A Numerical Model of Equatorial Waves

With Application to the Seasonal Upwelling in the Gulf of Guinea

by

Randall J. Patton

A.B., University of California at Berkeley

(1976)

SUBMITTED TO THE DEPARTMENT OF
METEOROLOGY AND PHYSICAL OCEANOGRAPHY
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS OF THE
DEGREES OF
MASTER OF SCIENCE IN
PHYSICAL OCEANOGRAPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August 1981

© Randall J. Patton 1981

The author hereby grants to M.I.T. permission to reproduce and to
distribute copies of this thesis document in whole or in part.

Signature of Author

[Signature]

Department of Meteorology and Physical Oceanography
August 24, 1981

Certified by

Mark A. Cane, Thesis Supervisor

Accepted by

R. Piechotta, Departmental Graduate Committee

MAR 22 1982

LIBRARIES
# TABLE OF CONTENTS

ABSTRACT ......................................................... 11

ACKNOWLEDGEMENTS ............................................. iv

CHAPTER 1: INTRODUCTION ....................................... 1

CHAPTER 2: EQUATORIAL DYNAMICS. ............................. 7

CHAPTER 3: BOUNDARY CONDITIONS FOR THE GULF OF GUINEA . . . . 18

  Section I: Reflected and transmitted waves. .................. 18
  Section II: Asymptotic description of mass conservation in
              Kelvin and anti-Kelvin modes ....................... 26
  Section III: The boundary layer at the zonal coast. .......... 29

CHAPTER 4: NUMERICAL SCHEME .................................. 34

  Section I: Finite difference equations and their solution .... 34
  Section II: Accuracy .......................................... 39
  Section III: Accuracy and behavior of individual modes ..... 41
  Section IV: Stability ......................................... 52

CHAPTER 5: MODEL RESULTS. ...................................... 57

  Section I: Introduction ....................................... 57
  Section II: Effects of non-uniform phase and amplitude in the
              wind forcing .......................................... 72
  Section III: Previous studies ................................ 76
  Section IV: Description of the wind field and model results .. 79
  Section V: Discussion ........................................ 105

CHAPTER 6: SUMMARY. ............................................ 115

REFERENCES ..................................................... 118
A NUMERICAL MODEL OF EQUATORIAL WAVES

WITH APPLICATION TO SEASONAL UPWELLING IN

THE GULF OF GUINEA

by

RANDALL J. PATTON

Submitted to the Department of Meteorology and Physical Oceanography
on August 25, 1981 in partial fulfillment of the
requirements for the Degree of Master of Science in
Physical Oceanography

ABSTRACT

A new and efficient numerical model of linear equatorial dynamics is
presented and used to simulate the pronounced seasonal variations found in
equatorial regions. The scheme takes advantage of the different propaga-
tion directions of the non-dispersive equatorial Kelvin and Rossby waves
that are obtained in the low frequency limit on an equatorial beta-plane.
This approximation requires special consideration of the boundary condi-
tions at the basin's meridional coasts. In particular, new results for
the conditions at the partial boundary represented by the model African
coast and the generation of reflected and transmitted equatorial waves
are derived. A boundary layer is found to exist along the model zonal
coast (i.e., Guinea coast). Some of its simple properties are derived
using a scaled version of the equatorial vorticity equation.

The finite difference equations and their method of solution, which
involves separating out the Kelvin solution, are then presented. The
Kelvin mode can be solved exactly along a characteristic, but the west-
ward propagating Rossby response, which is made up of meridional modes
traveling at different speeds, is solved by a finite difference technique.
This method is analogous to solving the single equation in $v$ familiar
in the studies of equatorial dynamics, and is used to derive a finite
difference version of the wave equation for each mode. This equation is
then used to analyze the behavior and accuracy for individual modes as the
frequency of the forcing and the step-sizes in $x$ and $t$ are varied.

A model of the Atlantic basin was forced with a uniform, periodic wind
and compared with analytic results in the literature. This simple case
was then extended in two experiments which examine the effect of longi-
dudinal variations in the amplitude and phase of the forcing. The results
of forcing the basin, including a model of the Gulf of Guinea, with a
realistic trade wind forcing, show very good agreement with observations.
The major features of seasonal surface temperature, particularly the
upwelling along the Guinea coast and the equator, are paralleled in the
model height contours. The boundary layer is also clearly evident and
explains the narrow scale and intensity of the coastal upwelling and eastward Guinea current. Both remote winds concentrated in the far west and winds in the eastern part of the basin appear to force the annual summer upwelling. A smaller winter upwelling is found to be forced by only the local variation of wind caused by the migration of the axis of the trade winds (ITCZ). These results contradict theories of upwelling involving remotely forced single Kelvin waves and purely nearcoastal mechanisms. The nearly in-phase response to the wind forcing found in the Gulf is very similar to the adjustment found for the uniform, periodic case (Cane and Sarachik IV, Philander and Pacanowski, 1981).

Thesis Supervisor: Dr. Mark Cane

Title: Assistant Professor of Meteorology
I would like to thank my advisor, Mark Cane, for his invaluable guidance, patience and good humor during the development of this thesis. Paul Schopf and his associates at the Goddard Space Flight Center provided the analytic curve-fit formulas for the wind forcing, which were a big help in producing model results which could be compared with observation. I would especially like to thank my very close friend, Linda Bulriss, who helped tremendously with typing and producing figures and who shared the ups and downs of graduate student life with me.
Chapter 1: Introduction

The equatorial regions are particularly interesting places to study the dynamics of wind forced ocean circulation. There are strong, distinct currents which vary considerably with the seasons and underlie two strong belts of winds, the Northeast and Southeast Trades. The region is also interesting from a purely theoretical point of view; the dynamical effect of the reversal in sign of the Coriolis parameter at the equator creates a waveguide which supports planetary waves unique to the tropics. Plane wave solutions to the linearized equations have a faster propagation speed near the equator than they do at higher latitudes. A low frequency baroclinic Rossby wave may take only a couple of months to travel the width of the equatorial Atlantic while the same type of wave at mid-latitudes may take a decade to travel the same distance. The waveguide also permits a unique equatorial Kelvin wave which can reflect energy eastward along the equator instead of having it be lost in a western boundary layer. At seasonal periods, linear equatorial waves could influence the entire basin and greatly modify the local response of the ocean. This thesis presents a numerical model of these waves and examples of their influence in the eastern Atlantic basin.

The mathematical structure of equatorial wave solutions have been studied extensively in the past decade or so (see references). There is convincing evidence for equatorially trapped waves in the inertia-gravity branch of the dispersion diagram, Fig. 1a, with periods of four to five days (Wunsch and Gill, 1976; Luther, 1980) and for Rossby-gravity waves along the equator with 31 day periods (Weisberg, Horigan and Colin, 1979). However, because of the lack of sufficiently long time series and the variability of surface
layers due to non-wave mechanisms, direct evidence of long, low frequency waves has not been found. The effects of these large waves might be observed in the structure and seasonal variability of the equatorial currents and surface topography. By using a simple linear model of equatorial physics, Busalacchi and O'Brien (1980) showed that many features of the thermocline slope and surface topography could be simulated. Other researchers have concentrated on the generation of long baroclinic Kelvin and Rossby waves by a sudden wind forcing and the effects they have in distant parts of the ocean. These studies have used this kind of remote forcing to attempt to explain the reversal of the Somali current during the onset of the seasonal monsoons in the Indian Ocean (Lighthill, 1969), the interannual intrusion of warm equatorial waters off the west coast of Peru during the El Nino event (McCreary, 1976; Hurlburt, et. al., 1976) and the summer upwelling of cold water along the equator and in the Gulf of Guinea in the Atlantic (Adamec and O'Brien, 1978; Moore, et. al., 1978). By attempting to relate the phase of Ekman pumping to isotherm displacement in the tropical north Pacific, Meyers (1979) and White (1977) found it necessary to modify the Ekman pumping balance to include geostrophic divergence and thus allow for a westward, non-dispersive Rossby wave. Their simple models allowed isotherm displacement generated at areas of strong Ekman pumping to be propagated westward by the Rossby wave mechanism. In their studies, they considered latitudes that were far enough away from the equator so that it could be argued that only a single Rossby wave would be significantly excited. Nearer the equator, the dynamics of isotherm displacement could be influenced by a sum of Rossby and Kelvin waves permitted by the meridional structure (Cane and Sarachik IV; Philander and Pacanowski, 1981).
As the previous studies suggest, the horizontal and temporal scales found in the equatorial current system seem to readily support wavelike solutions. By contrast, the circulation of the current systems in higher latitudes are dominated by the familiar Sverdrup balance

$$\beta v = \text{curl} \ (\vec{\tau})$$  \hspace{1cm} (1.1)

($\vec{\tau}$ = wind stress) which has the ocean in equilibrium with the wind forcing and no waves are excited. The next higher order balance is the potential vorticity equation (Pedlosky, 1979)

$$\frac{\partial}{\partial t} (v_x - u_y) + f(u_x + v_y) + \beta v = 0$$  \hspace{1cm} (1.2)

which does have Rossby wave solutions. These waves provide the mechanism for maintaining the intense western boundary currents found in mid-latitudes, but are otherwise secondary to the steady wind forced circulation. The situation near the equator is not the same, however. The Coriolis parameter is much smaller and the meridional scales are short compared to zonal scales (Wyrtki, 1974). The balances in the vorticity equation are thus different from those in the mid-latitude example. Table I contains typical values for the various scales associated with equatorial circulations. If these are substituted into equation (1.2), the $v_{xt}$ term can be neglected and the other terms are roughly equal.

The long zonal length scales, low frequency and modal meridional structure of the equatorial solutions to (1.2) allow them to be approximated by non-dispersive, westward propagating Rossby waves and an eastward propagating equatorial Kelvin wave. The other possible wave motions, namely inertia-gravity and eastward propagating Rossby waves will be neglected, their influence being expressed only in the initial and boundary conditions.
Table I  Equatorial Scaling - Terms in the Vorticity Equation

<table>
<thead>
<tr>
<th>Term</th>
<th>Approximate Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_y$</td>
<td>$\approx$ 300 km</td>
<td>width of equatorial current and internal Rossby radius of deformation</td>
</tr>
<tr>
<td>$\delta t = \omega$</td>
<td>$= 3 \times 10^{-7}$ s$^{-1}$</td>
<td>seasonal frequency</td>
</tr>
<tr>
<td>$H$</td>
<td>$\approx$ 100 m</td>
<td>depth of thermocline and vertical scale of motion</td>
</tr>
<tr>
<td>$U$</td>
<td>$\approx$ 1 ms$^{-1}$</td>
<td>speed of equatorial currents zonally</td>
</tr>
<tr>
<td>$V$</td>
<td>$\approx 1/20$ ms$^{-1}$</td>
<td>much smaller meridional speeds</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$\approx$ 20 m</td>
<td>change in layer thickness (sea surface and isotherm displacement)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\approx 2 \times 10^{-11}$ (ms)$^{-1}$</td>
<td>Coriolis parameter</td>
</tr>
</tbody>
</table>
The derivation of this approximation for seasonal, equatorial scales is reviewed in Chapter 2. The numerical model uses the difference in the direction of propagation of the two types of solutions (westward Rossby and eastward Kelvin) to solve for each separately. In the equations of motion, the part of the forcing that projects onto the Kelvin mode can be subtracted away so that the equations only excite westward Rossby waves. These waves can be solved for very efficiently by iterating from east to west in the numerical scheme. Similarly, the Kelvin mode component of the forcing can be used to excite only eastward traveling Kelvin waves. This mode obeys a simple wave equation whose solutions follow characteristics and can be solved for exactly in the numerical scheme. The finite difference equations, their solutions and an analysis of the accuracy, stability and behavior of the scheme are presented in Chapter 4.

In solving the equations of motion, the numerical scheme does not decompose the westward Rossby waves into meridional modes. That sort of spectral approach becomes cumbersome for basins with corners (e.g. western Gulf of Guinea) where the modes must be matched across the corner's longitude. The present scheme avoids this by solving the equations all at once so that only the boundary conditions need to be specified at each meridional boundary. The boundary conditions, however, must be derived in the context of the long, low frequency wave approximation; this is done in Chapter 3.

Chapter 5 presents the results of various model integrations. A uniform, periodic wind, varying at an annual frequency, is used to force a model of the Atlantic basin. The results are compared to analytic results in the literature in order to test the scheme's accuracy. Two simple experiments are also discussed which give examples of the effects of varying the wind's amplitude and phase longitudinally. An application to observations
is discussed in the last part of Chapter 5 where a model Atlantic basin, including the Gulf of Guinea, is forced with realistic winds. The results, which agree very well with observations, are compared with previous theories of eastern Atlantic seasonal variation. Special attention is also paid to the relation of various model features to the spatial and temporal variations of the winds. The long, low frequency waves discussed in the beginning of the introduction and which form the core of the physics modeled in the numerical scheme are invoked to explain the adjustment of the ocean to the changing wind pattern and the remarkable similarity over the whole basin between model results and observations. Chapter 6 summarizes the basic features of the model and the results obtained in the seasonal Atlantic simulations.
Chapter 2: Equatorial Dynamics

The equations appropriate to studying large scale seasonal motions are the linearized shallow water equations on an equatorial beta-plane

\[ u_t - \beta y v + g h_x = F \quad (2.1a) \]
\[ v_t + \beta y u + g h_y = G \quad (2.1b) \]
\[ h_t + \hat{H}(u_x + v_y) = Q \quad (2.1c) \]

where \( f=\beta y \) is the Coriolis parameter near the equator, \( g \) is the acceleration of gravity, \( F \) is a zonal wind forcing, \( G \) is a meridional forcing, \( Q \) is a source (or sink) of heat or mass and \( u \) and \( v \) are the zonal velocity and meridional velocity respectively. \( \hat{H} \) is the ocean depth and can be written \( \hat{H}(x,y,t) = \hat{h} + h(x,y,t) \), where \( h \) is the variation of sea surface height and \( \hat{h} \) the mean depth. These equations represent the motion of a stratified fluid if \( \hat{h} \) is considered as an equivalent depth, e.g.

\[ \hat{H}_n = \frac{N^2 D^2}{g m_n^2} \]

where \( N=\)buoyancy frequency, \( D=\)vertical scale of the motion and \( m_n=\)vertical wave number of the \( n \)th baroclinic mode. Typical equivalent depths for the first baroclinic mode are one meter or less. The equations \((2.1a,b,c)\) can be non-dimensionalized by defining the equatorial time and length scales \((\text{Matsuno, 1966})\)

\[ T_{eq} = (g H \beta^2)^{-1/4} \quad L_{eq} = (g H \beta)^{1/4} \]

Note that \( L_{eq} \) is the Rossby radius of deformation. The result is:
\[ u_t - y v + h_x = F \]  \hspace{1cm} (2.2a)
\[ v_t + y u + h_y = G \]  \hspace{1cm} (2.2b)
\[ h_t + u_x + v_y = Q \]  \hspace{1cm} (2.2c)

with \( F, G \) and \( Q \) now representing non-dimensional forcings. The corresponding boundary and initial conditions are:

\[ u=v=h=0 \quad \text{at} \quad t=0 \]  \hspace{1cm} (2.2d)
\[ u=0 \quad \text{at} \quad x=0, x_e \quad \text{at} \quad x_e \text{= eastern boundary} \]
\[ v=0 \quad \text{at} \quad y=y_n, y_s \quad \text{north and south boundaries (possibly at } \pm \infty) \]  \hspace{1cm} (2.2e)

If free solutions \( (F=G=Q=0) \) are considered to be zonal plane waves

\[ (u,v,h) \propto \phi(y)e^{i(kx-\omega t)} \]

it is possible to manipulate equations (2.2a,b,c) to form an equation in \( v \) only (Moore and Philander, 1976)

\[ \left( \frac{\partial^2}{\partial y^2} - y^2 + \omega^2 - \frac{k}{\omega} - k^2 \right)v = 0 \]  \hspace{1cm} (2.3)

This equation has solutions satisfying the boundary conditions (2.2e) only for

\[ \omega^2 - \frac{k}{\omega} - k^2 = 2\nu_m + 1 \]  \hspace{1cm} (2.4)

which is a dispersion relation for the plane waves and introduces meridional modes with eigenvalues \( \nu_m \). In a basin bounded in \( y \), the eigenfunctions for the \( m \)th mode are Hermite functions

\[ v_m(y) = \frac{1}{\sqrt{2\pi} \nu_m^{1/4}} e^{-y^2/2} H_m(y) \]
where $H_m(y)$ is the $m$th Hermite polynomial. For this case, $v$ goes to zero as $y$ goes to plus or minus infinity and the eigenvalue $\nu_m$ is an integer; $\nu_m = m$. In a basin with boundaries as $y = y_n$ above the equator and $y = y_s$ below, the eigenfunctions, which satisfy $v = 0$ at $y = y_n, y_s$, are parabolic cylinder functions with $\nu_m > m$ (Cane and Sarachik III). The solutions represented by equation (2.3) and dispersion relation (2.4) consist of high frequency gravity waves, low frequency Rossby waves and a Yanai or mixed Rossby gravity wave (for which $m = 1$). Another important solution to equations (2.2a,b,c) is the Kelvin mode with $v = 0$. It has the simple dispersion relation $\omega = k$, as shown in Fig. 1a. Its meridional structure is easily derived by considering the equations of motion with $v$ set equal to zero. A solution to this has

$$u_k(y) = h_k(y) = ce^{-y^2/2}$$

The normalization constant $c$ is determined by requiring

$$\int_{y_s}^{y_n} (u_k^2 + h_k^2) dy = 1$$

If $y_n \to \infty$ and $y_s \to -\infty$, then the solutions are Hermite functions and

$$h_k(y) = u_k(y) = \frac{1}{\sqrt{2\pi} \frac{1}{4}} e^{-y^2/2} = \psi_{-1}(y) \quad (2.5)$$

The other solution, $u = -h$, is the anti-Kelvin mode with $u(y) \propto e^{+y^2/2}$ (Cane and Sarachik III), and can only exist in a basin bounded in $y$. Both these solutions are in geostrophic balance, i.e. both modes satisfy $yu + h_y = 0$.

The waves shown in the dispersion diagram, Fig. 1a, represent all the possible solutions to equation (2.2). However, if the model ocean is forced at a low frequency (e.g. seasonal winds), the forced response will consist of the low frequency planetary wave branches of the dispersion diagram, near
Fig. 1a. Dispersion relation, eqn. 2.3, for equatorial waves (from Cane & Sarachik I)

Fig. 1b. Dispersion relation, eqn. 2.6, for low frequency and wavenumber, non-dispersive equatorial waves. The dashed line represents a particular frequency and its intersection with the solid curves shows the increasingly smaller wavelengths of higher modes. The anti-Kelvin wave can exist only in a basin bounded in y.
the k axis. The high frequency gravity waves, because of their higher propagation speeds, are only important in the short time needed for the ocean to adjust to a sudden onset of wind. Their fast propagation also poses numerical problems and it is desirable to filter them out before equations (2.2a-e) are solved. To do this, we can assume that the frequency of motion is very small, say \( \omega = O(\varepsilon) \), \( \varepsilon << 1 \). The two roots for the wavenumber k now lie in the Rossby wave branch and are either very small (westward propagating) or very large (eastward propagating). The large zonal spatial scales associated with a small k are a feature of the currents of the equatorial regions (Wyrtki, 1973). For \( k << 1 \) and \( \omega << 1 \), the dispersion relation equation (2.4) can be approximated by (see Fig. 1b)

\[
k = - \omega (2v_m + 1)
\]  

(2.6)

The higher modes for which \( v_m \) becomes large have their largest amplitude away from the equator so that this approximation remains valid near the equator.

The high wavenumber Rossby waves not represented by equation (2.6) have much slower group and phase velocities and cannot travel very far before being damped by frictional or inertial effects (Pedlosky, 1965). If they are generated at a western boundary, they will be concentrated in a boundary layer in a time short compared to seasonal variations (Lighthill, 1969). The simplified dispersion relation (2.6) can be used to obtain only low frequency and wavenumber solutions, which are typical of equatorial flows. This approximation filters not only the gravity waves, but also the short Rossby waves and the thin western boundary layers they generate.

Previous researchers have also used long, non-dispersive Rossby waves to model the equatorial environment by modifying equation (2.3), which can
be written non-dimensionally for non-zero forcing

\[ \nu_{ttt} - \nu_{xxt} - \nu_{yyt} + y^2 \nu_t - \nu_x = \tau_{tt}^{(y)} - y \tau_{t}^{(x)} - (\text{curl } \tau)_x \]

\[ \tau = \text{wind stress} \]

In studying the seasonal generation of baroclinic Rossby waves in the Indian Ocean, Lighthill (1969) used the large zonal scale of the wind forcing and the relatively low frequency of the motion to justify the neglect of the \( \nu_{xxt} \) and \( \nu_{ttt} \) terms from the above equation. The remaining solutions have the dispersion relation equation (2.6). If the \( \nu_{xxt} \) term is kept, short, eastward traveling Rossby waves can exist, but in an ever thinning region trapped to the eastern boundary. Lighthill used these properties to model the intensification of the Somali current at the western boundary. McCreary (1976) modeled the response of an equatorial ocean to a switched-on zonal forcing by calculating a forced interior solution (consisting of an eastward traveling equatorial Kelvin wave) and the response at the eastern boundary needed to make the zonal velocity zero there. This boundary solution could consist only of westward traveling waves, and so he also neglected the \( \nu_{xxt} \) and \( \nu_{ttt} \) terms to obtain the same approximation used throughout the basin in this model.

Rescaling equations (2.2a-e) with \( \omega = O(\varepsilon) \) and \( k = O(\varepsilon) \), \( \varepsilon << 1 \), we find from equation (2.2a) that

\[ \nu = O(\varepsilon u) \]

and equation (2.2b) becomes

\[ yu + h_y = G + O(\varepsilon^2) \]
The filtered, non-dimensional equations to be solved, corresponding to equations (2.2a,b,c), are then

\begin{align}
  u_t - yv + h_x &= F \quad (2.7a) \\
  yu + h_y &= G \quad (2.7b) \\
  h_t + u_x + v_y &= Q \quad (2.7c)
\end{align}

The remaining low frequency solutions are westward propagating, long Rossby waves, an eastward propagating equatorial Kelvin wave and, if the basin is bounded in y, a westward propagating anti-Kelvin wave. The associated initial and boundary conditions must be modified due to the absence of high frequency and wavenumber modes, as will be seen.

The solutions to equation (2.7) represent the long time asymptotic response to a slowly varying wind. The direction of propagation of the long wave solutions means that an eastward traveling Kelvin wave incident at an eastern boundary must be reflected as a combination of westward traveling long Rossby modes and anti-Kelvin waves (throughout this discussion, it should be kept in mind that the amplitude of the anti-Kelvin wave vanishes if the zonal boundaries are infinitely far away), while the reflection of the westward traveling modes at a western boundary must consist of only the Kelvin mode (Cane and Sarachik II). Let \( u_R \) denote the westward traveling waves allowed by the model approximations (Rossby and anti-Kelvin) and \( u_k \) denote the eastward traveling Kelvin wave. At the eastern boundary, the condition of zero total zonal mass flux requires (\( a_k = \text{amplitude of Kelvin mode}, x_e = \text{eastern boundary} \))
\[ u_r = -u_k = -a_k(x_e, t) \psi_{-1}(y) \]  \hspace{1cm} (2.8)

At the western boundary, the condition is not as straightforward. The meridional boundary current normally needed to redistribute the mass there is not resolved. This current would normally be expressed as a sum of eastward propagating, short Rossby waves which are not allowed. The explicit movement of mass meridionally by the Rossby modes into the outgoing Kelvin mode has to be implicitly taken into account by the boundary condition. This consists of setting the meridionally integrated zonal mass flux at the western boundary due to the long Rossby modes equal and opposite to the amplitude of the Kelvin mode there (Cane and Sarachik II, p. 404). Analytically, this is written

\[ \int_{y_s}^{y_n} a_k(0, t) \psi_{-1}(y) dy = -\int_{y_s}^{y_n} u_r(y) dy \]

or

\[ a_k(0, t) = -\int_{y_s}^{y_n} u_r(0, t, y) dy/1_k \]  \hspace{1cm} (2.9)

where \((1)_k\) is the projection of 1 on the Kelvin mode. Since \(u=0\) for the Kelvin wave, this also gives a boundary condition on \(h\). The boundary condition on \(h\) at the eastern boundary is found by considering equation (2.7b) with \(u=0\) (no normal flow). For no meridional forcing

\[ h_y = 0 \text{ implies } h = \text{constant} \]

Since the total height is the sum of the Rossby and Kelvin modes

\[ h = h_r + h_k = h_r + a_k \psi_{-1} = \text{constant} = c \]

The constant is determined by taking a projection of both sides onto the Kelvin mode.
\[ a_k(x_e, t) \int_{y_s}^{y_n} \psi_{-1}^2 (y) \, dy = c \int_{y_s}^{y_n} \psi_{-1} \, dy = c \cdot (1)_k \]

Thus,

\[ c = \frac{(1)_k}{a_k(x_e, t)} \quad (2.10) \]

The incident Kelvin wave height has to be redistributed meridionally so that the total height becomes a constant along the wall. In the real ocean, the modes responsible for this are made up of high meridional mode Rossby waves which have complex zonal wavenumbers (cf. equation (2.4) for \( \nu_m > \frac{1}{4 \, \omega^2} \)). They are thus trapped to the wall and behave like a coastal Kelvin wave (Moore, 1968). The modified dispersion relation, equation (2.6), allows only real wavenumber Rossby waves so that, like the western boundary current, this mechanism is unresolved and only implied in the boundary condition.

If \( h \) is given as the response to a meridional forcing only (\( G=\)forcing), a similar projection gives

\[ h(x_e, t, y) = \int_{y_s}^{y_n} G(y') \, dy' + \int_{y_s}^{y_n} G(y') \psi_{-1}(y) \, dy/\left(1\right)_k \quad (2.11) \]

When the forcings in equations (2.2a,b,c) are step-functions in time, part of the response will consist of high frequency gravity waves (cf. Cane and Sarachik I). These waves allow the ocean to adjust from the rapid variations produced by a sudden onset of the wind to the steady or low frequency motions closely approximated by the model's asymptotic solutions. As shown by Cane and Sarachik I, II, the asymptotic response to a switched-on meridional wind, \( G(x, y) \), with zero initial conditions, has \( v=0 \) and

\[ (u^{(2)}, h^{(2)}) = ( - \frac{3}{\partial y}, y) \, L^{-1}(G) \quad (2.12) \]
where the operator \( L = \frac{\partial^2}{\partial y^2} - y^2 \). Given the initial value of \( G \), the finite difference approximation to (2.12) can be solved to provide the initial values for the asymptotic solutions. If a zonal wind is suddenly switched-on over an ocean at rest, Cane and Sarachik II find that gravity waves set-up a steady component of \( v \) given by

\[
v^{(1)} = L^{-1}(yF + Qy)
\]

which is the initial condition on \( v \) needed by the model solutions.

When forced by the wind, the inviscid shallow water equations (2.2) produce equatorial waves which travel back and forth across the basin, eventually bringing the ocean into adjustment. Without damping, these motions will persist for several years. To diminish noise in the model and to cut down on the spin-up time needed for transient motions to settle down, the effect of friction can be easily introduced into the model equations by defining a Rayleigh friction term for \( u \) and \( h \) in equations (2.7a,c)

\[
\begin{align*}
  u_t - yv + h_x &= F - ru \\
  yu + h_y &= G, \\
  h_t + u_x + v_y &= Q - rh
\end{align*}
\]

It is assumed that \( rv \ll h_y \), so the Rayleigh friction term may be neglected in (2.7b). With the substitution

\[
\begin{align*}
  u^* &= u e^{rt} \\
  h^* &= h e^{rt}
\end{align*}
\]

the equations become, after multiplying by \( e^{rt} \)

\[
\begin{align*}
  u^*_t - yv^*_y + h^*_x &= F e^{rt} = F^* \\
  yu^* + h^*_y &= G e^{rt} = G^* \\
  h^*_t + u^*_x + v^*_y &= Q e^{rt} = Q^*
\end{align*}
\]
These are identical to equations (2.7) and can be solved in the same way (see Chapter 4). To obtain \((u,v,h)\) after a time \(\Delta t\) from the solutions \((u^*, v^*, h^*)\), simply take the reverse transformation

\[
\begin{align*}
    u &= u^* e^{-r\Delta t} \\
    h &= h^* e^{-r\Delta t}
\end{align*}
\]

The effect of multiplying the forcing by the factor \(e^{r\Delta t}\) can be incorporated as a constant proportional change in amplitude of the original forcings (which are in arbitrary units anyway). This means that friction can be modeled in the numerical scheme by simply multiplying the solutions obtained after each time step by \(e^{-r\Delta t}\). The effects of friction in the model results will be discussed in Chapter 5.
Chapter 3: Boundary Conditions for the Gulf of Guinea

Section I: Reflected and transmitted waves

The sharp corner used in the model to approximate the African coast (see Fig. 2) acts as a partial barrier to the planetary waves propagating zonally across the basin. The boundary conditions already described for the flow at the far eastern and western (full) boundaries must be carefully applied to this new case so as to ensure the conservation of mass.

The usual boundary conditions at a full boundary are that no normal flow exists at the boundary and, because of the modal nature of the solution, waves reflected at an eastern boundary (necessarily long Rossby waves for low frequency motions) have zero projection on the Kelvin mode. This second condition can be stated as (cf. equation (2.10))

\[ \int_{y_n}^{y_s} (u_r,0,h_r) \cdot (\psi_{-1},0,\psi_{-1}) \, dy = 0 \]  

(3.1)

which represents an inner product (with \( y_n \) and \( y_s \) the northern and southern extent of the basin, possibly at \( \pm \infty \)). For the long wave approximation, the meridional momentum equation (2.7b) implies that the boundary condition \( u=0, y>b \) may be written (cf. equations (2.8), (2.9) and (2.11), \( b= \)latitude of corner)

\[ u_r = -u_k \]  

(3.2)

\[ h_r = D - h_k + \int_{y}^{\infty} G \, dy \]  

(3.2b)

The constant \( D \) can be determined by applying equation (3.1). Since \( u=0 \) and no Kelvin wave is contained in the reflected solution, mass is conserved. This is the formulation used in Chapter 2 at the full eastern boundary.
Fig. 2. Diagram of partial boundary near equator used in calculating transmission coefficients of reflected planetary waves.

Fig. 3. Transmission coefficients of Kelvin mode and height constants along upper wall for partial boundary as functions of distance b from the equator to the zonal coast (see Fig. 2 above); (a) transmission coefficient for incident Kelvin wave, (b) transmission coefficient for incident Rossby modes with unit amplitude at corner, (c) height constant set-up for incident Kelvin wave, (d) height constant for incident Rossby waves.
The situation at a partial boundary is more subtle. If the boundary does not extend across all latitudes in the basin, equations (3.2a,b) do not apply in the areas that remain as open ocean and thus cannot determine \( u_r \) and \( h_r \) there. Since equation (3.1) depends on values of \( u_r \) and \( h_r \) for all \( y \), more information must be specified before the constant \( D \) in equations (3.2a,b) can be determined. There is also the possibility of Rossby waves propagating in from the east, underneath the boundary. When they pass the corner, the absence of a wall will require some sort of adjustment in the upper basin (e.g. coastal or anti-Kelvin wave turning the corner).

Consider a partial boundary at \( x = x_B \) extending from a latitude \( b \) at the south to infinity at the north as in Fig. 2. When a Kelvin wave encounters the boundary, the part of the wave north of \( b \) will be reflected as long Rossby waves as in the case at a full boundary. However, south of \( b \) and east of \( x_B \), it is possible to have a transmitted Kelvin wave and short Rossby waves which have eastward group velocities.

West of \( x_B \) we have

\[
\begin{align*}
{u^W} &= u_r + \psi_{-1} \\
{h^W} &= h_r + \psi_{-1}
\end{align*}
\]

while east of it

\[
\begin{align*}
{u^E} &= u_s + T\psi_{-1} \\
{h^E} &= h_s + T\psi_{-1}
\end{align*}
\]

where \( T \) is the amplitude of the transmitted Kelvin wave and \((u_s,v_s,h_s)\) is the sum of short Rossby waves. From Cane and Sarachik II, the short Rossby waves have the form

\[
(u_s,v_s,h_s) = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, y) \chi^s(x,y,t) \quad (3.3)
\]
Now $u$ and $h$ must be continuous at $x=x_B$ south of $y=b$; i.e.

\[ u^E = u^W, \quad h^E = h^W, \quad x = x_B \quad y < b \]

Hence

\[ u^S = u_r + (1 - T)\psi_{-1}; \quad h^S = h_r + (1 - T)\psi_{-1}. \]

Since the long Rossby waves and the Kelvin wave both satisfy

\[ yu + h_y = 0 \]

\((u^S, h^S)\) must also satisfy this at $x_B$; hence

\[ y(- \frac{\partial}{\partial y} \chi^S) + \frac{\partial}{\partial y}(y\chi^S) = 0 \]

which implies $\chi^S = 0$. Thus, only long Rossby and Kelvin waves are reflected.

Since $u = T\psi_{-1}$ at $y > b$, $u$ is bounded at $y=b$. As a result, an integration of equation (2.7b) across the boundary latitude, i.e.

\[ \int_{b_-}^{b_+} yudy + h(b_+) - h(b_-) = 0 \]

shows that there can be no jump in $h$ at $b$. The conditions that must be satisfied at a partial boundary are then:

a) $h$ is continuous in $y$ at the corner

b) $h$ and $u$ must be continuous in $x$ at the boundary longitude

c) the Rossby waves generated to the west of the boundary longitude must have zero projection on the Kelvin mode (cf. equation (3.1)).

As will be seen, these conditions imply that an incident Kelvin wave has a change in amplitude as it crosses the longitude of the boundary (i.e. the transmission coefficient for the wave is not equal to one) and that effects
due to the boundary are felt at all latitudes along that longitude (although they are asymptotically small far away from the boundary). The linearity of the model allows the solutions to the two possible cases of waves reflecting at a partial boundary, namely an incident Kelvin wave with no initial Rossby waves and incident Rossby waves from the east with no initial Kelvin wave, to be superimposed. The desired solutions are those which asymptotically approach the types of motion allowed by the approximations inherent in the model. The case of an incident Kelvin wave will be discussed first. The second case then follows easily.

Consider an equatorial Kelvin wave, of amplitude $a_k$, incident on the partial boundary in Fig. 2. Part of the wave is transmitted below the boundary with transmission coefficient $T^k$, the rest being reflected by Rossby modes. South of the boundary, the Rossby modes must conspire to cancel out the untransmitted part of the Kelvin wave to ensure continuity in $x$ (condition b). In this region then,

$$h_r = u_r = -(1 - T^k)a_k \psi_{-1} \quad (3.4)$$

Above the boundary, the conditions are the same as for a full boundary, namely, no normal flow and the total $h$ a constant

$$u_r = -a_k \psi_{-1} \quad (3.5a)$$

$$h_r = D^k - a_k \psi_{-1} \quad (3.5b)$$

Applying conditions (a) and (c) will determine the constants $T^k$ and $D^k$. Thus, at $y=b$, continuity requires

$$h_r^{\text{south}} = -(1 - T^k)a_k \psi_{-1}(b) = D^k - \psi_{-1}(b) = h_r^{\text{north}}$$

or

$$T^k a_k \psi_{-1}(b) = D^k \quad (3.6)$$
Zero projection of the Kelvin mode on the sum of the Rossby modes can be written, using equations (3.4) and (3.5),

$$ -2a_k \int_b^\infty \psi_{-1}^2 \, dy - 2a_k (1-T_k) \int_{-\infty}^b \psi_{-1}^2 \, dy + D_k \int_b^\infty \psi_{-1} \, dy = 0 $$

Using equation (3.6) and the normalization condition for the Kelvin mode, equation (2.5), equation (3.7) can be solved for $T_k$

$$ T_k = \frac{1}{2 \int_{-\infty}^b \psi_{-1}^2 \, dy + \psi_{-1}(b) \int_b^\infty \psi_{-1} \, dy} $$

The other possible situation is to have a set of Rossby waves propagate in from the east and encountering the corner. Since there is no incident zonal velocity above the boundary, we have simply

$$ u_r = 0, \quad h_r = D^r \quad y > b $$

Below the boundary, the possibility of a Kelvin wave being reflected back to the east can be written

$$ u_r = \hat{u}_r + T_r \psi_{-1} \quad y < b $$

$$ h_r = \hat{h}_r + T_r \psi_{-1} $$

where the hat denotes the values of the incident waves. Applying condition (a) gives

$$ D^r = \hat{h}_r(b) + T_r \psi_{-1}(b) $$

West of $x_b$, the Kelvin amplitude is zero. Projecting a Kelvin wave onto (3.9) and (3.10) and using (3.11) gives
\begin{align*}
\hat{h}_r(b) \int_b^\infty \psi_{-1} \, dy + T^r \psi_{-1}(b) \int_b^\infty \psi_{-1} \, dy + 2T^r \int_{-\infty}^b \psi_{-1}^2 \, dy &= 0
\end{align*}

Solving for \( T^r \), we have

\begin{equation}
T^r = \frac{-\hat{h}_r(b) \int_b^\infty \psi_{-1} \, dy}{2\int_{-\infty}^b \psi_{-1}^2 \, dy + \psi_{-1}(b) \int_b^\infty \psi_{-1} \, dy}
\end{equation} \tag{3.12}

Note that \( T^r \) and \( D^r \) depend only on the height of the Rossby modes at \( y=b \), i.e. \( \hat{h}(b) \). Combining the two cases and using the expressions for \( T^k, T^r, D^k \) and \( D^r \), the total response at a given time due to the presence of Kelvin and Rossby waves at the longitude of the boundary is given by

\begin{align*}
\mathfrak{u}_r^{\text{tot}} &= -a_k \psi_{-1}(y) \quad y > b \quad \text{\tag{3.13a}} \\
\mathfrak{h}_r^{\text{tot}} &= \hat{h}_r(b) + (T^k a_k - T^r) \psi_{-1}(b) - a_k \psi_{-1}(y) \quad y > b \quad \text{\tag{3.13b}} \\
\mathfrak{u}_r^{\text{tot}} &= \mathfrak{u}_r(y) - (T^r + (1-T^k) a_k) \psi_{-1}(y) \quad y < b \quad \text{\tag{3.14a}} \\
\mathfrak{h}_r^{\text{tot}} &= \hat{h}_r(y) - (T^r + (1-T^k) a_k) \psi_{-1}(y) \quad y < b \quad \text{\tag{3.14b}}
\end{align*}

When equation (3.8) is evaluated for various values of \( b \) (the latitude of the boundary), we obtain the surprising result that the Kelvin wave transmission coefficient is greater than one (see Fig. 3). The curve asymptotes to one from above as \( b \) becomes large and positive, but approaches infinity as \( b \) increases south of the equator. This behavior is really not so strange if the entire mechanism is considered. South of the equator, the transmitted Kelvin wave can be considered to be a coastal Kelvin wave traveling eastward along the coast at \( y=b \). With the change of variable \( y=b-\eta \) (\( \eta = \text{distance from coast} \)), we have for the equatorial Kelvin wave
\[ e^{-\frac{y^2}{2}} = e^{-\frac{b^2}{2}} e^{-b\eta} e^{-\frac{\eta^2}{2}} \]

For \( \eta \leq 1 \) and \( |b| >> 1 \), \( e^{-\frac{\eta^2}{2}} \) is very small, \( e^{-\frac{\eta^2}{2}} \approx 1 \) and \( 1/b \) is like a (scaled) radius of deformation, so that the wave amplitude behaves like a usual coastal Kelvin wave. The amplitude of this wave must be such as to make its height at the corner match the constant height set up along the boundary, as given by equation (3.6). This height is maximum when \( b \) goes to negative infinity, where it approaches the value it should have for a full boundary (cf. equation (2.11)). The coastal Kelvin wave is then never unusually large, but because it is calculated here as the tail end of an equatorial Kelvin wave and its value far away from the equator is asymptotically small, the amplitude of the corresponding equatorial Kelvin mode must be very large. This equatorial Kelvin wave does not actually exist north of \( y=b \).

The transmission coefficient and height constant for the case of incident Rossby waves with a combined height of unit amplitude at the corner, given by equations (3.11) and (3.12), are also plotted in Fig. 3. Away from the equator to the north, the height along the wall is simply the height of the incident Rossby modes at the corner and no Kelvin wave is reflected. This is to be expected since the height along the wall will have an increasingly smaller projection on the equatorial Kelvin mode as the boundary recedes from the equator. The eigenfunctions of the Rossby modes to the east of the boundary (in the smaller part of the basin), when combined with the constant height along the wall, project completely onto the eigenfunctions of the Rossby modes to the west (big basin). South of the boundary, \( T^r \) increases to infinity for the same reason as mentioned
for $T^k$, namely, a modest coastal Kelvin wave far from the equator, required by continuity, corresponds to a huge equatorial Kelvin wave.

Section II: Asymptotic description of mass conservation in Kelvin and anti-Kelvin modes

The westward moving solutions can also include the anti-Kelvin mode which cannot exist away from the boundary and so, unlike the Rossby modes, cannot travel past the corner into the interior. The mass flux associated with this mode must be transferred past the corner and eventually into the large basin. The implicit projection of the $u$ and $h$ fields due to the anti-Kelvin mode onto the transmitted Rossby modes west of the boundary will eliminate it from the interior solution. However, the speed of propagation of the Rossby modes is slower than that of the anti-Kelvin mode so that the flux of mass is decreased west of the boundary. This is complimented by the height produced along the wall which makes up the missing mass flux as the Rossby modes move away from the wall. This is the long wave version of the Kelvin wave turning the corner. This mass balance can be shown analytically if the corner at $b$ is sufficiently far away from the equator. We have for $v$

$$[v_{yy} - y^2 v_t]_x = 0$$  \hspace{1cm} (3.15)

If $b \gg 1$, then $y \gg 1$ along the wall, while $v$ is a smooth function of $y$ (i.e. $v_{yy} \leq O(1)$). Hence, we can approximate (3.15) by

$$y^2 v_t + v_x = 0$$  \hspace{1cm} (3.16)

and the speed of the wave is $c = -1/y^2$. The corresponding equation for $h$ is:
\[ h_{yyt} - \frac{2h_{yt}}{y} - y^2 h_t - h_x \]

The additional term \(-\frac{2h_{yt}}{y}\) is at least as small as the first for \(y\) large so that we can use the analogue to equation (3.16) to model \(h\) at these latitudes

\[ y^2 h_t + h_x = 0 \quad (3.17) \]

The effects of beta-dispersion, cf. Schopf, Anderson and Smith (1981), are negligible since the ray paths need only extend a little ways away from the boundary for this analysis. If a height, \(h_o\) = constant, is given along the boundary, these waves will propagate away from the boundary according to equation (3.17) and produce a mass field after a time \(t\) given by the height field integrated over \(x\) and \(y\). The solutions to equation (3.17) follow characteristics so that after a time \(t\) the mass north of \(y=b\) and west of \(x=x_B\) is given by

\[ M_R = \int_b^\infty \int_{-\infty}^{x_B} h_o H(x-x_B + \frac{t}{y^2}) dx dy = \int_b^\infty h_o \frac{t}{y^2} dy = h_o \frac{t}{b} \quad (3.18) \]

The form of the anti-Kelvin wave is \(h_{ak} = -u_{ak} = A_n e^{-y^2/2}\) where \(A_n\) is a normalization constant. A change of variable \(\eta = b-y\) gives

\[ u_{ak} = A_n \exp[\frac{b^2}{2} - b\eta + \frac{\eta^2}{2}] = A e^{-b\eta} e^{\eta^2/2} \quad (3.19) \]

For \(\eta\) small, the second exponential is close to one and equation (3.19) looks like a coastal Kelvin wave on an \(f\)-plane. The mass arriving at the boundary due to this wave can be calculated by integrating the zonal mass flux in \(y\) and \(t\). Taking equation (3.19), expanding the second exponential in its Taylor series and integrating in \(\eta\) and \(t\) gives
\[
\int_0^\infty A e^{-b\eta^2/2} \, d\eta = t A \left( \int_0^\infty e^{-b\eta} \, d\eta + \int_0^\infty e^{-b\eta^2} \, d\eta + \ldots \right)
\]

\[
= A \frac{t}{b} + A \frac{t}{b^3} + O\left(\frac{1}{b^5}\right) \quad (3.20)
\]

Equations (3.18) and (3.20) are equal to \(O(1/b^3)\) if \(h_o = A\). The anti-Kelvin wave can be thought of as turning the corner and transferring its mass flux into the Rossby modes by producing the constant height \(h_{ak} = h_o\) as it travels up the wall.

For an incident Kelvin wave at large values of \(b\), the height at the corner needed for mass conservation asymptotically approaches the amplitude of the Kelvin wave there, which implies a transmission coefficient of one as shown in the graph of \(T^k\). The mass contained in the Kelvin wave above the boundary can be computed by integrating the zonal mass flux in time and \(y\) as before. For a Kelvin wave given by

\[
u_k = \frac{1}{\sqrt{2\pi} \frac{1}{4}} e^{-y^2/2}
\]

the mass is

\[
M^k = \int_b^\infty \int_0^t \psi_{-1}(y) dy = \frac{t}{\sqrt{\pi}} \int_b^{\sqrt{2}} e^{-y^2} dy \quad (3.21)
\]

\[
= \frac{t}{\frac{1}{4}} \text{erfc}(b/\sqrt{2})
\]

A series expansion for the error function as \(b \to \infty\) has the first two terms.
\[
erfc(\frac{b}{\sqrt{2}}) = \frac{1}{2} e^{-\frac{b^2}{2}} \left( \frac{\sqrt{\pi}}{b} - \frac{\Gamma(3/2)}{\Gamma(1/2)} \frac{1}{(b/\sqrt{2})^3} + \ldots \right)
\]

\[
e^{-\frac{b^2}{2}} - \frac{1}{b^3 \sqrt{2} \cdot 8} e^{-\frac{b^2}{2}} + \ldots
\]

so that to lowest order, equation (3.21) becomes

\[
M^k = \frac{t}{b^{1/4} \pi^{1/4}} e^{-\frac{b^2}{2}} + O(1/b^3) = \frac{t}{b} \psi_1(b)
\] (3.22)

If \( h_o = \psi_1(b) \), then, to \( O(1/b^3) \), the values of \( M^k \) as given by equations (3.18) and (3.22) are the same and all the mass flux at the wall is returned by the Rossby modes there. This is also the condition for the height to be continuous so that the Kelvin wave does not need to change its amplitude in order to slip under the boundary.

Section III: The boundary layer at the zonal coast

The results from the model runs (Chapter 5) also show a thin, boundary layer-like region beneath the zonal coast. The relatively smooth meridional structure of solutions in the interior must adjust abruptly to insure that \( v=0 \) at the boundary; see Fig. 4 (Cane and Sarachik III). The thinness and coastally trapped nature of this feature suggests that it be examined by considering it as a boundary layer in the solution of the vorticity equation on an equatorial beta-plane. Taking the curl of the shallow water equations for zero forcing (the non-zero \( v \) of the wind forced interior solutions force the boundary layer solutions), and using continuity, we have
Fig. 4. Meridional section of height from Cane & Sarachik III showing the thin boundary layer-like adjustment near the zonal coasts. Their figures show the transient response during spin-up from rest for a zonal forcing $F = 1$. The model presented in this thesis shows boundary layers for periodic forcings.
\[ v = y h_t + u_y t \]  \hspace{1cm} (3.23)

The small scale of the boundary layer suggests that \( \partial / \partial y > y \) and introduces the scaling

\[ \frac{\partial}{\partial y} = o(\omega^{-1/2}), \quad y = o(1), \quad x = 1, \quad v = o(1), \quad h = o(1), \quad u = o(\omega^{-1/2}) \]

where \( \partial / \partial t = \omega < 1 \). The equations of motion become, with non-dimensional variables

\[ \omega^{1/2} u_t + y v + h_x = 0 \]
\[ \omega^{-1/2} u - \omega^{-1/2} y = 0 \]
\[ \omega h_t + \omega^{-1/2} u_t + \omega^{-1/2} v_y = 0 \]

The vorticity equation becomes

\[ v = \omega y h_t + u_y t \]

The continuity equation shows that to lowest order in \( \omega^{1/2} \), we can define a stream function for \( u \) and \( v \). Neglecting the \( h_y t \) in vorticity, we have

\[ \frac{\partial^2}{\partial y^2} \psi_t^B + \psi_x^B = 0 \]

where \( \psi^B \) is the boundary layer solution. If \( \psi^I \) is the interior solution, the boundary conditions are that \( \psi^I + \psi^B = 0 \) at the coast and \( \psi^B = 0 \) at the eastern boundary. The small parameter \( \omega \) represents frequency for a periodic forcing. If a Rayleigh friction is included (cf. Chapter 2), it introduces a term similar to the time derivative so that, for the general case, \( \partial / \partial t \) can be replaced by \( \omega + r = \delta \). The scaling is then
\[ u \approx \frac{\partial}{\partial y} = 0(\delta^{-1/2}), \quad v \approx h \approx x = 0(1), \quad \frac{\partial}{\partial t} = 0(\delta) \]

By defining \( \zeta = y e^{-y} \), \( \eta = x e^{-x} \), so that distances are measured positive from the zonal coast and the eastern boundary, the boundary layer can be described as

\[ -\delta \psi^B_{\zeta\zeta} + \psi^B_\eta = 0 \quad (3.24) \]

\[ \psi^B = -\psi^I(\eta) \quad \zeta = 0 \quad (3.25a) \]

\[ \rightarrow 0 \quad \zeta \rightarrow \infty \]

\[ \psi^B = 0 \quad \eta = 0 \quad (3.25b) \]

Taking the Laplace transform in \( \eta \), so that \( \partial / \partial \eta \rightarrow p \), we have

\[ \hat{\psi}^B_{\zeta\zeta} - \frac{p}{\delta} \hat{\psi}^B = 0 \]

which has the solution

\[ \hat{\psi}^B(\zeta, p) = -\hat{\psi}^I(p) e^{-\sqrt{p}/\delta} \zeta \]

If \( \hat{\psi}^I(\eta, \zeta=0) = H(\eta) \) for convenience, the inverse transform, with help from the convolution theorem, is

\[ \psi^B(\zeta, \eta) = \int_0^\infty \frac{\zeta}{\sqrt{\pi}} \sqrt{\frac{1}{\eta}} \frac{1}{\sqrt{\Gamma \eta}} e^{-\eta^2/4\delta \eta} \, d\eta \]

The change of variable \( s = \zeta/2\sqrt{\delta \eta} \) shows that this result can be written

\[ \psi^B(\zeta, \eta) = \text{erfc}(\frac{\zeta}{2\sqrt{\delta \eta}}) \]
The boundary layer amplitude goes to zero with increasing $\zeta$, but the decay scale increases with $\eta$ and $\delta$. This widening of the boundary layer is observed in the model results (Chapter 5).
Chapter 4: Numerical Scheme

Section I: Finite difference equations and their solution

As shown in Chapter 2, the solutions to equations (2.7a,b,c) and (2.8), (2.9) and (2.11) are a single eastward propagating equatorial Kelvin wave and westward propagating long, non-dispersive Rossby waves. The height and zonal velocity are thus split into two pieces, traveling in opposite directions. The scheme takes advantage of this by solving for each piece separately. If the part of the forcing which projects onto the Kelvin mode is subtracted away from equation (2.7), the resulting set of equations will have only westward traveling Rossby modes as forced solutions. This can be written analytically

\[ U_t - yU_x + h_x = F^* \]  
\[ yU_x + h_y = G^* \]  
\[ h_t + u_x + v_y = Q^* \]

where \( F^* = F \cdot f \cdot \Psi_1 \), \( G^* = Q \cdot f \cdot \Psi_1 \), and \( f \cdot \) is the projection of the total forcing on the Kelvin mode, i.e.,

\[ f_k(x,t) = \frac{1}{\sqrt{2}} \left( F, G, Q \right) \cdot \left( \Psi_1, \tilde{\omega}, \Psi_{-1} \right) \]

Only the boundary conditions at the east, equations (2.8) and (2.11), are needed to solve equation (4.1) for the Rossby modes. The Kelvin wave is given by (cf. equation (2.5))

\[ U_k = h_k = a_k(x,t) \Psi_1(y) \]

where the amplitude \( a_k \) obeys
\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) A_k (x, t) = f_k (x, t) \quad (4.3)
\]

which is obtained by projecting the Kelvin mode onto equations (2.7).

This wave equation is easily solved along characteristics with the initial condition, equation (2.9), at the western boundary. The analytic solution to (4.3) is

\[
A_k (x, t) = A_k (x, t, 0) + \int_0^t f_k (x, t', t - t') dt' \quad (4.4)
\]

The finite difference versions of equation (4.1a,b,c) are defined on a staggered grid,

<table>
<thead>
<tr>
<th>space:</th>
<th>u,h</th>
<th>F,G</th>
<th>u,h</th>
<th>j+1</th>
<th>time: n \rightarrow u,h,G</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>G</td>
<td>v</td>
<td>G</td>
<td>j+1/2</td>
<td>n-1/2 \rightarrow v,F,Q</td>
</tr>
<tr>
<td></td>
<td>u,h</td>
<td>F,Q</td>
<td>u,h</td>
<td>j</td>
<td>i</td>
</tr>
<tr>
<td></td>
<td>i</td>
<td>i+1/2</td>
<td>i+1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where n, i and j represent the indexes of time, longitude and latitude, respectively. The equations of motion take the form

\[
\frac{1}{2} \left[ u^{n+1}_{i+1,j} + u^n_{i,j} - u^{n-1}_{i-1,j} \right] = \frac{1}{2} \left[ v^{n-1}_{i,j+1} + v^n_{i,j} - v^{n+1}_{i,j-1} \right] + \frac{1}{2} \left[ h^{n+1}_{i,j} + h^n_{i,j} - h^{n-1}_{i,j} \right] = F^{n-1}_{i,j} \quad (4.5a)
\]

\[
\frac{1}{2} \left[ u^{n+1}_{i,j+1} + u^n_{i,j+1} - u^{n-1}_{i,j} \right] + \frac{1}{2} \left[ h^{n+1}_{i,j} + h^n_{i,j} - h^{n-1}_{i,j} \right] = C^n_{i,j} \quad (4.5b)
\]

\[
\frac{1}{2} \left[ h^{n+1}_{i+1,j} + h^n_{i+1,j} - h^{n-1}_{i-1,j} \right] + \frac{1}{2} \left[ v^{n-1}_{i,j} + v^n_{i,j} - v^{n+1}_{i,j} \right] + \frac{1}{2} \left[ u^{n+1}_{i,j+1} + u^n_{i,j+1} - u^{n-1}_{i,j+1} \right] = Q^{n+1}_{i,j} \quad (4.5c)
\]

The boundary conditions corresponding to equations (2.8), (2.9) and (2.11) are
\[ U_{n_k,i,j}^n = -a_{n_k}^n U_{n_k,i,j}^{n-1} \]  
(4.6)

\[ d_{i}^n = -\sum_{j=1}^{N_y} U_{n,i,j}^n \Delta y \]  
(4.7)

\[ h_{n_k,i,j}^n = H - a_{n_k}^n \Psi_{i,j} \Delta y + \frac{1}{2} \sum_{k'=1}^{K} G_{n_k,i,j}^n \Delta \tau \]  
(4.8)

respectively, where \( N_X \) is the total number of grid points in \( x \) and \( N_Y \) the total in \( y \). The constant \( H \) is determined by calculating a finite difference analogue to equation (2.11). These equations can be solved for

\[ U_{i,j}^n = \frac{\Delta t}{1 - \alpha^2} \left\{ R_1^i + V_j \gamma_y \left[ y_{j+1} - \frac{2 \gamma}{\Delta y} \right] + V_j \gamma_x \left[ y_{j+1} - \frac{2 \gamma}{\Delta y} \right] \right\} \]  
(4.9a)

where

\[ R_1^i = 2 \left[ F_j + \alpha Q_i \right] \left( -\frac{1}{\Delta t} \right) \frac{U_{i-1,j}^n}{U_{i,j}^n} - \left( \frac{1}{\Delta t} \right) \frac{U_{i+1,j}^n - U_{i-2,j}^n}{U_{i,j}^n} - \frac{2 \alpha^2}{\Delta t} \left[ U_{i,j}^n - U_{i-1,j}^n \right] \]  

\[ \Delta t = \frac{\Delta t}{1 - \alpha^2} \left\{ R_2^i + V_j \gamma_x \left[ \frac{2 \gamma}{\Delta y} + \Delta y_j \right] + V_j \gamma_y \left[ -\frac{2 \gamma}{\Delta y} + \Delta x_j \right] \right\} \]  
(4.9b)

where

\[ R_2^i = 2 \left[ Q_i + \alpha F_j \right] \left( -\frac{1}{\Delta t} \right) \frac{R_{i-1,j}^n}{R_{i,j}^n} - \left( \frac{1}{\Delta t} \right) \frac{R_{i+1,j}^n - R_{i-2,j}^n}{R_{i,j}^n} - \frac{2 \alpha^2}{\Delta t} \left[ R_{i,j}^n - R_{i-1,j}^n \right] \]  

and \( \alpha = \Delta t / \Delta x \) represents an inverse numerical phase speed. An equation in \( v \) only, the analogue of equation (2.3) can be found

\[ V_j \gamma_x \left[ y_{j+1}^v - \frac{2 \gamma}{\Delta y} \right] + V_j \gamma_y \left[ y_{j+1}^v + \frac{\gamma^2}{\Delta y^2} + 4 \Delta x \right] + V_j \gamma_y \left[ y_{j+1}^v - \frac{2 \gamma}{\Delta y} \right] = \hat{L} V_j \gamma_y = \frac{1}{L} RHS \]  
(4.10)

where

\[ RHS = 2 \left( -\frac{1}{\Delta t} \right) G_{j+1}^v + y_{j+1}^v \left[ y_{j+1}^v + y_{j+1}^v - \frac{2 \gamma}{\Delta y} \right] \]  

\[ - \frac{2 \gamma}{\Delta y} \left[ R_{j+1}^v - R_{j}^v \right] . \]

\( R_j^v \) and \( R_j^v \) are defined above and \( L \) is the finite difference analogue to the operator \( \partial^2 / \partial y^2 - y^2 \). Given the value of \( ( U_{n_k,i,j}^n, h_{n_k,i,j}^n ) \) at the
eastern boundary, the tridiagonal system equation (4.10) can be solved for \( v \) at \( N X \cdot \Delta x \) with the boundary condition \( v = 0 \) at the northern and southern boundaries. Equations (4.9a,b) then give \( u \) and \( h \) at \( (N X - 1) \), the procedure having moved the solution over to the left by one grid point. The scheme then continues calculating successive values of \( u_r \) and \( h_r \) to the left until the western boundary is reached.

The boundary condition (4.7) can now be used to determine \( \partial^N_{k,i} \) at the western boundary. Values of \( a_k \) across the basin to the right are obtained by using the finite difference form of equation (4.4). Thus,

\[
\partial^N_{k,i} = \partial^{N-1}_{k,i} + \sum_{\ell = i-1}^{i-\alpha} f_{k,\ell} \Delta x
\]

where the \( f_{k,\ell} \) is the projection of the forcing on the Kelvin mode given by (4.2) and \( \alpha \) is the number of grid points the Kelvin wave travels in one time step. In order to start the solution along the characteristics, values of \( a_k \) between the western boundary and the point \( i = \alpha \Delta x \) must be obtained. This is done by linearly interpolating between \( \partial^N_{k,i} \) and \( \partial^{N-1}_{k,i} \),

\[
\partial^{N + \Delta x}_{k,i} = \partial^{N-1}_{k,i} + \sum_{\ell = i-1}^{i-\alpha} f_{k,\ell} \Delta x = \partial^N_{k,i + \Delta x}
\]

The Kelvin amplitude at the eastern boundary, needed to calculate \( \lambda^N_{N X} \) and \( \lambda^R_{N X} \) at the beginning of a time step, is calculated as

\[
\partial^N_{N X} = \partial^{N-1}_{N X} + \sum_{\ell = N X - \alpha}^{N X} f_{k,\ell} \Delta x
\]

The Kelvin wave, traveling at unit (non-dimensional) speed in the model solution, covers a distance \( \alpha \Delta x \) in time \( \Delta t \). Thus, by following a characteristic, the amplitude of the Kelvin wave at a longitude \( x_o \) is given by the amplitude at \( x_o = \alpha \Delta x \) at the previous time step, plus the integral of the forcing in accordance with equation (4.4).

When the Gulf of Guinea is included in the basin, the boundary conditions and transmission coefficients must be calculated using the finite difference forms of the appropriate equations given in Chapter 3. The
transmission coefficients calculated in equations (3.8) and (3.12) must be added to the Kelvin wave at each step in time. This is done in the routine by calculating

$$\bar{A}_k = T_k + T_k^* \bar{A}_k$$

for \( \chi_0 \leq i \Delta x < \chi_0 + \alpha \Delta x \)

where \( \chi_0 \) is the boundary longitude. Only the part of the Kelvin wave that has passed under the corner is modified. The normalization constant in equation (2.5) must also be recomputed for the two parts of the basin since, when taking projections, it is assumed that

$$\sum_{j=1}^{N} \sum_{i=1}^{M} \psi_{ij}^2 \Delta y_j = \frac{\gamma}{2}$$

where \( N \) is the latitude of either the African coast or the top of the model basin.

The conservation of mass in the finite difference scheme can be checked by summing the continuity equation (4.5c) in \( x \) and \( y \) over the basin; for \( Q=0 \) (no source or sinks of mass), the mass due to the westward traveling modes is

$$\frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{M} (h_{i,n,j}^n + h_{i,n-1,j}^n - h_{i,n+1,j}^n - h_{i,n,j}^{n-1}) \Delta x_i \Delta y_j + \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{M} (u_{i,n,j}^n + u_{i,n-1,j}^n - u_{i,n+1,j}^n - u_{i,n,j}^{n-1}) \Delta y_j$$

where the \( v \) term vanishes since \( v_{NY} = v_1 = 0 \). If \( \bar{h} \) represents the average value of \( h \) at \( i+1/2 \) and \( \bar{u} \) represents an average in \( u \) and \( n-1/2 \), we have

$$\left( \sum_{i \neq \bar{h}_{i,j}} \bar{h}_{i,j}^n - \sum_{i \neq \bar{h}_{i,j}} \bar{h}_{i,j}^{n-1} \right) \frac{1}{\Delta t} = \lambda \sum_{j=1}^{N} \left( \bar{u}_{ij,n} - \bar{u}_{ij} \right)$$

The difference in mass between two time steps is the difference between the average flux into and out of the basin in that time. As the boundary
conditions (4.6) and (4.7) show, this flux at the boundaries is exactly
equal and opposite to the Kelvin wave flux. When the Kelvin contribution
is added to the above equation, the right hand side will be non-zero only
if the numerical Kelvin wave loses mass as it crosses the basin. The solu-
tion along the Kelvin wave characteristic is exact except for the points
near the western boundary which are linearly interpolated. Also, the value
of the Kelvin flux at the eastern boundary is taken as the amplitude which
intersects the coast at the particular time nΔt instead of an average over
the amplitudes that have hit the coast in the intervening time step. Calcu-
lations of the total mass in the basin at each time step during model runs
show that the inaccuracy involved in these approximations are negligible
in the mass balance. In the same way, the scheme approximately conserves
energy when there is no friction.

Section II: Accuracy

The accuracy of the finite difference analogues to the derivatives in
the equations of motion can be determined by using the Taylor series

\[ U(t + \Delta t) = U(t) + \frac{\partial U}{\partial t} U(t + \frac{\Delta t}{2}) \Delta t + \frac{\partial^2 U}{\partial t^2} U(t + \frac{\Delta t}{2}) \left( \frac{\Delta t}{2} \right)^2 \frac{1}{2} + \ldots \]

\[ U(t) = U(t + \frac{\Delta t}{2}) - \frac{\partial U}{\partial t} U(t + \frac{\Delta t}{2}) \Delta t + \frac{\partial^2 U}{\partial t^2} U(t + \frac{\Delta t}{2}) \left( \frac{\Delta t}{2} \right)^2 \frac{1}{2} + \ldots . \]

We can solve for

\[ \frac{\partial U}{\partial t} U(t + \frac{\Delta t}{2}) = \frac{U(t + \Delta t) - U(t)}{\Delta t} - \frac{\partial^2 U}{\partial t^2} U(t + \frac{\Delta t}{2}) \left( \frac{\Delta t}{2} \right)^2 + \ldots . \]

Taking

\[ U_{i}^{n} = U(t + \Delta t) \quad \text{if} \quad U_{i}^{n+1} = U(t + \frac{\Delta t}{2}) \]
we can write the previous result as
\[
\frac{\partial}{\partial t} U_i^n = U_i^n - U_i^{n-\frac{1}{2}} - \frac{\partial^3}{\partial t^3} U_i^{n-\frac{1}{2}} \left( \frac{\Delta t}{2} \right)^2 + \ldots .
\]

If we take the average of
\[
\frac{\partial}{\partial t} U_i^{n-\frac{1}{2}} + \frac{\partial}{\partial t} U_{i+1}^{n-\frac{1}{2}}
\]
to get an expression for \( \frac{\partial}{\partial t} U_{i+\frac{1}{2}}^{n-\frac{1}{2}} \) (i.e. centered at \( i+1/2 \)), we have
\[
\frac{\partial}{\partial t} U_{i+\frac{1}{2}}^{n-1} = \frac{1}{\Delta t} \left[ U_{i+1}^n - U_{i+1}^{n-1} + U_i^n - U_i^{n-1} \right] - \left[ \frac{\partial^3}{\partial t^3} U_{i+1}^{n-\frac{1}{2}} + \frac{\partial^3}{\partial t^3} U_i^{n-\frac{1}{2}} \right] \frac{(\Delta t)^2}{48} + \ldots .
\]

Retaining just the first expression in brackets gives an error of
\[
\mathcal{E} = -\frac{\Delta^3}{\Delta t^3} \left[ U_i^{n-\frac{1}{2}} + U_{i+1}^{n-\frac{1}{2}} \right] \left( \frac{\Delta t}{2} \right)^2
\]
(4.11a)

A similar analysis for the x-derivative terms shows that
\[
\frac{\partial}{\partial x} U_i^{n-\frac{1}{2}} = \frac{U_{i+1}^n - U_{i+1}^{n-1} + U_i^n - U_i^{n-1}}{2\Delta x} + \mathcal{E}
\]
where
\[
\mathcal{E} = -\frac{\Delta^3}{\Delta x^3} \left[ U_i^{n+\frac{1}{2}} + U_{i+1}^{n+\frac{1}{2}} \right] \left( \frac{\Delta x}{2} \right)^2
\]
(4.11b)

The accuracy of the scheme depends not only on the time step and the grid spacing, but also on the size of the derivatives in equations (4.11a) and (4.11b). The low frequency and long wavelength of the solution should ensure their small amplitude. For a periodic forcing of frequency \( \omega \), we have \( \partial^3/\partial t^3 \propto \omega^3 \). The period of the response is \( T = N\Delta t = 2\pi/\omega \), so that the error in the time derivative equation (4.11a) becomes
\[
\mathcal{E} = \left( \frac{\pi^3}{6 N^3 \Delta t} \right) \left[ U_i^{n-\frac{1}{2}} + U_{i+1}^{n-\frac{1}{2}} \right]
\]
For a yearly period, the first baroclinic mode in the Atlantic (see Table II) has \( T = 235.8 \) (non-dimensionally) so that \( N = 72 \) for \( \Delta t = 3.275 \), which corresponds to a time step of \( 1/6 \) a month. The error is less than \( 10^{-5} \).

The \( x \)-derivatives in equation (4.11b) represent the combined effect of the wavenumbers of all the meridional Rossby modes which make up the solution and are thus harder to evaluate. As mentioned before, the higher modes have smaller length scales and would be expected to contribute more to the overall error in the \( x \)-derivatives than lower modes. The next section clarifies this by looking at how accurately the scheme models each individual mode.

Section III: Accuracy and behavior of individual modes

The numerical scheme solves equations (2.7) without decomposing the solution into its meridional modes. It is informative, however, to see how well each mode is modeled and how it behaves as the scheme iterates across the basin. This can be done by looking at the wave equation governing the propagation of the Rossby modes. If we define the operator \( L \equiv \partial^2 / \partial y^2 - y^2 \), then the unforced version of equation (2.3) can be written non-dimensionally (with \( v = e^{i(kx-\omega t)} \), \( k < 0 \))

\[
\frac{\partial}{\partial t} L v + \frac{\partial}{\partial x} v = 0
\]  

(4.12)

The eigenvalues of \( L \) are \( 2\nu_m + 1 \) (cf. equation (2.4)), so that using the long wave dispersion relation, equations (2.6) and (4.12) become

\[
u_{m,t} + c_v \nu_{m,x} = 0
\]
Table II  Non-dimensional parameters for the Atlantic & Pacific

<table>
<thead>
<tr>
<th>Baroclinic Mode</th>
<th>Equivalent Depth (cm)</th>
<th>Phase Speed (m/sec)</th>
<th>( L_{eq} ) (km)</th>
<th>( T_{eq} ) (days)</th>
<th>Basin Width</th>
<th>Annual Frequency ( \omega )</th>
<th>Period T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atlantic:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>60</td>
<td>2.43</td>
<td>326</td>
<td>1.55</td>
<td>20.4</td>
<td>.0266</td>
<td>235.8</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>1.40</td>
<td>248</td>
<td>2.05</td>
<td>26.85</td>
<td>.0352</td>
<td>178.3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>.89</td>
<td>197</td>
<td>2.56</td>
<td>33.8</td>
<td>.0440</td>
<td>142.8</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>.63</td>
<td>166</td>
<td>3.05</td>
<td>40.1</td>
<td>.0524</td>
<td>119.8</td>
</tr>
<tr>
<td>Pacific:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>87</td>
<td>2.92</td>
<td>357</td>
<td>1.41</td>
<td>46.6</td>
<td>.0242</td>
<td>259.2</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
<td>1.63</td>
<td>267</td>
<td>1.89</td>
<td>62.4</td>
<td>.0325</td>
<td>193.4</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>1.04</td>
<td>214</td>
<td>2.38</td>
<td>77.8</td>
<td>.0409</td>
<td>153.6</td>
</tr>
<tr>
<td>4</td>
<td>06.5</td>
<td>.80</td>
<td>187</td>
<td>2.71</td>
<td>89.0</td>
<td>.0466</td>
<td>134.9</td>
</tr>
</tbody>
</table>

1 of equatorial Kelvin wave

2 i.e., actual width/\( L_{eq} \)
where \( c = -1/2v_m + 1 = \) phase speed of the \( m \)th mode. This simple wave equation in \( x \) and \( t \) is obeyed by all the meridional long Rossby modes (cf. Cane and Sarachik I). If the finite difference analogues to the time and space derivatives (defined in equations (4.1a,b,c)) are substituted into the corresponding wave equation for \( u \), the result can be manipulated to form the prediction equation

\[
U_{i+1}^{n+1} = U_{i+1}^{n} + \gamma_m \left( U_{i+1}^{n} - U_{i+1}^{n+1} \right) \tag{4.13}
\]

where

\[
\gamma_m = \frac{1 + \alpha C_m}{1 - \alpha C_m}.
\]

The parameter \( \alpha = \Delta t/\Delta x \) has been previously defined. In this form, equation (4.13) describes the behavior of a particular mode. By analyzing equation (4.13), the model's accuracy and stability for each mode can be examined as a function of \( \alpha \) and \( C_m \).

Equation (4.13) is centered at \( i+1/2, j, n-1/2 \). The values of \( u \) at the grid points surrounding these points can be found by expanding \( u = u(i+1/2, j, n-1/2) \) in a Taylor series. For example

\[
U_{i+1}^{n-1} = U_0 + \frac{\partial U_0}{\partial x} \frac{\Delta x}{2} - \frac{\partial^2 U_0}{\partial x \partial t} \frac{\Delta t}{2} + \frac{1}{2} \frac{\partial^3 U_0}{\partial x^3} \frac{\Delta x}{4} - \frac{1}{2} \frac{\partial^2 U_0}{\partial x \partial t^2} \frac{\Delta t}{4} + \frac{1}{2} \frac{\partial^2 U_0}{\partial x \partial t} \frac{\Delta x \Delta t}{4} + \ldots
\]

gives \( u \) at the point \((i+1, j, n-1)\). Hence \( u_{i+1}^{n-1} \) is forward of \( u_0 \) by \( \Delta x/2 \) in space, but lags \( u_0 \) by \( \Delta t/2 \) in time. If the various Taylor series are substituted into equation (4.13), they give

\[
\frac{\partial U_0}{\partial t} + C_m \frac{\partial U_0}{\partial x} + O(\Delta x^3, \Delta x \Delta t^2, \Delta x \Delta t^3, \Delta t^3) = 0
\]

Keeping track of higher terms in the series expansion shows that all even powers of \( \Delta t \) and \( \Delta x \) cancel, so that the next higher order of the error is
actually $O(\Delta x^5)$, etc.). The first error term is

$$\mathcal{E} = 2 \left( \frac{2^2\alpha}{2\alpha^2} \right) \left( \frac{\Delta x}{3!} + \frac{\Delta x}{3!} \right) + 2 \alpha \left( \frac{\Delta x}{3!} \right) \left( \frac{\Delta x}{3!} \right) \left( \frac{\Delta x}{3!} \right) \left( 1 - \frac{1}{(\alpha c_m)^2} \right)$$

Using $\partial u/\partial t = -c_m \partial u/\partial x$ and $\alpha = \Delta t/\Delta x$, this error can be written in two ways:

$$\mathcal{E} = \frac{1}{12^2} \left( \frac{\Delta x}{3!} \right)^3 \left( \frac{\Delta t}{3!} \right)^3 \left( 1 - \frac{1}{(\alpha c_m)^2} \right) \quad (4.14a)$$

or

$$\mathcal{E} = \frac{1}{12} \left( \frac{\Delta x}{3!} \right)^3 \left( 1 - \frac{1}{(\alpha c_m)^2} \right) \left( \frac{\Delta t}{3!} \right)^3 \left( 1 - \frac{(\Delta x)^2}{(\Delta t)^2 (\alpha c_m)^2} \right) \quad (4.14b)$$

These expressions show the importance of the parameter $\alpha c_m$. If it is $-1$, then the numerical phase speed matches that of the mode and the model solves the equation exactly along the characteristic with no error. Figs. 5a, b, and c show the error as a function of the three variables $\Delta t, \Delta x$ and $c_m$. Note that as $\Delta x \rightarrow 0$, $\mathcal{E} \rightarrow 3^2 u/3! \Delta t^3$ so that $\mathcal{E} \rightarrow 0$ as $\Delta x, \Delta t \rightarrow 0$ which is necessary for the finite difference equations to consistently approximate the continuous differential equation. The important feature is that for any fixed $\Delta x$ and $\Delta t$, the error increases as the phase speed gets small. Since the phase speed is given by the eigenvalues of equation (4.12) which increase with the mode number, the higher modes will be more and more inaccurately modeled.

The actual behavior of the scheme for different modes can also be described by equation (4.13). If the value of $u$ at a particular point is suddenly increased, as is the case at a sharp front, the values "downstream" in the direction of wave propagation are also affected immediately. Consider the situation where all the values of $u$ are negligible except at $i+1$, $n$, i.e. let

$$u_{i+1}^n = U_0 \quad \text{and} \quad u_{i}^{n-1} = u_{i+1}^{n-1} = 0$$
Fig. 5. Numerical scheme's error in approximating the m\textsuperscript{th} meridional mode, $\psi_m(x,t)$, as a function of a) $\Delta t$, b) $\Delta x$ and c) phase speed for m mode $c_m$. The error is zero when the speed of the iterations, $\alpha^{-1} = \Delta x/\Delta t$, is equal to the mode's phase speed, i.e. $\alpha c_m = -1$. 
where \( i+1 \) may be a boundary or area of sudden strong forcing. Since all previous values of \( u \) are negligible, equation (4.13) tells us

\[
U_i^m = -\gamma_m U_i^{m+1} = -\gamma_m U_o
\]

the next point to the left is

\[
U_{i-1}^m = -\gamma_m U_i^m = -\gamma_m (-\gamma_m U_o) = \gamma_m^2 U_o.
\]

Continuing to the left, there results

\[
U_{i-k}^m = (-1)^k \gamma_m^k U_o. \tag{4.15}
\]

This difference equation shows an exponential damping behavior since \( |\gamma_m| < 1 \), as seen earlier. Note that, for the fast modes, \( \gamma_m < 0 \) and

\[
U_{i-h}^m = |\gamma_m|^h U_o
\]

is a smooth exponential decay, whereas for the slow modes, \( \gamma_m > 0 \), and the solution oscillates in sign with each step as it decays (see Fig. 6). The decay is very rapid for \( \alpha \gamma_m \) close to \(-1\).

The situation for a propagating front is more complicated because the previous values of \( u \) may not be negligible. The degree of smoothness of a front can be thought of as rising or falling off of amplitude in time, like that of an oscillation varying at a particular frequency. To examine the scheme's ability to model variations in space and time, it is helpful to look at its response to periodic forcings. Since the Rossby waves propagate to the west, these forcings can be applied at an eastern boundary.

Let the values of \( u \) at the boundary have the periodic amplitude

\[
U_B = \sin (2\pi \beta n). \tag{4.16}
\]
Fig. 6. The numerical solution (solid line) is slightly ahead of the actual solution (dashed) in this example where the forcing frequency is $\beta = 1/10$ and the non-dimensional characteristic phase speed is $\alpha c = 1/2$. Note the exponential damping ahead of the wave front at $5^\circ$. 
For brevity in the notation, \( \beta \) really represents the non-dimensional frequency of \( \omega \Delta t \). For a mode traveling along a characteristic at speed \( c_m \), the solution at the first point away from the wall can be calculated exactly

\[
U_n^* (x, t) = \sin \left( 2\pi \beta \left( t - \frac{1}{c_m} x \right) \right) \quad t > \frac{c_m}{x}
\]

\[= 0 \quad t < \frac{c_m}{x}.
\]

Thus, for \( u_n^0 = u_B \), we have at the first point away from the wall,

\[
U_n^* = \sin \left( 2\pi \beta \left( n - \frac{1}{c_m} \right) \right).
\] (4.17)

To find an expression for \( u_n^1 \), from the finite difference scheme, consider equation (4.13) at the previous time step

\[
U_n^{1} = U_n^{0} + \delta_t (U_n^{0} - U_n^{-1})
\]

All previous values of \( u_n^1 \) can be calculated this way and when substituted back into equation (4.13), there results the series

\[
U_n^n = U_n^{n-1} + \delta_t (U_n^{n-2} - U_n^{n}) + \delta_t^2 (U_n^{n-3} - U_n^{n-1}) + \delta_t^3 (U_n^{n-4} - U_n^{n-2}) + \ldots
\]

or

\[
U_n^n = \sum_{k=1}^{n-1} \delta_t^k \left( U_n^{n-(k+1)} - U_n^{n-k} \right) + U_n^{n-1} \quad (4.18)
\]

This expression involves only values of \( u \) at the boundary, which are given.

Rewrite the difference inside the sum by letting

\[
U_n^{n-(k+1)} = \sin \left( 2\pi \beta \left( n-(k+1) \right) \right)
\]

and

\[
U_n^{n-(k-1)} = \sin \left( 2\pi \beta \left( n-(k-1) \right) \right).
\]
These can be expanded by trigonometric identities and used in equation (4.18).

Let \( s = \sin 2\pi \beta \), then

\[
U^n_1 = \text{Re} \left\{ 2s e^{i2\pi \beta n} \sum_{m=1}^{n-1} \left( \gamma_m e^{-i2\pi \beta m} \right) \right\}
\]

Since \( |\gamma_m e^{-2\pi \beta}| < 1 \), the series can be summed to obtain

\[
U^n_1 = \text{Re} \left\{ -2s e^{i2\pi \beta n} \left[ \frac{1 - \gamma^n e^{-i2\pi \beta n}}{1 - \gamma_m e^{-i2\pi \beta}} \right] \right\}
\]

Taking the real part, we obtain

\[
U^n_1 = -2s \left\{ \cos 2\pi \beta - \frac{1}{2} - \frac{\gamma_m}{1 - \gamma_m \cos 2\pi \beta} \right\} .
\]

(4.19)

Since \( |\gamma_m| < 1 \), for \( n \) large (a sufficiently spun-up ocean), \( \gamma^n_m \) and \( \gamma^{n+1}_m \) terms can be neglected. There are three limits which can be compared to the actual solution; \( |\alpha c| << 1 \), \( |\alpha c| = 1 \) and \( |\alpha c| >> 1 \). In each case, \( \beta \) will be considered small so that the trigonometric functions can be approximated by

\[
\cos 2\pi \beta = 1 \quad \sin 2\pi \beta = 2\pi \beta
\]

Case 1: \( |\alpha c| << 1 \). This corresponds to high modes which propagate very slowly compared to the numerical scheme's iterations. In this case

\[
\frac{2\pi \beta}{|\alpha c n|} \gg 2\pi \beta \quad \text{and} \quad \frac{1 - |\alpha c n|}{1 + |\alpha c n|} = O(1)
\]

Equation (4.19) can be approximated by (with \( S = 2\pi \beta \) and \( \gamma = \cos 2\pi \beta n = \sin 2\pi \beta n \))

\[
U_n = -2\lambda \cdot 2\pi \beta \left[ \frac{1 - S + \frac{d_3 \pi \beta}{(1 - S)^2}}{(1 - S)^2} \right] + 4\lambda \pi \beta + \lambda - \lambda 2\pi \beta
\]

(4.20)

or

\[
U_n = \lambda \left[ 1 - \frac{2\pi \beta}{\alpha c n} - \frac{1}{2} \left( \frac{2\pi \beta}{\alpha c n} \right)^2 + O(2\pi \beta) \right]. \quad \text{(numerical solution)}
\]
The actual solution can also be expanded to yield (from equation (4.17)),

\[ U_n = \chi \left[ 1 - \frac{2 \pi \beta}{|\alpha c_n|} + \frac{1}{2} \left( \frac{2 \pi \beta}{|\alpha c_n|} \right)^2 + O \left( \frac{2 \pi \beta}{|\alpha c_n|} \right)^3 \right] \quad \text{(actual solution)} \]

The numerical solution is thus the same as the actual solution up to order \( \left( \frac{2 \pi \beta}{|\alpha c_n|} \right)^2 \), but there it truncates and the error is then

**Case 1 error:** \[ \mathcal{E} = O \left( \left( \frac{2 \pi \beta}{|\alpha c_n|} \right)^3 \right) + O \left( 2 \pi \beta \right) \]

Thus it is important to keep the ratio \( \frac{\beta}{|\alpha c|} \) small for these slow modes. This can be done by making \( \alpha = \Delta t/\Delta x \) large as seen before, assuming \( \beta \) is already small.

**Case 2:** \( |\alpha c| \approx 1 \). This corresponds to modes which move nearly at the speed of the numerical iteration. The accuracy, of course, should be much better. For \( \beta \) small

\[ \frac{2 \pi \beta}{|\alpha c_n|} \ll 2 \pi \beta \ll 1 \quad , \quad \gamma_m \ll 1 \]

Here, all quadratic terms in \( \frac{2 \pi \beta}{|\alpha c|} \), \( 2 \pi \beta \), or \( \gamma \) are neglected and equation (4.19) can be rewritten

\[ U_n = \chi \left[ 1 - \frac{2 \pi \beta}{|\alpha c_n|} \right] \]

and the actual solution, equation (4.17),

\[ U_n = \chi \left[ 1 - \frac{2 \pi \beta}{|\alpha c_n|} \right] \]

their difference being very small since \( \gamma_m \ll 1 \) and \( |\alpha c| \approx 1 \).

**Case 3:** \( |\alpha c| \gg 1 \). The very fast waves corresponding to the lowest modes have
Now the \( \left( \frac{2 \pi \beta}{|\alpha c_e|} \right)^2 \) terms can be neglected as well so that equation (4.19) becomes

\[
U_n = -4 \lambda \pi \beta \left[ \frac{1 - \delta \nu_m + \delta \nu_m 2 \pi \beta}{(1 - \delta \nu_m)^2} \right] \lambda \pi^2 \beta + \lambda
\]

or with \( \frac{\delta \nu_m}{(1 - \delta \nu_m)^2} = -4 \left[ 1 - \frac{1}{(\alpha c_e)^2} \right] \) and \( \frac{1}{1 - \delta} = \frac{1}{2} \left( 1 + \frac{1}{(\alpha c_e)} \right) \) as in Case 1, there obtains

\[
U_n = \lambda \left[ 1 - \frac{2 \pi \beta}{|\alpha c_e|} - \frac{1}{2} (2 \pi \beta)^2 + O\left( \frac{2 \pi \beta}{|\alpha c_e|} \right) \right]
\]

where the \( (2 \pi \beta)^2 \) has been purposely retained. The difference from the actual solution is then

Case 3 error:

\[
\mathcal{E} = \lambda \frac{(2 \pi \beta)^2}{2}
\]

The fast modes then only require that \( \beta = \omega \Delta t \) be small, their error lying in the scheme's ability to handle the temporal term

\[
U_t \sim \omega \Delta t U.
\]

The slow modes also need a low frequency, but they need the parameter \( \frac{\beta}{|\alpha c_e|} \) to be small also. In summary, the scheme requires that the forcing frequency be small and also, for the slow modes only, that

\[
\frac{\beta}{|\alpha c_e|} \ll 1
\]

Since \( \beta = \omega \Delta t \) and \( \alpha c_e = \frac{\Delta t}{\Delta x} c_m \), \( \frac{\beta}{|\alpha c_e|} = \frac{\omega \Delta t}{c_m} \Delta x \),

trying to make \( \frac{\beta}{|\alpha c_e|} \) small by increasing \( |\alpha c_e| \) must be accomplished by decreasing \( \Delta x \).
An example of the response of the model equation (4.13) for a particular mode (phase speed), frequency and $\alpha$ is compared to the exact solution in Fig. 6. The actual mode (dashed line) is being forced periodically at the boundary and has had a chance to propagate a distance of 52 units into the basin. The solid line shows the scheme's ability to match the phase speed of the mode and its reaction to the sharp front. This shows a case where the mode's phase speed is less than the speed of the iterations; i.e. $|\alpha_c| < 1$. As described in this section, modes for which $|\alpha_c| > 1$ have an exponential damping after the front without the oscillation and are in general more accurately approximated.

Only a finite number of eigenfunctions can be resolved in the model with a finite number of grid points, which means that there will be a slowest mode present in the solution. By decreasing the number of grid points, the phase speed of this slowest mode will increase and the parameter $\frac{\beta}{|\alpha_c|}$ will become smaller, which increases the absolute accuracy of the model. There is thus a tradeoff in the adjustment of the resolution in $y$ between resolving small scale meridional structure (and calculating phase speeds) and eliminating high modes which are inaccurately modeled. Since high modes are trapped away from the equator, their importance is diminished there and tests of the resolution show that the model only requires the fairly low value of 45 grid points in $y$ to obtain good accuracy. In Fig. 7, the eigenvalues and eigenfunctions calculated numerically for the operator $\hat{L}$ in equation (4.10) are compared to actual eigenvalues of the parabolic cylinder functions from Cane and Sarachik III.

Section IV: Stability

The westward group velocity of the Rossby wave solutions to equation (4.13) suggest that the scheme should iterate in the same direction. To see
Fig. 7. First six eigenfunctions of model tridiagonal matrix for asymmetric basin; \( y_n = 1.7, y_s = -5.0 \)

Table III  Comparison of approximate and exact eigenvalues

Eigenvalues, \( \nu_n \), computed from the model tridiagonal matrix (cf. equation (4.10)) with \( N \) grid points.

\[ \varepsilon_N = \% \text{ relative error} = (\text{exact} - \text{numerical}) \cdot 100/\text{numerical} \]

<table>
<thead>
<tr>
<th></th>
<th>mode</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( N = 46 )</td>
<td>0.03</td>
<td>1.15</td>
<td>2.39</td>
<td>3.72</td>
<td>5.12</td>
<td>6.55</td>
<td>8.02</td>
<td>9.53</td>
<td>11.13</td>
</tr>
<tr>
<td>( \varepsilon_{46} )</td>
<td>6.08</td>
<td>5.94</td>
<td>5.99</td>
<td>5.93</td>
<td>6.21</td>
<td>6.49</td>
<td>7.12</td>
<td>8.04</td>
<td>9.3</td>
</tr>
<tr>
<td>( N = 61 )</td>
<td>0.04</td>
<td>1.17</td>
<td>2.43</td>
<td>3.79</td>
<td>5.21</td>
<td>6.68</td>
<td>8.19</td>
<td>9.76</td>
<td>11.43</td>
</tr>
<tr>
<td>( \varepsilon_{61} )</td>
<td>4.27</td>
<td>4.19</td>
<td>4.03</td>
<td>4.10</td>
<td>4.14</td>
<td>4.28</td>
<td>4.60</td>
<td>5.20</td>
<td>5.85</td>
</tr>
<tr>
<td>Exact</td>
<td>0.046</td>
<td>1.22</td>
<td>2.532</td>
<td>3.943</td>
<td>5.424</td>
<td>6.957</td>
<td>8.541</td>
<td>10.209</td>
<td>12.025</td>
</tr>
</tbody>
</table>
that this is essential, consider a disturbance occurring at time \( t = n\Delta t \) so that

\[
U_{i}^{n-1} = U_{i+1}^{n-1} = 0
\]
equation (4.13) becomes

\[
U_{i}^{n} = -Y_{m} U_{i+1}^{n}.
\]

Since \( \alpha > 0 \) and \( c_{m} < 0, |\gamma| < 1 \) (see Table III), a disturbance is damped if the scheme iterates from east to west, but grows from west to east.

The numerical scheme is implicit in that the most recent values of \( u \) and \( h \) are used to compute the \( x \)-derivatives. However, the scheme is not unconditionally stable as has just been shown. This is because the full equations have modal solutions in \( y \) which have only westward phase propagation. They introduce an asymmetry in the preferred solution direction which is absent in the usual implicit finite difference formulations.

This can be shown by a simple Von Neumann analysis, which determines the stability of wave solutions in terms of their spectral components. For example, the stability in time of solutions to equation (4.12) can be found by defining

\[
U_{l,j}^{n} = U_{l,j} e^{i\omega \Delta t} = U_{l,j} e^{in\theta} \tag{4.21}
\]

where \( l \) is now the \( x \)-increment (replacing \( i \) to avoid confusion). The non-dimensional parameter \( \theta = \omega \Delta t \) is the phase increment in time \( \Delta t \) for an arbitrary frequency \( \omega \). Substitution of (4.21) in (4.13) gives

\[
U^{n} = \left( \frac{1 + \alpha c \tan \frac{\theta}{2}}{1 - i\alpha c \tan \frac{\theta}{2}} \right) U^{n-1} = G U^{n-1}.
\]
If $G \leq 1$, the scheme is stable in time. This seems to be true since

$$|G|^2 = \left[ 1 - (\alpha \xi)^2 \tan^2 \Theta/2 \right] + \frac{\xi (\alpha \xi)^2 \tan^2 \Theta/2}{1 + (\alpha \xi)^4 \tan^2 \Theta/2} = 1$$

A similar analysis shows that the scheme appears to be stable in space also. However, this analysis is misleading because the eigenvalues $c_m$ for the phase speed have not been taken into account. If the wave form

$$\sum_{k,j} i(\xi - \eta \omega) = U_j e^{i(\xi \eta - \eta \omega)}$$

is substituted into (4.13), we have

$$\alpha c_m = \frac{\alpha}{\omega_m+1} = \frac{\tan \Theta/2}{\tan \Theta/2}$$

Since both $\alpha$ and $\omega_m$ are positive, this requires

$$\text{sgn}(\xi) = -\text{sgn}(\theta)$$

and equation (4.22) represents a westward traveling wave. If an eastward wave is present, so that $\text{sgn}(\xi) = \text{sgn}(\theta)$, we would have (for $\eta$ assumed real and positive)

$$-\frac{\tan \Theta/2}{\tan \Theta/2} > 0 \Rightarrow \tan \Theta/2 < 0$$

which is only true if $\xi$ is complex. The imaginary part of $\xi$ can cause exponential growth in equation (4.13) and thus make an eastward traveling wave unstable.

The equations (2.7a,b,c) still contain solutions which are not governed by equation (2.6). These are the Kelvin and anti-Kelvin waves for which $\nu=0$. The anti-Kelvin wave has a westward phase velocity and so is stable.
when equations (4.5) are solved. The Kelvin wave travels eastward and so is not stable in either time or space. The forcings in equations (2.7) have that part of them that projects onto the Kelvin mode subtracted, so that they do not excite a forced Kelvin wave. The unstable Kelvin mode is generated by small noise and truncation error and very quickly grows with each space and time step. This spurious mode must be filtered from the numerical solutions every few time and space steps so that it does not swamp them. This is done by calculating the projection of the Kelvin mode given by

$$\hat{a}_{k_i} = \sum_{j=1}^{NY} (U_{i,j}^n + h_{i,j}^n) \psi_{-,j} \Delta y$$

and subtracting it away

$$U_{i,j}^E = U_{i,j}^n - \hat{a}_{k_i} \psi_{-,j}$$

to obtain only the westward propagating modes. The proper Kelvin solution is added on at the end of the time step.
Chapter 5: Model Results

Introduction

The results of forcing a model Atlantic basin are presented in this chapter. In Section I, a uniform, periodic wind was used in order to compare model results to previous analytic results for that case (Cane and Sarachik IV, Philander and Pacanowski, 1981). The effects of friction and the presence of the Gulf are also examined. The second section expands the simple, uniform forcing to include longitudinal variations in amplitude and phase of the wind by presenting the results of two simple experiments run with the model. Section III discusses previous studies of the seasonal variability in the eastern Atlantic and, in particular, theories and models that have attempted to explain the strong upwelling found along the Guinea coast in the summer. The winds used to simulate the Atlantic trade wind system are presented in Section IV and the salient features of the resulting model height and zonal velocity fields are briefly described. Section V then discusses the relation of these features to the wind field and to the kinematics of long equatorial waves in a basin.
Section I: Periodic forcings of the Atlantic Basin

A model of the Atlantic basin including the Gulf of Guinea (see Fig. 8) was forced with a zonal wind at the yearly frequency for the first baroclinic mode (see Table II) and a meridional structure \( F = e^{-0.1y^2} \). A small Rayleigh friction, with \( r = 0.01 \), was added to the scheme as described in Chapter 4. Contours of the height and zonal velocity fields were generated each month (see Figs. 9a, b, c, d) and the height along the coast from 15°W to 10°E and along the longitude 2°E were plotted against time (Fig. 10a, b). The results were compared to those obtained by Cane and Sarachik IV, reproduced in Figs. 11a, b, which show the variation of phase and amplitude with latitude and longitude. A plot of the phase and amplitude of the model height along the equator, Fig. 12, shows the same features as the bottom of Figs. 11a, b. The rapid change in phase and the minimum in amplitude near the middle of the basin reflects the near equilibrium response to low frequency forcing; the slope of the height is balanced by the wind stress and the height field pivots up and down around the mode in the middle (Philander and Pacanowski, 1981). The fact that the change in phase does not occur at exactly one point shows that the solution is not exactly at equilibrium and zonal phase propagation is still needed to make adjustments (Cane and Sarachik IV). Away from the equator, the dynamic balances change (the influence of the Kelvin wave and lowest Rossby waves diminishes and higher Rossby waves become more important) and the slope of the height will not adjust as quickly to the zonal forcing. The resulting complexity in phase of the response is clearly shown in Fig. 11b.

The time series for the height along the latitude of the coast, Fig. 10 a, shows that the response of the height field is nearly in phase with the wind (the maximum in January and the minimum in July). This agrees with
Fig. 8. Atlantic basin showing the outline of the idealized model basin used in Chapter 5.
Fig. 9a. Height contours for uniform periodic forcing, \( F = \cos(wt) \cdot \exp(-0.1y^2) \), in model Atlantic basin. The African coast is the blank area in the upper right hand corner. The height and zonal velocity contours for July are exactly negative of those for January, shown above, because of the zero mean of the periodic forcing. The same relation holds for April and October. Note the widening boundary layer beneath the Guinea coast.
Fig. 9b. January zonal velocity (cf. Fig. 9a)
Fig. 9c. April height (cf. Fig. 9a). The corresponding closed contours in the lower left corner in October (when the height there is negative) are similar to the Angola dome.
Fig. 9d. April zonal velocity (cf. Fig. 9a)
Fig. 10a. Time series of height for uniform periodic forcing along Guinea coast. Note slight phase lag from east to west and increase in amplitude to east.
Fig. 10b. Time series of height for uniform, periodic forcing along 2°E from Guinea coast (top) to 5°S. Note boundary layer visible at top.
Fig. 11. Phase, (a), and amplitude, (b), of height for annual periodic forcing in an Atlantic basin from Cane and Sarachik IV. Superimposed are the longitude, 5°N, and latitudes, 15°W - 10°E, of the zonal coast in the Gulf of Guinea used in the present model.
Fig. 12a. Phase and amplitude for uniform, periodic forcing along equator in model Atlantic basin.

Fig. 12b. Same as 12a except zonal winds have been set to zero in the eastern 7/10ths of the basin. Slight bumps on graph of amplitude are real (not noise) and may be due to Rossby wave interference.
Fig. 11b which shows the zero phase contour in the vicinity of the model’s Gulf coast. The slight increase in phase towards the west there is also reflected in the model results. Note that the amplitude of the response also increases to the east, reflecting the greater excursions away from the node in the middle due to the seesaw effect.

The effect of the coast at $5^\circ$N is to add a narrow boundary layer, as shown in Fig. 10b. The response is, of course, perfectly symmetrical in $y$ when the basin is square and no Gulf corner is added. The influence of the boundary does not extend further south than about $1 1/2^\circ$. Note also that the amplitude variation is greatest near the equator and at the coast, but is diminished inbetween. The contours of $u$ and $h$ show the growth of the boundary layer (Fig. 9a) as a function of longitude, becoming wider towards the west ir accordance with the results derived in Chapter 3, Section III.

The similarity of the time derivative for a periodic response and the Rayleigh friction term in equation (3.24) suggests that the boundary layer should widen as the friction parameter $r$ is increased as well. The addition of such a small amount of friction, however, ($r=0.01$ compared to $\omega=0.026$) does not affect the thickness much. When the friction was increased to $r=0.02$, the boundary layer was observed to widen by about $30\%$. Adding a little bit of Rayleigh friction greatly reduced the numerical noise, particularly in the zonal velocity field $u$. The fluctuations in $u$ are caused by the model’s inaccuracy in approximating the higher Rossby modes. They have smaller spatial scales, but also a slower phase speed, so that adding a simple friction which diminishes the amplitude of each point every time step (cf. Chapter 4) will tend to get rid of these modes before they have had a chance to go very far. The addition of a slight amount of friction also allows the model to be spun-up faster. The difference friction makes for
the periodic forcing is shown by Fig. 13a which is a meridional cross-section of u for various latitudes with no friction, and by Fig. 13b which shows the same cross-section, but with friction added as explained in Chapter 2, Section IV. The corner of the Gulf is a natural source for sharp jumps as u must be a step-function in y there (Chapter 3). The u field, however, seems to be just as noisy in areas away from the corner. The improvement in the height field with friction is not as dramatic because h is not as intrinsically noisy as u (it is an integral in y of u; see Fig. 14). This allowed comparisons between the height fields generated with and without friction to be made in order to determine the right amount of friction to add so as not to change the results too much. A friction parameter of r=0.01 was found to be best.
Fig. 13. Meridional section of zonal velocity at 30°W for periodic forcing; a) without friction, b) with Rayleigh friction, $r=0.01$. 
Fig. 14. Meridional section of height at 30°W for periodic forcing; a) without friction, b) with Rayleigh friction, r=0.01.
Section II: Effects of non-uniform phase and amplitude in the wind forcing

As Fig. 11b illustrates, even the simplest periodic forcing gives rise to considerable phase and amplitude variation in the ocean. The reflection and interference of the long, low frequency equatorial waves create a response which may have little resemblance to the propagation of a single wave or local forcing. The increased complexity of the real winds also raises the question of to what extent are variations in the wind's own phase and amplitude redistributed by the ocean over the entire basin. The equilibrium adjustment of the height along the equator, for instance, would have to change from a simple seesaw to more complicated undulations if it were to try to respond to a shifting phase in the forcing. The adjustment in the wake of a Kelvin or low Rossby mode wave, excited in an area of strong forcing, would be carried into areas of less intense forcing (eg. Hurlburt, et. al., 1976), not to mention its reflections and the phase and amplitude structure they would evolve under periodic forcing.

To get an idea of the importance of including phase and amplitude variations in periodic forcings, two simple experiments were run with the model. The seasonal signal for the zonal wind field in the equatorial Atlantic is much larger in the western half of the basin than in the east (Hastenrath and Lamb, 1979). Adamec and O'Brien (1978) used this feature in a numerical model to excite a Kelvin wave and study its effects in the eastern side of the basin and the Gulf of Guinea. The model periodic winds were modified so that the zonal forcing was non-zero in the western 3/10ths of the basin only. Fig. 12b shows the phase of the height field for this case. Compared with Fig. 12a for a uniform amplitude, the difference is a shifting of the node of oscillation to near the center of the forcing. To the east of this node, the ocean responds in phase and includes most of the basin. The far
western part of the basin is $180^\circ$ out-of-phase, as would be expected. The change of the response from in-phase to out-of-phase takes place over roughly the same distance as in the uniform forcing case. The amplitude is diminished over the eastern part of the basin as compared to the uniform wind forcing which showed an increase in amplitude toward the east. The amplitude in Fig. 12b was multiplied by $10^{-2}$ in order to more easily compare it with the phase. If the original amplitude is plotted (not shown), two small areas centered at approximately $10^\circ$W and $6^\circ$E rise above the flattened curve by about 10%. These are barely visible in Fig. 12b.

To look at phase variation in the wind forcing, a simple linear phase shift was used to model the actual variation in phase of the yearly harmonic of the zonal wind stress in the Pacific, as shown by Meyers (1979). Because of the out-of-phase response of the thermocline slope to surface topography, the forcing was modeled as a cosine with zero phase in January, which is approximately $180^\circ$ out-of-phase with the annual frequency computed in Meyers' study. This produces a phase and amplitude of height which can be compared more easily with his results. If a model Pacific basin is forced with a uniform, periodic wind with no phase variation, the result is very similar to Fig. 12a for the Atlantic. If the phase shift is included, the result is very different. Fig. 15a,b compares the model results for this case with Meyers'. He noted a point of divergence near $130^\circ$E with eastward propagation to the east and westward propagation to the west. There is no sign of this in Fig. 12a, but Fig. 15b shows such a point at nearly the same longitude. The eastward propagation is again much too slow for a Kelvin wave, but is just about right in the model run. The amplitude also has a similar structure, particularly in the east. The results for the far western half of the model basin differ from Meyers' results, but other influences may help reconcile this discrepancy. First, the extreme width of the Pacific
Fig. 15.  a) Phase and amplitude of the annual harmonic of zonal wind stress and the depth of the 14°C isotherm in the Pacific from Meyers (1979). b) Phase and amplitude for height along the equator from the model forced at the annual frequency and uniform amplitude, but with a phase variation as shown. The height in the model results and the depth of the 14°C isotherm are out of phase in accordance with a deep level of no motion.
makes it hard to believe that Rossby waves travel all the way across, at least on a seasonal scale. The pattern in the east may be the result of the Kelvin waves forced in the middle of the basin and traveling towards the east where they interfere with their own Rossby wave reflections and locally forced Rossby waves. The far western region is also subject to monsoons (Wyrski, 1974) whose seasonal cycle are surely not fully represented in the model forcing. The effects of beta-dispersion (Schopf, Anderson and Smith, 1981) create a focus of Rossby waves near 180°, which would be an area of rapid phase variation (Cane and Sarachik IV). Its location might also be sensitive to the eastern boundary shape, which is idealized in the model as a meridian at 80°W. Finally, no attempt was made to model the wind's amplitude variation, which, being large in the west, may seriously affect the results there, as shown in the first numerical experiment. Even without stressing the comparison with Meyers' results, Fig. 15b at least points out the drastic change in character of the phase structure of the ocean in response to significant variation in the wind's phase.
Section III: Previous studies

The upwelling of cold water along the zonal coast of the Gulf of Guinea during summer and early fall is a well-known annual feature (Houghton, 1980; Bakun, 1978). Attempts to explain the upwelling as a result of the divergence of surface waters by Ekman transport generated by the local winds have been unsuccessful. Ekman transport seems to vary more with longitude along the coast than it does during the year (see Fig. 2 from Bakun, 1978). The correlation between wind stress and upwelling is not strong (Houghton, 1976). This implies that larger scale effects are important. These can be in the form of advection of colder waters into (or warm waters out of) the area by existing current systems or by the propagation of an upwelling signal by waves. Because of the lack of direct correlation between transport and isotherm displacement, Houghton (1980) concludes that the Equatorial Undercurrent (which requires more vertical resolution than the numerical model is capable of and is probably influenced by non-linear effects (Cane, 1976)) does not seem to be responsible for driving the upwelling even though it is a major feature of the circulation. Similarly, Hisard and Merle (1979) state that the onset of the summer coastal upwelling cannot be satisfactorily explained by current-induced upwelling. Bakun (1978) describes a second upwelling in the winter which is confined to the extreme far western part of the Gulf and notes its connection to a tropical front oscillating along the west African coast.

The equator in the eastern Atlantic also experiences an upwelling at about the same time of year. This event has been linked to wind-forced tilting of the dynamic height along the equator in the western and central part of the basin (Katz, et. al., 1977). This lead to speculation that the upwelling in the entire basin and along the coast is the result of an
equatorially trapped Kelvin wave originally generated by the strong, seasonal zonal wind stress in the western half of the basin. This is analogous to the mechanism which has been successful in explaining the interannual upwelling event (El Nino) in the Pacific (McCreary; Hurlburt). However, this explanation has serious shortcomings. The phase lag between upwelling along the equator and along the Gulf required by the propagation of a single wave is not clearly evident in the data (Moore, et. al., 1978). In fact, Merle, Fieux and Hisard (1979) note a westward phase propagation on the equator near the African coast during the onset of summer cooling and also find only a slight phase lag along the Gulf, observations which are both inconsistent with an eastward traveling Kelvin wave and its reflections as a coastal Kelvin wave. Clarke (1979) presents a simple linear model which does not include equatorial dynamics at all, but suggests that the upwelling is the result of a coastal Kelvin wave forced by the winds that are eastward of the upwelling. Because his wave is linear and includes a decay scale, the upwelling at a particular location is essentially the result of integrating the forcing along the characteristic of the simple wave equation (which naturally follows the coast) for a distance proportional to the decay scale. His theory is then able, by adjustment of the decay scale, to relate upwelling along the coast to previous intensification of the winds to the east. His results for reasonable values of the decay scale (1500-450 km) show qualitative agreement to Bakun's (1978) observations. The inability of coastally-trapped wave theory to predict teleconnections with the equatorial waveguide, which have been shown to have strong influence on low frequency equatorial motions, suggest that mechanisms which include the Gulf in the equatorial dynamical regime might still be sought (particularly since the Gulf is only 5° away from the equator). As will be seen, a realistically
forced Atlantic basin, modeled as an equatorial beta-plane, can account for
the major features of coastal upwelling, including its narrow scale, and
also help explain the response at the equator.
Section IV: Description of the wind field and model results

The winds used to force the model are an analytic curve fit to the zonal and meridional trade winds developed by Paul Schopf at the Goddard Space Flight Center. The analytic wind forcing includes the annual migration of the Intertropical Convergence Zone (ITCZ) which resides just north of the equator. Seasonal contours for the zonal wind are shown in Fig. 16. The relative maxima and minima about this mean simulate those observed in the real wind fluctuations (cf. Hastenrath and Lamb, 1977), i.e. a relative maximum in the winter and a minimum in the summer. It is these relative maxima and minima that will be analyzed in the model results. The model zonal winds have an easterly mean in the Gulf of Guinea while the mean of the observed Gulf during the summer and the strong positive zonal wind stress in the winds is slightly westerly. The effect of this mean in the model winds is a steady sea surface tilt to the west.

The zonal trade wind also has a relatively simple configuration; the ITCZ slants along an axis inclined at about 45° in the northern and western parts of the basin. This localization of strong (positive) wind forcing in the west is a well known feature of the Atlantic wind field (cf. Adamec and O'Brien, 1978). The meridional component of the trade wind forcing is shown for April and October in Fig. 17. These winds have a large positive amplitude in the Gulf of Guinea, but, as will be seen, are not as important in forcing the observed features in the Atlantic as are the zonal winds.

The results of forcing the model with only the zonal wind component are shown in the seasonal contours of sea surface height and zonal velocity of Figs. 18 a-h. A small amount of friction, $r=0.01$, was also included in the simulations. The upward tilt towards the west is seen throughout
Fig. 16a. Amplitude contours of the zonal component of model trade wind forcing (ms$^{-1}$).
The area between the -3 contours is a relative maximum and corresponds to the Intertropical Convergence Zone.
Fig. 17a. Amplitude contours of the meridional component of trade wind forcing.
Fig. 17b. Amplitude of meridional wind
Fig. 18a. Contours of height for model Atlantic basin forced with analytic zonal trade wind: January
Fig. 18b. Contours of zonal velocity for zonal wind: January
Fig. 18c. Contours of height for zonal wind: April
Fig. 18d. Contours of zonal velocity for zonal wind: April
Fig. 18e. Contours of height for zonal wind: July
Fig. 18f. Contours of zonal velocity for zonal wind: July
Fig. 18g. Contours of height for zonal wind: October
Fig. 18h. Contours of zonal velocity for zonal wind: October
the year. About this mean tilt, the height along the equator appears to be responding in phase to the annual signal of the wind, seesawing about a pivot near the center of the basin. The contours of height are closer together in the summer, particularly in the east, which is due to the increased westward wind stress and subsequent steepening of the slope of the sea surface height. The subsequent decrease in height in the east corresponds to the observed summer upwelling.

The contours of zonal velocity show an intense eastward flow in the western half of the basin around $3^\circ - 5^\circ$N during the summer, which die away during the winter. This corresponds closely to the North Equatorial Counter Current (NECC). A feature corresponding to the westward flowing South Equatorial Current (SEC), which straddles the equator, can also be seen in the western half of the basin. It is less well organized in the east, which is consistent with observations (Knauss, 1963). Both the NECC and the SEC have a pronounced northeast tilt to them, similar to the ITCZ.

An obvious feature in the contours of both height and zonal velocity is the thin boundary layer beneath the Guinea coast (cf. Chapter 3, Section III). The flow there is strongly eastward and is most intense in the summer when it also appears to connect with the NECC (Philander, 1973; note the $10^\circ$ contours in Fig.18f). This corresponds to the narrow, intense part of the Guinea current (LeMasson and Rebort, 1973, see Fig. 19). The width of the boundary layer is roughly $1/2^\circ - 1^\circ$ of latitude and it appears to widen to the west, as in the case for a uniform periodic wind (Section I).

A time series of height along the Guinea coast (corresponding to Fig. 10a, Section I) is shown in Fig. 21, where it is compared to Bakun's (1978) time series for sea surface temperature. The mean tilt due to the negative mean of the wind stress has been subtracted out in order to better reveal the relative amplitudes and allow comparison with the purely periodic forcing
Fig. 19. Meridional section showing vertical structure of currents near Guinea coast at Bassam ($\approx 4^\circ W$) in August. Note intense, narrow eastward surface flow near $5^\circ N$ corresponding to the Guinea current.

Fig. 20. Contours of height in square basin during spin-up ($t=64\approx 2$ months) showing distinctive Rossby wave pattern (from Cane and Sarachik III)
Fig. 21.
Top: Time series of height for zonal trade wind forcing along Guinea coast

Right: Time series of observed sea surface temperature (°C) along Guinea coast (after Bakun, 1978)

Far right: Schematic of Guinea coast (after Bakun, 1978)
case of Section I. As mentioned above, this also gives a closer correspondence to the actual conditions in the Gulf, where the mean is much smaller. There is clearly shown an overall lowering of dynamic typography in late May and June, with the minimum lasting through late September. This corresponds to the summer upwelling of cold water in the observations. There is a second lowering of height in late winter that is localized in the far western part of the coast. The summer upwelling is also less intense in the west. Both these features are reflected in Bakun's data (Fig. 21). A slight westward phase lag between 10°E and 15°W of about two to three weeks is discernible in the height and temperature fields, as was observed in the periodic case of Section I. A time series for height along 2°E from the coast at 5°N to 5°S is shown in Fig. 22. The thin boundary is visible at the top and the strong summer upwelling appears at all latitudes. The winter upwelling is absent this far east along the coast. The contours are symmetric about the equator except for within about 1° of the coast, which is the region of the BL. The westward bulge of the contours in Fig. 22 shows that the upwelling occurs earlier along the equator and the coast than at latitudes inbetween. The closed -45 contour about the equator also implies that the upwelling is most intense there. The closely packed contours in the boundary layer show that the upwelling has a local maximum there as well.

These features of the model's response can be compared to the contours of temperature at 20 m for summer and winter from Mazeika (1964), Fig. 23a,b. Other, less equatorially confined features are also evident. A minimum in height in the July contours off the northern African coast can be seen to correspond to the Guinea thermal dome in Fig. 23b. A similar feature, the Angola dome, is shown in Fig. 23a at around 10°S in the winter. This dome
Fig. 22. Time series of height for zonal wind forcing along 2°E
(eastern Gulf of Guinea)
Fig. 22. Temperature at 20 meters in a) northern winter, b) northern summer
(from Mazelka, 1967)
also appears as a depression in the height contours when the model is run without any friction. These features will be discussed more fully in the next section.

The foregoing results were achieved by forcing the model with the zonal component of the wind only. The addition of the meridional component enhances the correspondence between the height contours, Fig. 24 and Figs. 23a,b, though the modification to the overall pattern is slight. In addition to a mean upward tilt to the north, the summer minimum in the contours near the equator is shifted 1° - 2° southward with a slight tilt upward from west to east, as in Fig. 23b. The Guinea dome is also slightly better defined.
Fig. 24a. Contours of height for meridional and zonal forcing: January
Fig. 24b. Contours of zonal velocity for meridional and zonal forcing: January
Fig. 24c. Contours of height for meridional and zonal forcing: July
Fig. 24d. Contours of zonal velocity for meridional and zonal forcing:
July
Section V: Discussion

The foregoing brief description has shown that there is good agreement between the model results for a trade wind forcing and actual conditions observed in the Atlantic. The ability of the model to simulate the various observed features and their connection to the spatial and temporal variations of the wind field will be discussed in the context of the adjustment of low frequency equatorial waves in a basin and contrasted with previous models and theories.

The seasonal migration of the ITCZ and the entire trade wind system results in a large annual variation of wind stress near the equator. Though the zonal trade winds have an overall negative mean, the relative amplitudes are maximum in the winter and minimum in the summer (cf. Section IV, Fig. 16). The model sea surface height along the equator responds by increasing its negative slope in the winter and decreasing it in the summer, becoming nearly flat. This correspondence between wind and pressure gradient has been noted in the actual data (Katz, et. al., 1977). For points east of about 20°W, this seesaw effect produces height minimums and maximums along the equator which are in phase with the wind. The frequency of the annual signal is low enough to allow the fast, equatorially trapped waves excited by the changing wind a chance to bring the ocean surface into adjustment, i.e. the gradient of the height is balanced by the zonal wind stress. The adjustment along the equator is carried out essentially by the Kelvin wave and lowest mode Rossby wave (Cane, 1979).

Results for a uniform periodic forcing (Section I, Cane and Sarachik IV) showed that even for latitudes a few degrees away from the equator, the response of the ocean in the far eastern side of the basin is still nearly in phase with the annual signal (see Fig.11b). The adjustment for low
frequency forcing is faster in the eastern part of the basin than in the western part (Philander and Pacanowski, 1981) because the Kelvin wave is able to travel across the basin to the east faster than the Rossby waves can travel to the west. The equilibrium response which exists after the passage of these waves thus occurs later in the west. This is why the phase of the response in the eastern part of the basin in the experiment of Section II was still essentially zero even though the winds were concentrated in the far west. Latitudes near the Guinea coast are not quite in phase with the annual signal in the Atlantic as is reflected in the slight westward phase propagation there (cf. Section I).

The dominant feature of the zonal trade winds is the convergence zone or ITCZ in the western part of the basin. The intersection of this axis with the zonal coast during its seasonal north-south migrations causes distinct variations in the amplitude of the zonal wind there. Fig. 25 shows a time series for the model's zonal wind stress at several latitudes along the coast. Note the double maxima and minima in the far west at 15°W and, in contrast, the completely annual signal in the far east at 10°E. This is caused by the ITCZ, which has a relative maximum in the center of its axis. This maximum moves eastward along the coast during the southward migration of the ITCZ in winter, causing a December maximum in Fig. 25 at 15°W, but not at 10°E, which is too far to the east. The wind stress then goes through a minimum in February at 15°W as relatively weaker winds are felt in the wake of the eastward traveling axis. The ITCZ then reverses its direction in March, causing the axis to move westward, having reached no farther than about 0° in its eastward excursion. It again passes by 15°W, causing another relative maximum there in April. Because of the limited range of the migration of the ITCZ, the far east experiences only an annual variation in zonal wind amplitude while the far west feels a (smaller) semi-annual signal as well.
Fig. 25. Time series of zonal wind stress amplitude (ms\(^{-1}\)) along Guinea coast at various longitudes
Comparing Figs. 21 and 25 shows that the secondary upwelling in winter in the far west corresponds very closely to the double passage of the ITCZ described above. In the east, both the wind field and the height field display only an annual signal. The height field thus appears to be responding in near equilibrium to both the strong annual signal, which extends throughout the basin, and also to the smaller semi-annual signal in the far west, which is a relatively local effect of the ITCZ. The meridional section at 2°E, which is east of the furthest eastward excursion of the ITCZ, again reflects only the strong annual signal.

The foregoing results show that upwelling in the Gulf of Guinea is a near equilibrium response to the shifting pattern of the zonal trade wind system. The maximum of the zonal wind stress is strongest in the western part of the basin. As shown in Section II and mentioned above, this still allows the eastern part of the basin near the equator to respond in phase with the remote forcing. The model was run with the analytic zonal wind set to zero east of 25°W. The results showed a remotely forced summer upwelling still present along the Guinea coast. However, the winter upwelling in the western part of the coast was completely missing, while another winter minimum, localized in the east, appears. This new minimum also occurs at the equator at 2°E, decreasing with latitude, as in the summer upwelling case. When the model was run with the winds set to zero west of 25°W, the winter upwelling was left intact with a strong summer signal as well. In fact, the time series for height along the Gulf and along 2°E were almost the same as those for the full winds. The relative amplitude at 10°E at the coast for these two cases is compared with the case for the full wind forcing in Fig. 26. The time series for height was obtained after a two year spin-up with the friction parameter, r=0.01.
Fig. 26. Four time series of height at 10°E and Guinea coast (mean subtracted, arbitrary units) comparing response from winds concentrated in eastern and western halves of basin and their sum with full wind response. The sum of the two concentrated forcing cases is not exactly equal to full wind case because the basin was only spun-up for 2 years in these
The curves show that essentially annual signal which is characteristic of the far eastern part of the Gulf. The curve for the western half winds, however, has a significantly smaller amplitude than for the eastern half case. This shows that the influence of the western forcing is less than the relatively local winds in the east in generating the observed annual summer upwelling. There is also a slight dip in February in the western half case, noted above, which is absent in the other curves. Thus, the influence of the migration of the ITCZ is felt as a slight remote response in the far east, but not in the western part of the Gulf of Guinea. Perhaps this is due to the double passage of the ITCZ across the equator at that time of year which can only be felt in the east as a Kelvin wave, i.e. the signal in the east is strictly equatorial in origin and not felt directly at the coast. The amplitude of both the summer and eastern-intensified winter upwellings diminish with distance west along the coast, suggesting that coastal Kelvin wave reflections are not as efficient in transferring upwelling signals as previous researchers have assumed. Nonetheless, the observed western-intensified winter upwelling is clearly the result of the local passage of the ITCZ in that part of the Gulf. The summer upwelling is also mostly a response to the local annual signal, which, though less intense than in the west, is still important in the Gulf. These results show that the pattern of upwelling is a result of the adjustment of the ocean through long, low frequency equatorial waves to both local and remote forcing. The location of the Gulf in the eastern part of the basin and its proximity to the equator allow this adjustment to occur essentially in phase with the wind forcing.

Observations in the real Atlantic and the results from the model simulations are just the opposite that would be expected from standard
coastal upwelling theory. The usual model predicts that an increased positive zonal wind stress will cause an Ekman transport of water away from the coast and thus induce upwelling there. This failure of coastal upwelling theory to explain events observed along the Gulf was noticed by earlier researchers (Section III). The weak upwelling observed in 1968 was perplexing because it occurred during a year when the summer winds were anomalously strong (Houghton, 1976). This is not contradictory, of course, if the winds and height are in phase; a strong positive wind would tend to keep the sea surface from lowering in the east and thus reduce or even prevent the usual upwelling. This example also shows that the basin can respond in equilibrium to interannual wind events.

Since local winds did not seem able to produce the upwelling, at least by the mechanism of classical upwelling theory, previous investigators sought explanations in terms of remote forcing. The success of the theories of the El Nino in the Pacific and the intensification of the zonal winds in the western Atlantic prompted researchers to apply El Nino-like models to the Gulf of Guinea upwelling. This type of model assumes that the upwelling event is a response to an impulse in the wind forcing in the form of a single equatorial Kelvin wave front and its coastally trapped reflections at the eastern boundary (ignoring the low Rossby modes; Moore, et. al., 1978). This is in contrast to the low frequency periodic response seen in the model results (Sections I and II) which assumes that the ocean has time to approach equilibrium by generating reflected Rossby as well as Kelvin waves. As seen in Section III, these models have only very limited success.
The possibility of explaining the coastally trapped upwelling with a coastally trapped equatorial wave was also attractive. However, the scale of the trapping is an equatorial radius of deformation, or roughly 200 km near the Gulf, whereas the upwelling and the Guinea current are confined to a region about $1/2^\circ$ to $1^\circ$ wide, or somewhat less than 100 km (Clarke, 1979). As seen in the model results, the boundary layer beneath the coast has approximately this width. Section III of Chapter 3 described the boundary layer as the adjustment of the meridional modes to the condition of no normal flow at the zonal coast. Using the vorticity equation, the model of the boundary layer showed that it should widen to the west and that it did not depend solely on friction but could exist for periodic forcings as well. Both these features are observed in the model results. Because it is a response to the forcing in the interior, the model boundary layer reacts differently to variations in the forcing than the coastally trapped Kelvin waves generated at the eastern boundary considered by earlier researchers. For one thing, the thinness of the boundary layer implies that it must consist of higher Rossby modes as well. The rapid adjustment in the boundary layer is not the result of various waves propagating in from a source in the east (this would be a very slow process for the high Rossby modes), but rather can be thought of as a propagation southward in response to the imposed initial condition at the boundary. The thinness of the boundary layer allows this adjustment to the interior solution to set-up very quickly. This is reflected in the modified scaling for the boundary layer used in Chapter 3, Section III. The time derivatives become relatively less important in the equations of motion, showing that the boundary layer is able to adjust immediately to the forcing at the boundary. The model results show the upwelling along the coast (in the boundary layer regime) to be essentially in phase with the wind forcing, i.e. there is no
lag due to coastal waves traveling through. Thus, a strong negative wind stress, which tends to increase the negative sea surface slope, will cause an immediate lowering of height in the Gulf; the boundary layer cancels the interior velocity and lowers the height along the coast, in phase with the interior height and the wind forcing. The strong response of the height field in the boundary layer is geostrophically related to the intense coastal Guinea current (see Fig. 18a, b). The current and the height gradient are most intense during the summer upwelling, though they are present all year round.

Clarke (1979) argues that the roughly 200 km equatorial radius of deformation is too large to model the thin upwelling and current structure near the African coast and thus concludes that equatorial waves are not important there. However, this does not take into account the boundary layer possible at the coast, which is made up entirely of equatorial modes (see Fig. 7 and Table III) and is forced by non-local winds.

Regions away from the Gulf of Guinea and the equator also have a seasonal character that appears to be dominated by higher mode Rossby waves. Similarities can be seen between Fig. 20, which shows a square basin being spun-up (from Cane and Sarachik III), Fig. 18a, which is the contours of height for the zonal trade wind forcing and Fig. 23b, which shows Mazeika's (1967) 20m temperature contours. The distinctive pattern of contours bending back and forth between 5° and 14° due to Rossby waves appears in all three. When no friction is present, features similar to the Guinea and Angola domes appear in the height field near 10°N and 10°S respectively. The results of the model forced with a uniform periodic wind (see Fig. 9c) also show features which resemble the thermal domes. The Angola dome is missing from Fig. 18a because the small amount of Rayleigh friction, r=0.01, added to the scheme has damped it out. This shows that the feature associated
with the Angola dome is made up of high Rossby modes which are more attenuated by the effect of friction than the modes near the equator (cf. Chapter 2).

Another example of the importance of the Rossby wave reflections is also suggested in Fig. 23b which shows the summer minimum in temperature contours at 20 m. Imbedded in this minimum during the summer are two cold water domes near 2°S and at approximately 6°W and 4°E. These thermal domes are similar to the Guinea and Angola domes except that they are near the equator. The results of Section II showed a slight increase (≈10%) in the amplitude of the height field at approximately these latitudes. These areas would thus be slightly deeper depressions in the overall lowering of height associated with the summer equatorial upwelling. As mentioned in Adamec and O'Brien (1978), these depressions are caused by the reflection and interference of Rossby waves of different speeds from the eastern boundary due to the Kelvin wave excited during fluctuations in winds to the west.
Chapter 6: Summary

This thesis has presented a unique numerical model of linear equatorial physics and shown that it can be used as an efficient and accurate tool for studying seasonal variations in an equatorial basin. The scheme itself takes advantage of a simple low frequency approximation to the equations of motion on an equatorial beta-plane to provide very fast and stable integrations. The step size in time is on the order of days instead of hours. The special boundary conditions at the meridional coasts required by the simple physics of the model were extended to include the reflection and transmission of the long, non-dispersive Kelvin and Rossby wave solutions at a partial boundary, such as the one represented by the African coast. Thus, various basin configurations can be easily accommodated by the model. In all, the speed, accuracy and simplicity of the scheme allows many different cases, which provide excellent resolution of seasonal equatorial features, to be run.

The results of forcing the model with uniform, periodic winds agreed very well with previous analytic studies. These simple, periodic cases also provide a clear example of the adjustment process of the long waves in response to varying wind stress and can be used as a reference point for comparing results of more complicated, realistic winds. The effect of friction and the effect of introducing a zonal coast near the equator could also be examined. There was discovered in the results a thin, boundary layer like region, made up of equatorial meridional modes, which is an important feature at the coast.

To better understand the connection between the structure of the wind forcing and the ocean's response, the amplitude and phase of the wind were
varied in two separate experiments. The first concentrated a periodic zonal wind in the western 3/10ths of a model Atlantic basin and showed that the response in the far east was in phase with the remote forcing, but with diminished amplitude. The second experiment modeled the nearly linear shift in phase with longitude of annual zonal winds in the Pacific. The results showed that a point of divergence in the ocean's phase propagation along the equator near 130°W, shown in a study by Meyers (1979), could be explained in the model as a result of the wind's phase shift, which sets up a distinctive interference pattern of locally forced and reflected waves not seen for winds with no phase shift and not attributable to the propagation of a single wave.

By applying a realistic wind forcing to a model Atlantic basin, the basic features of the observed seasonal temperature and surface dynamic topography were found to be remarkably well simulated, particularly the upwelling at the Guinea coast. The results showed that the height field responds essentially in phase with the local wind forcing in the Gulf of Guinea. This produces the well known summer upwelling along the equator and along the Guinea coast, which is intensified by the boundary layer, and also a second winter upwelling, also found in the data, which is localized in the far western part of the coast. This winter upwelling is shown to be in phase with the local maximum in wind stress associated with the migratory Intertropical Convergence Zone (ITCZ) and emphasizes the importance of local forcing in the ocean's equilibrium response. When the winds are set to zero in the western half, with those in the eastern half left intact, the resulting time series for height is nearly identical to the full wind case, showing that the relatively local winds in the Gulf of Guinea are primarily responsible for both the large summer and smaller winter upwellings. When the opposite forcing is applied, with only remote
winds in the west, an attenuated summer event is still observed, but the winter event is now localized in the east instead of the west. The fast coastal waves created by the remotely forced Kelvin wave do not seem capable of effectively extending the equatorial upwelling signal along the coast. The upwelling at the coast seems instead to be a boundary layer like response to an interior solution made up of Rossby modes, which can be generated locally. The boundary layer reflects the intensity and narrow scale of the observed upwelling and current structure and shows that purely local, non-equatorial mechanisms are not necessary to explain them.

The overall pattern of the response also simulates features found at higher latitudes. There is a clear resemblance between numerical results showing the effects of Rossby waves and contours of actual dynamic topography and temperature in the Atlantic. In particular, two features which were clearly evident in the model at higher latitudes correspond to the Guinea and Angola thermal domes, which appear near 10°N in summer and 10°S in winter, respectively. The amplitude of the feature associated with the Angola dome was found to diminish greatly with only a small amount of Rayleigh friction, showing that it is comprised of higher, less equatorially trapped Rossby modes, which are more sensitive to friction. A direct effect of Rossby modes along the equator was suggested by a slight increase in the amplitude of the periodic height field near 5°W and 4°E, which correspond to small thermal domes imbedded in the summer upwelling.

The other prominent equatorial features, such as intense subsurface flows, will no doubt modify the simple linear results presented here and complicate the vertical structure. Local forcing of coastal waves and interannual events will also enhance the realism of the model. In the future, real winds will be used to force the model, so that better comparisons can be made.
REFERENCES


___ (1979b) Annual variation in the slope of the 14°C isotherm along the equator in the Pacific Ocean. J. Phys. Oc. 9: 885-891
____ (1979) Geophysical Fluid Dynamics. Springer-Verlag Publishers