SELF-STABILIZING MAGNETIC BEARINGS FOR FLYWHEELS

by

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ABSTRACT

Techniques are presented for stabilizing permanent-magnet, radially-active magnetic bearings using a control concept which requires only coils of wire for transducers. These systems sense the radial velocity in two orthogonal directions as control inputs. This velocity-feedback approach allows the rotor to seek its own natural equilibrium position. If the control system is located in the rotating frame of reference, then this scheme also allows the rotor to spin about its own center of mass. These characteristics dramatically reduce the steady-state power requirement for the control system, allowing it to be powered directly from the rotational energy in the spinning flywheel. Under these conditions the flywheel could be stabilized without the need for any external power source. A 7.4 kg magnetically-suspended flywheel was built using velocity-feedback control, and required only 100 mW of power in the steady state.

Generally-applicable methods for analyzing the radial dynamics of rotationally-symmetric magnetic structures are presented. A notation for combining two symmetric orthogonal axes into a single complex-valued function of the Laplace transform variable \( s \) is introduced. This allows the powerful methods of classical control theory to be applied to the design of radial magnetic bearings. In this description, the effect of spin on a rotating control system is simply to translate the poles and zeros associated with the controller up the imaginary axis by an amount equal to the spin speed.

Several useful intermediate results are derived, including tensor boundary conditions which directly relate first-order perturbation quantities which describe the movement of a surface through a known equilibrium magnetic field. Another result predicts the speed dependence of the radial magnetic stiffness in induction motors.

The analytical description of the test model flywheel predicts incipient instability at a spin speed of 13 Hz. Instability leading to a loss of free suspension does indeed occur between 13.0 and 13.4 Hz. This limitation is due to the flexibility of the purposefully-compliant stator. Stiff support will be required in high-speed applications of velocity-feedback control.

Thesis Supervisor: Richard D. Thornton
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TEST MODEL MAGNETICALLY-LEVITATED FLYWHEEL
CHAPTER 1

INTRODUCTION

1.1 BACKGROUND

Flywheels, by virtue of the high energy density obtainable in the form of rotational inertia, represent a promising alternative to batteries for the efficient storage of electrical power. Specific energies as high as 310 kJ/kg have been successfully demonstrated, and 120 kJ/kg rotors could be fabricated for as little as $1.50/kg. Unfortunately, these favorable specifications are burdened by the addition of the support, propulsion, vacuum, and electrical processing elements which are required for a complete system. A recent attempt by Rockwell to use conventional components for these peripheral functions resulted in parasitic losses sufficient to drain all of their flywheel's energy in only one hour.

Tremendously reduced spinning losses can be realized through the use of magnetic bearings. Magnetic levitation has been used in inertial guidance systems since 1953, and has more recently been developed for high-speed ground transportation. While early efforts employed electromagnets, power dissipation can be cut dramatically by using permanent magnets, or superconducting magnets in large-scale applications.

Soon after permanent magnets were first incorporated into magnetic bearing systems in 1970, it became clear to that in order to fully realize the power-reduction abilities of the permanent magnet design, some means for letting the suspended member come to rest at its own natural equilibrium position was necessary. Conventional position-feedback control could not insure that this would occur under conditions of varying load or drifting electrical component values. In 1971 the Cambridge Thermionic
Corporation delivered a magnetic bearing system to NASA which utilized a novel feedback control concept which they termed a "Virtually-Zero-Power" (VZP) system. The new technique was based on using velocity as the feedback variable.

Several magnetic bearing systems have been constructed since the introduction of VZP. Whenever power consumption has been a main concern, the VZP concept has been utilized in an axial-gap, one-dimension-of-control design. Examples include a 4 kg flywheel by Cambion in 1972, a 29 kg wheel by Sperry Flight Systems in 1974, a 6.8 kg wheel by MIT's Lincoln Lab in 1977, and a 173 kg flywheel by MIT's Lincoln Lab in 1980. All were designed for spacecraft except the most recent system, which is intended for terrestrial energy storage. Typical power consumption in these control systems is 50 mW/kg. In contrast, systems designed for high performance and ease of integration with motor-generator units have tended toward radially-active designs, using position-control feedback. Examples include General Electric's 12 kg wheel in 1973, Societe de Mechanique Magnetiques 1000 kg wheel in 1979, and Draper Laboratories' 5 kg flywheel in 1980. These designs require somewhat more control power, on the order of one watt per kilogram.

Magnetic suspension essentially eliminates the heat removal problem present in conventional support systems. Consequently, these bearings can be placed inside the flywheel's vacuum enclosure. This negates the need for rotating seals, eliminating the losses and leaks they introduce. The vacuum's integrity can then be preserved for very long periods, removing the power drain of continuous pumping. These concepts have been incorporated in the design of a flywheel energy storage system under development at MIT's Lincoln Lab, illustrated in Figure 1.1. Total
Figure 1.1 Lincoln Lab Residential Flywheel Energy Storage Unit.
parasitic losses as low as 40 milliwatts per levitated kilogram have been achieved.\textsuperscript{12}

In order to fully realize the energy storage capability of modern high-strength materials, it is important that the rotor not be subjected to stress concentrations or vibrations which promote fatigue. Stress concentrations are introduced when the flywheel is secured to a central axle. Tying the flywheel to a shaft can also introduce vibrations due to rotor imbalance, though this effect is minimized if the shaft is long and flexible.\textsuperscript{3} This approach, adopted in the Lincoln Lab design, allows the rotor to spin about its own center of mass. Care must be taken to avoid low frequency whirl oscillations when using this configuration.\textsuperscript{28} Several designs have been proposed which eliminate the need for an axle by distributing the magnetic levitation and propulsion interactions around the inner periphery of the rotor itself.\textsuperscript{15,16,24} Such an arrangement also reduces the required size of the vacuum containment structure.

Energy transfer to and from a magnetically-suspended flywheel is a difficult process which requires novel motor-generator design. The complications arise due to the wide speed range encountered, the requirement that no physical contact be made to the rotor, the need for low stand-by losses, and the avoidance of forces which could upset the magnetic suspension. Lincoln Lab has developed a permanent-magnet axial-gap machine for use with their system.\textsuperscript{18} It is driven by a variable-frequency inverter and uses a cycloconverter to translate the high-frequency output to 60 Hz. An ironless armature is used to minimize destabilizing forces and to cut the stand-by losses to less than 0.5\% of full output power. The anticipated production cost of this propulsion system is $350/kW, however, so improvement in this area is desirable. Other work on the flywheel
propulsion system has been performed by Draper Lab\textsuperscript{7} and General Electric\textsuperscript{17}.

A preliminary theoretical analysis of the issues pertinent to flywheel mechanics and magnetic levitation was performed in work by the author leading to the Master's degree, and that thesis document contains further details regarding these topics.\textsuperscript{1} The following sections of this chapter provide the essential information needed to understand the significance of radial magnetic bearings to flywheel energy storage.

1.2 FLYWHEEL MECHANICS

Flywheels store kinetic energy by virtue of their polar mass moment of inertia, \( J_p \),

\[
E = \frac{1}{2} J_p \omega^2 \tag{1.1}
\]

The maximum speed a given rotor can achieve is limited by the tensile stresses induced by the centrifugal force. The geometry which yields the highest practical specific energy storage capability is the thin rim configuration,\textsuperscript{1} for which

\[
E = \frac{1}{2} M v^2 \tag{1.2}
\]

where \( v \) is the tangential rotor velocity. The maximum allowable velocity is an intrinsic material property and is given by

\[
v_{max}^2 = \frac{\sigma_{max}}{\rho} \tag{1.3}
\]

where \( \sigma_{max} \) is the allowable tensile stress and \( \rho \) is the material density. Several thin-rim rotors of different materials can be nested as shown in Figure 1.2.\textsuperscript{21} One material then considered for flywheel energy storage is a composite of E-type fiberglass and epoxy. This material has a maximum working stress of 800 MN/m\textsuperscript{2} and a density of 2200 kg/m\textsuperscript{3}. If stress concentration and fatigue are eliminated then the material can be spun up
to a rim velocity of 600 m/s. Such high velocities emphasize the need for a vacuum enclosure.

Figure 1.2 Multi-Ring Rotor Configuration: Cut-Away View.

An important dynamic effect in flywheel mechanics is the transition from subcritical to supercritical operation as the flywheel passes through its first critical speed. Above the first critical speed, the support structure flexes to allow the flywheel to rotate about its own center of mass. If the lateral stiffness of the support structure can be lumped into an equivalent horizontal spring, then the first critical frequency can be given as

$$\omega_1 = \sqrt{\frac{K_{xx}}{M}}$$  (1.4)

where $K$ is the horizontal stiffness and $M$ is the rotor mass. At high speeds this mode can become unstable in the form of low-frequency whirl if
there is more mechanical hysteresis dissipation in the rotor than in the support structure.\textsuperscript{25}

Higher-order critical speeds correspond to more complex mode shapes in the flywheel and support structure, but the center of mass remains fixed.\textsuperscript{3} The most important of these is the first tilting mode. The frequency of this mode is split by a strong interaction with the spin speed through the gyroscopic effect and is given in simple configurations by

\[
\omega_2 = \frac{\Omega J_p}{2J_d} \pm \sqrt{\left(\frac{\Omega J_p}{2J_d}\right)^2 + \frac{T}{J_d}}
\]  \hspace{1cm} (1.5)

where \( J_p \) and \( J_d \) are the polar and diametrical moments of inertia, \( T \) is the torsional stiffness about the diametrical axis (N(rad)), and \( \Omega \) is the spin speed.\textsuperscript{25} The higher-frequency of the two modes can be unstable if \( \omega \) is ever less than \( \Omega \). This situation is avoided if the ratio of \( J_p/J_d \) is greater than unity. This restriction is satisfied if the aspect ratio of the rotor is set such that

\[
\left[\frac{h}{b}\right]^2 < 3\left(1 + \left[\frac{a}{b}\right]^2\right)
\]  \hspace{1cm} (1.6)

where \( h \) is the height of a rotor having inner and outer radius \( a \) and \( b \), respectively.

1.3 MAGNETIC LEVITATION

The primary function of the magnetic bearing system is to provide contact-free lift against the force of gravity. For maximum reliability and minimum power dissipation the lift should be generated using permanent magnets. Saturation demagnetization curves for several relevant materials are shown in Figure 1.3. Where size and weight are not restrictions, ceramic magnets are preferred on the basis of cost and the availability of the constituent materials. At current prices, the cost per kilogram
levitated using rare-earth or Alnico magnets is six times that of ceramic V.\(^1\)

![Graph showing constitutive relations for magnets.](image)

**Figure 1.3 Constitutive Relations for Magnets.**

Best use of the magnet material is made if the operating point is selected to maximize the BH energy product in the magnet. However, allowance for the presence of externally-produced demagnetizing fields often dictates a more conservative selection of the operating point.[9] For ceramic V a suitable operating point is in the linear portion of the curve where \(B_m = -2\mu_0 H_m\), giving \(B_m = .26\) Tesla, \(\mu_0 H_m = -.13\) Tesla, and \(B_m H_m = 26\)
kJ/m$^3$. In order to establish this operating point, the aspect ratio of the magnet must be properly matched to the magnetic circuit in which it is placed. Specifically, the reluctance of the region occupied by the magnet must be twice the reluctance of the magnetic circuit seen looking out from the poles of the magnet.

The development of a linear circuit analog for the magnet system is shown in Figure 1.4. Subscripts $m$, $l$, $i$, and $g$ refer to the magnet, leakage flux, iron magnetization, and the air gap. In terms of these parameters, the force introduced at the air gap is

$$f_i = -\frac{1}{2} \phi^2 \frac{\partial R}{\partial x_i} g$$

(1.7)

The displacements can be either perpendicular to the gap (normal, $x_n$) or transverse with respect to the gap ($x_T$).

The force can also be shown to depend directly on the magnet's BH energy product as

$$f_i = \frac{1}{2} \eta \frac{[BH]_m}{g} [Vol.]_{mag} f_i$$

(1.8)

where $\eta$ is the magnet flux efficiency, which can be given as

$$\eta = 1 - \frac{R_l}{R} - \frac{R}{R_g}$$

(1.9)

for high-efficiency arrangements. The gap width $g$ is introduced to allow the formation of a dimensionless gap-influence factor, $f$:

$$f_i = \frac{g}{R} \frac{\partial R}{\partial x_i}$$

(1.10)

This factor is unity for the case of opposing flat plates, and has lower values for other gap geometries.
Figure 1.4 Circuit Analogs for Permanent-Magnet Levitation Systems
In a similar fashion, the magnetic stiffness in the direction of lift can be expressed as

\[ K_{zz} = \frac{1}{2} \eta \left( \frac{[BH]}{g^2} \right)^m [\text{Vol.}]_{\text{mag}} f_{Sz} \]  

(1.11)

where \( f_{Sz} \) is the vertical stiffness gap-influence coefficient, which is also unity for the normal stiffness of opposing flat plates.

Toothed arrangements as shown in Figure 1.5 are necessary to obtain effective transverse magnetic shearing forces. For the radial-gap system the optimum value of \( f_T \) is .16, corresponding to \( t/g = 2.5 \), \( t/\lambda = .25 \), and \( \delta/t = .8 \). The associated value of \( f_{Sz} \) is .03 if ceramic magnets are assumed, and the reluctance of this arrangement is 2.75 times greater than if the slots were filled with iron.

The gap area required to provide a given amount of force can be expressed as

\[ f_i = \frac{1}{2} \frac{B_o^2}{\mu_0} \frac{f_i}{C} A_{gTOT} \]  

(1.12)

where \( A_{gTOT} \) is the total gap area utilized, \( B_o \) is the peak field intensity in the gap, and \( C \) is the reluctance multiplier to account for the presence of slots in a toothed structure (\( C = 2.75 \) in the previous example). Assuming a toothed radial-gap system with a 1 mm air gap, then .03 kg of ceramic magnets and 1 cm\(^2\) of gap area are needed for each kilogram of levitated mass.

Samuel Earnshaw proved in 1842 that any fixed arrangement of freely suspended point particles whose forces vary as the inverse square of distance cannot exist in a state of stable equilibrium. Later, in 1939, Braunbek used an energy argument to extend Earnshaw's result to conclude that the stable suspension of a finite body in a static magnetic (or
Figure 1.5 Toothed Gaps in Magnetic Bearings.
electric) field is impossible unless diamagnetic material is present. Superconductors are often included in this category, as their ability to shield out magnetic fields can be interpreted as an effective permeability of zero. Braunsbek's result, known as Earnshaw's theorem, applies only if all conductors in the system carry a constant current. If the current is allowed to change upon displacement of the body, then the theorem no longer applies and stability is possible. This is the more natural explanation for the stabilizing influence of superconductors, and this reasoning clearly extends to conventional conductors also. The need for displacement-controlled currents is often misinterpreted to imply that an external power source is required. In the case of flywheels, however, the energy stored in the spinning rotor represents a potential source for driving the dynamic currents directly. Levitation methods which provide their lift force using permanent magnets, which generate correction currents by drawing only on the kinetic energy of the spinning flywheel, and which consume a vanishingly small amount of power under steady-state conditions, will be termed "self-stabilizing". If, in addition, the control currents are implemented using only passive circuit elements (inductance, resistance, and capacitance), then the scheme will be referred to as providing "passive" stabilization. Concepts for obtaining passive stabilization are investigated in Chapter 3, while a potentially self-stabilizing active-control system is considered in Chapter 4.

In an effort to optimize efficiency and minimize complexity, a topology for magnetic levitation has been found which is not only reliable and inexpensive, but which has the added feature of actively controlling the troublesome radial dynamics while still allowing the flywheel to rotate about its own center of mass. Figure 1.6 illustrates the basic concept of
a. CENTERED HOMOPOLAR ROTOR

Conductors Located on Inner Rotor Surface

Homopolar Levitation Field

b. DISPLACED HOMOPOLAR ROTOR

Conductors Located on Inner Rotor Surface

Perturbed Levitation Field

Figure 1.6 Homopolar Magnetic Bearing Configuration.
the homopolar radial magnetic bearing. This geometry is inherently stable in the vertical direction, and by incorporating the radial-gap toothed structure shown in Figure 1.5, enough lift can be generated to fully levitate the rotor. As long as the rotor remains centered about the stator, conductors located on the rotor see a constant field and no currents flow. If the rotor is horizontally displaced, however, the conductors see a time-varying field as they rotate. This changing field can be used to generate a voltage in an electric circuit which then drives currents to force a return to equilibrium.

An imbalance due to an eccentric center of mass causes the rotor to wobble, as viewed from a stationary frame of reference. In the frame of reference of the rotor, however, the imbalance causes a static displacement of each point on the rotor with respect to the stator surface. Thus a point on the rotor which sees a larger air gap than normal due to imbalance, always sees that same gap. There is no time derivative of the flux for a coil mounted to the rotor in this situation, so no voltage is generated. Not only is the rotor allowed to rotate about its center of mass unimpeded, there is no power dissipation in the control system as a result of the imbalance. By designing the control circuit in a manner similar to the VZP control used in one-dimensional axial-levitation schemes, the power required to maintain stability during normal operation can be made vanishingly small.
CHAPTER 2

POSITIONAL STABILITY OF RADIAL-GAP CONFIGURATIONS

This chapter provides the theoretical developments which are necessary to evaluate the forces of magnetic origin which arise when an axisymmetric rotor and stator become misaligned. Many of the intermediate results apply to a wide range of situations and this generality is preserved and identified wherever possible. The method presented is a perturbation approach which yields linear equations of motion for electromechanical systems.

The major contribution presented in chapter is a method for analyzing the dynamic behavior of electromechanical systems which exhibit coupling of motion between two symmetric axes. This type of coupling is present in smooth-radial-gap magnetic structures due to currents induced in windings, and is present in all rotating systems as a result of the gyroscopic effect. Both of these phenomena are important in magnetic bearing design. A complex-variable notation is developed which allows the two-dimensional interaction to be represented by a single expression which, except for its complex variables, is completely analogous to a comparable one-dimensional relationship. The important techniques of classical stability analysis, including the root locus and Nyquist diagrams, are shown to be directly applicable to equations of motion written in terms of this complex notation, allowing the stability of rotating systems to be evaluated almost as easily as one-dimensional interactions.

In order to perform the stability analysis, it is first necessary to derive equations which properly describe the dynamic interaction of the magnetic field with movements of the rotor and stator. Section 2.1 uses
the Maxwell stress tensor to obtain, in cylindrical coordinates, the change in the force on an object due to small changes in the magnetic field surrounding it (equation 2.15). Sections 2.2 through 2.5 then proceed to analyze the perturbation magnetic field for both homopolar radial magnetic bearings and radial-gap motor-generators. In either system, changes in the magnetic field result directly from movements of the material boundaries. The perturbation magnetic field which results must satisfy certain boundary conditions which are directly related to the boundary displacements (equation 2.44). These boundary conditions are specialized for the case of the homopolar radial magnetic bearing in equation 2.59 and for the motor-generator in equation 2.61.

The displacement boundary conditions act like sources for the perturbation magnetic field. The field which is actually generated depends upon the specific magnetic topology and the presence of conducting material. The effect of conductors in which current can be induced is to present another boundary condition on the magnetic field, this time between the normal flux and the magnetic potential at the surface. Specific formulas are given for symmetric discrete windings (equation 2.96) and for the squirrel-cage induction rotor (equation 2.104). Using these boundary conditions, the air-gap field is solved for the homopolar radial magnetic bearing (Section 2.4) and the motor-generator (Section 2.5). Inserting these solutions into the force relations given in Section 2.1 yields a complex-valued Laplacian transfer function between the force of magnetic origin and the displacement (equations 2.122 and 2.140). Combining this relation with the mechanical equation of motion determines the behavior of the electromechanical system. Finally, classical techniques for stability analysis can be applied (Section 2.6).
Several nonconventional notations are used in this chapter which deserve advanced warning. Functions of time and their Laplace transform are both represented by the same variable. The distinction is made by explicitly indicating "t" as the argument for time functions, and "s" as the argument for the transform. To make certain that this does not lead to confusion, parentheses are used in equations only to indicate functional dependence and not to indicate the ordering of operations. Complex variables whose complexity is associated with position or direction in a two-dimensional space are indicated by a tilde, \( \sim \). Complex values which are associated with a specific dependence on both space and time are indicated by the circumflex, \( ^\wedge \). All other complex values and functions are distinguished by an underscore. Vectors are indicated by a single overbar, and tensors by a double overbar. The capital letter \( Z \) is used as an operator to represent integration over \( z \), while \( Z \) is used to represent circuit impedance. Other unusual notations may be applied locally in derivations and are defined where they occur.

2.1 MAGNETIC FORCES IN CYLINDRICAL SYMMETRY

The Maxwell stress tensor is a convenient means for finding the net force of magnetic origin acting on a volume and requires only the magnetic fields present at the surface of the volume. This section considers the force on a volume which can be enclosed by a cylindrical surface. In anticipation that the resulting expressions will be applied in a perturbation analysis, the change in the force due to small changes in the magnetic field is evaluated.

2.1.1 The Maxwell Stress Tensor

The magnetic force density in an incompressible and magnetically
linear material is

\[ \overline{F} = J \times B - \frac{1}{2} |\overline{H}|^2 \gamma \mu \] (2.1)

Using Gauss' law and vector identities this can be written as the divergence of a tensor:

\[ \overline{F} = \nabla \cdot \{ \mu \overline{H} \overline{H} - \frac{1}{2} \mu [\overline{H} \cdot \overline{H}] I \} \] (2.2)

where \( I \) is the identity matrix. The quantity in brackets is the Maxwell stress tensor for quasi-static magnetic systems which are incompressible and magnetically linear,

\[ \overline{T} = \mu \overline{H} \overline{H} - \frac{1}{2} \mu [\overline{H} \cdot \overline{H}] I \] (2.3)

where

\[ \overline{F} = \nabla \cdot \overline{T} \] (2.4)

The total force on the volume is found by integrating the force density over the volume.

\[ \overline{f} = \int_{v} \overline{F} \, dV = \int_{v} \nabla \cdot \overline{T} \, dV \] (2.5)

The divergence theorem allows the volume integral to be converted to a surface integral.

\[ \overline{f} = \int_{s} \overline{T} \cdot \overline{n} \, dS \] (2.6)

As long as the surface of integration lies in a magnetically linear and incompressible region then this expression is valid even if the material enclosed does not meet these criteria.\(^9\)

Consider an object which can be contained within a cylindrical surface such that at each point on the curved portion of the surface either \( \mu = \mu_0 \) or \( \overline{H} = 0 \). Furthermore, the two circular end caps are assumed to yield no net force contribution either due to symmetry or because the magnetic field on them is negligible. Such a model is suitable for most magnetic structures.
having a radial gap.

In terms of cylindrical coordinates, the net force acting on the material contained within the cylinder becomes

\[
\begin{align*}
\mathbf{f}_x &= r_o \int \int T_{rr} \cos \theta - T_{\theta r} \sin \theta \, d\theta \, dz \\
\mathbf{f}_y &= r_o \int \int T_{rr} \sin \theta + T_{\theta r} \cos \theta \, d\theta \, dz \\
\mathbf{f}_z &= r_o \int \int T_{zr} \, d\theta \, dz
\end{align*}
\]  

(2.7)

where

\[
T_{rr} = 1/2 \mu_0 \left[ H_r H_r - H_\theta H_\theta - H_z H_z \right]
\]

(2.8)

\[
T_{\theta r} = \mu_0 H_\theta H_r
\]

\[
T_{zr} = \mu_0 H_z H_r
\]

\(r_o\) is the radius of the integration surface.

The torque about the polar axis of the cylinder is similarly given by

\[
\tau_z = r_o^2 \int \int T_{\theta r} \, d\theta \, dz
\]

(2.9)

The \(x\) and \(y\) forces can be conveniently described by a single complex value;

\[
\mathbf{\tilde{f}} = \mathbf{f}_x + j\mathbf{f}_y = r_o \int \int [T_{rr} + jT_{\theta r}] e^{j\theta} \, d\theta \, dz
\]

(2.10)

where now \(\mathbf{\tilde{f}}\) represents both the magnitude and direction of the net force in the radial plane. This representation is useful in simplifying expressions involving the force in the radial plane (the radial force) whenever those expressions reflect rotational symmetry. In such cases a single equation involving the complex force \(\mathbf{\tilde{f}}\) may replace two separate equations in \(f_x\) and \(f_y\).
2.1.2 Axipersistent Fields

The force components can be written directly in terms of the fields by substituting the proper stress-tensor components from equation 2.8 into equations 2.7, 2.9, and 2.10,

\[ \vec{f} = \frac{1}{2} \mu_0 r_o \int \int \left\{ \left[H_z^r + j H_\theta^r \right]^2 - H_z^2 \right\} e^{j \theta} \, d\theta \, dz \]

\[ f_z = \mu_0 r_o \int \int H_z H_r \, d\theta \, dz \]

\[ \tau_z = \mu_0 r_o^2 \int \int H_\theta H_r \, d\theta \, dz \]  \hspace{1cm} (2.11)

Because the fields have a periodicity of \(2\pi\) in the coordinate \(\theta\), they can be expressed as Fourier series of the form

\[ H_1(\theta) = \sum_{m=0}^{\infty} \text{Re}(H_{1m} e^{-jm\theta}) \]  \hspace{1cm} (2.12)

This form can be introduced into the force expressions (equation 2.11), yielding a double summation over the Fourier components. Most of these terms vanish when the integration over \(\theta\) is performed, leaving

\[ \vec{f} = \mu_0 \pi r_o \sum_{m=1}^{\infty} \left\{ \left[H_z^r + j H_\theta^r \right] \left[H_z^r + j H_\theta^r \right] - H_z^2 \right\} \]

\[ + \frac{1}{2} \mu_0 \pi r_o^2 \sum_{m=1}^{\infty} \left\{ \left[H_z^r + j H_\theta^r \right] \left[H_z^r + j H_\theta^r \right] - H_z^2 \right\} \]

\[ f_z = 2\mu_0 \pi r_o^2 \sum_{m=1}^{\infty} \text{Re}(H_z^r H_z^r) \]  \hspace{1cm} (2.13)

\[ \tau_z = 2\mu_0 \pi r_o^2 \sum_{m=1}^{\infty} \text{Re}(H_z^r H_z^r) \]

where \(Z\) is used as an operator to represent integration over the axial coordinate. For fields independent of \(z\) this simply becomes a multiplier equal to the active axial length.
2.1.3 Perturbation Field Analysis

When small changes in the magnetic field occur there will be corresponding changes in the net force components. Let the magnetic field before perturbation be represented by $\vec{H}$, and small changes in the field by $\vec{h}$. The sum of both fields must then be inserted into the force expression. This substitution will yield force terms which depend

(1) on the dominant field $H$ only,
(2) on products of $H$ and $h$,
(3) on second-order terms in $h$.

Since the magnitude of $\vec{h}$ is presumed to be much less than that of $\vec{H}$, the second order terms in the perturbation quantity will be neglected. The two remaining terms correspond to the force before perturbation and the first-order force deviation, respectively.

Consider now the simplified situation where the field $\vec{H}$ has only a single dominant space-harmonic component:

$$H_1(\theta) \approx \text{Re}\{\vec{H}_{1p} e^{-jp\theta}\} \quad (2.14)$$

The perturbation fields may have any number of space harmonics, but only those at $p$ and $p \pm 1$ will affect the force expressions. The perturbation force expressions (equation 2.13) become:
\[
\tilde{f} = \begin{cases} 
\mu_0 \pi r_o^2 \{ [H_{r_0} + jH_{\theta_0}] [\tilde{h}_{r_0} + j\tilde{h}_{\theta_0}] - H_{z_o} \tilde{h}_{z_o} \} & ; p=0 \\
\mu_0 \pi r_o^2 \{ [\tilde{H}_{r_1} + j\tilde{H}_{\theta_1}][h_{r_1} + jh_{\theta_1}] - \tilde{H}_{z_1} h_{z_1} \} \\
+ \frac{1}{2} \mu_0 \pi r_o^2 \{ [\tilde{H}_{r_1} + j\tilde{H}_{\theta_1}][h_{r_2} + jh_{\theta_2}] - \tilde{H}_{z_1} \tilde{h}_{z_2} \} & ; p=1 \\
\frac{1}{2} \mu_0 \pi r_o^2 \{ [\tilde{H}_{r_1} + j\tilde{H}_{\theta_1}][\tilde{h}_{r_1} + j\tilde{h}_{\theta_1}] - \tilde{H}_{z_1} \tilde{h}_{z_1} \} \\
+ \frac{1}{2} \mu_0 \pi r_o^2 \{ [\tilde{H}_{r_1} + j\tilde{H}_{\theta_1}][\tilde{h}_{r_2} + j\tilde{h}_{\theta_2}] - \tilde{H}_{z_1} \tilde{h}_{z_2} \} ; p>1
\end{cases}
\]

(2.15)

\[
f_z = \begin{cases} 
2\mu_0 \pi r_o^2 \{ h_{r_o} h_{z_o} + h_{\theta_o} h_{r_o} \} & ; p=0 \\
\mu_0 \pi r_o^2 \text{Re}\{\tilde{h}_{r_p} \tilde{h}_{z_p} + \tilde{h}_{\theta_p} \tilde{h}_{r_p} \} & ; p>0
\end{cases}
\]

(2.16)

\[
\tau_z = \begin{cases} 
2\mu_0 \pi r_o^2 \{ h_{r_o} h_{\theta_o} + h_{\theta_o} h_{r_o} \} & ; p=0 \\
\mu_0 \pi r_o^2 \text{Re}\{\tilde{h}_{r_p} \tilde{h}_{\theta_p} + \tilde{h}_{\theta_p} \tilde{h}_{r_p} \} & ; p>0
\end{cases}
\]

(2.17)

Given both the original and perturbation magnetic fields on a cylindrical surface enclosing an object, the net magnetic force on the object can be calculated using the results of this section. This establishes the effect of the magnetic subsystem on the mechanical subsystem. In order to develop equations of motion for the coupled electromechanical configuration, the effect of mechanical motion on the magnetic fields must be determined.
2.2 BOUNDARY CONDITIONS FOR SMALL DISPLACEMENTS IN A MAGNETIC FIELD

Consider a magneto-quasi-static configuration which is composed of several regions each having uniform material properties. Throughout some period of time the locations of the boundary surfaces which separate the regions are presumed known, as are the magnetic field and its gradient on those surfaces. All currents are restricted to flow either along the boundaries as surface current or uniformly over a region. These modeling restrictions place the magnetic field analysis in the realm of lumped parameters, thereby permitting very simple models for a wide range of important magnetic structures. Any desired level of accuracy can be obtained by specifying a large enough number of regions. The fields and currents present in this fully-solved configuration will be referred to as the equilibrium values, though the system need not actually be in force equilibrium. Equilibrium quantities will be denoted by capital letters.

Deviations from the equilibrium condition can be investigated by allowing each element of material to undergo a small time-dependent displacement from its equilibrium trajectory. These displacements are assumed to be smooth functions of both space and time. With the restrictions given, there are two ways this movement can affect the magnetic field:

(1) Movement of Boundaries

Large changes in the magnetic field can occur locally in the small volume traversed by a moving boundary. This results in a small change in the magnetic field throughout the system. Note that because materials and currents are uniform within each region, movements within boundaries do not themselves alter the magnetic field.
(2) Induced Currents

Voltages induced in conductors due to flux variations caused by the displacements can result in small changes in the currents. These in turn produce a small change in the magnetic field throughout the system. The dependence of these currents on the displacements is generally a global phenomenon which may involve externally-imposed terminal relationships.

Throughout the following development, only linear terms in perturbation quantities will be retained. Consequently, the two effects described above are simply additive and can be evaluated independently. In both cases the perturbation values will be denoted by small letters. This section considers only the direct effect of displaced boundaries, with the perturbation currents retained only as unspecified quantities. In section 2.3 these currents are evaluated for several situations of interest. Once all contributions to the perturbation magnetic field are determined, the force of magnetic origin acting on a region can be found by evaluating the Maxwell stress tensor over its surface.

2.2.1 Jump Conditions at Displaced Boundaries

Let \( \vec{r} \) be a Lagrangian position vector whose time evolution describes the equilibrium trajectories of all elements of material in the equilibrium configuration. Then each element of material which lies at position \( \vec{r}_0 \) at time \( t=0 \) is located for all later times at \( \vec{r}(\vec{r}_0, t) \). It is assumed that all regions in the system contain traceable material so that this description is meaningful, although its density can be made arbitrarily small.

Now permit the presence of displacements from the equilibrium trajectories. An element of material which would have been found at \( \vec{F}(\vec{r}_0, t) \) is instead located at \( \vec{r}(\vec{r}_0, t) + \xi(\vec{r}, t) \). Thus \( \xi(\vec{r}, t) \) becomes a
displacement-vector field which assigns a time-dependent displacement to each point in space. It does not correspond to a particular element of material as does the equilibrium position vector \( \mathbf{r}(\mathbf{r}_0, t) \). For this reason the displacement vector has Eulerian characteristics as far as time and space derivatives are concerned. The reason for this hybrid combination of a Lagrangian description of the equilibrium situation and an Eulerian perturbation vector lies in a desire to describe the interaction of the boundaries (a Lagrangian phenomenon) with the magnetic field (an Eulerian quantity).

The particles which form the interface at \( t=0 \) continue to form the interface at all later times, given that there is no mixing across the boundary. Thus if \( \mathbf{r}_b \) represents the values of \( \mathbf{r}_0 \) which lie on the boundary at \( t=0 \), then \( \mathbf{r}(\mathbf{r}_b, t) \) describes the equilibrium boundary location throughout time. and \( \mathbf{r}(\mathbf{r}_b, t) + \xi(r(r_b, t), t) \) describes the displaced boundary.

The location of the boundaries is important because in order to satisfy Gauss' magnetic law and Faraday's law, the magnetic field must satisfy two "jump conditions" relating the field on one side of the interface to that on the other:

\[
\mathbf{n} \times \left[ \mathbf{H} \right] = \mathbf{K} \\
\mathbf{n} \cdot \left[ \mathbf{B} \right] = 0
\]

(2.18)

where \( \mathbf{n} \) is the surface normal vector, directed from region b into region a.

\( \left[ \mathbf{H} \right] = \mathbf{H}^a - \mathbf{H}^b \) is the "jump" in \( \mathbf{H} \).

\( \mathbf{K} \) is the surface current along the interface.

If \( \mathbf{n}(\mathbf{r}) \) is the surface normal vector in the equilibrium configuration for a particular location \( \mathbf{r} \) on the boundary at a given instant of time, then let \( \mathbf{n}'(\mathbf{r} + \xi) \) be the normal vector exhibited by the same boundary
material when the displacements are added. Similarly, let the material experience a modification of the field from \( \vec{H}(r) \) to \( \vec{H}'(r+\xi) \) and of the current from \( \vec{K}(r) \) to \( \vec{K}'(r+\xi) \). The two jump conditions must be satisfied in the displaced arrangement just as they were in the equilibrium configuration:

\[
\vec{n}'(r+\xi) \times \left[ \vec{H}'(r+\xi) \right] = \vec{K}'(r + \xi) \\
\vec{n}'(r+\xi) \times \left[ \vec{H}^\prime(\vec{r}+\vec{\xi}) \right] = 0
\]

As illustrated in Figure 2.1 for a simple one-dimensional case, the values of \( \vec{H}' \) on each side of the new boundary are expected to be close to the values obtained by linearly extrapolating the equilibrium fields within each material from \( \vec{r} \) to \( \vec{r} + \vec{\xi} \). In addition, \( \vec{H}' \) will differ from \( \vec{H} \) throughout the system by a small difference, \( \vec{h} \), due to the new boundary locations.

![Figure 2.1 Extrapolating the Fields to a Displaced Boundary.](image)

It is often convenient to describe the magnetic field using a coordinate system other than Cartesian space. This is certainly true of rotating machines which exhibit rotational symmetry. Consequently, the expressions which follow are derived for the general case of orthogonal curvilinear coordinates. Because the unit basis vectors change with
position in curvilinear space, several interesting complications arise which require careful treatment. For example, two vectors are considered to be equal if they both have the same magnitude and direction. This does not necessarily imply that they will have the same direction components, however, as illustrated in Figure 2.2.

\[ \overrightarrow{A} = A_o \overrightarrow{a}_r \quad \overrightarrow{A} = A_o \overrightarrow{a}_\theta \]

Figure 2.2 Two Equal Vectors Can Have Different Components.

In an orthogonal curvilinear coordinate system, each component of \( \overrightarrow{H} \) is a scalar function of the coordinates \((u_1, u_2, u_3)\). The value of each component at \( \overrightarrow{r} + \xi \) can be estimated by using a Taylor's series expansion about \( \overrightarrow{r} \),

\[
H^{a}_{i}(\overrightarrow{r} + \xi) \approx H^{a}_{i}(\overrightarrow{r}) + \Delta u_j \frac{\partial H^{a}_{i}}{\partial u_j} \bigg|_{\overrightarrow{r}} + h^{a}_{i}(\overrightarrow{r})
\]

\[
H^{b}_{i}(\overrightarrow{r} + \xi) \approx H^{b}_{i}(\overrightarrow{r}) + \Delta u_j \frac{\partial H^{b}_{i}}{\partial u_j} \bigg|_{\overrightarrow{r}} + h^{b}_{i}(\overrightarrow{r})
\]

(2.20)

where standard index notation is used to imply summation of repeated indices over the three coordinates. The perturbation field \( \overrightarrow{h} \) can be assigned to location \( \overrightarrow{r} \) instead of \( \overrightarrow{r} + \xi \) since the change in a perturbation field over a perturbation distance is second order.
If \((m_1, m_2, m_3)\) are the three metric coefficients for the coordinates \((u_1, u_2, u_3)\), then

\[
\Delta u_j = \frac{\xi_j}{m_j} \tag{2.21}
\]

and

\[
\hat{H}^i(r+\xi) \approx \hat{H}^i(r) + \frac{\xi_i}{m_j} \frac{\partial \hat{H}^i}{\partial u_j} + h_i(r) \tag{2.22}
\]

It is tempting to rewrite this in vector form as

\[
\tilde{H}^i(r+\xi) \approx \tilde{H}(r) + \xi \cdot \nabla \tilde{H} + \tilde{h}(r) \tag{2.23}
\]

but this form is equivalent only in Cartesian coordinates. The \(\nabla\) operator acts on the unit basis vectors as well as the components \(\hat{H}_i\), leading to a different result when the basis vectors are themselves position dependent.

Specifically, for an arbitrary vector field \(\vec{A}\),

\[
\xi \cdot \nabla \vec{A} = \frac{\xi_j}{m_j} \frac{\partial (A_i \vec{a}_i)}{\partial u_j} = \frac{\xi_j}{m_j} \frac{\partial A_i}{\partial u_j} \vec{a}_i + \frac{\xi_j}{m_j} A_i \frac{\partial (\vec{a}_i)}{\partial u_j} \tag{2.24}
\]

Now if a given vector \(\vec{A}(r)\) is "moved" to \(r+\xi\) and expressed in terms of the unit basis vectors at \(r+\xi\), then to first order

\[
\vec{A}(r; r+\xi) = \vec{A}(r) + \frac{\xi_j}{m_j} A_i \frac{\partial (\vec{a}_i)}{\partial u_j} \tag{2.25}
\]

where the parametric argument following the semicolon is used to indicate the location for evaluating the vector components when it differs from the location for evaluating the vector itself. The second term on the right is identical to the last term in the gradient expression above. Consequently, the proper form of equation 2.23 is

\[
\vec{H}(r+\xi) = \vec{H}(r; r+\xi) + \xi \cdot \nabla \tilde{H} + \tilde{h}(r) \tag{2.26}
\]
This result was predictable given the following reasoning. Gradient operators necessarily act at a single point in space. Thus they generate vectors which are expressed in terms of the unit basis vectors at that point. A vector expression which is derived using vector operators is therefore valid only at the point of application. If it is desired to describe a vector at some location removed from the point of application, then the result obtained must be expressed in terms of the unit basis vectors at the new location. In the example above, the use of gradient operators yields a result in terms of the unit vectors at \( \vec{r} \). The desired vector is one located at \( \vec{r} + \vec{\xi} \), so the entire result must be "relocated". The perturbation terms need not explicitly indicate their relocation, since for them it produces only a second-order difference. Based on this interpretation of the localized effect of the vector gradient operator, it is possible to more clearly interpret the often-used argument that a vector relation, once proven in Cartesian coordinates, is presumed valid in any orthogonal curvilinear system of coordinates. It is now apparent that such a statement applies only to vector relations which involve only a single point in space. Such arguments can be applied to expressions relating two different points only if the relocation of the result is specifically indicated.

For the particular case of cylindrical coordinates, the vector gradient is
\[ \vec{v} \vec{A} = \begin{bmatrix} \frac{\partial A_r}{\partial r} & \frac{\partial A_\theta}{\partial r} & \frac{\partial A_z}{\partial r} \\
\frac{1}{r} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r} & \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r} & \frac{1}{r} \frac{\partial A_z}{\partial \theta} \\
\frac{\partial A_r}{\partial z} & \frac{\partial A_\theta}{\partial z} & \frac{\partial A_z}{\partial z} \end{bmatrix} \tag{2.27} \]

and the coordinate transformation for relocation a small distance away is

\[ \vec{A}(\vec{r}; \vec{r} + \xi) = \vec{A}(\vec{r}) + \frac{\xi_\theta}{r} A_\theta \vec{a}_r - \frac{\xi_\theta}{r} A_r \vec{a}_\theta \tag{2.28} \]

The "extra" terms in the coordinate transformation are in fact seen to be exactly those necessary to cancel the extra terms in the gradient tensor when both are inserted in the expression for \( \xi \cdot \vec{V} \vec{H} \). This leaves only the scalar derivatives of the field components corresponding to the form which was originally derived (equation 2.22). With the help of the notations developed thus far, the boundary conditions (equation 2.19) become

\[ \vec{n}^-(\vec{r} + \xi) \cdot \left[ \vec{H}(\vec{r} + \xi) + \frac{\xi \cdot \vec{V} \vec{H}}{2} + \vec{b} \right] = \vec{K}(\vec{r} + \xi) \quad \text{(2.29)} \]

\[ \vec{n}^-(\vec{r} + \xi) \cdot \left[ \vec{B}(\vec{r} + \xi) + \frac{\xi \cdot \vec{V} \vec{B}}{2} + \vec{b} \right] = 0 \]

If an equilibrium surface current is present then it is presumed to remain attached to the surface when displacements occur. For this reason it is not enough simply to know the location of the boundary before and after the displacement. In order to determine what value of \( \vec{K} \) is present at a point on the displaced boundary, one must know where that point originated on the equilibrium surface. This information is provided by the vector field \( \vec{\xi} \). Thus the equilibrium current \( \vec{K}(\vec{r}) \) gets moved to \( \vec{r} + \vec{\xi} \) when
the displacement occurs. In addition, \( \overline{K} \) may change by a small amount \( \overline{k} \) due to the movement;

\[
\overline{K}'(\overline{r} + \overline{\xi}) = \overline{K}(\overline{r}; \overline{r} + \overline{\xi}) + \overline{k}
\] (2.30)

where again it has been necessary to express \( \overline{K} \) in terms of the coordinates present at \( \overline{r} + \overline{\xi} \).

The boundary conditions continue to contain both equilibrium and perturbation values. The goal is to obtain expressions relating only the perturbation quantities. To attain this goal, \( \overline{n}'(\overline{r} + \overline{\xi}) \) must be expressed as a perturbation to the equilibrium value, \( \overline{n}(\overline{r}) \).

The equilibrium boundary can be described by specifying the locus of all Eulerian Cartesian coordinates \((x_1, x_2, x_3)\) which satisfy a function \( F(x_1, x_2, x_3) = 0 \). In this form, the normal vector is given by

\[
\overline{n}(\overline{r}) = \frac{\nabla F(x_1, x_2, x_3)}{|\nabla F(x_1, x_2, x_3)|}.
\] (2.31)

If the following substitution is made in the function \( F \),

\[
[x_1, x_2, x_3] + [x_1 - \xi_1(\overline{r}), x_2 - \xi_2(\overline{r}), x_3 - \xi_3(\overline{r})]
\] (2.32)

then the values \((x_1, x_2, x_3)\) which are the solution of

\[
F(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3) = 0
\] (2.33)

now represent the points on the displaced boundary. The normal vector, in terms of the coordinates at the displaced boundary, becomes

\[
\overline{n}'(\overline{r} + \overline{\xi}) = \frac{\nabla F(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3)}{|\nabla F(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3)|}
\] (2.34)

Expanding the gradient using the chain rule for differentiation gives

\[
\nabla F(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3) = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} \overline{a}_i
\] (2.35)

Because the displacement terms are introduced functionally in direct linear
combination with the \( x \) terms,

\[
\frac{\partial F}{\partial \xi_j} = -\frac{\partial F}{\partial x_j} \quad (2.36)
\]

and

\[
\nabla F(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3) = \left( \frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_j} \frac{\partial \xi_j}{\partial x_1} \right) \vec{a}_1 \quad (2.37)
\]

The first term on the right yields exactly the same result as \( F(x_1, x_2, x_3) \) evaluated for the equilibrium configuration. The same is true of the \( \frac{\partial F}{\partial x_j} \) term, suggesting that this expression might be given conveniently in vector form as

\[
\nabla F(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3) = \nabla F(x_1, x_2, x_3) - \nabla \xi \cdot \nabla F(x_1, x_2, x_3) \quad (2.38)
\]

But, as before, this is only valid for Cartesian coordinates. In all other coordinate systems, this expression will yield a value in terms of the unit basis vectors at \( \vec{r} \) rather than at \( \vec{r} + \xi \). Thus a valid form is given by

\[
\nabla F(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3) = \nabla F(x_1, x_2, x_3; \vec{r} + \xi) - \nabla \xi \cdot \nabla F(x_1, x_2, x_3) \quad (2.39)
\]

where \( \nabla \xi \) is written together as a tensor to clarify the order of operations. Relocation of the second term to \( \vec{r} + \xi \) is unnecessary, as this represents a second-order effect.

Taking the magnitude of the gradient yields, to first order;

\[
|\nabla F(\vec{r} - \xi)| = |\nabla F(\vec{r}; \vec{r} + \xi)| \left( 1 - \frac{\nabla F(\vec{r}) \cdot \nabla \xi \cdot \nabla F(\vec{r})}{|\nabla F(\vec{r})|^2} \right) \quad (2.40)
\]

Thus the normal vector at the new location is given by combining equations 2.39 and 2.40;

\[
\vec{n}'(\vec{r} + \xi) = \vec{n}(\vec{r}, \vec{r} + \xi) - \{ \nabla \xi \cdot \vec{n} - \vec{n} \cdot [\nabla \xi \cdot \vec{n}] \} \quad (2.41)
\]

This form is presumed to be valid in any orthogonal curvilinear coordinate space. The last term above simply removes the normal component of the gradient from the first term in brackets, leaving what is often referred to
as the surface gradient, $\nabla_{\xi}$, so that finally,

$$
\vec{n}^* (\vec{r} + \vec{\xi}) = \vec{n} (\vec{r}; \vec{r} + \vec{\xi}) - \frac{\nabla_{\xi} \vec{n}}{\nabla_{\xi} \vec{n}}
$$

(2.42)

Inserting the expression for the normal vector into the boundary conditions for the field gives

$$
\vec{n} (\vec{r}; \vec{r} + \vec{\xi}) \times \left[ [\vec{H}(\vec{r}; \vec{r} + \vec{\xi}) + \vec{\xi} \cdot \nabla_{\xi} \vec{H} + \vec{h}] - [\nabla_{\xi} \vec{n} \times \vec{H}] \right] = \vec{k} (\vec{r}; \vec{r} + \vec{\xi}) + \vec{k}
$$

(2.43)

$$
\vec{n} (\vec{r}; \vec{r} + \vec{\xi}) \cdot \left[ \vec{B}(\vec{r}; \vec{r} + \vec{\xi}) + \vec{\xi} \cdot \nabla_{\xi} \vec{B} + \vec{b} \right] - [\nabla_{\xi} \vec{n} \cdot \vec{B}] = 0
$$

The equilibrium boundary conditions (equation 2.18), when expressed in terms of the coordinates at $\vec{r} + \vec{\xi}$, can be subtracted from the two equations, leaving the desired result;

$$
\vec{n} \times \left[ [\vec{\xi} \cdot \nabla_{\xi} \vec{H} + \vec{h}] - [\nabla_{\xi} \vec{n} \times \vec{H}] \right] = \vec{k}
$$

(2.44)

$$
\vec{n} \cdot \left[ [\vec{\xi} \cdot \nabla_{\xi} \vec{B} + \vec{b}] - [\nabla_{\xi} \vec{n} \cdot \vec{B}] \right] = 0
$$

These expressions represent a major new simplification in the perturbation analysis of magnetic fields. By directly equating the perturbation quantities, the pitfalls inherent in the linearization process can now be avoided in subsequent calculations.

It is illuminating to rewrite these conditions as

$$
\vec{n} \times \left[ \vec{H} \right] = \vec{k} + \left\{ [\nabla_{\xi} \vec{n} \times \vec{H}] - \vec{n} \times \left[ \vec{\xi} \cdot \nabla_{\xi} \vec{H} \right] \right\}
$$

(2.45)

$$
\mu_0 \vec{n} \cdot \left[ \vec{B} \right] = \sigma_m + \left\{ [\nabla_{\xi} \vec{n} \cdot \vec{B}] - \vec{n} \cdot \left[ \vec{\xi} \cdot \nabla_{\xi} \vec{B} \right] \right\}
$$

where $\sigma_m$ is the perturbation "magnetic charge" of the Chu formulation,\textsuperscript{11} related to the magnetization by

$$
\sigma_m = -\mu_0 \vec{n} \cdot \left[ \vec{m} \right]
$$

(2.46)

Written in this form, boundary displacements serve only to introduce imposed surface currents and magnetic charges into a perturbation field problem which otherwise has its own independent self-consistent solution.
Stated another way, the effect of boundary displacements on the field can be duplicated by leaving the system in its equilibrium configuration while introducing some combination of surface currents and magnetic charges at the boundaries. When the equilibrium configuration lends itself to an analytical solution, the equivalence is extremely useful. The Appendix presents an example where the boundary conditions just described are applied to a problem for which the exact solution is known, thus verifying many of the points raised in this section.

### 2.2.2 Edge Conditions

The previous discussion assumed at the outset that the magnetic field within a region was a smoothly-varying function of space so that its value at $\bar{r} + \bar{\xi}$ could be inferred from its value and derivatives at $\bar{r}$. This is generally true, but often there are locations where the field is rapidly varying and it is inconvenient to describe the field accurately. Such a situation arises where there are sharp edges in the material boundaries. A convenient way to analyze these situations is to use the integral forms of Ampere's and Gauss' laws over the entire area surrounding the edge, rather than on each face separately as the jump conditions do.

The use of the integral form for Ampere's law is shown in Figure 2.3. For each point along a sharp edge there is a plane which is perpendicular to the edge and which includes the normal vectors for both of the sides which come in contact. A contour can be drawn which lies in this plane, encloses the edge, and crosses each of the two boundaries in their normal directions. The dimensions of this contour are made small so that surfaces 1 and 2 are essentially flat in the region of interest, and any displacement occurs uniformly over the area.
The edge and its surroundings are now allowed to suffer a displacement \( \vec{\xi} \), as shown in Figure 2.4. Large changes in the field occur over the volume swept by the displacement, and there is an additional perturbation field \( \vec{h} \) throughout the region. The concentrated edge current may also be modified by an amount \( i_e \).

To first order in the perturbations, the contour integral becomes

\[
- \left[ \vec{\xi} \cdot \vec{n}_1 \right] \left[ \vec{n}_1 \cdot \left[ \vec{H}_1 \right] \right] + \left[ \vec{\xi} \cdot \vec{n}_2 \right] \left[ \vec{n}_2 \cdot \left[ \vec{H}_2 \right] \right] + \oint \vec{h} \cdot d\vec{\xi} = i_e \quad (2.47)
\]

In the perturbation field solution the first two terms above are equivalent to imposing an additional current to flow along the edge.

A volume for evaluating Gauss' law would be defined by the contour
just used, extended over a depth \( \Delta \) parallel to the edge.

\[ \int B \cdot da = 0 \]

\[ \vec{n}_s \]

\[ \vec{n}_1 \]

\[ \vec{n}_2 \]

\[ \Delta \]

\[ \text{Region a} \]

\[ \text{Region b} \]

Figure 2.5 Gauss' Law Near a Sharp Edge.

Evaluating the surface integral to first order in the perturbations yields

\[ -[\vec{n}_1 \times \left[ \vec{B}_1 \right] ] \cdot \vec{n}_s \left[ \xi \cdot \vec{n}_1 \right] \Delta + [\vec{n}_2 \times \left[ \vec{B}_2 \right] ] \cdot \vec{n}_s \left[ \xi \cdot \vec{n}_2 \right] \Delta + \int_{\text{edge}} b \cdot da = 0 \quad (2.48) \]

This is equivalent to introducing into the perturbation field solution a magnetic charge per unit length along the edge;

\[ \rho_m = [\vec{n}_1 \times \left[ \vec{B}_1 \right] ] \cdot \vec{n}_s \left[ \xi \cdot \vec{n}_1 \right] - [\vec{n}_2 \times \left[ \vec{B}_2 \right] ] \cdot \vec{n}_s \left[ \xi \cdot \vec{n}_2 \right] \quad (2.49) \]

The effect of a small displacement on the magnetic field near a sharp edge is seen to be duplicated by retaining the equilibrium position, but adding a combination of current and magnetic charge. This is consistent with the previous results for smooth boundaries. Considering all contributions, the added "sources" should follow the same rules as are normally obeyed by currents and charges. Namely, the currents should all close upon themselves in divergent-free solenoidal loops, and the net magnetic charge in any uniform region should sum to zero. This test provides a valuable check on the proper application of the boundary conditions proposed in this section.

2.2.3 Displacement of a Ferromagnetic Ring

The boundary conditions can by specialized for the useful case of a
rigid ring of high permeability surrounded by free space. Both surface and edge currents may be present, and the magnetic field \( \mathbf{H} \) is presumed to be negligible within the ferromagnetic material. The equilibrium magnetic field surrounding the ring is assumed known except perhaps for the small region near the sharp edges. A steady rotation of the ring about its polar axis is allowed, given that the equilibrium fields used correspond to this situation. The only effect such a rotation has on the perturbation solution is in determining the explicit form for the perturbation currents. Since in this section these currents remain unspecified, the results which follow are unchanged by the presence of rotation.

Any rigid translation of the ring can be broken into a uniform axial displacement \( \xi_z \) and a displacement in the \( r-\phi \) plane (radial displacement) having a magnitude of \( \xi_o \) in the direction \( \phi \), as shown in Figure 2.6.

![Figure 2.6 Displacement of a Rigid Ring.](image-url)
Each point on the surface therefore undergoes a displacement
\[ \tilde{\xi}(r) = \xi_r(\theta) a_r + \xi_\theta(\theta) a_\theta + \xi_z a_z \] (2.50)
where
\[
\begin{aligned}
\xi_r &= \text{Re}\{\tilde{\xi} e^{-j\theta}\} \\
\xi_\theta &= \text{Re}\{-j\tilde{\xi} e^{-j\theta}\}
\end{aligned}
\]
(2.51)

Thus \( \tilde{\xi} \) becomes a single complex value which represents both the instantaneous magnitude and direction of the radial displacement.

The perturbation boundary conditions must be separately evaluated for each of the ring's four surfaces and four edges. The high-permeability condition allows the boundary condition based on Faraday's law to be written directly in terms of the external fields, without the "jump" notation. Thus the tangential \( \vec{h} \) on the exterior surfaces can be found explicitly using only this one condition. The condition derived from Gauss' law is not needed since the internal fields are not of interest.

Solution of the boundary condition (equation 2.45a) on the inner and outer surfaces gives, for rigid translations
\[
\begin{aligned}
\vec{h}_\theta(r) &= \vec{h}_z - \frac{1}{r} \frac{\partial \vec{H}}{\partial \theta} - \xi_r \frac{\partial H_\theta}{\partial r} - \xi_\theta \frac{1}{r} \frac{\partial H_\theta}{\partial \theta} - \xi_z \frac{1}{r} \frac{\partial H_z}{\partial \theta} \\
\vec{h}_z(r) &= \vec{h}_\theta - \xi_r \frac{\partial H_z}{\partial r} - \xi_\theta \frac{1}{r} \frac{\partial H_z}{\partial \theta} - \xi_z \frac{\partial H_z}{\partial z}
\end{aligned}
\] (2.52)

where the upper signs are for the outer surface. Using \( \nabla \times \vec{H} = 0 \) in the space surrounding the ring, the radial derivatives can be replaced by more convenient surface derivatives;
\[
\begin{align*}
    h_\theta(r) &= \frac{1}{r} \frac{\partial H_r}{\partial \theta} + \xi_r \frac{1}{r} \frac{\partial H_\theta}{\partial \theta} - \xi_\theta \frac{1}{r} \frac{\partial H_r}{\partial \theta} - \xi_z \frac{1}{r} \frac{\partial H_z}{\partial \theta} \\
    h_z(r) &= \frac{1}{r} \frac{\partial H_\theta}{\partial z} - \xi_r \frac{3 H_r}{3 z} - \xi_\theta \frac{3 H_\theta}{3 z} - \xi_z \frac{3 H_z}{3 z}
\end{align*}
\] (2.53)

In a similar fashion the perturbation fields on the top and bottom surfaces become

\[
\begin{align*}
    h_r(r) &= \frac{1}{r} \frac{\partial H_r}{\partial r} + \xi_r \frac{1}{r} \frac{\partial H_\theta}{\partial r} + \xi_\theta \frac{1}{r} \frac{\partial H_z}{\partial r} \\
    h_\theta(r) &= \frac{1}{r} \frac{\partial H_r}{\partial \theta} + \xi_r \frac{1}{r} \frac{\partial H_\theta}{\partial \theta} + \xi_\theta \frac{1}{r} \frac{\partial H_z}{\partial \theta} - \xi_z \frac{1}{r} \frac{\partial H_z}{\partial \theta}
\end{align*}
\] (2.54)

These expressions are consistent with the earlier statement that the direct effect of boundary displacement can be represented by an effective surface current. Perturbations to the real equilibrium surface current then add to these effective currents to generate the total perturbation fields. The currents which represent boundary displacement, in the absence of tangential equilibrium fields, are shown in Figure 2.7. The edge currents shown are derived from equation 2.47. Note that these currents serve as end-return paths for the surface currents, thus insuring the required continuity.

For the special case of rigid-body translation considered here, a further simplification is possible. Let the perturbation field due to boundary displacement be given as the gradient of a scalar potential;

\[
\bar{h} = -\nabla \psi_d
\] (2.55)
In order to satisfy the boundary conditions for the tangential fields, this potential must be given at the surface of the ring by

$$\psi_d(\vec{r}) = \vec{\xi}(\vec{r}) \cdot \vec{H}(\vec{r}) + \psi^0_d(\vec{r})$$  (2.56)

where $\psi^0_d(\vec{r})$ is an arbitrary constant which is uniform over each surface. Examination of equations 2.53 and 2.54 shows that this expression is valid on all four surfaces of the ring.

Since the potential $\psi^0_d$ is uniform across any one of the four surfaces, it can only produce magnetic fields which have no azimuthal variation and are everywhere normal to the surface. The normal component of the field at each surface is not determined by the boundary conditions, which only specify the tangential field values. Thus it is not surprising that $\psi^0_d$ appears as an undetermined constant in this expression. In order to evaluate this homopolar surface potential, it is necessary to consider the

---

**Figure 2.7** Equivalent-Current Representation of Displacement with No Tangential Equilibrium Fields.
entire magnetic environment of the ferromagnetic ring in conjunction with the tangential field conditions (equations 2.53 and 2.54). If there are no homopolar tangential fields or edge currents, then the homopolar surface potential must be zero. Figure 2.8 illustrates the particular case of no tangential equilibrium fields or homopolar perturbation currents. Comparison with Figure 2.7 shows the advantage of the scalar-potential description.

\[ \xi = \xi_0 \bar{a}_x + \xi_z \bar{a}_z \]

\[ \psi = H_z \xi_z \]

\[ \psi = \xi_0 \cos \theta \, H_r \]

**Figure 2.8** Surface-Potential Representation of Displacement with No Tangential Equilibrium Fields.

The equilibrium magnetic field at the surface is necessarily periodic in \( \theta \) and can thus be given as a Fourier series of the form

\[ H_i(r, \theta, z, t) = \sum_{m=0}^{\infty} \text{Re}\{ \hat{H}_{im} (r, z, t) e^{-jm\theta} \} ; i=r, \theta, z \quad (2.57) \]

It follows that the perturbation fields and currents will also assume this form. The perturbation potential corresponding to boundary displacement
becomes

\[ \psi_d(r) = \frac{1}{2} \text{Re}\left\{ \sum_{m=0}^{\infty} \tilde{\xi} \tilde{H}_{r,m} e^{-j[m+1]\theta} \right\} \]

\[ + \frac{1}{2} \text{Re}\left\{ \sum_{m=0}^{\infty} \tilde{\xi}^* \tilde{H}_{r,m} e^{-j[m-1]\theta} \right\} \]

\[ + \text{Re}\left\{ \sum_{m=0}^{\infty} \xi_z \tilde{H}_{z,m} e^{-jm\theta} \right\} + \psi^o_d(r) \] \hspace{1cm} (2.58)

Axial displacements thus produce fields matching the \( \theta \) dependence of the equilibrium field, while radial displacements split each equilibrium component into two parts: one with an extra wavelength per revolution, another with one less.

The two cases of particular interest in this study are radial displacements of the homopolar radial magnetic bearing element (Section 2.4) and of the radial-gap motor-generator (Section 2.5). For the homopolar radial bearing there are no equilibrium surface currents, so the equilibrium fields are normal to the surface with azimuthal dependence \( m=0 \). None of the tangential surface fields or edge currents are homopolar, so \( \psi^o_d=0 \). The resulting perturbation potential corresponding to boundary displacement is zero for all axially-facing surfaces and is given on the radial surfaces by

\[ \psi_d(r) = \text{Re}\{\tilde{\psi} e^{-j\theta}\} \quad ; \quad \tilde{\psi} = \tilde{\xi} \tilde{H}_r(r) \] \hspace{1cm} (2.59)

For the motor-generator an axially-directed surface current is assumed which has end-return paths along the edges and a single dominant space-harmonic \( m=p \). The perturbation potential corresponding to boundary displacement is again zero on the axially-facing surfaces, and on the radial faces is given by
\[ \psi_d(r) = \text{Re}\{\tilde{\psi}_d^+ e^{-j[p+1]\theta} + \tilde{\psi}_d^- e^{-j[p-1]\theta}\} \]  

where

\[ \tilde{\psi}_d^+ = \frac{1}{2} \tilde{\xi}[\tilde{H}_r - j\tilde{H}_\theta] \]

\[ \tilde{\psi}_d^- = \begin{cases} 
  \frac{1}{2} \tilde{\xi}^* [\tilde{H}_r + j\tilde{H}_\theta] & ; p>1 \\
  0 & ; p=1 
\end{cases} \]  

In writing these expressions, the homopolar perturbation field has been taken to be zero. There will, in fact, be some homopolar field when \( p=1 \). However, after crossing the air gap this field must return to the inner member by a path external to the motor-generator. The reluctance of this path depends on the magnetic circuit surrounding the machine. In a conventional, isolated, motor the only return path is through the air around the ends of the machine. The reluctance of this path is generally large, so the homopolar field is small. Even when the motor-generator is a part of a magnetic suspension system, it is desirable to have the motor section magnetically isolated from the magnetic bearings to avoid undesirable interactions. Based on these arguments, only the case where the reluctance of the return path is large will be considered, and the homopolar surface potential has been set to zero.

Using these boundary condition the component of the perturbation field associated with surface displacement can be determined using either analytical or numerical techniques.

Departures from steady rotation can also be evaluated using the techniques developed thus far. Although not a major issue in this study, the rotational stability of machines is an important topic in its own right. An angular deviation from steady rotation, \( \gamma \), results in an
azimuthal displacement of all ring surfaces given by

\[ \xi_\theta = r \gamma \]  \hspace{1cm} (2.62)

This creates a perturbation surface potential of

\[ \psi_d(\tau) = r \gamma H_\theta(\tau) \]  \hspace{1cm} (2.63)

This section has described the mechanism by which mechanical motion affects the magnetic field and therefore the force acting on the material. The surface potentials found, however, only represent the driving source of the perturbation magnetic field. The actual value of the field they produce depends on the specific mechanical topology and on the presence of any perturbation currents which might be induced.

2.3 BOUNDARY CONDITIONS FOR WINDINGS IN MOTION

Conductive material located either on the rotor or stator can be subjected to a time-varying flux linkage either due to motion or to time-varying excitation of the magnetic field. The induced voltage which results will cause a current to flow in the conductor if a closed path is provided. The only motion explicitly considered in this section is a steady rotation of the rotor. All other motions of a displacement character; be they radial, axial, or rotational; are replaced by the equivalent surface potentials given in section 2.2.

The net magnetic field in the air gap due to the equilibrium and perturbation fields will be composed of several space harmonics. In this linear analysis, each component can be considered separately. The harmonic under consideration is designated as \( p \), where \( p > 0 \). Static homopolar fields (\( p=0 \)) may be present but are not included since they do not result in any induced currents in the windings.

The interaction of the conductors and the air-gap field can be modeled
as a conservative electromechanical coupling, treating all losses and leakage inductances as lumped-circuit elements attached to external terminal pairs. If the conductors themselves provide closed current paths, then these are the only elements associated with the terminals. If the conductor is a discrete wire, however, the ends of the wire provide the opportunity to connect any external electrical circuit in series with the loss and leakage elements.

When current does flow in the conductors, it creates an azimuthal variation in the magnetic potential of the ferromagnetic surface in its vicinity. Thus the presence of windings introduces a magnetic boundary condition which relates the normal field component to the surface magnetic potential by way of imposed terminal relations. The exact nature of the boundary condition depends on the symmetry and distribution of the conductors, as well as any externally-imposed constraints.

Discrete windings offer the greatest flexibility in adjusting the magnetic boundary condition by providing a port for connection to an electrical control system. Because the possibilities for connection are so varied, the voltage induced in a discrete winding and the effect caused by current in the winding will be considered separately. Connection of an impedance to the winding terminals is a special case which will be evaluated in particular. The many interconnected paths present in the squirrel-cage type of winding do not lend themselves conveniently to the analysis used for discrete windings, so this case is considered independently.

2.3.1 Induced Voltage

Consider a winding of N total turns located on the rotor of an
axisymmetric radial-gap device. A single turn is defined as one complete round-trip traversal of the active axial length of the device. All displacements from axisymmetry have been replaced by equivalent surface-potential boundary conditions. A stator winding can be evaluated using the same derivation which follows by simply setting the rotational speed \( \Omega \) to zero.

Let each individual turn of the winding be specified by the angular position \( \theta \) about which it is centered at time \( t=0 \) \( (\alpha_i) \), and the angular span of the turn \( (\gamma_i) \), as illustrated below.

![Figure 2.9 Winding-Factor Angles.](image)

The end-turns of the winding are assumed to be located sufficiently far from the gap that each individual turn links all of the flux passing between the rotor and stator over the active axial length. This treatment allows the incorporation of axial teeth. If \( R \) is the radius of the rotor surface at the axial location of minimum rotor-stator air gap, then

\[
R \int_0^L B_r(R,\theta,z,t)dz = \text{Re}(\bar{\Phi}_p(t)e^{-jp\theta})
\]  

(2.64)

making \( \bar{\Phi} \) the phasor flux per radian which links the winding. The total flux linking any one turn is then given by
\[ \phi_i(t) = \int_{a_i - [\gamma_i/2] + \Omega t}^{a_i + [\gamma_i/2] + \Omega t} \text{Re}\{\overline{\phi}_p(t)e^{-j\rho \theta}\}d\theta \]  

which is, for \( p \neq 0 \),

\[ \phi_i(t) = \text{Re}\{\frac{2}{p} \pi \frac{pY_i}{2} e^{-j\rho a_i} e^{-j\rho \Omega t} \overline{\phi}_p(t)\} \]  

In connecting the \( N \) individual turns in series, a reference direction for current is established which may or may not correspond to the reference for positive flux-linkage implied by the above expressions. Let \( a_i \) be +1 if positive current in turn "i" creates flux in the positive radial direction at position \( a_i \), and -1 if it creates flux in the negative radial direction. The total flux linked by the \( N \)-turn winding is then

\[ \lambda(t) = \sum_{i=1}^{N} a_i \phi_i(t) \]  

Define a complex normalized winding factor which depends only on the geometrical winding distribution;

\[ k_{wp} = \frac{1}{N} \sum_{i=1}^{N} a_i e^{-j\rho a_i} \sin\frac{pY_i}{2} \]  

The flux linkage is then expressed conveniently as

\[ \lambda(t) = \text{Re}\{\frac{2N}{p} k_{wp} e^{-j\rho \Omega t} \overline{\phi}_p(t)\} \]  

The voltage induced in this winding is the time derivative of the flux;

\[ V(t) = \text{Re}\{\frac{2N}{p} k_{wp} e^{-j\rho \Omega t} \frac{\partial}{\partial t} a_j \overline{\phi}_p(t)\} \]  

This voltage can be written to explicitly show the effect of rotation by
putting it in modulated time-phasor form as

$$V(t) = \text{Re}\{\hat{V}(t)e^{-jp\Omega t}\}$$

(2.71)

where

$$\hat{V}(t) = \frac{2N}{p} kwp \frac{3}{\partial t} - jp\Omega \hat{\Phi}_p(t)$$

(2.72)

This expression remains valid if there are additional windings located on the rotor or stator. The flux $\Phi$, of course, is due in part to currents flowing in all of the various windings.

2.3.2 **Current-Generated Surface Potential Variation**

Let the current in the $N$-turn winding be $I(t)$. As a function of $\theta$, the distribution of axially-directed current becomes

$$I_z(\theta, t) = \sum_{i=1}^{N} a_i I(t)[u_o(\theta - \Omega t - \alpha_i - \frac{Y_i}{2}) - u_o(\theta - \Omega t - \alpha_i + \frac{Y_i}{2})]$$

(2.73)

where the impulse $u_0$ is here taken to have units of rad$^{-1}$. The presence of this current will create an azimuthal variation in the surface magnetic potential given by

$$\frac{\partial \psi}{\partial \theta} = \hat{I_z}(\theta, t)$$

(2.74)

where the upper sign is for windings on the inner surface of the air gap.

The surface potential is periodic in $\theta$ and can be expressed as a Fourier series;

$$\psi(\theta, t) = \sum_{m=1}^{\infty} \text{Re}\{\tilde{\psi}_m(t)e^{-jm\theta}\}$$

(2.75)

Solving for the coefficients yields

$$\tilde{\psi}_m(t) = \frac{2}{\pi m} I(t) e^{im\Omega t} \sum_{i=1}^{N} a_i e^{jm\alpha_i} \sin\frac{mY_i}{2}$$

(2.76)

The summation in this equation is identified as the complex conjugate of
the complex winding factor introduced in equation 2.68, with \( p \) replaced by \( m \). Therefore,

\[
\tilde{\psi}_m(t) = \frac{2N}{\pi m} k^* \cdot I(t) \cdot e^{jm\omega t}
\]  

(2.77)

If several windings are present then their contributions to the surface potential are additive.

2.3.3 Laplace Transform Analysis of Modulated-Phasor Quantities

The voltage and current in a winding will generally be related by a linear differential equation having constant coefficients. This is certainly the case when the winding terminals are connected to linear lumped circuit elements. Such a relationship is most conveniently described in terms of the Laplace-transform variable \( s \). Throughout the following analysis, the initial conditions for all perturbation variables and their time derivatives are assumed to be zero. All equilibrium values are presumed to be either constants or in the sinusoidal steady state. These conditions are acceptable for use in a linear stability evaluation. Extension to other initial conditions is a straightforward application of the Laplace transform definition.\(^{10}\)

If voltage and current are to be described in terms of Laplace transforms, then the relationship between the voltage and the flux, and between the current and the surface potential, must be reevaluated in terms of the new variable \( s \). This determination requires the proper evaluation of the Laplace transform of modulated-phasor quantities which take the form

\[
A(t) = \text{Re}\{\tilde{A}(t)e^{j\omega t}\}
\]  

(2.78)

The complicating issue in the transformation of this equation is the presence of the real-part operator. Fortunately, the Laplace
transformation of complex values is well defined so that the expression can be expanded to

\[ A(s) = \frac{1}{2} \int \tilde{\mathcal{A}}(t)e^{j\omega t} \, dt + \frac{1}{2} \int \tilde{\mathcal{A}}^*(t)e^{-j\omega t} \, dt \]  \hspace{1cm} (2.79)

Written in terms of the transform definition, this becomes

\[ A(s) = \frac{1}{2} \int_{0}^{\infty} \tilde{\mathcal{A}}(t)e^{j\omega t} e^{-st} \, dt + \frac{1}{2} \int_{0}^{\infty} \tilde{\mathcal{A}}^*(t)e^{-j\omega t} e^{-st} \, dt \]  \hspace{1cm} (2.80)

or

\[ A(s) = \frac{1}{2} \int_{0}^{\infty} \tilde{\mathcal{A}}(t)e^{j\omega t} e^{-st} \, dt + \frac{1}{2} \int_{0}^{\infty} \tilde{\mathcal{A}}^*(t)e^{j\omega t} e^{-s^*t} \, dt \]  \hspace{1cm} (2.81)

The two terms now appear as complex conjugates except for the variable s. In the second term, s appears as \( s^* \) upon completion of the integral, then gets changed back to s when the entire expression is conjugated. Since neither conjugation affects the form of the expression in any way, this double reversal can be avoided by temporarily treating s as a real value while performing the transformation and subsequent conjugation. This treatment makes the two terms above perfect conjugates, so that

\[ A(s) = \text{Re}\left\{ \mathcal{F}[\tilde{\mathcal{A}}(t)e^{j\omega t}] \right\} \bigg|_{s \text{ real}} \]  \hspace{1cm} (2.82)

This expression is interpreted to mean that when taking the real part, s should be treated as if it was a real value. Once this operation is completed, s may again assume complex values. Though this method is somewhat artificial, it does yield the correct result and is a major simplification of both notation and effort.

In retrospect, the time function used in the derivation was arbitrary. It is possible to write a general statement valid for all complex time
functions, including differential operators;

\[ \mathcal{L}\{\text{Re}[f(t, \frac{\partial}{\partial t}, \ldots)]\} = \text{Re}[\mathcal{L}\{f(t, \frac{\partial}{\partial t}, \ldots)\}] \bigg|_{\text{s real}} \tag{2.83} \]

\[ \mathcal{L}\{\text{Im}[f(t, \frac{\partial}{\partial t}, \ldots)]\} = \text{Im}[\mathcal{L}\{f(t, \frac{\partial}{\partial t}, \ldots)\}] \bigg|_{\text{s real}} \]

Another result from Laplace transform theory is necessary to complete the analysis of modulated phasors. A well-known theorem involves the transform of the product of a time function and an exponential term:

\[ \mathcal{L}\{e^{-\beta t}f(t)\} = F(s+\beta) \tag{2.84} \]

This expression should be interpreted as stating that the transform of the product is the transform of \( f(t) \) with \( s \) replaced by \( s+\beta \). This theorem applies to complex as well as real values of \( \beta \), so that the Laplace transform of a modulated phasor becomes

\[ \mathcal{L}\{\text{Re}[\tilde{A}(t)e^{j\omega t}]\} = \text{Re}[\tilde{A}(s-j\omega)] \bigg|_{\text{s real}} \tag{2.85} \]

Using this notation, the voltage induced in a winding can be expressed in the \( s \)-domain as

\[ V(s) = \text{Re}\left[\frac{2N}{p} \frac{k}{\omega_p s} \tilde{\phi} \left( s+j\omega \right) \right] \bigg|_{\text{s real}} \tag{2.86} \]

and the surface potential due to current flow as

\[ \tilde{\psi}_m(s+jm\omega) = \mp \frac{2N}{\pi m} \frac{k^*}{\omega_m} I(s) \tag{2.87} \]

2.3.4 Symmetric Impedance Termination

The expressions derived in this section have so far considered only a single winding. There may, of course, be several windings present, all contributing to the flux and surface potential in the air gap. An important multiple-winding arrangement is the case of \( M \) identical \( N \)-turn windings evenly spaced by \( \pi/[pM] \) radians and each terminated in the same
impedance, \( Z(s) \). This model applies to the perturbation analysis of multiple-phase windings driven by independent current or voltage sources, as well as the analysis of windings whose current is entirely induced.

One winding can be arbitrarily selected as the reference winding and assigned the index of zero. The other \( M-1 \) windings are identical except for a spacial phase shift between each one of \( \pi/[pM] \) radians. Thus the complex winding factor (equation 2.68) for phase \( \ell \) at space harmonic \( m \) is

\[
k_{wml} = \frac{1}{N} \sum_{i=1}^{N} a_i e^{-jm[\alpha_i + \pi \ell/pM]} \sin \frac{mY_i}{2}
\]

or

\[
k_{wml} = e^{-j\pi m\ell/pM} k_{wmo}
\]

The net surface potential due to all \( M \) phases is the sum of all contributions;

\[
\tilde{\psi}_m(s+j\omega) = \mp \frac{2N}{p} k^*_{wmo} \sum_{\ell=0}^{M-1} e^{j\pi m\ell/pM} I_\ell(s)
\]

Similarly, the induced voltage in phase \( \ell \) is

\[
V_\ell(s) = \text{Re}\{\frac{2N}{p} k_{wpo} e^{-j\pi \ell/M} s\tilde{\phi}_p(s+jp\omega)\} \bigg|_{s \text{ real}}
\]

The current and the induced voltage are related by the termination impedance, \( Z(s) \). Since the current has been defined referenced into the winding, the relationship is

\[
V_\ell(s) = -I_\ell(s)Z(s)
\]

Combining these last three equations yields

\[
\tilde{\psi}_m(s+j\omega) = \frac{-2N}{\pi m} k^*_{wmo} \sum_{\ell=0}^{M-1} e^{j\pi m\ell/pM} \text{Re}\{\frac{2N}{p} k_{wpo} e^{-j\pi \ell/M} s\tilde{\phi}_p(s+jp\omega)\} \bigg|_{s \text{ real}}
\]

In many situations it is the space-harmonic \( p \) of the surface potential
which is primarily of interest, this being the direct mode of interaction
between the flux and the winding. The other values of \( m \) can then be
modeled as leakage inductance for the windings. Under these conditions,
the real-part expression can be simplified, leaving

\[
\tilde{\psi}_p(s+jp\Omega) = \frac{2MN^2}{\pi p^2} \left| k_{wp} \right|^2 \frac{Z(s)}{Z(s-jp\Omega)} \tilde{\phi}_p(s+jp\Omega) ; M>1
\]  

(2.94)

A simple substitution of variables allows the flux and potential to be
described directly in terms of their time dependence as viewed from a
stationary frame of reference;

\[ s + sjp\Omega \]  

(2.95)

This allows a simple boundary condition to be written which relates the
phasor potential to the phasor flux, thereby avoiding any concern with the
intricate details of the windings in subsequent calculations;

\[
\tilde{\psi}_p(s) = \frac{2MN^2}{\pi p^2} \left| k_{wp} \right|^2 [s-jp\Omega] \frac{Z(s-jp\Omega)}{Z(s-jp\Omega)} \tilde{\phi}_p(s) ; M>1
\]  

(2.96)

where, to summarize,

M is the number of equivalent phases.

p is the number of pole-pairs in the flux field.

N is the number of turns for each phase.

\( k_{wp} \) is the complex winding factor (equation 2.68).

\( \tilde{\phi}_p \) is the space-phasor flux per radian (equation 2.64).

\( \tilde{\psi}_p \) is the space-phasor surface potential.

\( Z(s) \) is the winding termination impedance.

\( \Omega \) is the rotational speed (rad/sec).

2.3.5 Squirrel-Cage Winding

The squirrel-cage rotor is an example of a winding for which the
separation between the lossless coupling mechanism and the lumped-impedance termination does not fit the discrete-winding model developed thus far. This type of winding is illustrated below.

![Diagram of Squirrel-Cage Rotor](image)

**Figure 2.10** Squirrel-Cage Rotor

The model used to evaluate the boundary condition imposed by a
squirrel cage is shown in Figure 2.11. Each of the N rods is represented by a lumped resistance and slot-leakage inductance in series. The rods are connected together at each end by end-rings which have a given resistance and leakage inductance between each rod. As before, the air-gap flux component considered is the space harmonic "p".

\[
\begin{align*}
\text{Rod } & \ell-1 & \text{Rod } \ell & \text{Rod } \ell+1 \\
Z_e(s) & \downarrow & Z_e(s) & \downarrow \\
Z_s(s) & \circlearrowleft_{\ell-1} & Z_s(s) & \circlearrowleft_{\ell} \\
Z_e(s) & \downarrow & Z_e(s) & \\
\bigg. & \bigg. & \bigg. & \bigg. \\
Z_s(s) = R_s + sL_s & Z_e(s) = R_e + sL_e \\
\end{align*}
\]

Figure 2.11 Squirrel-Cage Circuit Model.

The flux per radian which links the winding can be defined as in equation 2.64. One rod will become the reference point, and its position at t=0 denoted by \( \alpha_0 \). The other N-1 rods then have angular positions given by

\[
\alpha_{\ell} = \alpha_0 + \frac{2\pi \ell}{N} \quad ; \quad \ell = 0, 1, \ldots, N-1
\]  

(2.97)

A "window" is formed between rod \( \ell \) and rod \( \ell+1 \). The flux through this window is obtained by integrating the flux density between the rod locations. The Laplace transform of the result of this integration can be written as

\[
\lambda_\ell(s) = \text{Re}(\tilde{\lambda}_0(s)e^{-jp2\pi\ell/N}) \bigg|_{s \text{ real}}
\]

(2.98)
where
\[ \tilde{\chi}_o(s) = \frac{2}{p} \sin \frac{\pi p}{N} e^{-jp[\alpha_o + \pi/N]} \tilde{\Phi}_p(s+jp\alpha) \] (2.99)

Faraday's law for the circuit model shown in Figure 2.11 is
\[ i_{\chi}(s)2[Z_s(s)+Z_e(s)] - [i_{\chi+1}(s)-i_{\chi-1}(s)]Z_s(s) = s \lambda(s) \] (2.100)

The last two equations suggest that the currents will have solutions of the form
\[ i_{\chi}(s) = \text{Re}\{\tilde{i}_o(s)e^{-j\pi 2pN/N}\} \text{ s real} \] (2.101)

Substitution of this form into equation 2.100 shows that the proper result is
\[ \tilde{i}_o(s) = \frac{v_2}{s} \frac{\tilde{\chi}_o(s)}{Z_e(s)+Z_s(s)[1-\cos\frac{2\pi p}{N}]} \] (2.102)

These loop currents produce rod currents given by
\[ I_{\chi}(s) = i_{\chi}(s) - i_{\chi-1}(s) \] (2.103)

which in turn generate a variation in the surface magnetic potential. If the surface potential is described as a Fourier series in the angle \( \theta \) (equation 2.75) then the currents will create fields having space harmonics corresponding to values of \( m \) given by \( N\ell p \), where \( \ell \) is any integer which yields a positive result for \( m \). Assuming that \( N>p \), then the fields corresponding to \( \ell\neq 0 \) will be small compared to those corresponding to \( m=p \) and can be modeled as an additional leakage inductance, often referred to as the "zig-zag" leakage.\(^1\) The surface magnetic potential for the space harmonic \( p \) is then given by
\[ \tilde{\psi}_p(s) = \frac{\mp \frac{N}{p^2} [\sin \pi \frac{p}{N}]^2 [s - j\rho \Omega]}{Z_e(s - j\rho \Omega) + [1 - \cos \frac{2\pi p}{N}]Z_s(s - j\rho \Omega)} \]  

(2.104)

This expression provides the boundary condition between the phasor potential and the phasor flux which can be used to account for the effect of the cage in calculations of the magnetic field in the air gap. The continuum limit of the squirrel-cage winding would be a continuous sheet of conductor facing the air gap, connected to end-rings at both ends. These rings might be just an axial extension of the same conductor. Letting the ratio \( N/p \) become large in the expression above,

\[ \tilde{\psi}_p(s) = \frac{\mp \pi [s - j\rho \Omega]}{NZ_e(s - j\rho \Omega) + 2\pi^2 p^2 Z_s(s - j\rho \Omega)/N} \]  

(2.105)

But \( Z_E = N Z_e \) is the impedance of one complete trip around an end ring. \( Z_0 = Z_s/N \) is the impedance connecting the two end rings together. These two definitions are correct without regard to the specific value of \( N \) and are well-defined even in the continuum limit. Thus the boundary condition for this case is usefully expressed as

\[ \tilde{\psi}_p(s) = \frac{\mp \pi [s - j\rho \Omega]}{Z_E(s - j\rho \Omega) + 2\pi^2 p^2 Z_0(s - j\rho \Omega)} \]  

(2.106)

This section has provided expressions relating the magnetic field quantities of flux and surface potential to the circuit quantities of voltage and current for discrete windings rotating at a constant speed \( \Omega \). Two cases were investigated where "external" elements provided a relationship between the current and voltage in specific winding
configurations, thus establishing an effective boundary condition relating the surface potential to the flux. The expressions derived are valid for perturbation fields as well as the equilibrium field, given that the correct azimuthal dependence of the field under consideration (p) is inserted into the equations.

2.4 HOMOPOLAR RADIAL MAGNETIC BEARING

Radial-gap magnetic bearings must have axial teeth as described in Chapter 1 if they are to effectively provide a lift force to offset gravity. Consequently, it has been necessary to carefully include the possibility of axial dependence in all of the field analysis performed in this chapter. This section considers only homopolar equilibrium fields (p=0) and the dipole perturbation fields (p=1) which result from displacements in the radial plane.

Sections 2.2 and 2.3 provided the necessary boundary conditions for use in determining the perturbation magnetic fields in the air gap. In this section these conditions are used to estimate the fields present in a narrow air gap which has teeth which may be axially offset. In performing this analysis, the azimuthal fields are assumed to be much weaker than the radial and axial fields surrounding the teeth. The actual magnitude of the azimuthal field is less than the radial field by a factor which is on the order of the ratio of the gap width to the radius of the gap. This simplification permits a two-dimensional "arcs-and-lines" approximation of the field in the r-z plane. This solution can be combined with the force expressions from section 2.1 to relate the radial force of magnetic origin to the radial displacements.
2.4.1 Equilibrium Homopolar Field

Figure 2.12 shows the general form of the equilibrium magnetic field between a pair of axially offset teeth. Since this field is homopolar, this view is valid for all values of \( \theta \). The gap is assumed to be narrow enough that the radius \( r \) is essentially constant through the air-gap region. This allows the field mapping to be performed as if the gap was in a two-dimensional Cartesian plane. The resulting field is skew-symmetric in that the field is unchanged by a 180 degree rotation of the plane.

![Figure 2.12 Equilibrium Field Plot.](image)

A useful approximation of the field solution is shown in Figure 2.13. In this rendition, all field lines are drawn as a combination of straight lines and arcs of circles. As a set, these lines define a collection of flux tubes connecting the rotor to the stator. Within each tube the flux is conserved, and since the tubes are specifically constructed to have a
constant width, the flux density is also constant within each tube. Therefore the magnetic field along a field line is constant and is equal to the potential difference between the rotor and stator, divided by the total length of the field line. This flux-tube technique for field mapping is commonly used since it produces acceptable accuracy for many purposes with a minimum of effort.\textsuperscript{12} Most importantly, it yields analytical results which allow for variations in geometry.

![Diagram of approximate equilibrium field](image)

**Figure 2.13 Approximate Equilibrium Field.**

The analytical description of the radial magnetic field at mid-gap as a function of axial position is given below for the single-tooth case. This can be extended to the case of multiple noninteracting teeth as well. The results given for each range of \( z \) correspond to dividing the potential difference \( \Psi_o \) by the length of the field line which crosses the gap at axial location \( z \). The dependence shown is valid for all values of tooth offset so long as the offset, \( \delta \), is less than the tooth width, \( t \). Though the field extends to infinity in all directions, the validity of the
expressions breaks down for axial locations which are removed from the gap by a distance which is a significant fraction of the radius, since the Cartesian approximation is no longer appropriate. Fortunately, though this far-fringing field may be a significant factor in the total flux passing between rotor and stator, it has little effect on the force, which depends on the square of the field magnitude.

\[
H_r(z) \approx \begin{cases} 
\frac{H_0}{1 - \frac{\pi}{2} \frac{\delta}{g} - \frac{\pi}{g} \frac{z}{g}} ; & z < -\delta \\
\frac{H_0}{1 - \frac{\pi}{2} \frac{z}{g}} ; & -\delta < z < 0 \\
H_0 ; & 0 < z < t - \delta \\
\frac{H_0}{1 + \frac{\pi}{2} \frac{\delta}{g} + \frac{\pi}{2} \frac{z-t}{g}} ; & t - \delta < z < t \\
\frac{H_0}{1 + \frac{\pi}{2} \frac{\delta}{g} + \frac{\pi}{g} \frac{z-t}{g}} ; & z > t 
\end{cases}
\]  

(2.107)

where \(H_0 = \frac{\psi_0}{g}\) is the peak equilibrium magnetic field.

2.4.2 Displacement Field

Either the rotor or the stator may be displaced from their equilibrium position in the radial plane. For the purpose of analyzing the perturbation forces which result, it is only the relative position of the outer member with respect to the inner member which matters. In the
following development the outer member is held fixed and the relative displacement is assigned to the inner member.

The surface-potential boundary condition for a radially-displaced ferromagnetic ring can be used to determine the magnetic potentials given in Figure 2.14. The air-gap field shown is for the r-z plane corresponding to $\theta = \phi$, where $\phi$ is the azimuthal location where maximum gap narrowing occurs (Figure 2.6). The azimuthal field is assumed to be negligible in this narrow-gap situation. The resulting field lines are noticeably different from the equilibrium field distribution. Because the surface potential on the radial face of the inner member is a function of $z$, the field lines are not always normal to the surface. Furthermore, a distinct axial asymmetry exists due to the fact that the entire displacement potential has been assigned to the inner member.

Figure 2.14 Perturbation Displacement Field.

Once again an arcs-and-lines approximation to the field can be made, as illustrated in Figure 2.15. The lines shown are similar to the equilibrium field but only those field lines which originate on the radial
face of the inner member are significant. The perturbation field is assumed to be zero outside of this range.

\[ \rightarrow \mathbf{g} \rightarrow \kappa \]

Figure 2.15 Approximate Displacement Field.

The values of \( H_r \) on the inner radial face, which determine the potential difference across the field lines, can be taken from equation 2.107. Thus, an analytical description of the perturbation displacement field at mid-gap is given in terms of the complex radial displacement by

\[
\tilde{h}_r(z) \approx \begin{cases} 
0 & ; z \leq 0 \\
\frac{H_o \tilde{\xi}}{g} & ; 0 < z < t - \delta \\
\frac{H_o \tilde{\xi}}{g} \left[ \frac{1 + \frac{\pi \delta}{g} + \frac{\pi (z - t)}{2}}{\left( 1 + \frac{\pi \delta}{2g} + \frac{\pi (z - t)}{2g} \right)^2} \right] & ; t - \delta < z < t \\
0 & ; z \geq t
\end{cases}
\]  

(2.108)
where \( \tilde{\xi} = \xi_x + j\xi_y \) is the relative displacement of the inner member with respect to the outer member.

Integrating equation 2.108 over \( z \) provides a result for the phasor flux per radian due to a radial displacement;

\[
\tilde{\Phi}_d = \mu_0 R H_o \left[ \frac{t}{g} \right] \tilde{\xi} \tag{2.109}
\]

where

\[
\left[ \frac{t}{g} \right] \tilde{\xi} \approx \frac{t}{g} \tilde{\xi} \left[ \frac{\pi}{2} \frac{\delta}{g} \right] \frac{\delta}{g} \left[ 1 + \frac{\pi}{2} \frac{\delta}{g} \right] \tag{2.110}
\]

\( R \) is the radius of the gap.

\( d \) denotes the displacement component of the flux.

2.4.3 Perturbation–Current Field

In the homopolar radial magnetic bearing, radial displacements cause dipole perturbations in the magnetic field which can link to windings and result in currents which are linearly related to the displacement. These currents cause a surface-potential variation on the member to which the winding is attached. The dipole (m=1) contribution to the \( \Theta \)-dependence of this variation is the only portion which interacts with the homopolar equilibrium field to produce a net radial force (equation 2.15 for \( p=0 \)). Consequently, only this component is considered further.

The surface potential is uniform over the axial extent of the winding, and therefore applies to both the radial and axially-facing surfaces. The resulting magnetic field thus assumes the same shape in the \( r-z \) plane as the equilibrium field. If windings are located on both members, then their fields are additive. The net result still has the same \( r-z \) dependence, though the two fields may peak at different values of \( \Theta \) and thus must be
added as phasors. If is the net phasor potential difference of the inner member with respect to the outer member, then the approximate description for the perturbation magnetic field at mid-gap due to winding currents becomes

\[
\begin{align*}
\tilde{h}_p(z) \approx \begin{cases} 
\frac{\tilde{\psi}_o/g}{1 - \pi \frac{\delta}{2g} - \pi \frac{z}{g}} ; & z<\delta \\
\frac{\tilde{\psi}_o/g}{1 + \frac{\pi \delta}{2g} + \frac{\pi z-t}{g}} ; & t-\delta<z<t \\
\frac{\tilde{\psi}_o/g}{1 + \frac{\pi \delta}{2g} + \frac{\pi z-t}{g}} ; & z>t \\
\end{cases}
\end{align*}
\]

(2.111)

Integrating these expressions permits the permeance relation between the phasor potential and the phasor flux to be estimated.

\[
\tilde{\phi}_c = \mu_0 R_c \tilde{\psi}_o 
\]

(2.112)
where

\[
\left[ \frac{t}{g} \right]_c \approx \frac{t}{g} - \delta \frac{R}{g} + \frac{\pi r_{\text{max}}}{g} \left[ \pi \frac{g}{R} + \frac{\pi \delta}{2g} \right]
\]  \hspace{1cm} (2.113)

\( r_{\text{max}} \) is the radius of the farthest fringing-field line which links the winding.

\( R \) is the radius of the gap.

\( c \) denotes the current-produced contribution to the flux.

2.4.4 Magnetic Stiffness

The total force acting on the inner member can be found using the techniques of section 2.1. A cylindrical surface can be located at the mid-gap radius \( R \) and the Maxwell stress tensor can be integrated over its area. For the displaced homopolar radial bearing the equilibrium field has \( m=0 \), and the perturbation field of interest has an \( m=1 \) dipole dependence.

The appropriate result from section 2.1 is therefore

\[
\tilde{T} = \mu_0 \pi R \bar{Z} \left[ (H_{ro} + jH_{\theta_0})(\tilde{h}_{r_1} + j\tilde{h}_{\theta_1}) - H_{zo} \tilde{h}_{z_1} \right]
\]  \hspace{1cm} (2.114)

In the flux-tube approach used here, the radial field component at mid-gap dominates over both the axial and azimuthal parts. Retaining only the radial field terms, the net radial force is simplified to

\[
f \approx \mu_0 \pi R \bar{Z} \left[ H_{ro} \tilde{h}_{r_1} \right]
\]  \hspace{1cm} (2.115)

It is useful to consider the contributions to the perturbation field from displacements and from winding currents separately, since they have been shown to have different axial dependence. Each of these perturbation fields can be multiplied point-by-point with the equilibrium field to form two contributions to the integral over \( z \). The sum of both then represents the total force of magnetic origin. The displacement component is

\[
\tilde{T}_d = \mu_0 \pi R H_o^2 \left[ \frac{t}{g} \right]_d \tilde{z}
\]  \hspace{1cm} (2.116)
where the effective aspect ratio for displacement forces is given for a single tooth by

\[
[t/g]_d \approx \frac{t}{g} - \frac{\delta}{g} + \frac{\delta}{g} \frac{[1 + \frac{\pi}{4} \frac{\delta}{g}]}{[1 + \frac{\pi}{2} \frac{\delta}{g}]^2}
\]  
(2.117)

The winding-current component for the same situation is

\[
\tilde{\xi} = \mu_0 \pi \frac{d}{g} [t/g]_c \tilde{\psi}_o
\]  
(2.118)

where the effective aspect ratio for current-produced forces is given, again for the single-tooth case, by

\[
[t/g]_c \approx \frac{t}{g} + \frac{2}{\pi} - \frac{\pi}{2} \frac{[\delta]^2}{g}
\]  
(2.119)

The particular perturbation winding currents which will be present depends on how the windings are terminated or driven. Generally, the currents will be related to the displacement by a linear differential equation having constant complex coefficients so that in the Laplace s-domain,

\[
\tilde{\psi}_o(s) = -a(s) \tilde{\xi}(s)
\]  
(2.120)

The negative sign is included in equation 2.120 for future convenience. Making this substitution in the force expression (equation 2.118) and adding the displacement force (equation 2.116); the total force of magnetic origin on the inner member is given in the s-domain by

\[
\tilde{f}(s) = \mu_0 \pi \frac{d}{g} [t/g]_d \tilde{c}(s) - \mu_0 \pi \frac{d}{g} a(s) [t/g]_c \tilde{c}(s)
\]  
(2.121)

The constant relating the magnetic potential to the displacement has the awkward units of Amps/meter. It is helpful to redefine this constant as the product of the peak equilibrium field strength, \( H_0 \), and a dimensionless factor, \( \Gamma(s) \). Also, let \( \beta \) represent the relative size of the
aspect ratios for current compared to displacement. The total force then assumes the form

\[
\tilde{f}(s) = \mu_0 \pi R^2 \frac{t}{g} [1 - \beta \Gamma(s)] \tilde{\xi}(s)
\]  

(2.122)

where

\[
\Gamma(s) = \frac{-\psi_o(s)}{[H_o \tilde{\xi}(s)]}
\]

(2.123)

\[
\beta = \frac{[t/g]_c}{[t/g]_d}
\]

The force on the outer member due to the air-gap fields is just the negative of this same result. Given the transfer function \(\Gamma(s)\) between the currents and the displacement, this simple expression completely accounts for the effect of magnetic fields on the radial dynamics of the system.

The form of this force expression is similar to the mechanical equation for a spring, in that the force is linearly related to the displacement. It differs in two important ways: The force depends on time rates of change (\(s\)), and the complex-valued relationship actually describes a two-dimensional interaction. Nevertheless, it is convenient to describe the magnetic interaction as a complex, dynamic stiffness;

\[
\tilde{f}(s) = -K(s) \tilde{\xi}(s)
\]  

(2.124)

where

\[
K(s) = -\mu_0 \pi R^2 \frac{t}{g} [1 - \beta \Gamma(s)]
\]  

(2.125)

The negative sign is introduced so that a positive real spring constant corresponds to a force which tends to restore equilibrium.

The complex spring constant can be converted to relate the real variables it represents by expanding the force and displacement into their x and y components, then separately examining the real and imaginary parts of the expression. In taking the real and imaginary parts, \(s\) should be temporarily treated as a real value as explained in section 2.3.3. Thus the x and y forces can be given in matrix form as

80
If the relationship between winding currents and the displacement is known, then the effect of the magnetic fields on the radial motion of a homopolar radial magnetic bearing can be determined using the results of this section. In Chapter 3 the relationship used is the symmetric-impedance termination presented in section 2.3, while Chapter 4 investigates active-feedback-control possibilities.

2.5 RADIAL-GAP MOTOR-GENERATOR

When a radial displacement occurs in a smooth-air-gap machine having a highly-permeable rotor and stator, the perturbations to the equilibrium magnetic field have azimuthal dependences having one more and one less wavelength per revolution than the equilibrium field. These fields interact with the equilibrium field to generate a net radial force on the two members, as described in section 2.1.

This section considers the forces present when the equilibrium field is dominated by a single travelling wave with space harmonic \( p \) and frequency \( \omega \). The axial variation in the field which played such an important role in the analysis of section 2.4 is not included for the motor-generator, which is assumed to have an active axial length much longer than the gap width. As before, the gap width is assumed to be much less than the gap radius so that the radial magnetic field is uniform in the \( r-z \) plane between the rotor and stator.

Windings may be present on both the rotor and the stator and can either be driven externally or terminated to permit induced currents. With
the rotor and stator aligned concentrically, the driving currents produce the equilibrium magnetic fields. In general there will be an equilibrium torque produced and there may even be an equilibrium radial force. These forces can be obtained from the equilibrium fields using equation 2.13 of section 2.1. The analysis to follow in this section considers variations in the radial force due to small radial displacements for several winding configurations of interest.

2.5.1 Magnetic Stiffness

The perturbation magnetic field in a motor-generator is driven by surface potentials arising from boundary displacements. These potentials have azimuthal dependence given by

$$\psi(\theta) = \text{Re}\{\tilde{\psi}^+ e^{-j[p+1]\theta} + \text{Re}\{\tilde{\psi}^- e^{-j[p-1]\theta}\}\} \quad (2.127)$$

In the narrow-gap approximation used here only the relative displacement of the inner surface compared to the outer surface is important. If \(\psi_d\) is the potential due to displacement of the inner surface with respect to the outer surface, then from equation 2.61,

$$\tilde{\psi}_d^+ = 1/2 \left[ \hat{H}_r - j\hat{H}_0 \right] \tilde{\xi}$$

$$\tilde{\psi}_d^- = 1/2 \left[ \hat{H}_r + j\hat{H}_0 \right] \tilde{\xi}^*$$

(2.128)

where the phasor form of the equilibrium field corresponds to

$$\hat{H}_r(\theta,t) = \text{Re}\{\hat{H}_r(t) e^{-j\theta}\} \quad (2.129)$$

with $$\hat{H}_r(t) = \hat{H}_r e^{j\omega t}$$

The narrow-gap condition requires that the variation of the azimuthal field over the width of the gap be small compared to the magnitude of the
radial field. For a given difference in \( \tilde{H}_\theta \) between the rotor and the stator, this does not preclude the possibility of a large component common to both surfaces. Such a component is not, however, consistent with reality for the vast majority of applications. To maintain large azimuthal fields at both surfaces would require the currents on the rotor and stator to be almost perfectly in phase. There is certain to be a phase shift due to losses and leakage inductance which precludes this possibility. Thus the magnitude of \( \tilde{H}_\theta \) on both surfaces must be small compared to \( \tilde{H}_r \). This same argument applies to the perturbation field, so the azimuthal field will be neglected entirely.

In addition to the surface potential due to displacement, there will also be potentials resulting from perturbation currents flowing in windings located on the rotor and stator. Again utilizing the narrow-gap condition, the perturbation magnetic field is given by

\[
\tilde{h}_r^\pm = [\tilde{\psi}_d^\pm + \tilde{\psi}_c^\pm - \tilde{\psi}_o^\pm] / g
\]  

(2.130)

where \( \tilde{\psi}_c^\pm \) is the surface potential due to currents on the inner member, \( \tilde{\psi}_o^\pm \) is the surface potential due to currents on the outer member, and \( g \) is the gap width.

The displacement potential has been shown to be related directly to the displacement, but the perturbation winding currents result from voltages induced by the air-gap flux. In the Laplace-transform domain this relationship can be written as

\[
P_{ag} \tilde{\psi}_c^\pm(s) = -Q_i^\pm(s) \tilde{\phi}(s)
\]

(2.131)
where $Q(s)$ is the quality factor for the appropriate winding, here a complex function of $s$.

$P_{ag} = \mu_0 RL/g$ is the air-gap permeance where $L$ is the active axial length of the machine.

$\tilde{\phi}(s)$ is the air-gap flux per radian (equation 2.64).

By integrating equation 2.130 over $z$, the flux is seen to depend on all of the surface potentials,

$$\tilde{\phi}^\pm = [\tilde{\psi}_d^\pm + \tilde{\psi}_{ci}^\pm - \tilde{\psi}_{co}^\pm] P_{ag}$$

Substituting the assumed form for the perturbation current potentials (equation 2.131) and solving for the flux yields

$$\tilde{\phi}^\pm(s) = \frac{P_{ag}\tilde{\psi}_d^\pm(s)}{1 + Q_1^\pm(s) + Q_0^\pm(s)}$$

(2.133)

Relating the flux back to the radial magnetic field leaves

$$\tilde{h}^\pm_r(s) = \frac{\tilde{\psi}_d^\pm(s)/g}{1 + Q^\pm(s)}$$

(2.134)

The Laplace transform of the displacement surface potential, as required in the above expression, can be obtained from equations 2.128, 2.129, and the transform law for the product of a function and an exponential term;

$$\tilde{\psi}_d(s+j\omega) = 1/2 \hat{A}_r \tilde{\xi}(s)$$

(2.135)

$$\tilde{\psi}_d(s+j\omega) = 1/2 \hat{A}_r \tilde{\xi}^*(s)$$

The appropriate force expression is equation 2.15 from section 2.1 which for a travelling-wave field in a narrow air gap reduces to
\[ \tilde{\tilde{f}}(t) = \begin{cases} \frac{1}{2} \mu_0 \pi R L \hat{h}_r^* e^{-j\omega t} \tilde{h}_r^+(t) & ; p=1 \\ \frac{1}{2} \mu_0 \pi R L \hat{h}_r^* e^{j\omega t} [\tilde{h}_r^-(t)]^* \\ + \frac{1}{2} \mu_0 \pi R L \hat{h}_r^* e^{-j\omega t} \tilde{h}_r^+(t) ; p>1 \end{cases} \] (2.136)

Taking the Laplace transform while using the same law of products as above yields

\[ \tilde{\tilde{f}}(s) = \begin{cases} \frac{1}{2} \mu_0 \pi R L \hat{h}_r^* \tilde{h}_r^+(s+j\omega) & ; p=1 \\ \frac{1}{2} \mu_0 \pi R L \hat{h}_r^* [\tilde{h}_r^-(s+j\omega)]^* \\ s \text{ real} \\ + \frac{1}{2} \mu_0 \pi R L \hat{h}_r^* \tilde{h}_r^+(s+j\omega) ; p>1 \end{cases} \] (2.137)

where \( s \) must be held real for the complex conjugation, as explained in section 2.3.3. The appropriate perturbation fields follow from equations 2.134 and 2.135;

\[ \tilde{h}_r^+(s+j\omega) = \frac{\tilde{\psi}_d(s+j\omega)/g}{1 + Q^+(s+j\omega)} = \frac{1}{2} \hat{h}_r \tilde{\xi}(s)/g \]

\[ \tilde{h}_r^-(s+j\omega) = \frac{\tilde{\psi}_d(s+j\omega)/g}{1 + Q^-(s+j\omega)} = \frac{1}{2} \hat{h}_r \tilde{\xi}^*(s)/g \] (2.138)

except \( \tilde{h}_r^-=0 \) for \( p=1 \). Taking the complex conjugate of the latter expression, while holding \( s \) real, leaves

\[ [\tilde{h}_r^-(s+j\omega)]^* = \frac{1}{2} \hat{h}_r^* \tilde{\xi}(s)/g \\ s \text{ real} \\
\] (2.139)

Finally, these perturbation expressions can be substituted into the force
formula (equation 2.137).

\[
\tilde{f}(s) = \begin{cases} 
\frac{1}{4} \mu_o \pi R^2 \left| \hat{H}_r \right| 2 \frac{1}{1 + Q^+(s+j\omega)} \tilde{\zeta}(s)/g ; p=1 \\
\frac{1}{4} \mu_o \pi R^2 \left| \hat{H}_r \right| 2 \frac{1}{1+Q^+(s+j\omega)} + \frac{1}{1+Q^-(s+j\omega)}^{*} \tilde{\zeta}(s)/g ; p>1
\end{cases}
\]

The term for \( p>1 \) can be regarded as a general form for all values of \( p \) if \( Q^- \) is taken as approaching infinity for \( p=1 \). Given the appropriate values of \( Q^+(s) \), this expression completely describes the effect of the magnetic field on the radial dynamics of the system.

Just as for the homopolar radial bearing, the complex force has been related to the complex displacement by a complex, dynamic stiffness coefficient given by

\[
\tilde{K}(s) = -\left[ \frac{1}{4} \mu_o \pi R^2/g \right] \left| \hat{H}_r \right| 2 \frac{1}{1+Q^+(s+j\omega)} + \frac{1}{1+Q^-(s+j\omega)}^{*} \tilde{\zeta}(s)/g
\]

Expressions for the quality factor \( Q \) are derived for a few important types of windings in the remainder of this section.

2.5.2 Series Windings

For conventional motor-generators having \( M \) phases and \( p \) pole-pairs, an important winding configuration is for each phase to have a single \( N \)-turn winding per pole pair. The \( p \) windings which then constitute one phase can either be connected together in series or in parallel without any fundamental difference in their interaction with the equilibrium magnetic field. The distinction between series and parallel connections does, however, have a pronounced effect on the way the windings react to the perturbation field which arises due to a radial displacement.
Windings may be present on both the inner and outer members. Since these windings may be of different types, like a series stator and a squirrel-cage rotor, the analysis to follow considers the windings individually. Fortunately, the effects of multiple windings on the magnetic field are linearly additive, so that the appropriate values of \( Q_1(s) \) for the inner member and \( Q_0(s) \) for the outer member can be separately determined and then added to obtain the net \( Q(s) \) required in the stiffness expression (equation 2.141).

Specifically consider the case where all [pM] windings are identical. Furthermore assume that each winding is symmetrically wound about the pole pair to which it corresponds. If the p windings of each phase are connected in series, then both the equilibrium and perturbation currents in those windings are constrained to be the same for all pole pairs. Identical currents flowing in p identical windings necessarily produces fields having space harmonics only at integer multiples of p. Since the windings are also assumed to be symmetric about the poles, all even multiples of p are eliminated. Thus the symmetric series winding only links to fields whose spacial dependence corresponds to odd multiples of p. Since the perturbation fields which result from a radial displacement have space harmonics at \( p \pm 1 \), the symmetric series winding does not interact with the perturbation fields at all. Thus \( Q^s(s) = 0 \) for such a winding whether it is located on the inner or outer member.

The situation where \( p=1 \) is unique in that with only one winding per phase the distinction between series and parallel connections disappears. Nevertheless, the argument given above is valid for \( p=1 \) and shows that a symmetric \( p=1 \) winding will not link to the perturbation fields. Thus a symmetric 2-pole winding can be categorized as a series connection, making
Q(s)=0 for this case.

In the absence of winding effects, when Q=0, the stiffness is due to displacement alone and is given by

\[ K_o = \frac{1}{2} \left[ \mu_o \frac{R^2}{g} \right] \left| \frac{\hat{H}_r}{R} \right|^2. \]  \hspace{1cm} (2.142)

This is the same as would be expected for a homopolar equilibrium field having the same RMS field magnitude (equation 2.125).

2.5.3 Parallel Windings

Again consider M identical phases and p identical pole pairs, with a single symmetric N-turn winding per pole pair. This time, however, the p windings of each phase are connected in parallel prior to connecting the phase to any external circuit. The M phase windings (M>1) on each of the p pole pairs (p>1) meet the criteria for symmetric-impedance termination presented in section 2.3.4. The actual value of the impedance seen by each individual winding will be a combination of series and parallel elements from the other windings of the same phase and from any external impedance which may be attached to the phase as a whole. The contributions to the surface potential from the different pole pairs can be summed to yield the net surface potential due to winding current.

Consider a single set of M phases on one particular pole pair, located on the inner member. The surface potential generated by this winding is related to the perturbation air-gap flux by equation 2.96;

\[ \tilde{\phi}_{G1}^\pm(s) = \frac{-2MN^2}{\pi[p\pm1]^2} \left| k_w^\pm \right|^2 \frac{(s-j[p\pm1]\Omega)}{Z(s-j[p\pm1]\Omega)} \tilde{\phi}^\pm(s) \]  \hspace{1cm} (2.143)

where \( \tilde{\phi}^\pm(s) \) is the phasor perturbation flux per radian.
$K_w^\pm$ is the winding factor based on $m=p\pm 1$ (equation 2.68).

$Z(s)$ is the impedance seen by a single winding.

$\Omega$ is the rotational speed ($\text{rad}/\text{s}$) of the member to which the winding being evaluated is attached.

In the above expression it doesn't matter which phase is used to evaluate the winding coefficient since they all have the same magnitude. In fact, the expression is also independent of which particular pole pair is selected. Thus the net effect of all $p$ pole pairs is just $p$ times the surface potential generated by one pole pair:

$$\tilde{\psi}_{c1}(s) = \frac{2MN^2}{\pi[p\pm 1]^2} \left| \frac{k_w^\pm}{[s-j[p\pm 1]\Omega]} \right|^2 \tilde{\phi}(s)$$  \hspace{1cm} (2.144)

This expression relates the surface potential due to winding currents to the perturbation flux for a parallel winding on the inner member. For a winding on the outer member, simply take the negative of the same expression above. This sign reversal is matched in the definitions of $Q_1(s)$ and $Q_0(s)$ (equation 2.131). Thus the contribution to the total $Q(s)$ takes the same form for both members. For the case of identical parallel windings this contribution is

$$Q^\pm(s) = \frac{2MN^2\mu_0 R_x}{\pi[p\pm 1]^2 g} \left| \frac{k_w^\pm}{[s-j[p\pm 1]\Omega]} \right|^2 \frac{1}{Z(s-j[p\pm 1]\Omega)}$$  \hspace{1cm} (2.145)

The quality factor, $Q$, appears in the stiffness expression with the argument $s+j\omega$ (equation 2.141) so that the nature of the interaction is best interpreted when this dependence is introduced;

$$Q^\pm(s+j\omega) = \frac{2MN^2\mu_0 R_x}{\pi[p\pm 1]^2 g} \left| \frac{k_w^\pm}{[s+j\omega-j[p\pm 1]\Omega]} \right|^2 \frac{1}{Z(s+j\omega-j[p\pm 1]\Omega)}$$  \hspace{1cm} (2.146)
Even for a static displacement (s=0), the windings interact with the field. The effective frequency which governs the interaction is seen to be \( \omega' \pm \Omega \), where \( \omega' \) is the frequency seen by the moving winding in equilibrium. For windings on the stator (\( \Omega = 0 \)), the effective frequency is simply \( \omega \). Note that \( Q \) can have a substantial imaginary part, thus coupling motion in the x and y directions (equation 2.126).

2.5.4 Squirrel-Cage

A very important configuration which does not fit the criteria of section 2.5.2 or 2.5.3 is the squirrel cage rotor. If the equilibrium field has an azimuthal dependence dominated by space harmonic \( p \), then the perturbation fields due to a radial displacement will occur at \( p \pm 1 \). As before, \( p = 1 \) is an exception since it is assumed that no homopolar fields are generated (equation 2.61). These fields interact with the squirrel cage according to equation 2.104 of section 2.3.5:

\[
\psi^\pm(s) = \frac{N}{\pi[p \pm 1]^{2}} \left\{ \sin \frac{\pi[p \pm 1]}{N} \right\}^{2} \left\{ s-j[p \pm 1]\Omega \right\} \Omega^\pm(s)
\]

\[
\frac{N}{\pi[p \pm 1]^{2}} \left\{ \sin \frac{\pi[p \pm 1]}{N} \right\}^{2} \left\{ s-j[p \pm 1]\Omega \right\} \Omega^\pm(s) = \frac{2\pi[p \pm 1]}{Z_{e}(s-j[p \pm 1]\Omega)+(1-\cos \frac{\pi[p \pm 1]}{N})Z_{s}(s-j[p \pm 1]\Omega)}
\]  

(2.147)

This expression corresponds to a cage on the inner member. The negative of this same expression is appropriate for a squirrel cage located on the outer member. As before these signs are matched by the definition of the quality factors (equation 2.131) so that the contribution to the total \( Q(s) \) takes the same form for cages located on either the inner or outer member:

\[
Q^\pm(s) = \frac{\mu_{0}NRl}{\pi[p \pm 1]^{2}g} \left\{ \sin \frac{\pi[p \pm 1]}{N} \right\}^{2} \left\{ s-j[p \pm 1]\Omega \right\}
\]

\[
\frac{\mu_{0}NRl}{\pi[p \pm 1]^{2}g} \left\{ \sin \frac{\pi[p \pm 1]}{N} \right\}^{2} \left\{ s-j[p \pm 1]\Omega \right\} \Omega^\pm(s) = \frac{2\pi[p \pm 1]}{Z_{e}(s-j[p \pm 1]\Omega)+(1-\cos \frac{\pi[p \pm 1]}{N})Z_{s}(s-j[p \pm 1]\Omega)}
\]  

(2.148)

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where, as a reminder,

\( N \) is the number of rods.

\( Z_e(s) \) is the impedance connecting two rods together on one end.

\( Z_s(s) \) is the slot impedance of a rod.

In the continuum limit of large \( N \),

\[
\Omega^\pm(s) = \frac{\{s-j[p\pm 1]\Omega\} \mu_o Rl/g}{Z_E(s-j[p\pm 1]\Omega) + 2\pi^2[p\pm 1]^2 Z_o(s-j[p\pm 1]\Omega)} \tag{2.149}
\]

where \( Z_E(s) \) is the impedance of a complete trip around an end ring.

\( Z_o(s) \) is the net impedance connecting the two end rings together.

Induction motors represent such an important application of this theory that further discussion is in order. To simplify the analysis, only the continuum limit with \( 1 < p < 5 \) will be considered here. The stator is assumed to be series wound and thus not a factor.

First the quality factor is evaluated for the argument \( s+j\omega \),

\[
\Omega^\pm(s+j\omega) = \frac{\{s+j\omega-j[p\pm 1]\Omega\} \mu_o \pi Rl/g}{Z_E(s+j\omega-j[p\pm 1]\Omega) + 2\pi^2[p\pm 1]^2 Z_o(s+j\omega-j[p\pm 1]\Omega)} \tag{2.150}
\]

Presumably the motor is operating in steady state with a small value of slip so that \( \omega - p\Omega \ll \Omega \). Therefore,

\[
\Omega^\pm(s+j\omega) \approx \frac{\{s+j\Omega\} \mu_o \pi Rl/g}{Z_E(s+j\Omega) + 2\pi^2[p\pm 1]^2 Z_o(s+j\Omega)} \tag{2.151}
\]

Two cases are now of particular interest. The first arises when the rotor is not concentric with its own shaft. If this occurs, the rotor will execute a wobbling motion whereby the geometric axis of the rotor rotates...
about the axis of the stator at an angular speed of $\Omega$. This is equivalent to a time-varying complex rotor displacement given by

$$\tilde{\xi}(t) = \xi_0 e^{j\Omega t} \quad (2.152)$$

Taking the Laplace transform yields

$$\tilde{\xi}(s) = \frac{\xi_0}{s-j\Omega} \quad (2.153)$$

By the method of residues, the force $\tilde{f}(s)$ contains a steady-state part which corresponds to setting $s=j\Omega$ in the stiffness relation (equation 2.141). But when this substitution is inserted in the expression for $Q^\pm(s+j\omega)$ given above in equation 2.151, the result is $Q^\pm(s+j\omega)=0$. Thus the squirrel cage does not interact significantly with this type of motion.

The other case of interest is a static displacement caused by a misalignment of the rotor and stator axes. In small mass-produced motors this can represent a displacement of as much as 30% of the gap width. Static conditions correspond to $s=0$ so that

$$Q^\pm(j\omega) \approx \frac{-j\omega \mu_o \pi Rl/g}{Z_E(-j\omega) + 2\pi^2 [p\mp1]^2 Z_o(-j\Omega)} \quad (2.154)$$

But this represents a ratio of the magnetizing air-gap impedance to the rotor impedance. This ratio must be large in any motor designed for good speed regulation. Substituting this form into the stiffness expression (equation 2.141) gives

$$K(s=0) = \frac{1}{2} K_o \frac{Z_E(-j\Omega) + 2\pi^2 [p+1]^2 Z_o(-j\Omega)}{-j\omega \mu_o \pi Rl/g} + \frac{Z_E(-j\Omega) + 2\pi^2 [p-1]^2 Z_o(-j\Omega)}{-j\omega \mu_o \pi Rl/g} \quad (2.155)$$

where $K_o = -1/2 \mu_o \pi Rl |\hat{H}_p|^2/g$ is the stiffness without any perturbation currents in the squirrel cage.
If the impedances are split into a series resistance and leakage inductance, then

\[
K(s=0) = K_o \frac{L_E + 2\pi^2 [p^2 + 1]L_o}{\mu_o \pi R_l/g} + j \frac{R_E + 2\pi^2 [p^2 + 1]R_o}{\Omega \mu_o \pi R_l/g}
\]  
(2.156)

The first term in this expression shows that for a given offset the unbalanced magnetic pull at a given flux level is reduced by the presence of the cage. The factor by which the force in the direction of displacement is reduced is essentially the ratio of the leakage inductance of the cage to the magnetizing inductance of the air gap. This can easily result in an order of magnitude decrease in the unbalanced force.

The second term above indicates a coupling between the x and y axes. Specifically, it shows that for rotation in the positive \( \theta \) direction \( (\Omega > 0) \) a displacement of the inner member in the x direction produces a force on it in the positive y direction. Thus the cage encourages forward whirl.

This example of the effect of a squirrel-cage winding on the force of magnetic origin has demonstrated both a decrease in the unbalanced force for a given static offset and a quadrature coupling between the x and y directions. The next section will develop techniques for interpreting the complex stiffness factor under dynamic \( (s \neq 0) \) conditions so as to predict the stability or instability of radial-gap configurations.

### 2.6 Dynamic Analysis of Rotating Systems

The complete stability of a rigid body requires that six degrees of freedom be considered.\(^5\) In addition to the two radial directions, there can be axial motions, rotation about the z axis (polar rotation), and rotations about the x and y axes (cross-axis rotation). It is often
possible to decouple these different modes and consider them separately. Section 2.6.1 briefly discusses some of the issues associated with the different modes.

A large part of the discussion in this chapter has concentrated on the effect of radial displacement, which causes a parallel misalignment of the rotor and stator axes. There is good reason for this, since this is the mode of motion for which the magnetic field has its dominant destabilizing effect. Controlling this instability is the overriding concern in the design of a homopolar radial magnetic bearing and can even play an important role in conventional machine design, where the magnetic forces alter the mechanical resonances of the structure. Cross-axis motion can also be important since this is the mode influenced by the gyroscopic effect. Emphasis in this section will center on the radial mode with the understanding that cross-axis motion can be evaluated using completely analogous techniques.

The dynamics of motion in the radial plane are inherently two-dimensional. Whenever the two axes are symmetric, however, the complex notation developed earlier in this chapter can be applied to significantly reduce the difficulty of analyzing the system. Section 2.6.2 extends some techniques commonly used to analyze the stability of real-valued one-dimensional systems to incorporate the use of complex variables.

Finally, methods are introduced for analyzing the dynamic system response to external disturbances. Both frequency-response techniques and transient analysis are considered. In particular, oscillations driven by an imbalance in the rotor are examined.
2.6.1 Decoupling of Modes

There are six degrees of mechanical freedom for a rigid rotor relative to a Cartesian coordinate system having its origin at the rotor's center of mass. These are translation along any of the three axes, and rotation about these same axes. It is possible for a displacement in any one of the six variables to result in a force in any of the six "directions". Fortunately, this full generality is rarely necessary, and in the case of radial-gap devices only a few interactions between different axes are important. The following discussion concentrates on the interactions which result from the forces of magnetic origin, and assumes that any mechanical constraints placed on the rotor do not result in any coupling between different modes of motion. Cases may arise for which the decoupling of modes presented here is not appropriate, but such situations will almost certainly require numerical analysis.

Rotational symmetry prevents either axial translations or polar rotations from interacting to first order with either radial motions or cross-axis rotations. This is a widely-applicable rule which is even true for axial-gap systems. It can be justified by considering that axial translations and polar rotations produce perturbation fields having the same azimuthal dependence as the equilibrium field \( p \), and do not provide the fields at \( p \neq 1 \) which are needed to create radial forces or cross-axis torques. If, in addition, the system is symmetric in the axial direction with respect to the center of mass, then radial motions are decoupled from cross-axis rotations. In a magnetically-levitated structure this can never be completely true, since there would be no axial lift if it were not for the asymmetry introduced by a "sagging" of the rotor teeth below the stator teeth. Nevertheless, this criteria can be adequately met to permit the
decoupling in most situations. Finally, the coupling between axial
translation and polar rotation can be neglected since in magnetic bearings
the p=0 perturbation due to axial motion causes no torque, and in motor-
generators the axial magnetic field is ignorable because of uniformity in
the axial direction.

Having eliminated all of these various coupling terms by the proper
use of symmetry, the dynamic stiffness matrix can be written in four
separate parts;

\[
\begin{bmatrix}
    f_x(s) \\
    f_y(s)
\end{bmatrix} = -\begin{bmatrix}
    K_{xx}(s) & K_{xy}(s) \\
    K_{yx}(s) & K_{yy}(s)
\end{bmatrix}\begin{bmatrix}
    \xi_x(s) \\
    \xi_y(s)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    \tau_x(s) \\
    \tau_y(s)
\end{bmatrix} = -\begin{bmatrix}
    T_{xx}(s) & T_{xy}(s) \\
    T_{yx}(s) & T_{yy}(s)
\end{bmatrix}\begin{bmatrix}
    \phi_x(s) \\
    \phi_y(s)
\end{bmatrix}
\] (2.157)

\[
f_z(s) = -K_{zz}(s) \xi_z(s)
\]

\[
\tau_z(s) = -T_{zz}(s) \xi_z(s)
\]

With rotational symmetry, the same complex notation which has been
developed in this chapter for radial motions applies to the cross-axis
rotations, so that equation 2.157 becomes

\[
\tilde{f}(s) = -K(s) \tilde{\xi}(s) \quad f_z(s) = -K_{zz}(s) \xi_z(s)
\] (2.158)

\[
\tilde{\tau}(s) = -T(s) \tilde{\phi}(s) \quad \tau_z(s) = -T_{zz}(s) \phi_z(s)
\]

where

\[
\tilde{f} = f_x + jf_y \quad \tilde{\xi} = \xi_x + j\xi_y
\]

\[
\tilde{\tau} = \tau_x + j\tau_y \quad \tilde{\phi} = \phi_x + j\phi_y
\] (2.159)

Therefore the radial and cross axis motions can be decoupled from the
other modes and from each other, and analyzed separately if the necessary
symmetry is present in the device under evaluation. This represents a
significant simplification in the analysis. Before proceeding with a
complete analysis of these modes, however, it is worthwhile to examine the
general characteristics of the other modes present.

In most radial-gap configurations, stability in the axial direction
occurs passively without the need for any current-produced fields. A
positive axial stiffness occurs in homopolar radial bearings because the
axial force generated by offsetting the rotor and stator teeth is a
continuously-increasing function over the range of offsets, δ, which are
appropriate for providing a lift force against gravity (Section 1.3). Any
currents which might be induced in windings due to axial movements will
have the same frequency as the motion, and will tend to damp out axial
oscillations. Axial forces in radial-gap motor-generators tend to be small
because these devices lack the tooth structure which has been shown to be
necessary to obtain a significant axial stiffness. Some restoring force
does exist, and is sometimes used in induction motors in place of a thrust
bearing.

Polar rotations have been shown to be separable from all other modes
when symmetry is present. The spin speed, Ω, still affects the other
modes, but only as a parameter. The justification for assuming a steady
value for Ω in the radial-mode analysis is based on the fact that to first
order the rotor speed is independent of all other modes and has a time
constant which is long compared to other dynamical times of interest.

Cross-axis rotation can be evaluated by assuming that it is equivalent
to radially displacing each axial location by an amount
\[ \tilde{\xi}(z,s) = -j \tilde{\psi}(s)z \]  

(2.160)

where \( z \) is the axial location measured from the center of mass. This approximation is valid so long as the radial forces due to a rotation dominate over the axial forces. Since the axial forces are themselves dominated by a positive spring constant, it is unlikely that ignoring them will hide a source of instability. The cross-axis torque which results from the radial force generated in a small range of \( z \), identified by the subscript "i", is

\[ \tilde{\tau}_i(s) = j \tilde{\tau}_i(s) z_i \]  

(2.161)

Substituting the radial stiffness relation from equation 2.159,

\[ \tilde{\tau}_i(s) = -j K_i(s) \tilde{\xi}_i(s) z_i \]  

(2.162)

or, by combining this with equation 2.160,

\[ \tilde{\tau}_i(s) = - K_i(s) z_i^2 \tilde{\psi}(s) \]  

(2.163)

The total torque due to \( N \) axial regions is then

\[ \tilde{\tau}(s) = -\overline{T}(s)\tilde{\psi}(s) ; \overline{T}(s) = \sum_{i=1}^{N} K_i(s) z_i^2 \]  

(2.164)

For the homopolar radial magnetic bearing and similar systems, the radial forces are concentrated in a few small regions of \( z \) and the formula above is appropriate. For a conventional motor-generator, the radial forces are evenly distributed over a large range of \( z \) so that the integral equivalent of equation 2.164 is more suitable;

\[ \overline{T}(s) = \frac{1}{12} K(s) \ell^2 \]  

(2.165)

The basic effect of the magnetic field on the cross-axis mode is the same as for the radial mode. The only difference is that for parts of the system located at different axial levels, the net radial stiffness is just
the sum of the $K_i(s)$ whereas the net cross-axis stiffness weights the $K_i(s)$ by $z_f^2$. This only emphasizes the effect of some parts of the system more than others. It does not change the essential character of the magnetic interaction.

Although the cross-axis stiffness of magnetic origin is similar to that for radial displacements, there is an important difference in the mechanical equations of motion due to the gyroscopic effect. For a freely-suspended rotor, which in equilibrium spins about the $z$ axis, the mechanical equations of inertia can be expanded to first order terms in the displacement variables to yield

$$\ddot{\tilde{f}}(s) = s^2M\ddot{\tilde{z}}(s) \quad \quad f_z(s) = s^2M\ddot{z}(s)$$

$$\ddot{\tau}(s) = s^2J_d\dot{\tilde{\phi}}(s) - j\Omega J_p\dot{\tilde{s}}(s) \quad \quad \tau_z(s) = s^2J_p\phi_z(s)$$

(2.166)

where $M$ is the rotor mass.

$J_p$ is the polar mass moment of inertia.

$J_d$ is the diametrical mass moment of inertia

(the moment of inertia about the $x$ or $y$ axis).

All of the $s^2$ terms are familiar inertial acceleration terms. The gyroscopic effect enters into the cross-axis mode as an imaginary damping term which is proportional to the spin speed $\Omega$. This term is completely consistent with the coupled two-dimensional dynamics which the complex notation has been developed to describe. Thus the gyroscopic term can be thought of as a modification to $\ddot{T}(s)$ so that the equivalence of form between the radial and cross-axis modes is preserved. The remainder of this section concentrates on the radial mode, remembering that the techniques developed are equally applicable to an analysis of cross-axis stability.
2.6.2 Radial Stability

Consider the general linear case of a rotationally-symmetric magnetic interaction in the radial plane between a rotor and stator, given by

\[ \tilde{f}(s) = -K(s) \tilde{\xi}(s) \]  \hspace{1cm} (2.167)

where \( \tilde{f} = f_x + j f_y \) is the force on the inner member.

\( \tilde{\xi} = \xi_x + j \xi_y \) is the displacement of the inner member with respect to the outer member.

\( K(s) \) is the complex dynamic stiffness of magnetic origin.

Specific expressions for individual contributions to \( K(s) \) were given for the homopolar radial magnetic bearing (equation 2.125) and the radial-gap motor-generator (equation 2.141). The \( K(s) \) considered here is the sum of all the magnetic forces acting between the rotor and stator.

Temporarily assume that the rotor is the inner member. The location of the geometric center of the rotor and stator can each be measured relative to a fixed equilibrium position so that

\[ \tilde{\xi}(s) = \tilde{\xi}_R(s) - \tilde{\xi}_S(s) \]  \hspace{1cm} (2.168)

Therefore, the magnetic force on the rotor is

\[ \tilde{f}_R(s) = -K(s)[\tilde{\xi}_R(s) - \tilde{\xi}_S(s)] \]  \hspace{1cm} (2.169)

and the force on the stator is

\[ \tilde{f}_S(s) = -K(s)[\tilde{\xi}_S(s) - \tilde{\xi}_R(s)] \]  \hspace{1cm} (2.170)

If the rotor is instead placed as the outside member then the same expressions result. Therefore these expressions are valid without regard for which member is allowed to rotate.

In addition to the force of magnetic origin, the rotor and stator may be subject to mechanical forces. Only mechanical forces which are rotationally symmetric and contain no coupling between axes are considered here. Such forces can be represented by dynamic stiffness coefficients.
similar to those used for magnetic forces, except that the mechanical terms will only have real parts;

\[ \tilde{f}_R(s) = -K_{RS}(s) [\tilde{\xi}_R(s) - \tilde{\xi}_S(s)] \]

\[ \tilde{f}_S(s) = -K_{RS}(s) [\tilde{\xi}_S(s) - \tilde{\xi}_R(s)] - K_S(s) \tilde{\xi}_S(s) \]  

\[ (2.171) \]

In writing these expressions it is assumed that the only mechanical connections in the system are between the rotor and the stator, and between the stator and inertial ground. For a magnetically-suspended rotor there is no physical contact between the rotor and stator, so \( K_{RS} = 0 \).

The magnetic and mechanical forces act together to accelerate the rotor and stator masses. If the center of mass for both members coincides with their geometric center, then the magnetic forces act directly on the centers of mass and the equations of motion become

\[ s^2 M_R \tilde{\xi}_R = \tilde{f}_R = \tilde{f}_{R \text{mag}} + \tilde{f}_{R \text{mech}} \]

\[ s^2 M_S \tilde{\xi}_S = \tilde{f}_S = \tilde{f}_{S \text{mag}} + \tilde{f}_{S \text{mech}} \]  

\[ (2.172) \]

The case of rotor imbalance, wherein the center of mass is displaced from the center of geometry, is considered separately in section 2.6.3.

The equations of motion in the s-domain are obtained by combining the last three sets of expressions in matrix form;

\[
\begin{bmatrix}
 s^2 M_R + K_{RS}(s) + K(s) & -K_{RS}(s) - K(s) \\
 -K_{RS}(s) - K(s) & s^2 M_S + K_S(s) + K_{RS}(s) + K(s)
\end{bmatrix}
\begin{bmatrix}
 \xi_R(s) \\
 \xi_S(s)
\end{bmatrix}
= 
\begin{bmatrix}
 f_R(s) \\
 f_S(s)
\end{bmatrix}
\]  

\[ (2.173) \]

where now \( \tilde{f}_R \) and \( \tilde{f}_S \) represent externally-applied forces not accounted for on the left-hand side. These equations could be expanded to relate the real functions of s they represent, but the matrix would increase in size from 2\times2 to 4\times4. Thus there is a distinct advantage to retaining the complex notation.
Homogeneous solutions to the coupled equations of motion correspond to setting the determinant of the $2 \times 2$ matrix above to zero. The result is a polynomial in $s$ with complex coefficients. The roots of this equation then correspond to the values of the exponent coefficients in the time-domain solution. For example,

$$\tilde{\xi}_R(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots \quad (2.174)$$

The only difference between this approach and conventional s-domain analysis is that the complex roots no longer must appear as complex-conjugate pairs, and the coefficients $c_i$ can be complex quantities. In fact, all of the fundamental implications of s-plane analysis still hold, since the Laplace transform is equally well defined for complex as for real functions. Most importantly, the stability of the system can still be determined from the location of the poles in the s-plane. Therefore it is possible to perform the two-dimensional stability analysis using the complex variables as if they were real one-dimensional values. This makes available many analysis tools from linear system theory, the most useful being block diagram manipulation and root-locus evaluation. All of the expressions derived in this section can be verified, if desired, by expanding the complex notation into real form.

The general situation reflected by equation 2.171 can be expressed as a block diagram as shown in Figure 2.16. Since the system is linear, only one external input need be considered at a time. The block diagram can then be manipulated until it achieves the plant-feedback form shown in Figure 2.17.
Figure 2.16  Block Diagram Using Complex Variables.

Figure 2.17  Plant-Feedback Form.

Depending on what variable is selected as the output quantity, it is possible to divide the system as desired into plant and feedback components. This is most valuable for stability analyses, where the
closed-loop poles determine stability without regard to which variables are labeled as the input and output. Let the system be divided into a part which is considered fixed and unchangeable, and a part which is under investigation. Assign the fixed part to the plant and the uncertain part to the feedback. Root-locus techniques can then be used to evaluate the effect of changes in the feedback portion of the system.¹⁰

The usefulness of the root-locus method can be best understood by means of an example. Consider a conventional motor which has the following characteristics:

(1) Rotor mass $M_R$ and stator mass $M_S$.

(2) The magnetic field acts primarily to reduce the stiffness of the shaft, clearing a net positive stiffness, $K$.

(3) Deflections of the shaft cause small rate-dependent losses which are modeled by a simple damping coefficient, $B$.¹⁴

When this motor is secured rigidly to ground and run at a speed $\omega$ in excess of the critical speed $\omega_o$, where

$$\omega_o = \sqrt{\frac{K}{M_R}}$$  \hspace{1cm} (2.175)

then the rotor breaks into unstable forward whirl. It would be useful to know what sort of mounting arrangement for the stator, if any, would eliminate the whirling problem.

First the block diagram in Figure 2.16 is reduced to plant-feedback form with only the stator stiffness, $K_S(s)$, in the feedback leg, as shown in Figure 2.18.
Figure 2.18 Plant-Feedback Form for Evaluating Stator Stiffness.

The transfer relation for the plant can be simplified for this particular case by assigning

\[ K_{RS}(s) + K(s) = K + [s - j\Omega]B \quad \text{(2.176)} \]

Note that the effective frequency seen by the rotating damping mechanism is \( s - j\Omega \). This transformation to the rotating frame is familiar from section 2.3 and follows from the fact that a static displacement of the rotor axis causes an oscillatory stress in the shaft at frequency \( \Omega \).

The poles and zeros of the plant are shown in Figure 2.19 for \( M_R = M_S \), \( \Omega = 2\omega_o \), and \( \omega_o B/K = 0.1 \).

Figure 2.19 Root Locus with No Stator Damping.
If the stator is mounted to ground by a spring, then $K_S(s)$ is just a real constant. Adjustment of this stator stiffness from zero to infinity will cause the closed-loop poles to move from the open-loop poles to the open-loop zeros, tracing out the root-locus which was shown in the figure. The closed-loop zeros, if they are needed, will be the zeros of the plant and the poles of the feedback. With the aid of the root-locus diagram it is immediately apparent that no value of stator stiffness can prevent the instability.

Now consider adding damping to the stator along with the spring. Then

$$K_S(s) = K_S[1 + s\tau]; \tau = \frac{B_S}{K_S}$$

This places an open-loop zero on the negative real axis, resulting in the root-locus shown in Figure 2.20. This diagram shows that proper placement of the zero, combined with an intermediate value of $K_S$, can indeed stabilize the system.

![Root Locus with Stator Damping](image)

Figure 2.20 Root Locus with Stator Damping.
When drawing root-locus diagrams, it is important to recognize that many of the standard construction rules do not apply. However, the underlying rule remains: Every point in the plane for which the phase angle of $G_p(s)H_{eq}(s)$ is 180 degrees, is a part of the root locus. This rule can be used directly to determine entry and exit angles for poles and zeros, as well as the asymptotic behavior for large gain.

The use of the root locus as a tool for analyzing and designing systems described by complex variables has been presented. Another commonly used technique from classical linear system theory, the Nyquist criterion, will be considered in the next section which treats frequency response in rotating systems.

2.6.3 Frequency Response

Two-dimensional motions of a single-input, single-output linear system which exhibits rotational symmetry can be described by a complex transfer function,

$$\tilde{V}(s) = G(s) \tilde{U}(s)$$  \hspace{1cm} (2.178)

where $U(s) = U_x(s) + jU_y(s)$ is the input variable.

$\tilde{V}(s) = V_x(s) + jV_y(s)$ is the output variable.

$G(s)$ is the transfer function, representable as the ratio of two polynomials in s having complex coefficients.

The frequency response of the system corresponds to the behavior of the output when the input is driven in the sinusoidal steady state. In the two-dimensional system considered here, the amplitude and phase of the input drive can be different for the x and y directions. The time dependence of each direction can be written in time-phasor form as
\[ U_x(t) = \text{Re}\{\hat{U}_x e^{j\omega t}\} \]  
\[ U_y(t) = \text{Re}\{\hat{U}_y e^{j\omega t}\} \]  
(2.179)

In terms of the two phasor amplitudes, the complex input to the system can be given as
\[ \hat{U}(t) = \hat{U}^+ e^{j\omega t} + \hat{U}^- e^{-j\omega t} \]  
(2.180)

where
\[ \hat{U}^+ = \frac{1}{2} [\hat{U}_x + j\hat{U}_y] \]  
\[ \hat{U}^- = \frac{1}{2} [\hat{U}_x^* + j\hat{U}_y^*] \]  
(2.181)

The first term \((\hat{U}^+)\) corresponds to an excitation which "rotates" from axis to axis in the positive \(\theta\) direction with a rotational speed of \(\omega\) radians per second. The second term \((\hat{U}^-)\) corresponds to a rotation in the negative \(\theta\) direction. Therefore any sinusoidally-varying drive can be represented as a combination of a positive-frequency term and a negative-frequency term. When the variable being considered is a radial or cross-axis displacement, then the positive and negative frequencies can be directly associated with forward and retrograde whirl, respectively.

The output of the system is expected to have the same form as the input,
\[ V_x(t) = \text{Re}\{\hat{V}_x e^{j\omega t}\} \]  
\[ V_y(t) = \text{Re}\{\hat{V}_y e^{j\omega t}\} \]  
(2.182)

where the orthogonal-axis phasors are found from the positive and negative frequency terms using
\[ \hat{V}_x = \hat{V}^+ + j[\hat{V}^-]^* \]  
\[ \hat{V}_y = -j\hat{V}^+ - [\hat{V}^-]^* \]  

(2.183)

where

\[ \tilde{V}(t) = \hat{V}^+ e^{j\omega t} + \hat{V}^- e^{-j\omega t} \]  

(2.184)

The transfer function can always be used to derive the differential equation which relates input and output variables. In doing this, each appearance of \( s \) becomes a derivative in time. Since both the input and output are exponential functions of time, the derivative introduces a multiplier equal to the coefficient of \( t \) in the exponent. For the positive-frequency term this is \( j\omega \), while for the negative-frequency term it is \( -j\omega \);

\[ \tilde{V}(t) = G(j\omega) \hat{U}^+ e^{j\omega t} + G(-j\omega) \hat{U}^- e^{-j\omega t} \]  

(2.185)

Therefore the transfer function relating the positive and negative frequency phasors is

\[ \hat{V}^\pm = G(\pm j\omega) \hat{U}^\pm \]  

(2.186)

Thus the two functions \( G(j\omega) \) and \( G(-j\omega) \) completely characterize the system, and will be referred to as the positive and negative frequency response, respectively. Equivalently, the single function \( G(j\omega) \) represents a complete description if both positive and negative values of \( \omega \) are considered.

The Nyquist criterion provides a useful alternative to the root-locus technique for determining the stability of a system. As described in section 2.6.2, the block diagram of the system can be reduced to the plant-feedback form (Figure 2.17). For a frequency-response analysis of the closed-loop system, \( s \) is replaced by \( j\omega \). Positive values of \( \omega \) are then
used to relate the positive-frequency phasors and negative values of $\nu$ are used to relate the negative-frequency phasors. The general form for the system under these steady-state conditions is shown in Figure 2.21.

![Figure 2.21 Plant-Feedback Form for Frequency Response.](image)

The Nyquist criteria uses the frequency response of the open-loop system to determine the stability of the closed-loop system. In doing so it considers both positive and negative frequencies, so that instability of either the positive or negative mode is reflected as an instability of the system as a whole. To apply the criterion, both the magnitude and phase of the frequency response need to be known for the open-loop function $G(j\nu)H_{eq}(j\nu)$. This information is plotted in the complex plane for all frequencies from $-\infty$ to $+\infty$, forming a looping curve which closes upon itself. This procedure is identical to the manner in which the Nyquist method is applied to one-dimensional systems. The difference only shows up in that the Nyquist diagram for complex variables is generally not symmetric about the real axis as it is in the traditional case. Once this diagram is drawn, the Nyquist stability criteria can be evaluated: The closed-loop system is stable if and only if the net number of counterclockwise encirclements of the point $-1+j0$ by the Nyquist diagram equals the number of open-loop poles in the right-half plane.$^{10}$

The Nyquist criterion is most useful for determining what magnitude of
the open-loop transfer function will cause instability. Increasing the magnitude of any real multiplicative constant in the open-loop system simply changes the scale of the Nyquist diagram, moving the -1 point relative to the curves until an extra encirclement occurs. The Nyquist plot is also valuable for obtaining information on the relative stability of the system. The two most common measures of relative stability are the factor by which the gain can be adjusted before instability ensues (gain margin), and the phase angle at unity magnitude between the negative real axis and the closest approach of the Nyquist curve (phase margin). Both of these margins are well defined for complex-variable systems.

The Nyquist procedure can be applied to the example presented in section 2.6.2 to determine exactly how stiff the stator can be made before inducing unstable whirl. Initially assume that the stator stiffness is equal to the rotor stiffness and that the stator damping has a time constant given by $\omega_o \tau_s = 0.5$. This places the stator zero in the location shown in Figure 2.20. The parameter values previously assumed for this example will be retained, namely $M_R = M_S$, $\Omega = 2\omega_o$, and $\omega_o \tau_R = 0.1$. Substituting these values into the open-loop transfer relation (Figure 2.18) and setting $s = j\nu$ provides the frequency response necessary to generate the Nyquist diagram shown in Figure 2.22.

Examination of the diagram shows that for the parameter values selected there is one counter-clockwise encirclement of the -1 point. Since there is exactly one open-loop pole in the right-half plane (Figure 2.19), the closed-loop system is stable. The curve crosses the negative real axis at about -0.2. This implies a gain margin of 5 and suggests that for the given parameters, the stator can be made up to five times stiffer than the rotor. Larger values of stiffness will allow the whirl
instability to grow. The phase margin can also be obtained from the diagram. For equal stator and rotor stiffness, the phase margin is 15 degrees, and is less for both larger and smaller values of stiffness.

![Nyquist Diagram](image)

**Figure 2.22 Nyquist Diagram for Stator Damping.**

Often the most important point in the frequency-response spectrum of a rotating system is the positive frequency corresponding to the spin speed, \( \Omega \). It is at this frequency that the system is driven by rotor imbalance.\(^4\)

Consider the situation where the center of mass does not coincide with the geometric center of the rotor, as depicted in Figure 2.23. The location of the center of mass relative to the center of geometry is given at time \( t=0 \) by \( \tilde{\xi}_\Delta \), where

\[
\tilde{\xi}_\Delta = \xi_{\Delta x}(t=0) + j\xi_{\Delta y}(t=0)
\]  
(2.187)

As time progresses, the relative positions of the center of mass and of geometry change due to the polar rotation, \( \Omega t \). At any instant, the
location of the center of mass relative to the origin is specified by

\[ \tilde{\xi}_{\text{CM}}(t) = \tilde{\xi}_R(t) + \tilde{\xi}_\Delta e^{j\Omega t} \]  

(2.188)

Thus the imbalance introduces a disturbance which is characterized by forward whirl at frequency \( \Omega \). Because the whirl speed equals the spin speed, this type of motion is often referred to as synchronous whirl.

![Diagram showing center of geometry and center of mass with \( \tilde{\xi}_R \) and \( \tilde{\xi}_\Delta \) marked with arrows]

Figure 2.23 Rotor Imbalance Displacements.

All variables in the system can be assumed to carry the same steady-state time dependence, and can be written using the phasor notation developed in this section. In particular,

\[ \tilde{\xi}_{\text{CM}}(t) = \hat{\xi}_{\text{CM}} e^{j\Omega t} \]  

(2.189)

\[ \tilde{\xi}_R(t) = \hat{\xi}_R e^{j\Omega t} \]

so that

\[ \hat{\xi}_{\text{CM}} = \hat{\xi}_R + \hat{\xi}_\Delta \]  

(2.190)

The mechanical equation of inertia for radial displacements of the rotor is

\[ \ddot{\tilde{r}}_R(t) = M_R \frac{d^2 \tilde{\xi}_{\text{CM}}(t)}{dt^2} \]  

(2.191)
where \( \tilde{f}_R \) includes all external mechanical and magnetic forces acting on the rotor. Substituting the assumed exponential time dependence yields

\[
\tilde{f}_R = -M_R \Omega^2 [\xi_R + \tilde{\xi}_\Delta] \tag{2.192}
\]

The first term on the right is the usual inertia term, evaluated using \( s = j\Omega \). The second term acts like an external drive force at frequency \( \Omega \). Equation 2.192 can be written to reflect this interpretation;

\[
[s^2M_R]_{s = j\Omega} \dot{\xi}_R = \tilde{f}_R + M_R \Omega^2 \tilde{\xi}_\Delta \tag{2.193}
\]

In terms of a block diagram this becomes

![Block Diagram with Rotor Imbalance.]

Figure 2.24 Block Diagram with Rotor Imbalance.

Therefore steady synchronous whirl due to rotor imbalance can be analyzed using the same block diagrams developed for stability analysis by letting \( s = j\Omega \) and introducing an external drive force on the rotor equal to \( M_R \Omega^2 \tilde{\xi}_\Delta \).

### 2.6.4 Transient Analysis

The Laplace variable \( s \) has been retained throughout this chapter. This allows the transient response of the system to be analyzed directly from the transfer relation. There are two procedures which can be taken to obtain the time response for each axis. The first of these methods starts with the expanded real form of the transfer function, \( \bar{G}(s) \);

\[
\begin{bmatrix}
V_x(s) \\
V_y(s)
\end{bmatrix} =
\begin{bmatrix}
\text{Re}\{\bar{G}(s)\} & -\text{Im}\{\bar{G}(s)\} \\
\text{Im}\{\bar{G}(s)\} & \text{Re}\{\bar{G}(s)\}
\end{bmatrix}
\begin{bmatrix}
U_x(s) \\
U_y(s)
\end{bmatrix}
\tag{2.194}
\]
The input drives, \( U_x(s) \) and \( U_y(s) \), are obtained by taking the Laplace transform of the real time functions \( U_x(t) \) and \( U_y(t) \). The matrix product then yields a function of \( s \) with real coefficients for both \( V_x(s) \) and \( V_y(s) \). Taking the inverse Laplace transform then supplies the real time functions for the output variables.

Alternatively, the system can be described using complex notation;

\[
\tilde{V}(s) = \tilde{G}(s) \tilde{U}(s) \, .
\] (2.195)

The input drive \( \tilde{U}(s) \) can be obtained by combining the transforms of the two real-time input functions using \( U_x(s) + jU_y(s) \), or simply by taking the Laplace transform of the complex time function \( U_x(t) + jU_y(t) \). Multiplying by the complex transfer function yields a complex function of \( s \) for the output. The inverse Laplace transform of this complex function can be taken directly, yielding a complex function of time. The real part of this time function is \( V_x(t) \), and the imaginary part is \( V_y(t) \). This procedure for obtaining the transient response has the advantage that the complex form of the output emphasizes the rotational aspects of the response. When the output variable is a mechanical displacement this method directly indicates the forward and retrograde whirl components.

As an example of the use of transient analysis, consider a spinning rotor suspended with its center of mass fixed. Cross-axis rotations are constrained by an effective torsional spring constant, \( T \). The equation of motion for cross-axis rotations of this gyroscopic system is obtained by combining the spring stiffness with the inertia relation from equation 2.166;

\[
\tilde{\tau}(s) = \left\{ s^2 J_d - j\omega_J s + T \right\} \tilde{\phi}(s) \, .
\] (2.196)
where $\tilde{\tau}(s)$ now represents any externally-applied torques.

$J_p$ is the polar mass moment of inertia.

$J_d$ is the diametrical mass moment of inertia.

The transient response $\tilde{\phi}(t)$ can now be found for any applied torque $\tilde{\tau}(t)$. In particular, consider the case where an impulse of torque is applied at $t=0$ in the positive $\phi_x$ direction,

$$\tilde{\tau}(t) = I_o u_o(t)$$

(2.197)

The resulting function for $\tilde{\phi}(s)$ is

$$\tilde{\phi}(s) = \frac{I_o}{s^2 J_d - j\Omega J_p J_d + T}$$

(2.198)

This is the typical impulse response of a second-order system, except that the damping coefficient is imaginary. The complex time response is therefore given by

$$\tilde{\phi}(t) = \frac{I_o}{J_d \omega_d} e^{j\Omega J_p t/2J_d} \sin[\omega_dt] ; \ t>0$$

(2.199)

where

$$\omega_d = \sqrt{\frac{T}{J_d} + \left(\frac{\Omega J_p}{2J_d}\right)^2}$$

(2.200)

This very interesting response is plotted in Figure 2.25 for the case where

$$\frac{\Omega J_p}{2J_d} = \sqrt{\frac{T}{J_d}}$$

(2.201)

The order in which the lobes are traced is indicated in the figure. Note that the motion does not decay. This is to be expected since no loss mechanisms were included in the system description.
Figure 2.25  Impulse Response of a Gyroscopic System.

The response function can be written to emphasize that the displacements are the sum of two whirl modes;

\[ \tilde{\phi}(t) = \frac{-jI_o}{2J_d\omega_d} \left[ e^{ju^+t} - e^{-ju^-t} \right];\ t>0 \quad (2.202) \]

where

\[ u^+ = \frac{\Omega J_P}{2J_d} + \sqrt{\frac{T}{J_d} + \left[ \frac{\Omega J_P}{2J_d} \right]^2} \]

\[ u^- = -\frac{\Omega J_P}{2J_d} + \sqrt{\frac{T}{J_d} + \left[ \frac{\Omega J_P}{2J_d} \right]^2} \quad (2.203) \]

In the absence of spin, both modes converge on the same value;

\[ \omega_o = \sqrt{\frac{T}{J_d}} \quad (2.204) \]
As the spin speed is increased, the forward whirl mode increases in frequency, while the retrograde mode decreases in frequency. If \( J_p > J_d \) then the frequency of the forward whirl mode is always greater than \( \omega \). Unstable forward whirl induced by mechanical hysteresis and similar loss mechanisms on the rotor can result if this condition is not satisfied.\(^3\),\(^4\)

The transient response of a symmetric two-dimensional system can be determined directly from the complex transfer function. Methods for generating this function have been presented for several cases of interest, concentrating on issues appropriate for magnetic bearings and their associated motor-generators. To the extent that the motions of a rigid rotating system can be decoupled into one- and two-dimensional interactions, a thorough analysis of the coupled electromechanical dynamics can be performed using the techniques presented in this chapter.
CHAPTER 3

PASSIVE MAGNETIC STABILIZATION

This chapter investigates the capability of windings terminated in passive, lumped, circuit elements to stabilize a homopolar radial-gap magnetic suspension system. This possibility represents the simplest, most rugged form of self-stabilization, in that neither a power supply nor signal amplification is required. This concept relies entirely on inducing currents in the windings when a displacement from equilibrium occurs. If properly configured, these currents will tend to restore the equilibrium condition. Although it is shown that passive stabilization is possible under very restricted conditions, its application to practical magnetic bearings is unlikely.

3.1 DYNAMIC ANALYSIS OF SIMPLE SYMMETRIC-WINDING TERMINATIONS

The techniques developed in Chapter 2 can be applied to the specific case of windings terminated in simple series impedances. The two cases which will be presented in detail correspond to shorted coils and to capacitors in series with the coils. In both cases it is assumed that the windings are multi-phase and are arranged to couple effectively to the \( p=1 \) dipole field which is created in the air gap as a result of radial displacement. These windings then establish the boundary condition relating the air-gap flux and surface potential. The equilibrium field present is a static homopolar field, and so does not interact with the windings at all. Consequently, these passive connections have no current flow or power dissipation in the windings when the rotor and stator are concentrically aligned.
3.1.1 Resistive Wall

The simplest possible winding termination is to connect the ends of the coils together. This effectively puts only the resistance and leakage inductance of the winding across the terminals of the conservative electromechanical coupling. Thus this configuration will be referred to as the RL connection. In the continuum limit a large number of these windings approximates a continuous sheet of conductor in the air gap, forming a "resistive wall".

The RL windings can be located on either the rotor or stator, with very different results. In the stationary frame a resistive wall acts as a damper, but does not affect the negative-spring stiffness inherent in the magnetic field. In the rotating frame a resistive wall acts like a positive spring, tending to offset the static stiffness of the magnetic field, but introduces higher-order dynamics which are destabilizing. This dynamic destabilization by a resistive wall in motion is present in several of the schemes which have been proposed for magnetically-levitated trains.\(^2\) In the analysis to follow, the winding is assumed to be located on the rotor. The case of a stator winding can be obtained by setting \(\Omega = 0\) in the expressions.

The boundary condition imposed on the dipole perturbation field by a resistive wall is provided by equation 2.96 for a series of discrete coils, by equation 2.104 for a squirrel-cage structure, and by equation 2.106 for a continuous sheet of conductive material. Each of these expressions can be written in the general form

\[
\tilde{\psi}_R(s) = \frac{\pi [s - j\Omega] \tilde{\phi}(s)}{R + [s - j\Omega]L_R} \quad (3.1)
\]

where \(R\) is the effective rotor resistance per radian and \(L_R\) is the
effective rotor leakage inductance per radian. The upper sign is for windings on the inner surface of the air gap. Assuming that this is the only winding present, then the total flux is given in terms of the radial displacement by a combination of equations 2.109 and 2.112,

$$\tilde{\phi}(s) = \mu_0 RH_0 \left[ \frac{T}{g} \right]_d ^' \tilde{\xi}(s) \pm \mu_0 R \left[ \frac{T}{g} \right]_c ^' \tilde{\psi}_R(s)$$  (3.2)

Combining these last two equations gives a result for the rotor surface potential,

$$\tilde{\psi}_R(s) = \frac{[s - j\Omega] \mu_0 RH_0 \left[ \frac{T}{g} \right]_d ^'}{R + [s - j\Omega][L_g + L_\ell]} \tilde{\xi}(s)$$  (3.3)

where

$$L_g = \mu_0 R \left[ \frac{T}{g} \right]_c ^'$$  (3.4)

is the winding’s effective air-gap inductance per radian.

The complex dynamic magnetic stiffness follows from equation 2.125 as

$$K(s) = -K_m \{1 - \beta \Gamma(s)\}$$  (3.5)

with

$$K_m = 2\mu_0 \pi RH_0^2 \left[ \frac{T}{g} \right]_d$$  (3.6)

$$\Gamma(s) = \frac{[s - j\Omega][s - j\Omega][L_g \left[ \frac{T}{g} \right]_c ^'/ \left[ \frac{T}{g} \right]_d ^']}{R + [s - j\Omega][L_g + L_\ell]}$$  (3.7)

In this representation, $K_m$ is the magnitude of the total negative stiffness of the static equilibrium magnetic field acting on the ferromagnetic materials, and $\Gamma(s)$ is a "control-law" relationship which describes the dynamic dependence of winding currents on the radial
displacement. The added factor of 2 in equation 3.6 is to account for the assumed presence of two identical RL-equipped gaps, one where the flux goes from the stator to the rotor, and one where it returns.

The feedback-control aspect of the RL connection can be emphasized through the use of a block diagram representation as shown in Figure 3.1. For the case illustrated, the stator is assumed to be fixed to inertial ground and the mass of the levitated rotating body is M. This mass is assumed to possess the symmetry described in section 2.6.1 needed to consider the radial dynamics separately.

Figure 3.1 Expanded RL Block Diagram.

Although this $K_m$ block is physically more closely related to the control block than the mass block, a distinct advantage arises from combining the inertia and $K_m$ blocks as shown in Figure 3.2.
In this form, the forward path "plant" depends directly on the variable "s", while in the feedback path the dynamics always appear in the combination s-\(j\Omega\). This separation of the fixed-frame and rotating-frame dynamics permits a clear picture of the effect of rotation on the system. In particular, a root-locus diagram can be drawn in which the feedback poles and zeros are all shifted up the imaginary axis by an amount equal to the spin speed, \(\Omega\). Figure 3.3 shows the root locus for an intermediate value of spin.

Figure 3.2 Feedback-Control Block Diagram for RL Connection.

\[ \tau_R = \frac{L_g + L_2}{R} \]

\[ \alpha = \frac{\begin{bmatrix} t \\ g \end{bmatrix}_c \begin{bmatrix} t \\ g \end{bmatrix}_d'}{L_g + L_\ell} \]

Figure 3.3 Root Locus for RL on Rotor.
An evaluation of the entry and exit angles for the zeros and poles is by itself enough information to conclude that despite what values are selected for the system parameters, one closed-loop pole will always remain in the right-half-plane. This conclusion has been confirmed by numerically testing numerous specific cases.

An important special case is the effect of the RL windings on the static \( s=0 \) stiffness of the magnetic field. If the net static stiffness could be made positive then there is a chance that the addition of damping mechanisms to the stator might lead to a stable configuration. From Figure 3.3 the combined effect of the field and windings on the static stiffness is, for \( \Omega_r \gg 1 \),

\[
K(s = 0) = [\alpha - 1] K_m
\]  

(3.8)

Thus if \( \alpha \) were ever greater than unity the net static spring stiffness would become positive. Unfortunately, even if \( L_q \) approaches zero, an investigation of the various \([t/g]\) terms from section 2.4 shows that \( \alpha \) can never exceed 1, and will be considerably less than unity whenever a toothed gap appropriate for obtaining axial lift is employed. Typical values for \( \alpha \) lie near 0.6. Consequently it is unlikely that any modification of the stator structure supporting an RL-wound rotor will lead to a stable configuration.

3.1.2 Tuned Circuit

Another passive termination possibility is to place a capacitor in series with each winding. This option requires the use of discretely-wound coils and so is less desirable than the rugged squirrel-cage RL approach. Nevertheless, the connection is fairly simple to construct and does provide more control flexibility than the RL technique. By suitable selection of the capacitor value, the inductance of the rotor winding can be cancelled
over a limited range of frequencies, thus allowing much larger correction currents to flow in the coils.

Adding a capacitor to the symmetric termination impedance modifies the form of the flux-potential boundary conditions at the surface of the rotor, in accordance with equation 2.96. Again, the specific parameters of the winding can be expressed in a general form similar to equation 3.1,

\[
\tilde{\Psi}_R(s) = \frac{\frac{s}{\omega} \tilde{\phi}(s)}{1 + [s - j\omega]RC + [s - j\omega]^2 L_{\text{L}}^2 C}
\]

(3.9)

where \( R \) is the effective rotor resistance per radian.

\( C \) is the effective series capacitance per radian.

\( L_L \) is the effective rotor leakage inductance per radian.

Again assuming that this is the only winding in the air gap, the total flux is given by equation 3.2 so that the rotor potential becomes

\[
\tilde{\Psi}_R(s) = \frac{\frac{s}{\omega} L_G \left[ \frac{t}{g} \right]_d / \left[ \frac{t}{g} \right]_c \omega \tilde{\zeta}(s)}{1 + [s - j\omega]RC + [s - j\omega]^2 C[L_G + L_L]}
\]

(3.10)

where \( L_G \) is the air-gap inductance as defined in equation 3.3.

The resulting magnetic stiffness takes the same form as equation 3.5, with a new control law,

\[
\Gamma(s) = \frac{[s - j\omega]^2 L_G \left[ \frac{t}{g} \right]_d / \left[ \frac{t}{g} \right]_c}{1 + [s - j\omega]RC + [s - j\omega]^2 C[L_G + L_L]}
\]

(3.11)

This feedback implementation can be described using the block diagram representation shown in Figure 3.4.
The root-locus description of this connection is shown in Figure 3.5 for an intermediate value of spin and an LC tuned frequency which is slightly lower than the spin frequency. Just as for the RL connection, an evaluation of entry and exit angles shows that one closed-loop pole will always remain in the right-half-plane.
The primary distinction of the tuned-circuit connection is its effect on the static radial stiffness \(s=0\). Specializing from Figure 3.4, the net static stiffness acting on the levitated mass is, for \(\omega_T^T R >> 1\),

\[
K(s = 0) = K_m \left\{ \frac{\alpha \Omega^2}{\Omega^2 - \omega_T^2} - 1 \right\}
\]  

(3.12)

For spin speeds slightly greater than the tuned-circuit frequency, the net static stiffness is positive. For \(\alpha=0.6\), positive stiffness occurs over a speed range of 1.5 to 1.

This result is closely related to the role of tuned circuits in the stabilization of alternating-current suspensions such as are used in inertial-navigation instruments.\(^1\) In these devices a viscous liquid surrounds the levitated member and provides the damping necessary to obtain dynamic stability. Parente has studied the dynamic stability of AC suspensions and shows that, on a relative scale, a considerable amount of damping in required.\(^4\) Although it seems unlikely that this much damping can be obtained in a rotating system where no physical contact with the levitated member is permitted, an investigation of the possibilities does yield some interesting results.

Damping without physical contact can be realized by inducing currents in a conductor fixed in the stationary frame. An example would be an RL winding located on the stator. Care must be taken, however, not to introduce additional negative magnetic spring stiffness from the damper structure. An example of a configuration which does not is shown in Figure 3.6.
The axial magnetic field created by the permanent magnet on the rotor induces currents in the conductive material attached to the stator whenever there is a relative velocity between the two members. These currents interact with the magnetic field to create a damping force on the rotor. If the velocity is too high, however, the currents effectively screen out the field variation and the damping effect is greatly reduced. The frequency above which the shielding is a problem is determined by the effective L/R time constant of the induced currents. In order to avoid this effect in the arrangement illustrated, the damper conductor can only occupy a small portion of the axial air gap. This severely limits the amount of damping which can be obtained in a reasonably-sized structure. As many as ten of the damper sections illustrated might be required to obtain enough damping to stabilize the system. This would certainly add more mass than the levitation section could support in a normal-gravity
environment. Nevertheless, these constraints are based on material limitations and at least in theory it is possible to add an arbitrary amount of induced-current damping to the system.

A block-diagram representation which includes damping is shown in Figure 3.7. Since the damping depends only on $s$ and not $j\Omega$, it is appropriate to first combine this block with the mass-spring block. The root-locus for this combination is shown in Figure 3.8. If $B$ is large enough, then the RHP pole can be moved close to the origin and the LHP pole moves completely out of the picture. Thus the combination effectively yields a single closed-loop pole near the origin. If the control block is now combined with this pole, the root-locus diagram is as shown in Figure 3.9. This drawing suggests that there exists a combination of parameters which will yield a stable configuration, at least over a narrow range of spin speed. This conclusion has been verified by numerical calculation.

![Figure 3.7 Block Diagram for Stator Damping.](image)

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Having demonstrated the radial stability for the damped tuned-circuit configuration, the cross-axis stability must be considered. The block diagram for this mode is shown in Figure 3.10. The axial symmetry necessary to decouple the modes is again assumed, as is a fixed stator position.
Figure 3.10 Block Diagram for Cross-Axis Mode with Damping.

Unfortunately, the gyroscopic effect causes $\Omega$ to appear in the mass-spring block. The movement of the two mass-spring poles as a function of rotor speed is shown in Figure 3.11.

Figure 3.11 Gyroscopic Effect on Mass-Spring Poles.

For speeds fast enough for the gyroscopic effect to dominate, both of the poles lie on the imaginary axis and are approximately given by

$$s = j\Omega \frac{J_p}{J_d}; \quad j\frac{T_m}{\Omega J_p}$$  \hspace{1cm} (3.13)
When this block is acted on by the stator damping, the root-locus of Figure 3.12 results. For large values of damping, one closed-loop pole approaches the origin from the RHP while the other moves far into the LHP. This results in a single important pole which is located in essentially the same position as for the radial mode. Therefore the same parameters which stabilize the radial mode can stabilize the cross-axis mode.

![Root Locus Diagram]

Figure 3.12 Root Locus for Stator Damping of Cross-Axis Mode.

A situation has been described which appears to allow the stable magnetic levitation of a rotating body using only passive elements. Although very interesting from a theoretical standpoint, this configuration can only operate over a limited speed range and probably cannot be designed with existing materials to work in a normal-gravity environment.

Many modifications and combinations of the passive configurations presented in this section have been examined, but none have proven practical for free magnetic suspension. These techniques might prove useful in other situations where damping by physical contact is allowable. For magnetic bearings, however, the addition of just a few active
components can greatly improve the performance of the system. Obtaining this improvement without a major sacrifice in reliability is the topic of Chapter 4.

3.2 EXPERIMENTAL INVESTIGATION

Early in the thesis research, before the dynamic aspects of the magnetic interaction were fully understood, experiments were performed to measure the radial magnetic stiffness as a function of speed for the resistive-wall approach, with the expectation that at high speeds a net positive stiffness might result. Although unsuccessful in this regard, the data collected does provide a useful test of the theory presented in Chapter 2.

A finite-element computer simulation of the experiment was generated in an attempt to understand the observed behavior. The results of this program played an important role in the development of the analytical model presented in section 2.4, and are presented here as evidence in support of that model.

3.2.1 Apparatus

A benchtop-scale experiment was constructed to test the prediction of the rotating RL theory that the static magnetic stiffness between the rotor and stator should approach zero as the speed increases. A general plan of the apparatus is illustrated in Figure 3.13. The hardware can be conveniently described in three parts: the rotor, the stator, and the drive motor.

The rotor consists of a stack of silicon steel laminations clamped by two #1020 steel plates which form the teeth for providing an axial lift force. Attached to the inner surface of these teeth are 1.6mm-thick copper
Figure 3.13 Experimental Test Apparatus.

**Rotor**
- Mass: 45 kg
- Rotational Inertia: 0.63 kg m²
- Slow-down time constant in air: 600 sec.

**Stator**
- Mass: 11.7 kg
- Coil: 1225 turns, #19 wire
- Pendulum Length: 2.39 m
- Pendulum Frequency: 0.33 Hz
rings which form a true resistive wall. These rings were formed by rolling a copper sheet and welding the seam with silver solder. The rotor is held above a steel support plate by three bolts. The support plate has two ball bearings at its center which provide both radial and thrust support while allowing free rotation of the rotor structure.

The stator is hung from chains attached to the ceiling so that it is suspended in the center of the rotor and free to move only in one radial direction. The steel stator core is wound to create an electromagnet which generates a homopolar equilibrium field in both the upper and lower air gap. For all of the measurements presented here, the stator was positioned so that its teeth were aligned with the rotor teeth, making $\gamma = 0$. The magnetic gap width between teeth ($g$) is 5 mm, and the tooth width ($t$) is 1.3 cm.

The magnetic stiffness in the $x$ direction was measured in a novel fashion by comparing the pendulum frequency of the hanging stator with and without current applied to the electromagnet. The stiffness can be computed using

$$K_{xx} = \text{Re}\{K(s = 0)\} = (\omega^2 - \omega_0^2)M_g$$  \hspace{1cm} (3.14)

where $\omega$ is the radian frequency with current applied.

$\omega_0$ is the radian frequency without current.

$M_g$ is the equivalent suspended point-mass of the stator and chains (10.9kg).

In order to spin the rotor to various speeds, a pancake-type induction motor was built onto the steel base plate, using a sheet of aluminum attached to the rotor support plate as the induction element. The motor has three phases, six poles, and is designed for 60 Hz operation. A diagram of the motor's construction is shown in Figure 3.14, and the performance
Winding Diagram

A phase

B phase

C phase

18 steel pole-pieces

Radius (cm)

15 10 5 0 5 10 15

Side View

1.5 mm air gap

6 mm steel plate

6 mm aluminum plate

Windings

1.3 cm steel pole-pieces

1.3 cm steel base plate

Figure 3.14 Induction Motor Pictoral.
specifications are given in Figure 3.15. By making the rotor circuit lossy, variable speed control is possible by adjusting the voltage applied to the windings. Although not very efficient, the motor torque is adequate to drive the rotor over the full speed range.

3.2.2 Static Stiffness-Speed Characteristic

The apparatus is designed to measure only the static magnetic stiffness component in the direction of motion, $K_{xx}$. As a test of the theory presented in Chapter 2, the dependence of $K_{xx}$ on the rotational speed, $\Omega$, will first be predicted analytically and then these results will be compared to the observed dependence.

Equation 2.106 provides the boundary condition for a continuous sheet of conductor. Setting $s=0$ leaves

$$\tilde{\psi} = \frac{-j\pi \Omega \tilde{\phi}}{R_E - j\Omega L_E + 2\pi^2 [R_o - j\Omega L_o]} \quad (3.15)$$

In order to estimate the impedance terms it is necessary to conceptually separate the copper ring into an air-gap part and two end-ring parts. A reasonable approximation is to let the 1.3 cm of the ring which covers the pole-piece be the air-gap part which experiences axial current flow, and let the remaining 4.3 cm on each side act as end rings which support azimuthal current flow. The radius is 7.0 cm and the ring's thickness is 1.6 mm so that the two resistance values become

$$R_o = \frac{\rho}{\sigma A} = \frac{0.013}{(5.8 \times 10^{-7})(2\pi)(.07)(1.6 \times 10^{-3})} = .32 \ \mu\Omega \quad (3.16)$$

$$R_E = \frac{\rho}{\sigma A} = \frac{2\pi(.07)}{(5.8 \times 10^{-7})(.043)(1.6 \times 10^{-3})} = 110 \ \mu\Omega$$
Rotor

Conductor: 6061 Solid Aluminum Plate

Iron Backing: 1020 steel plate

Effective Q at 60 Hz: 3.0

Stator

Poles: 6  Phases: 3

Windings: 150 turns #24 wire each phase

Mutual Inductance: 30 mH per phase

Stator Leakage Inductance: 20 mH per phase

Losses at Peak Torque (30% slip)

Stator Copper 58%

Rotor 11%

Iron 5%

Net mechanical Power 26%

![Torque-Speed Graph](image1)

65 volts

![Torque-Speed Graph](image2)

2 amps per phase

Figure 3.15 Motor-Drive Specifications.
A calculation of the end-ring leakage inductance is based on the approximate flux plot in Figure 3.16. Those flux lines which fail to link both the rotor and stator poles constitute end-ring leakage.

![End-Ring Leakage Field](image)

Figure 3.16 End-Ring Leakage Field.

Assuming the curved paths near the ends are a negligible portion of the total path length, the inductance per radian is roughly

$$L_E = \frac{\mu_0 R}{2 e_E} = 5.1 \text{ nH}$$  \hspace{1cm} (3.17)

where $e_E$ is the height of the end-ring.

$R$ is the radius of the ring.

g is the gap width.

$L_o$ approaches zero in the narrow-gap limit when there are no slots, and is ignored in this treatment. The effective impedance of the rotor rings becomes, in terms of the notation in equation 3.1,

$$\mathcal{R} = 37 \mu \Omega/\text{rad}$$  \hspace{1cm} (3.18)

$$L_e = 1.6 \text{ nH/\text{rad}}$$
Using the formulas given in section 2.4, the effective tooth-width factors for this arrangement are

\[
\begin{bmatrix}
\frac{t}{g}
\end{bmatrix}_d = \frac{t}{g} = 2.6
\]

\[
\begin{bmatrix}
\frac{t}{g}
\end{bmatrix}_c = \frac{t}{g} + \frac{2}{\pi} \ln \left( 1 + \frac{\pi r_{\text{max}}}{g} \right) = 4.3
\] (3.19)

\[
\begin{bmatrix}
\frac{t}{g}
\end{bmatrix}_d = \frac{t}{g} = 2.6
\]

\[
\begin{bmatrix}
\frac{t}{g}
\end{bmatrix}_c = \frac{t}{g} + \frac{2}{\pi} = 3.2
\]

A value of 2 cm is used for \( r_{\text{max}} \) since this is the point at which the flux links only half of the copper rings. Using these values with equations 3.4, 3.6, and Figure 3.2,

\[
L_g = 0.36 \ \mu \text{H/rad}
\]

\[
\tau_R = 9.7 \ \text{ms}
\]

(3.20)

\[
\alpha = 0.76
\]

\[
K_m = 22 \ \text{N/m} \quad \text{at} \quad B_o = 5 \ \text{mT}
\]

For this particular situation, the rotor leakage inductance is completely negligible compared to the air-gap inductance, and has no effect on \( \alpha \) or \( \tau_R \).

The complex dynamic magnetic stiffness is given by substituting equation 3.7 into equation 3.6 and recognizing the definitions of \( \alpha \) and \( \tau_R \) from Figure 3.2,
\[ K(s) = -K_m \left\{ 1 - \frac{\alpha [s - j\Omega] \tau_R}{1 + [s - j\Omega] \tau_R} \right\} \quad (3.21) \]

The quantity measured by the apparatus, \( K_{xx} \), is given by

\[ K_{xx} = \text{Re}(K(s = 0)) \quad (3.22) \]

or

\[ K_{xx} = -K_m \left\{ 1 - \frac{\alpha [\Omega \tau_R]^2}{1 + [\Omega \tau_R]^2} \right\} \quad (3.23) \]

Since \( K_m, \alpha, \) and \( \tau_R \) have all been determined analytically, the dependence of \( K_{xx} \) on \( \Omega \) can be predicted.

The actual value of \( K_{xx} \) was determined experimentally using the pendulum-frequency method mentioned previously. The motor was adjusted to run the rotor at or slightly above the desired speed for each measurement. The speed was monitored using a stroboscopic tachometer, and drag was applied with a finger to maintain the correct speed. While the integrated speed error using this method is limited only by the accuracy of the tachometer, short-term fluctuations of up to 0.5 Hz were common.

Once the rotor speed was stabilized, the hanging stator was tapped with the free hand so as to initiate pendulum motion having a radial-displacement amplitude of around 1 mm. Great care was needed to avoid striking the stator too hard and causing it to contact the spinning rotor. When contact did occur the results were usually not destructive, but were quite noisy.

With the stator in motion, ten complete periods of oscillation were timed using an accurate stopwatch. This process was repeated ten times at
each speed, both with and without the electromagnet energized. The results of the experiment are plotted in Figure 3.17. The size of the data circles corresponds to a 90% statistical confidence interval based on an assumed normal distribution in the data. This data was taken with a 5 mT peak magnetic field in the air gap. Also shown is the curve predicted analytically for this same flux density.

![Graph showing experimental data and theoretical predictions.]

Experimental Data
Direct Theory
\[ \tau_R = 9.7 \text{ ms} \]
\[ \alpha = 0.76 \]
\[ K_m = 22 \text{ N/m} \]

Adjusted Theory
\[ \tau_R = 10.7 \text{ ms} \]
\[ \alpha = 0.84 \]
\[ K_m = 24 \text{ N/m} \]

Figure 3.17 Magnetic Stiffness Theory and Experiment.

Considering the number of approximations in the air-gap flux pattern which went into the analytical model, the resulting close correspondence is very encouraging. In fact, adjusting each of the parameters by only 10% yields
the dotted line shown, which almost exactly matches the data. The conclusion is that the accuracy to be expected from the model presented in Chapter 2 is about 10%.

3.2.3 Finite-Element Computer Simulation

When a radial displacement of the rotor occurs, the resulting configuration is no longer symmetric about the axis of rotation. This suggests that a computer solution of the magnetic fields must consider the complete three-dimensional space. Three-dimensional magnetic field solutions by computer are a formidable task even in simple cases, and are essentially impossible given existing programs when induced currents must be considered. Fortunately, the equivalent surface potential description for small displacements presented in Chapter 2 can be used to simplify the problem. With this description, the material boundaries become axisymmetric (no \( \theta \) dependence) and the magnetic field becomes axiperiodic (periodic in \( \theta \)).

A finite-element computer implementation which solves axiperiodic problems in the \( r-z \) plane, including induced currents, was written by J. Mallick of MIT in 1979.\(^3\) Using this program, the equilibrium magnetic field is solved by considering the case where \( p=0 \) and \( \omega=0 \). The driving source for this field is the electromagnet winding. Next the perturbation field is solved using \( p=1 \) and \( \omega=\Omega \). The driving source for this field is the magnetic potential on the stator surface given by equation 2.59. The induced currents in the rotor copper ring are self-consistently calculated as a part of the solution.

Additional analysis programs have been written which take the magnetic field solution and use the Maxwell stress tensor to find the radial stiffness. By running the entire series of programs for several different
values of $\Omega$, the stiffness-speed characteristic can be generated.

The computer analysis was applied to the experimental apparatus as depicted in Figure 3.13. The finite-element mesh which was used is shown in Figure 3.18. Only half of the system's cross-section needs consideration due to the symmetry about the plane $z=0$. The radial fields which determine the stiffness at zero speed are shown in Figure 3.19. These plots give the radial magnetic field as a function of axial position for the radius which corresponds to the middle of the air gap, 6.75 cm. The magnitude of the fields shown is much larger than actually used in the experiment, but the correspondence is simply a linear scaling. Figure 3.20 shows the perturbation field profile for two different rotor speeds. The equilibrium field is the same for all speeds.

The most important result of the computer analysis can be obtained by comparing the perturbation field at high speed to zero speed. The zero-speed field is essentially uniform over the face of the poles, and zero elsewhere. As speed increases there is a reaction field generated by the currents induced in the copper sheet. This field decreases the flux over the face of the poles, as expected. It also creates a fringing field which is represented by the negative areas to either side of the pole face in Figure 3.20. This fringing flux makes the effective tooth width larger for the current-produced field than for the displacement-produced field, resulting in a value of $\alpha$ less than unity.

Figure 3.21 shows the computed radial stiffness, along with the experimental data and the analytical prediction based on the modified parameter values corresponding to the dotted line in Figure 3.17. The computer acceptably matches the data at low speeds, and confirms the projection of the analytical curve out to high speeds.
Figure 3.19  Zero-Speed Mid-Gap Radial Magnetic Field Profile  
\( B_o = 0.6 \, \text{T, } \xi = 1 \, \text{mm} \)
Figure 3.20  Mid-Gap Radial Magnetic Field Profile
\( (B_0 = 0.6 \, \text{T}, \xi = 1 \, \text{mm}) \)
Figure 3.21 Speed Dependence of Magnetic Stiffness.

$K_{xx}$ (N/m at $B_0=5$ mT)

- Experimental Data
- Computer Simulation

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Theory (with parameters adjusted to match data):

$\tau_R = 10.7$ ms, $\alpha = 0.84$, $K_m = 24$ N/m
CHAPTER 4

VELOCITY FEEDBACK CONTROL

This chapter investigates the stabilization of a homopolar radial magnetic bearing through the use of an active feedback control circuit which uses only coils as input and output transducers. The reliability of these elements is combined with a circuit of minimum complexity to yield a reliable, low-power system.

The "Virtually-Zero-Power" system was introduced in Chapter 1 as an example of a control concept for magnetic levitation which requires only a velocity input. Although many VZP systems derive their velocity information by differentiation of a position signal, this same information can be obtained directly from the voltage induced in a coil moving through a static magnetic field. VZP control has previously been applied only to axial-gap systems having one controlled dimension. A generalization of this concept to radial-gap control dynamics, known here as "Velocity-Feedback-Control" (VFC), can be applied in either the stationary or rotating frame of reference. The fundamental performance of VFC and the relative merits of placing the control system on the rotor or the stator are considered in section 4.1.

A flywheel system which utilizes a control system in the rotating frame of reference has been built and successfully stabilized using velocity feedback. The control system requires so little power that it can be operated from 9-volt transistor-radio batteries. Incorporation of a low-power generator on the rotor could provide this power for as long as the flywheel remained spinning, creating a truly self-stabilizing magnetic suspension system.
4.1 PROPERTIES OF AN IDEAL VFC SYSTEM

Given the desirability of velocity feedback, this section considers the design of the velocity-fed control system needed to stabilize a radial-gap magnetic bearing. The simplest form of the solution for zero-speed operation is found to be identical to the VZP concept used in one-dimensional systems. Using techniques from Chapter 2 for analyzing the dynamics of rotating magnetic structures, the basic VFC concept is then applied to radial control both from the stationary and rotating reference frames.

4.1.1 Windings as Transducers

To insure simplicity in the design of the VFC system, only single-input, single-output, time-invariant control will be considered. Consequently, it is appropriate to employ classical Laplacian control theory and its extension to symmetric two-dimensional situations using the complex notation presented in Chapter 2. This section applies these techniques to find the transfer-function dependence for windings used as input and output transducers.

Separate, noninteracting coils are assumed for the input velocity sensing and the output force-producing current. This requires that the sensor and actuator windings be located in separate magnetic gaps. This allows the effect of each can be analyzed independently. In the development which follows it is assumed that the windings are located on the rotor, spinning at angular velocity \( \Omega \). For stator windings simply set \( \Omega = 0 \) in the expressions.

First consider the sensor windings. The voltage induced in a particular winding is given by equation 2.86, specialized here for \( p=1; \)
\[ V(s) = \text{Re}\{2N \frac{k_{\omega}}{s} \tilde{\phi}(s + j\Omega)\} \bigg|_{s \text{ real}} \quad (4.1) \]

Let there be two identical sensor windings displaced from each other by 90 degrees. Label one winding as "x" and the other as "y". A complex voltage can then be defined in the spirit of Chapter 2,

\[ \tilde{V}(s) = V_x(s) + jV_y(s) \quad (4.2) \]

where now

\[ \tilde{V}(s) = 2Nk_{\omega} s \tilde{\phi}(s + j\Omega) \quad (4.3) \]

or equivalently,

\[ \tilde{V}(s - j\Omega) = 2Nk_{\omega}[s - j\Omega] \tilde{\phi}(s) \quad (4.4) \]

In an ideal situation, the only flux linking the sensor windings is the perturbation flux due to displacements. This assumes that the sensor winding is terminated with a large impedance so that no current flows in the coil, and that extraneous magnetic fields have been excluded. Under these conditions the displacement flux from equation 2.109 can be inserted in equation 4.4 to yield

\[ \tilde{V}(s - j\Omega) = 2N \frac{k_{\omega}}{s} \frac{R_B}{b} \left[ \frac{t}{g} \right]' [s - j\Omega] \tilde{\xi}(s) \quad (4.5) \]

This relationship is of the general form

\[ \tilde{V}(s - j\Omega) = A_V[s - j\Omega] \tilde{\xi}(s) \quad (4.6) \]

where \( A_V \) is the sensor-voltage gain constant (volt-seconds/meter). If the coordinate origin \( \theta = 0 \) is selected to correspond to the magnetic axis of the x-coil at time \( t = 0 \), then \( k_{\omega} \) and hence \( A_V \) will be real numbers. As expected, the induced voltage is seen to be proportional to the effective radial velocity of the coil with respect to the fixed frame of reference.

Now consider the actuator windings. The contribution to the dipole magnetic surface potential by a single winding carrying a current \( I \) is,
from equation 2.87,

\[
\tilde{\psi}(s + j\Omega) = \pm \frac{2N}{\pi} \frac{k^*}{w} \tilde{I}(s)
\]  

(4.7)

where the upper sign is for windings located on the inner member. Let there be two identical orthogonal windings having the same azimuthal orientation as the \(x\) and \(y\) sensor windings. A complex current can then be defined as

\[
\tilde{I}(s) = I_x(s) + jI_y(s)
\]  

(4.8)

so that

\[
\tilde{\psi}(s + j\Omega) = \pm \frac{2N}{\pi} \frac{k^*}{w} \tilde{I}(s)
\]  

(4.9)

or equivalently,

\[
\tilde{\psi}(s) = \pm \frac{2N}{\pi} \frac{k^*}{w} \tilde{I}(s - j\Omega)
\]  

(4.10)

In an ideal situation where the permeability of the iron poles is effectively infinite and there are no induced eddy currents, the force resulting from the surface potential given above follows from insertion into equation 2.118,

\[
\tilde{f}(s) = 2NRB_o \left[ \frac{t}{g} \right]_c \frac{k^*}{w} \tilde{I}(s)
\]  

(4.11)

This has the general form

\[
\tilde{f}(s) = A_I \tilde{I}(s)
\]  

(4.12)

where \(A_I\) is the force-current gain constant (Newtons/Ampere), which will be a real value under the same conditions that make \(A_V\) real.

4.1.2 Controller Design for Zero Speed

To facilitate development of the VFC system, it is important that the controller provide stable levitation at zero speed. This is especially true
when the control system is mounted on the rotor, as it is difficult to
debug and optimize the electronics if they must be spinning in order to
stabilize the system. This section examines the issues associated with
stabilization of the radial mode at zero speed, given ideal system
conditions. This provides a base from which to embark on more complicated
analyses.

Only the radial mode is considered here explicitly. At zero speed the
cross-axis mode differs only in the effective mass and spring values. The
symmetry necessary to decouple the radial and cross-axis modes is assumed
to exist. Note that since separate gaps are needed for the sensor and
actuator windings, axial symmetry about z=0 requires a minimum of two
complete two-dimensional control systems.

A block-diagram representation of the control problem is shown in
Figure 4.1. A fixed stator is assumed, with levitated rotor mass \( M \) and
intrinsic magnetic stiffness \(-K_m\). The dynamics associated with the sensor
and actuator windings are counted as a part of the forward-path "plant"
since the use of coils as transducers is not subject to modification in
this treatment. The control system then takes the form of a generalized
complex admittance \( Y(s) \) which generates currents in the two actuators based
on the voltages from the two sensor coils.

![Block Diagram](image)

**Figure 4.1 Zero-Speed VFC Block Diagram.**
If the $x$ and $y$ actuators are independently controlled from their respective sensors, then the feedback function will be a real function of $s$ at zero speed. Since interconnecting the two axes introduces a complication which is not necessary for stability, only the case of independent axes of control will be considered further.

The root locations for the plant portion of the system shown in Figure 4.1 are shown in Figure 4.2.

![Figure 4.2 Zero-Speed Plant Poles and Zeros.](image)

The root locus for the closed-loop system can be constructed by adding to this diagram the poles and zeros of $Y(s)$ and solving for the 180-degree points in the plane. An inspection of the diagram shows directly that there is no control function $Y(s)$ which can yield a stable closed-loop system if the control function must have only left-half-plane (LHP) poles. The only way to put all of the closed-loop poles in the left-half-plane is to include an odd number of right-half-plane (RHP) poles in $Y(s)$, and a corresponding zero in the LHP for each one. Clearly the simplest approach is a single RHP pole and LHP zero, both on the real axis. This approach, in terms of its root locus, is shown in Figure 4.3. This control concept is identical to the VZP system first proposed by J. Lyman of Cambridge
Thermionic. 3

![Root Locus Diagram](image)

Figure 4.3 Root Locus for VZP Control.

The control admittance can be written in the form

$$Y(s) = \frac{A_C(1 + s\tau_C)}{\frac{R_a}{1 - s\tau_1}}$$

where $R_a$ is the resistance of an actuator windings.

$A_C$ is the controller gain constant (dimensionless).

$\tau_C$ is the lead-compensation time constant.

$\tau_1$ is the time constant of controller instability.

From the Hurwitz test for stability 4 it can be shown that there are two necessary conditions on the control parameters for obtaining a stable configuration,

$$\frac{A_C A_v A_I}{R a K m \tau_1} > 1$$

$$(4.14)$$

$$(4.14)$$

$$\frac{A_C A_v A_I \tau_C}{R a M} > 1$$

Provided that $\tau_C$ is selected to locate the LHP zero in the general vicinity of the LHP pole, and the RHP controller pole is located between the origin
and the rightmost pole, then the second condition will always be satisfied if the first one is. This first condition corresponds to a statement that the loop gain must be greater than 1 for frequencies between $1/\tau_1$ and $1/\tau_c$. Consequently this term will be referred to as the "mid-band gain", $A_{MB}$. This gain is seen to be the ratio of a part which depends only on the plant (including windings) and a part which depends only on the controller,

$$A_{MB} = \frac{A_v \tau_p}{\tau_1} \quad (4.15)$$

where

$$\tau_p = \frac{A_v A_I}{R K a_m} \quad (4.16)$$

is the effective time-constant of the plant. Making $A_{MB}>1$ provides an essential guide for the initial selection of system parameters.

4.1.3 Effects of Rotation

The same control design which was developed for zero-speed stabilization can be applied to radial control. While this probably does not represent an optimum strategy at operating speeds, it is adequate at zero speed and should operate effectively under rotating conditions at least for low speeds. Such a situation is particularly attractive from an experimental point of view, since stability at zero speed is certain to be the first stage of development.

There are two major new issues which arise in the application of VFC to rotating systems. One is the impact of placing the control system and windings in the rotating frame of reference and the other is the presence of the gyroscopic effect in the cross-axis dynamics. Thus there are four separate control problems to be considered, corresponding to radial and
cross-axis stability as controlled from either the stationary or rotating frame. The salient features of each will be examined for the ideal case so as to concentrate on the fundamental performance of this type of velocity-feedback control. As a standard basis of comparison, the following values are assumed for all four configurations:

\[
\omega_0 = 15 \text{ Hz} \\
1/\tau_i = 1 \text{ Hz} \\
1/\tau_c = 10 \text{ Hz} \\
\Omega/2\pi = 50 \text{ Hz}
\]

\[A_{MB} = 10\]

The block diagram for VFC of the radial mode is shown in Figure 4.4. This diagram is based on the transducer relations from section 4.1.1, and reduces to Figure 4.1 for windings located on the stator or for rotor windings at zero speed. Figure 4.5 shows the root locus for this mode for both a stationary or rotating controller.

\[s' = \begin{cases} s-j\Omega & \text{(rotor)} \\ s & \text{(stator)} \end{cases}\]

Figure 4.4 Ideal VFC System Block Diagram: Radial Mode.

The stationary controller is unaffected by spin and maintains the same dynamics as at zero speed. The rotating controller, however, goes unstable at 42 Hz for a mid-band gain of 10. Increasing the gain to 100 does extend the stable speed range to 145 Hz.
The block diagram for VFC of the cross-axis mode is shown in Figure 4.6. Note that \( \Omega \) appears in the gyroscopic term even if the controller is stationary.

A speed of 50 Hz is fast enough so that for ratios of \( J_p/J_d \) on the order of unity both of the plant poles lie on the positive imaginary axis. The resulting root-locus plot is shown in Figure 4.7 for a stationary controller. This mode is stable for all speeds. Note that by actively
controlling this mode the tendency for mechanical hysteresis to nudge the gyroscopic poles into the RHP is thwarted.

\[ \text{Closed-Loop Poles} \]
\[ \text{Speed} = 50 \, \text{Hz} \]

Figure 4.7 Ideal VFC Root Locus: Stationary-Frame Control of Cross Axis.

The situation for a rotating controller is somewhat different. Figure 4.8 shows the root-locus for a situation where \( J_p/J_d > 1 \), as is usually recommended to avoid whirl instabilities. This mode is unstable for all speeds above the point where the gyroscopic poles meet at the imaginary axis. In this particular case that speed is only 33 Hz. The instability occurs regardless of the amount of gain employed.

\[ \text{Closed-Loop Poles} \]
\[ \text{Speed} = 50 \, \text{Hz} \]
\[ J_p/J_d = 1.2 \]

Figure 4.8 Ideal VFC Root Locus: Rotating-Frame Control of Cross Axis.

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Interestingly, the situation is greatly improved if \( J_p/J_d < 1 \) is allowed, as shown in Figure 4.9. For this case, the cross-axis mode remains stable up to 93 Hz for a mid-band gain of 10, and beyond 200 Hz for a gain of 100.

![Diagram](image)

- Closed-Loop Pole Locations
- Speed = 50 Hz
- \( J_p/J_d = 0.8 \)

Figure 4.9 Ideal VFC Root Locus: Rotating Frame Control of Cross Axis.

From a stability standpoint, it is clear that a stationary controller is superior to a rotating controller. There are no speed restrictions with control from the stator and for modest loop gains the system dynamics remain unaffected by speed, even in the presence of the gyroscopic effect. However, rotating control has the compelling advantage that it does not dissipate any power as a result of rotor imbalance. This is apparent from the fact that rotor imbalance corresponds to a steady-state signal injection having \( s=j\Omega \). The sensor-voltage block yields a zero output at this frequency, so there is no input signal to the control system. Since rotor imbalance is the chief source of dynamic losses in most magnetic bearing systems, this is an important result.

Both rotor and stator control of a radial VFC magnetic suspension have their advantages. With adequate gain, however, the rotating control
appears to offer an adequate speed range while promising greatly reduced control losses. The experiment described in section 4.2 represents an attempt to determine the feasibility of this option under non-ideal conditions.

4.1.4 **One-Dimensional Implementation**

Since the zero-speed VFC system is closely related to the one-dimensional VZF system, a simple one-dimensional experiment was constructed to test the viability of this scheme and explore potential complications. A pictorial view of the levitation apparatus is shown in Figure 4.10. The system is composed of a DC electromagnet for providing lift, a long suspended steel member, and a much weaker DC electromagnet which provides flux through the sensor coil. Stability in the radial plane is obtained passively through the attraction of the suspended poles for the electromagnets. Cross-axis stability results from gravity acting on the suspended member's low center of mass made possible by the large separation between magnet poles. This separation also insures that the sensor and actuator coils are magnetically independent. The control coils are wound on the suspended member to better simulate a control system mounted to a levitated rotor, but they should work just as well if wound onto the stationary magnet cores.

The block diagram for this system is identical to that in Figure 4.1 except that all variables are real functions of \( s \). The actual values of the plant parameters must be determined differently than for the radial case, however. These values can be determined analytically using the dimensions shown in Figure 4.10.
Figure 4.10 Velocity-Feedback Single-Axis Test Apparatus.
First consider the intrinsic magnetic stiffness, $K_m$. The lift force is proportional to the square of the flux density in the air gap,

$$ f_z = \frac{B^2}{\mu_0} \text{ gap} $$

(4.17)

Since the lift magnet carries a constant DC current, the flux density is inversely proportional to the gap width,

$$ B = \frac{\mu_0 N I}{g - \xi} $$

(4.18)

where $\xi$ is the vertical displacement of the levitated section from its equilibrium gap width, $g$. Thus the change in the force for a given displacement $\xi$ is

$$ K_m = -\left. \frac{df_z}{d\xi} \right|_{\xi=0} = \frac{2B^2}{\mu_0 g} \frac{A \text{ gap}}{g} = \frac{2f_z}{g} = \frac{2Mg}{g} $$

(4.19)

where $g$ is the acceleration of gravity. For the experiment as illustrated, the stiffness is 3900 N/m, making the natural frequency of the intrinsic magnetic instability

$$ \omega_0 = \sqrt{\frac{K_m}{M}} = \sqrt{\frac{2g}{g}} = 10 \text{ Hz} $$

(4.20)

The sensor-voltage gain constant is found by estimating the flux linking the coil,

$$ \lambda_s = NB \frac{A \text{ gap}}{g} = 2.5 \text{ mV} \cdot \text{s} $$

(4.21)

This flux is inversely proportional to the gap width, $g+\xi$, so that the induced voltage becomes

$$ V_s = s\lambda_s = s\lambda_s (\xi = 0) \frac{\xi}{g} = s[0.5 \text{ V} \cdot \text{s/m}] $$

(4.22)

Therefore $A_V = 0.5 \text{ V} \cdot \text{s/m}$. 

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For the force-current gain factor, it is recognized that the actuator current adds to the equilibrium flux density in the gap an amount equal to $\mu_o N I_a / 2g$. The total force thus becomes

$$f_z = \frac{[B_{go} + \frac{\mu_o N I_a}{2g}]^2}{\mu_o} A_{\text{gap}}$$  \hspace{1cm} (4.23)

This expression contains the equilibrium lift force and a term proportional to the current,

$$f_z = B_{go} A_{\text{gap}} \frac{N I_a}{g} = [5 \text{ N/A}] I_a$$  \hspace{1cm} (4.24)

Therefore $A_I = 5 \text{ N/A}$. Given the location of the two plant poles at $\pm 10$ Hz, frequencies were selected for the controller dynamics at 7.2 Hz for the LHP zero and 0.72 Hz for the RHP pole. A circuit which realizes these dynamics is shown in Figure 4.11. The input RC defines $\tau_c$, while the positive feedback around the integrating op-amp provides the RHP pole. The second op-amp drives the power transistors in a voltage-to-current topology. This circuit has a controller gain constant of $A_C = 100$ and an effective actuator resistance of $R_a = 0.1$ Ohms.

Figure 4.11 Control Circuit for One-Dimensional Experiment.
The mid-band gain for the combination of this controller and the magnetic-suspension plant is, from equation 4.15,

\[
A_{MB} = \frac{A_C A_v A_I}{R a K_m \tau_i} = \frac{(100)(.5)(5)}{(.1)(3900)(.22)} = 2.9
\]  

(4.25)

As suggested by this result, the circuit did produce stable levitation.

Several complicating factors did become apparent while working with this apparatus. Perhaps the most significant is the problem of getting the system into the equilibrium configuration in the first place. Velocity feedback works fine as long as the suspended member is free to move in reaction to changes in the actuator current. During start-up, however, the levitated member is typically clamped to either the lift or sensor magnet. As soon as power is supplied to the control circuit, the output current grows exponentially with the time constant \( \tau_i \). If the direction of instability is such as to push the suspended member away from the clamped position, then free movement is restored and the system locks onto the equilibrium position. On the other hand, if the instability pulls the suspended member tighter against the magnet it is already clamped to, then the current will increase until it is limited by the peak output voltage of the op-amps. If the DC offset in the output of the op-amps is effectively nulled out then it is an even chance which direction the instability will grow. This ambiguity was resolved by intentionally allowing some DC offset in the output. The start up switch (S1 in Figure 4.11) was added to allow for rapid resetting of the controller in the event that latch-up should occur. It was also observed that if the instability grew too rapidly then the levitated member would overshoot the equilibrium point and clamp on the opposite side. This phenomenon places a lower limit on the value of \( \tau_i \).

Another unanticipated problem involves the DC offsets in the op-amps.
The DC gain of the control system is quite large, but the DC gain of the closed-loop system is zero. Thus a DC offset voltage at the input op-amp terminals gets multiplied by the controller DC gain and is consequently applied to the actuator windings. This results in a continuous power dissipation which serves no useful purpose. Unless great care is taken to null these offsets, this DC current can dominate the power expenditure of the system in equilibrium.

A valuable observation regarding this experiment involves the power requirements which can be expected for stabilization systems of the VFC type. From a design standpoint, the power-handling capabilities of the output electronics and actuator windings should be sized to satisfy the demands of the start-up transient, as this is the maximum power surge which can be expected. In sharp contrast, the power requirement under quiescent equilibrium conditions is limited only by external disturbances down to extremely low levels. DC offset, 60 Hz line noise, and vibrations transferred from the floor are each at least an order of magnitude above the random noise background in this experiment. This creates a fundamental problem with using a fixed power supply voltage in the output since in order to handle the peak power requirement, operation under quiescent conditions is extremely inefficient. Only the fact that the power demand is exceptionally small makes this approach viable.

One final unexpected result is worth noting. Over a period of several minutes a vibration would develop in the suspended rod at around 120 Hz. This vibration was the result of a tendency of the control system to reinforce motion which causes the sensor and actuator to move in opposite directions. A flexing of the rod causes just that to occur. Placing electrical tape around the shaft at its mid-point provided enough damping
of this mode to eliminate the vibration.

The construction of the model VFC system successfully demonstrated that an extremely simple controller, coupled only through coils, can both start up and maintain stable equilibrium. Experience gained from the apparatus pointed to several key areas where practical limitations are significant.

4.2 **FLYWHEEL IMPLEMENTATION OF ROTATING-FRAME CONTROL**

The characteristic nature of velocity feedback in the rotating frame to ignore rotor imbalance makes this option particularly interesting for applications where minimizing power losses is essential. This is certainly the case for flywheel energy storage, and this feature is especially important when fiber-composite flywheel materials are proposed, as these designs are subject to a shifting imbalance at high speeds. Rotating-frame control is also the most interesting VFC option from a conceptual point of view, as its analysis requires the most general case of the theory developed in Chapter 2. For these reasons a test model flywheel has been designed and built which incorporates rotating-frame control. After several modifications to the basic control electronics, stable levitation was achieved. Extensive data was collected under zero speed conditions. Stability was maintained up to a speed of 13 Hz when the rotor was accelerated by a built-in induction motor. This section describes the apparatus in detail and develops a model for the system which is capable of accurately predicting the speed at which instability occurs. The improved understanding of VFC made possible by this effort will allow future systems to be designed to offer greatly improved performance.
4.2.1 Apparatus

A pictorial half-section of the complete flywheel experiment is shown in Figure 4.12. The basic configuration is a 16 cm diameter cylindrical rotor made of #1020 steel and outfitted with VFC windings and electronics, spinning around a 10 cm diameter central stator which carries the permanent magnets and induction motor windings. The two sections are separated by a 2 mm air gap. The stator shaft is bolted to an aluminum plate which rests on rubber feet. A hole down the center of the shaft provides a passage way for wires connected to the 3-phase induction motor coils.

Radial needle bearings fit around the top and bottom of the shaft and are each attached to the rotor by three radial spokes. These touch-down bearings are larger in diameter than the shaft by 0.8 mm so that when the rotor is floating there is no contact between the bearings and shaft. When the radial stabilization system is inactive, these bearings maintain a minimum air gap of 1.6 mm, preventing saturation and subsequent magnetization of the steel pole pieces.

The magnetic design of the experiment represents the integration of three major components; the permanent-magnet flux for obtaining axial lift, the VFC stabilization system on the rotor, and the induction motor located at the axial midplane. These components have been combined in a manner which provides the necessary axial symmetry for decoupling the radial and cross-axis modes. Although the design process requires simultaneous consideration of all three components, the operation of each in the final arrangement can be described separately.

The primary aim in the permanent-magnet design is to generate as much lift as possible for a given magnet size. The two factors which are most responsible for successfully obtaining this objective are appropriate
Figure 4.12 Half-Section View of Test Model Flywheel.
matching of the magnet and gap reluctance and minimizing the magnet's leakage flux. Placing the magnet in the center and arranging for it to fill most of the space available forces essentially all of the magnet's flux to pass through the rotor. This results in a considerable improvement in magnet flux efficiency compared to axial-gap magnetic bearings, tending to offset the lower effectiveness of the radial-gap design for providing lift.

The optimum tooth geometry of $t/g=25$ is used in a single-tooth-per-pole configuration which provides the proper reluctance ratio of the magnet to the air gaps of 2 to 1. A fairly large air gap of 2 mm was chosen to allow observable radial displacements. The two Ceramic-V magnets together have a mass of 1.2 kg and can lift 18 kg. This lift ratio of 16:1 could be doubled if a 1 mm gap were used instead. The rotor itself is only 7.4 kg, leaving a substantial margin of reserve. As a result, the actual vertical displacement of the rotor with respect to the stator is only 1.1 mm, 22% of the tooth width. Some of the flux returns to the stator through the motor gap such that the measured peak flux densities are

$$ B_o = \begin{cases} .47 \text{ T in the actuator gaps} \\ .44 \text{ T in the sensor gaps} \\ .12 \text{ T in the motor gap} \end{cases} $$

The mid-section of the system is occupied by a radial-gap induction motor and the power supply for the control electronics. A radial-plane pictorial of this section is shown in Figure 4.13. The inner stator section has 18 slots which provide space for a three-phase 6-pole series-wound stator winding. The induction coil on the rotor is a single series-wound six-pole winding.
Figure 4.13 Motor Section of Test Model Flywheel.

Making both windings series-wound prevents interaction with the \( p=2 \) and \( p=4 \) fields which result from radial displacements (section 2.5). Furthermore, the \( p=3 \) symmetry insures that no coupling will occur with the homopolar flux present in the motor gap. Thus the effect of the induction motor on the positional dynamics is simply to add an additional negative
magnetic stiffness which is proportional to the square of the magnitude of the phasor air-gap flux density. This value is at most .2 T, and so does not have a major effect on the total intrinsic magnetic stiffness of the structure. This motor was included purely as a functional means of spinning the rotor, and does not represent an efficient or desirable propulsion system for flywheel energy storage.

Also mounted in the motor section are the capacitors and associated components which form the power supply for the control electronics. Sixteen of the 24 holes around the periphery are filled with 200μF, 25V electrolytic capacitors. The other eight locations are left clear to allow space for connectors which pass through the outer rotor wall, allowing power to be injected externally. Of the eight connector plugs available, four are used for an external power supply and four are used for externally-mounted batteries. Thus both a supply and batteries can be connected simultaneously, allowing a smooth transition between sources. Figure 4.14 shows the circuit diagram for the power-supply section. Note that the op-amps are protected by the Zener diodes from voltages in excess of ±15V.

![Circuit Diagram](image)

**Figure 4.14 Power Supply Circuit Diagram.**
As was shown in Figure 4.12, the windings for the control system are located with the actuators toward the outside and the sensors on the inside. This was necessitated by a lack of room in the interior for the bulkier actuators. This unfortunately places the sensor windings close to the motor, causing a significant disturbance signal in the control system. The actuator windings are 180 turns of #24 wire for each of the x and y channels, while the sensor windings are 360 turns per channel of #32 wire. There are a total of 24 slots in each rotor pole-piece into which coils are distributed. Heat-shrink tubing is used in the slots to protect against shorts to the steel.

A total of four identical control channels are required for the system, two axes for each bearing. The circuit diagram for the control electronics which were eventually mounted on the rotor is shown in Figure 4.15. This controller includes some dynamics not required in the one-dimensional experiment and which will be described in section 4.2.2. The basic concept remains, however, of a single op-amp to establish the important dynamics and a second one to drive the output power transistors.

Figure 4.15 Control Circuit Diagram.
The output transistors selected are 4A, 60V Darlington transistors which are heat-sinked directly to the rotor steel, though electrically isolated. The rest of the control system is divided between two crescent-shaped circuit boards for each bearing. These boards are sandwiched between the sensor and actuator windings on the rotor. A single quad op-amp is all that is needed for each bearing. The LM124AJ was selected for its low quiescent power consumption (.8mA) and its low input offset voltage (1mV). Individual offset-null resistors were selected for each channel, resulting in a typical DC output voltage of 5 mV across each actuator coil.

The series RC at the input is used to insure that when power is applied, the control circuit will start up by going unstable in the same direction every time. This particular solution to the start-up problem does not introduce any DC offset, and so is a considerable improvement over the one-dimensional experiment. The series RC in the negative feedback loop of the first op-amp determines the frequency of the LHP zero, while the 8.2 MΩ resistor provides the positive feedback necessary to create a RHP pole. The output amplifier is a voltage stage rather than a current driver as before. A voltage gain of 7.2 is built into this stage through its feedback-resistor ratio. This circuit has functioned as predicted and has operated without failure through dozens of trials.

4.2.2 Additions to the Simple Model

The idealized model is still appropriate for describing the fundamental nature of VFC stabilization as implemented in the experimental test apparatus, but several modifications to the analytical description are necessary to describe the observed behavior and predict limitations on performance. All of these modifications are based on interactions which can be predicted once a specific topology as selected, so that application
of the model to other designs is warranted. The complete block diagram for the extended model is shown in Figure 4.16. Each of the new additions is explained in this section.

![Block Diagram](image)

Figure 4.16 Complete VFC Block Diagram: Radial Mode.

One deviation from the idealized theory results from the relative ease with which the actuator winding can induce eddy currents in the solid steel pole pieces. The result is that the flux generated in the air gap, and therefore the radial force, is not proportional to the actuator current at high frequencies. Similarly, the current in the coil is not related to the applied voltage by any simple analytical relationship. However, it has
been found that the air-gap flux is still related to the applied voltage by a single-pole roll-off. Thus the radial force is related to the applied voltage by the same relationship which would be expected if there were no eddy currents;

\[
\tilde{f}(s) = \frac{[A_r/R_a] \tilde{V}_a}{1 + [s - j\Omega]r_a}
\]  
(4.26)

Since the dependence of the force on voltage is well determined, whereas its dependence on current is not, the control system was modified to impose a certain voltage rather than current on the actuator winding. This has two beneficial side-effects; it removes the need for space-consuming current-sensing resistors, and it improves the linearity of the transistor output stage. The additional LHP pole added by this modification is tolerable since it is located at a fairly high frequency (88Hz). This frequency could be raised further at the expense of efficiency by adding series resistance to the winding. The ultimate solution is to use laminated iron poles and return to current control.

Another dynamic effect which is uncovered through the use of voltage rather than current control is the influence of induced voltage in the actuator winding. The same motion which induces a voltage in the sensor winding also creates a "back-EMF" in the actuator. Thus the net effective voltage across the actuator is given by

\[
\tilde{V}_{a-net} = \tilde{V}_a - [s - j\Omega]A_{A} \tilde{\xi}(s)
\]  
(4.27)

where \(A_{A}\) is the actuator-voltage gain constant (V•s/m). It is this net voltage which applies in equation 4.26.

The single most disruptive effect in the test flywheel which was not present in the one-dimensional system is the presence of inductive coupling between the sensor and actuator windings of each channel. Coupling between
different channels was found to be negligible. This "mutual inductance" effect severely limits the amount of controller loop gain which can be employed without causing instability.

The presence of eddy currents affects the mutual-inductance coupling just as it affected the self-inductance of the actuator, only to a different extent. The result is that the voltage induced in the sensor coil due to current in the actuator coil is most accurately described in terms of the net voltage across the actuator,

\[
\tilde{V}_S = -\frac{[s - j\Omega]\tau_2}{1 + [s - j\Omega]\tau_m} \tilde{V}_{a-net}
\] (4.28)

where \(\tau_2\) is the linkage time constant and \(\tau_m\) is the mutual-inductance time constant. Note that without eddy currents, \(\tau_a\) and \(\tau_m\) would be expected to be the same. Instead they differ by a substantial amount, emphasizing the importance of including this modification in the model.

The impact of the mutual inductance coupling was found to be so severe that it made stabilization even at zero speed impossible. Stable levitation was achieved only after modifying the control electronics. The series RC across the 8.2 M\(\Omega\) resistor in Figure 4.15 adds to \(\tilde{V}_S\) a signal which is related to \(\tilde{V}_a\) in the same way as the mutual inductance coupling is related to \(\tilde{V}_{a-net}\). The polarities of the two are reversed, however, so that they tend to cancel each other out. There are two limitations which keep this technique from being more effective than it is. One is that the electrical feedback does not account for the back-EMF effect and the other is that the electrical signal, which is positive feedback, must never exceed the signal it is cancelling or instability will result. This limits the cancellation signal to about 80% of its counterpart, leaving 20% of the undesirable interference.
A side-effect of inserting the mutual-inductance cancellation network is that a LHP pole must be added to the control dynamics to stabilize the electronics against high-frequency parasitic oscillations. The .0022 μF capacitor was added to the circuit to perform this function. Figure 4.17 lists all of the parameters for the electrical control system which was shown in Figure 4.15.

\[
\begin{align*}
\tau_1 &= 59 \text{ ms} \\
\tau_c &= 11 \text{ ms} \\
\tau_F &= .46 \text{ ms} \\
\tau_m &= 1.05 \text{ ms} \\
\tau_b &= .46 \text{ ms}
\end{align*}
\]

\[
\begin{align*}
\tau_a &= 1.8 \text{ ms} \\
R_a &= 4.0 \\
A_C &= 100 \\
am &= .84
\end{align*}
\]

Figure 4.17 Electrical Control Circuit Parameters.

The last major modification which was included is a relaxation of the assumption that the stator is fixed. In the test model flywheel the stator sits on flexible rubber feet and has little more mass than the rotor. Consequently the dynamics of the stator play an important role in the overall system.

Because the stator and its support plate are effectively rigid compared to the compliance of the rubber feet, the simple mass-spring-damper model of the stator used in section 2.6 is appropriate. The block diagram shown in Figure 2.16 is therefore applicable, with \( K_{RS} = 0 \) and \( K_S(s) = K_S + sB_S \). The right-hand side of this diagram can be reduced to yield the form shown in Figure 4.18. In this representation \( \xi \) is the position of the rotor relative to the stator and \( f \) is the radial force of magnetic origin acting between the rotor and stator. The injection point for external forces acting on the rotor is retained to allow an analysis of
rotor imbalance. If the magnetic stiffness $K(s)$ is separated into the intrinsic stiffness, $-K_m$, and the control-system forces, then the block diagram shown previously in Figure 4.16 results.

![Block Diagram with Stator Compliance](image)

Figure 4.18 Block Diagram with Stator Compliance.

An equivalent derivation could be performed for the cross-axis mode, but movements of the support plate in reaction to this type of motion would couple to the radial mode, invalidating the analysis. Fortunately, the support plate is very stiff to twisting movements and remains effectively fixed for this mode. Thus the cross-axis analysis can be performed using the simple mechanical model given in Figure 4.6.

4.2.3 Parameter Estimation

All of the parameters shown in the system block diagram (Figure 4.16) can be both measured and predicted analytically with varying degrees of precision. Only the electromechanical terms will be considered here, as the techniques for determining the electrical parameters given in Figure 4.17 are widely known and available elsewhere. First the analytical values will be determined, then measurements which confirm these calculations will be presented. A complete list of the system parameters
is at the end of this section (Figure 4.21).

The intrinsic radial magnetic stiffness associated with each magnetic gap is found using equation 2.125,

\[ K_{m} = \frac{\pi R B_0^2}{\mu_0} \left[ \frac{t}{g} \right]_d \]  

(4.29)

Using the dimensions given in section 4.2.1 and the effective tooth width from equation 2.117,

\[ [t/g]_d = \begin{cases} 2.2; & \text{magnet gap} \\ 4.7; & \text{motor gap} \end{cases} \]

then the contributions to the magnetic stiffness are

\[ K_m = \begin{cases} 5.9 \text{ N/mm per actuator gap.} \\ 52.6 \text{ N/mm per sensor gap.} \\ 8.4 \text{ N/mm for the motor gap.} \end{cases} \]

The total intrinsic magnetic stiffness is therefore 230 N/mm for the radial mode.

To get the intrinsic cross-axis stiffness, each contribution above must be multiplied by the square of its axial position;

\[ 59.9 \text{kN/m} \cdot (0.049\text{m})^2 = 144 \text{ N} \cdot \text{m/rad per actuator gap.} \]

\[ T_m = \begin{cases} 52.6 \text{kN} \cdot \text{m} \cdot (0.025\text{m})^2 = 33 \text{ N} \cdot \text{m/rad per sensor gap.} \\ 8.4 \text{kN} \cdot \text{m} \cdot (0\text{m})^2 = 0 \text{ N} \cdot \text{m/rad for the motor.} \end{cases} \]

The total cross-axis stiffness from these radial contributions is 350 N·m/rad. In addition, there is a restoring torque due to the local vertical displacements which result from angular deflection, given by \[ T_o = 1/2 \ K_{zz} R^2 \]. The vertical stiffness, \[ K_{zz} \], can be determined from the natural frequency of vertical oscillations of the rotor (13Hz) to be 49 N/mm. Thus \[ T_o = 62 \text{ N} \cdot \text{m/rad and the net cross-axis stiffness is expected to be 290 N} \cdot \text{m/rad.} \]
The sensor-voltage gain constant, $A_V$, is given by equation 4.5,

$$A_V = 2Nk_w RB_o \left[ \frac{t}{g} \right]_d \quad (4.30)$$

The winding coefficient for the sensor windings is derived from equation 2.68 using $p=1$ to yield $Nk_w = 325$. The effective tooth width term is, for $\delta/g = .56$, equal to 2.24. Thus $A_V = 32 \text{ V} \cdot \text{s/m}$ for radial motion. For the cross-axis mode multiply by the moment arm, 2.5 cm, to get $A_V(\text{cross-axis}) = 0.8 \text{ V} \cdot \text{s/rad}$.

In a similar fashion the actuator-voltage gain constant, $A_A$, follows from equation 4.30. For this case, $Nk_w = 133$ and $B_o = .47 \text{ T}$ so that $A_A = 14 \text{ V} \cdot \text{s/m}$ for the radial mode. For the cross-axis mode the moment arm is 4.9 cm, giving $A_A(\text{cross-axis}) = .67 \text{ V} \cdot \text{s/rad}$.

Finally, the current-force constant $A_I$ follows from equation 4.11,

$$A_I = 2Nk^* w RB_o \left[ \frac{t}{g} \right]_c \quad \text{per actuator} \quad (4.31)$$

where $Nk_w = 133$ for the actuator and $[t/g]_c = 2.9$. This gives a value of 18 N/A per actuator or 36 N/A total for the radial mode. For the cross-axis mode, multiplication by the moment arm, 4.9 cm, gives $A_I(\text{cross-axis}) = 1.8 \text{ N} \cdot \text{m/A}$. Both of these numbers are based on the assumption of infinitely-permeable iron poles. In fact, the permeability of the #1020 steel was measured and found to be about $200 \mu$. This fairly low value of $\mu$ necessitates a correction factor to $A_I$. Since $A_I$ depends ultimately on the amount of flux produced in the air gap for a given actuator current, the addition of iron reluctance in the magnetic path will decrease $A_I$ in proportion to the total reluctance. The length of the iron path is 30 cm for 4 mm of air gap. Thus the expected decrease in $A_I$ is by a factor

$$\frac{4 \text{ mm}}{[4 \text{ mm} + (30 \text{ cm}/200)]} = .73$$
Therefore the corrected values for $A_T$ are 26 N/A for the radial mode and 1.3 N·m/A for the cross-axis mode.

The mass of the rotor and stator were measured directly as 7.4 kg and 8.6 kg, respectively. The values of the cross-axis inertia terms, however, must be calculated. Each element of mass should be weighted according to the square of its distance from the axis of rotation. The resulting polar moment of inertia is 0.039 kg m$^2$, while the diametrical moment of inertia is 0.031 kg m$^2$.

Only one of the parameters discussed so far was measured directly, that being $K_m$. An external radial force was applied to the rotor and its displacement with respect to the stator was measured using a micrometer. 70 N of force resulted in a displacement in the opposite direction of 0.28±0.03 mm. Thus the magnetic stiffness is negative and lies in the range 250±40 N/mm. This is in excellent agreement with the predicted value of 230 N/mm.

The two parameters $A_y$ and $A_T$ are difficult to verify separately, but their product can be inferred from the overall gain of the system. In particular, the block diagram in Figure 4.16 can be split conceptually into two parts. The "plant" part gives an output $\tilde{v}_s$ for an input $\tilde{v}_a$. This block includes the mechanical blocks and the transducer windings. The remaining "controller" part gives an output $\tilde{v}_a$ for an input $\tilde{v}_s$ and contains only electronic components with well-defined dynamics. By studying the measured frequency response of the plant, the values of several parameters can be inferred.

The frequency response for the radial mode of the plant is shown in Figure 4.19. This data was generated by injecting a sinusoidal signal into the x channels of both the top and bottom bearings. The magnitude and
Figure 4.19 Flywheel Plant Frequency Response: Radial Mode.
phase of both \( \tilde{V}_a \) and \( \tilde{V}_s \) were measured and their ratio calculated as a function of frequency.

First consider high frequencies. From the block diagram, the plant response as \( s \) approaches infinity should approach the ratio \( \tau_p/\tau_m \). The value observed, .46, is quite close to the value expected based on totally independent measurements of \( \tau_p \) and \( \tau_m \), which predict a ratio of .48.

For very low frequencies, the block diagram predicts that the response will go as

\[
G_p(s \to 0) = s \left( \tau_p - \tau_p \right)
\]

where the plant time constant was introduced in equation 4.16,

\[
\tau_p = \frac{A_V A_I}{R_a K_m}
\]

Using the predicted values for \( A_V \) and \( A_I \), and the confirmed values for \( R_a \) and \( K_m \), the predicted plant time constant is .91 ms for the radial mode. The observed slope at low frequencies of .39 ms suggests that the actual plant time constant is .39 ms + .46 ms = .85 ms. This means that the product of \( A_V \) and \( A_I \) is accurate to within 10%.

The observed response can also be used to determine \( K_S \) and \( B_S \). This was actually done by working analytically backwards from the plant response until only the mechanical response remained. Without detailing the procedure, the response in Figure 4.19 is predicted by the block-diagram model if values are chosen which make \( K_S = 770 \) N/mm and \( B_S = 700 \) N's/m.

The radial-mode response was generated by injecting the same signal into both \( x \) channels. If the signal to one bearing is inverted, however, then it is the cross-axis mode which is excited. The same procedure can be used to obtain the frequency response. This data is shown in Figure 4.20. Note the absence of the pole-zero pair around 50 Hz which was encountered in the radial response. This confirms the assumption that the stator is
Figure 4.20 Flywheel Plant Frequency Response: Cross-Axis Mode.
essentially fixed for cross-axis motions. The response profile does suggest that the natural frequency of the intrinsic stiffness is 13 Hz,

\[ \omega_{\text{cross-axis}} = \frac{T_m}{J_d} = 2\pi [13 \text{ Hz}] \]  (4.34)

This is in fair agreement with the value based on the predicted stiffness and inertia, 15 Hz.

Finally, the low frequency response suggests that the actual value of \( \tau_p \) for cross-axis motion is .99 ms, compared to the predicted value of .91 ms. Again the product of \( A_V \) and \( A_I \) appears to be accurate to within 10%.

The collected parameter values are shown in Figure 4.21 for later reference. Where available, the experimental value has been substituted for the predicted value, although it is reassuring to remember that the discrepancies between the measured and expected values are small.

<table>
<thead>
<tr>
<th>ELECTRICAL</th>
<th>MECHANICAL</th>
<th>ELECTROMECHANICAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_i ) = 59 ms</td>
<td>( M_R = 7.4 \text{ kg} )</td>
<td>( K_m = 250 \text{ N/mm} )</td>
</tr>
<tr>
<td>( \tau_c ) = 11 ms</td>
<td>( M_S = 8.6 \text{ kg} )</td>
<td>( T_m = 200 \text{ N\cdot m/rad} )</td>
</tr>
<tr>
<td>( \tau_s ) = .86 ms</td>
<td>( K_S = 770 \text{ N/mm} )</td>
<td>( A_V = 32 \text{ V\cdot s/m} )</td>
</tr>
<tr>
<td>( \tau_m ) = 1.05 ms</td>
<td>( B_S = 700 \text{ N\cdot s/m} )</td>
<td>( A_{V(CA)} = 0.8 \text{ V\cdot s/rad} )</td>
</tr>
<tr>
<td>( \tau_b ) = .46 ms</td>
<td>( J_d = .030 \text{ kg m}^2 )</td>
<td>( A_A = 14 \text{ V\cdot s/m} )</td>
</tr>
<tr>
<td>( \tau_a ) = 1.8 ms</td>
<td>( J_p = .039 \text{ kg m}^2 )</td>
<td>( A_{A(CA)} = .67 \text{ V\cdot s/rad} )</td>
</tr>
<tr>
<td>( a_m ) = .84</td>
<td></td>
<td>( A_I = 26 \text{ N/A} )</td>
</tr>
<tr>
<td>( R_a ) = 4.0</td>
<td>( \tau_p = .85 \text{ ms} )</td>
<td>( A_{I(CA)} = 1.3 \text{ N\cdot m/A} )</td>
</tr>
<tr>
<td>( A_C ) = 100</td>
<td>( \tau_p(CA) = .99 \text{ ms} )</td>
<td>( K_{zz} = 99 \text{ N/mm} )</td>
</tr>
</tbody>
</table>

Figure 4.21 Test Model Flywheel VFC Parameters.
4.2.4 Zero-Speed Stability

With the rotor not spinning it is possible to make detailed measurements of the control system which are not feasible when it is rotating. Figures 4.19 and 4.20 were obtained in this fashion. This section will examine the frequency response data to evaluate the relative stability of the system at zero speed.

One way to divide the system into plant-feedback form is to use $V_a$ and $V_s$ as the input and output variables. This "plant-controller" division is convenient from a measurement point of view since both $V_a$ and $V_s$ are readily accessible. The block diagram for this conceptual division is shown in Figure 4.22.

![Block Diagram](image)

Figure 4.22 Plant-Controller Division of the VFC System.

The frequency response of the plant has been shown for both the radial and cross-axis modes. The controller part contains only well-defined electronic components, so that its response can be determined analytically with good accuracy. The frequency response of this block is shown in Figure 4.23.
Figure 4.23 Control-Circuit Frequency Response.
The Nyquist criterion for stability, discussed in section 2.5, is based on the combined open-loop response of the system, \( G_p(\omega)H_0(\omega) \). This product of the plant and controller frequency response can be obtained by graphically adding the two curves. Figure 4.24 shows the result for the radial mode.

Since the Nyquist diagram is really just a polar representation of the magnitude and phase shown in this figure, an evaluation of the system's stability can be made directly from this plot, without actually generating the Nyquist contour. This is common practice in the analysis of minimum-phase systems. Since VFC is very definitely a nonminimum phase concept, extra care must be taken. It is helpful to sketch the general topology of the Nyquist contour to determine the number of encirclements of the \(-1\) point, but still use the log-gain and phase plots to obtain accurate measurements of the critical values. Figure 4.25 shows such a sketch for the radial plant-controller response for positive frequencies. The curve for negative frequencies is the reflection about the real axis of the curve shown.

The first step in evaluating the system's stability is to count the number of RHP poles in the open-loop system. In the idealized system, there are two, but as can be determined from Figure 4.23, the addition of the mutual-inductance cancellation network adds two more RHP poles to the controller block. Thus there are a total of four RHP poles in this open-loop representation of the system. There are also four counter-clockwise encirclements of the \(-1\) point in the Nyquist diagram, two for negative frequencies and two for positive frequencies. This confirms the observed fact that this mode is stable.
Figure 4.24  Plant-Controller Open-Loop Frequency Response: Radial Mode.
Figure 4.25 Plant-Controller Nyquist Topology for $\omega > 0$: Radial Mode.

The greatest value of the Nyquist criterion is the ability to estimate relative margins of stability. From Figures 4.24 and 4.25, the smallest phase margin occurs at the 70 Hz unity-gain point, where the phase margin is only 15 degrees. The smallest gain margin is determined by the gain at the 315 Hz 180-degree point, which is 1.5.

This same procedure can be repeated for the cross-axis mode, generating the open-loop frequency response shown in Figure 4.26 and the approximate Nyquist contour of Figure 4.27. As before, the Nyquist contour shown is for positive frequencies only. Thus there are again four counterclockwise encirclements of the -1 point and the stability of this mode is also confirmed.
Figure 4.26 Plant-Controller Open-Loop Frequency Response: Cross Axis Mode.
Figure 4.27  Plant-Controller Nyquist Topology for $\omega>0$: Cross-Axis Mode.

The phase margin for this mode is determined by the phase at the 10 Hz unity-gain point, which is again 15 degrees. The gain margin is set for this mode by the gain at the 8 Hz 180-degree point, which is 1.5. Note that while the radial-mode diagram did not indicate any stability limits for increases in the loop gain, the cross-axis mode has a 180-degree crossing at 12 Hz where the magnitude, now .35, must remain less than 1 for stability.

While the plant-controller division of the system provides useful information about the system as designed, it does not provide much help in setting individual parameter values, since there is so much interaction due to the mutual inductance effect and cancellation network. A consequence of this is that there is no single parameter which can adjust the open-loop gain in the plant-controller system without also changing the loop dynamics. In order to select an appropriate value for the voltage gain of
the controller's output op-amp, $A_o$, it is possible to divide the system as illustrated in Figure 4.28. An external signal is injected into the input of the output op-amp itself, and the signals are measured just after and just before this injection point. The ratio of these voltages gives a complete open-loop frequency response for the system which is conceptually much different from the plant-controller viewpoint. The advantage is that the open-loop gain is directly proportional to the output op-amp gain, which is easily and independently adjustable.

![Block diagram of system partitioning for evaluating op-amp gain.](image)

**Figure 4.28** System Partitioning for Evaluating Op-Amp Gain.

This data was collected for the radial mode, and is shown in Figure 4.29.

The approximate Nyquist contour for this response is shown for both positive and negative frequencies in Figure 4.30. For this division of the system there are only two RHP poles, so stability corresponds to two counter-clockwise encirclements of the -1 point. The smallest phase margin is 30 degrees, at the 20 Hz unity-gain crossover. The margin for gain increase is 1.47 at 70 Hz and the margin for gain decrease is 1.34 at 5.6 Hz. This confirms that the op-amp gain has been set correctly, near the middle of the stable range.
Figure 4.29  Open-Loop Frequency Response for Gain-Set Partitioning.
4.2.5 Effects of Rotation

When the flywheel is levitated and spinning, it is difficult to obtain detailed information on the dynamics of the system. Three observable phenomena were measured, and a comparison of those values with the predictions given by theory do provide some measure of confirmation for the analytical model. The three measurements made were of eddy-current losses, rotational imbalance, and the speed above which the VFC system goes unstable.

The easiest measurement to make on the test flywheel involves increasing the speed until instability ensues. When instability does occur, the rotor sets down on its radial needle bearings and coasts to a smooth stop. In ten separate trials the system went unstable between 13.0 Hz and 13.4 Hz. The system has never exceeded this limit.
Enlisting the aid of a computer program which finds the roots of polynomials having complex coefficients, the closed-loop pole locations were determined both for the complete eighth-order model of the radial mode shown in Figure 4.16, and for the sixth-order cross-axis mode. The parameters used were those in Figure 4.21. This task was performed using yet another division of the system into plant and feedback parts. In the mechanical-plant, electrical-controller view of the system, all of the blocks which contain $s'$ are combined into the electrical feedback part. What remains are the mechanical blocks and $K_m$, which are assigned to the plant. Thus the input to the plant is $\tilde{r}_o$, the current-produced force, and the output is $\tilde{\xi}$. The advantage of this formulation is that the roots of the electrical block need only be solved once, using $\Omega=0$. These roots then simply translate up the imaginary axis by an amount $j\Omega$ when spin is introduced. For the radial mode the mechanical plant is unaffected by spin, making this description especially convenient. For the cross-axis mode the mechanical plant poles move with changing speed due to the gyroscopic term.

Figure 4.31 shows the locations of the roots for the electrical feedback block. Both modes have an additional high-frequency pole and zero on the negative real axis which are not shown. It is interesting to see how this complex system compares to the idealized version in section 4.2.2. The use of voltage rather than current drive of the actuator adds a LHP pole at 88 Hz which corresponds to the L/R time constant of the winding. The other extra pole and the high-frequency zero are both due to an inability to completely cancel out the mutual inductance coupling.
Figure 4.31 Feedback Roots for Electrical-Mechanical Partitioning.

The roots of the mechanical plant are shown in Figure 4.32 for the radial mode. The compliance of the stator has introduced a pair of lightly-damped complex poles and zeros in addition to the usual pair of real poles. This same phenomenon has been observed by others.\(^2,5\) These roots remain fixed when spin is introduced.

![Diagram](image)

Figure 4.32 Mechanical Plant Poles and Zeros: Radial Mode.

Figure 4.33 shows the root-locus diagram for the radial mode at both zero speed and at a forward rotation of 25 Hz. The location of the closed-loop poles as solved by the computer are also shown. Note that an increase
Figure 4.33 Root Locus for Electrical-Mechanical Partitioning
in the loop gain will quickly cause the two outer poles to move into the RHP, even at zero speed. These poles are a problem as a direct result of the phase lag introduced by the mutual inductance effect. It is this effect, then, which limits the loop gain which can be employed. The complex pole-zero pair does not threaten the stability at zero speed, but with increasing speed these roots interfere with the movement of the RHP poles into the LHP. Thus it is this effect which is primarily responsible for limiting the range of stable speeds in this implementation of velocity-feedback control.

Finally, Figure 4.34 shows the location of the closed-loop roots as a function of speed for both modes. Only the critical roots below 30 Hz are shown. The cross-axis mode is projected to go unstable at 13.0 Hz, in complete agreement with the observed behavior.

![Critical Closed-Loop Pole Locations for Increasing Speed.](image)

Figure 4.34 Critical Closed-Loop Pole Locations for Increasing Speed.
The magnitude of the synchronous whirl resulting from rotor imbalance can be predicted analytically using the radial-mode block diagram (Figure 4.16). Just as in the ideal case, the control system does not react to synchronous whirl at all, since for this type of motion, $s'=0$. This considerably simplifies the dynamics, and all that remains is the mechanical plant with the imbalance injected where indicated in the figure.

The imbalance force on the rotor was given in Figure 2.24 as

$$
\hat{f}_{\text{IMBALANCE}} = M_R \Omega^2 \tilde{\xi}_\Delta
$$

(4.35)

Setting $s=j\Omega$ in the remainder of the system gives the radial displacement as a function of rotor speed,

$$
\hat{\xi} = \frac{-\Omega^2 M_R \tilde{\xi}_\Delta [-\Omega^2 M_S + j \Omega B_S + K_S]}{[-\Omega^2 M_S + j \Omega B_S + K_S][\Omega^2 M_R + K_m] - \Omega^2 M_R K_m}
$$

(4.36)

For low speeds the displacement is given by

$$
\hat{\xi} = -\frac{\Omega^2}{\omega_o^2} \tilde{\xi}_\Delta, \quad \text{where} \quad \omega_o^2 = \frac{K_m}{M_R}
$$

(4.37)

while at high speeds,

$$
\hat{\xi} = -\hat{\xi}_\Delta
$$

(4.38)

Thus at high speeds the rotor is allowed to rotate about its own center of mass.

Using the parameters for the test model flywheel, the displacement versus speed is as shown in Figure 4.35. Note that there is no "critical speed" as is encountered in positive-spring rotating systems. In fact, the magnitude of the rotor displacement never exceeds the eccentricity of the center of mass, despite the relatively small amount of damping present in the stator. Except in the immediate vicinity of the stator-compliance
modes, the displacement $\hat{\xi}$ is with 10 degrees of $-\tilde{\xi}_\Delta$.

![Graph showing speed dependence of imbalance displacement.](image)

**Figure 4.35** Speed Dependence of Imbalance Displacement.

The magnitude of the imbalance displacement was measured using a mechanical runout indicator using speeds up to 10 Hz. Figure 4.35 indicates that this was entirely in the quadratic low-speed regime. The data collected is shown in Figure 4.36. The displacement did grow quadratically, and the values imply a value of $\tilde{\xi}_\Delta = .36$ mm. Most of this imbalance is likely due to the batteries strapped to the outside of the flywheel. These measurements also indicate the presence of a .08 mm rotating eccentricity which is independent of speed. This is probably due to a remnant dipole magnetization in the rotor steel.

![Graph showing observed imbalance displacement.](image)

**Figure 4.36** Observed Imbalance Displacement.
The eddy-current losses were measured by observing the speed of rotation versus time for the free-spinning flywheel. For speeds below 5 Hz the speed decreased exponentially with time. This suggests that eddy-current power losses proportional to $\Omega^2$ are dominant over windage in this range. The measured time constant for the slow down is 400 seconds. Since $J_p = 0.039 \text{ kg} \cdot \text{m}^2$, the drag torque is

$$T_{\text{drag}} = J_p / \tau_{\text{Slow-Down}} = 9.5 \cdot 10^{-5} \Omega \text{ N} \cdot \text{m}$$  \hspace{1cm} (4.39)

and the power dissipation is

$$P_{\text{eddy}} = T_{\text{drag}} \cdot \Omega = 9.5 \cdot 10^5 \Omega^2 \text{ Watts}$$  \hspace{1cm} (4.40)

The eddy-current losses due to specific causes can be estimated analytically. Two potential sources will considered; the field variation in the stator iron due to the rotating eccentricity of the rotor, and the field variation in the stator iron due to the presence of slots in the rotor. In both cases it is first necessary to solve at least approximately for the magnetic field in the iron. The induced currents can then be solved by considering the paths the current must take and the flux linked by those paths. Although the test flywheel has solid steel poles, the analysis will include the possibility that the poles could be made from stacks of thin laminations.

The effects of both rotating eccentricity and rotor slots can be evaluated concurrently if the general case of a traveling magnetic wave in the air gap is evaluated. The rotating eccentricity then corresponds to a wave having space-harmonic $p=1$ and frequency $\Omega$, while the presence of rotor slots introduces a wave having $p=n$ and frequency $n\Omega$, where $n$ is the number of slots.

Although the magnetic field in the air gap is not uniform in the z direction, the magnetic field within the steel stator poles will be
essentially uniform over the thickness except within a couple of gap-widths of the edge. The magnetic field solution in the stator pole pieces can thus be solved by assuming uniformity in \( z \) and matching the radial normal flux in the iron to that in the air gap at the outer radius, \( R \). If the inner radius of the pole pieces is small compared to \( R \), then an inner radius of zero yields a good approximation. The magnetic field solution in the iron for this case is

\[
\overline{B}(r, \theta, t) = \text{Re}\{ \hat{B}_p \left[ \frac{\hat{R}}{R} \right]^{p-1} e^{j[\omega t - p \theta]} [\overline{a}_r - j\overline{a}_\theta] \} \quad (4.41)
\]

where \( \hat{B}_p \) is related to the air-gap phasor flux density per radian by

\[
\hat{B}_p = \frac{\hat{\phi}}{Rt} \quad (4.42)
\]

Faraday's law in the iron can be simplified by recognizing that spacial derivatives in the \( z \) direction will dominate over derivatives in the radial plane. This will be especially true if thin laminations are used. The remaining terms are

\[
\frac{\partial E_\theta}{\partial z} = \frac{\partial B_r}{\partial t} \quad ; \quad \frac{\partial E_r}{\partial z} = -\frac{\partial B_\theta}{\partial t} \quad (4.43)
\]

The appropriate boundary condition is that there can be no net current in the radial plane within an individual lamination. Let \( z=0 \) be the axial mid-plane for a given lamination. The induced electric field is therefore

\[
\overline{E} = -\text{Re}\{ \omega \hat{B}_p \left[ \frac{\hat{R}}{R} \right]^{p-1} z e^{j[\omega t - p \theta]} [\overline{a}_r - j\overline{a}_\theta] \} \quad (4.44)
\]

The power dissipation density follows as

\[
P_D = \sigma|\overline{E}|^2 = \sigma \omega^2 \left[ \frac{\hat{R}}{R} \right]^{2p-2} z^2 |\hat{B}_p|^2 \quad (4.45)
\]

This expression is only valid within the given lamination. Averaging over
the thickness d of the lamination yields

\[ \langle P_D \rangle_z = \frac{1}{12} \sigma d^2 \omega^2 \left[ \frac{r}{R} \right]^{2p-2} |\hat{B}_p|^2 \]  \tag{4.46}

This power dissipation can also be averaged over the radius R,

\[ \langle P_D \rangle_{r,z} = \frac{\sigma d^2 \omega^2 |B_p|^2}{12 p} \]  \tag{4.47}

Multiplying this value by the volume of iron in the pole piece gives the total power dissipation in that pole.

First evaluate the effect of rotor eccentricity. For this type of motion \( \omega = \Omega \), \( p = 1 \) and

\[ \hat{B}_p = B_o \hat{\xi} \left[ \frac{t}{g} \right]' \frac{1}{d} \]  \tag{4.48}

The power dissipation in a given pole piece is therefore

\[ P_{eddy} = \frac{1}{12} \sigma \Omega^2 B_o^2 |\hat{\xi}|^2 \left\{ \left[ \frac{t}{g} \right]' \right\}^2 \left[ \frac{d}{t} \right]^2 [\text{Vol.}] \]  \tag{4.49}

Summing the contributions from all five poles in the test model flywheel and using the observed displacement at low speed of .08 mm, the total power loss due to this factor is

\[ P_{eddy} = 3.6 \cdot 10^{-7} \Omega^2 \text{ Watts} \]

where a conductivity of 4.0 \( \cdot 10^6 \) S/m has been used. This factor accounts for only 0.4\% of the observed dissipation.

Now consider the effect of the rotor slots. Direct measurements using a Hall-effect probe give a variation in the field which has a periodicity \( p = 24 \) and a fundamental-component amplitude of \( \hat{B}_p = .05 \text{ T for the magnet poles and } \hat{B}_p = .012 \text{ T for the motor. Let } \omega = 24[\Omega] \text{ and } p = 24 \text{ so that the total losses are} \]

\[ P_{eddy} = 7.3 \cdot 10^{-5} \Omega^2 \text{ Watts} \]
This factor successfully accounts for over 75% of the observed dissipation and certainly is the dominant effect. Some additional loss is likely from eddy currents induced in the motor rotor due to the stator's slots.

The power dissipation due to rotor slots in the test model flywheel is far too great for an energy-storage system. However, these losses can be greatly reduced by laminating the stator magnet poles. The use of standard 0.36 mm (14 mil) laminations would reduce the eddy-current losses by a factor \( [d/t]^2 = 200 \). This would increase the spin-down time constant associated with this effect to 22 hours. The possibility of eliminating the slots by securing the actuator and sensor wires directly to the surface of a smooth rotor deserves consideration.

4.2.6 System Performance

Those aspects of the test model flywheel which are important to the design and modeling of VFC systems in general have been presented in previous sections. There remain a number of unrelated facts which are peculiar to this experiment. They are presented here both as general background for anyone who might attempt to work in this field, and in the unlikely event that someone might actually attempt to work with this particular experiment in the future. Most of the information which follows deals in some way with the performance of the experiment under realistic laboratory conditions.

Starting the experiment from its rest position latched to the side of the touch-down bearing is by no means a sure thing. The four control system's outputs grow exponentially at different rates due to differences in the mutual-inductance coupling, and the large surge of current from the power supply can couple into the op-amps, occasionally causing one to go unstable in the wrong direction. The success rate for start-ups depends
heavily on the value of $\tau_4$. Too fast and the system overshoots, locking onto the opposite side of the bearing. Too slow and the rotor walks around the bearing instead of launching into the middle. Incidentally, it is not possible to position the rotor by hand and then apply power. The magnetic stiffness is simply too great.

The flywheel's stability is actually fairly robust when operated with a stiff external power supply. It has been observed that the system can sustain a blow by hand strong enough to cause the rotor to bounce off of its touch-down bearing. If a force is applied and held, however, the negative stiffness takes over and adjusts the position of the rotor to find force equilibrium. It doesn't take much force of this type to cause the rotor to move until it contacts the touch-down bearing. When this happens the feedback loop is broken and the controller latches up.

The entire assembly can be lifted off of the table without causing instability. In fact, the assembly can be turned on its side and still float. The rotor decreases the gap on the bottom side to provide the necessary lift force.

In order to spin the rotor, batteries must be clamped to the rotor to power the electronics. It wasn't planned that way, but the original concept for generating on-board power proved too weak to do an adequate job. Two 9V NiCad batteries in parallel are used for each supply. The high current capability of these cells is essential to being able to spin the flywheel. Alkaline cells were tried, but the slightest touch of the rotor demanded more current than they could deliver. Care must be taken when operating on batteries, since a latch-up condition is nearly like putting a short circuit across the battery terminals.

It was mentioned that due to space constraints the sensor windings had
to be located near the motor. The result is that the motor field couples strongly into the control system. At a typical motor current of 3 A peak per phase, the control system draws about 400 mA per supply to combat this mistaken disturbance. Consequently, the batteries only provide about 10 minutes of operation under these conditions. In contrast, the power drain during quiescent conditions is only 6 mA.

The motor current cannot be turned up any higher than 4 A peak per phase when operating with 9V batteries. This is not because of the increased magnetic stiffness introduced by the motor, but because the control electronics saturate from the parasitic coupling. Any future effort should strive to eliminate this interaction.

Finally, anyone seeking to work in this area should be warned of the frustrations involved in trying to debug a system which is open-loop unstable. Until the controller works there is nothing to measure, so there is no information with which to decide what is needed to make the system work. It was only desperate speculation that led to the mutual-inductance cancellation network which proved to be the key element in obtaining stable operation with this experiment. Hopefully the analytical models presented in this chapter will help make the process a bit less dependent on luck in the future.
CHAPTER 5
CONCLUSIONS

Velocity-feedback control of a homopolar magnetic bearing has been demonstrated to be capable of providing stable levitation with excellent reliability and minimum complexity. By actively controlling the radial mode of motion, the problem of unstable whirl found in radially-passive systems is eliminated. Because this approach eliminates the need for a shaft and allows the rotor to spin about its own center of mass, this system offers the best available opportunity to realize in practice the high energy densities and low cost which flywheels are capable of in theory. Furthermore, the homopolar radial bearing topology has been shown to be compatible with a conventional radial-gap motor-generator and to allow the efficient use of inexpensive ceramic permanent magnets to provide the lift force.

In the process of developing appropriate analytical models for magnetic levitation and stabilization, several results were obtained which are applicable to a broad range of problems. The radial dynamics of rotating machines can be analyzed using the techniques in section 2.5. These relationships are important in selecting shaft and bearing sizes. The methods presented represent a significant improvement over the existing techniques, which are primarily empirical.

The complex notation introduced in Chapter 2 can greatly simplify the design of rotationally-symmetric systems since it allows the versatile methods from classical control theory to be utilized. This is especially important when coupling between axes is present, as with the gyroscopic effect. The simple picture of poles and zeros translating by \( j\omega \) when spin
is introduced to a system is particularly appealing.

The perturbation-tensor boundary conditions which relate the perturbation magnetic field to the small displacements which produce it are very general in scope, and include applications in the continuum electromechanics of liquids and deformable solids, as well as being useful for rigid-body dynamics. The linearization process normally required in magnetic stability problems is avoided by directly applying the tensor expressions, which relate only first-order perturbation quantities. This saves a significant amount of time and removes many of the pitfalls present in the linearization process, especially when dealing with coordinate systems other than Cartesian.

More work is needed before radial magnetic bearings with velocity-feedback control can be used in practical devices. The biggest problem is the inability of the present control implementation to operate near or above the resonant frequency of the stator support structure. The apparent solution is to stiffen the stator by rigidly securing it to a large fixed mass, such as a concrete slab. The dominant resonances will then be much higher in frequency, and keeping them above the expected speed range should be achievable. Fortunately, while these higher-frequency resonances will tend to have essentially no damping, this is not a problem with velocity control, which pulls the poles of the resonance into the left-half plane (Figure 4.33).

It also will be necessary to increase the loop gain to make the present feedback concept stable at high spin speeds (Figure 4.5). In order to do this the undesirable mutual-inductance coupling of the actuator and sensor winding will have to be substantially reduced. This might require the use of some sensing scheme other than a simple coil of wire, perhaps
using capacitance. Ultimately it would be useful to incorporate some means of position sensing to facilitate start-up and to permit restarting after unintentional touch-downs at high speed.

The analysis presented in Chapter 4 did not consider the possibility of interconnecting the two axes. In terms of the complex notation, this would result in the possibility of a complex coefficient in the feedback block. This added flexibility would allow the formation of controller poles and zeros anywhere in the s-plane, without requiring symmetry about the real axis. Since the direction of spin can be predetermined, this feature could be used to advantage in the control-loop design. Finally, an intelligent controller, perhaps microprocessor-based, could result in a substantial improvement in the system's performance. The processor would be able to change control algorithms depending on the speed of the rotor to optimize the stability of the system. Perhaps most importantly, the advantages of exercising control from the rotating frame of reference could be obtained using a control system and windings located in the stationary frame. The processor could simulate the rotating-control response by combining the sensor inputs from both axes in a manner which would be a function of the speed, as measured with a tachometer. This would allow the electronics to be mounted externally, avoiding the harsh environment of high-speed rotation in a vacuum.

Another aspect of the system which deserves attention is the motor-generator section. This component should be designed so that there is no more flux present in the gap at any given time than is needed to transfer the necessary power. This stipulation is needed in order to keep the eddy-current losses in the motor-generator section to a minimum. This criteria seems to indicate that a synchronous variable-reluctance type of machine is
best suited to the task. Such a machine would also dissipate most of its heat on the stator where it can be removed easily, rather than on the rotor where heat must be removed by radiation.

Flywheels promise so much in comparison to batteries in terms of lifetime, performance, maintenance, and environmental impact, that their continued development is an essential part of any scheme involving electrical energy storage on a large scale. Potential applications include electric vehicles, storage for intermittent energy sources like photovoltaics and wind, and uninterruptable power supplies for everything from computers to hospitals. The VFC concept for self-stabilizing magnetic bearings appears to be capable of significantly enhancing the posture of flywheels as an energy storage medium.
CHAPTER 1


REFERENCES

CHAPTER 2


REFERENCES

CHAPTER 3


CHAPTER 4


APPENDIX

SMALL DISPLACEMENTS OF A CYLINDER CARRYING SURFACE CURRENT

Figure A.1 Cylinder Carrying Surface Current.

Consider the two-dimensional model of a long cylinder carrying surface current, shown above. The cylinder is allowed to have a different magnetic permeability than its surroundings. The magnetic field can be solved exactly for this situation, and is given in cylindrical coordinates by

\[
\overline{H}(r,\theta) = \begin{cases} 
-K_o \frac{\mu_2}{\mu_1 + \mu_2} \{\sin \theta \overline{a_r} + \cos \theta \overline{a_\theta}\}; & \text{region 1} \\
-K_o \frac{\mu_1}{\mu_1 + \mu_2} \frac{R^2}{r^2} \{\sin \theta \overline{a_r} - \cos \theta \overline{a_\theta}\}; & \text{region 2}
\end{cases}
\]

This field can also be expressed in terms of Cartesian coordinates;

\[
\overline{H}(x,y) = \begin{cases} 
-K_o \frac{\mu_2}{\mu_1 + \mu_2} \overline{a_y}; & \text{region 1} \\
-K_o \frac{\mu_1}{\mu_1 + \mu_2} \frac{R^2}{[x^2 + y^2]^2} \{2xy \overline{a_x} + [y^2 - x^2] \overline{a_y}\}; & \text{region 2}
\end{cases}
\]

Now let the cylinder move a small distance \( \xi_x \) in the positive-\( x \) direction, with no change in the surface current. The new magnetic field is given exactly by replacing \( x \) with \( x - \xi_x \) in equation A.2;
\[ \bar{H}'(x,y) = \begin{cases} 
-K_o \frac{\mu_2}{\mu_1 + \mu_2} \bar{a}_y &; \text{region 1} \\
-K_o \frac{\mu_1}{\mu_1 + \mu_2} \frac{R}{[(x-x_x) + y^2]^2} \{2(x-x_x) y \bar{a}_x + [y^2 - (x-x_x)^2] \bar{a}_y \} &; \text{region 2}
\end{cases} \] (A.3)

This new field can be expressed as the sum of the original field and a perturbation field;
\[
\bar{H}' = \bar{H} + \bar{h}
\] (A.4)

To first order in the displacement \( \xi_x \), the perturbation field is reliably determined from equation A.3 to be
\[
\bar{h}(x,y) = \begin{cases} 
0 &; \text{region 1} \\
-K_o \frac{\mu_1}{\mu_1 + \mu_2} \frac{R^2 \xi_x}{[x^2 + y^2]^3} \{[3x^2 - y^2] y \bar{a}_x + [3y^2 - x^2] x \bar{a}_y \} &; \text{region 2}
\end{cases}
\] (A.5)

Returning to cylindrical coordinates yields a simpler form;
\[
\bar{h}(r,\theta) = \begin{cases} 
0 &; \text{region 1} \\
K_o \frac{\mu_1}{\mu_1 + \mu_2} \frac{R^2 \xi_x}{r^3} (\sin 2\theta \bar{a}_r - \cos 2\theta \bar{a}_\theta) &; \text{region 2}
\end{cases}
\] (A.6)

Two different perturbation approaches to this same problem will be tried and tested against this result. The first is a blind application of the boundary-condition approach given by Melcher\(^*\), which yields an incorrect answer. The boundary conditions given in equation 2.44 are then applied to yield the correct result.

A.1 Traditional Method, Incorrectly Applied

After the displacement has occurred, the location of the surface is correctly described to first order in the perturbation displacement by

\[ r = R + \xi_x \cos \theta \]  \hspace{1cm} (A.7)

The normal vector to this surface, again to first order, is

\[ \bar{n}' = \bar{a}_r + \frac{\xi_x}{R} \sin \theta \bar{a}_\theta \]  \hspace{1cm} (A.8)

The equilibrium magnetic field (equation A.1) is evaluated at the new boundary location (equation A.7). To first order in the displacement these boundary fields are

\[ \bar{H}(R + \xi_x \cos \theta, \theta) = \begin{cases} 
-K_o \frac{\mu_2}{\mu_1 + \mu_2} \{\sin \theta \bar{a}_r + \cos \theta \bar{a}_\theta\}; & \text{region 1} \\
-K_o \frac{\mu_1}{\mu_1 + \mu_2} \left[1 - \frac{2\xi_x}{R} \cos \theta\right] \{\sin \theta \bar{a}_r - \cos \theta \bar{a}_\theta\}; & \text{region 2} 
\end{cases} \]  \hspace{1cm} (A.9)

The two jump conditions, evaluated at the radius corresponding to the displaced interface, become

\[ \bar{n}' \times \left[\bar{H}(R + \xi_x \cos \theta, \theta) + \bar{h}\right] = K_o \cos \theta \]  \hspace{1cm} (A.10)

\[ \bar{n}' \cdot \left[\mu[\bar{H}(R + \xi_x \cos \theta, \theta) + \bar{h}]\right] = 0 \]

Inserting equation A.8 and A.9 into these expressions yields two non-trivial perturbation boundary conditions,

\[ \left[\begin{array}{c} \bar{h}_\theta \\
\end{array}\right] = -\frac{K_o}{\mu_1 + \mu_2} \frac{\xi_x}{R} \{[\mu_1 - \mu_2] \sin^2 \theta - 2\mu_1 \cos^2 \theta\} \]  \hspace{1cm} (A.11)

\[ \left[\begin{array}{c} \mu h_r \\
\end{array}\right] = -K_o \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \frac{2\xi_x}{R} \sin 2\theta \]

The second condition, on the radial component, matches the field given in equation A.6; but the tangential-field condition does not. The procedure followed led to an incorrect result. The origin of this error

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can be understood in terms of Figure A.2, below.

![Figure A.2 Radial-Displacement Description of the Boundary.](image)

As indicated in the figure, the translation \( \xi_x \) moves the boundary at a given angle \( \theta \) from radius \( R \) to \( R + \xi_x \cos \theta \). In writing the boundary conditions at point "a" (equation A.10), the equilibrium values were evaluated for the same value of \( \theta \), corresponding to point "b". But the surface current which actually ends up at "a" originated at point "c". The difference in the surface current between points "b" and "c" is first order in the perturbations; hence the error.

### A.2 Perturbation-Tensor Boundary Conditions

The boundary conditions relating only perturbation quantities were given in equation 2.44:

\[
\bar{n} \times \left[ \bar{\nabla} \cdot \bar{V} \bar{H} + \bar{h} \right] - \left[ \bar{\nabla} \cdot \bar{\nabla} \xi \right] \cdot \bar{n} \times \left[ \bar{V} \bar{H} \right] = \bar{k}
\]

\[
\bar{n} \cdot \left[ \bar{\nabla} \cdot \left[ \bar{\nabla} [\mu \bar{H} + \mu \bar{h}] \right] - \left[ \bar{\nabla} \cdot \bar{\nabla} \xi \right] \cdot \bar{n} \cdot \left[ \mu \bar{H} \right] \right] = 0
\]

(A.12)

The displacement vector is the same for all points and is given in cylindrical coordinates by

\[
\bar{\xi} = \xi_x \cos \theta \bar{a}_r - \xi_x \sin \theta \bar{a}_\theta
\]

(A.13)

As expected for a uniform displacement field,

\[
\bar{\nabla} \cdot \bar{\xi} = 0
\]

(A.14)

Using the expression for the vector gradient in cylindrical coordinates

\[
\bar{V} \nabla \xi = \nabla \xi \cdot \bar{a}_r + \xi_x \left(\nabla \xi \cdot \bar{a}_\theta \right)
\]
given in equation 2.27, and the equilibrium field given in equation A.1,

\[
\nabla \overline{H}(r, \theta, z) = \begin{cases} 
0 & ; \text{region 1} \\
\frac{\mu_1}{K_0 \left( \frac{\mu_1}{\mu_1 + \mu_2} \right)} \frac{2}{R} \begin{bmatrix} 
\sin \theta & -\cos \theta & 0 \\
-\cos \theta & -\sin \theta & 0 \\
0 & 0 & 0 
\end{bmatrix} & ; \text{region 2} 
\end{cases}
\]  \hspace{1cm} (A.15)

Inserting these expressions in equation A.12 with \( k=0 \) yields two non-trivial boundary conditions,

\[
\begin{bmatrix} 
\mathbb{h}_\theta \\
\mathbb{u}_\theta 
\end{bmatrix} = K_0 \frac{\mu_1}{\mu_1 + \mu_2} \frac{2 \xi_x}{R} \cos 2\theta
\]  \hspace{1cm} (A.16)

\[
\begin{bmatrix} 
\mathbb{u}_r \\
\mathbb{h}_r 
\end{bmatrix} = -K_0 \frac{\mu_1}{\mu_1 + \mu_2} \frac{2 \xi_x}{R} \sin 2\theta
\]

Both of these conditions agree with the exact solution (equation A.6) at the boundaries, and consequently lead to the correct perturbation field throughout the plane.
BIOGRAPHICAL NOTE

Paul Basore was born in Albuquerque, New Mexico in 1956. He lived in Bethesda, Maryland before moving to Stillwater, Oklahoma where he graduated from C.E. Donart High School in 1974. Paul attended Oklahoma State University in his home town, receiving the B.S. degree in Electrical Engineering and graduating first in his class. While an undergraduate, he worked as Chief Engineer and as an announcer for radio station KSPI AM-FM in Stillwater. Just prior to graduation Paul married Nancy Starks, whom he had known since grade 10. Upon notification that he had been selected as a Tau Beta Pi Fellow, Paul accepted entry into graduate school at MIT. After one year he submitted a proposal to International Harvester based on the research topic of his Master's thesis, flywheel energy storage. He was selected, and received fellowship support for two years. Upon graduation from MIT with the Ph.D., Paul will join the faculty at Iowa State University in Ames, where his wife will attend graduate school.