AN EQUILIBRIUM MODEL OF RARE EVENT PREMIA

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Abstract

In this paper, we study the asset pricing implication of imprecise knowledge about rare events. Modeling rare events as jumps in the aggregate endowment, we explicitly solve the equilibrium asset prices in a pure-exchange economy with a representative agent who is averse not only to risk but also to model uncertainty with respect to rare events. Our results show that there are three components in the equity premium: the diffusive-risk premium, the jump-risk premium, and the “rare event premium.” While the first two premia are generated by risk aversion, the last one is driven exclusively by uncertainty aversion. To dis-entangle the “rare event premium” from the standard risk-based premia, we examine the equilibrium prices of options with varying degree of moneyness. We consider models with different levels of uncertainty aversion – including the one with zero uncertainty aversion, and calibrate all models to the same level of equity premium. Although observationally equivalent with respect to the equity market, these models provide distinctly different predictions on the option market. Without incorporating uncertainty aversion, the standard model cannot explain the extent of the premia implicit in options, particularly the prevalent “smirk” patterns documented in the index options market. In contrast, the models incorporating uncertainty aversion can generate significant premia for at-the-money option prices, as well as pronounced “smirk” patterns for options with different degrees of moneyness.

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1 Introduction

Sometimes, the weirdest things happen and the least expected occurs. In financial markets, the mere possibility of extreme events, no matter how unlikely, could have a profound impact. One such example is the so-called “peso problem,” often attributed to Milton Friedman for his comments about the Mexican peso market of the early 1970s.\textsuperscript{1} The existing literature acknowledges the importance of rare events by adding a new type of risk (event risk) to the traditional models, while keeping the investor’s preference intact.\textsuperscript{2} Implicitly, it is assumed that the existence of rare events affects the investor’s portfolio of risks, but not their decision-making process.

This paper begins with a simple yet important question: Could it be that investors treat rare events somewhat differently from the common, more frequent events? Models with the added feature of rare events are easy to build, but much harder to estimate with adequate precision. After all, rare events are infrequent by nature. How could we then ask our investors to have full faith in the rare-event model we build for them?

Indeed, some decisions we make just once or twice in a lifetime – not much room to learn from experiences, while some we make everyday. Naturally, we treat the two differently. Likewise, in financial markets, we see daily fluctuations, and we see rare events of extreme magnitudes. In dealing with the first type of risks, one might have reasonable faith in the model built by financial economists. For the second type of risks, however, one cannot help but feeling a tremendous amount of uncertainty about the model. And if the market participants are uncertainty averse in the sense of Knight (1921) and Ellsberg (1961), then the uncertainty about rare events will eventually find its way into financial prices in the form of a premium.

To formally investigate this possibility of “rare-event premium,” we adopt an equilibrium setting with one representative agent and one perishable good. The stock in this economy is a claim to the aggregate endowment, which is affected by two types of random shocks. One is diffusive in nature, capturing the daily fluctuations in the fundamentals; the other is pure jump, representing events with low frequency and sudden occurrence. While the probability laws of both types of shocks can be estimated using the existing data, the precision for the rare events is much lower than that for the normal shocks. As a result, in addition to balancing between risk and return according to the estimated probability law, the investor factors into his decision the possibility that the estimated law for the rare event may not be correct. His asset demand therefore depends not only on the tradeoff between risk and return, but also on the tradeoff between uncertainty and return.

\textsuperscript{1}Since 1954, the exchange rate between the U.S. dollar and Mexican peso has been fixed. At the same time, the interest rate on Mexican bank deposits exceeded that on comparable U.S. bank deposits. In the presence of the fixed exchange rate, this interest rate differential might seem to be an anomaly to most people. But it was fully justified when in August 1976, the peso was allowed to float against the dollar and its value fell 46%. See, for example, Sill (2000) for a more detailed description.

\textsuperscript{2}For example, in an effort to explain the equity premium puzzle, Rietz (1988) introduces a low probability crash state to the two-state Markov-chain model used by Mehra and Prescott (1985). Naik and Lee (1990) add a jump component to the aggregate endowment in a pure-exchange economy, and investigate the equilibrium property. More recently, Liu, Longstaff, and Pan (2002) and Das and Uppal (2001) examine the effect of event risk on investor’s portfolio allocation problem, and Dufresne and Hugonnier (2001) study the impact of event risk on pricing and hedging of contingent claims.
We model the investor’s decision-making concerning both risk and uncertainty by adopting the robust control approach of Anderson, Hansen, and Sargent (2000). In this framework, the agent has two tasks. First, to protect himself against the unreliable aspects of the reference model estimated using existing data, he evaluates his future prospects under alternative models. Second, acknowledging the fact the reference model is indeed the best statistical characterization of the data, he penalizes the choice of the alternative model by how far it deviates from the reference model.

The equilibrium is solved in closed form. Our results show that the total equity premium has three components: the usual risk premia for the diffusive and jump risks, and the uncertainty premium for rare events. While the first two components are generated by the investor’s risk aversion, the last one is linked exclusively to the representative agent’s uncertainty aversion to rare events. These predictions of our model, however, are impossible to test by using the equity return data alone, since the effects of aversions to risk and uncertainty are observationally equivalent in the equity returns.

To investigate the empirical relevance of our model, we turn our attention to the options market. Because of their differential sensitivities to rare events, options — particularly options with varying degrees of moneyness — provide a wealth of information for us to separately identify the three components of the equity premium. Specifically, empirical studies on the S&P 500 index options indicate that options, including at-the-money options, are typically priced with a premium [Jackwerth and Rubinstein (1996)], and this premium component is more pronounced for out-of-the-money puts than for at-the-money options, generating a “smirk” pattern in the cross-sectional plot of option-implied volatility against the option’s strike price [Rubinstein (1994)]. Using joint time-series data on the S&P 500 index and options, Pan (2002) shows that to explain this spectrum of premia implicit in options across moneyness, one has to allow the jump risk to be priced significantly higher than the diffusive price risk. In other words, if risk aversion is the only source for the premia implicit in options, then one has to use a risk-aversion coefficient for the jump risk that is significantly higher than that for the diffusive price risk. By introducing a crash aversion component to the standard power utility framework, Bates (2001) recently proposes a model that can effectively provide a separate coefficient for jump risk, disentangling the market price of jump risk from that of diffusive risk. The economic source of such a crash aversion, however, remains to be explored.

Applying our equilibrium results to the options market, we conduct calibration exercises to examine the extent to which our model can accommodate the high premia associated with the jump risk without having to incorporate an exaggerated risk aversion coefficient for the jump risk. Specifically, we consider models with different levels of uncertainty aversion, and compare them with the standard model without uncertainty aversion. The models are calibrated so that the total equity premium is the same across different models. We find that while the model without uncertainty aversion cannot even begin to explain the extent of the premia implicit in options, the models with uncertainty aversion can generate significant premia for at-the-money option prices, as well as pronounced “smirk” patterns for options with different degrees of moneyness.

Our approach to model uncertainty falls under the general literature that accounts for imprecise knowledge about the probability distribution with respect to the fundamental

First, our investor is worried about model mis-specifications with respect to rare events, while feeling reasonably comfortable with the diffusive component of the model. This differential treatment with respect to the nature of the risk sets our approach apart from that of Anderson, Hansen, and Sargent (2000) in terms of methodology as well as empirical implications. Second, we provide a more general version of the distance measure between the alternative and reference models. The “relative entropy” measure adopted by Anderson, Hansen, and Sargent (2000) is a special limit of our proposed measure. This extended form of distance measure is important in handling uncertainty aversion toward the jump component.3

The rest of the paper is organized as follows. Section 2 sets up the framework of robust control for rare events. Section 3 solves the optimal portfolio and consumption problem for an investor who exhibits aversions to both risk and uncertainty. Section 4 provides the equilibrium results. Section 5 examines the implication of rare event uncertainty on option pricing. Section 6 concludes the paper. Technical details, including proofs of all three propositions, are collected in the appendices.

2 Robust Control for Rare Events

Our setting is that of a pure exchange economy with one representative agent and one perishable consumption good [Lucas (1978)]. As usual, the economy is endowed with a stochastic flow of the consumption good. For the purpose of modeling rare events, we adopt a jump-diffusion model for the rate of endowment flow \( \{Y_t, 0 \leq t \leq T\} \). Specifically, we fix a probability space \((\Omega, \mathcal{F}, P)\) and information filtration \((\mathcal{F}_t)\), and suppose that \(Y\) is a Markov process in \(\mathbb{R}\) solving the stochastic differential equation

\[
dY_t = \mu Y_t dt + \sigma Y_t dB_t + \left( e^{Z_t} - 1 \right) Y_{t-} dN_t,
\]

where \(Y_0 > 0\), \(B\) is a standard Brownian motion and \(N\) is a pure-jump process. In the absence of the jump component, this endowment flow model is the standard geometric Brownian motion with constant mean growth rate \(\mu \geq 0\), and constant volatility \(\sigma > 0\). The jump component adopted here is identical to that in Naik and Lee (1990). Specifically, jump arrivals are dictated by the Poisson process \(N\) with intensity \(\lambda > 0\). Given jump arrival at time \(t\), the jump amplitude is controlled by \(Z_t\), which is normally distributed with mean \(\mu_J\) and standard deviation \(\sigma_J\). Consequently, the mean percentage jump in the endowment flow is \(k = \exp(\mu_J + \sigma_J^2/2) - 1\), given jump arrival. To the spirit of robust control over worse-case

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3Specifically, under the “relative entropy” measure, the robust control problem is not well defined for the jump case. For pure-diffusion models, however, our extended distance measure is equivalent to the “relative entropy” measure.
scenarios, we focus our attention on undesirable event risk. Specifically, we assume \( k \leq 0 \). At different jump times \( t \neq s \), \( Z_t \) and \( Z_s \) are independent, and all three types of random shocks \( B, N, \) and \( Z \) are assumed to be independent.

We deviate from the standard approach by considering a representative agent who, in addition to being risk averse, exhibits uncertainty aversion in the sense of Knight (1921) and Ellsberg (1961). The infrequent nature of the rare events in our setting provides a reasonable motivation for such a deviation. Given his limited ability to assess the likelihood or magnitude of such events, the representative agent considers alternative models to protect himself against possible model mis-specifications. For this purpose, we adopt the robust-control approach of Anderson, Hansen, and Sargent (2000).

To focus on the effect of jump uncertainty, we restrict the representative agent to a pre-specified set of alternative models that differ only in terms of the jump component. Letting \( P \) be the probability measure associated with the reference model (1), the alternative model is defined by its probability measure \( P(\xi) \), where \( \xi_T = dP(\xi)/dP \) is its Radon-Nikodym derivative with respect to \( P \),

\[
d\xi_t = \left( e^{a+bZ_{t-} - bm_j - \frac{1}{2}b^2\sigma_j^2} - 1 \right) \xi_t \, dN_t - (e^a - 1) \lambda \xi_t \, dt ,
\]

where \( a \) and \( b \) are constants and \( \xi_0 = 1 \). By construction, the process \( \{\xi_t, 0 \leq t \leq T\} \) is a martingale of mean one. The measure \( P(\xi) \) thus defined is indeed a probability measure.

Effectively, \( \xi \) changes the agent’s probability assessment with respect to the jump component, without altering his view about the diffusive component. To be more specific, under the alternative measure \( P(\xi) \) defined by \( \xi \), the jump arrival intensity \( \lambda^\xi \) and the mean jump size \( k^\xi \) deviate from their counterparts in the reference measure \( P \) by

\[
\lambda^\xi = \lambda e^a , \quad 1 + k^\xi = (1 + k) e^{b\sigma_j^2} .
\]

A detailed derivation of (3) can be found in Appendix A.

The agent operates under the reference model (1) by choosing \( a = 0 \) and \( b = 0 \), and ventures into other models by choosing some other values for \( a \) and \( b \). Let \( \mathcal{P} \) be the entire collection of such models defined by \( a, b \in \mathbb{R} \). We are now ready to define our agent’s utility when robust control over the set \( \mathcal{P} \) is his concern. For ease of exposition, we start our specification in a discrete-time setting, leaving the derivation of its continuous-time limit to the next section. Fixing the time period at \( \Delta \), we define his time-\( t \) utility recursively by

\[
U_t = \frac{c_t^{1-\gamma}}{1-\gamma} \Delta + e^{-\rho \Delta} \inf_{P(\xi) \in \mathcal{P}} \left\{ \frac{1}{\phi} \psi(U_t) E_t^\xi \left\{ h \left( \ln \frac{\xi_{t+\Delta}}{\xi_t} \right) \right\} + E_t^\xi (U_{t+\Delta}) \right\} \text{ and } U_T = 0 ,
\]

where \( c_t \) is his time-\( t \) consumption, \( \rho > 0 \) is a constant discount rate, and \( \psi(U_t) \) is a normalization factor to be defined shortly.

The specification in (4) implies that any chosen alternative model \( P(\xi) \in \mathcal{P} \) can affect the representative agent in two different ways. On the one hand, in an effort to protect

\footnote{It is also important to notice that while the agent is free to deviate his probability assessment about the jump component, he cannot change the state of nature. That is, an event with zero probability in \( P \) remains so in \( P(\xi) \). In other words, our construction of \( \xi \) in (2) ensures \( P \) and \( P(\xi) \) to be equivalent measures.}
himself against model uncertainty associated with the jump component, the agent evaluates his future prospect $E^\xi_t (U_{t+1})$ under alternative measures $P(\xi) \in \mathcal{P}$. Naturally, he focuses on other jump models that provide worse prospects than the reference models $P$. Hence the infimum over $P(\xi) \in \mathcal{P}$ in equation (4). On the other hand, he knows that statistically $P$ is the best representation of the existing data. With this in mind, he penalizes his choice of $P(\xi)$ according to how much it deviates from the reference $P$. This discrepancy or distance measure is captured in this paper by $E^\xi_t [h(\ln (\xi_{t+1}/\xi_t))]$, where for some $\beta > 0$ and any $x \in \mathbb{R}$,
\begin{equation}
    h(x) = x + \beta (e^x - 1). \tag{5}
\end{equation}
Intuitively, the further away the alternative model is from the reference model $P$, the bigger is this distance measure. Conversely, when the alternative model is the reference model, we have $\xi \equiv 1$ with a distance measure of 0. Finally, to control this tradeoff between “impact on future prospects” and “distance from the reference model,” we introduce a constant parameter $\phi > 0$ in (4). With a higher $\phi$, the agent puts less weight on how far away the alternative model is from the reference model, and, effectively, more weight on how it would worsen his future prospect. In other words, an agent with higher $\phi$ exhibits higher aversion to model uncertainty.

By specifying the agent’s utility in terms of (4), we adopt the robust control framework of Anderson, Hansen, and Sargent (2000). Our approach, however, differs from theirs in two important ways. First, we restrict the agent to a pre-specified set $\mathcal{P}$ of alternative models that differ from the reference model only in their jump components. As a result, the uncertainty aversion exhibited by the agent only applies to the jump component of the model. This distinction becomes important as we later take the model to option pricing because options are sensitive to diffusive shocks and jumps in different ways.

In fact, we can further apply this idea and modify the set $\mathcal{P}$ so that the agent can express his uncertainty aversion toward one specific part of the jump component. For example, by restricting $b = 0$ in the definition (2) of $\xi$, we build a subset $\mathcal{P}^a \subset \mathcal{P}$ of alternative models that are different from the reference model only in terms of the likelihood of jump arrival. Applying this subset to the utility definition of (4), we effectively assume that the agent has doubt about the jump-timing aspect of the model, while he is comfortable with the jump-magnitude part of the model. Similarly, by letting $a = 0$ in (2), we build a class $\mathcal{P}^b$ of alternative models that are different from the reference model only in terms of jump size. An agent who searches over $\mathcal{P}^b$ instead of $\mathcal{P}$ finds the jump-magnitude aspect of the model unreliable, while having full faith in the jump-timing aspect of the model. Finally, by letting $a = 0$ and $b = 0$, we reduce the set $\mathcal{P}^{ab}$ to a singleton that contains only the reference model. Effectively, this is the standard case of a risk-averse investor with no uncertainty aversion.

Second, we extend the discrepancy (or distance) measure of Anderson, Hansen, and Sargent (2000) to a more general form. Specifically, our “extended entropy” measure reduces to their “relative entropy” when $\beta$ approaches to zero. Given that $h(x)$ is convex and $h(0) = 0$, the result of Wang (2001) can be used to provide an axiomatic foundation for our specification (his Theorem 5.1, part a). As it will become clear later, this extended form of distance measure is important in handling uncertainty aversion toward the jump component. In particular, the robust control problem specified in (4) does not have an interior global
minimum for the “relative entropy” case. For pure diffusion models, however, it is easy to show that our extended distance measure is equivalent to the “relative entropy” case.

Finally, following Maenhout (2001), we introduce a normalization factor $\psi(U)$ for analytical tractability. To keep the penalty term positive, we let $\psi(U) = (1 - \gamma)U$ for $\gamma \neq 1$, and $\psi(U) = 1$ for the log-utility case.

3 The Optimal Consumption and Portfolio Choice

As in the standard setting, there exists a market where shares of the aggregate endowment are traded as stocks. At any time $t$, the dividend payout rate of the stock is $Y_t$, and the ex-dividend price of the stock is denoted by $S_t$. In addition, there is a riskfree bond market with instantaneous interest rate $r_t$. The investor starts with a positive initial wealth $W_0$, and trades competitively in the securities market and consumes the proceeds. At any time $t$, he invests $\theta_t$ fraction of his wealth in the stock market, $1 - \theta_t$ in the riskfree bond, and consumes $c_t$, satisfying the usual budget constraint.

Having the equilibrium solution in mind, we consider stock prices of the form $S_t = A(t)Y_t$ and constant riskfree rate $r$, where $A(t)$ is a deterministic function of $t$ with $A(T) = 0$. Under the reference measure $P$, the stock price follows,

$$dS_t = \left(\mu + \frac{A'(t)}{A(t)}\right)S_t dt + \sigma S_t dB_t + (e_t^Z - 1) S_t dN_t$$

(6)

And the budget constraint of the investor becomes

$$dW_t = \left[r + \theta_t \left(\mu - r + \frac{1 + A'(t)}{A(t)}\right)\right] W_t dt + \theta_t W_t \sigma dB_t + \theta_t - W_t - (e_t^Z - 1) dN_t - c_t dt.$$  

(7)

Given this budget constraint, our investor’s problem is to choose his consumption and investment plans $\{c, \theta\}$ so as to optimize his utility. Let $J_t$ be the indirect utility function of the investor,

$$J(t, W) = \sup_{\{c, \theta\}} U_t,$$

(8)

where $U_t$ is the continuous-time limit of the utility function defined by (4). The following proposition provides the Hamilton-Jacobi-Bellman (HJB) equation for $J$.

**Proposition 1** The investor’s indirect utility $J$, defined by (8), has the terminal condition $J(T, W) = 0$ and satisfies the following HJB equation,

$$\sup_{c, \theta} \left\{ u(c) - r J(t, W) + A J(t, W) + \inf_{a, b} \left\{ \lambda e^a \left(E^{Z(b)}[J(t, W + (1 + (e^Z - 1) \theta))] - J(t, W)\right) \right. \right.$$  

$$+ \frac{1}{\phi} \psi(J) \lambda \left[1 + \left(a + \frac{1}{2} b^2 \sigma^2 - 1\right) e^a + \beta \left(1 + \left(e^{a+b^2 \sigma^2} - 2\right) e^a\right)\right] \left. \right\} = 0,$$

(9)

5Roughly speaking, the penalty function in Anderson, Hansen, and Sargent (2000) is not strong enough to counter-balance the “loss in future prospect” for an agent with risk-aversion coefficient $\gamma > 1$. As a result, the investor’s concern about a mis-specification in the jump magnitude makes him go overboard to the case of total ruin.
where $E^Z(b)(\cdot)$ denotes the expectation with respect to $Z$ under the alternative measure associated with $b$. That is, for any function $f$

$$E^Z(b)(f(Z)) = E\left(e^{bZ - b\mu J - \frac{1}{2}b^2J} f(Z)\right).$$

(10)

The term $AJ(t,W)$ in the HJB equation (9) is the usual infinitesimal generator for the diffusion component of the wealth dynamics,

$$AJ = J_t + \left[ r + \theta \left( \mu - r + \frac{A'(t) + 1}{A(t)} \right) \right] W J_W - c J_W + \frac{\sigma^2}{2} \theta^2 W^2 J_{WW},$$

(11)

where $J_t$ is the derivative of the indirect utility $J$ with respect to $t$, and $J_W$ and $J_{WW}$ are its first and second derivatives with respective to $W$.

The intuition behind the HJB equation (9) parallels exactly that of its discrete time counterpart, equation (4). Specifically, compared with the standard HJB equation for jump-diffusions, the HJB equation in (9) has two important modifications. First, the risk associated with the jump component is evaluated at all possible alternative models indexed by $(a,b)$, reflecting the investor’s precaution against model uncertainty with respect to the jump component. Second, it incorporates an additional term in the second line of (9), penalizing the choice of the alternative model by its distance from the reference model. The following proposition provides the solution to the HJB equation.

**Proposition 2** Assuming existence, the solution to the HJB equation is given by

$$J(t,W) = \frac{W^{1-\gamma}}{1-\gamma} f(t)^\gamma,$$

(12)

where $f(t)$ is a time-dependent coefficient satisfying the ordinary differential equation (B.4) in Appendix B with terminal condition $f(T) = 0$. The optimal consumption plan is given by $c^*_t = W^*_t/f(t)$, where $W^*$ is the optimal wealth process. Finally, the optimal solutions $\theta^*$, $a^*$ and $b^*$ satisfy

$$\left( \mu - r + \frac{1 + A'(t)}{A(t)} \right)\theta - \lambda e^a \left( 1 + (e^Z - 1)\theta \right)^{-\gamma} (e^Z - 1) = 0, \quad (13)$$

$$\frac{1}{\phi} \left( a + \frac{1}{2} b^2 \sigma^2 \right) + 2\beta \left( e^{a+b^2\sigma^2} - 1 \right) + E^Z(b) \left[ (1 + (e^Z - 1)\theta)^{1-\gamma} - 1 \right] = 0, \quad (14)$$

$$\frac{1-\gamma}{\phi} a \sigma^2 \left( 1 + 2\beta e^{a+b^2\sigma^2} \right) + \frac{\partial}{\partial b} E^Z(b) \left[ (1 + (e^Z - 1)\theta)^{1-\gamma} \right] = 0, \quad (15)$$

where $E^Z(b)(\cdot)$ defined in (10) is the expectation with respect to $Z$ under the alternative measure associated with $b$.

4 Market Equilibrium

In equilibrium, the representative agent invests all of his wealth in the stock market $\theta_t = 1$ and consumes the aggregate endowment $c_t = Y_t$ at any time $t \leq T$. The solution to market equilibrium and the pricing kernel are summarized by the following proposition.
Proposition 3 In equilibrium, the total (cum-dividend) equity premium is
\[
\text{total equity premium} = \gamma \sigma^2 + \lambda k - \lambda^Q k^Q,
\] (16)
where \(k = \exp(\mu J + \sigma^2_J/2) - 1\) is the mean percentage jump size of the aggregate endowment, and \(\lambda^Q\) and \(k^Q\) are defined by\(^6\)
\[
\lambda^Q = \lambda \exp \left(-\gamma \mu J + \frac{1}{2} \gamma^2 \sigma^2_J + a^* - b^* \gamma \sigma^2_J \right), \quad k^Q = (1 + k) \exp ((b^* - \gamma) \sigma^2_J) - 1,
\] (17)
and \(a^*\) and \(b^*\) are the solution of the following non-linear equations:
\[
a + \frac{1}{2} b^2 \sigma^2_J + 2 \beta \left(e^{a^* + b^* \sigma^2_J} - 1\right) + \frac{\phi}{1 - \gamma} \left(\left(1 + k\right) e^{(b^* - \frac{1}{2} \gamma) \sigma^2_J}\right)^{1-\gamma} - 1 = 0 \quad (18)
\]
\[
b \left(1 + 2 \beta e^{a^* + b^* \sigma^2_J}\right) + \phi \left(1 + k\right) e^{(b^* - \frac{1}{2} \gamma) \sigma^2_J} - \left(1 - \frac{1 - \gamma}{\phi} \lambda \left[1 + \left(a^* + \frac{1}{2} (b^*)^2 \sigma^2_J - 1\right) e^{a^*} + \beta \left(1 + \left(e^{a^* + (b^*)^2 \sigma^2_J} - 2\right) e^{a^*}\right)\right]\right)^{1-\gamma} = 0 \quad (19)
\]
The equilibrium riskfree rate \(r\) is
\[
r = \rho + \gamma \mu - \frac{1}{2} \gamma \left(\gamma + 1\right) \sigma^2 + \lambda^* \left(1 - (1 + k^*)^{-\gamma} e^{\frac{1}{2} \gamma (1 + \gamma) \sigma^2_J}\right)
\]
\[
- \frac{1 - \gamma}{\phi} \lambda \left[1 + \left(a^* + \frac{1}{2} (b^*)^2 \sigma^2_J - 1\right) e^{a^*} + \beta \left(1 + \left(e^{a^* + (b^*)^2 \sigma^2_J} - 2\right) e^{a^*}\right)\right],
\] (20)
where \(\lambda^* = \lambda \exp(a^*)\) and \(k^* = (1 + k) \exp ((b^*)^2 \sigma^2_J) - 1\). Finally, the equilibrium pricing kernel is given by
\[
d\pi_t = - r \pi_t dt \quad - \gamma \sigma \pi_t dB_t + \left(e^{a^* + (b^* - \gamma) Z - b^* \mu_J - \frac{1}{2} (b^*)^2 \sigma^2_J} - 1 - \pi_t \pi_t^- dN_t \right)
\]
\[
- \lambda \left(e^{a^* - (\mu_J + b^* \sigma^2_J) + \frac{1}{2} \gamma^2 \sigma^2_J} - 1\right) \pi_t dt.
\] (21)

To understand how the investor’s uncertainty aversion affects the equilibrium asset prices, let’s first take away the feature of uncertainty aversion by setting \(a \equiv 0\) and \(b \equiv 0\), or \(\phi \to 0\). Our results in (16) and (20) are then reduced to those of Naik and Lee (1990) — the standard case of a risk-averse investor with no uncertainty aversion. In this case, the total equity premium is attributed exclusively to risk aversion:
\[
\text{diffusive risk premium} = \gamma \sigma^2; \quad \text{jump risk premium} = \lambda k - \lambda \bar{k}, \quad (22)
\]
where \(\bar{\lambda}\) and \(\bar{k}\) are the counterparts of \(\lambda^Q\) and \(k^Q\) when the uncertainty aversion \(\phi\) is set to zero:
\[
\bar{\lambda} = \lambda \exp \left(-\gamma \mu J + \frac{1}{2} \gamma^2 \sigma^2_J \right), \quad \bar{k} = (1 + k) \exp (-\gamma \sigma^2_J) - 1.
\] (23)
Quite intuitively, both types of risk premia approach zero when the risk-aversion coefficient \(\gamma\) approaches zero, and are positive for any risk-averse investors (\(\gamma > 0\)).

\(^6\)As will become clear in the next section, \(\lambda^Q\) and \(k^Q\) are the risk-neutral counterparts of \(\lambda\) and \(k\).
When the investor exhibits uncertainty aversion ($\phi > 0$), there is one additional component in the equity premium:

$$\text{rare event premium} = \bar{\lambda} \bar{k} - \lambda^Q k^Q.$$  \hfill (24)

It is important to emphasize that while the magnitude of this part of equity premium depends on the risk aversion parameter of the investor, it is the uncertainty aversion of the investor that gives rise to this premium. Specifically, the rare event premium remains positive even when we take the limit $\gamma \to 0$, while it becomes zero when the investor’s model uncertainty aversion $\phi$ approaches zero. The following two examples highlight this feature of the rare event premium by considering the extreme case where the investor is risk neutral ($\gamma = 0$).

In the first case, the investor is worried about model mis-specification with respect to the jump arrival intensity, i.e., how frequent the jumps occur. He performs robust control by searching over the subset $\mathcal{P}^a$ defined by $a \in R$ and $b \equiv 0$. Setting $b = 0$ and $\gamma = 0$, equation (18) reduces to

$$a + 2\beta (e^a - 1) + \phi k = 0. \hfill (25)$$

Focusing our discussion for the case of adverse event risk ($k < 0$), we can see from (25) that $a^* > 0$ if and only if the investor exhibits uncertainty aversion ($\phi > 0$). The rare event premium in this case is

$$\bar{\lambda} \bar{k} - \lambda^Q k^Q = \lambda k (1 - e^{a^*}),$$

which is positive if and only if $\phi > 0$.

In the second case, the investor is worried about model mis-specification with respect to the jump size. This time, he performs robust control by searching over the subset $\mathcal{P}^b$ defined by $b \in R$ and $a \equiv 0$. Setting $a = 0$ and $\gamma = 0$, equation (19) reduces to

$$b = -\frac{\phi}{1 + 2\beta e^{b\sigma_j^2}}, \hfill (26)$$

which indicates that $b^* < 0$ when there is uncertainty aversion ($\phi > 0$). The rare event premium in this case is

$$\bar{\lambda} \bar{k} - \lambda^Q k^Q = \lambda (1 + k) e^{b^* \sigma_j^2},$$

which is again positive if and only if $\phi > 0$.

These two cases are the simplest examples of our more general results. Other than serving to provide some important intuition behind our results, they also deliver a quite important point. That is, the aversion toward model uncertainty is independent of that toward risk, and the effect of uncertainty aversion becomes most prominent with respect to rare events. Indeed, the fact that our model allows such separate of total equity premium into risk and rare event components is crucial for our analysis. As emphasized in the introduction, our contention is that rare events are treated differently from the more common events by investors and such differential treatment will be reflected in asset prices. The decomposition of equity premium characterized in Proposition 3 allows us to study the effect on prices and can potentially lead to empirically testable implications with respect to the different components of the equity premium.
To elaborate on the last point and set the stage of the next section, we note that if there is no model uncertainty or if the investor is uncertainty neutral ($\phi = 0$), then according to equations (22) and (23), both diffusive and jump risk premia are linked by just one risk-aversion coefficient $\gamma$. This constraint can in fact be tested using securities with different sensitivities to the diffusive and jump risks. Indeed, using a joint time-series of spot and option prices at the aggregate level (the S&P 500 index and option), Pan (2002) shows that the jump risk is priced quite differently from the diffusive risk. In particular, the “data-implied $\gamma$” for the jump risk is considerably larger than that for the diffusive risk, indicating a premium structure beyond that generated by the standard expected utility model.

5 The Rare Event Premia in Options

To further disentangle the rare event premia from the standard risk premia, we turn our attention to the options market. Using the equilibrium pricing kernel $\pi$ (Proposition 3), we can readily price any derivative securities in this economy. Specifically, let $Q$ be the risk-neutral measure defined by the equilibrium pricing kernel $\pi$ such that $\pi_T = dQ/dP$. It can be shown that the risk-neutral dynamics of the ex-dividend stock price follows:

$$dS_t = (r-q)S_t dt + \sigma S_t dB^Q_t + (e^{Z_t} - 1) S_t dN_t - \lambda^Q k^Q dt,$$

(27)

where $r$ is the riskfree rate and $q$ is the dividend payout rate, and where under $Q$, $B^Q$ is a standard Brownian motion and $N_t$ is a Poisson process with intensity $\lambda^Q$, and given jump arrival at time $t$, the percentage jump amplitude is log-normally distributed with mean $k^Q$. Both risk-neutral parameters $\lambda^Q$ and $k^Q$ are defined earlier in (17). European-style option pricing for this model is a modification of the Black and Scholes (1973) formula, and has been established in Merton (1976). For completeness of the paper, the pricing formula is provided in Appendix C.

What makes the option market valuable for our analysis is that, unlike equity, options have different sensitivities to diffusions and jumps. For example, a deep out-of-the-money put option is extremely sensitive to negative price jumps, but exhibits little sensitivity to diffusive price movements. This non-linear feature inherent in the option market enables us to disentangle the three components of the total equity premium (Proposition 3) that are otherwise impossible to separate using equity returns alone. This “observational equivalence” with respect to equity returns is further illustrated in Table 1.

Table 1 details a simple calibration exercise with parameters for the reference model $P$ set as follows. For the diffusive component, the volatility is set at $\sigma = 15\%$; for the jump

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7 For the rest of our analysis, we will set the riskfree rate at $r = 5\%$, and the dividend yield at $q = 3\%$. In other words, we are not using the equilibrium interest rate and the dividend yield. This is without much loss of generality. Specifically, the parameter $\rho$ can be used to match the desired level of $r$. The dividend payout ratio $q$ is slightly more complicated, since it is in fact time-varying in our setting. For an equilibrium horizon $T$ that is sufficiently large compared to the maturity of the options to be considered, we can use the result for the infinite horizon case, and take $q = 1/\alpha$, where $\alpha$, given by (B.6), can be calibrated by the free parameter $\mu$. Finally, as our analysis focuses on comparing the prices of options with different moneyness, the effect of $r$ and $q$ will be minor as long as the same $r$ and $q$ are used to price all options.
Table 1: The three components of the equity premium, jump case 1

<table>
<thead>
<tr>
<th>jump parameters</th>
<th>aversion</th>
<th>diffusive risk</th>
<th>jump risk</th>
<th>rare event premium</th>
<th>the total premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 1/3 )</td>
<td>( \phi )</td>
<td>3.47</td>
<td>7.80</td>
<td>0.20</td>
<td>0</td>
</tr>
<tr>
<td>( \mu_J = -1% )</td>
<td>10</td>
<td>3.15</td>
<td>7.09</td>
<td>0.19</td>
<td>0.72</td>
</tr>
<tr>
<td>20</td>
<td>2.62</td>
<td>5.91</td>
<td>0.15</td>
<td>1.94</td>
<td></td>
</tr>
</tbody>
</table>

component, the arrival intensity is \( \lambda = 1/3 \), and the random jump amplitude is normal with mean \( \mu_J = -1\% \) and standard deviation \( \sigma_J = 4\% \). Given this reference model, three different scenarios are considered for the representative agent’s risk aversion \( \gamma \) and uncertainty aversion \( \phi \). As shown in Table 1, each scenario corresponds to an economy with a distinct level of uncertainty aversion \( \phi \), and yields a distinct composition of the diffusive-risk premium, the jump-risk premium, and the rare event premium. For example, the rare event premium is zero when the representative agent exhibits no aversion to model uncertainty, and increases to 1.94% per year when the uncertainty aversion coefficient becomes \( \phi = 20 \). These predictions of our model, however, cannot be tested if we focus only on the equity return data. As shown in Table 1, for a fixed level of uncertainty aversion \( \phi \), one can always adjust the level of risk aversion \( \gamma \) so that the total equity premium is fixed at 8% a year, although the economic sources of the respective equity premium differ significantly from one scenario to another. To be able to break the total equity premium into its three components, we need to take our model one step further to the options data.

To examine the option pricing implication of our model, we start with the same reference model and the same set of scenarios of uncertainty aversion as those considered in Table 1. For each scenario, we use our equilibrium model to price one-month European-style options, both calls and puts, with the ratio of strike to spot prices varying from 0.9 to 1.1. As it is standard in the literature, we quote the option prices in terms of Black-Scholes implied volatility (BS-vol), and plot them against the respective ratios of strike to spot prices. The first panel of Figure 1 reports the “smile” curves generated by the three equilibrium models with varying degrees of uncertainty aversion. We can see that although all three scenarios are observationally equivalent with respect to the equity market, their implications on the options market are notably different.

Focusing first on the case of zero uncertainty aversion, we recall from Table 1 that, in this case, the equity premium has only two components, both of which are driven by the representative agent’s risk aversion \( \gamma \). To understand the impact of risk aversion on the equilibrium option prices, we use the case with zero aversion to risk or uncertainty as a reference. For the at-the-money (ATM) option (both puts and calls with strike-to-spot ratio of 1), both cases yield a BS-vol of 15.2%, which is very close in magnitude to the total market volatility \( \sqrt{\sigma^2 + \lambda (\mu_J^2 + \sigma_J^2)} = 15.2\% \).

As we move to out-of-the-money (OTM) put options, however, the impact of the repre-

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8These jump parameters are close to those reported by Pan (2002) for the S&P 500 index. Alternative jump parameters will be considered in later examples.
Figure 1: The equilibrium “smile” curves.
sentative agent’s risk aversion is slightly more notable. In particular, for a 10% OTM put option (with strike-to-spot ratio of 0.9), the equilibrium model with only risk aversion (that is, $\phi = 0$) sets its price at 15.6% BS-vol, while the reference model (with $\gamma = 0$ and $\phi = 0$) sets its price at 15.5%. By incorporating risk aversion, the equilibrium model does generate a “smile” curve that is slightly more skewed than the reference model. Compared with the pronounced “smirk” pattern observed for options on the aggregate market,9 however, this magnitude is remarkably low. To generate a more pronounced “smirk” pattern, one could increase the level of risk aversion $\gamma$, but at the same time, it would imply an extremely high level of diffusive-risk premium for the underlying equity. In other words, one could potentially use risk aversion to explain the premium exhibited in the aggregate equity market, or the premia implicit in options with varying degrees of moneyness. But to explain both markets simultaneously, the equilibrium model with risk aversion alone runs into trouble.

Next we consider the two cases that incorporate the representative agent’s uncertainty aversion. As shown in Table 1, in both cases the total equity premium has three components, two of which are driven by the representative agent’s risk aversion $\gamma$, one—the rare event premium—is driven by his uncertainty aversion $\phi$. Comparing the case of $\phi = 20$ with the previously discussed case of $\phi = 0$, our first observation is that, even for at-the-money options, the two models generate different equilibrium prices. Specifically, for the case of zero uncertainty aversion, the BS-vol implied by an ATM option is 15.2%, but for the case of uncertainty aversion $\phi = 20$, the BS volatility implied by an ATM option is 15.5%.

This implies that, while both cases are observationally equivalent when viewed using equity prices, the model incorporating uncertainty aversion ($\phi = 20$) predicts a premium of about 2% for one-month ATM options. This result is indeed consistent with the empirical fact that options, even those that are at the money, are priced with a premium.10 Our results show that, to a large extent, this premium is linked exclusively to the investor’s uncertainty aversion toward rare events. In fact, this premium becomes even more pronounced as we move to OTM puts—options that are highly sensitive to adverse rare events. Specifically, the first panel in Figure 1 shows that a 10% OTM put option is priced at 17.2% BS-vol, compared with 15.6% BS-vol in the case of $\phi = 0$. That is, for every dollar invested in a one-month 10% OTM put option, typically used as a protection against rare events, the investor is willing to pay 10 cents more because of his uncertainty aversion toward the adverse rare events.

As shown in Pan (2002), both empirical facts—ATM options priced with a premium, and OTM put options priced with an even higher premium, resulting in a pronounced “smirk” pattern—are indeed closely connected. If only risk aversion is employed to explain these empirical facts, one direct implication is that the “data-implied $\gamma$” for the jump risk has to be considerably larger than that for the diffusive risk. By incorporating uncertainty aversion in this paper, however, we are able to explain these empirical facts without having to incorporate an exaggerated risk aversion coefficient for the jump risk. By doing so, we offer a simple explanation for the significant premium implicit in options, especially those

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9Rubinstein (1994) was one of the earlier papers in literature documenting the “smile” patterns in the option market. Among others, the cross-sectional “smile” patterns of index options are further examined by Bates (2000) and Bakshi, Cao, and Chen (1997).

10See, for example, Jackwerth and Rubinstein (1996) and Pan (2002).
put options that are deep out of the money. That is, when it comes to rare events, the investors simply do not have a reliable model. They react by assigning rare event premia to each financial security that is sensitive to rare events. Options with varying moneyness are sensitive to the rare events in a variety of ways, bearing different levels of rare event premia. Our analysis shows that a significant portion of the pronounced “smirk” pattern can be attributed to this varying degree of rare event premia implicit in options.

Table 2: The three components of the equity premium, jump case 2

<table>
<thead>
<tr>
<th>jump parameters</th>
<th>aversion</th>
<th>premia (%)</th>
<th>the total premium</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi$</td>
<td>$\gamma$</td>
<td>diffusive risk</td>
</tr>
<tr>
<td>$\lambda = 1/25$</td>
<td>0</td>
<td>3.47</td>
<td>7.81</td>
</tr>
<tr>
<td>$\mu_J = -10%$</td>
<td>10</td>
<td>2.88</td>
<td>6.47</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.61</td>
<td>3.62</td>
</tr>
</tbody>
</table>

Table 3: The three components of the equity premium, jump case 3

<table>
<thead>
<tr>
<th>jump parameters</th>
<th>aversion</th>
<th>premia (%)</th>
<th>the total premium</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi$</td>
<td>$\gamma$</td>
<td>diffusive risk</td>
</tr>
<tr>
<td>$\lambda = 1/100$</td>
<td>0</td>
<td>3.47</td>
<td>7.81</td>
</tr>
<tr>
<td>$\mu_J = -20%$</td>
<td>10</td>
<td>2.36</td>
<td>5.31</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.68</td>
<td>1.54</td>
</tr>
</tbody>
</table>

Finally, to show the robustness of our results, we modify the two key jump parameters, $\lambda$ and $\mu_J$, in the reference model considered in Table 1. In Table 2, we consider jumps that happen once every 25 years, with a mean magnitude of $-10\%$, capturing the magnitude of major market corrections. In Table 3, jumps happen once every 100 years with a magnitude of $-20\%$, capturing the magnitude of an event as rare as the 1987 crash. The option pricing implications of these models are reported in the lower two panels in Figure 1. As we can see, although all three reference models incorporate rare events that are very different in intensity and magnitude, the impact of uncertainty aversion remains qualitatively similar.

6 Conclusion

Motivated by the observation that models with rare events are easy to build but hard to estimate, we developed in this paper a framework to formally address the following questions with respect to rare events. First, is it possible that rare events are treated somewhat differently by investors than the more common shocks are? If so, what are the economic sources for this differential treatment? Could it be because of the model uncertainty associated with rare events? If so, what are the equilibrium asset-pricing implications?
We modified the standard pure-exchange economy by adding jumps as rare events, and by allowing the representative agent to perform robust control (in the sense of Anderson, Hansen, and Sargent (2000)) as a precaution against possible model mis-specification with respect to rare events. We provided an explicitly solved equilibrium. In our equilibrium, the total equity premium has three components: the diffusive risk premium, the jump risk premium, and the rare event premium. In such a framework, the standard model with only risk aversion becomes a special case with over-identifying restrictions on the three components of the total equity premium. Specifically, in the standard model, the first two components of the total equity premium are tied together by one parameter: the representative agent’s risk aversion, while the rare event component is absent. This presents an interesting testable implications for both the equity premium and the premia for options with different degrees of moneyness. Obviously, a full empirical investigation is beyond this paper. Nevertheless, the calibration exercises performed in this paper suggest that the model uncertainty inherent in rare events, coupled with the investor’s aversion toward such uncertainty, plays an important role in explaining the premia implicit in the options market. In particular, without incorporating uncertainty aversion, our model, which reduces to the standard model, cannot generate the magnitude of the “smirk” patterns that are prevalent in the index options market. By contrast, incorporation of uncertainty aversion toward rare events contributes significantly to the steepness of the spectrum of premia implicit in options of varying degrees of moneyness, while simultaneously matching the level of total equity premium in the data.
Appendices

A Changes of Probability Measures for Jumps

We first derive the arrival intensity \( \lambda^\xi \) of the Poisson process under the new probability measure \( P(\xi) \). Let

\[
dM = dN_t - \lambda dt
\]

be the compensated Poisson process, which is a \( P \)-martingale. Applying the Girsanov theorem for point processes (see, for example, Elliott (1982)), we have

\[
dM^{P(\xi)} = dM_t - E \left[ \exp(a + bZ_t - \frac{1}{2} b^2 \sigma^2_t - 1) \right] \lambda dt = dM_t - (e^a - 1) \lambda dt = dN_t - \lambda^\xi dt
\]

where \( \lambda^\xi = \lambda \exp(a) \), as given in (3).

Next we derive the mean percentage jump size \( k^\xi \) under \( P(\xi) \). Let

\[
dM = (e^Z - 1) S_t dN_t - k S_t \lambda dt
\]

be the compensated pure-jump process, which is a \( P \)-martingale. Applying the Girsanov theorem, we have

\[
dM^{P(\xi)} = dM_t - E \left[ \left( e^{a+bZ_t - \frac{1}{2} b^2 \sigma^2_t} - 1 \right) \right] S_t \lambda dt = (e^Z - 1) S_t dN_t - k^\xi S_t \lambda^\xi dt
\]

where \( k^\xi = (1 + k) \exp(b \sigma^2_t) - 1 \), as given in (3).

B Proofs of Propositions

Proof of Proposition 1: Given zero bequest motive, it must be that \( J(T, W) = 0 \). The derivation of the HJB equation involves applications of Ito’s lemma for jump-diffusion processes. The derivation is standard except for the penalty term. In particular, we need to calculate the continuous-time limit of the “extended entropy” measure. For this, we first let

\[
E_t^\xi \left[ h \left( \ln \left( \frac{\xi_{t+\Delta}}{\xi_t} \right) \right) \right] = E_t \left[ \frac{\xi_{t+\Delta}}{\xi_t} h \left( \ln \left( \frac{\xi_{t+\Delta}}{\xi_t} \right) \right) \right] = E_t \left( \frac{\xi_{t+\Delta}}{\xi_t} \ln \left( \frac{\xi_{t+\Delta}}{\xi_t} \right) \right) + \beta E_t \left( \frac{\xi_{t+\Delta}}{\xi_t} \ln \left( \frac{\xi_{t+\Delta}}{\xi_t} - 1 \right) \right)
\]

\[
= \frac{1}{\xi_t} E_t \left( \xi_{t+\Delta} \ln \xi_{t+\Delta} - \xi_t \ln \xi_t \right) + \beta \frac{1}{\xi_t^2} E_t \left( \xi_{t+\Delta}^2 - \xi_t^2 \right), \quad (B.1)
\]

where we used the martingale property \( E_t (\xi_{t+\Delta}) = \xi_t \) of the Radon-Nikodym process \{\xi\}. Applying Ito’s lemma to the processes \{\xi \ln \xi\} and \{\xi^2\} separately, it is a straightforward calculation to show that

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \frac{1}{\xi_t} E_t \left( \xi_{t+\Delta} \ln \xi_{t+\Delta} - \xi_t \ln \xi_t \right) = \lambda \left( 1 + \left( a + \frac{1}{2} \sigma^2_t b^2 - 1 \right) e^a \right)
\]

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \frac{1}{\xi_t^2} E_t \left( \xi_{t+\Delta}^2 - \xi_t^2 \right) = \lambda \left( 1 + \left( e^{a+b^2 \sigma^2_t} - 2 \right) e^a \right).
\]
Proof of Proposition 2: We conjecture that the solution to the HJB equation is indeed of the form (12). The first order condition for \( c \) becomes
\[
c = f^{-1}(t)W
\] (B.2)
Substituting (12) and (B.2) into the HJB equation, we have
\[
\sup_{c, \theta} \left\{ \gamma \frac{1 + f'(t)}{1 - \gamma} f(t) - \frac{\rho}{1 - \gamma} + r + \theta \left( \mu - r + \frac{A'(t) + 1}{A(t)} \right) - \frac{1}{2} \gamma \sigma^2 \theta^2 
+ \inf_{a, b} \left\{ \frac{1}{\phi} \lambda \left[ 1 + \left( a + b^{*2} \sigma_j^2 - 1 \right) e^{a*} + \beta \left( 1 + \left( e^{a*+b^{*2} \sigma_j^2} - 2 \right) e^{a*} \right) \right] 
+ \frac{1}{1 - \gamma} \lambda e^{a*} \left( E^{Z(b)} \left[ (1 + (e^{Z} - 1)\theta)^{1-\gamma} \right] - 1 \right) \right\} \right\} = 0.
\] (B.3)
The first order conditions in \( \theta, a, \) and \( b \) give equations (13), (14) and (15), respectively. Substituting the solutions \( \theta^*, a^* \) and \( b^* \) back to equation (B.3), we obtain the ordinary differential equation for \( f(t) \),
\[
\frac{\gamma}{1 - \gamma} \frac{1 + f'(t)}{f(t)} - \frac{\rho}{1 - \gamma} + r + \theta^* \left( \mu - r + \frac{A'(t) + 1}{A(t)} \right) - \frac{1}{2} \gamma \sigma^2 \left( \theta^* \right)^2 
+ \frac{1}{\phi} \lambda \left[ 1 + \left( a^* + \frac{1}{2} (b^*)^{2} \sigma_j^2 - 1 \right) e^{a^*} + \beta \left( 1 + \left( e^{a^*+(b^*)^2 \sigma_j^2} - 2 \right) e^{a^*} \right) \right] 
+ \frac{1}{1 - \gamma} \lambda e^{a^*} \left( E^{Z(b^*)} \left[ (1 + (e^{Z} - 1)\theta^*)^{1-\gamma} \right] - 1 \right) = 0.
\] (B.4)
Proof of Proposition 3: Applying the equilibrium condition \( \theta = 1 \) to the first order conditions (14) and (15), we immediately obtain the equations (18) and (19) for the optimal \( a^* \) and \( b^* \).
Next, the equilibrium conditions of \( S_t = W_t \) and \( c_t = Y_t \) imply \( A(t) = f(t) \). The ordinary differential equation (B.4) becomes
\[
A'(t) = \frac{A(t)}{\alpha} - 1.
\] (B.5)
where the constant coefficient \( \alpha \) is defined by
\[
\frac{1}{\alpha} = \rho - (1 - \gamma) \mu + \frac{\sigma^2}{2} \gamma (1 - \gamma) - \lambda e^{a^*} \left( e^{(1-\gamma)(\mu_j+b^* \sigma_j^2)} + \frac{1}{2} (1-\gamma)^2 \sigma_j^2 - 1 \right) 
- \frac{1 - \gamma}{\phi} \lambda \left[ 1 + \left( a^* + \frac{1}{2} (b^*)^{2} \sigma_j^2 - 1 \right) e^{a^*} + \beta \left( 1 + \left( e^{a^*+(b^*)^2 \sigma_j^2} - 2 \right) e^{a^*} \right) \right].
\] (B.6)
Under the terminal condition $A(T) = 0$, $A(t)$ can be solved uniquely,

$$A(t) = \alpha \left( 1 - \exp \left( -\frac{T-t}{\alpha} \right) \right).$$

The first order condition (13) evaluated at $\theta = 1$ gives,

$$\mu + \frac{1}{\alpha} = r + \gamma \sigma^2 - \lambda e^{\alpha x} \left( e^{(1-\gamma) b^* \sigma^2 + (1-\gamma) j \frac{\sigma^2}{2} + (1-\gamma) \mu_j} - e^{-\gamma b^* \sigma^2 + \gamma j \frac{\sigma^2}{2} - \gamma \mu_j} \right). \quad (B.7)$$

Using equations (B.5) and (B.7), it is a straightforward calculation to show that the equity premium (cum-dividend) and the risk-free rate are as given in (16) and (20).

Finally, to see that $\pi$ is indeed a pricing kernel, one can first show, via a straightforward deviation, that $\pi$ produces the equilibrium riskfree rate and the total equity premium for the stock. Next one can solve the same equilibrium problem by adding a derivative security (non-linear in stock) with zero-net supply, and show that the equilibrium risk premium for the derivative security can indeed be produced by $\pi$.

### C The Option-Pricing Formula

The following result can be found in Merton (1976), and is included for the completeness of the paper. Let $C_0$ denote the time-0 price of a European-style call option with exercise price $K$ and time $\tau$ to expiration. It is a straightforward derivation to show that

$$C_0 = e^{-\lambda' \tau} \sum_{j=0}^{\infty} \frac{(\lambda' \tau)^j}{j!} BS\left(S_0, K, r_j, q, \sigma_j, \tau\right), \quad (C.8)$$

where $\lambda' = \lambda Q (1 + kQ)$, and for $j = 0, 1, \ldots$,

$$r_j = r - \lambda Q kQ + \frac{j \ln (1 + kQ)}{\tau}, \quad \sigma_j^2 = \sigma^2 + \frac{j \sigma_j^2}{\tau},$$

and where $BS\left(S_0, K, r, q, \sigma, \tau\right)$ is the standard Black-Scholes option pricing formula with initial stock price $S_0$, strike price $K$, riskfree rate $r$, dividend yield $q$, volatility $\sigma$, and time $\tau$ to maturity. To price a put option with the same maturity and strike price, one can use the put-call parity.
References


