A LOW-FREQUENCY INSTABILITY MECHANISM
IN A COAXIAL DUMP COMBUSTOR

by

JOHN ADAM KEKLAK

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

January 1982

© Massachusetts Institute of Technology 1982

Signature of Author

Department of Mechanical Engineering
January 14, 1982

Certified by

Professor T. Y. Toong
Thesis Supervisor

Accepted by

Chairman, Department Thesis Committee

JUN 7 1982
A LOW-FREQUENCY INSTABILITY MECHANISM

IN A COAXIAL DUMP COMBUSTOR

by

JOHN ADAM KEKLAK

Submitted to the Department of Mechanical Engineering on
January 14, 1982 in partial fulfillment of the requirements
for the Degree of Master of Science in Mechanical Engineering

ABSTRACT

The instability mechanism believed to be the source of large-
amplitude, low-frequency pressure fluctuations in a coaxial dump
combustor is modelled as a coupling between processes in the combustion
zone and the choked nozzle: pressure disturbances are produced as
entropy waves convect through the nozzle, and entropy waves are generated
as the pressure disturbances propagate through the combustion zone.
Closed-form expressions for limiting values of entropy wave and pressure
wave amplitudes, as well as for instability frequency, are derived from
the theory. Also, the conditions for sustenance of the instability are
identified.

The theory is compared to the results of an experimental program
consisting of high-speed filming and instrumented prototype combustor
testing; the agreement is good.

Theoretical predictions and experimental observations are used to
develop guidelines for the design of instability-free combustors. Most
of the suggestions relate to flame stabilization, and are based on the
reasoning that improved flame stability makes the combustion process
more resistant to flow fluctuations that stretch the flame and produce
entropy waves.

Thesis Advisor: T. Y. Toong
Title: Professor of Mechanical Engineering
ACKNOWLEDGEMENTS

I would like to express my appreciation to the people who contributed to this effort, in particular, Professor T. Y. Toong for his suggestions, encouragement and guidance, and to Dr. George E. Abouseif, who provided invaluable assistance during the initial part of this work.

Also I wish to thank my office mates Miles Greiner, Andrew Tangborn, Victor Filipenco and Asher Sharoni for their support and comments during the course of this work.

Finally, I wish to thank Ms. Patricia A. R. Murray for her typing skills and patience during the preparation of this thesis.

This work was done in conjunction with Air Force Research Grant AFOSR-78-3662.
NOMENCLATURE

\( a \)
acoustic velocity

\( A_c \)
combustor cross-sectional area

\( c_p \)
specific heat at constant pressure \((c_p = 6260 \text{ ft}-\text{lb}f/\text{lbm}^\circ\text{R})\)

\( f \)
frequency

\( f_N \)
defined frequency of \( N^{\text{th}} \) mode

\( f'_N \)
measured frequency of \( N^{\text{th}} \) mode

\( h_n \)
head of the \( n^{\text{th}} \) entropy spot

\( L_c \)
combustor length

\( \dot{m} \)
mass flow rate

\( M \)
Mach number

\( \bar{M}_{x_n^-}, \bar{M}_{x_n^+} \)
mean Mach number at points \( x_n^- \) and \( x_n^+ \)

\( N \)
mode number

\( p \)
pressure

\( p_n^-, p_n^+ \)
\( n^{\text{th}} \) rarefactions and compressions originating on the upstream side of the nozzle

\( p_b \)
nozzle back pressure
$\overline{P}_c$  mean combustor pressure

$P_o$  total pressure

$R$  specific gas constant ($R = 1450 \text{ ft}-\text{lbf}/\text{lbmo}_R$)

$s$  entropy

$s_o$  reference level of entropy

$s_p$  entropy of combustion products

$s_R$  entropy of reactants

$t$  time

$t_n$  tail of $n^{th}$ entropy spot

$T$  temperature

$T_{ad}b$  adiabatic flame temperature

$\overline{T}_i$  mean inlet temperature

$T_o$  total temperature

$u$  flow velocity

$\dot{V}$  volume flow rate

$x$  laboratory spatial coordinate (positive in flow direction)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{X}$</td>
<td>normalized position along combustor ($x = x/L_c$, $x = 0$ at the dump plane and $x = 1$ at the choke plane)</td>
</tr>
<tr>
<td>$\hat{X}_s$</td>
<td>point of flame separation</td>
</tr>
<tr>
<td>$x_n$</td>
<td>position of compact nozzle</td>
</tr>
<tr>
<td>$x_n^-, x_n^+$</td>
<td>positions of the inlet and outlet planes of the nozzle</td>
</tr>
<tr>
<td>$z$</td>
<td>spatial coordinate in wave reference frame (negative in the direction of propagation)</td>
</tr>
</tbody>
</table>

Greek

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>specific heat ratio ($\gamma = 1.3$)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>transition zone thickness</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>represents a finite change</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>integration variable</td>
</tr>
<tr>
<td>$\zeta_n$</td>
<td>$n^{th}$ flame separation</td>
</tr>
<tr>
<td>$\eta$</td>
<td>combustion efficiency (defined as $\bar{T}/T_{adb}$)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>wavelength</td>
</tr>
<tr>
<td>$\xi$</td>
<td>position variable (see Appendix A.1)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
</tr>
<tr>
<td>$\sigma_n$</td>
<td>entropy spot</td>
</tr>
</tbody>
</table>
\( \tau_N \) \hspace{1cm} \text{characteristic instability time of mode } N \\
\( \tau_\delta \) \hspace{1cm} \text{characteristic entropy wave deterioration time} \\
\( \tau_f \) \hspace{1cm} \text{characteristic flow time} \\
\( \tau_D^N \) \hspace{1cm} \text{flame re-establishment time for mode } N \\
\( \phi \) \hspace{1cm} \text{equivalence ratio} \\

superscripts \\
( )' \hspace{1cm} \text{denotes perturbation} \\
( \sim ) \hspace{1cm} \text{denotes mean quantity} \\
( )^* \hspace{1cm} \text{a quantity associated with the occurrence of choking} \\
( )^G, ( )^P \hspace{1cm} \text{quantities associated with entropy and pressure waves, respectively} \\
( )^{\text{max}} \hspace{1cm} \text{the theoretical maximum value of a quantity} \\

subscripts \\
( )_{\text{nozzle}} \hspace{1cm} \text{a quantity associated with the nozzle, the value of which is governed by the choking condition} \\
( )_{x^-_{n}}, ( )_{x^+_{n}} \hspace{1cm} \text{the values of quantities at the points } x^-_{n} \text{ and } x^+_{n}, \text{ respectively} \\
( )_o, ( )_l, ( )_{\text{ref}} \hspace{1cm} \text{values of quantities at reference conditions}
\( ( \sigma, p_p ) \) quantities associated with entropy and pressure waves, respectively

\( \text{peak-to-peak} \) a peak-to-peak value

\( \text{predicted} \) a predicted value
### TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>2</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>3</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>4</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>9</td>
</tr>
<tr>
<td>I  Introduction</td>
<td>11</td>
</tr>
<tr>
<td>II Theory of Waves and Wave-Boundary Interactions</td>
<td></td>
</tr>
<tr>
<td>II.1 Introduction</td>
<td>15</td>
</tr>
<tr>
<td>II.2 Normalization and Coordinate Conventions</td>
<td>15</td>
</tr>
<tr>
<td>II.3 Entropy Waves</td>
<td>16</td>
</tr>
<tr>
<td>II.4 Temporally-Invariant Large-Amplitude Entropy Waves</td>
<td>19</td>
</tr>
<tr>
<td>II.5 Periodic Entropy Fluctuations</td>
<td>21</td>
</tr>
<tr>
<td>II.6 Acoustic Waves</td>
<td>21</td>
</tr>
<tr>
<td>II.7 Wave Analysis Technique</td>
<td>24</td>
</tr>
<tr>
<td>II.8 Wave-Boundary Interactions</td>
<td>27</td>
</tr>
<tr>
<td>II.9 Entropy Wave-Choked Nozzle Interaction</td>
<td>28</td>
</tr>
<tr>
<td>II.10 Assumptions about the Entropy Wave-Nozzle Interaction</td>
<td>28</td>
</tr>
<tr>
<td>II.11 Method of Solution</td>
<td>31</td>
</tr>
<tr>
<td>II.12 Wavelet Analysis of the Entropy Wave-Nozzle Interaction</td>
<td>32</td>
</tr>
<tr>
<td>II.13 Large-Amplitude Entropy Wave-Nozzle Interaction</td>
<td>36</td>
</tr>
<tr>
<td>II.14 Qualitative Analysis of Pressure Wave-Flame Interaction</td>
<td>38</td>
</tr>
<tr>
<td>II.15 Rarefaction-Flame Interaction</td>
<td>39</td>
</tr>
<tr>
<td>II.16 Compression-Flame Interaction</td>
<td>41</td>
</tr>
<tr>
<td>II.17 Concluding Remarks</td>
<td>42</td>
</tr>
<tr>
<td>III The Low-Frequency Instability Model</td>
<td></td>
</tr>
<tr>
<td>III.1 Introduction</td>
<td>43</td>
</tr>
<tr>
<td>III.2 Coupling between Wave-Boundary Interactions</td>
<td>43</td>
</tr>
<tr>
<td>III.3 A Necessary Condition for Sustenance, and Limits of Growth</td>
<td>45</td>
</tr>
<tr>
<td>III.4 Sustained Coupling of Wave-Boundary Interactions</td>
<td>46</td>
</tr>
<tr>
<td>III.5 Low-Frequency Instability Modes</td>
<td>48</td>
</tr>
<tr>
<td>III.6 Instability Frequency</td>
<td>50</td>
</tr>
<tr>
<td>IV  Experimental Program</td>
<td></td>
</tr>
<tr>
<td>IV.1 Introduction</td>
<td>54</td>
</tr>
<tr>
<td>IV.2 High-Speed Photography</td>
<td>54</td>
</tr>
<tr>
<td>IV.3 Prototype Combustor Testing</td>
<td>55</td>
</tr>
</tbody>
</table>
V Comparison between Theoretical Predictions and Experimental Results
  V.1 Introduction 57
  V.2 Comparison between Observations from High-Speed Films and the Proposed Model 57
  V.3 Instability Frequency 60
  V.4 Pressure Fluctuation Amplitude 61
  V.5 Effects of Variations in Combustor Geometry and Operating Conditions 64
  V.6 Concluding Remarks 67

VI Guidelines for Suppression of the Low-Frequency Instability
  VI.1 Introduction 69
  VI.2 Experimentally-Verified Guidelines 69
  VI.3 Other Instability Suppression Methods 71
  VI.4 Recommendations for Future Work 72

VII Summary and Conclusions 75

References 77

Appendices
  A.1 Proof of Constant-Velocity Condition across an Entropy Wave 79
  A.2 Entropy Wave Relations 80
  B Acoustic Wave Relations 85
  C.1 Derivation of the Entropy Wave-Pressure Wave Relationship 87
  C.2 The Characteristic Deterioration Time 88
  D Calculation of Flow Parameters and Frequency in Filmed Tests 91
  E.1 Sample Frequency Calculation 93
  E.2 Sample Pressure Fluctuation Amplitude Calculation 93

Figures 95

Tables 132
I INTRODUCTION

This thesis considers the problem of a low-frequency instability in a coaxial dump combustor. The instability, an oscillatory phenomenon believed to be the result of coupling between hydrodynamic and combustion processes, produces pressure fluctuations of significant amplitude which provide energy to vibratory systems elsewhere in the device utilizing the dump combustor, and shorten the life of the combustor itself through fatigue. An understanding of the instability mechanism is a prerequisite for the formulation of guidelines to suppress this instability, and thus to make propulsion systems incorporating dump combustors feasible.

A dump combustor is shown in Figure I.1. It receives a compressed mixture of fuel and air from the inlet across the dump plane (a sudden expansion in duct area), provides suitable conditions for a rapid and complete burning of the fuel and air, and expels the products of combustion through the nozzle.

The particular geometry selected for this study consists of a coaxial configuration: the inlet, the combustor and the nozzle are centered on a common axis (see Figure I.1). The flow enters from the right through the inlet, crosses the dump plane, reacts in the combustion zone and exits through the nozzle, which, under most circumstances, is choked. The coaxial configuration allows the process to occur without requiring a change in flow direction and consequently, total pressure losses associated with turns in high velocity internal flows are avoided. Since total pressure losses in propulsion devices are translated ultimately
as thrust losses, the dump combustor is one of the most efficient means for extracting mechanical work from the energy provided by the fuel-and-air reaction.

The most novel feature of the dump combustor is the dump plane, which is a sudden expansion of the inlet. The sudden change in duct area causes recirculation zones to form along the outer portions of the combustion chamber near the inlet (see Figure I.1) which serve to stabilize the flame. As a result, a flameholder, which adds weight to an engine and causes a total pressure loss by obstructing the flow, is not necessary for the operation of a ramjet. However, studies [3] have shown that use of a small flameholder in conjunction with the recirculation zones substantially improves the efficiency of the ramjet (see Figure I.2) and therefore, many designs of dump combustors include a small, lightweight flameholder mounted at the dump plane (see Figure I.1).

Despite the attractive features of dump combustors, the device has been plagued by instabilities, one of which is mentioned above. Figure I.3 presents pressure-time traces of the two most common instabilities observed in dump combustors: (1) low-frequency instabilities, which are associated with a rumbling noise consisting of large-amplitude pressure fluctuations having a frequency between 100 hz and 250 hz, and (2) high-frequency instabilities, which are high-pitched (frequencies above 1000 hz) and of relatively small amplitude [1,2,3].

Studies done on combustor instabilities [2,3,4,5,6] have shown that the high-frequency instabilities are caused by one particular class of instability mechanisms and the low-frequency instabilities are caused by
another. In the high-frequency case, Marble, et al. [2] found the instability to be transverse in nature. Others [1,3,4] have speculated that the high-frequency instability is a radial or transverse acoustic resonance. Readers interested in the high frequency instabilities should consult the references mentioned above.

In the low-frequency case, however, Keklak [7] has shown through frequency and phase correlations that the instability is longitudinal in nature. In addition, he proposes a wave-interaction model of the instability mechanism, which will be analyzed and modified in this thesis.

Although the low- and high-frequency instability mechanisms appear to be completely different, their simultaneous existence is not guaranteed. Experiments [3] have shown that the low-frequency instability is dominant under one set of operating conditions, and the high-frequency instability is dominant under another. Figure I.4 shows such a transition from the low-frequency to the high-frequency instability as the equivalence ratio is lowered, with all other parameters held constant. This phenomenon is not studied in the present work, with the exception of the development of a sustenance requirement for the low-frequency instability that, in effect, specifies the conditions at which the low-frequency instability cycle is broken, and other instabilities are allowed to develop, if the potential for such instabilities exists.

The scope of this thesis is twofold: (1) to refine the model of the low-frequency instability proposed by Keklak [7] and (2) to develop guidelines for combustor design and operation which reduce the possibility of occurrence of the instability. The first portion of this work focuses
on an analysis of the physics of the instability phenomenon, including thorough treatments of waves and wave-boundary interactions. This analysis extends the theory proposed in References [8,9] by including expressions for large-amplitude wave-boundary interactions at the nozzle, and proposing a more realistic, though qualitative, analysis of acoustic wave-flame interactions.

The second portion of this work lists a number of measures that, when implemented, should decrease the rate of instability occurrence or eliminate the instability entirely. This listing includes both experimentally-verified measures as well as those predicted by the proposed theory, but yet untested experimentally.
II THEORY OF WAVES AND WAVE-BOUNDARY INTERACTIONS

II.1 Introduction

The low-frequency instability model presented in this thesis asserts that a coupling occurs between two processes, one located in the combustion zone and the other at the choked nozzle. The processes involved in this coupling are (1) an interaction between pressure waves and the flame in the combustion zone, which produces entropy waves in the exhaust flow, and (2) an interaction between convected entropy waves and the choked nozzle which produces pressure waves on both the upstream and downstream sides of the nozzle. Prior to development of the theory of coupling, however, each of the interactions must be studied individually, giving consideration to the characteristics of both the wave and the boundary involved in the interaction. This chapter analyzes these two interactions (termed "wave-boundary interactions"), beginning with a general discussion of entropy and acoustic waves, and concluding with analyses of entropy wave-nozzle and acoustic wave-flame interactions.

II.2 Normalization and Coordinate Conventions

The variable normalization conventions adopted in this thesis, listed in Table II.1, were chosen for two reasons. First, these conventions conform with the normalization techniques used in many of the references. Second, this normalization technique conveniently allows wavelet theory to represent changes in many properties across wavefronts as movement in
state planes along lines of slope \( \pm 1 \) (see Figure II.1).

The spatial coordinate convention adopted in this thesis, however, does not conform with the conventions used in most of the references. Since the experimental data is presented with the flow direction being from right to left, this direction is used to define the positive orientation of the spatial coordinate in the laboratory reference frame, denoted by \( x \), to avoid confusion in comparisons between theoretical and experimental results.

Another spatial coordinate, \( z \), is defined for the purpose of describing events in the reference frame of the wave; this coordinate is negative in the direction of propagation of the wave.

Figure II.2 illustrates these coordinate conventions.

II.3 Entropy Waves

During the operation of a jet engine or an industrial burner, the combustion process may experience an abrupt change in intensity due to, perhaps, a change in the oncoming flow of reactants or a change in the conditions within the combustion chamber. Such a change in the burning intensity results in a convected transition zone in the exhaust flow which, in general, is finite in length, but can be treated as a contact discontinuity for small changes in the burning intensity.\(^\dagger\); the flow

\(^\dagger\) The large-amplitude transition zone may be viewed as a superposition of many small-amplitude contact discontinuities; this fact is used later in the analysis of the large-amplitude transition zones.
conditions ahead of this zone correspond to the original burning rate, while the conditions behind it correspond to the new rate. Figure II.3 schematically shows this event for the case when the heat of reaction of the oncoming flow suddenly changes. (The reader should note that pressure waves are generated along with the transition zone when the change in heat release occurs, but these pressure waves, which travel at a much higher velocity than the flow, do not affect the convection of the transition zone except for the initial acceleration of the flow as the pressure waves propagate from the reaction zone. For more details of this phenomenon, the reader should consult Reference [10].)

Since the change in heat release is accompanied by a change in the rate of entropy production, the specific entropy level ahead of the transition zone differs from the level behind the transition zone. For this reason, such a transition zone, which is shown in Figure II.4, is termed an entropy wave. Since the transition zone is frozen to the flow, the propagation velocity of an entropy wave is the flow velocity.

The strength of an entropy wave is defined as the difference between the two levels of specific entropy normalized by the specific heat of the fluid at constant pressure:

\[ \frac{s - s_o}{c_p} \]  

(II.1)

where \( s_o \) and \( s \) are the entropy levels before and after the passage of an entropy wave. A positive value for the strength of an entropy wave indicates that the specific entropy increases across the transition zone
in the positive z-direction. In the example provided above, the entropy wave generated by the sudden increase in heat release produces a positive entropy wave (see Figure II.3).

A fundamental property of entropy waves may be brought to light by considering an entropy wave of small amplitude (wavelet), which can be viewed as an entropy contact discontinuity. The term "contact discontinuity" implies that the static pressure does not vary across an entropy wavelet. A direct consequence of this condition is that the flow velocity is also constant across an entropy wavelet. Since a large-amplitude entropy wave can be represented as the superposition of a large number of entropy wavelets, the pressure and velocity changes across a large-amplitude entropy wave are both zero. A proof of this condition, valid for both large-amplitude entropy waves and entropy wavelets, is presented in Appendix A.1.

The changes in fluid properties across an entropy wave (for both wavelets and large-amplitude waves) may be derived simply from the entropy wave amplitude in conjunction with the isobaric condition discussed above. For example, the density change can be determined by considering entropy as a function of pressure and density; since the entropy and pressure changes are known, the problem reduces to the exercise of solving for the density change.

Expressions for the changes in fluid properties and flow parameters across entropy wavelets and large-amplitude entropy waves are derived in Appendix A.2 and are listed in Table II.2. The method used for obtaining these closed-form expressions is outlined later in this chapter.
II.4 Temporally-Invariant Large-Amplitude Entropy Waves

An entropy wave of small amplitude \((\Delta s/c_p \ll 1)\) involves only a small temperature discontinuity, so heat transfer effects across the discontinuity may be neglected. However, an entropy wave of large amplitude \((\Delta s/c_p \gg 1)\) may represent an extremely large temperature difference over a very short distance if the associated mean spatial entropy gradient, \(\langle \frac{\partial (s/c_p)}{\partial z} \rangle\), is large, resulting in a rapid deterioration of the steep wavefront through heat conduction which continues until the temperature and entropy gradients become small enough to allow heat transfer effects to be neglected (see Figure II.5). At this point the entropy wave takes the form of a spatially-continuous distribution of entropy which may be treated as temporally invariant.

The criterion which determines when an entropy wave may be considered constant in time is

\[
\frac{\tau_\delta}{\tau_f} \gg 1,
\]

where \(\tau_\delta\) is the characteristic deterioration time of an entropy wave, given by

\[
\tau_\delta = \frac{\langle \frac{\partial (s/c_p)}{\partial z} \rangle}{\frac{d}{dt} \langle \frac{\partial (s/c_p)}{\partial z} \rangle},
\]

or equivalently,
\( \tau_{\delta} = \frac{\delta}{d\delta/dt} \)  \hspace{1cm} (II.3b)

(\( \delta \) is the transition zone thickness; see Appendix C for the derivation of Eq. (II.3b).), and \( \tau_{\delta} \) is the characteristic flow time. Therefore, an entropy wave is considered temporally-invariant during transit between two points in space if the amount of time that the flow takes to convect the entropy wave from one point to the other is much smaller than the time required to produce a significant change in the entropy distribution in the entropy wave through heat transfer.

The definition of the amplitude of the entropy wave is not changed by these considerations; it remains:

\[
\frac{s - s_0}{c_p}.
\]  \hspace{1cm} (II.4)

A consequence of this definition is that the sign of the entropy gradient in the z-direction is the same as the sign of the entropy wave amplitude. This observation facilitates the discussion of large-amplitude entropy wave-nozzle interactions and the low-frequency instability model.

The entropy waves encountered in the present problem of low-frequency instability satisfy the criterion given by Eq. (II.2) because the flow time is very short and the transitions between regions of different levels of entropy are relatively smooth. Thus, the analyses of entropy wave-nozzle interactions and coupling between the wave-boundary interactions at the nozzle and the combustion zone will assume that the entropy waves do not change in shape as they are convected by the flow along the length of
the combustor.

II.5 Periodic Entropy Fluctuations

The rate of entropy generation in some combustion processes may fluctuate periodically, resulting in a spatially-periodic distribution of entropy transition zones in the exhaust flow. Such a process would occur, for example, if the heat of reaction of the oncoming flow in the example above were to rise and fall periodically, generating a train of alternating positive and negative entropy waves (see Figure II.6a). (The reader should recall that a positive entropy wave consists of a rise in entropy in the positive z-direction, while a negative entropy wave consists of a fall in entropy in the positive z-direction.)

To simplify discussion of such a train of waves, one terms the positive entropy waves heads and the negative entropy waves tails. Also, the region of fluid between a head and the following tail constitutes an entropy elevation, more commonly known as an entropy spot, while the region of fluid between a tail and the following head is termed an entropy depression.

These definitions are illustrated in Figure II.6b.

II.6 Acoustic Waves

In many situations, a quiescent region of fluid may be subjected to a propagating pressure discontinuity termed an acoustic wave. The disturbance may take the form of, for example, a sudden acceleration of
a surface bounding the fluid (e.g., a piston), a sudden change in the heat release rate (in the case of steady-state heat flow) or a sudden change in the mass injection rate (in the case of steady-state fluid flow).

An acoustic wave, shown schematically in Figure II.7a is defined as an isentropic discontinuity of pressure which propagates relative to the fluid at the local sound speed. The isentropic specification leads to the result that the local sound speed is given by

\[ a = \sqrt{\frac{\gamma P}{\gamma \rho}} \]

\[ = \sqrt{\gamma RT} \quad \text{(II.5)} \]

Since acoustic waves travel at the acoustic velocity relative to the fluid, the apparent wave velocity in situations where the fluid is flowing equals the vector sum of the flow velocity and the acoustic velocity. Therefore in a flow-field with uniform acoustic velocity, acoustic waves appear to propagate in the laboratory reference frame at various velocities, depending on the position of the wave and direction of propagation. In the case of one-dimensional slug flow in a duct, acoustic waves travelling parallel to the flow direction appear to propagate at velocity \( a+u \) when travelling with the flow, and at velocity \( a-u \) when travelling against the flow (see Figure II.7b).

In this thesis, the strength of an acoustic wave is defined by the difference in static pressure across the pressure discontinuity normalized
by the product of the static pressure ahead of the wave and the ratio of the specific heats, $\gamma$. A positive amplitude indicates that the static pressure increases across the wave-front in the positive $z$-direction, while a negative amplitude denotes a static pressure drop across the wave-front. Furthermore, a pressure wave with a positive amplitude is termed a compression, while one with a negative amplitude is termed a rarefaction.

In order to satisfy mass and momentum conservation requirements, the pressure discontinuity must be associated with a flow velocity discontinuity. The amplitude of this velocity change, defined as the difference in flow velocity across the pressure discontinuity, normalized by the acoustic velocity ahead of the pressure wave, may be calculated from the amplitude of the pressure wave and the isentropic constraint. For acoustic waves of small amplitude ($p'/\gamma \tilde{p} \ll 1$), the velocity change is very nearly

$$\frac{u'}{a} = \pm \frac{p'}{\gamma \tilde{p}}.$$  \hspace{1cm} (II.6)

The positive and negative signs are associated with wave propagation in the positive and negative $x$-directions, respectively. (The reader is reminded that the positive $x$-direction is the direction of the flow.)

The changes in the other fluid properties across a pressure wave can be computed from the amplitudes of the pressure wave and associated velocity change and the isentropic constraint. The expressions for these changes are derived in Appendix B and are listed in Table II.3.

Although expressions for changes in fluid properties across large-amplitude pressure waves (shock waves) may be obtained, these expressions
are not necessary for the development of the low-frequency instability model because the amplitudes of the pressure waves produced by the instability fall in a range where the acoustic approximation suffices. Therefore only the expressions associated with small-amplitude pressure waves are considered in this thesis.

Figure II.8 illustrates the validity of the acoustic approximation by comparing the relations between the pressure ratio across a pressure wave and the associated velocity ratio from acoustic and shock wave theory; in the range of pressure wave amplitudes considered, the two relations are virtually identical.

For a more complete discussion of both small- and large-amplitude pressure waves, one should consult References [11,12,13,14,15 and 16].

II.7 Wave Analysis Technique

The discussion on entropy and pressure waves in the previous sections points out that the amplitude and associated constraint ($\Delta p/\gamma p = 0$ in the case of entropy waves, and $\Delta s/c_p = 0$ in the case of acoustic waves) provide complete information for determining changes in fluid properties across a wave, but it does not outline any methods for obtaining closed-form expressions for these changes. A technique for deriving these expressions is now presented.

This technique consists of two steps: (1) derivation of the changes in fluid properties across a wavelet, and (2) integration of these wavelet expressions to obtain expressions for changes across large-
amplitude waves. The first step of this process is performed by writing all of the variables as the sum of a mean and perturbation quantities, and eliminating all but zeroth and first order terms. For example, the mass flow rate relationship

\[
\left( \frac{\dot{m}}{A} \right) = \rho u
\]

(II.7)

is expressed in perturbation form as

\[
\left( \frac{\dot{m}/A}{\dot{m}/A} \right)' = \frac{\rho'}{\rho} + \frac{u'}{u}
\]

(II.8)

where the perturbation quantities \(( \_ )'\) are much smaller than the mean quantities \(( \_ )\).

The second step involves the integration of the perturbation forms, treating the perturbation quantities as differentials, and the mean quantities as the values of the variables. Hence Eq. (II.8) is written in differential form as

\[
\frac{d(\dot{m}/A)}{\dot{m}/A} = \frac{d\rho}{\rho} + \frac{du}{u}
\]

(II.9)

For the purpose of demonstration, the integration is performed over an entropy wave, where \(( \_ )_0\) refers to the conditions ahead of the entropy wave (see Figure II.9). Since the velocity is constant across an entropy wave and the density change \(d\rho/\rho\) is related to the entropy change \(ds/c_p\) by
\[ \frac{d\rho}{\rho} = - \frac{ds}{c_p} \quad \text{(II.10)} \]

(see Appendix A.2), Eq. (II.9) becomes

\[ \frac{d(\dot{m}/A)}{(\dot{m}/A)} = - \frac{ds}{c_p} \quad \text{(II.11)} \]

Integrating Eq. (II.11) from the conditions ahead of the entropy wave to some general conditions with entropy \( s \) and mass flow rate per unit area \((\dot{m}/A)\)

\[ \int_{(\dot{m}/A)_o}^{(\dot{m}/A)} \frac{d(\dot{m}/A)}{(\dot{m}/A)} = - \int_{s_o}^{s} \frac{ds}{c_p} \quad \text{(II.12)} \]

produces the result

\[ \frac{(\dot{m}/A) - (\dot{m}/A)_o}{(\dot{m}/A)_o} = e^{-\int_{s_o}^{s} \frac{ds}{c_p}} - 1 \quad \text{(II.13)} \]

for a semi-perfect gas \((c_p = c_p(T))\), and

\[ \frac{(\dot{m}/A) - (\dot{m}/A)_o}{(\dot{m}/A)_o} = e^{-\frac{(s-s_o)}{c_p}} - 1 \quad \text{(II.14)} \]

for a perfect gas (constant \(c_p\)).

This technique is applied in Appendix A.2 to derive expressions for entropy wavelets and large-amplitude entropy waves, and the first step of the process is used in Appendix B to derive acoustic wave relations. The results of these derivations are summarized in Tables II.2 and II.3.
The reader should be warned that this technique breaks down in the case of large-amplitude pressure wave because such waves steepen very quickly and form shock waves. Since shock waves generate entropy, the definition of acoustic waves given earlier is violated, and shock theory must be used to obtain expressions for fluid property changes. However, as mentioned in the discussion of acoustic waves, the amplitudes of the pressure waves produced by the low-frequency instability are moderate \((0.08 \leq \frac{\Delta p}{\gamma p} \leq 0.26)\), and the acoustic analysis suffices.

II.8 Wave-Boundary Interactions

When a wave in a bounded region of fluid is incident on one of the boundaries, a phenomenon termed a wave-boundary interaction occurs. The outcome of such an interaction, which may be an alteration of the incident wave or production of entirely new waves, is dependent on the nature of the incident wave and on the characteristics of the boundary involved.

A simple example of a wave-boundary interaction, shown in Figure II.10, is the reflection of an acoustic wave from a solid surface. The amplitude of the acoustic wave propagating away from the surface depends on the amplitude of the incident wave and on the specification that the wall is rigidly fixed.

Two wave-boundary interactions, which are somewhat more complicated than the interaction in the previous example, form the basis of the model of the low-frequency instability mechanism presented in this thesis. The first type of interaction involves entropy waves and a choked nozzle,
which generates pressure waves as entropy waves are convected through the nozzle. The other interaction is between acoustic waves and a flame, which produces entropy waves under the proper conditions. The physics of both of these wave-boundary interactions is discussed in detail in the following sections.

II.9 Entropy Wave - Choked Nozzle Interaction

The first of the wave-boundary interactions mentioned above involves the convection of an entropy wave through a choked nozzle. The changes in fluid properties across an entropy wave alter the mass flow rate allowed through the nozzle, creating a mismatch in both mass and volume flow rates upstream and downstream of the nozzle [8,9]. The volume flow rate mismatches constitute acoustic waves which propagate away from the nozzle.

This process would occur in the example dealing with a sudden change in the heat release rate in a combustion process (see "Entropy Waves") if the exhaust duct contained a flow constriction (or nozzle) small enough to choke the flow. As the entropy wave passed through the nozzle, acoustic waves would be produced on both sides by the readjustment of the mass flow rate through the nozzle to maintain the choked condition. (For an analysis of flow through a choked nozzle, the reader should consult Reference [12].)

II.10 Assumptions about the Entropy Wave-Nozzle Interaction

To simplify the task of relating the strength of the convected
entropy wave to the strengths of the acoustic waves produced, a number of assumptions are made. Among these assumptions are: (1) constant choking of the nozzle, (2) compactness of the nozzle, and (3) reversible and adiabatic flow upstream of the sonic plane.

First, the ratio between the back pressure \( p_b \) and the minimum total pressure at any point in the flow at any time \( p_{o,\text{min}} \) is assumed to be less than the ratio necessary for choking \( (p_b/p_o)_{\text{c}} \) at all times. This condition guarantees that the nozzle remains choked, and hence the mass flow rate through the nozzle (per unit area of the choke plane) is specified solely by the total temperature and total pressure of the oncoming flow.

Secondly, the nozzle length in most applications is much less than the wavelengths of the entropy waves impinging on the nozzle. Consequently, the nozzle can be considered compact, which implies that the nozzle can be treated as a discontinuity in duct area. An additional implication of this assumption is that mass cannot be stored in the nozzle, a condition that proves to be quite useful in the entropy wave-nozzle interaction analysis.

Finally, the flow upstream of the sonic plane is assumed to be reversible and adiabatic. This condition implies that entropy waves generated upstream of the nozzle are unmodified as they are convected by the flow towards the nozzle. The reader should note that this assumption disallows the presence of shock waves in the flow upstream of the sonic plane because shock waves generate entropy, and hence violate the reversibility condition.
These assumptions are expressed in the following mathematical forms (see Figure II.11) for definitions of notation):

constant choking condition:

\[
(\dot{m}/A)_{\text{nozzle}} = P_o \sqrt{\frac{\gamma}{RT_o}} \left( \frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}}
\]  
(II.15)

where \( P_o \) and \( T_o \) are the total pressure and total temperature of the fluid at the sonic plane

nozzle compactness:

\[
\hat{m}_{\text{nozzle}} = \hat{m}_{x_n^-} = \hat{m}_{x_n^+}
\]  
(II.16a)

\[
\frac{x_n^+ - x_n^-}{\lambda} \ll 1
\]  
(II.16b)

reversible flow:

\[
\frac{s-s_o}{c_p} (x,t) = \frac{s-s_o}{c_p} (x_n^-, t + \int_{x_n^-}^{x_n^+} \frac{d\zeta}{u(\zeta)})
\]  
(II.17a)

\[
= \frac{s-s_o}{c_p} (x_n^+, t + \int_{x_n^-}^{x_n^+} \frac{d\zeta}{u(\zeta)})
\]  
(II.17b)

or alternately\(^{\dagger}\),

\(\dagger\)This equation states that the total pressure of the flow is unchanged as the fluid passes through the nozzle; upstream of the nozzle, however, the total pressure may be changed (e.g., by a pressure wave) without violating the isentropic flow assumption.
\[ P_{o, x_n^-} = P_{o, x_n^+} \]  \hspace{1cm} (II.18)

adiabatic flow:

\[ T_o(x, t) = T_o(x_n^-, t + \int_{x_n^-}^{x_n^+} \frac{d\tau}{u(\zeta)}) \]

\[ = T_o(x_n^-, t + \int_{x_n^-}^{x_n^+} \frac{d\tau}{u(\zeta)}) \]  \hspace{1cm} (II.19)

Since the nozzle is assumed to be compact, the quantities

\[ \int_{x_n^-}^{x_n^+} \frac{d\tau}{u(\zeta)} \]  and \[ \int_{x_n^-}^{x_n^+} \frac{d\tau}{u(\zeta)} \] are essentially equal. Therefore, Eq. (II.17) and Eq. (II.19) become

\[ \frac{s-s_o}{c_p}(x, t) = \frac{s-s_o}{c_p}(x_n, t + \int_{x_n^-}^{x_n^+} \frac{d\tau}{u(\zeta)}) \]  \hspace{1cm} (II.20)

\[ T_o(x, t) = T_o(x_n, t + \int_{x_n^-}^{x_n^+} \frac{d\tau}{u(\zeta)}) \]  \hspace{1cm} (II.21)

respectively, where \( x_n \) is the position of the nozzle.

In the forthcoming analysis, the symbols \( x_n^+ \) and \( x_n^- \) refer to positions just downstream and upstream of the point \( x_n \).

II.11 Method of Solution

The technique used for solving for the relationships between the convected entropy wave and the generated acoustic waves is identical to the method outlined earlier that is used to compute property changes
across waves of large amplitude. First the interaction is considered on
the wavelet level; then, the results of this analysis are integrated to
obtain expressions relating large-amplitude entropy waves and the
generated acoustic waves.

In most of the cases considered in this thesis, a mismatch between
the back pressure \( p_b \) (see Figure II.11) and the pressure in the flow
exiting from the nozzle exists, resulting in the formation of expansion
and shock waves downstream of the sonic plane. Since these waves
overshadow any acoustic waves produced downstream of the nozzle, and, in
addition, the acoustic waves produced downstream of the nozzle do not
play a part in the proposed instability model, the derivation of the
amplitude relationship between the convected entropy wave and the
downstream acoustic wave is forgone in this thesis. Interested readers
should consult References [8 and 9] for this analysis.

II.12 Wavelet Analysis of Entropy Wave-Nozzle Interaction

When an entropy wavelet is convected through a choked nozzle, the
levels of total pressure and total temperature in the nozzle change
slightly, modifying the mass flow rate required by the choked nozzle
condition. This change in mass flow rate is given by

\[
\left( \frac{B^\prime}{\rho^\prime} \right)^\sigma_{\text{nozzle}} = \left( \frac{p^\prime}{\rho^\prime} \right)^\sigma_o - \frac{1}{\sigma} \left( \frac{\rho^\prime}{\rho^\prime} \right)^\sigma_o
\]  

\[
= -\frac{1}{\sigma} \frac{(1 + \gamma M^2) s^\prime}{1 + \frac{\gamma - 1}{2} M^2 \frac{c^\prime}{c}}
\]  

(II.22a)

(II.22b)
(super- and subscripts are defined in "Nomenclature").

After the passage of an entropy wave, the mass flow rate incident on the \( x_n^- \)-plane, \( \dot{m}_{x_n^-} \), (see Figure II.12a) is also altered. This change, equal to the change in mass flow rate across an entropy wave, is given by

\[
\left( \frac{\dot{m}'}{\bar{m}} \right)_{x_n^-}^\sigma = - \frac{s'}{c_p} \cdot \left( \frac{s'}{c_p} \right)_{x_n^-} ^\sigma \tag{II.23}
\]

Clearly, the mass flow rate required by choked nozzle condition does not equal the mass flow rate entering the nozzle. To remedy this situation, the flow entering the nozzle must be accelerated by a pressure wave propagating against the flow; this change in the mass flow rate is given by

\[
\left( \frac{\dot{m}'}{\bar{m}} \right)_{x_n^-}^p = - \left( 1 - \frac{1}{\gamma^p} \right) \frac{p'}{\gamma^p} \tag{II.24}
\]

(One should recall that the ultimate objective of the present analysis is to determine the value of the quantity \( p'/\gamma^p \).)

However, the pressure wave propagating against the flow alters the total temperature and total pressure of the flow through a work interaction, and hence the mass flow rate through the nozzle must once again be altered to meet the choked nozzle condition. This change is given by

\[
\left( \frac{\dot{m}'}{\bar{m}'} \right)_{\text{nozzle}} = \left( \frac{\dot{p}'}{\bar{p}'} \right)_{\text{nozzle}} - \frac{1}{2} \left( \frac{\dot{t}'}{\bar{t}'} \right)_{\text{nozzle}} \tag{II.25a}
\]
To satisfy mass continuity across the inlet of the nozzle (i.e., the $x_n^-$-plane; see Figure II.12a), the total change in mass flow rate through the nozzle required to maintain the choked condition must equal the total change in mass flow rate incident on the $x_n^-$-plane. Therefore the following condition holds:

\[
\left(\frac{\dot{m}'}{\dot{m}}\right)^c_{x_n^-} + \left(\frac{\dot{m}'}{\dot{m}}\right)^p_{x_n^-} = \left(\frac{\dot{m}'}{\dot{m}}\right)^c_{\text{nozzle}} + \left(\frac{\dot{m}'}{\dot{m}}\right)^p_{\text{nozzle}}
\] (II.26a)

or

\[
-\frac{s'}{c_p} - \frac{1-M}{M} \frac{p'}{\gamma p} = -\frac{1}{2} \frac{(1+\gamma M^2)}{1 + \frac{\gamma-1}{2} M^2} \frac{s'}{c_p} + \frac{1}{2} \frac{(\gamma+1)(1-M)}{1 + \frac{\gamma-1}{2} M^2} \frac{p'}{\gamma p}
\] (II.26b)

Solving for the amplitude of the generated pressure wave, one obtains:

\[
\frac{p'}{\gamma p} = -\frac{1}{2} \frac{\bar{M}}{1 + \frac{\gamma-1}{2} \bar{M}} \frac{s'}{c_p}
\] (II.27)

Details of the above analysis are presented in Appendix C.

The negative coefficient in Eq. (II.27),

\[
-\frac{1}{2} \frac{\bar{M}}{1 + \frac{\gamma-1}{2} \bar{M}}
\] (II.28)
lends an inverting property to a choked nozzle: a positive entropy wave produces a rarefaction, and a negative entropy wave produces a compression. This result can be verified by the following argument.

Since the velocity of the flow incident on the $x_n^-$-plane does not change after an entropy wave passes through the nozzle, the volume flow rate incident on the $x_n^-$-plane does not change. However, the volume flow rate required at the inlet of the nozzle to maintain the choked nozzle condition does change after the passage of an entropy wave, and consequently, a mismatch in volume flow rate is produced at the inlet of the nozzle. In the case of a positive entropy wave, the changes in total pressure and total temperature associated with the entropy wave are such that the volume flow rate required at the inlet of the nozzle increases after the passage of the entropy wave; hence, a rarefaction propagating against the flow is generated to accelerate the oncoming flow in the positive $x$-direction to meet the volume flow rate requirement at the nozzle entrance. Since a rarefaction is a negative pressure wave, this analysis shows that the "inverting" property embodied in the negative coefficient of Eq. (II.27) is indeed consistent with the physics of the entropy wave-nozzle interaction.

Figure II.12b presents a graphical representation of the fractional mass and volume flow rate changes that occur during the process. The changes shown under the labels "n" and "nozzle" indicate changes that occur in flow rates (1) incident on the $x_n^-$-plane, and (2) required to maintain the choked nozzle condition, respectively; the quantities enclosed by $G$ and $P$ refer to changes that are associated with
(1) the passage of an entropy wave, and (2) formation of a pressure wave on the upstream side of the nozzle, respectively. Beginning at State 1, the flow rates follow the paths indicated in the diagram to reach State 2 where, once again, the flow rates required to keep the nozzle choked equal the flow rates incident on the inlet plane of the nozzle, and the flow through the nozzle returns to steady-state.

II.13 Large-Amplitude Entropy Wave-Nozzle Interaction

The wavelet analysis of the entropy wave-nozzle interaction may be extended to large-amplitude entropy waves by considering a series of entropy wavelets that are convected through a choked nozzle. As each of the wavelets passes through the nozzle, the static pressure at the $x_n$-plane is changed by the amount given by Eq. (II.27), and it remains at the new level until another entropy wavelet passes through the nozzle and, once again, modifies the pressure level at the $x_n$-plane.

For sufficiently small amplitudes, the entropy wavelets superimpose to form a large-amplitude entropy wave, the amplitude of which equals the sum of the amplitudes of the entropy wavelets. Also, since the pressure changes at the inlet plane of the nozzle is dependent only on the amplitudes of the entropy wavelets (not on the frequency of incidence), the total change in pressure is a function simply of the amplitude of the large-amplitude entropy wave. Therefore, the amplitude of the generated pressure wave does not depend on the entropy distribution in entropy transition zone.

However, the initial pressure distribution in the generated pressure
wave is dependent on the entropy distribution of the incident large-amplitude entropy wave. If the entropy gradient in the transition is large, the resulting initial pressure distribution is steep; if the entropy gradient is moderate, the initial pressure distribution varies more gradually. This dependence, illustrated in Figure II.13, is a result of the relationship between the level of entropy of the fluid at the sonic plane and the static pressure on the upstream side of the nozzle.

As the pressure waves propagate from the nozzle, the pressure distribution is altered by dispersion which normally occurs in pressure waves. This dispersion is exaggerated somewhat in Figure II.13 to illustrate the temporal evolution of pressure waves (i.e., steepening of compressions, and flattening of rarefactions).

Using the integration technique outlined earlier in this chapter, Eq. (II.27) may be integrated to obtain the amplitude of the pressure wave generated by the interaction of a large-amplitude entropy wave and a choked nozzle. This procedure yields

\[
\frac{p_{-1} - p_{1}}{\gamma p_{1}} = \frac{1}{\gamma} \left[ e^{\left[ \int_{S_1}^{S} \frac{1}{1 + \frac{\gamma - 1}{2} M(s)} \frac{ds}{c_p} \right]} - 1 \right]
\]

(II.29)

where \((\quad)_{1}\) denotes the conditions at the nozzle entrance before the passage of the entropy wave. The integral in the exponent may be evaluated if average values of the Mach number and specific heat are assumed, allowing the integrand to be treated as a constant. Thus, Eq. (II.29) becomes
\[ \frac{p-p_1}{\gamma p_1} = \frac{1}{\gamma} (e - 1) . \] 

II.14 Qualitative Analysis of Pressure Wave-Flame Interaction

The second wave-boundary interaction introduced earlier involves the propagation of a pressure wave through a flame. The hydrodynamic effects associated with a pressure wave affect the combustion process and, under the proper conditions, produce entropy waves.

The boundary, in this case, consists of the flame, which is defined to be a zone in which the flow changes in chemical composition. Although the flame is not a boundary in the usual sense (such as a rigid wall), it is termed a boundary because, like the nozzle, it separates two distinctly different regions of flow (see Figure II.14).

This analysis considers only pressure waves which enter the flame from the downstream side because the model of the low-frequency instability mechanism does not involve pressure waves which enter the flame from the upstream side. However, the author suggests that a comprehensive theory of pressure wave-flame interactions be developed in future research efforts to provide the groundwork for other problems dealing with flames subjected to pressure disturbances from both the upstream and downstream sides.

The two types of pressure wave-flame interaction studied in this thesis are: (1) rarefaction-flame and (2) compression-flame interactions. The reader is reminded that the waves propagate against the exhaust flow and enter the flame through its downstream end.
The two types of interactions are studied separately because each produces very different results. The rarefaction stretches and breaks the flame, disrupting the entropy generation and producing a negative entropy wave in the exhaust flow. The compression, on the other hand, has little effect on the burning process unless the flame is operating near the blow-off condition. In this case, the compression decelerates the oncoming reactant flow, stabilizing the flame and improving the combustion efficiency of the flame. An improvement of combustion efficiency (and hence an increase in the heat released by the reaction) results in a positive entropy wave in the exhaust flow.

II.15 Rarefaction-Flame Interaction

A rarefaction propagating through a flame produces a rapid change in the combustion process by virtue of its associated positive velocity gradient. The gradient stretches individual fluid particles and decreases the density of the reacting fluid, resulting in a rapid reduction in reaction intensity. The drop in reaction intensity is reflected as a region of flow containing a negative entropy gradient convected by the exhaust flow, constituting a negative entropy wave (see Figure II.14).

The amplitude of this entropy wave cannot be specified exactly by the theory presented in this thesis because a method for determining the final level of reaction intensity is not developed. However, an upper bound for this amplitude may be obtained by considering the maximum change in the rate of entropy production that the flame can
experience. This maximum change corresponds to the case when the flame is extinguished entirely, giving the entropy wave an amplitude equal to

\[
\frac{s_R - s_p}{c_p}
\]  (II.31)

where \(s_p\) is the specific entropy of the reactants, and \(s_R\) is the specific entropy of the products assuming complete combustion.

After the rarefaction reduces the reaction intensity in the flame, one of two conditions exists: (1) the flow velocity of the oncoming reactants, which was increased by the passage of the rarefaction, is near the blow-off velocity, or (2) the flow velocity of the reactants is well below the blow-off velocity, even after the increase in velocity produced by the rarefaction\(^+\). These two conditions are illustrated in Figures II.15 and II.16.

In the first case, the reaction does not return to its original intensity unless the flow velocity is decreased or some other means of stabilization is introduced. In the other case, the reaction returns to its original intensity, without requiring any stabilizing influences. The difference between these two cases will be later shown to form the

\(^+\)The first condition has been observed experimentally through high-speed photography of a quartz combustor (see Figure II.16). The rarefaction increases the velocity of the oncoming reactant stream to a level where burning occurs only behind the recirculation zone. The resulting flame is toroidal in shape, which permits a stream of unburned gases to pass through the combustion zone.
basis for the difference between two of the low-frequency instability modes.

In some instances, when the reaction intensity is high and the rarefaction is relatively weak, the effect of the stretching of the flame is minor, and no entropy wave is produced. As the strength of the rarefaction is increased, one finds a threshold level which does produce separation of the flame, leading to the generation of an entropy wave. This limiting value of rarefaction strength, below which entropy waves are not produced, is an important quantity when the conditions for sustenance of the low-frequency instability are investigated in Chapter III. An expression for the threshold amplitude is not given by the present theory, but clearly, this amplitude is below the amplitudes of the pressure waves produced by the instability.

II.16 Compression–Flame Interaction

Unlike the rarefaction, the compression has little effect on a flame which is burning stably, and plays a significant role only when the combustion is occurring near blow-off conditions (see Figure II.15). In this situation, the pressure wave decreases the velocity of the oncoming reactants, increasing the stability of the flame and hence improving the efficiency of combustion. The increase in efficiency causes a change in the rate of heat release by the reaction, producing a positive entropy wave in the exhaust flow (see Figure II.17).

The magnitude of the entropy wave is once again not given by the present theory, but the upper bound of the amplitude may be estimated
from the maximum difference in the rate of entropy production that the reaction can achieve in developing from near-blow-off conditions to stable burning. This theoretical maximum amplitude, which is equal in magnitude to the maximum amplitude that an entropy wave produced by a rarefaction-flame interaction but opposite in sign, is given by

$$\frac{s_p - s_R}{c_p},$$ (II.32)

where $s_p$ and $s_R$ are the specific entropies of products (assuming complete combustion) and reactants, respectively.

II.17 Concluding Remarks

The analyses in this chapter provide insight into the types of events that occur during the operation of a combustor. The following chapter presents a coupling between the two wave-boundary interactions that leads to a low-frequency instability.
III THE LOW-FREQUENCY INSTABILITY MODEL

III.1 Introduction

This chapter presents a model of the low-frequency instability which occurs during operation of a coaxial dump combustor under certain conditions. The model, based on the theory of wave-boundary interactions developed in the previous chapter, asserts that coupling between two wave-boundary interactions leads to a sustained instability cycle. This coupling first is described in detail, followed by a discussion of the necessary condition for sustenance and the limits of growth of the instability. Finally, this chapter presents a description of two modes that the instability may assume, and derives corresponding frequency expressions based on the proposed model.

III.2 Coupling between Wave-Boundary Interactions

Up to this point, the two wave-boundary interactions have been treated independently. However, if the nozzle and the combustion zone are in close proximity, the possibility exists for the waves produced by one of the interactions to serve as the input for the other.

The analysis of the wave-boundary interactions have considered situations where (1) entropy waves enter a nozzle and produce pressure waves on the upstream side of the nozzle, and (2) pressure waves enter a combustion zone through the downstream side and generate entropy waves that are convected with the flow. These considerations dictate that the
relative positioning of the combustion zone and nozzle be such that entropy waves produced in the combustion zone pass through the nozzle, and pressure waves produced on the upstream side of the nozzle propagate into the combustion zone through its downstream end. The positioning that satisfies this requirement places the nozzle downstream of the combustion zone (see Figure III.1).

The wave-boundary interactions may couple in two ways. First consider the case of a sudden change in heat release which generates an entropy wave in the exhaust flow. By the specified geometry, the entropy wave is convected through the nozzle and, by the mechanism discussed in the previous chapter, a pressure wave is generated (see Figure III.1a). Thus, the combustion zone couples with the nozzle by providing entropy disturbances that excite the pressure wave-generating mechanism manifested in the choked nozzle.

Second, consider a process that causes the nozzle to produce a strong pressure fluctuation on its upstream side; such a process may result, for example, from a sudden change in the total temperature or total pressure of the oncoming flow. The pressure disturbance propagates against the flow, enters the combustion zone through its downstream side, and, depending on the type and strength of the pressure wave, an entropy wave may be produced (see Figure III.1b). In this manner, pressure waves produced by the nozzle drive the entropy wave-generation mechanism embodied in the reacting flow.

Conceivably, the communication between the two wave-boundary interactions may lead to a mutually-driven coupling. A condition for
sustenance of such a coupling is specified in the following section.

III.3 A Necessary Condition for Sustenance, and Limits of Growth

The coupling described in the previous section can be sustained only if the following condition is met: the rarefactions produced at the nozzle must be strong enough to separate the flame and form entropy waves. This condition implicitly requires that the amplitudes of the positive entropy waves produced must be sufficiently large to produce rarefactions above the threshold strength as the entropy waves pass through the nozzle. If this condition is not met, the flow between the combustion zone and the nozzle returns to its original undisturbed state, or another type of instability (e.g., the high-frequency instability discussed earlier), which is overshadowed by the relatively large-amplitude fluctuations in pressure and velocity associated with the low-frequency instability, may develop. In the latter case, the operating conditions at which the low-frequency instability cycle is broken correspond to the conditions for transition from the low-frequency instability to another type of instability.

In addition, for an unchoked nozzle, Marble [9] shows that the relationship between the amplitudes of the pressure waves generated on the upstream side of the nozzle and the incident entropy waves is given by

\[
\frac{dP}{\gamma P} = -\frac{\frac{(\tilde{M}_{x_+} - \tilde{M}_{x_-})}{\tilde{M}_{x_n}}}{\left(1 - \tilde{M}_{x_-}\right)\left(1 + \frac{\gamma - 1}{2}\left(\tilde{M}_{x_+} + \tilde{M}_{x_-}\right)\right)} \frac{s'}{c_p} .
\] (III.1)
The absolute value of the coefficient is an increasing function of $\bar{M}_{xn}^+$ (with $\bar{M}_{xn}^-$ held constant), and hence, pressure waves produced when the nozzle is unchoked ($\bar{M}_{xn}^+ < 1$) are weaker than if the nozzle is choked ($\bar{M}_{xn}^+ = 1$). Therefore, if the nozzle becomes unchoked, the instability mechanism may cease to be sustained because the amplitudes of the rarefactions produced at the nozzle are no longer strong enough to separate the flame and generate entropy waves. However, this condition is not considered in the present analysis because the nozzle is assumed to remain choked at all times.

If the necessary condition for instability is met, the amplitudes of entropy and pressure waves become larger with each subsequent cycle. However, the growth of this instability is limited by the maximum value that the amplitude of the entropy waves may assume, namely the amplitude based on the difference in specific entropy between the unreacted and completely-reacted fluid under constant pressure. High-speed films and calculations based on pressure-time traces show that this growth is very rapid (the final amplitude is reached within two or three cycles), and that the final value of the entropy wave amplitude is generally only slightly below the maximum value predicted by the theory.

III.4 Sustained Coupling of Wave-Boundary Interactions

As stated above, if the sustenance conditions for instability are met, the coupling between the wave-generation mechanisms embodied in the nozzle and combustion zone proceeds indefinitely. The events comprising
this coupling must, however, occur in a particular sequence. This sequence can be revealed by following the instability through an entire cycle.

Figure III.2 presents a sustained coupling of the two wave-boundary interactions in the t-x plane. A positive entropy wave \((h_1^-)\) is convected by the flow, generating a rarefaction \((p_1^-)\) as the entropy wave passes through the nozzle. This pressure wave propagates against the flow towards the combustion zone, and as it passes through the flame, it stretches the combusting fluid region, quenching the reaction and producing a negative entropy wave \((t_1^-)\) at the separation point \((\xi_5)\). (For the purpose of discussion, the increase in reactant velocity associated with the rarefaction is assumed to result in near blow-off conditions.) This entropy wave is convected through the nozzle, producing a compression \((p_1^+)\) that propagates against the flow toward the flame which, at this point, is toroidal in shape and is releasing only a small amount of heat to the fluid. As the wave passes through the combustion zone, it reduces the velocity of the oncoming reactant, allowing the reaction to re-intensify and forming a positive entropy wave \((h_2^+)\) through an increase in the heat release rate. This entropy wave is carried out of the combustion zone by the flow, bringing the instability cycle to its original state.

If the rarefaction \((p_1^-)\) does not increase the reactant flow velocity to a value near the blow-off level (corresponding to Case 2 in Figure II.15), the compression \((p_1^+)\) is not needed to re-establish the flame. Thus, the instability cycle under this condition differs slightly from the description above in the sense that the second entropy wave \((h_2^-)\)
forms spontaneously and is not directly associated with compression \( p_1^+ \). The compression, however, plays a role in decreasing the inlet flow velocity after the passage of a rarefaction to return the combustion process to its initial state after a complete cycle of the coupling.

Using the terminology defined in the previous chapter, the positive and negative entropy waves are referred to as heads and tails, respectively. Also the region of fluid following a head and preceding a tail is termed an entropy spot, and the region following a tail and preceding a head is termed an entropy depression. The reader should note that entropy spots are denoted in figures by the symbol \( \sigma \).

III.5 Low-Frequency Instability Modes

The instability mechanism described above can assume one of two modes, depending on the nature of the rarefaction-flame interaction. This section defines each of these modes and describes in detail the processes which characterize each mode.

One of the modes, shown in Figure III.2, consists of the wave-boundary interaction cycle presented in the previous section in which the \( n \)th cycle generates the \( (n+1) \)th cycle. The head \( (h_1) \) of entropy spot \( (\sigma_1) \) convects through a choked nozzle, generating rarefaction \( (p_1^-) \). This rarefaction propagates to the flame and quenches the reaction through stretching, creating separation \( (\xi_2) \), which forms the tail of entropy spot \( (\sigma_1) \). As tail \( (t_1) \) is converted through the nozzle, compression \( (p_1^+) \) is generated, which propagates to the combustion zone and restabilizes the combustion process, producing head \( (h_2) \) of entropy.
spot \((\sigma_2)\).

The other mode, presented in the \(t-x\) plane in Figure III.3, involves a somewhat more complicated coupling of wave-boundary interactions that is comprised of two oscillations, one in which the \(n^{th}\) cycle initiates the \((n+2)^{th}\) cycle, and another in which the \((n+1)^{th}\) cycle initiates the \((n+3)^{th}\) cycle. For the purpose of discussion, the two oscillations are termed "odd" and "even", depending on the numbers assigned to the entropy spots involved in the oscillation.

This mode of the instability proceeds as follows. The odd cycle is initiated by head \((h_1)\) of entropy spot \((\sigma_1)\), which passes through the nozzle and generates rarefaction \((p_1^-)\). This pressure wave propagates into the flame zone and stretches the flame, producing separation \((\zeta_3)\) and forming tail \((t_2)\) of entropy spot \((\sigma_2)\). The flame quickly re-establishes itself because, unlike the first mode, no stabilizing influence is necessary to return the combustion intensity to its original level, and head \((h_3)\) of entropy spot \((\sigma_3)\) forms spontaneously. This head is convected by the flow toward the nozzle, and the odd cycle returns to its original state.

Meanwhile the even oscillation goes through a similar series of events, but lags the odd oscillation by one-half of a period. Head \((h_2)\) generates rarefaction \((p_2^-)\), which stretches the flame and produces separation \((\zeta_4)\). The result of this separation is tail \((t_3)\) and head \((h_4)\). This head is convected toward the nozzle, bringing the even cycle to its original state.

One of the major differences between the entropy wave trains in
the two modes lies in the width of the entropy depressions. Since the time for re-establishment of the flame in the second mode is much less than in the first mode (the first mode requires the stabilizing effect of a compression wave produced at the nozzle, while the second mode does not), the entropy depressions in the second mode are thinner. This difference proves useful for distinguishing between the two modes in the high speed films.

Another major difference between the two modes is that the compression waves \( p_n^+ \) produced by the tails \( t_n \) do not play an active role in the instability cycle in the second mode. The compressions serve only to decelerate the flow (which was accelerated by the previous rarefaction \( p_n^- \)) to return the conditions in the inlet and combustor to the state before the cycle occurred.

III.6 Instability Frequency

The frequency of a steady-state oscillation is given by the inverse of the amount of time required for the oscillation to go through one cycle

\[
f = \frac{1}{\tau} \quad \text{(III.2)}
\]

In the case of the first instability mode, the time required for one cycle is the sum of the times required for: (1) the convection of a head \( h_n \) from the separation point in the flame zone \( \mathcal{S}_s \), (2) the propagation of a rarefaction \( p_n^- \) from the nozzle to the flame zone, (3) the convection of the generated tail \( t_n \) from the flame zone to the
nozzle, and (4) the propagation of a compression \( p_n \) from the nozzle to the flame zone to produce another head \( h_{n+1} \) (see Figure III.2).

These times are given by

\[
\tau_\sigma = \frac{L_c (1-\hat{x}_s)}{\bar{u}} \quad (\text{III.3a})
\]

\[
\tau_p = \frac{L_c (1-\hat{x}_s)}{(\bar{a} - \bar{u})} \quad (\text{III.3b})
\]

where \( \tau_\sigma \) and \( \tau_p \) are the transit times for entropy and pressure waves, respectively, \( \hat{x}_s \) is the point at which the flame separates, \( \bar{u} \) and \( \bar{a} \) are mean flow and acoustic velocities, and \( L_c \) is the combustor length.

The sum of the times required for a complete cycle of the instability in the first mode is

\[
\tau_I = \frac{2L_c (1-\hat{x}_s)}{\bar{u}} + \frac{2L_c (1-\hat{x}_s)}{(\bar{a} - \bar{u})} \quad (\text{III.4a})
\]

\[
= \frac{2L_c (1-\hat{x}_s)\bar{a}}{\bar{u}(\bar{a} - \bar{u})} \quad (\text{III.4b})
\]

According to Eq. (III.2), the frequency of this mode is given by

\[
f_I = \frac{\bar{u}(\bar{a} - \bar{u})}{2L_c \bar{a}(1-\hat{x}_s)} \quad (\text{III.5})
\]

For the second mode of the instability, the time required for one cycle of the instability is defined to be the time between the formation
of the \(n^{th}\) separation and the \((n+2)^{th}\) separation (see Figure III.3).

Note that the frequency derived from this definition is twice as low as the actual measured frequency (defined to be the frequency at which compression waves \(p^+_n\) are generated at the nozzle), a fact which should be taken into account when comparing observed frequencies to predicted frequencies.

By the definition above, the time required for one cycle of the instability in the second mode is comprised of the time necessary for

1. the convection of head \(h_n\) from the separation point \(\hat{x}_s\) to the nozzle,
2. the propagation of a rarefaction from the nozzle to the flame zone to produce a separation \(\zeta_{n+2}\), and
3. the re-establishment of the flame after the passage of the rarefaction to produce head \(h_{n+2}\).

This sum is expressed as:

\[
\tau_{II} = \frac{L_c(1-\hat{x}_s)}{\bar{u}} + \frac{L_c(1-\hat{x}_s)}{\bar{u}(\bar{a}-\bar{u})} + \tau_{D}^{II}
\]  

\[= \frac{-\tilde{a}L_c(1-\hat{x}_s) + \bar{u}\tau_{D}^{II}(\bar{a}-\bar{u})}{\bar{u}(\bar{a}-\bar{u})}
\]  

\[(III.6a)\]

\[(III.6b)\]

where \(\tau_{D}^{II}\) is the re-establishment delay time (see Figure III.3).

Inverting Eq. (III.6b) yields an expression for the frequency of the instability in the second mode:

\[
f_{II} = \frac{\bar{u}(\bar{a}-\bar{u})}{-\tilde{a}L_c(1-\hat{x}_s) + \bar{u}\tau_{D}^{II}(\bar{a}-\bar{u})}
\]

\[\text{(III.7)}\]
A general expression for the frequency may be obtained by noting that, in the first mode, the delay time for re-establishment of the flame after the passage of a rarefaction is simply the sum of times required for the latter two traverses, namely the traverses of tail ($t_n$) and compression ($p_n^+$) (see Figure III.2) Thus the general expression is

$$f_N = \frac{\bar{u}(\bar{a}-\bar{u})}{\bar{a}L(1-\bar{a}) + \bar{u}r_D^{N}(\bar{a}-\bar{u})}$$

$$N = I, II$$

(III.8)

where $\tau_D^{N}$ is the delay time for the $N^{th}$ mode of the instability.

Furthermore, the frequency of a mode ($f_N$) is related to the measured frequency ($f'_N$) by the relation

$$f'_N = nf_N$$

(III.9)

where

$$n = \begin{cases} 
1 & \text{for } N = I \\
2 & \text{for } N = II 
\end{cases}$$

This relation is derived by noting the frequency of the wave generation process at the nozzle.
IV EXPERIMENTAL PROGRAM

IV.1 Introduction

The experimental program investigating the low-frequency instability consists of two parts: (1) high speed films of the combustion process in a quartz combustor, and (2) instrumented combustor testing over a wide range of combustor geometries and operating conditions. The results of this program are used in the following chapters to verify predictions made by the proposed theory and to develop methods for suppression of the low-frequency instability.

References [3 and 17] should be consulted for details of the experimental methods.

IV.2 High-Speed Photography

The objective of the high-speed filming of a dump combustor was to provide insights into the characteristics of the combustion process under various operating conditions. The films proved useful to this research effort because they revealed two modes of the low-frequency instability.

The apparatus used for the filming consists of a high-speed camera (500 frames per second) and a quartz combustor 2 feet in length and 0.5 feet in diameter. The combustor was operated at a mean pressure of 1 atmosphere and with a mass flow rate of 2 lbm/sec.
Despite lack of a complete list of operating conditions, the values of other parameters can be calculated from the information provided and the slopes of the entropy wave trajectories (which equal the flow velocity). These calculations are presented in Appendix D.

Figure IV.1 presents a typical frame from one of the high-speed films and a corresponding schematic representation of the gas flowing through the combustor. The bright spots represent completely-combusted gases or entropy elevations, and the dark regions are uncombusted gases termed entropy depressions.

An unfortunate aspect of this portion of the experimental program is that the filming was performed entirely without instrumentation. Consequently, pressure fluctuations associated with the motion of the hot and cool regions of gas were not recorded. Nevertheless, the films prove useful because they confirm the existence of entropy waves in the combustor flow, and reveal the mechanism by which the entropy waves are formed.

IV.3 Prototype Combustor Testing

The objective of the second portion of the experimental program was to test a prototype dump combustor. One aspect of the testing involved monitoring low-frequency, large-amplitude pressure oscillations that occurred during operation. The purpose of this part of the testing was to determine the effects of changes in combustor geometry and operating conditions on the instability producing the pressure fluctuations. Baseline conditions, listed in Table IV.1, were used to compare the
effects of individual changes in combustor geometry and operating conditions on the instability.

The apparatus used for the testing is shown in Figure IV.2. It consists of a main burner, fuel injection system, the dump combustor, and an exhauster.

The pressure fluctuations were recorded using Kistler pressure transducers mounted in various points on the combustor. Figure IV.3 shows the transducer positions.

The signals from the pressure transducers were recorded in the form of oscillograms. Fast Fourier transforms (FFT) were performed on the recorded waveforms to determine the dominant frequencies and corresponding amplitudes. Figure IV.4 presents a typical record and its corresponding FFT.

Other data concerning the operation of the combustor was collected by means of a Mod Comp II computer-controlled data acquisition system and stored in files, each corresponding to a particular combination of combustor geometry and operating conditions. This information included mass flow rate, mean combustor pressure, mean temperatures in the inlet and combustor, quantities necessary for computation of predicted frequencies and amplitudes.
V.1 Introduction

In this chapter, a comparison is made between the predictions of the proposed low-frequency instability theory and the results of the experimental program. The topics considered include: (1) visually-observable aspects of the instability, (2) instability frequency, (3) entropy and pressure wave amplitudes and (4) effects of changes in combustor geometry and operating conditions on the instability.

The agreement between predictions and observations in each of these areas is good.

V.2 Comparison between Observations from High-Speed Films and the Proposed Model

Figures V.1a-i present a series of frames from a high-speed film of the instability (500 frames/sec). The schematic representation is defined in Chapter IV.

Clearly visible are (1) spatial periodicity in the brightness (and hence, entropy) of the flow, (2) convection to the left (i.e., with the flow) of entropy spots and (3) stretching of the flame and the formation of entropy depressions (shaded regions) by flame separation. Less obvious is the fact that a separation occurs about one frame after a head of an entropy spot passes through the nozzle, an important element of the proposed instability theory.
A more convenient method of representation of a series film frames is the t-x diagram, which is used in the development of the proposed theory. This method converts the observed instabilities into a form that can be easily compared to the idealized form of the instability, and allows plotting of pressure wave trajectories from theoretical considerations.

Figures V.2 and V.3 present such t-x diagrams of the movements of entropy spots constructed from two series of high-speed film frames. The heads of the entropy spots are indicated by h's and the tails by t's; the movement of the heads and tails are designated by dashed lines connecting the symbols.

The wide entropy depressions in the wave diagram in Figure V.2 suggest that this mode is the first mode of the instability (see Figure III.2). Indeed, when one completes the wave diagram with compressions (p⁺) and rarefactions (p⁻) drawn from theoretical considerations, the wave motion exactly resembles that of the first mode: a head generates a rarefaction at the nozzle, which propagates against the flow, separates the flame, and produces the tail of the first entropy spot as well as the head of the next entropy spot.

On the other hand, the narrow entropy depressions in Figure V.3 liken this wave pattern to that of the second mode (see Figure III.3). After inserting hypothetical pressure waves, one finds that the cycle of events matches the series of wave-boundary interactions characterizing the second mode, including "odd" and "even" sub-cycles.

These two diagrams illustrate a major difference between the two
modes: the delay time associated with the re-establishment of the flame after the passage of a rarefaction is considerably longer for the first mode. In fact, the delay time exactly equals the total time required for the convection of a tail (t) from the separation point to the nozzle and the propagation of a compression (p⁺) from the nozzle to the flame zone. This difference may be attributed to the large difference in flow velocity in the combustor. The mean flow velocity for the case in Figure V.2 is much higher than the mean flow velocity for the second case shown in Figure V.3, and hence, one may hypothesize that the conditions at the flame in the former case are closer to blow-off conditions than in the latter. Therefore, a rarefaction (p⁻) may increase the flow velocity at the flame in the first case to the point where a stream of uncombusted gas passes through the flame zone, decreasing the combustion efficiency greatly; this condition persists until a compression (p⁺) decreases the flow velocity and re-stabilizes the flame.

Meanwhile, in the second case, the flow velocity is much farther below the blow-off velocity than in the first case. Consequently, a rarefaction stretches and separates the flame, but the flame returns to a state of high combustion intensity after the passage of the rarefaction without requiring the re-stabilizing effect of a compression because the flow velocity after the passage of the rarefaction remains below the range where the flame becomes toroidal and very inefficient (see Figure II.15).

The delay time discussed in the definition of frequency is the amount of time required for the re-stabilization of the flame. At
present, the delay time for the second mode cannot be derived analytically, and hence the values used for frequency calculations were based on observed delay times in the high speed movies. In most cases when the second mode was observed, the delay time was approximately the time between two consecutive frames (i.e., $t_D^\text{II} \approx 0.002$ sec).

V.3 Instability Frequency

Since the conditions in the combustor tests with instrumentation are similar to the conditions which produce the first mode in the high-speed films of the quartz combustor, the frequency predictions are based on the frequency expression for the first mode (Eq. (III.5)). The major factors leading to this assumption are the relatively high mean combustor temperatures and large flow velocities in both the quartz combustor tests in which the first mode occurs and the prototype combustor tests.

Figure V.4 presents a comparison between the frequencies observed in the prototype combustor tests and the frequencies predicted from the operating conditions and Eq. (III.5). Over the entire range of observed frequencies, the agreement between theory and experiment is good.

Also, the relationship between instability frequency and the combustor length is apparent in Figure V.4. The frequencies corresponding to the longest combustor ($L_c = 3.0$ ft) fall in the lower portion of the observed frequency range, while the frequencies associated with the shorter combustor lengths, $L_c = 2.0$ ft and $L_c = 1.5$ ft, fall in the central and upper portions, respectively. This trend is consistent
with the low-frequency instability model, which, through Eq. (III.5), specifies that the frequency of the instability is inversely proportional to the combustor length when all other parameters are held constant.

Any discrepancy between the observed and predicted frequencies may be attributed to (1) idealizations made by the model, and (2) experimental error. The error may occur, in the first case, through the use of average values of mean flow and acoustic velocities; perhaps, in many instances, an accurate prediction of frequency requires consideration of changes in flow and acoustic velocities along the length of the combustor.

In the second case, the error may enter through the methods of gathering data. First, the measurements of the operating conditions for each test is specified by the experimenters to be accurate only to within ±10%. Second, much of the information concerning the operating conditions was calculated indirectly from more easily measured quantities (e.g., the total temperature in the combustor was calculated from the mean thrust produced by the combustor); this process introduces uncertainty not only through measurement error, but also through idealizations made in the calculation.

A sample frequency calculation is presented in Appendix E.1.

V.4 Pressure Fluctuation Amplitude

According to the proposed theory, the maximum amplitudes that the entropy waves in a combustor may assume is given by
\[
\frac{s_p - s_R}{c_p} = \frac{2n T_{ad} R}{T_R}.
\]

This maximum amplitude is given by the ideal gas relation

\[
\frac{s_p - s_R}{c_p} = 2n \frac{T_{ad} R}{T_R}.
\]

Where \(T_{ad}\) is the adiabatic flame temperature, and \(T_R\) is the temperature of the oncoming reactants.

Inserting this value into Eq. (II.30), one obtains the peak-to-peak* pressure fluctuation amplitude that can be produced at the nozzle at the given conditions. These fluctuations propagate from the nozzle and are felt throughout the combustor, including the combustion zone.

Figure V.5 presents a comparison between the amplitudes of observed pressure fluctuations and the maximum amplitudes allowed by the theory. The diagram is interpreted as follows.

If the observed amplitude coincides with the predicted maximum value, the entropy wave amplitude is at the maximum level predicted by the proposed model, and consequently, the amplitude of pressure waves produced at the nozzle is at its corresponding maximum level. However, if the observed amplitude is below the diagonal line, the amplitude of the entropy waves is below the maximum value. This situation is expected

*Since \(s_p - s_R / c_p\) is the peak-to-peak amplitude of the entropy fluctuations, the calculated pressure fluctuation amplitude is a peak-to-peak value as well.
because (1) the adiabatic flame temperature is usually not reached, and
(2) the entropy level in the entropy depressions is slightly higher than
that in the oncoming reactant flow since the flame is never extinguished
entirely and always produces some entropy.

However, the observed amplitude of the pressure fluctuations should
not exceed the theoretical maximum value; such a case would imply one
of the following: (1) the amplitude of the entropy waves produced in
the combustion zone exceed the value given by Eq. (II.31), (2) the
pressure waves produced at the nozzle are amplified as they propagate
from the nozzle, or (3) the entropy wave-nozzle interaction involves
effects that contribute to the amplitude of the generated pressure wave,
but are neglected by the present theory. The first condition would
require an entropy source besides the combustion process that adds
entropy only to regions in which the entropy is already at a relatively
high level, a situation which is unlikely. The latter two possibilities,
however, are quite plausible.

First, Toong, Abouseif, et al., [18,19,20,21] have shown that
acoustic disturbances in a reacting mixture, under the proper conditions,
have the capability to amplify through chemi-acoustic interactions.
Conceivably, the conditions in the exhaust flow from the combustion zone,
which in most circumstances contains reacting regions of gas (the
proposed model assumes that the reaction is frozen after the fluid leaves
the combustion zone), may be conducive to such amplification, accounting
for pressure fluctuation amplitudes which are above the theoretical
maximum amplitude predicted by the theory presented earlier.
Second, under certain conditions, effects that are considered negligible in the analysis of the entropy wave-nozzle interaction become important and contribute to the amplitude of the generated pressure wave. For example, the geometry of the nozzle, a variable neglected by the compact nozzle assumption, may be under certain conditions a considerable influence on the amplitude of the generated pressure waves.

For most of the prototype combustor tests, the observed pressure fluctuation amplitudes are below the maximum value allowed by the theory. In a few cases, though, the observed pressure fluctuation amplitudes are above the theoretical maximum level, a condition that implies that the factors neglected by the theory proposed in this thesis are important in these cases. Refinements of the entropy wave-nozzle interaction made should be focused on identification of these factors and the conditions under which they become important.

A sample pressure fluctuation amplitude calculation is presented in Appendix E.2.

V.5 Effects of Variations in Combustor Geometry and Operating Conditions

The conditions conducive to the sustenance of an instability depend greatly on the geometry of the combustor and the operating conditions. These factors must combine in a manner which satisfies the requirement for sustenance of an instability. If this requirement is not met, a link in the instability cycle is removed and the instability subsides. Knowledge of the combinations of combustor geometries and operating
conditions which break the instability cycle is of paramount importance for the development of guidelines for the suppression of the instability, a topic discussed at length in the next chapter.

Figures V.6, V.7 and V.8 show the percent rates of occurrence of the low-frequency instability in tests over a range of flameholder types, inlet diameter-combustor diameter ratios, and inlet temperatures. The percent rate of occurrence is defined as the fraction of tests from a group of tests with a particular geometry and set of operating conditions which demonstrate instability.

The effect of the presence and size of the flameholder can be seen in all three figures. Figure V.6, which compares tests without flameholders to tests with small (25% blockage) and large (35% blockage) flameholders, clearly shows that the presence of even a small flameholder has a marked effect on the instability (except when the inlet diameter-to-combustor diameter ratio is large), and that the large flameholder is extremely effective in breaking the instability cycle regardless of the diameter ratio.

In Figure V.7, this effect can be seen from a different viewpoint, but the conclusion is the same as above: use of a flameholder decreases the rate of instability occurrence significantly.

Finally, in Figure V.8, two sets of tests, one with a flameholder and one without, which investigate the effect of changes in inlet temperature show that use of a flameholder decreases the rate of instability occurrence regardless of the inlet temperature.

The decreases in the rates of instability occurrence associated
with the use and size of flameholders can be explained quite easily using the theory proposed in this thesis. The flameholder is a much better flame-stabilization mechanism than the recirculation zones, making the combustion process more resistant to velocity fluctuations associated with pressure waves. Therefore, a flameholder raises the threshold level of rarefaction strength (see Chapter II for a discussion of threshold rarefaction strength) and decreases the likelihood that an instability develops. Also, a larger flameholder stabilizes the flame better than a small one, and hence the threshold level associated with the large flameholder is higher.

Another variation in combustor geometry considered is the ratio between the inlet diameter and the combustor diameter. Figures V.6 and V.7 show that as this ratio is decreased, the rate of instability, in general, decreases. The sole exception is the increase as the diameter ratio is decreased from 0.67 to 0.59 without a flameholder; however, this increase may be attributed to the small number of samples rather than some physical reason that makes conditions more conducive to instability as the diameter ratio is decreased in this range.

As before, the theory presented in this thesis provides a reasonable explanation for this behavior. The decrease in the diameter ratio increases the relative size of the recirculation zones, increasing the stability of the combustion process and making it less susceptible to velocity fluctuations. Hence, decreasing the inlet diameter-to-combustor diameter ratio produces the same effect as increasing the size of a flameholder: the threshold strength of rarefactions required for
sustenance of a low-frequency instability is increased.

Finally, a variation in operating conditions is considered in Figure V.8. As the inlet temperature, the temperature of the oncoming reactants, is increased, the rate of occurrence of the low-frequency instability decreases. This phenomenon, also, may be explained by the proposed instability theory. The initial reaction rate, which is strongly dependent on the temperature of the reactants, increases quickly as the reactant temperature increases. Consequently, reactants at a higher temperature are consumed more quickly, resulting in a shorter combustion zone. A decreased combustion zone length renders the combustion process more resistant to flame stretching, and hence the minimum value of the strength of rarefactions required to produce an entropy wave increases. Thus, in cases where the threshold rarefaction strength rises above the maximum value of rarefaction amplitude that the entropy wave-nozzle interaction can produce after the reactant temperature is increased, the instability is eliminated.

V.6 Concluding Remarks

The agreement between the proposed low-frequency instability theory and the experimental results is generally good. First, the high-speed films reveal drifting regions of combusted and uncombusted gases, which constitute entropy waves. Secondly, the frequency and amplitude correlations indicate that the model predicts both of these aspects of the instability very reliably. Finally, the experimental variations in combustor geometry and operating conditions produce the effects that
the model predicts. This overwhelming agreement tends to indicate that the model indeed captures the essence of the low-frequency instability.
VI GUIDELINES FOR SUPPRESSION OF THE LOW-FREQUENCY INSTABILITY

VI.1 Introduction

Although the low-frequency mechanism appears to be fairly-well understood, as evidenced by the previous chapter, this understanding is not the ultimate goal of this research effort. The primary purpose of this program is to develop a set of guidelines for the suppression of the low-frequency instability. This chapter presents a number of suggestions which, according to the proposed theory, should reduce the rate of occurrence of the instability or eliminate it entirely.

VI.2 Experimentally-Verified Guidelines

This section reiterates the three effects which suppress the low-frequency instability that are discussed in the previous chapter, and makes recommendations for design of combustors that tend to be free of low-frequency instabilities. These three effects, which are verified by the results of the experimental program, include decreases in the rate of occurrence of the instability with (1) use of a flameholder, (2) decreases in the ratio of the inlet diameter to the combustor diameter, and (3) increases in the temperature of the oncoming reactants.

First, flame stabilization is an important issue in eliminating the low-frequency instability; accordingly, a flameholder, one of the most effective methods of stabilizing a combustion process, should be included in the design of an instability-free combustor. Since total
pressure loss minimization is a major concern in many applications of
dump combustors, a small flameholder should be used to avoid dissipation
of total pressure beyond permissible levels. In effect, a balance must
be achieved between conservation of total pressure and flame stabilization.

Another means of flame stabilization, also shown to reduce the rate
of instability occurrence, is the use of a small inlet diameter-to-
combustor diameter ratio. Therefore, the change in duct area across the
dump plane should be as large as possible to provide adequate space for
the formation of large recirculation zones, which help stabilize the
combustion process and reduce the size of the flameholder needed to
deactivate the entropy wave generation mechanism. A drawback of a reduced
inlet size is that pressure losses increase with decreased duct size at a
constant volume flow rate. These losses should be considered when the
optimum diameter ratio is determined.

Finally, both theory and experiment indicate that increases in the
temperature of the oncoming reactant decrease the likelihood of instability
occurrence by increasing the initial reaction rate of the combustion
process and making the flame more resistant to velocity fluctuations.
Thus, to suppress the entropy wave generation mechanism embodied in the
flame, the inlet temperature should be maintained at or above a specified
level. Preheating the reactants, however, introduces the need for a
heat source which may add weight and volume to the device utilizing the
dump combustor concept, which ultimately decreases the effectiveness of
the device through increased drag. A possible heat source, which minimizes
the increase in weight by not requiring that additional fuel for heating
the reactants be carried by the device, is the dump combustor itself; the heat shed by the walls of the combustor can be transported upstream by some means and added to the reactants before they enter the combustion zone.

VI.3 Other Instability Suppression Methods

This section presents three additional low-frequency instability suppression methods predicted by the proposed model, but yet untested experimentally. These methods include: (1) reduction of the approach Mach number to decrease the strength of pressure waves generated by convected entropy waves, (2) lengthening of the combustor to provide more time for temperature gradients to reduce entropy depressions through heat conduction, and (3) use of a fuel which minimizes the extinguishing effects of flame stretching, and reignites quickly to fill in entropy depressions.

The first method is based on the observation that the theoretical pressure wave amplitude is an increasing function of the nozzle approach Mach number (see Eq. (II.30)). Therefore, to reduce the amplitude of the pressure waves generated at the nozzle to a level below the threshold strength, the approach Mach number should be decreased, perhaps, through an increase in the combustor diameter. Note that increasing the combustor diameter while holding the inlet diameter constant also decreases the diameter ratio, helping stabilize the flame by providing space for a large recirculation zone, and consequently rendering the conditions in the
combustion zone less susceptible to flame separation. A disadvantage of this suggestion is that a larger combustor diameter leads to a larger amount of frontal drag, reducing the effectiveness of the device.

Another method for suppressing the low-frequency instability is to increase the length of the combustor to provide time for temperature gradients to reduce the amplitudes of the entropy waves by conducting entropy from entropy spots into entropy depressions. A clear disadvantage of this method is the increased weight and volume of the device, which ultimately leads to effectiveness losses through drag.

Finally, use of a fuel (or fuel additive) which makes the flame relatively insensitive to velocity fluctuations could conceivably raise the threshold level of rarefaction strength above the maximum level that an entropy wave-nozzle interaction could produce, preventing the conditions in the combustor from meeting the sustenance requirement. The difficulty with this method, which, unlike all of the previous suggestions, does not introduce effectiveness losses through mechanical means, lies in the development of such a substance.

VI.4 Recommendations for Future Work

Although the theory of low-frequency instability in its present form is a useful tool for developing ideas for suppression of the instability, many aspects of the problem remain unsolved. For example, the interaction between rarefactions and flames presently can be dealt with only on a qualitative basis, and calculations of wave amplitudes are restricted to
estimates from the maximum value that a wave amplitude can assume. The author would like to suggest a number of topics for future research which will lead to a better overall understanding of the low-frequency instability.

The largest void in the present theory is an absence of a quantitative pressure wave-flame interaction analysis. Since this behavior is an extremely important aspect of the proposed model, future research should, in part, concentrate on developing a theory parallel to the entropy wave-nozzle theory developed in this thesis. However, the proposed problem is much more difficult to solve analytically because the method of obtaining perturbation expressions and integrating to obtain corresponding large-amplitude expressions may not work as easily as for the entropy wave-nozzle analysis. The author suggests an approach which numerically integrates the highly nonlinear set of conservation and combustion equations to determine the behavior of a flame as large-amplitude pressure waves pass through the combustion zone. As well as providing insight into the low-frequency instability, this work would be useful to research in other problems which involve the propagation of a pressure wave through a flame.

Upon completion of a thorough analysis of the pressure wave-flame interaction, the results of this work should be incorporated into the composite low-frequency instability model. A quantitative understanding of the behavior of the flame should permit specification of the conditions that are unfavorable for generation of entropy waves, the most direct approach to breaking the instability cycle.
Also, a number of areas require experimental investigation. First, the theory of the two wave-boundary interactions should be verified to ensure that the wave-generating mechanism are properly understood. Second, a series of tests with a quartz combustor using simultaneous high-speed filming (at speeds no lower than 1000 frames per second) and instrumentation should be performed to verify that the wave-boundary coupling proposed in this thesis is correct. Finally, a study of the instability suppression methods developed from theoretical considerations should be made to determine their effectiveness.
VII SUMMARY AND CONCLUSIONS

The low-frequency instability mechanism is modelled as a coupling between entropy wave and pressure wave sources embodied in the combustion zone and choked nozzle, respectively. Pressure disturbances, produced as entropy waves (spatial entropy fluctuations that are convected by the flow) pass through the nozzle, propagate from the nozzle to the combustion zone and affect the combustion process, generating entropy waves as the rate of entropy production by the flame is disturbed. Closed-form expressions for the maximum amplitude that the entropy waves and pressure waves can assume, as well as for the frequency of two modes of coupling, are derived from the theory.

The theory is compared to the results of an experimental program (conducted at the Wright Aeronautical Laboratories, Wright-Patterson Air Force Base in Dayton, Ohio) which consisted of two parts: (1) high-speed filming (500 frames/sec) of a quartz combustor, and (2) instrumented testing of prototype combustors of various geometries operated over a range of conditions. The t-x diagrams constructed from the films are similar in form to the two modes predicted by the coupling theory, and the predicted frequencies agree well with those observed experimentally. Also, the observed amplitudes, with the exception of a few cases, fall below the maximum value that should be expected; amplitudes greater than the theoretical maximum value are attributed to effects not included in the model (e.g., chemi-acoustic interactions and the effect of nozzle geometry).

The predictions and observations are used to develop guidelines for
suppressing the instability. These guidelines include recommendations for:
(1) use of a large ratio between the inlet and combustor diameters to
provide space for larger recirculation zones, (2) use of a small flame-
holder to aid the recirculation zones in stabilizing the flame, hence
making the flame more resistant to velocity fluctuations, and (3) pre-
heating the oncoming reactants to increase the initial reaction rate,
thereby reducing the length of the combustion zone and rendering the
flame less susceptible to flame-stretching.

Despite apparent success with this model, additional work is required
to completely characterize the behavior of the low-frequency instability.
This work includes: (1) a quantitative analysis of the pressure wave-
flame interactions, (2) incorporation of the quantitative pressure-wave-
flame theory with the proposed low-frequency instability model, and
(3) experimental investigations to test the validity of the analyses of
the wave-boundary (i.e., entropy wave-nozzle and pressure wave-flame)
interactions and the coupling theory, and to determine the effectiveness
of the proposed solutions for suppressing the instability.
REFERENCES


11. Rayleigh, J.W.S. Theory of Sound. (Dover, 1945), Volumes I and II.


APPENDIX A

A.1 Proof of the Constant-Velocity Condition across an Entropy Wave

A flow containing an entropy wave satisfies the following expressions (see Figure A.1 for definition of notation):

**mass:**

\[ \rho_1 \frac{d\xi}{dt} - \rho_2 \frac{d\xi}{dt} - \rho_1 u_1 + \rho_2 u_2 = 0 \]  \hspace{1cm} (A.1.1)

**momentum:**

\[ \rho_1 u_1 \frac{d\xi}{dt} - \rho_2 u_2 \frac{d\xi}{dt} - \rho_1 u_1^2 + \rho_2 u_2^2 = 0 \]  \hspace{1cm} (A.1.2)

Solving Eq. (A.1.1) and (A.1.2) for \( \frac{d\xi}{dt} \),

\[ \frac{d\xi}{dt} = \frac{\rho_1 u_1 - \rho_2 u_2}{\rho_1 - \rho_2} \]  \hspace{1cm} (A.1.3a)

\[ \frac{d\xi}{dt} = \frac{\rho_1 u_1^2 - \rho_2 u_2^2}{\rho_1 u_1 - \rho_2 u_2} \]  \hspace{1cm} (A.1.3b)

Equating these two expressions yields

\[ \frac{u_2^2}{u_1^2} - 2\left(\frac{u_2}{u_1}\right) + 1 = 0 \]  \hspace{1cm} (A.1.4)

note the lack of dependence on \( \rho_1 \) or \( \rho_2 \).

The only solution to Eq. (A.1.4) is the double root
\[ \frac{u_2}{u_1} = 1 \quad \text{(A.1.5a)} \]

or

\[ u_2 = u_1 \quad , \quad \text{(A.1.5b)} \]

demonstrating that the flow velocity does not change across an entropy wave.

### A.2 Entropy Wave Relations

The amplitude of an entropy wave \((\Delta s/c_p)\) and the uniform pressure constraint provide complete information for computing property changes across an entropy wave. These property changes are determined by transforming the exact expressions into perturbation form, and then integrating these expressions to obtain expressions for large-amplitude entropy waves.

**Exact equations:**

**entropy:**

\[ s - s_1 = c_p \ln \left( \frac{T}{T_1} \right) - R \ln \left( \frac{p}{p_1} \right) \quad \text{(A.2.1a)} \]

**state:**

\[ p = \rho RT \quad \text{(A.2.1b)} \]
mass:

\[ \frac{\dot{m}}{A} = \rho u \]  

(A.2.1c)

Mach number:

\[ M = \frac{u}{a} \]  

(A.2.1d)

total temperature:

\[ T_o = (1 + \frac{\gamma - 1}{2} M^2) T \]  

(A.2.1e)

total pressure\(^\dagger\):

\[ P_o = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma - 1}} p \]  

(A.2.1f)

Assuming constant pressure, the entropy relation Eq. (A.2.1a) in perturbation form is

\[ \frac{s'}{c_p} = \frac{T'}{T} - (\gamma - 1) \frac{p'}{\gamma p} \]

\[ = \frac{T'}{T} \]  

(A.2.2)

\(^\dagger\)Although this equation assumes that entropy remains constant for relating \( P_o \) to \( p \), its use in this context is valid for the following reason: the equation is applied on either side of an entropy wave to calculate the total pressure change across an entropy wavefront; however, the fluid particles to which the equation is applied never experience a change in entropy (i.e., \( \frac{D s}{D t} = 0 \)).
The state equation in perturbation form is given by:

\[
\frac{p'}{p} = \frac{T'}{T} + \frac{\rho'}{\rho} ; \tag{A.2.3}
\]

combining Eqs. (A.2.2) and (A.2.3) yields

\[
\frac{\rho'}{\rho} = -\frac{s'}{c_p} . \tag{A.2.4}
\]

The fractional variation in mass flow rate is expressed as

\[
\frac{\dot{m}'}{\dot{m}} = \frac{\rho'}{\rho} + \frac{u'}{a} \frac{1}{M} . \tag{A.2.5}
\]

For an entropy wave, \(u'/\bar{a} = 0\), and the density change is given by Eq. (A.2.4); hence

\[
\frac{\dot{m}'}{\dot{m}} = -\frac{s'}{c_p} \tag{A.2.6}
\]

The Mach number, given by Eq. (A.2.1d), is written in perturbation form as

\[
\frac{M'}{M} = \frac{u'}{a} \frac{1}{M} - \frac{1}{2} \frac{T'}{T} ; \tag{A.2.7}
\]

substituting the entropy relations for \(u'/\bar{a}\) and \(T'/\bar{T}\) yields

\[
\frac{M'}{M} = -\frac{1}{2} \frac{s'}{c_p} . \tag{A.2.8}
\]
The fractional variations of total temperature and total pressure, Eqs. (A.2.1e) and (A.2.1f), are expressed as

\[
\frac{T'}{T_0} = \frac{(P_\infty') - \rho' \bar{u}'}{\gamma P_\infty} + (\gamma-1) \frac{\bar{M}}{a} \frac{u'}{\bar{a}} \\
1 + \frac{\gamma-1}{2} \frac{\bar{M}^2}{a}
\]  
(A.2.9)

\[
\frac{P'}{P_0} = \frac{\bar{M} \frac{u'}{a} + \frac{\gamma \bar{M}^2}{2} \frac{\rho'}{\rho} + \gamma(1 - \frac{1}{2} \bar{M}^2) \frac{P'}{\gamma P_0}}{1 + \frac{\gamma-1}{2} \frac{\bar{M}^2}{a}}
\]
(A.2.10)

Once again, noting that \( p'/\bar{p} \) and \( u'/\bar{a} \) are zero, and \( \rho'/\bar{\rho} \) is given by Eq. (A.2.4), these expressions become

\[
\frac{T'}{T_0} = \frac{1}{1 + \frac{\gamma-1}{2} \frac{\bar{M}^2}{a} \frac{s'}{c_p}}
\]  
(A.2.11)

\[
\frac{P'}{P_0} = -\frac{\frac{1}{2} \frac{\gamma \bar{M}^2}{2} \frac{s'}{c_p}}{1 + \frac{\gamma-1}{2} \frac{\bar{M}^2}{a}}
\]
(A.2.12)

These equations are now integrated from some reference condition denoted by \((\ )_1\). The resulting expressions are:

\[
\frac{(s-s_1)}{c_p} = \frac{T-T_1}{T_1} = e - 1
\]  
(A.2.13a)

\[
\frac{(s-s_1)}{c_p} = \frac{\rho-\rho_1}{\rho_1} = e - 1
\]  
(A.2.13b)
\[
\frac{m - m_1}{m_1} = e^{-\frac{(s-s_1)}{c_p} - 1} \quad (A.2.13c)
\]

\[
\frac{M - M_1}{M_1} = e^{-\frac{(s-s_1)}{2c_p} - 1} \quad (A.2.13d)
\]

\[
\frac{\int_{s_1}^{s} \frac{1}{1 + \frac{\gamma - 1}{2} M^2(s)} ds}{T_{o,1} - T_{o,1}} = e^{-1} \quad (A.2.13e)
\]

\[
\frac{\int_{s_1}^{s} \frac{\gamma}{2} \frac{M^2(s)}{1 + \frac{\gamma - 1}{2} M^2(s)} ds}{p_{o,1} - p_{o,1}} = e^{-1} \quad (A.2.13f)
\]

These results are summarized in Table II.2.
APPENDIX B

Acoustic Wave Relations

An acoustic wave is defined as a pressure wave across which the pressure (and velocity) changes isentropically. Property changes across an acoustic wave may be obtained from the perturbation forms of Eqs. (A.1.1a)-(A.1.1f), noting that for an acoustic wave

\[ \frac{u'}{a} = \pm \frac{P'}{\gamma P} \]  \hspace{1cm} (B.1a)

and

\[ \frac{s'}{c_p} = 0 \]  \hspace{1cm} (B.1b)

(In Eq. (B.1a), the (t) and (−) signs refer to waves propagating in the positive and negative x-directions, respectively.)

This yields:

\[ \frac{T'}{T} = (\gamma - 1) \frac{P'}{\gamma P} \]  \hspace{1cm} (B.2a)

\[ \frac{\rho'}{\rho} = \frac{P'}{\gamma P} \]  \hspace{1cm} (B.2b)

\[ \frac{\dot{m}'}{\dot{m}} = \frac{\dot{M} + 1}{\dot{M}} \frac{P'}{\gamma P} \]  \hspace{1cm} (B.2c)
\begin{align}
\frac{M'}{\bar{M}} &= \pm 1 - \frac{1}{2} (\gamma - 1) \frac{\bar{M}}{\bar{M}} \frac{p'}{\gamma p} \\
& \quad \text{(B.2d)}
\end{align}

\begin{align}
\frac{T'_o}{T_o} &= \frac{(\gamma - 1)(1 + \bar{M})}{1 + \frac{\gamma - 1}{2} \bar{M}^2} \frac{p'}{\gamma p} \\
& \quad \text{(B.2e)}
\end{align}

\begin{align}
\frac{P'_o}{P_o} &= \frac{\gamma (1 + \bar{M})}{1 + \frac{\gamma - 1}{2} \bar{M}^2} \frac{p'}{\gamma p} \\
& \quad \text{(B.2f)}
\end{align}

These results are summarized in Table II.3.
APPENDIX C

C.1 Derivation of the Entropy Wave-Pressure Wave Relationship

Four important quantities used in relating the strength of the entropy wave incident on the nozzle to the amplitude of the pressure wave generated on the upstream side of the nozzle are the fractional fluctuations of total temperature and total pressure associated with an entropy wave and a leftward-travelling acoustic wave: $(T'_o/T_o)^G$, $(P'_o/P_o)^G$, $(T'_o/T_o)^P$ and $(P'_o/P_o)^P$; these quantities are derived in Appendices A and B (Eqs. (A.2.11), (A.2.12), (B.2e) and (B.2f)).

The analysis of the process at the nozzle is essentially a mass balance between the amount of mass flow required to keep the nozzle choked (the assumption that $(p_b/P_o) < (p_b/P_o)^*$ at all times guarantees that the nozzle remains choked) and the mass flow rate incident on the inlet plane of the nozzle. This mass balance is expressed in Eq. (II.26) in terms of the components of change due to the passage of an entropy wave and the generation of an acoustic wave.

The fractional variation in the mass flow rate required for choking due to the passage of an entropy wave, is given by

$$
\left( \frac{\dot{m}'^G}{\dot{m}} \right)_{\text{nozzle}} = \left( \frac{P'_o}{P_o} \right)^G - \frac{1}{2} \left( \frac{T'_o}{T_o} \right)^G ; \quad (II.22a)
$$

the expressions for the fractional variations in total pressure and total temperature are given as stated above, by Eqs. (A.2.11) and
(A.2.12).

The fractional variation in the mass flow rate required to choke the nozzle due to the formation of a pressure wave on the upstream side of the nozzle is

\[
\left( \frac{\dot{m}'}{\dot{m}} \right)^p_{\text{nozzle}} = \left( \frac{p'}{p_0} \right)^p_{\text{o}} - \frac{1}{2} \left( \frac{T'}{T_0} \right)^p_{\text{o}} ; \quad \text{(II.25a)}
\]

the expressions for the total pressure and total temperature fluctuations are given by the equations listed above.

In addition, the fractional variations in mass flow rate associated with entropy waves and acoustic waves are derived in Appendices A and B (Eqs. (A.2.5) and (B.2c)).

Equating the fractional variations in both the mass flow rates required to keep the nozzle choked to the mass flow rates incident on the nozzle yields the relationship between the strength of the incident entropy wave and the amplitude of the generated pressure wave

\[
\frac{p'}{\gamma p} = -\frac{\frac{1}{2}M}{1 + \frac{\gamma - 1}{2} \frac{\dot{M}}{\dot{M}}} \frac{s'}{c_p} . \quad \text{(II.27)}
\]

C.2 The Characteristic Deterioration Time

The characteristic deterioration time of an entropy wave is given by
\[ \tau_{\delta} = \left| \frac{\partial(s/c_p)}{\partial z} \right| \left| \frac{\partial(s/c_p)}{\partial z} \right| \right| \right| \right| \right| ; \quad (\text{II.3a}) \]

\[ \left\langle \frac{\partial(s/c_p)}{\partial z} \right\rangle \] is the spatial mean entropy gradient defined by

\[ \left\langle \frac{\partial(s/c_p)}{\partial z} \right\rangle = \frac{1}{\delta} \int_{z_o}^{z_o+\delta} \frac{\partial(s/c_p)}{\partial z} \, dz \quad (\text{C.2.1}) \]

where \( \delta \) is the entropy transition zone thickness and \( z_o \) is the point just ahead of the entropy wave (see Figure (II.5)).

The integral on the right-hand side of Eq. (C.2.1) may be evaluated, yielding

\[ \left\langle \frac{\partial(s/c_p)}{\partial z} \right\rangle = \frac{1}{\delta} \frac{(s-s_o)}{c_p} \quad . \quad (\text{C.2.2}) \]

Taking the time derivative of Eq. (C.2.2) to obtain the rate of change of the spatial mean entropy gradient

\[ \frac{d}{dt} \left\langle \frac{\partial(s/c_p)}{\partial z} \right\rangle = \frac{1}{\delta} \frac{d}{dt} \left( \frac{(s-s_o)}{c_p} \right) - \frac{1}{\delta^2} \frac{(s-s_o)}{c_p} \frac{d\delta}{dt} \quad , \quad (\text{C.2.3}) \]

and noting that \( \frac{d}{dt}(s-s_o)/c_p \) is zero yields

\[ \frac{d}{dt} \left\langle \frac{\partial(s/c_p)}{\partial z} \right\rangle = -\frac{1}{\delta^2} \frac{(s-s_o)}{c_p} \frac{d\delta}{dt} \quad . \quad (\text{C.2.4}) \]
Forming the quotient defined by Eq. (II.3a) using Eqs. (C.2.2) and (C.2.4), and taking the absolute value gives

\[
\tau_\gamma = \left| \frac{1}{\delta} \left( \frac{s-s_o}{c_p} \right) \right| - \frac{1}{\delta^2} \left( \frac{s-s_o}{c_p} \right) \frac{d\delta}{dt}
\]

\[
= \frac{\delta}{d\delta/dt}
\]  

(C.2.5)
APPENDIX D

Calculation of Flow Parameters and Frequency in Filmed Tests

The information supplied about the high-speed films includes:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass flow rate</td>
<td>2 lbm/sec</td>
</tr>
<tr>
<td>combustor pressure</td>
<td>2150 lbf/ft²</td>
</tr>
<tr>
<td>combustor length</td>
<td>2 ft</td>
</tr>
<tr>
<td>combustor area</td>
<td>0.2 ft²</td>
</tr>
</tbody>
</table>

Estimating an average slope for the trajectories of the entropy waves (see Figures V.2 and V.3) yields the mean flow velocity in the combustor.

Figure V.2: \( \bar{u} = 315 \text{ ft/sec} \)
Figure V.3: \( \bar{u} = 200 \text{ ft/sec} \)

Reference [7] gives an approximate expression for the mean flow velocity in the combustor:

\[
\bar{u} \approx \frac{\dot{m}RT}{P_c A_c} \tag{D.1}
\]

Solving for the temperature and inserting values for \( \dot{m}, P_c, R \) and \( A_c \), one obtains:

Figure V.2: \( \bar{T} = 2600^\circ \text{R} \)
Figure V.3: \( \bar{T} = 1000^\circ \text{R} \)

The sound speed is related to the temperatures by
\[ \bar{a} = \sqrt{\gamma RT} \]  \hspace{1cm} \text{(D.2)}

which yields:

Figure V.2: \[ \bar{a} = 2230 \text{ ft/sec} \]

Figure V.3: \[ \bar{a} = 1350 \text{ ft/sec} \]

Finally, substituting these values into the frequency formulas for the first and second modes for Figures V.2 and V.3, respectively,

Figure V.2: \[ f = 150 \text{ Hz} \] (first mode assumed, \( \hat{\beta}_s = 0.4 \);

observed frequency: 131 hz)

Figure V.3: \[ f = 181 \text{ Hz} \] (second mode assumed, \( \hat{\beta}_s = 0.2, \)

\[ \tau_D = 0.002 \text{ sec}; \text{ observed frequency: 167 hz}. \]
APPENDIX E

E.1 Sample Frequency Calculation

Given data:

- mass flow rate ($\dot{m}$) 0.18 slug/sec
- mean combustor pressure ($\bar{P}_c$) 3600 lbf/ft$^2$
- mean combustor temperature ($\bar{T}$) 3274°R
- mean inlet temperature ($\bar{T}_i$) 750°R
- combustor length ($L_c$) 2.0 ft
- combustor area ($A_c$) 0.35 ft$^2$

Equations (D.1) and (D.2) give the mean flow and acoustic velocities:

$$\bar{u} = 687 \text{ ft/sec}$$
$$\bar{a} = 2479 \text{ ft/sec}$$

Substituting these values into Eq. (III.5), one obtains

$$f = 177 \text{ hz}$$

($\hat{\sigma}_s = 0.3$ assumed; observed frequency: 189 hz).

E.2 Sample Pressure Fluctuation Amplitude Calculation

The mean Mach number in the example above is
\[ \dot{M} = 0.277 \]

Assuming a combustion efficiency (defined as \( \eta = \frac{\dot{I}}{T_{\text{ad}}} \)) of 75\% yields an adiabatic flame temperature of \( 4365^\circ R \). Substituting this value into Eq. (V.1) to calculate the maximum entropy fluctuation, one obtains

\[ \frac{\Delta s}{c_p} = 1.76 \]

Equation (II.30) is now used to compute the maximum peak-to-peak pressure fluctuation that should be observed as the entropy wave is convected through the nozzle:

\[ \left| \frac{p'}{\gamma p} \right|_{\text{peak-to-peak}}^{\text{max}} = 0.264 \]

The observed amplitude in this test is

\[ \left| \frac{p'}{\gamma p} \right|_{\text{peak-to-peak}} = 0.236 \]
L/D = 3, T_{o,i} = 1000^\circ R, WALL INJECTION

no FH - no flameholder
0.25 Y - Y-shaped flameholder, 25% blockage
0.25 AW - annular " 25% "
0.35 Y - Y-shaped " 35% "
0.35 AW - annular " 35% "

Figure I.2: Measured Combustor Efficiency Illustrating the Effect of a Flameholder
Figure I.3: Pressure-Time Traces of Fluctuations Produced by (a) Low-Frequency and (b) High-Frequency Instabilities
Figure I.4: Transition from the Low-Frequency Instability to the High-Frequency Instability Resulting from a Decrease in Equivalence Ratio
\[
\begin{align*}
\frac{s'}{c_p} &= 0 \\
\frac{u'}{a} &= 0, \quad \frac{\rho'}{\rho} = 0, \quad \frac{T'}{T} = 0, \quad \frac{p'}{\gamma p} = 0
\end{align*}
\]

\[
\begin{align*}
\frac{d(p')}{\gamma p} &= \pm 1 \\
\frac{d(u')}{a} &= \pm 1
\end{align*}
\]

\(\Gamma^+\) - propagation in positive x-direction

\(\Gamma^-\) - propagation in negative x-direction

\(\Gamma^+\) pertains only to \(u'/a\);
\(T'/T\) and \(\rho'/\rho\) related to \(p'/\gamma p\) along \(\Gamma^+\)

\[
\begin{align*}
\frac{d(T')}{T} &= 1 \\
\frac{d(s')}{c_p} &= 1 \\
\frac{d(\rho')}{\rho} &= -1 \\
\frac{d(s')}{c_p} &= -1
\end{align*}
\]

Figure II.1: Graphical Representation of State Changes across
(a) Acoustic and (b) Entropy Wavefronts
Figure II.2: Spatial Coordinate Definition

leftward propagating wave (acoustic and entropy waves)  rightward propagating wave (acoustic only)
Figure II.3: Formation of an Entropy Transition Zone through a Change in Heat Release Rate
Figure II.4: A Convected Entropy Wave; for Small Amplitude Waves, the Transition Zone May Be Modelled as a Contact Discontinuity
Figure II.5: Deterioration of a Large-Amplitude Entropy Wave Over Time as Viewed in the Reference Frame of the Wave; the Entropy Distribution at $t = t_2$ Represents the Temporally-Invariant Form of an Entropy Wave of Amplitude $(s - s_o)/c_p$ with Its Associated Spatial Mean Entropy Gradient $\langle \delta s/\delta z \rangle$ during a Period of Time Equal to $\tau_f$
Figure II.6: (a) Generation of a Train of Entropy Waves, and (b) Definition of Entropy Elevations and Depressions.
Figure II.7: (a) Definition of an Acoustic Wave, and (b) Leftward and Rightward Propagating Acoustic Waves
Figure II.8: The Rankine-Hugoniot Curve vs. the Isentropic Acoustic Relation, Showing the Validity of the Acoustic Approximation in the Range of Operation (Source: Reference [12], p. 122)
Approximate: \[ \frac{(\dot{m}/A) - (\dot{m}/A)_0}{(\dot{m}/A)_0} = e^{-\sum_{j=1}^{n} \left( \frac{\Delta s}{c_p} \right)_j} - 1 \]

Exact: \[ \frac{(\dot{m}/A) - (\dot{m}/A)_0}{(\dot{m}/A)_0} = e^{-\int_{s_o}^{s} \frac{ds}{c_p(T)}} - 1 \]

Figure II.9: Illustration of the Integration Technique
Figure II.10: A Simple Example of a Wave-Boundary Interaction: the Reflection of a Plane Acoustic Wave by a Rigid Surface
Figure II.11:  (a) Flow with Variable Total Pressure, Total Temperature and Entropy Impinging on a Choked Nozzle and (b) the Compact Nozzle Assumption
Figure II.12: (a) The Passage of an Entropy Wave through a Choked Nozzle Generating a Pressure Wave, and (b) a Graphical Representation of Fractional Mass and Volume Flow Rate Changes during the Convection of a Positive Entropy Wave through a Choked Nozzle.
Figure II.13: The Dependence on the Entropy Distribution of the Initial Pressure Distributions in the Generated Pressure Waves for (a) Positive and (b) Negative Large-Amplitude Entropy Waves
Figure II.14: Rarefaction-Flame Interaction; (a) a Rarefaction Propagates against the Flow into the Combustion Zone, (b) Produces a Separation in the Flame by Reducing the Reaction Intensity through Stretching, and (c) Generates a Negative Entropy Wave
case 1: The rarefaction increases the reactant velocity to a level near the blow-off velocity.

case 2: The reactant velocity remains well below the blow-off velocity during the passage of a rarefaction through the flame.

Figure II.15: Effect of Passage of a Rarefaction through the Combustion Zone; the Zone of Low Flame Efficiency Corresponds to the Condition when a Stream of Unburned Gases Passes through the Flame
Figure II.16: (a) Strong Combustion before the Arrival of a Rarefaction, and (b) Flame Separation and the Resulting Toroidal Flame
Figure 11.16: (a) Strong Combustion before the Arrival of a Rarefaction, and (b) Flame Separation and the Resulting Toroidal Flame.
Figure II.17: (a) A Compression Wave Impinging on a Weak Toroidal Flame, (b) Re-Stabilization of the Weak Flame by a Decrease in the Reactant Velocity by the Passage of the Compression, and (c) Formation of a Positive Entropy Wave
Figure III.1: (a) Generation of a Pressure Wave by Convection of an Entropy Wave Produced by the Combustion Zone, and (b) Generation of an Entropy Wave by a Pressure Wave Produced at the Nozzle
Figure III.2: Idealized Representation of the First Mode of the Low-Frequency Instability; the Curve Upstream of the Separation Point ($\tilde{x}_s$) Connecting the ($t_n$) and ($h_{n+1}$) Trajectories Represents the Downstream Tip of the Flame Following Separation.
\[ \frac{p'}{\gamma p} = \frac{-k \bar{M}}{1 + \frac{\gamma-1}{2} \bar{M} c_p} s' \]

Figure III.3: Idealized Representation of the Second Mode of the Low-Frequency Instability
Figure IV.1: (a) A Typical Frame from a 500 frame/sec Movie of the Combustion Process in a Cylindrical Quartz Combustor, with (b) Schematic Representation
Figure IV.1: (a) A Typical Frame from a 500-frame, 20-shot Movie of the Combustion Process in a Cylindrical Quartz Combustor, with (b) Schematic Representation
Figure IV.3: Transducer Positions
Figure IV.4: (a) Pressure-Time Trace of an Observed Instability and (b) the Corresponding FFT
Figure V.1: Series of Frames from a High-Speed Movie (500 frames/sec) in Schematic Form; the Dashed Lines Represent Boundaries of Regions of Combusted Gas, and the Shading Represents Relatively Darker Zones in the Flow which Are Assumed to be Unreacted Gas
Flow Conditions

\( \dot{m} = 2.0 \text{ slug/sec} \)
\( \bar{P} = 2150 \text{ lbf/ft}^2 \)
\( T = 2600^\circ R \)
\( \bar{u} = 315 \text{ ft/sec} \)
\( \bar{a} = 2230 \text{ ft/sec} \)

observed frequency: 131 Hz
predicted frequency: 150 Hz

Figure V.2: The \( t-x \) Diagram for One Mode of the Low-Frequency Instability Constructed from a Series of Frames of a High-Speed Movie
Flow Conditions
\[ \dot{m} = 2.0 \text{ slug/sec} \]
\[ \bar{P} = 2150 \text{ lbf/ft}^2 \]
\[ T = 1000^\circ R \]
\[ \bar{u} = 200 \text{ ft/sec} \]
\[ \bar{a} = 1350 \text{ ft/sec} \]

observed frequency: 167 Hz
predicted frequency: 181 Hz

Figure V.3: The t-x Diagram for Another Observed Mode of the Low-Frequency Instability
\[ f_{\text{predicted}} = \frac{(\bar{a} - \bar{u}) u}{2L_c \bar{a}(1 - \bar{x}_s)} \]

Figure V.4: Comparison between Predicted and Observed Values of the Frequencies of the Low-Frequency Instability (\( \bar{x}_s = 0.3 \))
Figure V.5: Comparison between Predicted and Observed Values of the Peak-to-Peak Pressure Fluctuations Produced by the Low-Frequency Instability
Figure V.6: Effect of Flameholder Type on the Percent Occurrence of the Low-Frequency Instability
Figure V.7: Effect of Inlet Diameter - Combustor Diameter Ratio ($D_i/D_c$) on the Percent Occurrence of Low-Frequency Instability
### Operating Conditions

**Baseline with \( \frac{D_i}{D_c} = 0.59 \)**

---

**Figure V.8:** Effect of Inlet Temperature on the Percent Occurrence of the Low-Frequency Instability
Figure A.1.1: An Entropy Wave Passing through a Control Volume
<table>
<thead>
<tr>
<th>Flow Variable</th>
<th>Wavelet</th>
<th>Large-Amplitude Wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>entropy</td>
<td>( \frac{s'}{c_p} )</td>
<td>( \frac{s-s_1}{c_p} )</td>
</tr>
<tr>
<td>pressure</td>
<td>( \frac{p'}{\gamma p} )</td>
<td>( \frac{p-p_1}{\gamma p_1} )</td>
</tr>
<tr>
<td>velocity</td>
<td>( \frac{u'}{a} )</td>
<td>( \frac{u-u_1}{a_1} )</td>
</tr>
<tr>
<td>density</td>
<td>( \frac{\rho'}{\rho} )</td>
<td>( \frac{\rho-\rho_1}{\rho_1} )</td>
</tr>
<tr>
<td>temperature</td>
<td>( \frac{T'}{T} )</td>
<td>( \frac{T-T_1}{T_1} )</td>
</tr>
<tr>
<td>mass flow rate</td>
<td>( \frac{\dot{m}'}{\dot{m}} )</td>
<td>( \frac{\dot{m}-\dot{m}_1}{\dot{m}_1} )</td>
</tr>
<tr>
<td>Mach number</td>
<td>( \frac{M'}{M} )</td>
<td>( \frac{M-M_1}{M_1} )</td>
</tr>
<tr>
<td>total temperature</td>
<td>( \frac{T_o'}{T_o} )</td>
<td>( \frac{T_o-T_{o,1}}{T_{o,1}} )</td>
</tr>
<tr>
<td>total pressure</td>
<td>( \frac{p'}{p_o} )</td>
<td>( \frac{p_o-p_{o,1}}{p_{o,1}} )</td>
</tr>
</tbody>
</table>

Table II.1: Flow Variable Normalization Conventions (( )' Denotes Perturbation Quantity, ( ) Denotes Mean Quantity, ( )_1 Denotes a Reference Condition)
<table>
<thead>
<tr>
<th>Flow Variable</th>
<th>Wavelet</th>
<th>Large-Amplitude Wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>entropy ( \frac{s'}{c_p} )</td>
<td>( s-s_1 ) ( \frac{c_p}{c_p} )</td>
<td></td>
</tr>
<tr>
<td>pressure ( \frac{p'}{\gamma p} = 0 )</td>
<td>( \frac{p-p_1}{\gamma p_1} = 0 )</td>
<td></td>
</tr>
<tr>
<td>velocity ( \frac{u'}{a} = 0 )</td>
<td>( \frac{u-u_1}{a_1} = 0 )</td>
<td></td>
</tr>
<tr>
<td>density ( \frac{\rho'}{\rho} = \frac{s'}{c_p} )</td>
<td>( \frac{\rho-\rho_1}{\rho_1} = e^{\frac{c_p}{c_p}} - 1 )</td>
<td></td>
</tr>
<tr>
<td>temperature ( \frac{T'}{T} = \frac{s'}{c_p} )</td>
<td>( \frac{T-T_1}{T_1} = e^{\frac{c_p}{c_p}} - 1 )</td>
<td></td>
</tr>
<tr>
<td>mass flow rate ( \frac{\dot{m}'}{\dot{m}} = -\frac{s'}{c_p} )</td>
<td>( \frac{\dot{m}-\dot{m}_1}{\dot{m}_1} = e^{\frac{c_p}{c_p}} - 1 )</td>
<td></td>
</tr>
<tr>
<td>Mach number ( \frac{M'}{M} = -\frac{1}{2} \frac{s'}{c_p} )</td>
<td>( \frac{M-M_1}{M_1} = e^{\frac{2c_p}{c_p}} - 1 )</td>
<td></td>
</tr>
<tr>
<td>total temperature ( \frac{T'}{T_o} = \frac{1}{1+\frac{\gamma-1}{2} M^2} \frac{s'}{c_p} )</td>
<td>( \frac{T_o-T_{o,1}}{T_{o,1}} = e^{\frac{1}{1+\frac{\gamma-1}{2} M^2} \frac{ds}{c_p}} - 1 )</td>
<td></td>
</tr>
<tr>
<td>total pressure ( \frac{p'}{p_o} = -\frac{1}{2} \frac{\gamma M^2}{1+\frac{\gamma-1}{2} M^2} \frac{s'}{c_p} )</td>
<td>( \frac{p_o-p_{o,1}}{p_{o,1}} = e^{\frac{1}{1+\frac{\gamma-1}{2} M^2} \frac{ds}{c_p}} - 1 )</td>
<td></td>
</tr>
</tbody>
</table>

Table II.2: Entropy Wave Relations
<table>
<thead>
<tr>
<th>Flow Variable</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>pressure</td>
<td>$\frac{p'}{\gamma p}$</td>
</tr>
<tr>
<td>velocity</td>
<td>$\pm \frac{u'}{a}$</td>
</tr>
<tr>
<td>entropy</td>
<td>$\frac{s'}{c_p} = 0$</td>
</tr>
<tr>
<td>density</td>
<td>$\frac{\rho'}{\rho} = \frac{p'}{\gamma p}$</td>
</tr>
<tr>
<td>temperature</td>
<td>$\frac{T'}{T} = (\gamma - 1) \frac{p'}{\gamma p}$</td>
</tr>
<tr>
<td>mass flow rate</td>
<td>$\frac{\dot{m}'}{\dot{m}} = (\frac{M \pm 1}{M}) \frac{p'}{\gamma p}$</td>
</tr>
<tr>
<td>Mach number</td>
<td>$\frac{M'}{M} = (\frac{\gamma - 1}{2 M^2}) \frac{p'}{\gamma p}$</td>
</tr>
<tr>
<td>total temperature</td>
<td>$\frac{T_0'}{T_0} = \frac{(\gamma - 1)(1 \pm M)}{1 + \frac{\gamma - 1}{2} M^2} \frac{p'}{p}$</td>
</tr>
<tr>
<td>total pressure</td>
<td>$\frac{P_0'}{P_0} = \frac{\gamma (1 \pm M)}{1 + \frac{\gamma - 1}{2} M^2} \frac{p'}{\gamma p}$</td>
</tr>
</tbody>
</table>

Table II.3: Acoustic Wave Relations; in Cases with the Symbol $(\pm)$, the $(+)$ sign and the $(-)$ Sign Are Associated with Pressure Waves Propagating in the Positive and Negative $x$-Directions, Respectively
nozzle diameter - combustor diameter ratio \((D_n/D_c)\) 0.70
combustor length - combustor diameter ratio \((L_c/D_c)\) 3.0
inlet temperature \((\bar{T}_i)\) 1000\(^{\circ}\)R

mass flow rate \((\bar{m})\)
\[
\begin{align*}
L_c &= 1.5 \text{ ft} & 0.12 \text{ slug/sec} \\
L_c &= 2.0 \text{ ft} & 0.17 \ " \\
L_c &= 3.0 \text{ ft} & 0.26 \ "
\end{align*}
\]

combustor pressure \((P_c)\)
\[
\begin{align*}
L_c &= 1.5 \text{ ft} & 4800 \text{ lbf/ft}^2 \\
L_c &= 2.0 \text{ ft} & 3600 \ " \\
L_c &= 3.0 \text{ ft} & 2400 \ "
\end{align*}
\]

Table IV.1 Baseline Operating Conditions