \xi^2 \\
\sigma_{yy} = \beta_2 + \beta_5 \xi^1 \\
\sigma_{xy} = \beta_3
\]

(4-5-13)

Here, we can see that the introduction of \( U^\alpha \) leads to the completeness of polynomial expansion for \( U^\alpha \). On the other hand only two \( \lambda \) essentially plays the roles of internal parameters to satisfy the equilibrium equations. To seek the physical implication of the other two constraint equations, we shall consider a single finite element subject to beam bending type of loading as depicted in figure 25b. The equilibrium equation so given by

\[
\frac{\partial M_{xx}}{\partial \xi} - \ddot{\Omega}_x + \eta \sigma_{xy} \bigg|_{-1}^1 = 0
\]

(4-5-14)

where \( \ddot{\Omega}_x = \frac{1}{2} \left[ \ddot{\Omega}_x(\xi=+1) + \ddot{\Omega}_x(\xi=-1) \right] \)

Substituting eq. (4-5-5) into eq. (4-5-15), we get

\[
M_{xx} = \frac{2}{3} \beta_3 \quad , \quad \ddot{\Omega}_x = 2 \beta_3
\]

Eq. (4-5-14) then becomes

\[
2 \beta_3 \xi = 0 \quad \Rightarrow \quad \beta_3 = 0
\]

(4-5-16)
To preserve the symmetry property in stress assumptions, applying the same argument for y axis leads to

\[ \beta_y = 0 \quad (4-5-17) \]

Now we can see clearly that the two additional constraint equations are introduced to incorporate the quality of bending into the finite element.

4.5.2. Wilson's eight-node 3-D solid element and hybrid stress eight-node 3-D solid element

The stress distributions for 8-node 3-D solid element can be assumed as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz} \\
\end{bmatrix}
= \begin{bmatrix}
\rho_0 & \rho_0 & \rho_0 & \rho_0 \\
\rho_0 & \rho_0 & \rho_0 & \rho_0 \\
\rho_0 & \rho_0 & \rho_0 & \rho_0 \\
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_{24}
\end{bmatrix}
\quad (4-5-18)
\]

where the polynomial assumptions of \( \rho_0 \) can be given by

\[ \rho_0 \approx [ 1 \quad x \quad y \quad z ] \]

The approach of the determination of appropriate stress assumptions is the same as the case of 4 node plane stress element. But the introduction of three additional terms
\[ \sigma_x^a = \beta_{26} \eta \xi' \]
\[ \sigma_y^a = \beta_{26} \xi' \eta' \]
\[ \sigma_{xy}^a = \beta_{27} \xi' \zeta' \]

(4-5-19)

is necessary to prevent kinematic modes.

To begin with, let us consider the 3-D brick with slight distortion shown in figure 26.

The interpolation function for the displacement of eight node 3-D solid finite element is given by

\[ N_i = \frac{1}{8} (1 + 5_1 \xi) (1 + \eta) (1 + \xi \eta) \]
\[ j, i = 1, 2, \ldots, 8 \]

(4-5-20)

Again consider a slightly distorted brick element with mapping function between global coordinates and natural coordinates given by

\[
\begin{bmatrix}
\xi' \\
\eta' \\
\zeta'
\end{bmatrix} = \sum_{i=1}^{8} N_i(\xi, \eta, \zeta) \begin{bmatrix}
\xi_i \\
\eta_i \\
\zeta_i
\end{bmatrix} + \Delta \xi_0 (\xi = \eta = 0) \xi(1 - \xi^2)
+ \Delta \eta_0 (\xi = \eta = 0) \eta(1 - \eta^2)
\]

(4-5-21)

where \( \xi_0, \eta_0 (\xi = \eta = 0) \) are the covariant basis vectors along \( \xi \) and \( \eta \) axes at the origin and \( \Delta \xi, \Delta \eta \) are arbitrary small perturbation parameters. Now the critical term in the finite element formulation of assumed stress finite elements, is given by
\[ I^* = \int_V (\mathbf{D} \mathbf{C})^T \mathbf{u}_d dV = \int_{t_1}^{t_2} \int_{s_1}^{s_2} (\mathbf{D} \mathbf{C})^T \mathbf{u}_d dS dV \] (4-5-22)

In order to express the equilibrium equations in terms of natural coordinate system, one has to invert Jacobian matrix. Carrying out the inversion of the matrix leads to

\[
[\mathbf{J}]^{-1} = \frac{1}{J} \begin{bmatrix}
J_{11}^* & J_{12}^* & J_{13}^* \\
J_{21}^* & J_{22}^* & J_{23}^* \\
J_{31}^* & J_{32}^* & J_{33}^*
\end{bmatrix}
\] (4-5-23)

where

\[
[\mathbf{J}] = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix} ; \quad J = \text{det}[\mathbf{J}]
\] (4-5-24)

and

\[
\begin{align*}
J_{11}^* &= J_{22} J_{33} - J_{23} J_{32} \\
J_{12}^* &= J_{13} J_{32} - J_{12} J_{33} \\
J_{13}^* &= J_{12} J_{23} - J_{13} J_{22} \\
J_{21}^* &= J_{23} J_{31} - J_{21} J_{33} \\
J_{22}^* &= J_{11} J_{33} - J_{13} J_{31} \\
J_{23}^* &= J_{13} J_{21} - J_{11} J_{23} \\
J_{31}^* &= J_{21} J_{32} - J_{22} J_{31} \\
J_{32}^* &= J_{12} J_{31} - J_{11} J_{32} \\
J_{33}^* &= J_{11} J_{22} - J_{12} J_{21}
\end{align*}
\] (4-5-25)
The concrete expression of eq. (4-5-22) becomes

\[
I^* = \sum_{\alpha=1}^{1} \sum_{\beta=1}^{1} \sum_{\gamma=1}^{1} \left\{ \left( J_{11}^* \frac{\partial}{\partial x} + J_{12}^* \frac{\partial}{\partial y} + J_{13}^* \frac{\partial}{\partial z} \right) \delta_{\alpha} \right. \\
+ \left( J_{21}^* \frac{\partial}{\partial x} + J_{22}^* \frac{\partial}{\partial y} + J_{23}^* \frac{\partial}{\partial z} \right) \delta_{\gamma} \\
+ \left. \left( J_{31}^* \frac{\partial}{\partial x} + J_{32}^* \frac{\partial}{\partial y} + J_{33}^* \frac{\partial}{\partial z} \right) \delta_{\gamma} \right\} u^\lambda \\
+ \left\{ \left( J_{11}^* \frac{\partial}{\partial x} + J_{12}^* \frac{\partial}{\partial y} + J_{13}^* \frac{\partial}{\partial z} \right) \delta_{\alpha} \right. \\
+ \left. \left( J_{21}^* \frac{\partial}{\partial x} + J_{22}^* \frac{\partial}{\partial y} + J_{23}^* \frac{\partial}{\partial z} \right) \delta_{\gamma} \right\} v^\lambda \\
+ \left\{ \left( J_{11}^* \frac{\partial}{\partial x} + J_{12}^* \frac{\partial}{\partial y} + J_{13}^* \frac{\partial}{\partial z} \right) \delta_{\alpha} \right. \\
+ \left. \left( J_{21}^* \frac{\partial}{\partial x} + J_{22}^* \frac{\partial}{\partial y} + J_{23}^* \frac{\partial}{\partial z} \right) \delta_{\gamma} \right\} w^\lambda \int d^2 d^2 \eta \right]
\]

Substituting the expression for stress assumptions into eq. (4-5-20)

leads to

\[
I^* = \sum_{\alpha=1}^{1} \sum_{\beta=1}^{1} \sum_{\gamma=1}^{1} \left\{ \left( J_{11}^* \beta_{\alpha} + J_{12}^* \beta_{\beta} + J_{13}^* \beta_{\gamma} \right) + \left( J_{21}^* \beta_{14} + J_{22}^* \beta_{15} + J_{23}^* \beta_{16} \right) \right. \\
+ \left( J_{31}^* \beta_{22} + J_{32}^* \beta_{23} + J_{33}^* \beta_{24} \right) \} \cdot u^\lambda \\
+ \left\{ \left( J_{11}^* \beta_{14} + J_{12}^* \beta_{15} + J_{13}^* \beta_{16} \right) + \left( J_{21}^* \beta_{22} + J_{22}^* \beta_{23} + J_{23}^* \beta_{24} \right) + \left( J_{31}^* \beta_{36} + J_{32}^* \beta_{37} + J_{33}^* \beta_{38} \right) \} \cdot v^\lambda \\
+ \left\{ \left( J_{11}^* \beta_{22} + J_{12}^* \beta_{23} + J_{13}^* \beta_{24} \right) + \left( J_{21}^* \beta_{36} + J_{22}^* \beta_{37} + J_{23}^* \beta_{38} \right) + \left( J_{31}^* \beta_{40} + J_{32}^* \beta_{41} + J_{33}^* \beta_{42} \right) \} \cdot w^\lambda \int d^2 d^2 \eta \right]
\]

\[(4-5-27)\]
When we assume

$$\mathcal{U} = (1 - \xi^2)\lambda_1 + (1 - \eta^2)\lambda_2 + (1 - \xi^2)\lambda_3$$

$$\mathcal{V} = (1 - \xi^2)\lambda_4 + (1 - \eta^2)\lambda_5 + (1 - \xi^2)\lambda_6$$

$$\mathcal{W} = (1 - \xi^2)\lambda_7 + (1 - \eta^2)\lambda_8 + (1 - \xi^2)\lambda_9$$

(4-5-28)

and we employ the expressions for Jacobian components:

$$\begin{bmatrix} J_{11} \\ J_{12} \\ J_{13} \end{bmatrix} = \sum_{i=1}^{8} \frac{1}{8} \delta_i^2 (1 + \xi_i \delta_i) (1 + \xi_i \xi_i) \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} + \sum_{i=1}^{8} \frac{1}{8} \delta_i^2 \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} \Delta \delta \left(1 - \eta^2\right)$$

$$\begin{bmatrix} J_{21} \\ J_{22} \\ J_{23} \end{bmatrix} = \sum_{i=1}^{8} \frac{1}{8} \eta_i (1 + \xi_i \delta_i) (1 + \xi_i \xi_i) \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} - \sum_{i=1}^{8} \frac{1}{4} \xi_i \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} \Delta \xi \left(1 - \xi^2\right)$$

$$+ \sum_{i=1}^{8} \frac{1}{8} \eta_i \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} \Delta \eta \left(1 - \xi^2\right)$$

$$\begin{bmatrix} J_{31} \\ J_{32} \\ J_{33} \end{bmatrix} = \sum_{i=1}^{8} \frac{1}{8} \xi_i (1 + \xi_i \delta_i) (1 + \eta_i \xi_i) \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} - \sum_{i=1}^{8} \frac{1}{4} \eta_i \begin{bmatrix} \xi_i \\ \eta_i \\ \delta_i \end{bmatrix} \Delta \eta \left(1 - \delta^2\right)$$

(4-5-29)
it follows that, as the result of the variations of $I^*$ with respect to $\mathbf{N}_i (i=1,2,\ldots,9)$, nine constraint equations for stress equilibrium equations can be obtained; thus only 15 stress parameters are independent. This procedure can be carried out automatically in computational procedure in practice. But, in order to illustrate this procedure, we consider regular brick element with slight distortion. The original brick element is assumed to have the dimension of $2\times 2\times 2$. In this case, the Jacobian components become

$$J_{11} = 1 + \Delta \xi (1-\eta^2), \quad J_{12} = J_{13} = 0$$
$$J_{31} = -2 \Delta \xi \xi, \quad J_{32} = 1 + \Delta \eta (1-\xi^2), \quad J_{33} = 0$$
$$J_{31} = 0, \quad J_{32} = -2 \Delta \eta \xi, \quad J_{33} = 1$$

(4-5-30)

Then we have

$$J_{11}^* = 1 + \Delta \eta (1-\xi^2)$$
$$J_{12}^* = 0$$
$$J_{13}^* = 0$$
$$J_{31}^* = -2 \Delta \xi \xi$$
$$J_{32}^* = 1 + \Delta \xi (1-\eta^2)$$
$$J_{33}^* = 0$$
$$J_{31}^* = 2 \Delta \xi \Delta \eta \xi \eta^2 \xi$$
$$J_{32}^* = 2 \Delta \eta \eta \xi + 2 \Delta \xi \Delta \eta \eta \xi \xi (1-\eta^2)$$
$$J_{33}^* = 1 + \Delta \xi (1-\eta^2) + \Delta \eta (1-\xi^2) + \Delta \xi \Delta \eta (1-\eta^2)(1-\xi^2)$$

(4-5-31)
Substituting eq. (4-5-31) into eq. (4-5-27) gives

\[
I^k = \int \left[ \sum_{1}^{i} \sum_{1}^{j} \left[ \beta_{2} + \Delta \eta (1-S^2) \beta_{14} - 2 \Delta S \eta \beta_{15} + \beta_{15} + \Delta S (1-\eta^2) \beta_{15} \right. \right. \\
+ \left. \left. 4 \Delta S \eta \beta_{2} + 2 \Delta \eta \beta_{23} + 2 \Delta \eta \eta \beta_{23} (1-\eta^2) \beta_{23} \right. \right. \\
+ \left. \left. \beta_{24} \left[ 1 + \Delta S (1-\eta^2) + \Delta \eta (1-S^2) + \Delta \eta (1-S^2) (1-S^2) \right] \right] \right] dS \eta dS
\]

\[
+ \left[ \beta_{14} + \Delta \eta (1-S^2) \beta_{14} - 2 \Delta S \eta \beta_{16} + \beta_{16} + \Delta S (1-\eta^2) \beta_{16} \right. \\
+ \left. 4 \Delta S \eta \beta_{18} + 2 \Delta \eta \beta_{19} + 2 \Delta \eta \eta \beta_{19} (1-\eta^2) \beta_{19} \right. \\
+ \left. \beta_{20} \left[ 1 + \Delta S (1-\eta^2) + \Delta \eta (1-S^2) + \Delta \eta (1-S^2) (1-S^2) \right] \right] \right] dS \eta dS
\]

\[
+ \left[ \beta_{22} + \Delta \eta (1-S^2) \beta_{22} - 2 \Delta S \eta \beta_{23} + \beta_{23} + \Delta S (1-\eta^2) \beta_{23} \right. \\
+ \left. 4 \Delta S \eta \beta_{25} + 2 \Delta \eta \beta_{26} + 2 \Delta \eta \eta \beta_{26} (1-\eta^2) \beta_{26} \right. \\
+ \left. \beta_{27} \left[ 1 + \Delta S (1-\eta^2) + \Delta \eta (1-S^2) + \Delta \eta (1-S^2) (1-S^2) \right] \right] \right] \right] dS \eta dS
\]

(4-5-32)

Since the higher order perturbation obviously yields, as the result of variational procedure,
\[ \beta_{12} = \beta_{20} = \beta_{24} = 0 \quad (4-5-33) \]

we can simplify the expression of \( I^\ast \) by omitting higher order terms in the form:

\[
I^\ast = \int \int \int \int \left[ \beta_a + \Delta \eta (1-5^2) \beta_0 - 2 \Delta \xi \Delta \eta \beta_{14} + \beta_{15} + \Delta \xi (1-5^2) \right] d\xi d\eta
+ 2 \Delta \eta (1-\xi^2) \beta_{24} + \beta_{24} \left[ 1 + \Delta \xi (1-\eta^2) + \Delta \eta (1-\xi^2) \right] \right] \cdot \vec{u}
+ \left[ \beta_{14} + \Delta \eta (1-\xi^2) \beta_{14} - 2 \Delta \xi \Delta \eta \beta_6 + \beta_{17} \right] \cdot \vec{v}
+ 2 \Delta \eta (1-\xi^2) \beta_{19} + \beta_{19} \left[ 1 + \Delta \xi (1-\eta^2) + \Delta \eta (1-\xi^2) \right] \right] \cdot \vec{w}
+ \left[ \beta_{3a} + \Delta \eta (1-\xi^2) \beta_{3a} - 2 \Delta \xi \Delta \eta \beta_{18} + \beta_{19} + \Delta \xi (1-\eta^2) \beta_1 \right]
+ 2 \Delta \eta (1-\xi^2) \beta_{11} + \beta_{11} \left[ 1 + \Delta \xi (1-\eta^2) + \Delta \eta (1-\xi^2) \right] \right] \cdot \vec{w}
\]

\[
(4-5-34) \]

By using eq. (4-5-28) and computing each term of eq. (4-5-34) we get

\[
I^\ast = \left[ \frac{16}{3} \beta_2 + \frac{33}{4} \Delta \eta \beta_a + \frac{16}{3} \beta_{15} + \frac{33}{4} \Delta \xi \beta_15 + \frac{16}{3} \beta_{24} + \frac{33}{4} \Delta \xi \beta_{24} + \frac{33}{4} \Delta \eta \beta_{24} \right] \cdot \vec{v}
+ \left[ \frac{16}{3} \beta_2 + \frac{33}{4} \Delta \eta \beta_a + \frac{16}{3} \beta_{15} + \frac{16}{3} \beta_{24} + \frac{64}{15} \Delta \xi \beta_{24} + \frac{33}{4} \Delta \eta \beta_{24} \right] \cdot \vec{w}
+ \left[ \frac{16}{3} \beta_2 + \frac{64}{15} \Delta \eta \beta_a + \frac{16}{3} \beta_{15} + \frac{33}{4} \Delta \xi \beta_{15} + \frac{16}{3} \beta_{24} + \frac{33}{4} \Delta \xi \beta_{24} + \frac{64}{15} \Delta \eta \beta_{24} \right] \cdot \vec{w}
\]
\[
+ \left[ \frac{16}{3} \beta_{14} + \frac{32}{9} \Delta \eta \beta_{14} + \frac{16}{3} \beta_{14} + \frac{32}{9} \Delta \xi \beta_{14} + \frac{16}{3} \beta_{30} + \frac{32}{9} \Delta \eta \beta_{30} \right] \lambda_4 \\
+ \left[ \frac{16}{3} \beta_{14} + \frac{32}{9} \Delta \eta \beta_{14} + \frac{16}{3} \beta_{14} + \frac{64}{15} \Delta \xi \beta_{14} + \frac{16}{3} \beta_{30} + \frac{64}{15} \Delta \xi \beta_{30} + \frac{32}{9} \Delta \eta \beta_{30} \right] \lambda_5 \\
+ \left[ \frac{16}{3} \beta_{14} + \frac{64}{15} \Delta \eta \beta_{14} + \frac{16}{3} \beta_{14} + \frac{32}{9} \Delta \xi \beta_{14} + \frac{16}{3} \beta_{30} + \frac{32}{9} \Delta \xi \beta_{30} + \frac{64}{15} \Delta \eta \beta_{30} \right] \lambda_6 \\
+ \left[ \frac{16}{3} \beta_{2a} + \frac{32}{9} \Delta \eta \beta_{2a} + \frac{16}{3} \beta_{19} + \frac{32}{9} \Delta \xi \beta_{19} + \frac{16}{3} \beta_{1a} + \frac{32}{9} \Delta \xi \beta_{1a} + \frac{32}{9} \Delta \eta \beta_{1a} \right] \lambda_7 \\
+ \left[ \frac{16}{3} \beta_{2a} + \frac{32}{9} \Delta \eta \beta_{2a} + \frac{16}{3} \beta_{19} + \frac{64}{15} \Delta \xi \beta_{19} + \frac{16}{3} \beta_{1a} + \frac{64}{15} \Delta \xi \beta_{1a} + \frac{32}{9} \Delta \eta \beta_{1a} \right] \lambda_8 \\
+ \left[ \frac{16}{3} \beta_{2a} + \frac{64}{15} \Delta \eta \beta_{2a} + \frac{16}{3} \beta_{19} + \frac{32}{9} \Delta \xi \beta_{19} + \frac{16}{3} \beta_{1a} + \frac{32}{9} \Delta \xi \beta_{1a} + \frac{64}{15} \Delta \eta \beta_{1a} \right] \lambda_9 \\
\]

\((4-5-35)\)

When the variation of \( I^* \) with respect to \( \lambda_i (i=1,2,\ldots,9) \) is carried out and setting

\[
\delta I^* = \frac{\partial I^*}{\partial \lambda_i} \delta \lambda_i = 0 \quad (4-5-36)
\]

nine constraint equations on stress parameters can be obtained. Then the following equations results;
\[
\begin{align*}
\begin{cases}
\beta_2 + \beta_{15} + \beta_{24} = 0 \\
\beta_{15} + \beta_{24} = 0 \\
\beta_2 + \beta_{24} = 0
\end{cases} \quad \Rightarrow \quad \beta_2 = \beta_{15} = \beta_{24} = 0
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\beta_{14} + \beta_7 + \beta_{20} = 0 \\
\beta_7 + \beta_{20} = 0 \\
\beta_{14} + \beta_{20} = 0
\end{cases} \quad \Rightarrow \quad \beta_7 = \beta_{14} = \beta_{20} = 0
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\beta_{32} + \beta_{19} + \beta_{12} = 0 \\
\beta_{19} + \beta_{12} = 0 \\
\beta_{32} + \beta_{12} = 0
\end{cases} \quad \Rightarrow \quad \beta_{12} = \beta_{19} = \beta_{32} = 0
\end{align*}
\]

\[\therefore \quad \beta_2 = \beta_7 = \beta_{12} = \beta_{14} = \beta_{15} = \beta_{19} = \beta_{20} = \beta_{32} = 0 \quad (4-5-37)\]

Now when we add three additional terms indicated in eq. (4-5-19) in order to suppress the kinematic modes, the final form of stress assumption becomes

\[
\sigma_x = \beta_1 + \beta_2 \eta + \beta_3 \xi + \beta_4 \eta \xi
\]

\[
\sigma_y = \beta_5 + \beta_6 \xi + \beta_7 \eta + \beta_8 \xi \eta
\]

\[
\sigma_z = \beta_9 + \beta_{10} \xi + \beta_{11} \eta + \beta_{12} \xi \eta
\]
\[ \begin{align*}
\delta y &= \beta_{13} + \beta_{14} \xi \\
\delta y &= \beta_{15} + \beta_{16} \xi \\
\delta x &= \beta_{17} + \beta_{18} \eta
\end{align*} \] (4-5-38)

It should be noted that Wilson's version does not have additional terms to suppress the kinematic modes. Thus from the present development we can conclude that Wilson's version should have three kinematic modes. It is noted that for brick element the assumed stresses given above is identical to that used by Loikkanen [33] and Pian, Chen and Kang [17].
5. Use of new version of $\mathcal{N}_R$ or $\mathcal{N}_G$ to derive new Invariant Hybrid stress elements.

The term "Invariance" in this investigation has two different implications:

(1) An element becomes invariant with respect to the reference coordinate system.

(2) The influence to an element due to distortion is minimized.

In practical finite element applications the elements used are of distorted by necessity. It is thus desirable to construct finite elem which will have minimum discretization errors when element geometries are distorted from a regular shape. Another desirable feature for fin elements is the invariance property with respect to the reference coordinate system. The chapter presents a general discussion of geome representation of finite elements in natural coordinates and the formu tion of a four-node plane stress element which is invariant, and is apparently also not too sensitive to distortions.

It is noted that an axisymmetric solid element based upon natural coordinated system will be considered in Appendix 2.

5.1 Geometric representation of finite element

One can define $S$ as a bounded open subset of $\mathbb{R}^n$ (n=2,3); i.e. $S \subset \mathbb{R}^n$ and its boundary $\partial S$ and $S$ forms a finite element.

$$
\bar{S} = \partial S \cup S
$$

(5-1-1)
where $\overline{S}$ is a bounded closed subset of $\mathbb{R}^n$. Let us define the undistorted finite element $S_0$ indicated in figure 29 as a square element for $\mathbb{R}^2$ and a regular cube for $\mathbb{R}^3$.

Every point of $\overline{S}_0$ can be mapped into a distorted finite element. The mapping with one to one correspondence can be written as

$$F : \mathbf{x} \in \overline{S}_0 \rightarrow F(\mathbf{x}) \in \overline{S}$$

$$\overline{S} = F(\overline{S}_0) ; \text{image of } \overline{S}_0$$

(5-1-2)

where $$F(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_n(\mathbf{x}))$$.

When one defines the space of polynomials of order $m$ as

$$P_m$$

(5-1-3)

$F$ can be classified into two categories;

a: linear mapping

$$F_i \in P_1, 1 \leq i \leq n$$

(5-1-4)

b: nonlinear mapping; there exists at least one mapping of

$$F_i \in P_m, 2 \leq m$$

(5-1-5)

Here, one must make sure that $F^{-1}$ is also a mapping with one to one correspondence; hence

$$F^{-1}F(\mathbf{x}) = \mathbf{x} \quad, \quad |F| > 0$$

(5-1-6)

Now, one can get a set $\{ \mathbf{x}_i \}_{i=1}^N$ of $N$ points of $\mathbb{R}^n$ and a finite-dimensional
space $P$ of real-valued functions with $\dim P = N$. For any real scalars $a_i, 1 \leq i \leq N$, there exists one function $f \in P$ such that $f(\alpha_i) = a_i$, $1 \leq i \leq N$.

Then, for any set $\{\alpha_i\}_{i=1}^{N}$ there exists one mapping $F : x \in \bar{S}_o \rightarrow F(x) \in \bar{S}$ such that $F_i \in P$, $1 \leq i \leq n$ ; $F(\alpha_i) = a_i$, $1 \leq i \leq N$.

The mapping is given by

$$F = \sum_{i=1}^{N} F_i a_i$$  \hspace{1cm} (5-1-7)

where $F_i \in P$ ; $F_i(\alpha_j) = S_{ij}$, $1 \leq j \leq N$.

and $f_i$ represents the basic functions of a finite element space. In the isoparametric formulation, the same mapping is applied for both coordinates and displacements.

It is possible to describe mathematically the distortion effect of the domain of a finite element $\bar{S}$ by determining the spatial positions $\chi_i$ of each particle $\xi_i$. Therefore, it can be described as

$$\chi_i = \chi_i(\xi_i)$$  \hspace{1cm} (5-1-8)
where \( \chi_i \) is assumed to be a single-valued function differentiable with respect to the arguments. Further, one must assume that the determinant of the transformation is positive:

\[
\left| \frac{\partial \chi_i}{\partial \xi^j} \right| > 0
\]

and, therefore, eq. (5-1-8) has a unique inverse.

The derivative of \( \chi_i \) with respect to the material coordinates defines the "geometry" gradient

\[
\mathbf{J}_{ij}(\xi^k) = \frac{\partial \chi_i}{\partial \xi^j}
\]

(5-1-9)

The vector-valued function

\[
\mathbf{Z}_i(\xi^j) = \chi_i(\xi^j) - \xi^j
\]

(5-1-10)

is the "geometry" vector.

Suppose that \( \xi_k \) denotes an orthonormal system of basis vectors tangent to the spatial coordinates. Then the natural basis vectors tangent to \( \xi^j \) are the vectors

\[
\xi^j_i = \frac{\partial \chi^j}{\partial \xi^i} \quad ; \quad j = 1, 2, 3
\]

or

\[
\xi^j_i = J_{ij} \xi^j = (\delta_{ij} + Z_{ij}) \xi^j
\]

(5-1-11)

where

\[
Z_{ij} = \frac{\partial Z}{\partial \xi^j}
\]
The nonsingular $J \equiv J^j_i \xi_i \xi_j$ has the polar-decomposition.

$$
\tilde{J} = R \cdot \tilde{Z} = I + \tilde{V} \cdot \tilde{Z}
$$

(5-1-12)

where

$$
\begin{align*}
\tilde{Z} &\equiv z : \varepsilon_i^j \\
\tilde{V} &\equiv \frac{\partial}{\partial x^i} \omega_i^j
\end{align*}
$$

and

$Z$ is a symmetric, positive-definite tensor.

$I$ is the identity tensor and $R$ is the orthogonal rotation tensor.

such that $R^T = R^{-1}$.

Therefore, the distortion effect can be included in the term of $\tilde{V} \cdot \tilde{Z}$.

5.2 Finite element formulation based upon natural coordinate system

5.2.1 Governing equations based upon natural coordinate system

In order to seek an "Invariant Hybrid" stress model, the general formuation of hybrid stress model based upon natural coordinate system is investigated.

In three dimensional isoparametric formulation, the geometry of one element is represented in the following way;

$$
\begin{align*}
\tilde{X} &\equiv x^i \varepsilon_i^j \quad (i = 1, 2, 3) \\
\chi^i &\equiv \chi^i_j \quad (j = 1, 2, \ldots, n)
\end{align*}
$$

(5-2-1) (5-2-2)
where
\[ \mathbf{L} = \text{position vector} \]
\[ \mathbf{x}^i = \text{global coordinate system} \]
\[ \mathbf{e}_j^i = \text{orthonormal basis vector of global coordinate system} \]
\[ I_j^i = \text{interpolation functions} \]
\[ \mathbf{x}^j_i = \text{global coordinate values corresponding to node j} \]
and \[ \mathcal{N} = \text{the number of nodes of one element} \]

In order to obtain the covariant basis vector at one point inside of one element, differentiating eq. (5-2-1) with respect to natural coordinate system \( \xi^i \) leads to
\[ \xi^i = \frac{\partial \mathbf{x}^i}{\partial \xi^i} = \frac{\partial \mathbf{x}^i}{\partial \xi^j} \mathbf{e}_j^i = \left[ I_j^i \right] \mathbf{e}_j^i \equiv I_j^i \mathbf{e}_j^i \] \hspace{1cm} (5-2-3)

where \( \xi^j_i \) = covariant basis vectors
and
\[ J_j^i = I_j^i \mathbf{x}^j_i \]

The metrics \( \xi^i_\xi^j \) are given by
\[ \xi^i_\xi^j = \delta^i_j \left[ J_j^k \mathbf{e}_k^i \right] \left[ J_j^p \mathbf{e}_p^j \right] = J_j^i J_j^p \] \hspace{1cm} (5-2-4)

and
\[ \mathbf{S} = \det \xi^i_\xi^j = J^2 \] \hspace{1cm} (5-2-5)

where \( J \) represents Jacobian determinant. Then the contravariant basis vector can be obtained by
\[ \mathbf{e}^i = \mathbf{e}_j^k \frac{\xi^k_\xi^i}{\sqrt{\mathbf{S}}} = \frac{1}{J} \mathbf{e}_j^k \xi^k_\xi^i \times \xi^j_i \quad (i,j,k \text{ no sum}) \]
\[ \Rightarrow \mathbf{e}^i = \frac{1}{J} \mathbf{e}_j^k \mathbf{e}_p^r J_j^p J_j^r \mathbf{e}^i \equiv G^i_r \mathbf{e}^r \] \hspace{1cm} (5-2-6)
Since
\[ \bar{\nabla} \bar{u} = \bar{\xi} \cdot (u_k \bar{e}_k) \]
\[ = \bar{\xi} \cdot \left( \frac{\partial u_k}{\partial \xi} \right) (u_k \bar{e}_k) \]
\[ = \bar{e}^i \cdot \left( \frac{\partial u_k}{\partial \xi} \right) \bar{e}_i \bar{e}_k \]

we obtain
\[ \bar{\xi} = \frac{1}{\bar{e}} \left[ G_k^k \frac{\partial u_k}{\partial \xi} + G_k^k \frac{\partial u_k}{\partial \xi} \right] \quad (5-2-7) \]

Next we shall derive the equilibrium equations. The equilibrium equations are given by
\[ \bar{\nabla} \cdot \bar{\varepsilon} = 0 \quad (5-2-8) \]

where body forces are omitted for the sake of simplicity. Expressing the stress tensor in terms of local stresses based upon contravariant basis vectors, we get
\[ \bar{\varepsilon} = \sigma^{ij} \bar{e}_i \bar{e}_j = \tau^{ij} \bar{\xi}_i \bar{\xi}_j \quad (5-2-9) \]

where
$\gamma^i_j$ = tensor components of stresses based upon local covariant basis vectors

and $\sigma^i_j$ = tensor components of stresses based upon global coordinates.

Therefore the stress equilibrium equations become

$$\nabla \cdot \sigma = G_i^j \frac{\partial}{\partial x^i} \left( \gamma^{kl} \frac{\partial}{\partial x^l} \right)$$

$$= G_i^j \frac{\partial}{\partial x^i} \left( \gamma^{kl} J_k^p \delta^p_l \right)$$

$$= G_i^j \frac{\partial}{\partial x^i} \left( \gamma^{kl} J_k^i \delta^i_j \right)$$

$$= G_i^j \frac{\partial}{\partial x^i} \left( \gamma^{kl} \delta^i_j \right)$$

$$+ G_i^j J_k^i \gamma^{kl} \frac{\partial}{\partial x^i}$$

(5-2-10)

When we employ

$$G_i^j J_k^i = S_k^i$$

(5-2-11)

$$G_i^j \frac{\partial J_k^i}{\partial x^j} = \frac{1}{J} \frac{\partial J}{\partial x^k}$$

(5-2-12)

$$\frac{\partial}{\partial x^i} \delta^i_j = \Gamma^i_{jk} \delta^i_j$$

(5-2-13)

eq. (5-2-10) results in

$$\nabla \cdot \sigma = \frac{\partial \gamma^{kl}}{\partial x^k} \delta^i_j + \frac{1}{J} \frac{\partial J}{\partial x^k} \gamma^{kl} \delta^i_j + \gamma^{kl} \Gamma^p_{lk} \delta^i_p$$

$$= \left[ \frac{\partial (J \gamma^{kl})}{\partial x^k} + \gamma^{kl} \Gamma^p_{lk} \delta^i_p \right] \delta^i_j$$

$$\equiv 0$$

(5-2-14)
or \[ \frac{\partial (\mathbf{T} \mathbf{\tau}^{k l})}{\partial \mathbf{s}^{l k}} + \mathbf{\tau}^{m l} \frac{\partial \mathbf{m}^{l k}}{\partial \mathbf{m}^{l k}} = 0 \]

The relationship between \( \sigma^{i f} \) and \( \tau^{i f} \) can be obtained by

\[
\sigma^{i f} \mathbf{e}^i \mathbf{e}^f = \tau^{i f} \mathbf{e}^i \mathbf{e}^f
\]

\[
= \tau^{i f} (J_i^m \mathbf{e}^m)(J_f^n \mathbf{e}^n)
\]

\[
= J_i^m J_f^n \tau^{i f} \mathbf{e}^m \mathbf{e}^n
\]

\[ \therefore \quad \sigma^{i f} = J_i^m J_f^n \tau^{i f} \mathbf{e}^m \mathbf{e}^n \]

(5-2-15)

The constitutive relation for isotropic material can be written in the form

\[
\gamma_{mn} = \mathbf{S}^{*}_{mnpq} \tau^{pq}
\]

(5-2-16)

where

\( \gamma_{mn} \) strains based upon covariant basis vectors and

\[
\mathbf{S}^{*}_{mnpq} = \frac{1+v}{E} (\mathbf{S}_{mp} \mathbf{S}_{nq} + \mathbf{S}_{mq} \mathbf{S}_{np}) - \frac{\nu}{E} \mathbf{S}_{mn} \mathbf{S}_{pq}
\]

(5-2-17)

\( E = \) Young's modulus, \( \nu = \) Poisson's ratio

For orthonormal coordinate system, we have
\[ \varepsilon_{mn} = S_{mnpq} \sigma^{pq} \]  
(5-2-18)

where

\[ \varepsilon_{mn} = \text{strains based on global coordinate system} \]

and

\[ S_{mnpq} = \frac{1+\nu}{E} (\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}) - \frac{\nu}{E} \delta_{mn} \delta_{pq} \]  
(5-2-19)

5.2.2 Finite element formulation

When the complete satisfaction of equilibrium equations given by eq. (5-2-14) is difficult, it is possible to introduce only partial satisfaction in the finite element formulation. For example, we employ the following assumptions for stresses;

\[ \gamma^{ij} = P(\xi^{k}) \beta \]  
(5-2-20)  
(version 1)

or

\[ J \gamma^{ij} = P(\xi^{k}) \beta \]  
(5-2-21)  
(version 2)

Then the partial satisfaction of the stress equilibrium equations can be carried out.

But it can be seen from eq. (5-2-15) that the stresses based upon global Cartesian coordinate system cannot accommodate constant stress
condition which is critical for passing the first order patch test. Therefore, we may modify the stress assumptions by using the values of Jacobian at \( \xi = \eta = 0 \);

\[
\sigma^i = J^i_{(0,0)} J^{i^i (0,0)} \tau^{kl}
\]

\[
J = J_{(0,0)}
\]

Because of this modification, the patch test can be passed. Concrete examples will be given later.

5.3 The creation of invariant hybrid stress four node plane stress element

For a general quadrilateral element the isoparametric coordinates \( \xi \) and \( \eta \) are used to express stresses and displacements. For the stresses, the following complete linear terms are used:

\[
\sigma = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
1 & \xi & \eta \\
1 & \xi & \eta \\
1 & \xi & \eta
\end{bmatrix} \begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_9
\end{bmatrix} \quad (5-3-1)
\]

For the element displacements \( u_q \) are expressed by bilinear inter-
oplations in the isoparametric coordinates \( \xi \) and \( \eta \).

\[
\begin{bmatrix}
u_q \\
v_q
\end{bmatrix} = \sum_{i=1}^{4} \left( 1 + \xi_i \xi (1 + \eta_i \eta) \right) \begin{bmatrix}
u_i \\
u_i
\end{bmatrix} \quad (5-3-2)
\]
where \( u_i \) and \( v_i \) are nodal displacements. The coordinates \( x \) and \( y \) are expressed in terms of the coordinates of the four corner nodes by

\[
\begin{bmatrix}
  x \\
y
\end{bmatrix} = \sum_{i=1}^{4} (1 + s_i \xi)(1 + \eta_i \zeta) \begin{bmatrix}
x_i \\
y_i
\end{bmatrix}
\]

(5-3-3)

Let the Jacobian be

\[
[J] = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix} = \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\]

(5-3-4)

where from Eq. (5-3-3)

\[
\begin{align*}
J_{11} &= a_1 + a_2 \eta \\
J_{12} &= b_1 + b_2 \eta \\
J_{21} &= a_3 + a_2 \xi \\
J_{22} &= b_3 + b_2 \xi
\end{align*}
\]

(5-3-5)

and

\[
\begin{align*}
a_1 &= \frac{1}{4} (-x_1 + x_2 + x_3 - x_4) \\
a_2 &= \frac{1}{4} (x_1 - x_2 + x_3 - x_4) \\
a_3 &= \frac{1}{4} (-x_1 - x_2 + x_3 + x_4) \\
b_1 &= \frac{1}{4} (-y_1 + y_2 + y_3 - y_4) \\
b_2 &= \frac{1}{4} (y_1 - y_2 + y_3 - y_4) \\
b_3 &= \frac{1}{4} (-y_1 - y_2 + y_3 + y_4)
\end{align*}
\]

(5-3-6)
Then
\[
\begin{bmatrix}
\frac{\partial}{\partial x}
\frac{\partial}{\partial y}
\end{bmatrix}
= [J]^{-1}
\begin{bmatrix}
\frac{\partial}{\partial x}
\frac{\partial}{\partial y}
\end{bmatrix}
= \frac{1}{|J|}
\begin{bmatrix}
J_{22} - J_{12}
-J_{11}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial x}

\frac{\partial}{\partial y}
\end{bmatrix}
\]  
(5-3-7)

The term for the constraint of equilibrium equations then is
\[
I^* = \int_V \left( (D^{T}D) \right)^T \sigma \, dV = \int_1^1 \int_1^1 \left[ \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) u + \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) v \right] |J| d\xi d\eta
\]
\[
= \int_1^1 \int_1^1 \left[ \left( J_{22} \frac{\partial}{\partial x} - J_{12} \frac{\partial}{\partial y} \right) \sigma_x + \left( -J_{21} \frac{\partial}{\partial x} + J_{11} \frac{\partial}{\partial y} \right) \tau_{xy} \right] u \, d\xi d\eta
\]
\[
+ \left[ \left( J_{22} \frac{\partial}{\partial x} - J_{12} \frac{\partial}{\partial y} \right) \tau_{xy} + \left( -J_{21} \frac{\partial}{\partial x} + J_{11} \frac{\partial}{\partial y} \right) \sigma_y \right] v \, d\xi d\eta
\]  
(5-3-8)

or
\[
I^* = \int_1^1 \int_1^1 \left\{ \left[ (b_3 + b_2 \xi) \beta_3 - (b_1 + b_2 \xi) \beta_3 - (a_3 + a_2 \xi) \beta_3 \right.
\]
\[
+ (a_1 + a_2 \xi) \beta_3 \right] u \, d\xi d\eta
\]
\[
+ \left[ (b_3 + b_2 \xi) \beta_3 - (b_1 + b_2 \xi) \beta_3 - (a_3 + a_2 \xi) \beta_3 \right.
\]
\[
+ (a_1 + a_2 \xi) \beta_3 \right] v \, d\xi d\eta
\]  
(5-3-9)

It is expected here that in the limiting case of rectangular elements the present formulation should yield the 5-\( \beta \) hybrid stress element [32]. Thus, the four internal displacement terms used are the same as the one
used by Wilson, et al., [29] i.e.,

\[ \nu_{\lambda} = \lambda_1 (1 - \xi^2) + \lambda_2 (1 - \eta^2) \]  
(5-3-10)

\[ \nu_{\alpha} = \lambda_3 (1 - \xi^2) + \lambda_4 (1 - \eta^2) \]

Substituting these into eq. (5-3-9) and integrating, one obtains

\[ J^* = (b_3 \beta_2 - b_1 \beta_3 - a_3 \beta_5 + a_1 \beta_6) (\lambda_1 + \lambda_2) \]

\[ + (b_3 \beta_8 - b_1 \beta_9 - a_3 \beta_5 + a_1 \beta_6) (\lambda_3 + \lambda_4) \]  
(5-3-11)

Thus, this result is, in fact, the same as the use of only one \( \lambda \) term each for \( u \) and \( v \) and hence it is possible to reduce only to 7 independent stress parameters.

This situation is similar to the development in ref. [31]. In that case it was necessary to make the small perturbation of the element geometry. Let us consider a quadrilateral element (figure 28b) with the mid-sides of 23 and 14 distorted by small distance \( \Delta \) along the \( \xi \)-axis. The \( x \) and \( y \) components of this perturbation are \( \pm \Delta_x \) and \( \pm \Delta_y \) as shown. By using isoparametric representation of the element with two additional nodes 5 and 6 one can show the Jacobian now becomes

\[ [J] = \begin{bmatrix}
    a_1 + a_2 \eta_2 + 4x(1 - \eta^2) & b_1 + b_2 \eta_2 + 4y(1 - \eta^2) \\
    a_3 + a_2 \xi_2 - 2 \Delta x \xi \eta & b_3 + b_2 \xi - 2 \Delta y \xi \eta
\end{bmatrix} \]  
(5-3-12)

One also observes that

\[ \frac{\Delta y}{\Delta x} = \frac{b_1}{a_1} \]  
(5-3-13)
The following integral $I_u^*$, thus becomes

$$I_u^* = \int_A \left( \frac{\partial \sigma}{\partial x} + \frac{\partial \tau y}{\partial y} \right) u_d dA$$

$$= \int_1^1 \int_1^1 \left\{ \left( b_3 \beta_2 - b_1 \beta_3 - a_3 \beta_8 + a_1 \beta q \right) + \left( b_2 \beta_2 - a_2 \beta_8 \right) \xi + (- b_2 \beta_3 + a_2 \beta q) \eta \\
+ 2 \left(-a_1 \beta_2 + A_x \beta q \right) \xi \eta \\
+ (A_x \beta q - A_y \beta_3) (1 - \eta^2) \right\} \left[ \lambda_1 (1-\xi^2) + \lambda_2 (1-\eta^2) \right] d\xi d\eta$$

(5-3-14)

Integrating and setting the coefficients for $\lambda_1$ and $\lambda_2$ to zero one obtains the following two independent equations:

$$b_3 \beta_2 - b_1 \beta_3 - a_3 \beta_8 + a_1 \beta q = 0$$

(5-3-15)

and

$$\Delta x \beta q - A_y \beta_3 = 0$$

(5-3-16)

which may also be written as

$$a_1 \beta q - b_1 \beta_3 = 0$$

(5-3-17)

From eq. (5-3-15) then

$$b_3 \beta_2 - a_3 \beta q = 0$$

(5-3-18)

Similarly by considering the integral $I_v^* = \int_A \left( \frac{\partial \tau x}{\partial x} + \frac{\partial \sigma y}{\partial y} \right) v_d dA$
one obtains \[ b_1 \beta_4 - a_1 \beta_6 = 0 \] (5-3-19)

and \[ b_3 \beta_8 - a_3 \beta_5 = 0 \] (5-3-20)

Let \( \beta_3 \) and \( \beta_5 \) be the independent \( \beta \)'s then

\[
\beta_4 = \frac{b_1}{a_1} \beta_3 \quad ; \quad \beta_6 = \left( \frac{b_1}{a_1} \right)^2 \beta_3
\] (5-3-21)

\[
\beta_8 = \frac{a_3}{b_3} \beta_5 \quad ; \quad \beta_2 = \left( \frac{a_3}{b_3} \right)^2 \beta_5
\]

The assumed stresses are rearranged in matrix form as follows

\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & a_1^2 \eta & a_3^2 \xi \\
0 & 1 & 0 & b_1^2 \eta & b_3^2 \xi \\
0 & 0 & 1 & a_1 b_1 \eta & a_3 b_3 \xi
\end{pmatrix} \begin{pmatrix}
\beta_1 \\
\beta_5
\end{pmatrix}
\] (5-3-22)

In the case of a rectangular element with \( \xi \) and \( \eta \) in parallel with \( x \) and \( y \), \( b_1 \) and \( a_3 \) vanish and the 5-\( \beta \) terms of reference [31] is obtained. The present method, thus, provide a rational procedure for obtaining the assumed stresses for geometrically distorted elements. With Equation (5-3-22), the assumed stress hybrid element can be formulated by the old method. It should be emphasized that since the present formulation is based on complete stresses the resulting element stiffness matrix is an invariant.
One can also show that the resultant element stiffness matrix has sufficient rank so that all zero-energy deformation modes are suppressed. A procedure for examining the rank of the element stress matrices has been given in ref. [34]. The element deformation energy $U_d$ due to assumed stresses is defined by

$$2U_d = \int_V \mathbf{G} : \mathbf{E} \, dV = \int_V \mathbf{G} : \mathbf{D} \mathbf{u} \, dV = \mathbf{G}^T \mathbf{G}_d \mathbf{u}$$

(5-3-23)

where $\mathbf{G}_d$ are independent deformation modes. The ideal situation is that $\mathbf{G}_d$ is a square matrix and the condition for the absence of zero-energy deformation mode is that all diagonal terms in $\mathbf{G}_d$ are non-zero and no columns in $\mathbf{G}_d$ are linearly dependent. For the present 4-node plane stress element the displacements $u_q$ and $v_q$ may be expressed as

$$u_q = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta$$
$$v_q = \alpha_5 + \alpha_6 \xi + \alpha_7 \eta + \alpha_8 \xi \eta$$

(5-3-24)

and the strain components can be written as

$$|J| \varepsilon_x = (J_{22} \frac{\partial}{\partial \xi} - J_{12} \frac{\partial}{\partial \eta}) u_q$$
$$|J| \varepsilon_y = (-J_{21} \frac{\partial}{\partial \xi} + J_{11} \frac{\partial}{\partial \eta}) v_q$$
$$|J| \gamma_{xy} = (-J_{21} \frac{\partial}{\partial \xi} + J_{11} \frac{\partial}{\partial \eta}) u_q + (J_{22} \frac{\partial}{\partial \xi} - J_{12} \frac{\partial}{\partial \eta}) v_q$$

(5-3-25)

The use of Eq. (5-3-24) yields

$$|J| \varepsilon_x = \gamma_1 + \gamma_2 \xi + \gamma_3 \eta$$
$$|J| \varepsilon_y = \gamma_4 + \gamma_5 \xi + \gamma_6 \eta$$
$$|J| \gamma_{xy} = \gamma_7 + \gamma_8 \xi + \gamma_9 \eta$$

(5-3-26)
where
\begin{align*}
\gamma_1 &= b_3 \alpha_2 - b_1 \alpha_3 \\
\gamma_2 &= b_2 \alpha_2 - b_1 \alpha_4 \\
\gamma_3 &= b_3 \alpha_4 - b_2 \alpha_3 \\
\gamma_4 &= a_1 \alpha_7 - a_3 \alpha_6 \\
\gamma_5 &= a_1 \alpha_8 - a_4 \alpha_6 \\
\gamma_6 &= a_2 \alpha_7 - a_3 \alpha_8 \\
\gamma_7 &= -a_3 \alpha_2 + a_1 \alpha_3 + b_3 \alpha_6 - b_1 \alpha_7 \\
\gamma_8 &= -a_2 \alpha_2 + a_1 \alpha_4 + b_2 \alpha_6 - b_1 \alpha_8 \\
\gamma_9 &= -a_3 \alpha_4 + a_2 \alpha_3 + b_3 \alpha_8 - b_2 \alpha_7
\end{align*}

Using Eqs. (5-3-22) and (5-3-26), the matrix \( G_d \) is reduced to the form

\[
G_d \approx \frac{4}{3} \begin{bmatrix}
3 \gamma_1 & 0 & 0 \\
0 & 3 \gamma_4 & 0 \\
0 & 0 & 3 \gamma_7 \\
\gamma_3 & \left( \frac{b_1}{a_1} \right)^2 \gamma_6 & \left( \frac{b_1}{a_1} \right) \gamma_9 \\
\left( \frac{a_2}{b_3} \right)^2 \gamma_2 & \gamma_5 & \left( \frac{a_3}{b_3} \right) \gamma_8
\end{bmatrix}
\]

(5-3-28)

Here \( G \) has 6 columns one of which is linearly dependent on the others.
This is in agreement with the fact that there are only 5 independent deformation modes. By deleting this column the resulting 5x5 $G_d$ matrix is

$$G_d = \frac{4}{3} \begin{bmatrix}
3b_3 & 0 & -3b_1 & 0 & 0 \\
0 & 3a_1 & 0 & 0 & 0 \\
-3a_3 & -3b_3 & 3a_1 & 0 & 0 \\
0 & b_1 d_{12}/a_1^2 & d_{12}/a_1 & d_{13}/a_1 & b_1 d_{13}/a_1^2 \\
a_3 d_{23}/b_3 & 0 & 0 & a_3 d_{13}/b_3 & d_{13}/b_3
\end{bmatrix}$$

where

$$d_{12} = b_1 a_2 - a_1 b_2$$
$$d_{23} = a_3 b_2 - b_3 a_2$$
$$d_{13} = a_1 b_3 - b_1 a_3$$

It can be seen that for the $\xi, \eta$ and $x,y$ axes shown in fig. 28a,

$$a_1 > 0 \ , \ b_3 > 0 \ , \ d_{13} > 0$$

and the five columns in Eq. (5-3-26) are linearly independent for any element geometry.

5.4 Examples, evaluation and discussion

Several problems that have been used in references [28], [29] and [35] are solved by using the elements based upon the formulations given
in 5.2 and 5.3. The results are presented in tables 1 and 2 and figures 30 and 31.

For the stresses based upon covariant basis vectors of natural coordinate system, as discussed in 5.2, the following stress assumptions are employed;

for $\xi$ version 1

$$\tau^\xi = \beta_1 + \beta_4 \eta$$
$$\tau^\eta = \beta_2 + \beta_5 \xi$$
$$\tau^{\eta \xi} = \beta_3$$

$$\Rightarrow \frac{\partial \tau^\xi}{\partial \xi} + \frac{\partial \tau^{\eta \xi}}{\partial \eta} = 0$$
(5-4-1)

for $\xi$ version 2

$$J \tau^\xi = \beta_1 + \beta_4 \eta$$
$$J \tau^\eta = \beta_2 + \beta_5 \xi$$
$$J \tau^{\eta \xi} = \beta_3$$

$$\Rightarrow \frac{\partial (J \tau^\xi)}{\partial \xi} + \frac{\partial (J \tau^{\eta \xi})}{\partial \eta} = 0$$
(5-4-2)

The results in table 1 indicates that the two versions don't pass the patch test. It is seen in table 2 that the displacements under bending moment are in good agreement with the exact solutions while the stress results are poor. In order to see the behavior under bending moment clearly, the example given in figure 30 was solved. Within the range of 0 < a < 2, the behavior under distortion is not affected much. On the other hand, the larger distortion causes the deterioration in the solution. But such a distortion will not be used in computation anyway. The corresponding stress results for a = 1.75 are shown in
figure 3.1. It can be seen that both versions yield comparatively good solutions for stresses.

Since the versions don't pass the patch test, the stresses based upon global Cartesian coordinate system are used. Each stress assumptions can be listed as follows;

for $Q^-(\xi^d)$

\[
\begin{align*}
\sigma_x &= \beta_1 + \beta_4 \xi \\
\sigma_y &= \beta_2 + \beta_5 \xi \\
\sigma_{xy} &= \beta_3
\end{align*}
\] (5-4-3)

for $Q^-(\chi^d)$

\[
\begin{align*}
\sigma_x &= \beta_1 + \beta_4 \gamma \\
\sigma_y &= \beta_2 + \beta_5 \chi \\
\sigma_{xy} &= \beta_3
\end{align*}
\] (5-4-4)

for $Q^-(\xi^d)$ with $\gamma$ and $\beta$

\[
\begin{align*}
\sigma_x &= \beta_1 + \beta_4 \xi + \beta_6 \xi \\
\sigma_y &= \beta_2 + \beta_5 \xi + \beta_7 \xi \\
\tau_{xy} &= \beta_3 - \beta_7 \xi - \beta_6 \xi
\end{align*}
\] (5-4-5)

and $Q^*(\xi^d)$ indicates a newly created invariant hybrid element whose stress distribution is given by eq. (5-3-22). On the other hand, the modification indicated in eq. (5-2-22) can be applied to pass the patch test. The element is denoted as $Q^{*-}(\xi^d)$.

Also the results obtained by some displacement models are included in tables 1 and 2.
For case 1 in table 1, all the elements except $\mathcal{C}$ versions 1 and 2 pass the patch test. For case 2, the new method yields the most accurate solution for the displacement. The last two cases are used to examine the effect of element distortion. Here the new method yields a little better result in displacements than that by the quasi-conforming element while for stresses it yields comparable accuracy as that by the original hybrid stress model ($\mathcal{C}(x^d)$). The new method is definitely the one which provides excellent solutions for both displacements and stresses. It should be remarked that the original hybrid element is not an invariant and, for the present mesh pattern of cases 3 and 4 the second and third elements from the left which are of the same element geometry, actually all have additional zero energy deformation modes. The new hybrid elements with stresses in $\xi-\zeta$ coordinates is, however, invariant and is free of zero energy deformation modes for the same problem. As shown in figure 29, the property of invariance was examined. Indeed the element is invariant. The present approach is thus the rational method for the formulation of invariant hybrid stress elements. Another illustration of the advantage of the new hybrid element over the original one is given by figure 30 and 31. This is a cantilever beam under pure bending analyzed by using two plane stress elements. The degree of geometric distortion of these elements is represented by the dimension "a." It is seen that although the hybrid stress element for stresses in x-y coordinates yields the exact solution when there is no geometric distortion; under severe distortion it becomes almost as rigid as the assumed displacement compatible element. On the other hand, the new hybrid element with stresses in $\xi-\zeta$ coordinates is less sensitive to geometric distortions.
Discussions

The use of the internal displacements is to improve the equilibrium conditions. For a geometrically distorted element with stresses expressed in natural coordinates, in general, the equilibrium conditions cannot be satisfied pointwise. By using many terms in \( u_\alpha \), the equilibrium conditions can be better approximated. However, the number of terms in \( u_\alpha \) must be limited in order to maintain sufficient rank of the resulting element stiffness matrix.

It should be remarked that in the present formulation of the quadrilateral element the internal displacements \( u_\alpha \) may be chosen differently, yet the resulting element stiffness matrix will be the same. In \( u_q \) given by Eq. (5-3-24) the constant, \( \xi, \eta \) and \( \xi \eta \) terms are already included. Thus in \( u_\alpha \), higher order should be used. In fact, by using

\[
\begin{align*}
U_\alpha &= \lambda_1 \xi^2 + \lambda_2 \eta^2 \\
V_\alpha &= \lambda_3 \xi^2 + \lambda_4 \eta^2
\end{align*}
\]

one should be able to obtain the same result given by Eq. (5-3-22). But the displacement functions by Eq. (5-3-10) are physically more idealistic.

The element \( Q(\xi) \) and the new element \( \tilde{Q}(\xi) \) turned out to be identical. Considering the Jacobian components at \( \xi=\eta=0 \), we obtain

\[
J_{11} = a_1, \quad J_{12} = b_1, \quad J_{21} = a_3, \quad J_{22} = b_3
\]

Then the transformation indicated in eq. (5-2-22) yields

\[
\begin{align*}
\sigma_x &= \zeta^2 a_1^2 + 2 \zeta^2 a_1 a_3 + \zeta^2 a_3^2 \\
\sigma_y &= \zeta^2 b_1^2 + 2 \zeta^2 b_1 b_3 + \zeta^2 b_3^2 \\
\sigma_{xy} &= \zeta^2 a_1 b_3 + \zeta^2 (a_1 b_3 + a_3 b_1) + \zeta^2 a_3 b_3
\end{align*}
\]  
(5-4-6)
Substituting eq. (5-4-1) into eq. (5-4-6), it follows that

\[ \sigma_x = \beta_1 + a_1 x \beta_4 + a_3 \xi \beta_5 \]
\[ \sigma_y = \beta_2 + b_1 y \beta_4 + b_3 \xi \beta_5 \]
\[ \sigma_{xy} = \beta_3 + a_1 b_1 y \beta_4 + a_3 b_3 \xi \beta_5 \]  \hspace{1cm} (5-4-7)

This stress distribution is obviously equivalent to eq. (5-3-22). Therefore the element \( \mathcal{L}_2 \) is a special case derived from the new method. Since the patch test is required to be satisfied, the new element can be interpreted as an element which is so designed to yield a good performance under bending and at the same time satisfy the patch test.

The same approach can be extended to 8-node hexahedral elements. As having indicated in 4.5.2, for a regular brick element three quadratic stress terms are needed to suppress the kinematic deformation modes. By considering these terms as nonessential for solution accuracy it is entirely logical to use complete stresses only up to the linear terms. In this case, nine internal displacement terms are used and the 24 independent linear stress terms are reduced to 15, and the total number of independent \( \beta \)'s is 18.
6. **Suggestions for further research and conclusions**

6.1 **Conclusions**

The following conclusions can be stated from the present investigation:

1. New versions of variational principles based upon Hellinger-Reissner principle and Hu-Washizu principle provides the most rational and versatile way of constructing finite elements by hybrid stress method.

2. Hybrid semiLoof shell elements by the use of natural coordinate system and new version of modified Hu-Washizu principle are well-balanced and efficient elements. The difficulty in satisfying the equilibrium equations for thin shells can be relieved by introducing only partial satisfaction of the equilibrium equations. Such schemes also permit the uncoupling of membrane mid-surface displacements and lateral deflection hence to accommodate the important fundamental modes of rigid body motion, momentless membrane state and inextensional bending.

3. For rectangular plane stress and brick elements, the similarity and equivalence between Wilson's incompatible displacement models and hybrid stress elements were established through the use of new versions of variational principles.
4. The quasi-conforming element is closely associated with the modified Hu-Washizu principle. However, in its formulation the physical stresses are not explicitly employed. Thus, the present formulation by modified Hu-Washizu principle conceptually differs from that of quasi-conforming element in its explicit introduction of stress equilibrium conditions.

5. A new method for the formulation of hybrid stress elements by the new version of Hellinger-Reissner principle has been established by expanding the essential terms of the assumed stresses as complete polynomials in the natural coordinates of the element. The equilibrium conditions are imposed in a variational sense through the internal displacements which are also expanded in the natural coordinates. The resulting element possesses all the ideal qualities; it is invariant, it is least sensitive to geometric distortion, it contains a minimum number of stress parameters, and it provides accurate stress calculations. Being a hybrid stress element it can always pass the patch test. This new method can be derived and implemented in a routine manner. Thus, it opens the door for a great future in hybrid stress elements.

6.2 Suggestions for further research

From the present investigation, the following research problems can be suggested for further studies:

1. The suggested method of using internal displacement as a logic procedure for the formulation of invariant hybrid stress element has been demonstrated only for the four-node plane stress element and the eight-
node solid element. In these cases the guidelines for the choice of $u_A$ can be made either from the mathematical viewpoint of making the displacements complete or by incorporating the desirable physical constraints. Further investigations are required to establish a rational procedure for the choice of and of appropriate physical constraint equation for other types of elements.

2. In the formulation of plate and shell elements with $C^1$ continuity requirement, there is a need of a systematic method for the suppression of kinematic deformation mode.

3. The development of hybrid semiLoof elements in terms of natural coordinates should be extended to general shells.

4. The large deflection analysis for plates and shells by hybrid semiLoof elements should be extended to updated, convected Lagrangian formulation.

5. The variational principles used in the construction of invariant hybrid stress elements should be extended to the solution of other types of continuum mechanics problems.
References


32. T. H. H. Pian; On the Equivalence Between Incompatible Displacement and Hybrid Stress Element, to be published in Applied Mathematics and Mechanics, Chungking, China.


34. T. H. Pian and D. P. Chen; On the Suppression of Zero Energy Deformation Modes, to be published in Int. J. for Num. Meth. in Engng.


Figure 1 Conventional Variational Principles in Solid Mechanics
\( \sigma^* \) = Stresses Which Satisfy Equilibrium Equations.

Figure 2  Variational Principles for Finite Element Methods
Figure 3  Reference Coordinate Systems and Natural Coordinate System
Any Desirable Physical Information in Terms of

\[ \tilde{C}^u = 0 \]

\[ \tilde{U} ; \text{resolution of interelement compatibility} \]

Any Desirable Physical Information in Terms of

\[ C^u = 0 \]

Figure 4  General Feature of Hybrid Stress Element
Figure 5  Hybrid SemiLoof Shallow Shell Elements
Figure 6  Bending of a Square Plate With Clamped Edges Under a Concentrated Load
Note: $N_B = 37, 24$ are triangular elements and $N_B = 59$ is a quadrilateral element.

Figure 7a Displacement Distributions in Pinched Cylindrical Shell 
(4 x 4 Mesh)
Note: $N_B = 37, 24$ are triangular elements and $N_B = 59$ is a quadrilateral element.

Figure 7b Stress Distributions in Pinched Cylindrical Shell
Figure 7c Stress Distributions in Pinched Cylindrical Shell
E = 3.0 \times 10^6 \text{ lb/sq. in}, \nu = 0, g = 90 \text{ lb/sq. ft.}

R = 25 \text{ ft}, L = 50 \text{ ft}, \phi = 40^\circ, h = 0.25 \text{ ft.}

Figure 8  Cylindrical Shell Roof
Figure 9a Stress Distributions Along CD in Cylindrical Shell Roof
Figure 9b  Stress and Displacement Distributions Along CD in Cylindrical Shell Roof
Figure 10  Convergence of Vertical Deflection at D

Vertical Deflection at D (in)

Exact (Shallow Shell)

3.70331

598s

100  200  300  400  500 (DOF)

Cowper, Lindberg and Olson

Quadrilateral Element

Triangular Element

\( \eta_{mHW} \) and \( \eta_{mR} \)
(by Natural Coordinates)
Figure 11  Thin Shell Element Based Upon Natural Coordinate System and Global, Shell Coordinate Systems
Figure 12  Hybrid SemiLoof Cylindrical Shell Element
Figure 13a: Convergence of Deflection $w_0$ in Cylindrical Shell Roof Problem.
Figure 13b Convergence of Inplane Displacement $u_{\theta}$ in Cylindrical Shell Roof Problem
(1) Analytical Solution by Shallow Shell Theory
(2) Analytical Solution by Deep Shell Theory

○ Degenerated Shell with Reduced Integration
△ Dawe's 54 DOF Triangular Element
□ Hybrid SemiLoof 32 DOF 368 7λ
□ Hybrid SemiLoof 32 DOF 388 7λ

Figure 14 Convergence of Deflection at D in Cylindrical Shell Roof Problem
\[ M_\phi \] (ft.Kip/ft)

at \( y = 0 \)

* Figure 15a  Stress Distribution in Cylindrical Shell Roof Problem*
Figure 15b  Stress Distribution in Cylindrical Shell Roof Problem
Figure 16  Cylindrical Shell Under Internal Pressure:
R = 25 ft, E = 4.32 x 10^8 lb/sq.ft, ν = 0.0, P = 90.0 lb/sq.ft
\begin{align*}
v &= 0.3 \\
F_3 &= F_2 \cot \frac{1}{2} \phi \\
\frac{h}{R} &= 10^{-2}, \quad \frac{L}{R} = 5 \times 10^{-2}
\end{align*}

Figure 17 Slit Cylinder
Figure 18a The Relationship Between the Angle $\phi$ and Torsional Coupling $M_{12}$ in Slit Cylinder Problem
Figure 18b The Relationship Between the Angle $\phi$ and Deflection $w$ in Slit Cylinder Problem
This Deformation is Inextensional Bending Mode, i.e.
\[ \varepsilon_{\alpha\beta} = 0 \]

Figure 19  Cantilever Plate Under Edge Shear Load
Figure 20a  The Relationship Between Edge Shear Load and Lateral Deflection in Cantilever Plate Problem
Figure 20b  Stress Distribution for Cantilever Plate Under Edge Shear Load:

- $P^* = 7.52765 \times 10^{-3}$ for 20 DOF Element
- $P^* = 7.50000 \times 10^{-3}$ for SemiLoof Element
Data:

\[ L = 100 \text{ mm} \]
\[ E_1 = 26.2 \times 10^6 \text{ psi} \]
\[ E_2 = 1.49 \times 10^6 \text{ psi} \]
\[ \nu_{12} = 0.28 \]
\[ G_{12} = 1.04 \times 10^6 \text{ psi} \]
\[ \alpha_1 = -0.059 \times 10^{-6}/^\circ F \]
\[ \alpha_2 = 14.2 \times 10^{-6}/^\circ F \]
\[ h = 0.02126 \text{ in} \]

Figure 21  Unsymmetric Cross Ply Laminate
Figure 22  The Schematic Description of the Behavior of Unsymmetric Cross Ply Laminate Under Thermal Load
Figure 23  The Relationship Between Temperature Difference and the Deflection at A
Figure 24 The relationship between temperature difference and the deflections $W_A$ and $W_C$ with imperfection.
Figure 25a  A finite element with geometric perturbation

Figure 25b  A finite element as beam
Figure 26 Eight node 3-D solid element with geometric perturbation
Figure 27 Mapping
Figure 28b Quadrilateral element with geometric perturbation
Case A: regular shape
Case B: distorted shape
E = 1500, ν = 0.25

Figure 29a Coordinate rotation and four node plane stress finite elements
The first non zero eigenvalues of stiffness matrices

\[ \omega_1 \times 10^2 \]

rotation angle

- \( a \) present invariant hybrid stress element
- \( \sigma (\xi_2) \) 5B with stresses expressed in terms of natural coordinates

Case A: regular shape
Case B: distorted shape

Figure 29b Effect of coordinate rotation
<table>
<thead>
<tr>
<th>element</th>
<th>patch test</th>
<th>case 1 ($U_A$)</th>
<th>case 2 ($-V_A$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_4$</td>
<td>0</td>
<td>6,000</td>
<td>17.00</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>X</td>
<td>6,701</td>
<td>19.66</td>
</tr>
<tr>
<td>$Q_{M6}$</td>
<td>0</td>
<td>6,000</td>
<td>17.61</td>
</tr>
<tr>
<td>$Q_{p6}$</td>
<td>0</td>
<td>6,000</td>
<td>17.61</td>
</tr>
<tr>
<td>$Q_{c5}$</td>
<td>0</td>
<td>6,000</td>
<td>17.40</td>
</tr>
<tr>
<td>$Q_{c6}$</td>
<td>0</td>
<td>6,000</td>
<td>17.61</td>
</tr>
<tr>
<td>$\sigma(\xi^a)$</td>
<td>0</td>
<td>6,000</td>
<td>17.64</td>
</tr>
<tr>
<td>$\tau$ version 1</td>
<td>X</td>
<td>6,000</td>
<td>19.98</td>
</tr>
<tr>
<td>$\tau$ version 2</td>
<td>X</td>
<td>6,642</td>
<td>20.07</td>
</tr>
<tr>
<td>$\sigma(X^a)$</td>
<td>0</td>
<td>6,717</td>
<td>17.70</td>
</tr>
<tr>
<td>present one</td>
<td>0</td>
<td>6,000</td>
<td>17.64</td>
</tr>
<tr>
<td>exact</td>
<td></td>
<td>6,000</td>
<td>18.00</td>
</tr>
</tbody>
</table>

Note: The numerical integration is carried out by 2x2 Gaussian quadrature.

$Q_5$ is presented in Ref. 28.

Table 1
<table>
<thead>
<tr>
<th>element</th>
<th>case 3</th>
<th>case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \nu_A )</td>
<td>( \sigma_x ) at B</td>
</tr>
<tr>
<td>Q_4</td>
<td>45.7</td>
<td>-1761</td>
</tr>
<tr>
<td>Q_6</td>
<td>98.4</td>
<td>-2427.5</td>
</tr>
<tr>
<td>Q_{c5}</td>
<td>63.5</td>
<td>-2045.5</td>
</tr>
<tr>
<td>Q_{c6}</td>
<td>96.07</td>
<td>-2438.9</td>
</tr>
<tr>
<td>( \tau(\xi^0)5_B )</td>
<td>77.5</td>
<td>-2775</td>
</tr>
<tr>
<td>( \tau ) version 1</td>
<td>98.4</td>
<td>-4695</td>
</tr>
<tr>
<td>( \tau ) version 2</td>
<td>98.4</td>
<td>-3695</td>
</tr>
<tr>
<td>( \sigma(\xi^0)7_B )</td>
<td>62.4</td>
<td>-2530</td>
</tr>
<tr>
<td>( \sigma(X^0)5_B )</td>
<td>93.6</td>
<td>-2974</td>
</tr>
<tr>
<td>present one</td>
<td>96.2</td>
<td>-3014</td>
</tr>
<tr>
<td>exact</td>
<td>100</td>
<td>-3000</td>
</tr>
</tbody>
</table>

table 2
Figure 30  Cantilever beam (ν=0) under edge moment

Note: Error of δTOP is plotted except a new invariant element.
Figure 31 Stress distributions for cantilever beam under edge moment \([a = 1.75]\)
Figure A1  Four node axisymmetric solid element with geometric perturbation.
Figure A2 Axisymmetric solid element based upon
natural coordinate system
Appendix 1. Consideration on the false shear phenomenon in four node axisymmetric solid element

The phenomena of false shear stresses which were observed in both incompatible model [36] and stress hybrid model [37] is considered. Since the introduction of internal displacement parameters implies the constraint on stress equilibrium equations, we can see the effect of such parameters by checking the equilibrium equations.

A1.1 The false shear phenomenon in incompatible model

The four node axisymmetric solid element is illustrated in figure 7. Following the analogy to four node plane stress element, we introduce four additional incompatible modes.

The assumption of displacements can be given by

\[ \begin{align*}
U &= \sum_{i=1}^{4} N_i u_i + (1 - \xi^2) \lambda_1 + (1 - \eta^2) \lambda_2 \\
V &= \sum_{i=1}^{4} N_i v_i + (1 - \xi^2) \lambda_3 + (1 - \eta^2) \lambda_4
\end{align*} \]

or

\[ \begin{align*}
U &= \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta + (1 - \xi^2) \lambda_1 + (1 - \eta^2) \lambda_2 \\
V &= \alpha_5 + \alpha_6 \xi + \alpha_7 \eta + \alpha_8 \xi \eta + (1 - \xi^2) \lambda_3 + (1 - \eta^2) \lambda_4 \quad (A1-1)
\end{align*} \]

Since the strain-displacement relation is

\[ \begin{align*}
\varepsilon_\xi &= \frac{\partial w}{\partial \xi} \quad , \quad \varepsilon_\eta = \frac{\partial u}{\partial \eta} \\
\gamma_{\xi\eta} &= \frac{\partial u}{\partial \eta} + \frac{\partial w}{\partial \xi}
\end{align*} \quad (A1-2)

eq. (A1-1) yields
\[ \sigma_r = \alpha_2 + \alpha_4 \xi - 2 \xi \lambda_1 \]
\[ \sigma_\theta = \frac{1}{\gamma_0 + \xi} [\alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta] \]
\[ \sigma_\phi = \alpha_1 + \alpha_2 \xi - 3 \eta \lambda_4 \]
\[ \tau_{r\phi} = \alpha_3 + \alpha_4 \xi + \alpha_6 + \alpha_8 \eta - 2 \eta \lambda_2 - 2 \xi \lambda_3 \]  
\text{(A1-3)}

The equilibrium equations can be written as

\[ \frac{\partial \sigma_r}{\partial \xi} + \frac{\partial \tau_{r\phi}}{\partial \eta} + \frac{\sigma_r - \sigma_\theta}{\gamma_0 + \xi} = 0 \]
\[ \frac{\partial \sigma_\phi}{\partial \eta} + \frac{\partial \tau_{r\phi}}{\partial \xi} + \frac{\tau_{r\phi}}{\gamma_0 + \xi} = 0 \]  
\text{(A1-4)}

Substituting eq. (A1-3) into eq. (A1-4) gives

\[ -2 \lambda_1 + \alpha_8 - 2 \lambda_2 + \frac{\alpha_2 + \alpha_4 \xi - 2 \xi \lambda_1}{\gamma_0 + \xi} - \frac{\alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta}{(\gamma_0 + \xi)^2} = 0 \]

\[ -2 \lambda_4 + \alpha_4 - 2 \lambda_3 + \frac{1}{\gamma_0 + \xi} (\alpha_3 + \alpha_4 \xi + \alpha_6 + \alpha_8 \eta - 2 \eta \lambda_2 - 2 \xi \lambda_3) = 0 \]  
\text{(A1-5)}

When the translational movement along \( R \) and is considered, we have

\[ \alpha_1 = \text{const} \quad \alpha_i = 0 \quad ; \quad i = 2, 3, \ldots, 8 \]  
\text{(A1-6)}

Then eq. (A1-5) becomes

\[ -2 \lambda_1 - 2 \lambda_2 + \frac{-2 \xi \lambda_1}{\gamma_0 + \xi} - \frac{\alpha_1}{(\gamma_0 + \xi)^2} = 0 \]
\[ -2 \lambda_4 - 2 \lambda_3 + \frac{1}{\gamma_0 + \xi} (-2 \eta \lambda_2 - 2 \xi \lambda_3) = 0 \]  
\text{(A1-7)}
Realizing \( \frac{1}{2\pi} \int_{\gamma} d\gamma = \int_{0}^{1} r \left( r_{0} + \xi \right) |J| d\xi d\eta \) [here, assume |J| = const],

and \( \int_{\gamma} \frac{r^{2m+1}}{r^{2m+1}} d\gamma = \int_{\gamma} \frac{r^{2m+1}}{r^{2m+1}} d\gamma = 0; \ m = \text{integer}, \)

the constraint condition implies the coupling of \( \lambda_{1}, \lambda_{2} \) and \( \alpha_{1} \).

It can be seen in eq. (A1-7) that this coupling effect will be magnified where \( r_{0} \) is small. To eliminate the effect of coupling in \( \gamma_{r}^{2} \) distribution, it is obvious from eq. (A1-3) that one should set \( \lambda_{2} = 0 \). Then "false shear" caused by the term of \( \lambda_{2} (1 - \gamma^{2}) \) can be resolved, as indicated in Cook's paper.

Even though \( \zeta \) in the constitutive relation was assumed to be indentity, the nature of coupling effect can't be changed for \( \zeta \neq I \).

The primary cause of false shear in axisymmetric solid is the fact that \( U = \text{const} \) doesn't mean rigid body motion.

Al.2 The false shear phenomenon in hybrid stress model

When the element geometry is perturbed as shown in figure A1, we have

\[
\begin{align*}
\left\{ \begin{array}{c}
\tau \\
\eta \\
\end{array} \right\} = \sum_{i=1}^{4} \frac{1}{4} (1 + \xi_{i} \xi) (1 + \eta_{i} \eta) \left\{ \begin{array}{c}
\tau_{i} \\
\eta_{i} \\
\end{array} \right\} + \Delta \xi (1 - \eta^{2})
\end{align*}
\]

\( (A1-3) \)

where the perturbation vector is given by

\[
\Delta = \Delta x \xi_{1} + \Delta y \xi_{2}
\]

The coordinate transformation is represented as
\[
\begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{bmatrix} =
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial r}{\partial \xi} \\
\frac{\partial r}{\partial \eta}
\end{bmatrix}
\]  \hspace{1cm} (A1-4)

where

\[J_{11} = a_1 + a_2 \eta + 4x (1-\eta^2)\]
\[J_{12} = b_1 + b_2 \eta + 4y (1-\eta^2)\]
\[J_{21} = a_3 + a_4 \xi - 2A \xi \eta \]
\[J_{22} = b_3 + b_4 \xi - 2A \xi \eta \]

and

\[a_1 = \frac{1}{4} (\tau_1 + \tau_2 + \tau_3 - \tau_4)\]
\[a_2 = \frac{1}{4} (\tau_1 - \tau_2 + \tau_3 - \tau_4)\]
\[a_3 = \frac{1}{4} (-\tau_1 - \tau_2 + \tau_3 + \tau_4)\]
\[a_4 = \frac{1}{4} (-\tau_1 + \tau_2 + \tau_3 - \tau_4)\]

\[b_1 = \frac{1}{4} (\tau_1 + \tau_2 + \tau_3 + \tau_4)\]
\[b_2 = \frac{1}{4} (\tau_1 - \tau_2 + \tau_3 - \tau_4)\]
\[b_3 = \frac{1}{4} (-\tau_1 - \tau_2 + \tau_3 + \tau_4)\]

Then the critical term can be described by

\[
\mathcal{I}^* = \int_{-1}^{1} \int_{-1}^{1} \left[ \frac{1}{J} (J_{22} \frac{\partial \eta}{\partial \xi} - J_{12} \frac{\partial \eta}{\partial \eta}) \partial \tau + \frac{1}{J} (-J_{21} \frac{\partial r}{\partial \xi} + J_{11} \frac{\partial r}{\partial \eta}) \partial \eta + \frac{\partial r}{r} \right] u^2 d\xi d\eta
\]
\[
+ \left[ \frac{1}{J} (J_{22} \frac{\partial \eta}{\partial \xi} - J_{12} \frac{\partial \eta}{\partial \eta}) \partial \tau + \frac{1}{J} (-J_{21} \frac{\partial r}{\partial \xi} + J_{11} \frac{\partial r}{\partial \eta}) \partial \eta + \frac{\partial r}{r} \right] v^2 d\xi d\eta
\]
\[
= \int_{-1}^{1} \int_{-1}^{1} \left[ \frac{r (J_{22} \frac{\partial \eta}{\partial \xi} - J_{12} \frac{\partial \eta}{\partial \eta}) \partial \tau + r (-J_{21} \frac{\partial r}{\partial \xi} + J_{11} \frac{\partial r}{\partial \eta}) \partial \eta + J \frac{\partial \eta}{\partial \xi} \right] u^2 d\xi d\eta
\]
\[
+ \left[ \frac{r (J_{22} \frac{\partial \eta}{\partial \xi} - J_{12} \frac{\partial \eta}{\partial \eta}) \partial \tau + r (-J_{21} \frac{\partial r}{\partial \xi} + J_{11} \frac{\partial r}{\partial \eta}) \partial \eta + J \frac{\partial \eta}{\partial \xi} \right] v^2 d\xi d\eta
\]  \hspace{1cm} (A1-7)

where
\[ J = J_{11} J_{22} - J_{12} J_{21} \]
\[ = \left\{ a_1 + a_2 \xi + a_3 (1-\eta^2) \right\} \left\{ b_1 + b_2 \xi - 2 \Delta y \xi \eta \right\} \]
\[ - \left\{ b_1 + b_2 \eta + \Delta y (1-\eta^2) \right\} \left\{ a_3 + a_2 \xi - 2 \Delta x \xi \eta \right\} \]  (A1-8)

Now we assume the stress distributions in the following form:

\[ \sigma_\xi = \beta_1 + \beta_2 \xi + \beta_3 \eta \]
\[ \sigma_\eta = \beta_4 + \beta_5 \xi + \beta_6 \eta \]
\[ \sigma_3 = \beta_7 + \beta_8 \xi + \beta_9 \eta \]
\[ \tau_{x\eta} = \beta_{10} + \beta_{11} \xi + \beta_{12} \eta \]  (A1-9)

When we consider a regular four node axisymmetric solid element whose size is 2x2 and perturbate it by

\[ \Delta = \Delta x \xi \eta \]

we obtain

\[ J_{11} = 1 + \Delta x (1-\eta^2) \]
\[ J_{12} = 0 \]
\[ J_{21} = -2 \Delta x \xi \eta \]
\[ J_{22} = 1 \]
\[ J = 1 + \Delta x (1-\eta^2) \]  (A1-10)

and

\[ \tau = \tau_\xi + \xi + 2 \Delta x \xi (1-\eta^2) \quad \tau_\xi = \tau_{\xi \eta} = \eta = \Delta \]

Substituting eqs. (A1-9) and (A1-10) into eq. (A1-7), the expression for \( I^* \) becomes
\[ I^* = \int \sum_{i=1}^{4} \left\{ \left( \tau_0 + \xi + 2 \Delta x \xi (1 - \eta^2) \right) \left( \beta_2 + \Delta x \xi \beta_{11} + \left( 1 + \Delta x (1 - \eta^2) \right) \beta_{12} \right) \\
+ \left( 1 + \Delta x (1 - \eta^2) \right) \left( \beta_1 + \Delta x \xi \beta_3 \eta - \beta_4 - \beta_5 \xi - \beta_6 \xi \right) \right\} \mathcal{U}^2 \\
+ \left\{ \left( \tau_0 + \xi + 2 \Delta x \xi (1 - \eta^2) \right) \left( \beta_{11} + \Delta x \xi \beta_{12} \eta + \left( 1 + \Delta x (1 - \eta^2) \right) \beta_9 \right) \\
+ \left( 1 + \Delta x (1 - \eta^2) \right) \left( \beta_{10} + \beta_{11} \xi + \beta_{12} \eta \right) \right\} \mathcal{V}^2 \right\} d\xi d\eta \] (A1-11)

When we assume that

\[ \mathcal{U}^2 = (1 - \xi^2) \mathcal{A}_1 + (1 - \eta^2) \mathcal{A}_2 \]
\[ \mathcal{V}^2 = (1 - \xi^2) \mathcal{A}_3 + (1 - \eta^2) \mathcal{A}_4 \] (A1-12)

eq (A1-11) takes the form;

\[ I^* = \left[ \frac{8}{3} \tau_0 \beta_2 + \frac{2}{3} \tau_0 \beta_{12} + \frac{32}{15} \tau_0 \Delta x \beta_{12} + \frac{8}{3} \beta_1 + \frac{32}{15} \Delta x \beta_1 \\
- \frac{8}{3} \beta_4 - \frac{32}{15} \Delta x \beta_4 \right] \mathcal{A}_1 \\
+ \left[ \frac{8}{3} \tau_0 \beta_2 + \frac{8}{3} \tau_0 \beta_{12} + \frac{16}{q} \tau_0 \Delta x \beta_{12} + \frac{8}{3} \beta_1 + \frac{16}{q} \Delta x \beta_1 \\
- \frac{8}{3} \beta_4 - \frac{16}{q} \Delta x \beta_4 \right] \mathcal{A}_2 \\
+ \left[ \frac{8}{3} \tau_0 \beta_{11} + \frac{8}{3} \tau_0 \beta_9 + \frac{32}{15} \tau_0 \Delta x \beta_9 + \frac{8}{3} \beta_{10} + \frac{32}{15} \Delta x \beta_{10} \right] \mathcal{A}_3 \\
+ \left[ \frac{8}{3} \tau_0 \beta_{11} + \frac{8}{3} \tau_0 \beta_9 + \frac{16}{q} \tau_0 \Delta x \beta_9 + \frac{8}{3} \beta_{10} + \frac{16}{q} \Delta x \beta_{10} \right] \mathcal{A}_4 \] (A1-13)

Taking the variation of \( I^* \) with respect to \( \lambda_i \) \((i=1,\ldots,4)\) leads to
\[
\begin{align*}
\{ \begin{align*}
\tau_0 \beta_{12} + \beta_1 - \beta_4 &= 0 \\
\tau_0 \beta_{2} + \tau_0 \beta_{12} + \beta_1 - \beta_4 &= 0 \\
\tau_0 \beta_{3} + \beta_{10} &= 0 \\
\tau_0 \beta_{21} + \tau_0 \beta_{9} + \beta_{10} &= 0
\end{align*} \tag{A1-14}
\end{align*}
\]

from which

\[
\begin{align*}
\beta_2 &= \beta_{11} = 0 \\
\beta_{12} &= \frac{\beta_1 - \beta_4}{\tau_0} \\
\beta_{10} &= -\tau_0 \beta_9 
\end{align*} \tag{A1-15}
\]

Therefore we obtain

\[
\begin{align*}
\sigma_\tau &= \beta_1 + \beta_2 \eta \\
\sigma_\theta &= \beta_3 + \beta_4 \xi + \beta_5 \eta \\
\sigma_\beta &= \beta_6 + \beta_{17} \xi + \beta_8 \eta \\
\sigma_\xi &= -\tau_0 \beta_9 + \frac{(\beta_3 - \beta_1) \eta}{\tau_0} 
\end{align*} \tag{A1-16}
\]

As observed in incompatible model, if \( \lambda_2 \) is excited, the phenomenon of "false shear" happens. Therefore, the elimination of \( \lambda_2 \) term in eq. (6-4-53) is necessary to avoid the phenomenon. Then we have

\[
\begin{align*}
\sigma_\tau &= \beta_1 + \beta_4 \xi + \beta_8 \eta \\
\sigma_\theta &= \beta_3 + \beta_4 \xi + \beta_5 \eta \\
\sigma_\beta &= \beta_6 + \beta_7 \xi + \beta_8 \eta \\
\sigma_\xi &= -\tau_0 \beta_9 + \left( \frac{\beta_3 - \beta_1}{\tau_0} \right) \eta 
\end{align*} \tag{A1-17}
\]
Appendix 2. Axisymmetric solid element based upon natural coordinate system

As shown in figure A2, we consider an axisymmetric solid element whose shape of the cross section is rectangular and the size is 2a x 2b. The position vector at an arbitrary point inside of the element is given by

\[ \mathbf{\bar{r}} = \mathbf{\bar{r}}_0 + \mathbf{\bar{r}}_\xi \]
\[ = (r_0 + a\xi) \cos \theta \mathbf{e}_1 + (r_0 + a\xi) \sin \theta \mathbf{e}_2 + (\alpha + b\xi) \mathbf{e}_3 \]  
(A2-1)

where the position vector of the origin of the element \( \mathbf{\bar{r}}_0 \) is defined as

\[ \mathbf{\bar{r}}_0 = r_0 \cos \theta \mathbf{e}_1 + r_0 \sin \theta \mathbf{e}_2 + \beta \mathbf{e}_3 \]  
(A2-2)

and the position vector of an arbitrary point of the element from the origin is

\[ \mathbf{\bar{r}}_\xi = a\xi \cos \theta \mathbf{e}_1 + a\xi \sin \theta \mathbf{e}_2 + b\xi \mathbf{e}_3 \]  
(A2-3)

and \( \mathbf{e}_i \) = unit basis vectors of global coordinate system. When the stresses and the strains are defined with respect to natural coordinate system, we have

\[ \mathbf{\bar{\sigma}} = \gamma \mathbf{\bar{e}}_i \mathbf{\bar{\sigma}}_i \mathbf{\bar{e}}_j \]  
(A2-4)
\[ \mathbf{E} = \nabla_i \xi^i \xi^j \]  
(A2-5)

where \( \xi^i \) = covariant basis vectors of natural coordinate system 
and \( \xi_i \) = contravariant basis vectors of natural coordinate system.

Using eq. (A2-1), the covariant and contravariant basis vectors can be obtained by

\[
\begin{align*}
\xi_r &= \frac{\partial r}{\partial \xi} = a \cos \theta \xi_1 + a \sin \theta \xi_2 \\
\xi_\theta &= \frac{\partial r}{\partial \theta} = -r \sin \theta \xi_1 + r \cos \theta \xi_2 \\
\xi_\lambda &= \frac{\partial r}{\partial \lambda} = b \xi_3
\end{align*}
\]  
(A2-6)

and

\[
\begin{align*}
\xi_r &= \frac{1}{a} (\cos \theta \xi_1 + \sin \theta \xi_2) = \frac{\xi_r}{a^2} \\
\xi_\theta &= \frac{1}{r} (-\sin \theta \xi_1 + \cos \theta \xi_2) = \frac{\xi_\theta}{r^2} \\
\xi_\lambda &= \frac{1}{b} \xi_3
\end{align*}
\]  
(A2-7)

When the gradient tensor and displacements are defined by

\[
\begin{align*}
\nabla &= \xi_r \frac{\partial}{\partial \xi} + \xi_\theta \frac{\partial}{\partial \theta} + \xi_\lambda \frac{\partial}{\partial \lambda} \\
\mathbf{u} &= u_r \xi^r + u_\theta \xi^\theta + u_\lambda \xi^\lambda
\end{align*}
\]  
(A2-8)

and the following formula is used;

\[
\begin{align*}
\frac{\partial \xi_r}{\partial \theta} &= \frac{a}{r} \xi_\theta \\
\frac{\partial \xi_\theta}{\partial \theta} &= -\frac{r}{a} \xi_r \\
\frac{\partial \xi_r}{\partial \lambda} &= \frac{r}{a} \xi_\theta \\
\frac{\partial \xi_\theta}{\partial \lambda} &= -\frac{a}{r} \xi_r
\end{align*}
\]  
(A2-9)
the strain tensor becomes

\[
\varepsilon = \frac{1}{2} \left( \dot{\varepsilon} R + R \dot{\varepsilon} \right) \\
= \frac{\partial u_r}{\partial \xi} \dot{\xi} \dot{\xi} \dot{r} + \frac{1}{2} \left( \frac{\partial u_3}{\partial \xi} + \frac{\partial u_r}{\partial \eta} \right) \dot{\xi} \dot{\eta} \dot{r} + \frac{1}{2} \left( \frac{\partial u_3}{\partial \xi} + \frac{\partial u_3}{\partial \eta} \right) \dot{\eta} \dot{r} \dot{r} \\
+ \frac{\partial u_3}{\partial \eta} \dot{\xi} \dot{\eta} \dot{r} + \frac{r}{a} u_r \dot{\xi} \dot{\eta} \dot{r} \dot{r}
\]  

(A2-10)

Comparing eq. (A2-5) to eq. (A2-10), we get

\[
\gamma_r = \frac{\partial u_r}{\partial \xi}, \quad \gamma_\theta = \frac{r}{a} u_r \\
\gamma_3 = \frac{\partial u_3}{\partial \eta} + 2 \gamma_3 = \frac{\partial u_3}{\partial \xi} + \frac{\partial u_r}{\partial \eta}
\]  

(A2-11)

We now assume the displacements as

\[
\begin{align*}
&u_r = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \\
&u_\eta = \alpha_5 + \alpha_6 + \eta + \alpha_7 \xi
\end{align*}
\]  

(A2-12)

where one rigid body mode is removed from \( u_\eta \). Substituting eq. (A2-12) into eq. (A2-11) leads to

\[
\begin{align*}
\gamma_r &= \alpha_2 + \alpha_4 \\
\gamma_\theta &= \frac{r}{a} \left( \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \right) \\
\gamma_3 &= \alpha_6 + \alpha_7 \xi \\
2 \gamma_3 &= \alpha_5 + \alpha_7 \eta + \alpha_3 + \alpha_4 \xi
\end{align*}
\]  

(A2-13)

For the axisymmetric solid the deformation energy is given by
\[ 2\mathcal{U}d = \int_{\mathcal{V}} \gamma^{ij} \epsilon_{ij} d\mathcal{V} = \int_{\mathcal{V}} \gamma^{ij} \epsilon_{ij} d\mathcal{V} \]
\[ = 2\pi \int_{-1}^{1} \int_{-1}^{1} \gamma^{ij} (\hat{s}, \eta) \gamma_{ij} (\hat{s}, \eta) \tau J d\hat{s} d\eta \]  \hspace{1cm} (A2-14)

where \( J = ab \).

The evaluation of \( 2\mathcal{U}d \) is essentially the same as
\[ I^* = \int_{-1}^{1} \int_{-1}^{1} \gamma^{ij} \gamma_{ij} \tau d\hat{s} d\eta \]
\[ = \int_{-1}^{1} \int_{-1}^{1} \gamma^{ij} \gamma_{ij} (\tau_0 + a\hat{s}) d\hat{s} d\eta \]  \hspace{1cm} (A2-15)

The use of eq. (A2-13) yields
\[ I^* = \int_{-1}^{1} \int_{-1}^{1} \left( \gamma^{ij} \gamma_{ij} \tau_0 + \gamma^{ij} \gamma_{ij} \tau_1 + 2 \gamma^{ij} \gamma_{ij} \tau_2 \right) (\tau_0 + a\hat{s}) d\hat{s} d\eta \]
\[ = \int_{-1}^{1} \int_{-1}^{1} \left( \gamma^{ij} \gamma_{ij} (\alpha_2 + \alpha_4 \eta) \right) + \frac{1}{8} \left( \gamma^{ij} \gamma_{ij} (\alpha_3 + \alpha_5 \eta) \right) \\
+ \frac{1}{8} \left( \gamma^{ij} \gamma_{ij} (\alpha_4 + \alpha_6 \eta) \right) \right) (\tau_0 + a\hat{s}) d\hat{s} d\eta \]  \hspace{1cm} (A2-16)

where it is noted that \( \tau_0 \geq 0, \alpha_2 > 0 \) and \( \tau_0 \geq \alpha_1 \).

When we apply one \( \alpha - \)one \( \beta \) approach [34] to suppress kinematic modes, we can obtain two of the possible stress assumptions with \( \gamma \beta \);  
\[ \gamma^T = \beta_2 + \beta_4 \eta \]
\[ \gamma^\theta = \beta_1 \]
\[ \gamma^3 = \beta_6 + \beta_9 \hat{s} \]
\[ \gamma^{\tau_2} = \beta_3 + \beta_5 \hat{s} \]  \hspace{1cm} (A2-17)
or

\[ \tau_r = \beta_2 + \beta_4 \tau \]
\[ \tau_\theta = \beta_1 + \beta_3 \tau \]
\[ \tau_\phi = \beta_6 + \beta_7 \phi \]
\[ \tau_r^2 = \beta_5 \]  \hspace{1cm} (A2-18)

It is noted that the stress equilibrium equations can be expressed by

\[ \vec{\nabla} \cdot \tau = \left( \frac{\partial \tau_r}{\partial \tau} + \frac{3 \tau_\theta}{\partial \tau} + \frac{3 \tau_\phi}{\partial \tau} \right) \cdot \left( \tau_r^2 \xi_r + \tau_\theta \xi_\theta + \tau_\phi \xi_\phi + \tau_r \tau_\theta \xi_r \xi_\theta + \tau_r \tau_\phi \xi_r \xi_\phi + \tau_\theta \tau_\phi \xi_\theta \xi_\phi \right) \]
\[ + \frac{\partial \tau_r}{\partial \tau} \xi_r + \frac{\partial \tau_\theta}{\partial \tau} \xi_\theta + \frac{\partial \tau_\phi}{\partial \tau} \xi_\phi + \frac{\partial \tau_r}{\partial \tau} \xi_r + \frac{\partial \tau_\theta}{\partial \tau} \xi_\theta + \frac{\partial \tau_\phi}{\partial \tau} \xi_\phi \]
\[ - \frac{\tau}{a} \tau_\theta \xi_r + \frac{\tau}{a} \tau_\phi \xi_\phi \]
\[ \equiv 0 \]

or

\[ \frac{\partial \tau_r}{\partial \tau} + \frac{\partial \tau_\theta}{\partial \tau} + \left( \frac{a}{r} \tau_r - \frac{r}{a} \tau_\theta \right) \]
\[ - \frac{\partial \tau_r}{\partial \theta} + \frac{\partial \tau_\theta}{\partial \tau} + \frac{a}{r} \tau_\theta \gamma = 0 \]  \hspace{1cm} (A2-19)

where body forces are omitted for the sake of simplicity. From eq. (A2-19), we can see that when the order of \[ \frac{a}{r} \] is \[ a \approx O\left(\frac{d}{r}\right) \] like axisymmetric shell the terms \[ \frac{a}{r} \tau_r \] and \[ \frac{a}{r} \tau_\theta \] are order of \[ O(\Delta \tau^r) \] and \[ O(\Delta \tau^\theta) \] respectively. Then the simplified expressions for eq. (A2-19) are given by
\[
\left\{ \begin{array}{c}
\frac{\partial \tau^r}{\partial s} + \frac{\partial \tau^r}{\partial \eta} - \frac{r}{A} \tau^\theta \approx 0 \\
\frac{\partial \tau^r}{\partial s} + \frac{\partial \tau^s}{\partial \eta} \approx 0
\end{array} \right. \tag{A2-20}
\]

On the other hand, when \( T \) becomes small, the order of a term \( \frac{1}{A} \) also becomes small. The simplified equilibrium equations, therefore, are

\[
\left\{ \begin{array}{c}
\frac{\partial \tau^r}{\partial s} + \frac{\partial \tau^r}{\partial \eta} + \frac{a}{r} \tau^r \approx 0 \\
\frac{\partial \tau^r}{\partial s} + \frac{\partial \tau^s}{\partial \eta} + \frac{a}{r} \tau^r \approx 0
\end{array} \right. \tag{A2-21}
\]
Biographical Sketch

Kiyohide Sumihara was born on January 29, 1955, in Fukuoka, Japan. In March 1973, he graduated from the National High School attached to Tokyo Liberal Arts University, and in April 1973, he was admitted to Science 1 course, College of General Education, Tokyo University. In March 1977, he received the degree of Bachelor of Engineering with a major in aeronautics from Tokyo University. He received the degree of Master of Engineering in March 1979. He continued his work toward the degree of Doctor of Engineering until August 1980, when he was nominated by Murate Overseas Scholarship Foundation as a 1980-1982 all expense award grantee for graduate study in the United States. He enrolled in the Graduate School of Massachusetts Institute of Technology on leave of absence from Tokyo University.