CORRELATION TRACKING

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This dissertation addresses what are called correlation tracking problems. Correlation tracking problems can be defined as nonlinear filtering problems wherein the measurement nonlinearity is a function of the difference between the state vector of process being estimated and a control variable. This control variable then can be thought of as a centering point for the measurement nonlinearity. A solution to the correlation tracking problem comprises a control or centering strategy and a measurement processing algorithm.

In this dissertation the correlation tracking problem is conceptualized and its basic properties discussed. Some of the pertinent theoretical tools are developed and discussed, and several approximate solutions are given to a particular example problem.
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CHAPTER 1

INTRODUCTION

1.0 Introduction

This chapter introduces what will be called correlation tracking problems. There are three elements to this introductory chapter. First a simple example of a correlation tracking problem is given. Second, the traditional solution approaches are applied to this sample problem. In this way the limitations of these approaches can be visualized. The third element of this introductory chapter is an outline of the contents of the remainder of this dissertation.

1.1 An Illustrative Example

The correlation trackers which are easiest to describe and understand are optical trackers, of which the hypothetical one-dimensional star tracker depicted in Figure 1.1 is an example. This star tracker consists of a telescope, some hardware for processing the light entering the telescope, and a mechanism for driving the telescope elevation. The tracker is called one-dimensional because the telescope only rotates in a vertical plane. The angle between the horizon and the centerline of the telescope is called the elevation angle and is denoted in Figure 1.1.
by "d". The drive mechanism will move the telescope to any commanded elevation angle, and is considered sufficiently powerful that its dynamics may be neglected. Thus the commanded and actual elevations may be considered equal. The light entering the telescope is focused onto a solar cell and amplified. The output of this amplifier is from two sources. First there is a current flow (or lack of it) if the star is present in (or absent from) the telescope field. In addition there is a random current due to thermal noise in the electrical components. The final important element of the star tracking problem is that the star is assumed to be moving in some unknown manner. This is, perhaps, not a very realistic assumption in a star tracking problem, but this target motion is an important part of many correlation tracking problems. For example, if the target is an aircraft instead of a star, then tracking its maneuvers may be important. Star motion would enter the star tracking.
example naturally if the telescope base were moving. Then, in the
telescope reference frame the star would move. For navigating the
telescope base, this relative motion of the star in the telescope
reference frame would be one of the fundamental variables in the problem.

The correlation tracking problem, in this example, is to devise
an algorithm which effectively commands the telescope and processes the
resulting amplifier output to determine the star's elevation as it
evolves in time.

To effectively determine the star's elevation it will be necessary
to be able to move the telescope so as to keep the star centered in the
telescope's field.

1.2 **Distinguishing Features of Correlation Tracking Problems**

So far there is no intuitive basis for applying the name "cor-
relation tracking" to this star tracking problem. A problem will be
introduced in Chapter 3 to which this name more clearly applies. That
problem, which is estimation of the time of arrival of a radio navigation
signal, provided the initial motivation for this work, but many problems
(including the star tracker) have similar mathematical features and the
name "correlation tracking" has come to be applied to any one of these
problems.

The distinguishing mathematical feature of a correlation tracking
problem is the nature of its measurement nonlinearity. In the star
tracking example, the integrated amplifier output, denoted by $z_t$,
satisfies
\[ dz_t = h(\theta(t) - d(t)) \, dt + d\beta_t \] (1.1)

where

\[ h(x) = \begin{cases} 
1 & |x| \leq \delta/2 \\
0 & |x| > \delta/2 
\end{cases} \]

and \( \beta_t \) is a Brownian motion which models the thermal noise present on the amplifier output. The measurement nonlinearity \( h(\theta(t) - d(t)) \) is nonlinear in the star elevation \( \theta(t) \), making the estimation of the star's elevation given the measurement history \( z_t^0 = \{z_s: 0 \leq s \leq t\} \) a nonlinear problem. More than that, however, the nonlinearity also depends on \( d \), the telescope elevation. Furthermore, since \( h(x) = 0 \) for \( |x| > \delta/2 \), no algorithm which passively collects measurements will be effective no matter how cleverly the measurements are processed. It is necessary both to collect and process measurements and to determine a strategy for controlling the telescope's elevation. It is important that the nonlinearity depend on the difference \( \theta - d \). The nonlinearity in the star tracking case is symmetric about zero. In many problems the nonlinearity is the result of a correlation operation (as is the case with the time of arrival tracking problem described in detail in Chapter 3), and the nonlinearity is an autocorrelation function. Although this symmetry is present in many problems, it does not appear to be essential.
1.3 **Traditional Approaches**

The problem of estimating the star elevation is traditionally solved by designing two algorithms, one for use when the errors are large compared to the telescope field, and one for use when they are small. The large error algorithm can be thought of as a coarse estimation and would generally be used to begin the system operation.

The coarse estimation algorithm in the star tracking example divides the circle up into bins whose width equals the telescope field width, then methodically searches these bins and inspects each bin long enough to decide whether or not the star is located there. The decision about whether the star is in the bin currently being inspected is made using classical hypothesis testing methods. If the decision is made that the star is absent from a given bin, then the next bin is inspected. If it is decided that the star is present in a given bin, then the small error or fine tracking algorithm is entered.

The fine tracking algorithm operates by comparing the amplifier output when the elevation is slightly greater than the suspected star elevation to that when it is the same amount smaller. If, for example, the comparison favors the greater elevation, then the estimated star elevation should be adjusted upwards. The result of the comparison then indicates the error in the estimated star elevation and a feedback loop can be closed using this error indication, as Figure 1.2(b) indicates.

There are two methods of mechanizing the comparison hardware. One way is to construct a second telescope and signal conditioning circuitry
as indicated in Figure 1.2(a). The second telescope is then mounted in the same bracket as the first, but with its centerline offset by one field width $\delta$ from that of the first telescope. The difference between the outputs of the amplifiers associated with these two telescopes indicates the amount of error present and may be used to close a feedback loop. The hardware drawn in Figure 1.2(a) is modelled by the comparator, forward loop nonlinearity and disturbing noise $n(t)$ in Figure 1.2(b). The transfer function $G(s)$ in Figure 1.2(b) models the processing done to the error signal to arrive at a command for the telescope drive. To
understand how the hardware in Figure 1.2(a) leads to the model in
Figure 1.2(b), suppose that the star is not moving and is several field
widths higher in elevation than either of the telescopes. The star
elevation minus the elevation of the center line between the two tele-
scopes is a large negative number. This corresponds to a point at the
far left end of the graph of the forward loop nonlinearity in Figure
1.2(b). As the assembly is scanned upward the higher telescope encounters
the star first. Its output changes from zero plus noise to one plus
noise as the star enters its field. This happens when the center of
the assembly is lower than the star by one field width. The output
of the difference circuits in Figure 1.2(a) then changes from zero plus
noise to minus one plus noise since the signal from the upper telescope
is subtracted in the comparator. The upward motion of the telescope
assembly equates to an increase of the input to the nonlinear block in
Figure 1.2(b). The graph of the forward loop nonlinearity in Figure 1.2(b)
is swept out from left to right as the telescopes scan up to and past the
fixed star. The same sort of result may be obtained with a single tele-
scope by looking first above, then below the estimated star location and
subtracting the measurements obtained in these two locations. Using two
(or more) telescopes is called a parallel sensor or parallel correlator
approach. Using one telescope is called a dithered sensor approach.

1.4 Problems With Traditional Approaches

The solution of the star tracking problem by two algorithms, a
course acquisition algorithm and a fine tracking algorithm, typifies the
traditional approach to nonlinear problems. (See [Van Trees (2)] for example.) During the initial stages of system operation the errors are generally so large that the nonlinearities must be treated globally. Usually a reasonable, ad-hoc procedure is devised for coarse acquisition. The coarse acquisition procedure reduces the errors sufficiently that some type of linearization is valid. Typically the coarse acquisition procedure cannot reduce the errors much further. Once the system can be considered linear, operation is handed over to a fine tracking algorithm. This algorithm is generally designed using the tools of linear system theory and, so long as the linearization is valid, it performs well.

Even if the star tracking system of Figure 1.2(b) were really linear, the errors in the best estimate of the star's location would grow as the noise and dynamics grew. In the nonlinear case the errors also grow and as they do the fine tracking algorithm functions less effectively since the linearization implicit in its design becomes progressively less valid. The magnitude of noise and dynamics for which the fine tracking system fails to operate is called its threshold. In correlation tracking problems, both the coarse estimator (usually called acquisition algorithm) and the fine tracking algorithm have deficiencies which result in system failure when the disturbing noise or target dynamics are severe. The deficiency in the fine tracking algorithm is most apparent.

The star tracking system of Figure 1.2(b) fails to operate in high dynamics and noise because of the small range of errors over which its forward loop nonlinearity is functional. The forward loop nonlinearity
goes to zero for errors larger in magnitude than the telescope field width \( \delta \). Since the system cannot tolerate errors greater than \( \delta \), there is an envelope of noise levels and star dynamics levels outside of which the tracking algorithm is ineffective.

There are basically two ways to construct an error indicating device which is effective over a broader range of errors. One way is to build more telescopes and the other is to dither a single telescope over a wider range. If more telescopes are used, a scheme has to be worked out for managing the greater number of measurements.

In Figure 1.3 is shown a forward loop nonlinearity which could be obtained in the star tracking problem by using four telescopes instead of two.

![Diagram](image)

**Figure 1.3** Forward loop nonlinearity with extended tracking range.

The function displayed in Figure 1.3 is obtained by mounting the four telescopes next to one another, each one offset by one field width from its neighbor. The outputs of the highest and lowest elevation telescopes are weighted by plus and minus two respectively. The outputs of the next highest and lowest are weighted by plus and minus one respectively. The sum of the weighted telescope outputs, as a function of the offset
between the star and center of the array of telescopes, behaves like the
time delay function shown in Figure 1.3. The utility of the four telescope approach
just described is diluted by what happens to the noise present at each
telescope output. Suppose that the noise on the four telescope outputs
are independent; then the noise power in the error indication is increased
over that of an individual telescope by the sum of the squares of the
weights applied to the individual telescope outputs. In the four tele-
scope example this means a multiplier of $2^2 + 1^2 + 1^2 + 2^2 = 10$. In terms
of noise, extending the effective range of the forward loop nonlinearity
is expensive. This extra noise has lead previous authors [Spilker (1)]
[Schiff (1)] to reject extended range nonlinearities. The usual reason
for this rejection goes as follows. Suppose that the system closed
around a two-telescope nonlinearity can track with a mean square error
of $\epsilon$, which is much smaller than $\delta^2$. Increasing the range of the forward
loop nonlinearity by using four telescopes as indicated earlier, leaving
everything else fixed, will increase the mean square tracking error to
$5\epsilon$ since the noise power goes up by 5 over the noise in the two-telescope
system. If a single nonlinearity has to be decided upon, the two-telescope
system may be preferable because it yields smaller errors in benign con-
ditions. This problem can be overcome by making the weights applied to
the telescopes' outputs vary with the input noise and dynamics levels.
Input noise and dynamics level dependent weighting procedures for parallel
telecope systems are considered in Chapters 4 and 5.

A broad range dither system has not been previously considered
for tracking, although the traditional coarse acquisition scheme can be
thought of as a broad range dither scheme. The traditional coarse acquisition scheme does not, even during acquisition, make the most effective use of the single sensor. In Chapter 7 a more effective scheme is given and is shown to outperform the traditional scheme by providing faster, more reliable acquisition. The scheme presented in Chapter 7 appears to be usable for tracking as well.

1.5 Description of What Follows

Earlier in this chapter it was mentioned that the star tracking problem is called a correlation tracking problem because it bears a mathematical resemblance to some problems where correlations naturally occur. The common mathematical features of these problems can be abstracted and used to define a class of problems. Problems in this class will be called correlation tracking problems.

In Chapter 2 the correlation tracking problem is posed mathematically. In terms of complexity, the problem is then understood to lie somewhere between a nonlinear filtering problem and a partially observable stochastic control problem. The correlation tracking problem is also shown to be a special type of feedback estimation problem. Feedback communication, another special case of feedback estimation, is mentioned and results available in these areas are cited.

Chapter 3 develops the mathematical models for a particular correlation tracking system. It is shown that time-of-arrival tracking of a radio navigation signal can be considered as a correlation tracking problem in the sense that it has the mathematical features outlined in
Chapter 2. The procedure used is to correlate the received navigation signal with a stored replica of the signal. The stored replica is identical to the received signal except for a shift in time. If the shift is zero then the correlation is maximum. The correlation decreases toward zero as the shift increases. This situation is analogous to the star tracking problem. The shift between the input signal and the stored signal is analogous to the difference in elevation between the telescope center and the star, and the behavior of the correlation as the shift changes is similar to that of the telescope output as the elevation difference changes.

The radio navigation system considered is the Global Positioning System (GPS). In this system a receiver computes its location by determining the time of arrival of signals from several beacon satellites in known locations. As the vehicle moves there are changes in the arrival times of the signals from some of the satellite beacons. These changes are analogous to the relative motion between the telescope center and the star.

Time-of-arrival tracking of Global Positioning System signals provided the motivation for this work and the procedures presented here are presented in that context. The reason for using the name correlation tracking will be clear in this context.

The same procedures suggested for the star tracking problem can (and have) been used in the time of arrival tracking problem. For fine tracking, for example, a procedure analogous to the two-telescope procedure has traditionally been used.
In place of the upper and lower telescopes are correlations of the input with two stored signals which differ from each other only by a time shift. The difference between the outputs of these two correlators can then be used to close a tracking loop analogous to the one in Figure 1.2(b). In Chapter 3 the extended Kalman filter design formalism is used to arrive at a linearized fine tracking system. The resulting system is analogous to the system block diagrammed in Figure 1.2(b). This system has the form of a traditional fine tracking system, although the extended Kalman filter design procedure has not been used in practice to design fine tracking feedback loops. The extended Kalman filter, in some sense, represents the best possible traditional design. Monte Carlo simulation of the extended Kalman filter is used to demonstrate the limitations inherent in the traditional design.

It was suggested earlier that for the star tracking problem the limitations of the traditional fine tracking system might be overcome by using additional telescopes to increase the range of the forward loop nonlinearity. It was also pointed out that the noise present on the telescopes' outputs complicates the design of a tracking loop using extra sensors for range extension. Both the increased error handling capability and the increased noise of an extended range nonlinearity have to be considered for a meaningful design to result.

In Chapter 4, Fokker-Planck techniques are used to quantify the effect of an extended range nonlinearity on tracking loop performance. Among nonlinear problems, correlation tracking problems are particularly
amenable to Fokker-Planck treatment. This is due to the fact that the measurement nonlinearity can be made to be a function of the tracking error; in other words, that the nonlinearity is in the forward loop. Normally the measurement is a function of the input variable alone.

An optimization procedure is used to produce the optimum extended range nonlinearity. A tracking loop incorporating this nonlinearity is simulated and the results are compared with those obtained for the extended Kalman filter. The algorithm derived by Fokker-Planck techniques performs just as predicted. It has the limitation, however, that the solution can only be obtained when the target motion is a Brownian motion. (Actually, solutions are obtained for a few other types of target motions. Generality sufficient to include many reasonable target motion models is not achieved, however.)

In Chapter 5 some of the limitations of the designs obtained in Chapter 4 are overcome by taking a different approach to the problem of determining the optimum forward loop nonlinearity. It is shown that the problem may be viewed as a nonlinear filtering problem with a deformable measurement nonlinearity. Chapter 5 gives a multiple sensor design scheme based on approximate solution (Gaussian approximation) to the nonlinear filtering equations which arise. This algorithm has the advantage that it is easily applied to any order input process. This scheme is simulated and several comparisons are drawn between this algorithm and the one obtained by Fokker-Planck techniques. In addition to comparing the simulations of these schemes, analytical comparisons are made.

Chapter 6 shows how the multiple sensor methods developed in Chapters
4 and 5 may be applied to noncoherent code tracking for the Global Positioning System.

In Chapters 4, 5, and 6 the attention is on systems employing many sensors. In Chapter 7 the attention is on systems which employ only one sensor. It is first shown that the most obvious transfer of the approach in Chapter 4 will not work. Next, the problem is simplified by considering the signal time of arrival to be constant. This is analogous to assuming that the star does not move. This simplification makes it possible to solve exactly for the nonlinear filter and to then develop a dynamic sensor allocation strategy which maximizes the information contained in the resulting measurements. This method is compared to the traditional coarse acquisition strategy and justification is given for using the procedure even when there is some small motion of the star.

Chapter 8 contains a summary and recommendations for future work.
CHAPTER 2

MATHEMATICAL DESCRIPTION OF CORRELATION TRACKING PROBLEMS

2.0 Introduction

In Chapter 1 an example of a correlation tracking system was given, and it was indicated that the pertinent features of the problem could be abstracted mathematically. In this chapter the correlation tracking problem is stated mathematically. After this is done it will be seen that correlation tracking fits naturally into a broader class of problems—feedback estimation problems. Other feedback estimation problems have received recent treatment and some representative results are reported. Some of the tools of nonlinear filtering which will be used are also reported.

2.1 Abstract Model of the Correlation Tracking Problem

There are three elements in an abstract statement of the correlation tracking problem. They are a model for the motion of the target being tracked, a description of the measurements available, and a performance index. In the star tracking example of Chapter 1 the attention was on the measurement model. The measurement model is the distinguishing and novel feature of a correlation tracking problem.
In Chapter 1 the measurement equation for the star tracking problem is given as

\[ dz_t = h(\theta(t) - d(t)) dt + dB_t \]  \hspace{1cm} (2.1)

where

\[ h(x) = \begin{cases} 1 & \text{if } |x| \leq \delta/2 \\ 0 & \text{if } |x| > \delta/2 \end{cases} \]  \hspace{1cm} (2.2)

In Eqn. (2.1) \( z_t \) is the integrated measurement at the output of the telescope's processing circuitry. The increment \( dz_t \) is either \( dt + dB_t \) if the star is present in the telescope field, or \( dB_t \) if the star is absent. The process \( B_t \) models noise which is present no matter what the orientation of the telescope is. In the general correlation tracking problem it will be supposed that an array of measurements \( z_t^i \) is available and that the \( z_t^i \) satisfy

\[ dz_t^i = h_i(x-u)dt + r_i dB_t^i \]  \hspace{1cm} (2.3)

In Eqn. (2.3) \( B_t^i \) is a standard Brownian motion, \( h_i(\cdot) \) is a given function, \( x \) is the target position, \( u \) is a control variable, and \( r_i \) is the intensity of the additive measurement noise.

In many of the problems which can be addressed using the techniques presented in this dissertation, the functions \( h_i(\cdot) \) will be, or will have
some of the properties of, autocorrelation functions. For example, the
term from whence the name correlation tracking is derived, which
is described in the next chapter, has a measurement function \( h(\cdot) \) which
is an autocorrelation function. The nonlinearity for the star tracking
problem is not an autocorrelation function but it is symmetric about
zero, positive, unimodal, and goes to zero outside some finite range.
With a nonlinearity of this nature the control variable can be thought
of as a centering which needs to be determined so as to maximize the
output of the nonlinearity. This was the case with the star tracker and
it will be the case with the system described in Chapter 3.

In the star tracking example of Chapter 1 the argument of the
measurement nonlinearity \( h(\cdot) \) is a scalar – the difference between the
star elevation and the telescope elevation. In general the argument will
be a vector. For example, a realistic star tracking problem would
require pointing the telescope in both azimuth and elevation. Specifi-
cation of the target position would then require a two vector. There
is more than a single measurement because many correlation systems have
several measurements of the form in Eqn. (2.3) available. For example,
the two-telescope tracking system proposed in Chapter 1 had two adjacent
measurements available and could have had more.

Other optical systems have arrays of measurements available. The
array may be two dimensional instead of one dimensional. In the star
tracker example the basic sensing element (telescope) had a field width
\( \delta \) over which it was effective. In order to build a system effective over
a broad range it is necessary to combine many of these basic sensing elements.

The most apparent way to combine the elements is to position them one next to another with a separation equal to their field width. In a two dimensional problem the sensing elements are usually finite in both dimensions. In this case, to obtain a broad range, a two dimensional array might be constructed.

The individual elements in the array may either be individually controllable or they may be locked in fixed relation to one another and only their center left controllable. Eqn. (2.3) indicates that a single control enters all the measurement in the array, and it is assumed in this dissertation that the sensing elements are locked in fixed relation to one another. Methods analogous to using many adjacent telescopes to obtain an effective estimation scheme are the subjects of Chapters 4 and 5.

It will be assumed that the target position is part of a larger state vector which evolves as the solution to a differential equation forced by white noise. In practice this seems to be an adequate characterization of the target motion.

The vector $x(t)$, as before, is the target position and the vector $y(t)$ is the vector containing the remaining state variables (e.g., velocity and acceleration) necessary to describe the target's motion. These two vectors, $x(t)$ and $y(t)$, will be adjoined in a larger vector...
\[
x(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
\]

The vector \( X(t) \) is then assumed to satisfy

\[
dx = f(X,t)dt + g(X,t)\gamma_t
\]

where \( f(X,t) \) and \( g(X,t) \) are uniformly Lipschitz and \( \gamma_t \) is a standard Brownian motion.

To complete the problem specification, a measure of performance needs to be decided upon. In a normal control problem the performance measure would include a penalty on the state vector \( X \) and on the control \( u \). In this problem, however, the state (i.e., target) evolves independent of any action taken by the estimator or controller. Therefore a penalty on the state vector does not have any place here.

There are problems in much this same context wherein the actions of the estimator/controller influence the evolution of the target trajectory. Suppose, for example, the target were an aircraft trying to evade detection and the sensing system employed a spotlight to illuminate the target. Problems of this sort will not be considered here.

In the star tracking example no performance index was stated concretely, but it was stated verbally that the measurement processing
and telescope elevation command policy should be designed to determine the star's location as accurately as possible. A performance index which embodies this intuitive notion is the following

$$E \int_{0}^{T} L(X_t - E_{F_t}X_t)dt + E\phi(X_T - E_{F_T}X_T) \quad (2.5)$$

In Eqn. (2.5) the operator $E_{F_t}$ is the conditional expectation given the measurement history $Z_{0}^{t}$, where

$$Z_{0}^{t} = \{z_s : 0 \leq s \leq t\} \quad (2.6)$$

In Eqn. (2.6) $z_s$ is the vector of measurement composed of the $z_i$ defined in Eqn. (2.3). The quantity $(X_t - E_{F_t}X_t)$ appearing in Eqn. (2.5) is the error in estimating $x_t$. The performance index is then the expected value of the integral of some function, $L(\cdot)$, of the estimation error plus the expected value of some function, $\phi(\cdot)$, of the estimation error at the final time.

The correlation tracking problem is to find the measurement control $u(t)$ in Eqn. (2.3) which depends on the past history of the measurements and which minimizes the performance index in Eqn. (2.5).

### 2.2 Correlation Tracking as a Special Case of Feedback Estimation

The manner in which the control variable $u$ affects the measurement nonlinearity in Eqn. (2.3) is characteristic of a correlation tracking
problem, but there are many problems in which there is some control over the measurement. This broader class of problems in which the measurement is controllable is called feedback estimation.

Another example of a feedback estimation problem, besides the correlation tracking problem, is the feedback communications problem. This problem has received attention in the past and is closely enough related to the correlation tracking problem to be worth mentioning here.

The usual model for a feedback communications problem is one where a satellite, which has the capability of adjusting its modulation, sends a signal to a ground station. Mathematically this is stated by writing an equation for the transmitted signal $ds_t = h(x_t, u_t)dt$ where $h(\cdot)$ is the modulation, $x_t$ is the message to be sent, and $u$ indicates the control which can be exerted on the modulation. Satellites typically operate at low power levels and the signal received on the ground is corrupted with noise. Denote the received signal by $z_t$, then

$$dz_t = ds_t + d\beta_t$$  \hspace{1cm} (2.7)

or substituting for $s_t$

$$dz_t = h(x_t, u)dt + d\beta_t$$  \hspace{1cm} (2.8)

In Eqn. (2.8) $\beta_t$ is a standard Brownian motion. The signal received on the ground is transmitted back to the satellite. The ground transmitter is a high power transmitter and so the signal received at the
satellite can be considered to be uncorrupted. Thus the satellite knows the signal being received on earth and can use this knowledge to adjust its modulation so as to optimize system performance in some sense. The satellite uses the history of measurements received at the ground $Z^t_0 = \{z_s: 0 \leq s \leq t\}$ to adjust its modulation. This means that the transmitted signal depends on the received signal, or that $u_t$ is some functional of $Z^t_0$.

The similarity between the correlation tracking problem and the feedback communications problem is that in both of these problems the measurement nonlinearity depends on the state and on a control. The control may be any functional of the measurement history.

The difference between these two problems is that the measurement control variable in the correlation tracking problem subtracts directly from a portion of the state vector. In feedback communications problems the measurement control enters in some other way.

As complexity is concerned, feedback communication and correlation tracking problems seem to be specializations of the usual stochastic control problem with partial observations. The usual [Davis and Varaiya (1) and Elliot (1)] stochastic control problem has a steady vector $x_t$ satisfying

$$\dot{x}_t = f(x_t, u_t)dt + g(x_t, t)d\zeta$$ (2.9)

and a measurement $z_t$ satisfying
\[ dz_t = h(x_t, u_t)dt + d\beta_t \quad (2.10) \]

The problem is to find the control \( u_t \) which depends on the past measurements \( Z_0^t \) and to minimize the performance index

\[ J = E \int_0^T M(X_t, u_t)dt + E\Omega(X_t, u_t) \quad (2.11) \]

The feedback estimation problem by comparison involves a message vector \( x_t \) (or for correlation tracking a target state vector) which is assumed to satisfy

\[ dx_t = f'(x_t, t)dt + g'(x_t, t)dY_t \quad (2.12) \]

and a measurement

\[ dz_t = h'(x_t, u)dt + d\beta_t \]

The feedback estimation problem is to choose the measurement control \( u \) which depends on the measurement history \( Z_0^t \) and minimizes

\[ E \int_0^T L(x_t - E_{F_t}x_t)dt + E\phi(x_t - E_{F_t}x_t) \quad (2.13) \]

The stochastic control problem can be specialized to obtain a feedback estimation problem as follows.
Let the control in the stochastic control problem be

\[
\begin{bmatrix}
u^1_t \\
u^2_t
\end{bmatrix}
\]

(2.14)

and suppose that the state equation does not depend on \( u_t \)

\[
f(x_t, u_t) = f(x_t)
\]

(2.15)

Suppose that the first element \( u^1_t \) of the control vector affects the measurement

\[
h(x_t, u_t) = h(x_t, u^1_t)
\]

(2.16)

and that the second element \( u^2_t \) enters the performance index

\[
m(x_t, u_t) = m(x_t - u^2_t)
\]

and

\[
\Omega(x_t, u_t) = \Omega(x_t - u^2_t)
\]

(2.17)

with the restriction that

\[
u^2_t = E_{x_t} x_t
\]

The resulting stochastic control problem is the feedback estimation problem given in Eqns. 2.9, 2.12, and 2.13.
2.3 **Background for Feedback Estimation**

Feedback estimation is not a new problem. An early discussion of the problem is given in [Greer (1)]. Early work focuses on the problem where the message is a constant, that is, where

\[ dx_t = 0 \]  \hspace{1cm} (2.18)

For example, for the discrete time analog of this problem [Schalkwijk and Kailath (1)] and [Schalkwijk (1)] give a feedback scheme based on the Robbins Monro stochastic approximation scheme. They show that their scheme achieves the channel capacity of a white Gaussian channel with an average power constraint on their transmitted signal. The exploitation of the feedback link reduces the complexity and decoding delay required to achieve channel capacity. More recently [Kadota, Zakai, and Ziv (1), (2)] showed that, in the continuous time case, the presence of a noiseless feedback link does not change channel capacity and they give a formula for the amount of mutual information between the message and the output of a white Gaussian channel with noiseless feedback.

The most complete result along these lines is in [Liptser and Shiryaev (2)]. The problem they solve is the following. Suppose that the received signal satisfies

\[ dz_t = h(x, z_0^t, t) dt + d\beta_t \]  \hspace{1cm} (2.19)
and that the message $x$ is a Gaussian random variable. Suppose also that the measurement equation has a unique strong solution with

$$P \int_0^T h^2(x, z^t_0, t) dt < \infty = 1 \quad (2.20)$$

Impose a finite power constraint

$$\frac{1}{t} \int_0^t E h^2(x, z^s_0, s) ds < C \quad (2.21)$$

for some constant $C$. [Liptsen and Shiryayev (2)] show that the optimum modulation is given by

$$h(x, z^t_0, t) = \sqrt{\frac{C}{\gamma}} e^{\frac{Ct}{2}} (x - \hat{x}_t) \quad (2.22)$$

where $\gamma = E(x - \hat{x}_t)^2$. ($\hat{x}_t$ is defined here by $\hat{x}_t = E x_t$). This modulation scheme is optimum in the sense that it achieves a mean square error $\Delta(t)$ which satisfies

$$\Delta(t) = \inf E (x - \hat{x}_t)^2 \quad (2.23)$$

where the infemum is taken over all admissible modulation schemes. This is an unusually strong result. It might be expected from ordinary optimal control that one modulation might achieve mean square error equal to $\Delta(t_1)$ at time $t_1$ and another modulation might achieve the infemum at a different
time $t_2$. The result of Liptser and Shiryayev shows that one modulation achieves the minimum at all times. They also obtain the optimal modulation scheme for the case where $x_t$ is a first order linear process.

These feedback communication results do not all apply to the correlation tracking problem. Some of the general ones do. For example, the mutual information formula [Kadota, Zakai, and Ziv (1)] is quite general and will be used later. The results of Liptser and Shiryayev, however, do not apply. The reason is that their constraint for a feedback communication problem is finite average power

$$\frac{1}{t} \int_0^t \mathbb{E} h^2(x, Z_{0,s}) ds < C \quad (2.24)$$

In the correlation tracking problem the constraint is that

$$h(\cdot, Z_{0,t}) \in F \quad (2.25)$$

where $F$ is the set of all translations of a given function. The average power constraint admits the linear functions

$$h(x, Z_{0,t}) = a(t)x + b(t) \quad (2.26)$$

but the correlation tracking constraint does not generally admit the linear functions.
2.4 Tools from Nonlinearity Filtering Theory

It has been observed that in a correlation tracking problem a measurement is available which is a nonlinear function of the target state and which depends on a control variable. Since the control variable may depend on the measurement history, then the measurement nonlinearity itself may be random. This situation is beyond the scope of normal nonlinearity filtering theory [Jazwinski (1)] wherein the measurement nonlinearity is only a function of the target state.

There are two results from the theory of filtering with a deterministic measurement nonlinearity which will be needed later for correlation tracking problems. The first is a set of stochastic differential equations, usually called Kushner's equations. The second is a representation of the conditional density of the target state given the measurements. This representation is usually called the Bucy Representation Theorem.

The nonlinear filtering problem normally considered is the following. Suppose that the state vector $x_t$ satisfies

$$dx_t = f(x_t, t)dt + G(x_t, t)d3_t$$

(2.27)

and that the observation vector $z_t$ satisfies

$$dz_t = h(x_t, t)dt + d\eta_t$$

(2.28)
In Eqns. (2.27) and (2.28), $\beta_t$ and $\eta_t$ are Gaussian independent increment processes with zero mean and having

$$\mathbb{E} \beta_t \beta_s = \int_0^{\min(t,s)} Q(\sigma) d\sigma$$

(2.29)

$$\mathbb{E} \eta_t \eta_s = \int_0^{\min(t,s)} R(\sigma) d\sigma$$

(2.30)

Assume that the statistics of the initial state $x_0$ are known. Let $z_t^* = \{z_s : 0 \leq s \leq t\}$, as usual, and denote conditional expectation with respect to $z_t^*$ by $\hat{\cdot}$. That is, for a function $\phi(x_t)$

$$\hat{\phi}(x_t) \triangleq \mathbb{E} \{\phi(x_t) | z_t^*\}$$

(2.31)

If $\phi(\cdot)$ is twice continuously differentiable then a differential equation for $\hat{\phi}(\cdot)$ can be determined. This equation is called Kushner's equation. It is shown in [Jazwinski (1)] that

$$d\hat{\phi}(x_t) = \mathcal{C}_x f + \frac{1}{2} \text{tr} (G_t G_t^T \mathcal{C}_x^2) dt +$$

$$+ (\hat{\phi} - \hat{\phi}_t)^T R^{-1} (dz - \hat{h} dt) \quad t \geq 0$$

(2.32)
This result is very useful both practically and theoretically.

The representation theorem gives an expression for the conditional density $p(x_t, t \mid z_0^t)$. Define

$$\zeta_t = \int_0^t h_s R^{-1}(s)dz_s - \frac{1}{2} \int_0^t h_s R^{-1}(s)h_s ds$$

(2.33)

then [Bucy and Joseph (1)] and [Jazwinski (1)] show that the conditional density $p(x_t, t \mid z_0^t)$ satisfies

$$p(x_t, t \mid z_0^t) = \frac{\mathbb{E}^t\{\exp(\zeta_t) \mid x_t\} p(x_t, t)}{\mathbb{E}^t\{\exp(\zeta_t)\}}$$

(2.34)

In Eqn. (2.34) $p(x_t, t)$ is the a priori density of the state $x_t$ at time $t$ and the expectation $\mathbb{E}^t$ is over the process $\{x_t, t \in [0, t]\}$ for fixed $z_0^t$.

Now suppose that the filtering problem is altered by replacing the measurement equation (Eqn. (2.28) by

$$dz_t = f(x_t, t, z_0^t)dt + d\eta_t$$

(2.35)

That is, that the measurement nonlinearity $h$ is changed to a function depending on the past measurements. It is shown in [Fujisaki, Kallianpur, and Kunita (1)] and reported in [Clark (1)] that a twice differentiable function $\phi(\cdot, \cdot)$ satisfies Eqn. (2.32) with $h$ replaced by $f(\cdot, \cdot, \cdot)$ in
Eqn. (2.35). It is shown in Appendix A that the conditional density satisfies Eqn. (2.34) with $h$ replaced by $f$. 
CHAPTER 3

CORRELATION TRACKING MODEL FOR PSEUDO RANDOM NUMBER CODE
ARRIVAL TIME TRACKING AND TRADITIONAL TRACKING ALGORITHMS

3.0 Introduction

The purpose of this chapter is to introduce pseudo random number code tracking, and to show that pseudo random number code tracking can be considered to be a correlation tracking problem. This subject is introduced because a pseudo random number code is used for ranging purposes in the Global Positioning System (GPS). Design of tracking systems for the GPS pseudo random number code provided the motivation for the work reported here and the resulting designs fill the next four chapters. An introduction to these codes is, thus, necessary.

3.1 Description of Pseudo Random Number Code

The pseudo random number code (PRN) is a piecewise constant waveform built from a sequence of pseudo random numbers. A pseudo random number sequence is a sequence of numbers which can be generated systematically, but which has some of the desired properties of a sequence of random numbers. A familiar example of such a sequence is computer-generated noise. The method for building a PRN code from a PRN sequence is as follows. Suppose \( \{a_i\}_{i \geq 0} \) is a sequence of pseudo random numbers.
Choose a length of time \( T \) and define a PRN code \( S(t) \) by

\[
s(t) = a_i \quad iT \leq t < (i+1)T \tag{3.1}
\]

The length of time \( T \) is usually called one chip. The inverse of \( T \) is usually called the chipping rate. For example, for a particular code with a chipping rate of 10 MHz, \( T = 10^{-7} \) sec.

A particularly useful class of the PRN codes is the class of linear feedback shift register codes. In practice such a code is generated by a pseudo random number sequence in which the numbers are binary. These sequences look like coin toss sequences except that they repeat at some low frequency. Binary sequences are useful in practice because they can be generated by simple digital circuits.

Linear feedback shift register sequences are generated by circuits of which the circuit shown in Figure 3.1 is an example.

![Circuit Diagram](image)

Figure 3.1 Example of linear feedback shift register for 0,1 code.
The cells in the shift register shown store ones or zeros until a clock pulse comes along. When the pulse arrives, the number in a cell moves on to the right. The new number stored in the left-most cell is the mod 2 sum of the contents of the cells connected to the summer. This convention applies if the contents of the shift register are viewed as being 0 or 1. Another way to view the shift register is as containing plus ones and minus ones. With this convention the combiner is a multiplier as in Figure 3.2.

![Shift Register Diagram]

Figure 3.2 Example of a linear feedback shift register for -1, +1 code.

These two shift registers generate the same codes if +1 in the second is identified with 0 in the first.

The shift register length can be varied and so can the location and number of taps taken to develop the feedback signal. The tap placement determines the randomness properties of the code. Work in the area of shift register codes concentrates on the algebraic structure of these codes and its relation to their randomness. [Gold (2)] and
[Golomb (1)] give surveys of these methods and summarize the useful code properties in tables. These tables allow quick design of a shift register. [Gold (2)] also shows that a reasonable choice of shift register and tap placement yields a PRN waveform with the following properties.

Consider \( S(t) \) to be a binary random (or pseudo random) number code. That is, suppose that \( S(\cdot) \) is piecewise constant with

\[
S(t) = S(i) \quad \text{if } i \leq t < i + 1 \tag{3.2}
\]

and \( i \) an integer

and that \( S(t) \) is equal to either plus one or minus one. In this equation \( t \) is not time, but time multiplied by the chipping rate. The code autocorrelation \( R_{ss}(\tau) \) defined by

\[
R_{ss}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} S(t)S(t+\tau)dt \tag{3.3}
\]

has the form

\[
R_{ss}(\tau) = 1 - |\tau| \quad \text{for } |\tau - NT_R| \leq 1 \tag{3.4}
\]

for \( N \) any integer

\[
= \varepsilon(\tau) \quad \text{otherwise}.
\]
where $T_R$ is the code repeat time and $\varepsilon(\tau)$ is a function which is small compared to one. Figure 3.3 shows how $R_{SS}(\tau)$ might look. It will be

$$R_{SS}(\tau) = \begin{cases} 1 - |\tau| & |\tau| < 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.5)$$

This assumption is justified, as far as the consequent estimation procedure is concerned, if the a priori probability of the state (range) is concentrated on an interval $2T$ wide.

In many practical systems the repeat time $T_R$ may be very much larger than a priori timing uncertainties. For example, with GPS code the repeat time $T_R = 200$ days, while initial timing uncertainties might be a few microseconds.

In ranging and communication systems a PRN code is used to modulate a carrier wave. Many different types of modulation are possible. One type is called frequency shift keying (FSK). In this modulation scheme
two frequencies are chosen. If the code takes value one then the first frequency is sent; if it takes value zero then the second is sent. Another type of modulation is called phase shift keying (PSK). In this scheme the carrier is either phase shifted (usually 180°) or not phase shifted, depending on the state of the code. When the carrier is switched between 0° and 180° the modulation is called bi-phase shift keying (BPSK). BPSK can be viewed as multiplication of the carrier by a ±1 code.

3.2 Correlation Model

A particular example using a PRN for PSK is the Global Positioning System (GPS). The code in this system is used for ranging between a satellite transmitter and an arbitrarily located receiver. The GPS satellite sends a signal $r_s(t)$ equal to

$$ r_s(t) = S(\omega t) \sin \omega_c t $$

(3.6)

where $t$ is time in seconds, $\omega$ is the code chipping rate in chips per second, and $\omega_c$ is the carrier frequency in radians per second. The function $S(*)$ is a ±1 pseudo random code having an autocorrelation as described in Eqn. (3.5). The GPS receiver receives a signal $r(t)$ equal to

$$ r(t) = S[\omega t + \theta(t)] \sin [\omega_c t + \theta_c(t)] + n(t) $$

(3.7)

where $\theta(t)$ is the delay between the satellite and the receiver in chips, $\theta_c(t)$ is a phase shift in the carrier, and $n(t)$ is an additive channel noise.
Now suppose that the phase and frequency in the carrier are known. (The case where the carrier phase is not known will be dealt with later.) Knowing the carrier makes it possible to remove the carrier from the received waveform. This leaves only the PRN modulation \( S(\omega t + \theta) \). Removal of the carrier is accomplished by multiplying the received waveform by the known carrier waveform.

The modulating code \( S(\omega t + \theta) \) can be used to estimate the delay \( \theta \). To accomplish this a code \( S(\omega t + \hat{\theta}) \) is generated and correlated with \( S(\omega t + \theta) \). If \( \theta \), the delay in the input code, and \( \hat{\theta} \), the estimated delay used to generate a code, are equal then the resulting correlation takes its maximum value. If \( \theta \) and \( \hat{\theta} \) disagree then the resulting correlation takes some lesser value. The value that the correlation takes may be used to estimate the discrepancy between \( \theta \) and \( \hat{\theta} \). The correlation of the two codes \( S(\omega t + \theta) \) and \( S(\omega t + \hat{\theta}) \) is accomplished by multiplying these signals together and low pass filtering the result.

The input waveform is then first multiplied by \( \sin(\omega_c t + \theta_c(t)) \) to remove the carrier, then by \( S(\omega t + \hat{\theta}) \) to estimate the discrepancy between \( \theta \) and \( \hat{\theta} \). The effect of these multiplications can be determined by the following manipulations. Suppose that the input \( r(t) \) is multiplied by \( 2S(\omega t + \hat{\theta}(t)) \sin(\omega_c t + \theta_c(t)) \). (A gain of 2 is included so that the results are simpler.) The result of this multiplication is
\[ r(t)2S[\omega t + \hat{\theta}(t)] \sin [\omega_c t + \theta_c(t)] = \]

\[ = S[\omega t + \theta(t)]S[\omega t + \hat{\theta}(t)] 2 \sin^2 [\omega_c t + \theta_c(t)] + \tilde{n}(t) \]  

In this equation \( \tilde{n}(t) \) is a new noise which is shown in [Van Trees (2)] to be white. The double frequency components of \( \sin^2 [\omega_c t + \theta_c(t)] \) can be filtered, leaving only the bias of one-half. This makes

\[ r(t)2S[\omega t + \hat{\theta}(t)] \sin [\omega_c t + \theta_c(t)] = \]

\[ = S[\omega t + \theta(t)] S[\omega t + \hat{\theta}(t)] + \tilde{n}(t) \]  

The first term on the right hand side (RHS) of this equation displays a similarity to the integrand in the definition of the autocorrelation function. It seems reasonable to expect that the RHS of the equation above and \( R_{ss}[\theta(t) - \hat{\theta}(t)] + \tilde{n}(t) \) have equal time integrals, or, equivalently, that the outputs of the circuits shown in Figure 3.4(a) and 3.4(b) have equal time integrals. If these two outputs, shown in Figure 3.4(a) and (b), have equal time integrals then a measurement processor which acts as a low pass filter will have the same response to one as to the other. The processors to be proposed in later chapters will act as low pass filters and so modelling the physical situation in Figure 3.4(a) by the block diagram of Figure 3.4(b) will be valid.
In Appendix C it is shown rigorously that if the dynamics of $\theta(t)$ and $\hat{\theta}(t)$ are small, and if the chipping rate $\omega$ is large, then the correlation model is accurate in the sense that a time integral of the multiplier output in Figure 3.4(a) will be approximately equal to the time integral of the output signal in Figure 3.4(b). Appendix C shows additionally that the two signals which result from correlating the input signal with two different shifts of the feedback code will be corrupted by noises which are independent.
The essence of the arguments above is that, after some multiplications by known signals, the input signal may be taken to be a function $z(t)$ satisfying

$$z(t) = R_{ss} [\theta(t) - \hat{\theta}(t)] + \tilde{n}(t)$$

This measurement is in the form of the measurement taken by the telescope in Chapter 1. The only differences are that $R_{ss}$ is triangular whereas the telescope nonlinearity was square, and that $\hat{\theta}$, an estimate of $\theta$, appears where a general centering function appeared. Neither of these differences is fundamental. The basic understanding derived from the star tracking problem applies directly to the GPS code tracking problem.

The input signal was correlated with a single known code to produce the measurement $z(t)$. Several such correlations of the input can be performed against several shifts of the known code (i.e., against codes $S[\omega t + d_1(t)]$, $S[\omega t + d_2(t)]$, ...). This produces several measurements $z_i(t)$

$$z_i(t) = R_{ss} [\theta(t) - d_i(t)] + \tilde{n}_i(t)$$

where $d_i(t)$ is the time shift used to generate the $i^{th}$ shift of the known code and the noises $n_i$ are independent of one another. The situation here is analogous to having many telescopes making simultaneous measurements for star tracking. How to effectively use these multiple measurements will be shown later.
3.3 Extended Kalman Filter Derivation of Delay Lock Loop

In Chapter 1 a system using the outputs of two adjacent telescopes was proposed for tracking star elevation. The output of the lower telescope was subtracted from that of the upper one. The signal resulting from this subtraction gave an indication of tracking error and was used to close a feedback loop.

In this section the limitations of tracking systems of this type will be illustrated by some experiments with PRN code tracking loop designs. In particular, an extended Kalman filter will be designed for use in tracking PRN codes. The starting point for this extended Kalman filter design will be a measurement formed by taking the difference between two correlator outputs. The result of this subtraction will give an indication of error, just as in the star tracking example. The extended Kalman filter formalism is used to design the linear filter to follow the error indicator.

![Diagram of Code Loop Discriminator](image)

Figure 3.5 Code loop discriminator.

Figure 3.5 shows a block diagram of the circuitry used to develop an error signal. The two multipliers shown in Figure 3.5 provide the two
correlations required. The common input to these two multipliers is the input signal. The known code inputs to the correlators are time shifted by a fixed amount relative to one another. These time shifted codes are obtained by putting the code, out of the code generator, into a tapped delay line (shift register). The distance between the taps determines the offset between the codes on these taps.

Suppose that the code in the center position of the shift register in Figure 3.5 is believed to be in synchronism with the input code. That is, the code \( S(\omega t + \hat{\Theta}(t)) \) is in the center of the shift register. The code one shift to the right is then \( S(\omega t + \hat{\Theta} - 1) \) and that one shift to the left is \( S(\omega t + \hat{\Theta} + 1) \). The results of multiplying each of these with the code (disregarding input noise) on the input signal are modelled, according to earlier analysis, by \( R_{ss}(\Theta - \hat{\Theta} + 1) \) and \( R_{ss}(\Theta - \hat{\Theta} - 1) \) respectively. The difference between the two correlations with the input signal yields a measurement \( \frac{dz(t)}{dt} \) which satisfies

\[
\frac{dz}{dt} = (R(\Theta - \hat{\Theta} - 1) - R(\Theta - \hat{\Theta} + 1))dt + \beta_t \quad (3.10)
\]

The device shown in Figure 3.5 is called an early-late detector. "Early-late" refers to the fact that the correlations are with local codes which are earlier and later than the expected input. Figure 3.6 shows the mathematical model of the device in Figure 3.5.
Figure 3.6 Mathematical model of code loop discriminator.

Now suppose that $\theta_t$, the input delay process, is the first element of a vector process $x_t$ satisfying

$$dx_t = Ax_t dt + Bd\gamma_t$$

(3.11)

where $A, B$ are constant matrices and $\gamma_t$ is a vector Brownian motion. Let $\hat{x}_t$ denote the estimate of the vector $x_t$. Then the usual formalism [Jazwinski(1)] leads to the following extended Kalman filter

$$dx = A\hat{x} dt + PH^T \frac{1}{r} dz$$

(3.12)

$$\dot{\hat{P}} = AP + PA^T + Q - PH^T H \frac{1}{r}$$

(3.13)

where

$$r = \frac{1}{t} E(\beta_t^2), \quad Q = BB^T, \quad \text{and} \quad H = [1; 0]$$

(3.14)
Notice that the filter gains \( (PH^T_r)^{-1} \) have no data dependence. This is generally the case when extended Kalman filtering is applied to correlation tracking problems with linear dynamics. Suppose, for example, that the measurement nonlinearity is \( h(x-\hat{x}) \); then for the extended Kalman filter the linearized measurement matrix is

\[
H = \begin{bmatrix}
\frac{\partial h_i(x-\hat{x})}{\partial x_j}
\end{bmatrix}_{x=\hat{x}} = h'(0) \tag{3.15}
\]

This does not depend on the estimate \( \hat{x} \) because the nonlinearity is a function of the difference \( x - \hat{x} \).

The filter given above is in the form usually applied to the delay tracking problem. The usual tracking system passes the noisy error measurement \( dz/dt \) through a filter to arrive at an estimate of \( \hat{O} \). The resulting estimate is used to generate comparison codes, thus closing the tracking loop. This procedure is analogous to the one considered for the star tracking example in Chapter 1. In the delay tracking problem the system represented by Eqn. (3.12) is called a delay lock loop by analogy to the better known phase lock loop used for phase tracking of a sinusoidal signal. Delay lock loops are, for example, discussed in [Gill (1)], [Monzingo (1)], [Spilker & Magill (1)] and [Spilker (1), (2)] although the Kalman filter formalism has not been used to arrive at the delay lock loop.

The structure of the tracking system depicted by Eqn. (3.12) is perhaps better illustrated by the block diagram in Figure 3.7. The
extended Kalman filter formalism has yielded the forward loop transfer function for the delay lock loop. The design differs from those accomplished by other procedures in that this design gives optimal transient loop gains. Other design procedures suggested for this application are steady-state in nature.

![Block diagram of extended Kalman filter for early-late detector.](image_url)

Figure 3.7 Block diagram of extended Kalman filter for early-late detector.

In the next section the limitations of the delay lock loop will be illustrated by considering the particular case where the input delay motion is a Brownian motion.

3.4 **Limitations of Delay Lock Loop**

To demonstrate the difficulty which the delay lock loop suffers, some Monte Carlo runs have been made. For these purposes the following first order plant process is assumed.

\[ d \theta_t = d \gamma_t \]  

(3.16)

where \( \gamma_t \) (and thus the phase process \( \theta_t \)) is a Brownian motion.
This specialization yields the following filter equations

\[ d\theta_t = d\gamma_t \]  \hspace{1cm} (3.17)

\[ dz_t = [R(\theta-\hat{\theta}+1)-R(\theta-\hat{\theta}-1)]dt + d\beta_t \]  \hspace{1cm} (3.18)

\[ d\hat{\theta} = \frac{P}{r} dz_t \]  \hspace{1cm} (3.19)

\[ \hat{\beta} = q - \frac{P^2}{2r} \]  \hspace{1cm} (3.20)

Consider the predicted steady state error variance for this linearized filter. By solving the algebraic Riccati equation which results from setting \( \dot{\beta} = 0 \) the steady state covariance is found to satisfy

\[ p = \sqrt{2rq} \]  \hspace{1cm} (3.21)

Some Monte Carlo runs were made in order to see how well this predicted steady state covariance agrees with the error which the filter actually yields. This computer simulation was run as follows. Since the forward loop nonlinearity is only linear in the range \(-1 \leq e \leq 1\) it was expected that when the predicted error grew to one that the actual filter would be in trouble. A variety of situations was simulated. The conditions in each were chosen so that the extended Kalman filter was in steady state. Runs were made corresponding to each of several values of predicted one sigma error. Since the product qr parameterizes the
predicted steady state performance it seemed necessary only to vary one or the other of \( r \) and \( q \). For this reason \( q \) was fixed at 1.0 and \( r \) was varied to yield the desired steady state one sigma errors. In fact, ten values of \( r \) were chosen

\[
x_n = \frac{(1 \cdot n)^4}{2} \quad 0 < n \leq 10
\]  

(3.22)

For each value of \( r \) ten runs were made of the extended Kalman filter. The initial conditions and numbers in the noise sequences were obtained from Gaussian random number generators. Since the filter changes bandwidth with changes in noise statistics, the loop time constant was used to normalize the results. That is, the step size used for the numerical (Euler or rectangular) solution of the loop equation was taken to be \( 1/20 \) the closed loop time constant, and each sample output was computed for six loop time constants. Ten runs were made at each choice of steady-state error and corresponding value for the noise \( r \). The empirical standard deviation of the error is plotted versus the predicted standard deviation in Figure 3.8. That figure clearly indicates that the extended Kalman filter is inadequate for this correlation tracking problem when its predicted one sigma error grows past approximately one-half of a chip.
Figure 3.8 Simulation results versus results predicted by extended Kalman filter.
3.5 Previous Attempts to Overcome the Limitations of the Delay Lock Loop

It has long been understood that the finite range of the forward loop nonlinearity is the root cause of the failure of the delay lock loop for large errors. It was demonstrated in Section 3.4 that large errors could be induced by worsening the statistics of the noises injected into the delay lock loop. The delay lock loop is also ineffective in benign noise if it is given a large initial error. This ineffectiveness is the reason that a coarse acquisition algorithm has to be employed to initialize the delay lock loop. The range of the forward loop nonlinearity can be broadened by using additional measurements. Figure 3.9(a) shows one procedure for achieving this extra range. Figure 3.9(a) shows, on a single graph, the outputs of several early and several late correlators. Adding all the early correlations and subtracting all the late correlations yields a broadened forward loop nonlinearity as shown in Figure 3.9(b). This method is suggested, for example, in [Spilker (1)] and in [Schiff (1)]. The difficulty with adding these extra correlations in this manner is that each additional correlation brings with it additional noise. The measurement resulting from adding more than two correlations is then noisier than the traditional early measurement. This means that in the cases where the traditional scheme is able to track with small errors if it is given a good initial condition, the extended range scheme will experience relatively larger errors. Because of this, the traditional
Figure 3.9(a) Outputs of individual correlators.

Figure 3.9(b) Nonlinearity obtained by subtracting all early correlations from all late correlations.

Wisdom is that extra correlators are not useful for tracking. The fallacy with this is that the reasoning only applies in benign conditions where the traditional scheme can track with small errors. If conditions are severe enough to cause the failure of the traditional scheme then benefit can be derived by including extra correlations to form the forward loop nonlinearity. This fact will be proven in Chapter 4. Since these extra correlations are useful in some conditions but not in others, then if conditions are changing it is necessary to vary the detector shape in order to achieve optimum performance. Algorithms for varying the forward loop nonlinearity and the forward loop gains will be presented in the next two chapters.
3.6 Summary

In this chapter it has been demonstrated (with reference to complete proofs in Appendices) that the delay tracking problem can be thought of as a correlation tracking problem. That is, it has the measurement structure used to define correlation tracking problems in Chapter 2. The early-late detector has been introduced and the extended Kalman filter formalism has been used to arrive at tracking loop design based on the early-late detector. The limitations of this design were illustrated in an example problem.

In the next chapters several designs will be offered which surmount the difficulties encountered with the early-late detector and extended Kalman filter. The starting point for these designs will be the correlation model derived early in this chapter. It will turn out that both the early-late detector and extended Kalman filter must be scrapped in order to accomplish a useful design.
CHAPTER 4

POKKER-PLANCK APPROACH TO
MULTIPLE CORRELATOR MANAGEMENT

4.0 Introduction

It was seen in the last chapter that a GPS code tracking loop closed around the traditional early-late correlator will not function in a high measurement noise, high dynamics environment. It is desired to have a system which will operate in severe conditions, and it will be shown in this chapter that the outputs of extra correlators (besides one early and one late correlator) can be used to achieve this goal.

It is pointed out in the previous chapter that the problems of the traditional loop are partly the fault of its forward loop nonlinearity. The traditional nonlinearity, recall, is unresponsive to errors larger in magnitude than two chips. In this chapter the outputs of extra correlators will be used to build forward loop nonlinearities which respond to larger errors.

It will be assumed in this chapter that the outputs of many correlators are simultaneously available. This is analogous to having many telescope outputs available in the star tracking problem discussed in Chapter 1. It will be seen in this chapter that these correlator
outputs can be weighted and summed so as to yield any one of a broad class of forward loop nonlinearities. The complication is that each correlator used in building the forward loop nonlinearity brings with it some noise. The extra pull-in power of the nonlinearity with broad range has to be balanced against this extra noise. In this chapter Fokker-Planck techniques are used to quantify the effect of broadened range and increased noise on a tracking system performance.

Fokker-Planck techniques are useful in this context for two reasons. One is that the nonlinearity operates on the tracking error. This simplifies some of the mathematics. The second reason is that the Fokker-Planck solution depends on the global character of the nonlinearity. As a comparison, the extended Kalman filter depends on the local behavior of the nonlinearity. For example, the nonlinearities shown in Figures 4.1(a), 4.1(b), and 4.1(c) all have the same small signal gain and so the extended Kalman filter could only decide between them on the basis of their post-detection noise levels.

This chapter begins by demonstrating, in Section 4.1, that a broad family of forward loop nonlinearities can be built by weighting and summing adjacent correlator outputs. The relation between shape of the nonlinearity and the noise level at its output is given.
Figure 4.1 Three nonlinearities with the same small signal gain.
4.1 Generalized Weighted Sum of Correlations

Suppose that the outputs of many adjacent correlations are available. These multiple correlator outputs are diagrammed in Figure 4.2. Figure 4.2(a) shows the actual implementation while Figure 4.2(b) gives the autocorrelation model which the analysis in Chapter 3 suggests.

The shape and noise level of the forward loop nonlinearity determine its performance in a particular tracking loop.

Fokker-Planck techniques are used in Section 4.2 to determine the steady state probability density of the error achieved using an arbitrary forward loop nonlinearity. This probability density is obtained for a particular type of input motion and tracking loop. Since the solution is obtained for an arbitrary forward loop nonlinearity, a straightforward optimization can be done to yield that nonlinearity which minimizes the mean squared tracking error. This optimization is done in Section 4.3 and some interesting properties of the optimum nonlinearity are given. A tracking loop using this optimum forward loop nonlinearity is simulated and found to be able to track in conditions where the extended Kalman filter in Chapter 3 was unable to track.

The results in Sections 4.2 and 4.3 assume a particular type of input motion and tracking loop. Specifically, they assume a Brownian motion input and a first-order tracking loop. Sections 4.4, 4.5, and 4.6 discuss more general types of input motions and tracking loops.

Section 4.7 discusses the advantages and limitations of the approach taken in this chapter.
The stored code in Figure 4.2(a) is shown going into a shift register. The shift register is being clocked at the code chipping rate and so the contents of adjacent cells of the shift register are time shifted relative to one another by exactly one chip. (One chip, recall, is the basic length of time over which the code waveform is constant.) In the center of the shift register is a cell whose contents are denoted by $S(\omega t + \hat{\theta})$. The code $S(\omega t + \hat{\theta})$ is sometimes called the on-time code. It is desired to construct a system which will drive the code $S(\omega t + \hat{\theta})$ into coincidence with the code on the input signal.

![Diagram](image)

(a) Parallel multiplier circuit

![Diagram](image)

(b) Correlation model for parallel multiplier

Figure 4.2 Multiple correlator outputs.
The traditional tracking loop as described in Chapter 3 operates by subtracting the correlation of the input signal with the code one chip earlier than \( S(\omega t + \hat{\theta}) \) (i.e., the content of the cell to the right of the one containing \( S(\omega t + \hat{\theta}) \)) from the correlation of the input with the code one chip later than \( S(\omega t + \hat{\theta}) \). The early-late detector, as the two correlator and comparator circuit is sometimes called, was found in Chapter 3 to lead to system failure in severe noise and dynamics. This failure was blamed on the finite range of errors for which the detector output was nonzero.

The hardware configuration in Figure 4.2(a) presents correlations with many different shifts of the stored code. In this chapter these additional correlations will be used to design a system which will not suffer some of the problems that the traditional system suffers.

One method of using the outputs of these correlators to configure a delay tracking system is to weight these outputs, then sum the weighted outputs. The resulting weighted sum is some nonlinear function of the tracking error \( \theta - \hat{\theta} \). The shape of this nonlinear function of \( \theta - \hat{\theta} \) depends on the weights chosen. This function is closed in a feedback loop as the early-late detector was when only two correlations were available. As an example, suppose that all the outputs corresponding to multiplications by codes later than on-time (those to the left of \( S(\omega t + \hat{\theta}) \) in Figure 4.2(a)) are simply added, and those earlier are multiplied by minus 1 and then added. Adding together the outputs of adjacent correlators gives a forward loop nonlinearity whose behavior as a function of error can be understood by adding together the individual correlation
triangles involved. Figure 4.3(a) attempts to make this more clear. In Figure 4.3(a) are drawn, all on one graph, the outputs of each of the individual correlators as a function of tracking error. The early correlations, those corresponding to a negative error, are shown inverted because they have all been multiplied by minus one, according to the plan above. If these triangles are all added together then the nonlinearity shown in Figure 4.3(b) results. This method of combining the extra correlator outputs results in a detector whose output stays at ±1 as the errors increase past one in magnitude. The output still goes to zero outside some finite range, but the range is broader than in the simple early-late case. Schemes of this sort have been considered before [Spilker (2)].

(a) INDIVIDUAL CORRELATOR OUTPUTS

(b) RESULTING FORWARD LOOP NONLINEARITY

Figure 4.3 Nonlinearity resulting from adding all late chips and subtracting all early chips.

Since the extra correlators introduce extra noise it is clear that in high signal-to-noise ratio conditions a loop closed around this detector will suffer larger steady-state errors than one closed around
a standard detector. This consideration has led to the rejection of
this scheme as a tracking and acquisition system [Spilker (2)]. The
situation is different, however, if instead of choosing a single fixed
detector breadth, the breadth is left controllable. To control the
detector breadth, the number of early correlations and late correlations
that go to form the nonlinearity can be controlled. If one early and
one late correlation are used then the standard detector results. If
ten early and ten late correlations are used then a detector ±10 side
results. Then, with such a controllable breadth, one could imagine
the detector being extended during acquisition and narrowed as a tracking
status is reached. To optimize this type of system requires a nonlinear
approach. The analysis in [Bowles and Roth (1)] indicates that the
detector breadth should be made proportional to the expected one-sigma
error. Thus during acquisition, when the expected errors are large,
the detector range is broad, and during tracking, when the errors have
been reduced, the detector is narrow. This indicates that the use of the
usual ±1 detector during tracking is a limiting solution, valid when the
errors are small.

Many different choices of weights are possible. For example,
if each correlator is weighted by its local code's offset from the on-
time code, then an extended linear range results. Figure 4.4 shows
what the forward loop nonlinearity would look like in this case.
This type of extension is suggested in [Schiff (1)] and analyzed in
[Bowles and Roth (1)]. Although approximate nonlinear optimization sug-
gests that a loop with a variable range will outperform the standard
loop, there is still a problem with these systems.
In [Bowles and Roth (1)] two types of variable range detectors are considered. One is the flat top detector pictured in Figure 4.3(b); the other is the extended linear detector pictured in Figure 4.4. The only degree of control which can be exercised in either one of these cases is the number of early and late correlations, say \( N \). The number, \( N \), is an integer which indicates the breadth of the detector. For a flat top detector, \( \pm N \) is the range over which the detector is flat. For the extended linear detector, \( \pm N \) is the range over which the detector is linear. In [Bowles and Roth (1)] then, an approximate nonlinear filtering technique is used to determine, as a function of time and of the input noise and dynamics levels, the optimal range extension, \( N \), for the flat top detector and the linear detector. Analyses presented in [Bowles (1)] and [Konop (1)] tend to indicate that even with optimization both of these two variable range loops can be unstable in some circumstances. More specifically, if the input dynamics and the measurement noise are strong enough, the error covariance of the optimal variable range loop grows without bound. It will be shown here that this difficulty can be overcome in many cases, and its resolution will suggest a new optimization procedure.
The approaches above may be generalized by considering weights \( w_i \) applied to the correlator shifted by \( i \) bits with respect to the on-time bit. The result of this is a nonlinearity composed of straight line segments connecting the points \((-n, 0), (-n, w_{-n}), (-n, w_{-(n-1)}), \ldots, (-1, w_{-1}), (0, w_0), \ldots, (m, w_m) (m+1, 0)\) where the \( w_i \) are the arbitrary weights. This nonlinearity will be denoted by \( f(e) \). Notice that for an integer \( i \), \( f(i) = w_i \). An example of such a detector nonlinearity is shown in Figure 4.5. The detector drawn there would probably not be a useful one. The point is, however, that very general shapes are obtainable.

![Figure 4.5 A hypothetical detector which is obtainable using weights \( w_i \).](image)

Noise is an essential consideration in determining the optimum detector shape. It was shown in Chapter 3 that the noises present on the different correlator outputs are independent, zero mean, and white with equal covariances. Let \( n_i(t) \) be the noise present on the output of the \( i^{th} \) correlator and let \( \mathbb{E}[n_i(t)n_i(\tau)] = r_\delta(t-\tau) \). The spectral
density of the noise on the weighted sum of the correlator outputs is then

\[ E \left\{ \sum_{i=-n}^{m} w_i n_i(t) \sum_{i=-n}^{m} w_i n_i(\tau) \right\} = r \sum_{i=-n}^{m} w_i^2 \delta(t-\tau) \]

The post-detection noise variance is then proportional to \( \sum_{i=-n}^{m} w_i^2 \). The noise gain of a detector \( f(\cdot) \) will be denoted by \( K_f = (\sum w_i^2)^{1/2} \), where the weights \( w_i \) are the ones used to construct the detector \( f(\cdot) \). The \( w_i \) must be chosen so that their signal detection assets outweigh their noise liabilities. In the next two sections a method will be given to determine the optimum weights \( w_i \) for a particular type of input delay process \( \theta(t) \).

4.2 Steady-State Fokker-Planck Solution for Brownian Input

In this section a tracking loop will be closed around a general nonlinearity of the form shown in Figure 4.5. Then, for a specific input delay process \( \theta(t) \) the tracking errors will be determined. In this section it will be assumed that the input delay process \( \dot{\theta}(t) \) is a Brownian motion. (Recall that \( \dot{\theta}(t) \) is the argument of \( S(\omega t + \dot{\theta}(t)) \) which is the code present on the input signal.) The Brownian motion is a very simple random process and the input process \( \dot{\theta}(t) \) would in most cases be better modeled by a more complicated process. In most applications it seems that \( \dot{\theta}(t) \) is well enough modeled by the output of a linear system driven by white noise. Models of this sort will be considered in Sections 4.3 and 4.4 and in Chapter 5. The Brownian input allows very explicit solution and for that reason will be considered in this section.
The subject of this section will be the system diagrammed in Figure 4.6. In that diagram, the input is shown as $\sqrt{q} \beta_t$, where $q$ is a constant and $\beta_t$ is a sample path of a standard Brownian motion. That is, $E(\beta_t) = 0$; $\beta_0 = 0$ with probability one; $\beta_t$ is continuous with probability one; and $E(\beta_t \beta_s) = \min(t,s)$. Checking the loop response to this type of input is the stochastic equivalent of checking its response to a ramp input, as is often done in deterministic analysis. The post-detection noise is represented in Figure 4.6 by the expression $\sqrt{r} K_f \frac{dY_t}{dt}$.

The constant $r$ characterizes the received noise power density and will, along with $q$, be carried through the analysis. The noise gain $K_f$ depends on the detector weights chosen as was indicated in the last section.

The notation $\frac{dY_t}{dt}$ represents the formal derivative of a standard Brownian motion. That is, $\frac{dY_t}{dt}$ is a white noise with unit spectral density.

To simplify later optimization of $f(e)$, two approximations are made. First, it is assumed that any continuous function $f(e)$ is obtainable, not just those which are piecewise linear and of finite extent.
The second assumption involves the noise gain $K_f$. As indicated in the last section, $K_f$ is given by

$$K_f^2 = \sum_{i=-\infty}^{\infty} f^2(i)$$

but it will be assumed that $K_f$ is well approximated by

$$K_f^2 = \int_{-\infty}^{\infty} f^2(e)\,de \tag{4.1}$$

For the flat top and linear detectors considered above, this is a reasonable approximation. The linear detector pictured in Figure 4.4 has weights $w_i = i$ for $|i| \leq N$. For this detector

$$K_f^2 = \sum_{i=-N}^{N} i^2 = \frac{2N(N+1)(2N+1)}{6} \tag{4.2}$$

The approximation indicated would compute $K_f^2$ to be

$$K_f^2 = \frac{2N^3}{3} + \frac{2N^2}{3} \tag{4.3}$$

For large $N$ this is close to the exact $K_f^2$. The flat top detector in Figure 4.3 has weights $w_i = 1$ for $0 < i \leq N$ and $w_i = -1$ for $-N \leq i < 0$. For this detector

$$K_f^2 = 2N \tag{4.4}$$
The approximation indicated would give

\[ K_{\text{f}}^2 = 2(N-1) + 4/3 = 2N - 2/3 \]  \hspace{1cm} (4.5)

Again, the approximation is reasonable for large \( N \).

The loop error statistics are characterized by the time-varying probability density of the error \( e(t) \). To analyze this loop, the steady state probability density will be computed. To obtain this density, we write the differential equation for the error as indicated by the block diagram in Figure 4.6. In Itô form that equation is

\[ de = -K_{\text{f}}(e) dt - \sqrt{K_{\text{f}}} \, K_{\text{f}} \, d\gamma_t + \sqrt{\alpha} \, d\beta_t \]  \hspace{1cm} (4.6)

The error probability density then satisfies a Fokker-Planck equation which can be determined from Eqn. (4.6) [Jazwinskie (1)]. The Fokker-Planck equation in this case is

\[ \frac{\partial p(e,t)}{\partial t} = - \frac{\partial (-K_{\text{f}}(e)p(e,t))}{\partial e} + \frac{1}{2} \left( \kappa K_{\text{f}}^2 + \alpha \right) \frac{\partial^2 p(e,t)}{\partial e^2} \]  \hspace{1cm} (4.7)

Assume that \( p(e,t) \) approaches a limit as \( t \to \infty \). Then, in steady state, \( \frac{\partial p(e,t)}{\partial t} \to 0 \). Denote the steady-state probability density, if it exists, by \( p(e) \). That is,

\[ \lim_{t \to \infty} p(e,t) = p(e) \]  \hspace{1cm} (4.8)
if the limit exists. The steady-state probability density then satisfies

\[- \frac{\partial (-K f(e) p(e))}{\partial e} + \frac{1}{2} (rK^2 k^2 + q) \frac{\partial^2 p(e)}{\partial e^2} = 0 \]  

(4.9)

Integrating this equation once with respect to \( e \) and rearranging it gives

\[ \frac{\partial p}{\partial e} = - \frac{2K}{rK^2 k^2 + q} f(e) p(e) + C_1 \]  

(4.10)

where \( C_1 \) is a constant of integration. Assume both \( f(e) \) and \( p(e) \) are uniformly continuous. Then since

\[ \int_{-\infty}^{\infty} f^2(e) \, de = k_f^2 \]  

(4.11)

and

\[ \int_{-\infty}^{\infty} p(e) \, de = 1 \]  

(4.12)

it must be that

\[ \lim_{e \to \infty} f(e) = \lim_{e \to \infty} p(e) = 0 \]  

(4.13)

It follows that

\[ \lim_{e \to \infty} \frac{\partial p}{\partial e} = C_1 \]  

(4.14)

and so it must be true then that \( C_1 = 0 \). The resulting equation for \( p(e) \) may be solved to give
\[ p(e) = \exp \left\{ -b \int_0^e f(u) \, du \right\} p(0) \quad (4.15) \]

where

\[ b = \frac{2K}{rK^2 K_f^2 + q} \quad (4.16) \]

A first question to ask is whether there exists any function \( f(e) \) which gives finite error variance. In terms of the probability density just derived, the question is whether or not there exists any function \( f(e) \) for which both

\[ \int_{-\infty}^{\infty} e^2 p(e) \, de = \int_{-\infty}^{\infty} e^2 \exp \left\{ -b \int_0^e f(u) \, du \right\} p(0) \, de < \infty \quad (4.17) \]

and

\[ \int_{-\infty}^{\infty} f^2(e) \, de < \infty \quad (4.18) \]

In words, the condition in Eqn. (4.17) is that the steady state variance be finite. The condition in Eqn. (4.18) is that the detector noise gain \( K_f \) be finite. Eqn. (4.17) indicates that

\[ \lim_{e \to \infty} \int_0^e f(u) \, du = \infty \quad (4.19) \]
which might appear to contradict Eqn. (4.18). However for functions which behave for large $|e|$ like $1/|e|^\alpha$ with $1/2 < \alpha < 1$, both these conditions are satisfied. In that case (for large $|e|$) $\int_0^e f(u)du$ behaves like $e^{-\alpha + 1}$ which is unbounded as $|e| \to \infty$. It is not difficult to verify (again for $1/2 < \alpha < 1$) that $\exp\{-be^{-\alpha + 1}\}$ goes to zero faster than any polynomial in $e$ and so

$$\int_{-\infty}^{\infty} e^2 p(e) de$$

(4.20)

On the other hand, $\int_0^e f^2(u)du$ behaves for large $e$ like $e^{-2\alpha + 1}$ plus a constant. It may also be verified that $f(e) \sim c/e$ will work with a careful choice of the constant $c$.

It should be pointed out that no detector of finite extent will yield finite steady-state variance under the conditions considered here. This is both an indictment of finite range detectors and an indication of how severe a theoretical Brownian motion is.

4.3 Optimum Extended Range Detectors

The next step is to find the nonlinearity $f(\cdot)$ which gives minimum mean square error. That is, choose $f(e)$ to minimize

$$\int_{-\infty}^{\infty} e^2 p(e) de = \int_{-\infty}^{\infty} e^2 \exp\{-b\int_{0}^{e} f(u)du\} p(0)de$$

(4.21)
with the constraints

\[
\int_{-\infty}^{\infty} p(0) \exp \left\{-b \int_{0}^{e} f(u) du \right\} de = 1 \tag{4.22}
\]

and

\[
\int_{-\infty}^{\infty} f^2(e) de = K_f^2 \tag{4.23}
\]

This last constraint is necessary to insure that \( \int f^2 < \infty \). The numerical value for \( K_f^2 \) is not known ahead of time, but will be discovered later as part of the optimization. There is an obvious redundancy in this problem if forward loop gain, \( K \), is allowed to vary. It turns out to be computationally convenient to fix \( b \) and \( K_f \) and to choose the gain \( K \) so that the fixed parameter \( b \) corresponds to the worst possible \( q \) and \( r \) conditions. This will be explained in more detail in Section 4.3.1.

To facilitate this minimization let

\[
g(e) = \int_{0}^{e} f(u) du \tag{4.24}
\]

Then \( g'(e) = f(e) \). The minimization problem, after some rearrangement, is to minimize

\[
\int_{-\infty}^{\infty} e^2 \exp \{-bg(e)\} de \tag{4.25}
\]
with the constraints

\[ \int_{-\infty}^{\infty} \exp\{-b g(e)\} de = 1/p(0) \]  \hspace{1cm} (4.26)

\[ \int_{-\infty}^{\infty} g'^2(e) de = k_f^2 \]  \hspace{1cm} (4.27)

The Euler-Lagrange equation for this problem is

\[ g''(e) = -\frac{b}{2\lambda_2} (\lambda_1 + e^2) \exp\{-b g(e)\} \]  \hspace{1cm} (4.28)

where \( \lambda_1 \) and \( \lambda_2 \) are Lagrange multipliers associated with the constraints.

To obtain the optimal detector shape, that is the one which gives minimum mean square error, this differential equation must be solved for arbitrary \( \lambda_1 \) and \( \lambda_2 \). The multipliers \( \lambda_1 \) and \( \lambda_2 \) are then chosen so that the two constraints are satisfied.

### 4.3.1 Numerical Solution for the Optimal Detector

This section deals with the numerical solution of the Euler-Lagrange equation for the optimal detector shape. The reader who is uninterested in the numerics should skip over to Figures 4.8 and 4.9 which display the optimal detector and resulting probability density. The next section (Section 4.3.2) discusses some of the qualitative properties of the optimal detector shape.
In the absence of a closed form solution to the Euler-Lagrange equation, numerical solution would seem to be a reasonable approach. Since, in the performance index, $E[e^2], e^2$ is symmetric in $e$, it seems reasonable that the minimizing $p(e)$ should also be symmetric. This implies that $g(e)$ is also symmetric and, therefore, that $g'(e)$ is antisymmetric. From this it follows that $g'(0) = 0$ and from the definition of $g(e)$ it follows that $g(0) = 0$.

This symmetry assumption is validated when, with these initial conditions, a solution to the first order necessary conditions is obtained. Such a solution is produced below. With these conditions, the Euler-Lagrange equation can be solved as an initial value problem. The direct numerical approach is to choose $\lambda_1, \lambda_2$ and integrate the differential equation until the areas under $g'^2(e)$ and $p(e)$ stop growing. If the constraints are violated, then $\lambda_1$ and $\lambda_2$ are adjusted and the process repeated. When the constraints are satisfied to some reasonable accuracy, the operation halts. This procedure is not effective for the following reason.

When trying this direct numerical approach, two problems arise. Figure 4.7 shows typical plots of two solutions ($g'(e)$) for fixed $\lambda_1$ and different $b, \lambda_2$. As these behaviors illustrate, the values for one or both of the constraint functions is $+\infty$. The slightest error in $b, \lambda_2$, or $\lambda_1$ seems to cause one of the constraints to be violated in this way. This is a difficulty in itself and causes a problem in deciding when to stop the numerical integration. It is difficult to estimate the area remaining under $g'^2(e)$ and $p(e)$. A different approach will avoid these difficulties.
Figure 4.7 Two typical \( g'(e) \) behaviors obtained for different values of \( b \) and \( \lambda_2 \).

To solve the Euler-Lagrange equation, a combination, a combination numerical-asymptotic technique can be used. Notice that for large \( e \) the function

\[
g(e) = \frac{4}{b} \ln e + \frac{1}{b} \ln \frac{b^2}{8} \quad (4.30)
\]

is a solution to the Euler-Lagrange equation (Eqn. 4.28). That is, if in the Euler-Lagrange equation, Eqn. (4.28), \( \lambda_1 \) is neglected in favor of \( e^2 \), then the function in Eqn. (4.30) is a solution to the resulting differential equation. This means that the function in Eqn. (4.30) is an asymptotic solution to the Euler-Lagrange equation which is valid for large \( e \). This can be used to provide large \( e \) behavior, and the numerical solution used to provide small \( e \) behavior.
Before proceeding further, the differential equation should be inspected to find out in what ranges the parameters $b$, $\lambda_1$ and $\lambda_2$ should lie and to determine which of these parameters should be searched. The integrand of the performance index is $e^2 p(e)$. Since $e^2$ is monotone increasing it seems reasonable that $p(e)$ should be monotone decreasing. This means that $g(e)$ must be monotone increasing, and that $g'(e)$ for $e \neq 0$. The function $g'(e)$, however, must approach zero as $e$ and $g'(0) = 0$. These conditions imply that $g''(e)$ starts out positive, then changes to negative and remains negative, but approaches zero. It must be, then, that $\lambda_1 < 0$, $\lambda_2 > 0$, and $b < 0$. Since $e = \sqrt{-\lambda_1}$ is the point where $g''(e) = 0$ then it is the point where $g'(e)$ reaches its maximum. It is reasoned that some constant, say eight, times $\sqrt{-\lambda_1}$ is a sufficiently large value for $j$ that the asymptotic approximation in Eqn. (4.30) is valid. It seems reasonable then to fix $\lambda_1$; that is, to fix the point at which the solutions are matched and adjust $b$ and $\lambda_2$. In this way the differential equation can be solved as a two-point boundary value problem.

The initial conditions $g(0) = 0$ and $g'(0) = 0$ are used, along with a guess for $b$ and $\lambda_2$, to generate a numerical solution out to, say, $e = 8\sqrt{-\lambda_1}$. The numerical solution at this point is compared with the asymptotic solution given in Eqn. (4.30). The guesses for $b$ and $\lambda_2$ are adjusted and new numerical solutions are generated until the numerical and asymptotic solutions agree at the point $e = 8\sqrt{-\lambda_1}$. 
What about the constraints of the original minimization problem? If \( b, \lambda_1, \lambda_2 \) are all specified, then the constraints may be violated. It turns out that the only thing of importance in the constraints is that

\[
\int_{-\infty}^{\infty} g'(e)^2 \, de < \infty \quad (4.31)
\]

and

\[
\int_{-\infty}^{\infty} \exp\{-bg(e)\} \, de < \infty \quad (4.32)
\]

The precise values of these integrals can be handled analytically as a simple scaling to the final solution. Since \( b \) has been exchanged for \( \lambda_1 \) as a manipulable parameter, then it results that \( q \) and \( r \) have to be determined after the computation of the optimal \( g(e) \).

The algorithm used for finding the optimal \( g(e) \) is then summarized as follows. First choose \( \lambda_1 \). This will determine the spread of the probability density. In fact, the numerical examples will show that the minimized value of the performance index is equal to \(-\lambda_1\) (that is, \( \sigma^2 = -\lambda_1 \)). Second, choose some point \((e = 8\sqrt{-\lambda_1} \text{ works well enough})\) to match the asymptotic and numerical solutions. Third, select a value of \( b \) and \( \lambda_2 \) and integrate the equation to the matching point. Fourth, compute values of \( g' \) and \( g \) using the asymptotic formulae and compare to the numerical values. If the values are not close, adjust \( b \) and \( \lambda_2 \) and return to the third step. If the asymptotic and numerical values for \( g \) and \( g' \) are close, then compute the constraining integrals using the numerical curves.
for small \( e \) and the asymptotic curves for large \( e \). Denote the values thus obtained for the constraint integrals by

\[
\int_{-\infty}^{\infty} \exp\{-bg_0(e)\} \, de = \frac{1}{p(0)} \tag{4.33}
\]

and

\[
\int_{-\infty}^{\infty} g_0^2(e) \, de = K_f^2 \tag{4.34}
\]

where \( g_0 \) is the combination of the solution to the two-point boundary value problem and the asymptotic solution to the Euler-Lagrange equation.

This algorithm yields the optimal detector shape, albeit in a backwards fashion – backwards because the conditions for which the solution is optimal are an output from the algorithm instead of an input to it. This can be handled by generating a family of solutions for a range of different \( \lambda \)s. These would correspond to a range of bs. Given the input conditions, the solution can be looked up in this family of solutions. The value of the parameter \( b \) determines the achievable system performance. The larger \( b \) is, the smaller the resulting mean square error will be. It follows that the forward loop gain, \( K \), should be specified to maximize \( b \) for given input parameters \( q \) and \( r \).

The constant \( b \) was determined to solve the two-point boundary value problem and the equation
\[ b = \frac{2K}{rK^2_f + q} \equiv h(K) \]  

(4.35)

can be used to find \( r \) and \( q \). As just indicated, \( K \) is determined so as to maximize \( h(K) \). To this end, find \( K \) such that

\[ \frac{dh}{dK} = \frac{2}{rK^2_f + q} - \frac{4K^2 rK^2_f}{(rK^2_f + q)^2} = 0 \]  

(4.36)

\[ (rK^2_f + q) = 2K^2 rK^2_f \]  

(4.37)

or

\[ K^2 = \frac{q}{rK^2_f} \]  

(4.38)

This gives the maximum for \( h(K) \) to be

\[ h \left( \sqrt{\frac{q}{rK^2_f}} \right) = \frac{2}{rK^2_f + q} \frac{\sqrt{q} rK^2_f}{q} = \sqrt{\frac{1}{q} rK^2_f} \]  

(4.39)

Setting this equal to \( b \) and solving for \( qr \) gives

\[ qr = \frac{1}{b^2 K^2_f} \]  

(4.40)

The determinant of loop performance is then the \( q-r \) product.
Figures 4.8 and 4.9 are plots of the optimal detector, $g'(e)/K_f$, and the associated probability density, $\exp\{-bg\}p(0)$. The two plots are for different values of $\lambda_1$ and as a result they exhibit different loop errors $E(e^2)$. The associated constants $b$, $\lambda_2$, $q$ and $r$ are determined as indicated in the last paragraph and are displayed on the graphs.

These numerical results tend to indicate several things. One is that the optimal $E(e^2)$ is equal to $-\lambda_1$. Another is that the optimal $E(e^2)$ goes like $(qr)^2$ instead of $(qr)^{1/2}$ which would be the case for a linear system. The asymptotic approximation is validated by the numerical results. For a fixed matching point the errors between the asymptotic solution and the numerical solution cannot be made zero. (Remember that both the numerical and asymptotic solutions are functions of $b$ and $\lambda_2$). This could imply that at the match, the two solutions are not independent (certainly a desirable situation). By taking the matching point to be larger the relative errors can be reduced. That is, if $g_a$ is the asymptotic solution and $g_n$ the numerical solution, then

$$\min_{b, \lambda_2} \left[ \left( \frac{g_a(e) - g_n(e)}{g_a(e)} \right)^2 + \left( \frac{g_a'(e) - g_n'(e)}{g_a(e)} \right)^2 \right]$$

gets smaller as $e$ increases. Finally, the similarity of the detector shapes obtained in these examples tends to indicate that any two optimal detectors $g_1(e)$ and $g_2(e)$ are related by

$$g_1(e) = \alpha_1 g_2(\alpha_2 e)$$

where $\alpha_1$ and $\alpha_2$ are some constants.
Figure 4.8 Plots of optimal detector curve and resulting probability density $\sigma^2 = .01$. 

\[
\begin{align*}
    \theta &= 296.003 \\
    \lambda_2 &= 1.0969 \\
    qr &= 3.97599 \times 10^{-3}
\end{align*}
\]
Figure 4.9 Plots of optimal detector curve and resulting probability density $\sigma^2 = 1$. 

\begin{align*}
  b &= 7.222 \\
  \lambda_2 &= 6.527 \\
  qr &= 3.9641 \times 10^{-2}
\end{align*}
4.3.2 Properties of the Optimal Detector Shape

The detector shapes shown in Figure 4.8 and 4.9 have several interesting and useful features which should be indicated. The peaks on these detectors occur at the optimal one-sigma error. This means that all the errors between plus and minus one sigma see a gain which is roughly linear. The optimal one-sigma error, \( \sigma_{\text{opt}} \), depends on the product \( qr \). That is the product of the dynamics and the noise variance parameters. The one-sigma error is linear in \( qr \). That is,

\[
\sigma_{\text{opt}} = C \cdot qr
\]

and numerical estimates yield

\[
C = 25.2
\]

Different detectors are required for different values of \( qr \), but observation of the curves in Figures 4.8 and 4.9 indicate that those two curves, at least, are just stretched, scaled versions of one another. It will be proven later in this section that this observation is generally correct. It will be proven that the optimal detector shape (denoted by \( f_{\sigma_{\text{opt}}}(*) \) where \( \sigma_{\text{opt}} = 25.2 \cdot qr \cdot r_1 \)) is related to the shape in Figure 4.9 (denoted by \( f_{r_1}(*) \) since in that case 25.2 \( qr = 1 \)) by

\[
f_{\sigma_{\text{opt}}} (e) = f_{r_1} (e/\sigma_{\text{opt}}).
\]
The optimum forward loop gain under these conditions is, as was shown in the last section,

\[ K = \sqrt{\frac{q_1}{r_1 K_{f_1}^2}} \]

where

\[ K_{f_1}^2 = \int_{-\infty}^{\infty} f_{e_1}(e) de \]

These last five expressions completely characterize the optimal tracking loop and predict its steady state error. Since all the detector shapes are merely a stretch of gain shape, only one such shape has to be stored in order to implement this tracking system.

The rest of this section will be devoted to proving these properties. Those not interested in the details of these proofs may wish to turn to Section 4.3.3 where the simulation of this system is described.

To prove that optimal detectors are just scaled stretches of one another, suppose that \( g(e) \) solves

\[ g''(e) = -\frac{b}{2\lambda_2} (\lambda_1 + e^2) \exp\{-bg(e)\} \quad (4.43) \]

Now suppose that \( q_1 \) is a solution of Eqn. (4.43) for a particular set of parameters \( b, \lambda_1, \lambda_2 \) for which \( q_1 \) satisfies the given constraints. That is

\[ \int_{-\infty}^{\infty} \exp\{-bg_1(e)\} de = 1/\varphi(0) \quad (4.44) \]
and

$$\int_{-\infty}^{\infty} g_1^2(e) de = k_f^2$$  \hspace{1cm} (4.45)$$

Suppose also that $g_2(e)$ is another function which is related to $g_1$ by

$$\alpha_1 g_2(\alpha_2 e) = g_1(e)$$  \hspace{1cm} (4.46)$$

where $\alpha_1$ and $\alpha_2$ are scalars. From Eqns. (4.46) and (4.43) it follows that

$$g_2''(x) = -\frac{\tilde{b}}{2\lambda_2 \alpha_1^2 \alpha_2^4} (\lambda_1 \alpha_2^2 + x^2) \exp\{ -b \alpha_1 g_2(x) \}$$  \hspace{1cm} (4.47)$$

Define new parameters by

$$\tilde{b} \equiv b \alpha_1$$

$$\tilde{\lambda}_1 \equiv \lambda_1 \alpha_2^2$$  \hspace{1cm} (4.48)$$

$$\tilde{\lambda}_2 \equiv \alpha_1^2 \alpha_2^4 \lambda_2$$

Then

$$g_2''(x) = -\frac{\tilde{b}}{2 \lambda_2} (\tilde{\lambda}_1 + x^2) \exp\{ -\tilde{b} \tilde{g}_2(x) \}$$  \hspace{1cm} (4.49)$$

Further

$$\int_{-\infty}^{\infty} g_1^2(e) de = k_f^2 = \frac{1}{\tilde{g}_1^2} (\alpha_1 \alpha_2)^2 \int_{-\infty}^{\infty} g_2^2(\alpha_2 e) de$$  \hspace{1cm} (4.50)$$

$$= \alpha_1^2 \alpha_2^2 \tilde{k}_2^2$$
Similarly
\[ p_1(0) \int_{-\infty}^{\infty} \exp\{-bg_1(e)\} \, de = 1 \quad (4.51) \]

\[ = p_1(0) \int_{-\infty}^{\infty} \exp\{-b\tilde{\alpha}_1 g_2(\tilde{\alpha}_2 e)\} \, de \quad (4.52) \]

\[ = \frac{p_1(0)}{\tilde{\alpha}_2} \int_{-\infty}^{\infty} \exp\{-b\tilde{g}_2(e)\} \, de \]

so \( p_2(0) = \frac{p_1(0)}{\tilde{\alpha}_2} \). The two probability densities are related by
\[ p_1(0) \exp\{-bg_1(e)\} = \tilde{\alpha}_2 p_2(0) \exp\{-b\tilde{\alpha}_1 g_2(\tilde{\alpha}_2 e)\} \quad (4.53) \]

Eliminate the dependence on \( \tilde{\alpha}_1 \) by specifying how the detector height will change as it expands along the error axis. A reasonable dependence is no change at all. That is
\[ \tilde{\alpha}_1 \tilde{\alpha}_2 = 1 \quad (4.54) \]

This choice gives
\[ g_2'(\tilde{\alpha}_2 e) = g_1'(e) \quad (4.55) \]

Direct manipulation gives that \( \alpha_2 = \frac{\tilde{\alpha}_2}{q\tilde{r}} \), and substitution relates the probability densities as
\[ p_1(0) \exp\left\{ - \frac{1}{q_r K^2} q_1(e) \right\} = p_2(0) \exp\left\{ - \frac{q_r}{(q_r)^2 K^2} g_2\left( \frac{\ddot{q}r}{q_r} e \right) \right\} \quad (4.56) \]

Eqn. (4.56) then gives all the optimal probability densities as the product \( q_r \) is varied. This means that a single detector shape \( f(\cdot) \) corresponding to a particular input condition can be used to determine the proper shape for any input condition.

4.3.3 Simulation of a Tracking Loop Using an Optimal Detector

In this section a simulation of the tracking loop with optimum detector is performed. The simulation is similar to that for the Kalman filter and early-late detector in Chapter 3. It will be seen that the tracking system proposed here outperforms the one in Chapter 3.

In addition to demonstrating improved performance, the simulation results presented here validate the numerical computations done in Section 4.3.1. It will be observed that the errors this optimum loop achieves are quite close to those predicted. The principal difference between this simulation and the one previously described to evaluate the extended Kalman filter approach is that the optimal nonlinearity, just determined, is used in place of the \( t_1 \) detector, which was used with the extended Kalman filter. Since the object of incorporating the additional correlations was to increase the range of errors for which the loop is effective, much larger one-sigma errors are chosen as test points for the design using parallel correlators. Several desired one-sigma errors are
chosen. For each of these one-sigma errors a corresponding qr product is computed. As before, q is arbitrarily set to one, leaving r to be computed to achieve the given qr product. The forward loop gain, K, is chosen in accordance with q, r, and $k^2_f$. One hundred Monte Carlo runs are made for each choice of conditions. A random initial condition is used to start each run. The initial condition has a Gaussian density with variance equal to the theoretical steady-state variance. The simulation is then run for six time constants, with a time step of one-twentieth of a time constant. The loop time constant is defined using the slope of the detector function at zero error. The statistics of the loop errors at six time constants are shown in Figure 4.10.

Figure 4.10 indicates that the proposed tracking loop and detector shape are successful in two ways. First, the predicted variance and the observed variance agree, which validates the mathematics and the numerical solutions. Second, the variance is bounded, which means that this system betters the performance of the ±1 chip detector and extended Kalman filter.

Another comparison can be drawn between the proposed tracking loop and the extended Kalman filter. Since both filters indicate that the qr product determines their steady state performance, then the plots of their predicted steady state errors versus qr product can be compared, as is done in Figure 4.11. That plot shows that the tracking loop suggested here conservatively estimates its errors for errors greater than about .45 chips. The cross-over is near the point where the extended Kalman filter begins to experience difficulty. The suggested tracking
Figure 4.10 Simulated results versus analytical prediction for Extended Range Detector.
Figure 4.11 Comparison of performance predictions for extended Kalman filter and nonlinear filter.
loop apparently outperforms the extended Kalman filter's predicted performance below .45 chips.

It should be pointed out that the optimum detector required to achieve predicted errors of much less than one chip would violate some of the approximations made in Section 4.3 where the conditions for the optimum detector were derived. It was assumed there that any continuous function could be well approximated by weighting and summing correlator outputs. It has been pointed out that the optimum detector takes its maximum value at an argument equal to the predicted one-sigma error. If the predicted one-sigma error is much less than one, the resulting optimal detector shape is not well approximated by a function which is linear over the range [0,1]. A detector shape constructed from correlations separated by one chip from one another must be linear over the range [0,1]. The conclusion is then that those predictions at the low end of the curve in Figure 4.11 cannot be achieved using the scheme proposed here.

4.4 Solution with a Correlated Input Process

Several extensions of the results in 4.3 are possible. For one thing, the same computation may be carried out when the input process is a low pass process. That is, suppose the input phase process satisfies

\[ d\theta = -\frac{1}{\tau} \theta dt + d\theta_z \]  \hspace{1cm} (4.57)

and that the tracking loop is as drawn in Figure 4.12.
Then the output satisfies

\[ d\hat{\theta}(t) = -\frac{1}{\tau} \hat{\theta}(t)dt + \frac{K}{\tau} [f(e)dt + K_f d\gamma_t] \]  \hspace{1cm} (4.58)

Define the error \( e \triangleq \theta(t) - \hat{\theta}(t) \) then combining Eqns. (4.57) and (4.58) gives

\[ de = -\frac{1}{\tau} edt + d\beta_t - \frac{K}{\tau} [f(e)dt + K_f d\gamma_t] \]  \hspace{1cm} (4.59)

In the Brownian input problem it was indicated that the forward loop gain provided an extra degree of freedom. There is no loss of generality if we impose the constraint \( K_f = 1 \). Doing this and rearranging Eqn. (4.59) yields

\[ de = -\frac{1}{\tau} [e + K_f e]dt + d\beta_t - \frac{K}{\tau} d\gamma_t \]  \hspace{1cm} (4.60)

The Fokker-Planck equation for the error probability density is
\[ \frac{\partial p}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial e} [e + Kf(e)]p + \frac{1}{\tau^2} (q + K^2 r) \frac{\partial^2 p}{\partial e^2} \]  

(4.61)

In this equation \( q = \frac{\tau^2}{t} E[\beta_t^2] \), and \( r = \frac{1}{t} E[\gamma_t^2] \). The steady-state error density satisfies

\[ \frac{\partial p}{\partial e} = -\frac{1}{\tau} \frac{[e + Kf(e)]}{(q + K^2 r)} p + C_1 \]  

(4.62)

For \( p(*) \), a probability density, it must be that \( C_1 = 0 \), then

\[ \frac{\partial p}{\partial e} = -\frac{\tau [e + Kf(e)]}{q + K^2 r} p \]  

(4.63)

Then

\[ p_f(e) = p(0) \exp \left\{ -\frac{e^2}{q + K^2 r} \left[ \frac{e^2}{2} + K \int_0^s f(s) \, ds \right] \right\} \]  

(4.64)

The subscript \( f \) is a reminder that the steady-state probability density depends on the function \( f(*) \) chosen. Now impose the constraints that

\[ \int_{-\infty}^{\infty} p_f(e) \, de = 1 \]  

(4.65)

and

\[ \int_0^{\infty} n[f(e)] \, de = 1 \]  

(4.66)
where

\[ h(s) = s^2 \quad (4.67) \]

for the parallel-detector loop. Now find \( f(\cdot) \) satisfying the constraints of Eqns. (4.65) and (4.66) and that minimizes

\[ J(f) = \int_{-\infty}^{\infty} e^2 P_f(e) \, de \quad (4.68) \]

To transform this problem into the form of a classical calculus of variations problem, define a new function \( g(\cdot) \) by

\[ g(e) = \int_{0}^{E} f(c) \, dc \quad (4.69) \]

Physical considerations again indicate that the optimal \( f(\cdot) \) is antisymmetric and therefore the optimal \( g(\cdot) \) is symmetric. This allows all the integrals involved here to be converted to ones over \([0, \infty]\) instead of \([-\infty, \infty]\). Rewriting the optimization equations (4.65), (4.66), and (4.68) with these substitutions gives the problem as

\[ \text{Minimize } J(g') = \int_{0}^{\infty} e^2 P(0) \exp \left\{ -\frac{\tau}{q + K^2 r} \left[ \frac{e^2}{2} + K g(e) \right] \right\} \, de \quad (4.70) \]
subject to the constraints

$$
\int_0^\infty \exp \left\{ - \frac{T}{q + K^2 r} \left[ \frac{e^2}{2} + Kg(e) \right] \right\} \, de = \frac{1}{2p(0)} \tag{4.71}
$$

and

$$
\int_0^\infty h(g'(e)) \, de = 1 \tag{4.72}
$$

This problem is in the form of a classical calculus of variations problem and will not be pursued further here. A numerical solution is not produced here, but some of its qualitative properties are mentioned.

The solution $p_f(e)$, which is shown above, indicates that the phase estimation will be stable regardless of what the forward loop non-linearity is (so long as it is admissible, $f^2 < \infty$). This is because the variance of the correlated input process itself is bounded as a function of time. By contrast, the variance of the Brownian input process considered in Section 4.3 grows like $t$. Because of the inherent boundedness of the correlated input process, the detector shape $g'(e)$ for this process goes to zero much faster in this case than in the Brownian input case. In this case it goes like $e^{\exp -e^2}$ for large $e$.

Recall that in the Brownian input case it went like $1/e$. In the Brownian input case the large $e$ behavior of the detector shape was critical to system stability (i.e., to the existence of steady-state probability densities solving the Fokker-Planck equation). In the case of the low pass input, the optimal detector places less emphasis on large errors and presumably more emphasis on small errors.
Both the Brownian input and the low pass input resulted in solvable problems because the associated Fokker-Planck equations were in a single independent variable. In the next section higher order input processes will be considered and it will be seen that the situation is more difficult.

4.5 **Difficulty with Higher Order Input Process**

Suppose that the input process is the first element $x_1$ of the vector $x$ where

$$dx = Ax_d + Bd_Y$$

and that an observation

$$dz = h(x_1 - \hat{x}_1)dt + d\theta$$

is available where $x_1$ is the first element of the estimated state vector $x$. By analogy to the Kalman filter the estimate is required to satisfy

$$d\hat{x} = A\hat{x} + Kdz$$

where $K$ is a vector of gains. Let $e$ denote the error vector

$$e = x - \hat{x}$$

then $e$ satisfies

$$de = [Ae - Kn(e_1)]dt + d\theta - KdY$$
The probability density of the error is now a function of several variables - each component of the vector \( e \). The Fokker-Planck equation can be obtained for the error probability density

\[
\frac{\partial p}{\partial t} = - \sum_{i=1}^{n} \frac{\partial p [A e - K h(e)]_i}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 p [I + KK^T]_{ij}}{\partial x_i \partial x_j} \tag{4.78}
\]

or in the steady state

\[
0 = - \sum_{i=1}^{r} \frac{\partial p [A e = K h(e)]_i}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 p [I + KK^T]_{ij}}{\partial x_i \partial x_j} \tag{4.79}
\]

If \( e \) is more than one dimensional the steady-state Fokker-Planck equation is a partial differential equation, not an ordinary differential equation as was the case in the last two sections. This means that to carry out the procedure used earlier, a partial differential has to be solved in terms of functionals of the detector shape. There does not seem to be much hope for this procedure.

Resort to the methods of control of infinite dimensional systems does not seem to lead anywhere either. Although some infinite dimensional problems have been treated [Lions (1)] where the control enters multiplicatively (as the detector shape enters the Fokker-Planck equation) the results only apply to the case where the differential equation is to be solved on a bounded domain and where the equation has a unique solution. Neither of the conditions holds for the Fokker-Planck equation in question.
here. The domain for that equation is the whole Euclidean space containing the state vector and the equation may not have a solution satisfying $p = 1$. The infinite domain also complicates numerical solution of the Fokker-Planck equation.

4.6 Perturbation Solution for Weakly Second Order Input Process

Although closed form Fokker-Planck solutions cannot be obtained to problems of order higher than one, it seems that many problems may admit accurate approximations. Perturbation approximations are attractive in this application because the optimization procedure requires an approximation representing the solution analytically, as functionals of the detector shape. (See [Nayfeh (1)] for general discussion.)

Perturbation approximations have been used before to obtain approximate probability densities for randomly perturbed systems [Caughey and Payne (1)], [Donsker and Varadhan (1)], [Ventsel and Freidlin (1)], [Zakai (1)]. The approximate densities achieved by perturbation methods distinguish themselves by their closed form natures.

In this section it will be seen that perturbation methods can be used to obtain an approximate probability density for the randomly perturbed system which arises in tracking a particular type of second order input delay process. The first step taken in this section is to postulate a particular type of second order input dynamics. Next a tracking loop with an arbitrary extended range nonlinearity is suggested. Perturbation methods are then used to solve for the steady-state probability density of the errors in this tracking. This density is given as a
functional of the detector shape and, in this form, can be used to find the optimum detector shape. This optimization is the last thing in this section.

Suppose that the input code is of the form $S[\alpha t + \Theta(t)]$ where $\alpha$ is the nominal frequency and $\Theta(t)$ satisfies

$$
\begin{align*}
\,d\Theta_t &= \,\phi dt + d\delta_t \\
\,d\phi_t &= -a\phi_t dt + \varepsilon d\alpha_t
\end{align*}
$$

(4.80)

Here $\varepsilon$ is a small parameter presaging a perturbation solution. The process $\phi_t$ can be considered a small random disturbance in frequency. Change notation by letting $\omega = \phi/\varepsilon$. Then the model for the input process becomes

$$
\begin{align*}
\,d\Theta &= \varepsilon \omega dt + d\delta_t \\
\,d\omega &= -a\omega dt + d\alpha_t
\end{align*}
$$

(4.81)

The measurement is taken to be

$$
\,dz_t = f(\Theta - \hat{\Theta}) dt + K_f d\gamma_t
$$

(4.82)

The forward loop filter is chosen to match the input phase dynamics in the same manner as a Kalman filter does.
\[
\begin{bmatrix}
d\hat{\theta} \\
d\hat{\omega}
\end{bmatrix} = \begin{bmatrix}
0 & \varepsilon \\
0 & -a
\end{bmatrix} \begin{bmatrix}
\hat{\theta} \\
\hat{\omega}
\end{bmatrix} dt + \begin{bmatrix}
K_1 \\
\varepsilon K_2
\end{bmatrix} dz
\tag{4.83}
\]

The errors then satisfy

\[
d \begin{bmatrix}
\theta - \hat{\theta} \\
\omega - \hat{\omega}
\end{bmatrix} = \begin{bmatrix}
0 & \varepsilon \\
0 & -a
\end{bmatrix} \begin{bmatrix}
\theta - \hat{\theta} \\
\omega - \hat{\omega}
\end{bmatrix} dt + \begin{bmatrix}
K_1 f(\theta - \hat{\theta}) \\
\varepsilon K_2 f(\theta - \hat{\theta})
\end{bmatrix} dt - \begin{bmatrix}
K_1 f \\
\varepsilon K_2 f
\end{bmatrix} d\gamma_t
\tag{4.84}
\]

Let \( \theta_e = \theta - \hat{\theta} \) and \( \omega_e = \omega - \hat{\omega} \), then the steady state Fokker-Planck equation [Jazwinski (1)] for Eqn. (4.84) is

\[
0 = -\frac{\partial}{\partial \theta_e} \left[ (-K_1 f + \varepsilon \omega_e) p \right] - \frac{\partial}{\partial \omega_e} \left[ (-aw_e - \varepsilon K_2 f) p \right]
\tag{4.85}
\]

\[
+ \frac{1}{2} \left( q_1 + K_1^2 f \right) \frac{\partial^2 p}{\partial \theta_e^2} + \varepsilon K_1 K_2 f \frac{\partial^2 p}{\partial \theta_e \partial \omega_e} + \frac{1}{2} \left( q_2 + \varepsilon^2 K_2^2 f \right) \frac{\partial^2 p}{\partial \omega_e^2}
\]

where \( q_1, q_2, \tau \) are the power densities of \( \frac{d\hat{\theta}}{dt}, \frac{d\omega}{dt}, \frac{d\gamma}{dt} \) respectively. The results of a regular perturbation series expansion (see Appendix D) and comparison with a linear example lead one to consider an exponential form for \( p(\theta_e, \omega_e) \). To obtain an exponential solution assume that

\[
p(\theta_e, \omega_e) = \exp \left\{ g(\theta_e, \omega_e) \right\}
\tag{4.86}
\]
Substitution of this expression, Eqn. (4.86), into the Fokker-Planck equation, Eqn. (4.85), yields a partial differential equation for \( g \).

\[
0 = \exp \{g\} \left[ K_1 f' + K_1 f \frac{3\theta}{3e} - \varepsilon \omega e \frac{3\theta}{3e} + a + a \omega e \frac{3\theta}{3e} + \right.
\]
\[
+ \varepsilon K_2 f \frac{3\theta}{3e} + \frac{1}{2} \left(q_1 + K_1 K_2 r\right) \left[ \frac{3^2 \theta}{3e} + \left(\frac{3\theta}{3e}\right)^2 \right] + \varepsilon K_1 K_2 K_2 r \left[ \frac{3^2 \theta}{3e} + \left(\frac{3\theta}{3e}\right)^2 \right]
\]

Since \( \exp\{g\} \neq 0 \) it must be that what is inside the outermost square brackets in Eqn. (4.87) is equal to zero. To obtain a perturbation expansion for \( g \), the first step is to assume that for small \( \varepsilon \) \( g \) can be expressed as

\[
g(\theta, \omega) = g_0(\theta, \omega) + \varepsilon g_1(\theta, \omega)
\]  

(4.88)

The next step is to substitute Eqn. (4.88) into Eqn. (4.87) and to assume that the coefficient of each power of \( \varepsilon \) is equal to zero. Setting each one of these coefficients to zero yields the following set of differential equations...
\[ \varepsilon^0: 0 = k_1 f' + k_1 f \frac{\partial g_0}{\partial \theta} + \frac{1}{2} \left( q_1 + k_1^2 k_2 \right) \left[ \frac{\partial^2 g_0}{\partial \theta^2} + \left( \frac{\partial g_0}{\partial \theta} \right)^2 \right] + \]

\[ + a + a \omega \theta + \frac{q_2}{2} \left[ \frac{\partial^2 g_0}{\partial \omega^2} + \left( \frac{\partial g_0}{\partial \omega} \right)^2 \right] \]

\[ (4.89) \]

\[ \varepsilon^1: 0 = k_1 f \frac{\partial g_1}{\partial \theta} + \frac{1}{2} \left( q_1 + k_1^2 k_2 \right) \left( \frac{\partial^2 g_1}{\partial \theta^2} + 2 \frac{\partial g_1}{\partial \theta} \frac{\partial g_0}{\partial \theta} \right) + \]

\[ + a \omega \theta + \frac{q_2}{2} \left( \frac{\partial^2 g_1}{\partial \omega^2} + \frac{2 \partial g_1}{\partial \omega} \frac{\partial g_0}{\partial \omega} \right) - \omega \theta \frac{\partial g_0}{\partial \theta} + \]

\[ + k_2 f \frac{\partial g_0}{\partial \omega} + k_1 k_2 k_2 f \left( \frac{\partial^2 g_0}{\partial \omega \partial \theta} + \frac{\partial g_0}{\partial \omega} \frac{\partial g_0}{\partial \theta} \right) \]

\[ (4.90) \]

The \( \varepsilon^0 \) equation can be solved by using the results of a regular expansion (see Appendix D). Thus

\[ g_0(\theta, \omega) = - c_1 \int_0^\theta f dx - c_2 \omega^2 \]

\[ (4.91) \]
Substitution into Eqn. (4.89) gives

\[ 0 = K_1 f' - C_1 K_1 f^2 + \frac{1}{2} \left( q_1 + K_1^2 K_f f^2 \right) \left[ -C_1 f' + C_1 f^2 \right] + \]
\[ + a - 2C_2 \omega_e^2 + \frac{q_2}{2} \left( -2C_2 + 4C_2^2 \omega_e^2 \right) \]  

\[ (4.92) \]

which is solved by

\[ C_1 = \frac{2K_1}{q_1 + K_1^2 K_f f^2} \]

\[ C_2 = \frac{a}{q_2} \]  

\[ (4.93) \]

The zeroth order solution is then

\[ g_0(\theta_e, \omega_e) = - \frac{2K_1}{q_1 + K_1^2 K_f f^2} \int_0^{\theta_e} f(x) dx - \frac{a}{q_2} \omega_e^2 \]  

\[ (4.94) \]

Again, the result of regular perturbation (see Appendix D) indicates the proper form for \( q_1 \). The form of that solution should be

\[ g_1(\theta_e, \omega_e) = \omega_e h(\theta_e) \]  

\[ (4.95) \]
where the function $h(\theta_e)$ is yet to be determined. Substitution of Eqn. (4.95) into Eqn. (4.90) yields

$$0 = \omega e f \frac{\partial h}{\partial \theta_e} + \frac{1}{2} \left( q_1 + K_{11}^2 K_{2}^2 \right) \left( \omega e \frac{\partial^2 h}{\partial \theta_e^2} + 2 \omega e \frac{\partial h}{\partial \theta_e} \frac{\partial^2 h}{\partial \theta_e^2} \right) +$$

$$+ a \omega e f + q_2 h \frac{\partial g_0}{\partial \omega} - \omega e \frac{\partial g_0}{\partial \theta_e} + K_{1f}^2 f \frac{\partial g_0}{\partial \omega} + K_{1f} K_{2f}^2 \frac{\partial g_0}{\partial \theta_e} \frac{\partial g_0}{\partial \theta_e} \frac{\partial g_0}{\partial \theta_e}$$

(4.96)

Substitution of Eqn. (4.94) into Eqn. (4.96) and division by $\omega e$ yields

$$0 = K_{1f} \frac{\partial h}{\partial \theta_e} + \frac{1}{2} \left( q_1 + K_{11}^2 K_{2}^2 \right) \left( \frac{\partial^2 h}{\partial \theta_e^2} - \frac{4K_{1f}}{q_1 + K_{11}^2 K_{2}^2} \frac{\partial h}{\partial \theta_e} \right) +$$

(4.97)

$$+ a h - q_2 h \frac{2a}{q_2} + \frac{2K_{1f}}{q_1 + K_{11}^2 K_{2}^2} f + 2K_{2f} f \cdot \frac{a}{q_2} + K_{1f} K_{2f}^2 \frac{4K_{1f}}{q_1 + K_{11}^2 K_{2}^2} \frac{a}{q_2}$$

Some simplification gives the differential equation for $h(\theta_e)$ to be

$$\frac{d_1}{2} \frac{\partial^2 h}{\partial \theta_e^2} - K_{1f} \frac{\partial h}{\partial \theta_e} - a h + d_2 f = 0$$

(4.98)

where

$$d_1 = q_1 + K_{11}^2 K_{2}^2$$

(4.99)
and
\[
d_2 = \frac{2K_1}{d} - \frac{2a}{q_2} K_2 + \frac{4K_1^2 K_2 f r a}{q_2 d_1}
\]  
(4.100)

Using the results for the Brownian input, where
\[f(\theta_e) \sim 1/\theta_e \text{ as } |\theta_e| \to \infty\] reveals that
\[h(\theta_e) \sim 1/\theta_e \text{ as } |\theta_e| \to \infty\]  
(4.101)

which yields an integrable approximate density. The approximate density is summarized as
\[p_2(\theta_e, \omega_e) = \exp \left\{ -C_1 \int_0^{\theta_e} f(x) \, dx - C_2 \omega_e^2 + \varepsilon \omega_e h(\theta_e) \right\} \]  
(4.102)

To determine what detector shape is optimum, a performance index must be specified. It seems reasonable to choose the expected value of some quadratic form in the steady-state phase and frequency errors. Such an expected value can be computed using the expression Eqn. (4.102) for the steady-state probability density for the errors.
\[p_2(\theta, \omega) = \exp \left\{ -C_1 \int_0^{\theta} f(x) \, dx - C_2 \omega^2 + \varepsilon \omega h(\theta) \right\} \]  
(4.103)

The expected value of a general quadratic function of the phase error, $\theta_e$, and frequency error, $\varepsilon \omega_e$, then, takes the form
\[ \int \left( \theta_e^2 + a\varepsilon_\theta \theta_e + b(\varepsilon_\theta)^2 \right) p(\theta_e, \omega_e) d\theta_e d\omega_e \]  

(4.104)

In this expression \( a \) and \( b \) are some arbitrary constants. The \( \omega_e \) integration may be accomplished analytically by completing squares.

\[ c_2 \omega_e^2 + \varepsilon_\theta h(\theta_e) = -c_2 \left( \omega_e - \frac{\varepsilon_\theta h(\theta_e)}{c_2} \right)^2 + \frac{1}{c_2} \left( \frac{\varepsilon_\theta h(\theta)}{2} \right)^2 \]

(4.105)

so that

\[ \int \theta^2 p(\theta, \omega) d\theta d\omega = \]

\[ = \varepsilon^2 \exp \left\{ -c_1 \int_0^\theta f(x) dx + \frac{1}{c_2} \left( \frac{\varepsilon_\theta h(\theta)}{2} \right)^2 \right\} d\theta \int \exp \left\{ -c_2 \left( \omega - \frac{\varepsilon_\theta h(\theta)}{2c_2} \right)^2 \right\} d\omega \]

\[ = \sqrt{\frac{\pi}{c_2}} \int \theta^2 \exp \left\{ -c_1 \int_0^\theta f(x) dx + \frac{1}{c_2} \left( \frac{\varepsilon_\theta h(\theta)}{2} \right)^2 \right\} d\theta \]  

(4.106)
Similarly,

\[ \iint a \epsilon \omega p(\theta, \omega) d\theta d\omega = \]

\[ = \sqrt{\frac{\pi}{c^2}} \epsilon^2 \frac{a \theta}{2} \frac{h(\theta)}{c^2} \exp \left\{ -c_1 \int_0^\theta f(x) dx + \frac{1}{c^2} \left( \frac{\epsilon h(\theta)}{2} \right)^2 \right\} d\theta \]  

(4.107)

and

\[ \iint \epsilon^2 \omega^2 b p(\theta, \omega) d\theta d\omega = \]

\[ = \sqrt{\frac{\pi}{c^2}} \epsilon^2 \left[ \frac{1}{2c^2} + \left( \frac{\epsilon h(\theta)}{2c^2} \right)^2 \right] b \exp \left\{ -c_1 \int_0^\theta f(x) dx + \frac{1}{c^2} \left( \frac{\epsilon h(\theta)}{2} \right)^2 \right\} d\theta \]

(4.108)

It is seen that to order \( \epsilon \) the performance index P.I. is given by

\[ \text{P.I.} = \sqrt{\frac{\pi}{c^2}} \epsilon^2 \exp \left\{ -c_1 \int_0^\theta f(x) dx \right\} \]

(4.109)

This is the same performance index as considered in the Brownian input case and it has the same dependence on the detector shape. Therefore to order \( \epsilon \) the detector obtained for the Brownian input is optimal in this second order case.
4.7 **Summary and Discussion of the Fokker-Planck Approach**

The developments in Sections 4.1, 4.2 and 4.3 demonstrate conclusively that a tracking loop employing an extended range nonlinearity can track in cases where the traditional tracking loop cannot. The particular case for which this is demonstrated is that of a Brownian motion input delay process. In Sections 4.4 through 4.6 the attempt is made to find the optimal detector shape for other types of input processes. The attempt meets with varying degrees of success. The optimum detector shape for a general linear input process is seen, in Section 4.5, to be difficult to find.

This lack of a general solution is one of two drawbacks to the Fokker-Planck approach taken in this chapter. The second drawback is that the solutions in this chapter are steady-state solutions and as such probably sacrifice transient performance. The two drawbacks will be overcome by the techniques proposed in the next chapter.

It should be pointed out that Fokker-Planck approaches to sinusoidal phase lock problems have been used in the past. For the phase lock loop, Viberbi [Viterbi (1)] derived the first order solution analogous to the first solution derived here. The procedure for solution is similar to this one except that in the phase lock problem the nonlinearity is specified as sinusoidal before the Fokker-Planck equation is solved. The difference between the Fokker-Planck approach to the correlation tracking problem and the Fokker-Planck approach to the phase lock problem exemplifies the difference between a feedback estimation
problem and the more usual nonlinear estimation problem. In a feed-
back estimation problem, a nonlinear estimation problem has to be
solved over the admissible class of nonlinearities. Then, the non-
linearity is resolved by choosing the nonlinear filter yielding the most
desirable solution to the estimation problem. In the correlation tracking
problem just considered, the estimation part of the problem was solved
by solving the Fokker-Planck equation. It had to be solved for a
general nonlinearity. The problem of choosing the most desirable solution
to the Fokker-Planck equation was solved as a variational problem to
yield the optimum steady-state detector shape.

The second order Fokker-Planck solution for the phase lock
problem is not known. Approximation techniques have been suggested
[Lindsey (1)] for these higher-than-first-order problems. The procedures
suggested are successive approximations, however, and as such are useless
in many of these correlation tracking problems. Their limitation is
that they do not yield the probability density as a functional over the
detector shape. This type of analytical representation is necessary in
order that the optimization step proceed expeditiously.

The perturbation approximation technique suggested here is new,
to this author's knowledge, and seems extensible in two directions. It
seems that fairly concrete results can be obtained for the case where
the phase process \( \Theta_t \) satisfies
\[ \text{d}\theta_t = -\alpha\theta_t \text{d}t + \epsilon C \cdot \Omega_t + \text{d}\beta_t \quad (4.110) \]

where \( C \) is an \( n \)-vector of constants and \( \Omega_t \) is an \( n \)-vector satisfying

\[ \text{d}\Omega_t = A\Omega_t \text{d}t + \text{d}\gamma_t \quad (4.111) \]

with \( A \) a stable matrix and \( \gamma_t \) a vector Brownian motion. A generalization of a theoretical nature would be to solve for the probability density corresponding to

\[ \text{d}\theta_t = f(\theta_t) \text{d}t + \epsilon \omega_t + \text{d}\beta_t \quad (4.112) \]

\[ \text{d}\omega_t = g(\omega_t) \text{d}t + \text{d}\gamma_t \quad (4.113) \]

where \( \theta_t \) and \( \omega_t \) are both first order. This second problem would be more complicated because the eigenfunctions associated with the zeroth order differential operators would be more difficult to obtain.
CHAPTER 5

APPROXIMATE NONLINEAR FILTER APPROACH
TO MULTIPLE SENSOR MANAGEMENT

5.0 Introduction

In Chapter 4 Fokker-Planck techniques were used to find the optimal extended range detector. Optimal detector shapes were found for steady state operation with a Brownian motion for the input process. In addition, the necessary steps were given for finding optimal detector shapes with a correlated input process, and perturbation techniques were used to approximate the optimal detector shape for a particular weakly second order input process. A solution for the general input process was not obtained.

Besides not providing steady state answers for many interesting input processes, the approach taken in Chapter 4 does not provide optimal transient behavior. This means that the multiple correlator acquisition is probably slower than it has to be. The purpose of this chapter is to provide a design procedure which will not have these two deficiencies. That is, it will be easy to apply to arbitrary order linear input processes and it will provide for transient operation.
The design procedure suggested in this chapter is based on results from nonlinear filtering theory. It has been seen in previous sections that the sum of weighted correlations provides a nonlinear measurement of tracking error. It has also been seen that by varying the weights on the various correlations, the shape of the error measurement nonlinearity can be varied. The problem of determining the optimum weights can then be viewed as a nonlinear filtering problem in which the measurement nonlinearity is deformable. Conceptually the optimum weights can be determined by finding the optimum estimate of the target state for each fixed detector shape history, and then choosing the detector shape history which yields the best overall performance. Since all the detector shapes are nonlinear, the first step in this procedure (i.e., determination of an optimum estimate for fixed detector shape history) is a nonlinear filtering problem. The well-known complexity of these problems indicates that some form of approximation will be required to take the first step. In this section Gaussian approximation, as described later, is used to arrive at an approximate nonlinear filter. Once this step is taken it is easy to see what the optimum detector evolution should be in order to optimize the approximate nonlinear filter performance.

5.1 Gaussian Approximation with a Brownian Input

Suppose that the input delay process $\delta_t$ satisfies

$$d\delta_t = dB_t$$
and that the nonlinear measurement, built from weighted correlator outputs, satisfies

\[ dz_t = f(\theta_t - \hat{\theta}_t)dt + d\gamma_t \]  \hspace{1cm} (5.1)

The processes \( \beta_t \) and \( \gamma_t \) are Brownian motions. The process \( z_t \) is the measurement and \( f(\cdot) \) the measurement nonlinearity obtained as a weighted sum of correlation triangles. There is no dependence of the measurement noise magnitude on \( f \) because in this section \( f(\cdot) \) is constrained to have \( \int f^2 d\theta = 1 \). The nonlinearity, \( f, \) is undetermined beyond that. The desired procedure is to design a filter with \( f(\cdot) \) unspecified, then to choose \( f(\cdot) \) to optimize system performance.

Since the complete filtering problem cannot be solved analytically, an approximation will be used. The conditional moments \([Fujisaki \, et \, al \, (1)]\) and \([Clark \, (1)]\) satisfy

\[ d\hat{\theta}_t = (\hat{\theta}_f - \hat{\theta}_f) \frac{1}{r} (dz - \hat{\theta} dt) \]  \hspace{1cm} (5.2)

\[ dp = \left[-(\hat{\theta}_f - \hat{\theta}_f)^2 \frac{1}{r} + q\right] dt + \left\{\hat{\theta}_f^2 - \hat{\theta}_f^2 - 2\hat{\theta}_f^2 + 2\hat{\theta}_f^2 \right\} \frac{1}{r} (dz - \hat{\theta} dt) \]  \hspace{1cm} (5.3)

One useful approximation for truncating these moment equations is called Gaussian approximation. This approximation assumes that the conditional density is close enough in shape to a Gaussian density that the conditional expectations \((\cdot)\) may be carried out using a Gaussian density. The mean
and covariance are assumed to satisfy the resulting differential equation. If the detector shape \( f(*) \) is assumed to be antisymmetric then Gaussian approximation yields

\[
\frac{d\hat{C}}{dz} = \frac{g(p)}{r} dz
\]

\[
\frac{dp}{dt} = -\frac{g^2(p)}{r} + q
\]

where

\[
g = \frac{1}{\sqrt{2\pi p}} \int_{-\infty}^{\infty} ef(e) \exp \left\{ -\frac{e^2}{2p} \right\} de
\]

This can be changed into a more recognizable form by defining \( h(p) \) to be

\[
h(p) = \frac{g(p)}{p}
\]

\[
= \frac{1}{p} \frac{1}{\sqrt{2\pi p}} \int_{-\infty}^{\infty} ef(e) \exp \left\{ -\frac{e^2}{2p} \right\} de
\]

The function \( h(p) \) is then the describing function gain for the nonlinearity \( f(*) \). If \( ph(p) \) is substituted for \( g(p) \), the filter equations become

\[
\frac{d\hat{C}}{dz} = \frac{ph(p)}{r} dz
\]

\[
\frac{dp}{dt} = -\frac{p^2 h(p)^2}{r} + q
\]
These equations can be recognized as the Kalman filter linearized with the describing function gains. Gaussian approximation will generally have this interpretation as a Kalman filter linearized using a describing function except that there will usually be a data dependence in the covariance equation.

Two elements of the problem under consideration combine to remove data dependence in the covariance equation. The measurement nonlinearity is a function of the estimation error, not of the state alone, and it is an antisymmetric function. Recall that the measurement nonlinearity being a function of the estimation error, not just the target state, is the key property of a correlation tracking problem.

5.2 Gaussian Approximation for General Input

Gaussian approximation is easily generalized to the case where the delay process \( \theta_t \) is the first element of an \( n \)-vector \( x_t \) satisfying

\[
dx_t = Ax_t + Bd\gamma_t
\]

where \( A \) and \( B \) are \( n \times n \) matrices, and \( \gamma_t \) is an \( n \)-vector Brownian motion. The measurement equation is the same as before (Eqn. 5.1). In this case the mean equation is

\[
d\hat{x}_t = A\hat{x}_t + \hat{x}f (dz - \hat{f}dt) \cdot \frac{1}{r}
\]
and the covariance equation is

\[
\begin{align*}
[dP]_{ij} &= [AP + PA^T + BQB^T]_{ij} - (\hat{x}_i \hat{f} - \hat{x}_i \hat{x}_j \hat{f} - \hat{x}_j \hat{f}) \frac{dt}{r} + \\
&\quad + (\hat{x}_i \hat{x}_f - \hat{x}_i \hat{x}_j \hat{f} - \hat{x}_i \hat{x}_j \hat{f} - \hat{x}_i \hat{x}_j \hat{f} + 2 \hat{x}_i \hat{x}_j \hat{f})(dz_t - \hat{f} dt) \frac{1}{r}
\end{align*}
\]  

(5.13)

With the same assumptions as in the first order case, that is, that the conditional density is Gaussian and the nonlinearity antisymmetric, the filter equations become

\[
dx_t = Ax_t + P e_1 \frac{h(P)}{r} dz_t 
\]

(5.14)

\[
\frac{dP}{dt} = AP + PA^T + BQB^T - Pe_1 \frac{h^2(P)}{r} e_1 P 
\]

(5.15)

where \( e_1 = [100 \ldots 0] \) and \( h(P) \) is

\[
h(P) = \frac{1}{P_{11}} \frac{1}{\sqrt{2\pi P_{11}}} \int_{-\infty}^{\infty} e \Phi(e) \exp \left\{ -\frac{e^2}{2P_{11}} \right\} de
\]

(5.16)

\( P_{11} \) is the 1,1 element of the matrix \( P \). Notice that \( [h(P)00\ldots0] \) is the describing function gain for this detector so that the Gaussian approximation has the same interpretation as in the first order case.

In order to get the form presented above, the following manipulation is required.
Suppose that $E\{xf(y)\}$ is to be computed where $x$ and $y$ are jointly Gaussian with covariance matrix

$$\mathbf{P} = \begin{bmatrix} p_{xx} & p_{xy} \\ p_{xy} & p_{yy} \end{bmatrix} \quad (5.17)$$

Then

$$E\{xf(y)\} = \frac{1}{2\pi|\text{det} \mathbf{P}|^{1/2}} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} x \cdot$$

$$\cdot \exp \left\{ -\frac{1}{2(p_{xx}p_{yy} - p_{xy}^2)} \left( p_{yy} x^2 - 2p_{xy}xy + p_{xx}y^2 \right) \right\} \, dx \, dy \quad (5.18)$$

The inner integral may be computed by completing squares to yield

$$E\{xf(y)\} = \frac{p_{xy}}{p_{yy} \sqrt{2\pi p_{yy}}} \int_{-\infty}^{\infty} y f(y) \exp \left\{ -\frac{y^2}{2p_{yy}} \right\} \, dy \quad (5.19)$$

$$= p_{xy} h(p_{yy}) \quad (5.20)$$

The mean and covariance equations represent an approximate solution to the filtering problem for an arbitrary nonlinearity $f$. The complete problem will be solved when the nonlinearity is selected to yield optimum filter performance. Inspection of the covariance equation reveals that only one term is affected by the choice of the
nonlinearity. That term is the last term, the one corresponding to the quadratic term in the usual Kalman filter. To minimize the covariance then the best strategy is to maximize \( h^2(P) \). Doing this makes the derivative of the covariance as small as possible. The optimum \( f^* \) detector then satisfies

\[
\max_{\|f\|_2 = 1} \left[ \int_{-\infty}^{\infty} ef(e) \exp \left( -\frac{e^2}{2p_{11}} \right) \, de \right]^2 = \\
= \left[ \int_{-\infty}^{\infty} ef^*(e) \exp \left( -\frac{e^2}{2p_{11}} \right) \, de \right]^2
\]  

(5.21)

This is equivalent to solving the unconstrained problem

\[
\max_{g} \frac{\left[ \int_{-\infty}^{\infty} eg(e) \exp \left( -\frac{e^2}{2p_{11}} \right) \, de \right]^2}{\int_{-\infty}^{\infty} g^2(e) \, de}
\]  

(5.22)

and setting

\[
f^*(e) = g^*(e) \left[ \int_{-\infty}^{\infty} g^*2(e) \, de \right]^{-1/2}
\]  

(5.23)

The Schwartz inequality may be used to solve for \( g^* \).

\[
\left[ \int_{-\infty}^{\infty} eg(e) \exp \left\{ -\frac{e^2}{2p_{11}} \right\} \, de \right]^2 \leq \int_{-\infty}^{\infty} g^2(e) \, de \cdot \int_{-\infty}^{\infty} e^2 \exp \left\{ -\frac{e^2}{p_{11}} \right\} \, de
\]  

(5.24)
Then
\[
\left[ \int_{-\infty}^{\infty} eg(e) \exp \left\{ -\frac{e^2}{2P_{11}} \right\} \, de \right]^2 \leq \int_{-\infty}^{\infty} e^2 \exp \left\{ -\frac{e^2}{P_{11}} \right\} \, de
\]  \hspace{1cm} (5.25)

and equality holds if
\[
g^*(e) = e \exp \left\{ -\frac{e^2}{2P_{11}} \right\}
\]  \hspace{1cm} (5.26)

Since
\[
\int_{-\infty}^{\infty} g^*^2(e) \, de = \int_{-\infty}^{\infty} e^2 \left\{ -\frac{e^2}{P_{11}} \right\} \, de
\]  \hspace{1cm} (5.27)

\[
= \sqrt{\pi P_{11}} \frac{P_{11}}{2}
\]  \hspace{1cm} (5.28)

then it follows that
\[
f^*(e) = \left( \frac{P_{11}}{2} \right)^{-1/2} \left( \pi P_{11} \right)^{-1/4} \ e \exp \left\{ -\frac{e^2}{2P_{11}} \right\}
\]  \hspace{1cm} (5.29)

5.3 **Comparison with Fokker-Planck Method**

This shape, as a function of \( e \), is strikingly similar in appearance to the shape derived earlier. It has its maximum at the one-sigma point. It is different in that it drops off much more rapidly as \( e \to \infty \) than the other shape did. The analysis in Chapter 4 indicates that a nonlinearity which goes to zero this rapidly will not yield a finite steady state variance. The implications of this will be discussed later.
Consider for a moment that the phase process $\theta_t$ satisfies

$$d\theta_t = a\theta_t dt + dB_t \quad (5.30)$$

Later "a" will be set to zero to compare the situation here with the one derived in Chapter 4. With "a" unspecified its qualitative effect on the estimation process may be determined.

The covariance equation with the optimal shape is

$$\frac{dp}{dt} = 2ap + q - p^2 \frac{h^2(p)}{r} \quad (5.31)$$

Now

$$h^2(p) = \left[ \frac{1}{p} \frac{(\pi p)^{-1/4}}{\sqrt{2\pi p}} \left( \frac{p}{2} \right)^{-1/2} \int_{-\infty}^{\infty} e^2 \exp \left\{ -\frac{e^2}{p} \right\} \, de \right]^2 \quad (5.32)$$

$$= \frac{\pi^{-1/2}}{4} \, p^{-3/2} \quad (5.33)$$

The covariance equation then becomes

$$\frac{dp}{dt} = 2ap + q - \frac{\pi^{-1/2}}{4} \frac{p^{1/2}}{r} \quad (5.34)$$

Figure 5.1 below depicts graphically the steady state solutions to this equation. It is shown that if $a \leq 0$ then there is a unique, positive steady state solution to this equation and that $p > 0$ is its domain of attraction. If $a < 0$ there exist either two or zero steady
Figure 5.1 Portrayal of covariance behavior for different values of $a$. 
state solutions. If there are two, then one is stable and one is unstable. The domain of attraction for the stable singularity is a region of \( 0 < p < p_2 \), where \( p_2 \) is the second singular point. All trajectories having \( p(t) > p_2 \) for any \( t \) are monotone increasing with time.

These stability results indicate that this approximate solution agrees with the stability results given earlier by the Fokker-Planck method. They agree in the sense that they show asymptotic filter stability for \( a \leq 0 \) and monotone divergence for \( a > 0 \).

Consider, for a time, the case where \( a = 0 \). In this case the input is a Brownian motion. The filtering and detector equations are, in this case,

\[
\hat{d}_t = p \frac{h(p)}{r} \, dz_t
\]

\[
\frac{dp}{dt} = q - p^2 \frac{h^2(p)}{r}
\]

\[
 f^*(e) = \left( \frac{p}{2} \right)^{-1/2} (\pi p)^{-1/4} e \exp \left\{ -\frac{e^2}{2p} \right\}
\]

Substituting

\[
h(p) = \frac{\pi}{2} p^{-3/4}
\]

gives

\[
\hat{d}_t = \frac{\pi}{2} \frac{p}{r} \frac{1}{4} \, dz_t
\]

\[
\frac{dp}{dt} = q - \frac{\pi}{4} \frac{p}{r} \frac{1}{2}
\]
The steady state covariance is then given by

\[ p^{1/2} = 4 \ 1/2 \ q r \]  

(5.41)

\[ = 7.0898 \ q r \]  

(5.42)

This expression can be compared with that derived for the optimal
detector without Gaussian approximation. Call the covariance for the
optimal detector \( p_0 \). Then as was indicated earlier

\[ p_0^{1/2} = 25.2 \ q r \]  

(5.43)

This comparison suggests that Gaussian approximation does not work well
with a Brownian input. This conclusion follows from the fact that the
optimal detector was chosen to yield minimum steady state error. Since
the Gaussian approximation leads to a significantly smaller prediction
for steady state errors, it must be that the approximation is not very
good. The inadequacy of Gaussian approximation may be demonstrated on
the acquisition run used earlier. A plot of expected and achieved
one sigma error is shown in Figure 5.2. The results there show that for
this problem Gaussian approximation does not work well. The covariance
does not behave as it was predicted to behave. It has long been recog-
nized by users of describing function theory that first order linear
filtering in a loop with significant nonlinearity is inadequate to
justify the Gaussian distribution at the nonlinearity input.*

* This was pointed out by Dr. W. E. Van der Velde.
Figure 5.2 Acquisition performance of Gaussian approximation.

Figure 5.2 may mislead the reader somewhat. The tendency is to believe that a particular trajectory will behave roughly like the one-sigma trajectory. This is not the case, however. A sampling of the error trajectories indicates that for about 80 percent of the runs the errors behave as predicted. The remaining 20 percent fall far outside the
predicted errors. An average computed on the basis of these errors falls somewhere between the trajectories which are not captured and those that are. A given error trajectory, then, looks either somewhat worse or much better than the experimental one-sigma plot which is shown.

The reason for this inability of the Gaussian detector to pull in the 20 percent of signals which are left out is that in the tails this detector goes to zero too rapidly. It was shown in Chapter 4 that the fastest the detector could go to zero is like 1/e. The Gaussian detector, however, goes like \( e^{-\frac{e^2}{2}} \). The Gaussian approximation scheme is perhaps still useful in spite of this deficiency.

The problem with the system, as it is now configured, is that the actual errors do not affect the covariance. Even when the actual errors are very large, the predicted covariance, as plotted in Figure 5.2, is quite small. Somehow the actual errors must factor into the covariance computation. The reason that the measurement does not affect the covariance is that the measurement nonlinearity is antisymmetric. In the covariance equation, Eqn. (5.3) for example, the coefficient for the measurement is

\[
\frac{1}{r} (\hat{\theta} - \hat{\theta})^2 (\hat{f} - \hat{f}) = 0
\]

(5.44)

where (\(^\wedge\)) means conditional expectation and \( f \) is the measurement nonlinearity. If a Gaussian density is assumed, then any antisymmetric function of \( (\hat{\theta} - \hat{\theta}) \) has expected value zero. For this reason the data
dependence drops out of the covariance equation and the covariance
equation runs open loop. That is, the covariance is not responsive
to actual errors.

This situation can, perhaps, be corrected by taking another
measurement. The antisymmetric nonlinearity is constructed by adding
weighted correlations and a symmetric nonlinearity may be constructed
by the same technique. Two measurements will then be available. Call
the symmetric nonlinearity $g(*)$ and suppose that it is constrained to
have $\int g^2 = 1$. The two measurements $z_1^1$ and $z_2^2$ then satisfy

$$dz_1^1 = f(e)dt + d\theta_1^1$$

$$dz_2^2 = g(e)dt + d\theta_2^2$$

The measurement $z_1^1$ taken with the antisymmetric nonlinearity will enter
the mean computation and the one with the symmetric nonlinearity will
enter the covariance computation. For example, in the Brownian input case
Eqn. (5.3) for the error covariance would be modified by the addition
of the term

$$\frac{1}{r} (\hat{e}-\hat{e})(g-\hat{g})dz_2^2$$

It might be reasonable to make the symmetric nonlinearity in the shape
of a Gaussian density. The resulting measurement would then give an
an indication when the signal left the linear range of the anti-symmetric detector, and would cause the covariance to grow and recapture the signal.

More work is required to determine if this idea will result in more reasonable performance.
CHAPTER 6

APPLICATION OF MULTIPLE SENSOR METHODS

TO NONCOHERENT GPS CODE TRACKING

6.0 Introduction

In Chapter 3, where the correlation models for PRN code tracking are derived, it is assumed that the carrier waveform, onto which the PRN code is modulated, is known perfectly. In interesting practical cases the carrier sinusoid is not known. Generally neither the carrier phase nor frequency is known exactly. In order to track codes in these circumstances a system slightly different from the one described in Chapter 3 has to be used. The circuit used for code tracking without perfect knowledge of carrier is called a noncoherent correlator and employs power or envelope detection circuitry. The purpose of this section is to show that, with some reasonable approximations, the ideas developed in Chapters 4 and 5 can also be applied to design a code tracking loop using noncoherent correlators.

A noncoherent detector as used for GPS code tracking is block diagrammed in Figure 6.1(b). Basically the noncoherent correlator in Figure 6.1(b) provides a correlation of the input with a single feedback code. A single coherent correlator as shown in Figure 6.1(a) does not
have low pass filters or squarers and does not have a branch in which a $\cos \omega_c t$ multiplication is performed. The operation of the noncoherent correlator will be carefully described later, but roughly, this device indicates how much power is present near zero frequency after the code and carrier multiplications. If the input and feedback codes are in synchronism and the input and carrier frequencies are close, then all the signal power is at zero frequency (or within the LPF bandwidth); if not, then only a small fraction is. This device is insensitive to
carrier phase errors. If the frequency errors, \(|\omega_c - \tilde{\omega}_c|\), larger than the low pass filter bandwidth, then the device output is zero.

The element pictured in Figure 6.1(b) provides one correlation. Figure 6.2 shows the construction of a code tracking loop using many parallel noncoherent correlators. It will be demonstrated that the nonlinearity which results from this noncoherent configuration can be analyzed in the same way as the coherent one was. Only integer shifts between adjacent feedback codes are considered here. It may be that spacing other than integer spacing would yield better results, but this issue will not be addressed. To analyze and design an extended range detector using this element, the noise level on its output must be computed. Suppose that the input signal is

\[ r(t) = S \omega t + \Theta(t) \sin \omega_c t + n_o(t) \]  

(6.1)

where \( S(\omega t + \Theta(t)) \) is the input code, the sinusoid, \( \sin \omega_c t \), is called the carrier wave, and \( n_o(t) \) is an additive white Gaussian noise present on the input signal. The noise \( n_o(t) \) is assumed to have spectral density \( r/2 \). The result of multiplying the input signal by a feedback code, \( S(\omega t + \Theta) \), is denoted in Figure 6.1(b) by \( x(t) \). The signal \( x(t) \) may be characterized as

\[ x(t) = R(e) \sin \omega_c t + n(t) \]  

(6.2)
Figure 6.2 A noncoherent code tracking loop with variably weighted detector.
where \( e = \theta - \hat{\theta} \), \( R(\cdot) \) is the code autocorrelation, and \( n(t) \) is a white Gaussian noise. The spectral density of \( n(t) \) is also \( \tau/2 \).

Two products of the signal \( x(t) \) are formed next: the products \( x(t) \) times \( 2 \sin (\omega_c t + \varepsilon) \), and \( x(t) \) times \( 2 \cos (\omega_c t + \varepsilon) \). Each of these products results in a signal which is the sum of three terms. Two of the terms are sinusoids, one at frequency \( \omega_c - \tilde{\omega}_c \) and one at frequency \( \omega_c + \tilde{\omega}_c \). It will be assumed that the frequency error \( \omega_c - \tilde{\omega}_c \) is within the bandwidth of the low pass filter and the frequency \( \omega_c + \tilde{\omega}_c \) is well outside the low pass filter bandwidth. The third term in each of these products is a product of the noise \( n(t) \) with a sinusoid. In [Van Trees (1)] it is shown that the product of a white noise and a sinusoid is just another white noise. These considerations lead to the following models of the signals denoted in Figure 6.1(a) by \( y_1(t) \) and \( y_2(t) \).

\[
y_1 = R(e) \cos (\varepsilon) + n_1(t) \quad (6.3)
\]

\[
y_2 = R(e) \sin (\varepsilon) + n_2(t) \quad (6.4)
\]

for the upper and lower branches in Figure 6.1(b), respectively. The noises \( n_1, n_2 \) are independent and have spectral levels 4. The constant \( \varepsilon \) is the unknown carrier phase offset. Suppose that the LPF is first order with unity D.C. gain and that its pole is at \( \omega \) rad/sec. In steady state the outputs of the LPFs may be characterized by their autocorrelations which are
\[ R_{z_i}(\tau) = C_i^2 + \frac{\omega r}{2} e^{-\omega |\tau|} \quad i = 1, 2 \] (6.5)

where

\[ C_1 = R(e) \cos (\epsilon) \]
\[ C_2 = R(e) \sin (\epsilon) \]

The autocorrelation of the output of the squarer is then [Thomas (1)]

\[ R_{z_i}^2(\tau) = C_i^4 + C_i^2 \omega r e^{-\omega |\tau|} + \frac{(\omega r)^2}{2} e^{-2\omega |\tau|} + \left(\frac{\omega r}{2}\right)^2 \] (6.6)

The noise of interest here contributes

\[ R_{M_i}(\tau) = C_i^2 \omega r e^{-\omega |\tau|} + \frac{(\omega r)^2}{2} e^{-2\omega |\tau|} \] (6.7)

to Eqn. (6.6) for \( R_{z_i}^2 \) in the above expression. Only this part will contribute to the post-detection noise. Two terms appearing in \( R_{z_i}^2 \) are missing from \( R_{M_i} \). They are \( C_i^4 \) and \( \left(\frac{\omega r}{2}\right)^2 \). These two terms are dropped for different reasons. The bias \( C_i^4 \) is not noise but signal and so will be dropped for this noise computation. The \( \frac{\omega r}{2}^2 \) term will be common to all channels and so, if an antisymmetric detector is used, it will drop out of the final result. The zero frequency spectral density corresponding to \( R_{M_i}(\tau) \) is
\[
\int_{-\infty}^{\infty} R_M(\tau) = 2 \int_{0}^{\infty} \left[ c_i^2 \omega r e^{-\omega |\tau|} + \frac{(\omega r)^2}{2} e^{-2\omega |\tau|} \right] d\tau
\]  
(6.8)

\[
= 2 \left[ c_i^2 r + \frac{\omega r^2}{4} \right]
\]  
(6.9)

The noise in the output of the nonlinear correlator is the sum of the noises in the upper and lower branches. The spectral level of the sum is

\[
2 \left[ R^2(e) \sin^2(\omega) + R^2(e) \cos^2(\omega) \right] r + \omega r^2 = 2R^2(e)r + \omega r^2
\]  
(6.10)

To compute the noise level out of the extended range detector, recognize that each of the correlator outputs, corresponding to shifts of \(i\) relative to on-time, is multiplied by weight \(\omega_i\) and then all these weighted correlations are summed. This gives the total noise to be

\[
\sum_{i=-\infty}^{\infty} \omega_i^2 \left[ 2R^2(e + i)r + \omega r^2 \right]
\]  
(6.11)

Two simplifications can be made to this expression. First, \(R^2(e + i)\) is non-zero for at most two adjacent values for \(i\), and so the corresponding \(\omega_i\) can be upper bounded by the maximum value that \(\omega_i\) takes. Furthermore if \(e\), the error, wanders about some (as it surely does during acquisition) then the contribution of \(R^2(e + i)\) can be replaced by its average value, 2/3. Second, if the continuous detector shape is denoted by \(f(\cdot)\) then the standard approximation is to take
\[ \sum_{i=-\infty}^{\infty} \omega_i^2 = \int_{-\infty}^{\infty} f^2(u) \, du \] (6.12)

These substitutions yield the noise in the noncoherent extended range detector to be

\[ \frac{4}{3} f^2(e) r + \int_{-\infty}^{\infty} f^2(u) du \omega r^2 \] (6.13)

which is upper bounded by

\[ \frac{4}{3} \max_e f^2(e) \, r + \int_{-\infty}^{\infty} f^2(u) \, du \omega r^2 \] (6.14)

The noise level is a more complicated function of the detector shape than the level in the coherent case. Notice however that

\[ \sum_{i=-\infty}^{\infty} \omega_i^2 \, 2R^2(e+i) r + \omega r^2 \leq (2r + \omega r^2) \sum_{i=-\infty}^{\infty} \omega_i^2 \] (6.15)

This is more like the coherent noise level expression. It is the product of a spectral density and the sum of the squared weights. The fact that the noise level can be upper bounded by an expression of this form implies that stability is assured under the circumstances for which stability was proven in the coherent case. There is a different steady state variance than in the coherent case. The variance is different for two
reasons. First, in the coherent case the measurement noise had spectral density \( r \). Now it has spectral density \( 2r + \omega r^2 \). Second, the basic nonlinearity was \( R(e) \) in the coherent case and now is \( R^2(e) \). In both cases the area, and not the precise shape, is the important thing about the detector. This suggests that the basic nonlinearity be approximated as indicated in Figure 6.3. The approximate triangle is chosen to have the same area as the squared triangle. This is obtained by taking

\[
R^2(e) \approx \frac{2}{3} R(e)
\]  

(6.16)

and suggests that in addition to the difference in the noise due to squaring, there is a signal suppression in the amount of \( 2/3 \). The achievable steady-state variance for the noncoherent case can be upper bounded by using the formulae derived earlier with the value of noise computed with the noncoherent formula and multiplied by \( (3/2)^2 \) to account for signal suppression in the squarer.

Using the detector shape and noise level just computed, a noncoherent extended range tracking loop design can now be obtained. Either the Fokker-Planck procedure or the Gaussian approximation procedure may be applied with very little modification. The only changes required are that the noncoherent noise level be used and that the basic correlation triangle be replaced by a triangle of height \( 2/3 \) to reflect the signal suppression due to squaring.
Figure 6.3 Approximation to squarer output.
CHAPTER 7

DESIGN OF TRACKING SYSTEMS

EMPLOYING A SINGLE CORRELATOR

7.0 Introduction

Recall that in Chapter 1 the star tracker problem was introduced as having a single telescope (or analogously, a single correlator). The focus in Chapters 4, 5 and 6, however, has been on systems employing many correlators. In the introduction it was shown how two telescopes might be used for fine tracking, but in Chapter 3 the two-correlator PRN code tracking system was shown to be hampered by the finite range of errors to which a two-correlator system responds. It was suggested that the proper use of many correlations might overcome the limitations of the two-correlator system. Chapters 4 and 5 have given some effective methods for using the outputs of many correlators. In this chapter the focus returns to the case where a single correlator is available.

There are several reasons for considering single-correlator systems even though there is inherently more information available in a many-correlator system. One reason is that many systems having a single correlator have already been built. Reconfiguring such a system to incorporate many correlators is difficult. A change in the signal processing, however, is not so difficult, because the processing is typically
done in a general purpose computer. Another reason for considering single-correlator systems is that correlators can be expensive and bulky in some applications. In these cases, then, a design requiring many correlators is impractical. Besides the practical considerations, single-correlator systems exhibit some interesting requirements not found in multiple-correlator systems. These requirements will become clearer later in this chapter.

The first thing attempted in this chapter is to apply the techniques of Chapters 4 and 5 to the design of a single-correlator system. Recall that the multiple-correlator designs given in Chapters 4 and 5 were based on a natural generalization of the standard two-correlator design. A similar generalization of standard single-correlator tracking systems is suggested in this chapter, but certain technical difficulties render the approaches of Chapters 4 and 5 invalid in the generalized single-correlator systems.

A successful approach to the single-correlator problem is suggested in Section 7.3. The approach taken there is to simplify the problem to the point that the nonlinear filter may be determined explicitly. A reasonable policy for centering the single-measurement device is then determined and the resulting design is compared to current designs.

7.1 Traditional Approach to a Single Correlator Tracking System

The traditional approach to correlation tracking with a single correlator is very much analogous to the two correlator approach described in Chapter 3. Instead of comparing the output of an early correlation
with that of a late correlation, the input to a single correlator is
dithered back and forth between early and late codes. Many people use
the Greek letter tau to denote code delay and for this reason the usual
single correlator approach is called a tau-dither (or T-dither) approach.

The tau-dither approach can be visualized as a time shared version
of the two-correlator technique, and is used in some existing GPS code
tracking loops. A block diagram of the standard, coherent, tau-dither
tracking loop is shown in Figure 7.1.

![Diagram of a τ-dither loop]

Figure 7.1 A schematic of a τ-dither loop.

Figure 7.1 illustrates how a τ-dither loop operates. The two
switches shown operate together. When the early bit from the shift
register is being multiplied by the input, the correlator output is
multiplied by minus one and passed through the loop filter. The effect
is to retard the code generator when the early signal correlates with
the input. When the late signal is being multiplied by the input the
correlator output is multiplied by plus one. This speeds up the output code when it is running too slowly.

The tau-dither tracking system can be generalized just as the two-correlator tracking system was. Instead of connecting the correlator to one of only two codes the correlator may be connected to one of many codes, as shown in Figure 7.2. As the correlator input is switched, the weight applied to the correlator output is also switched.

In the next section it will be shown that if the switching rate is high, the generalized tau-dither system can be modelled by a time invariant system which is very similar in form to the nonlinear feedback loops considered in Chapters 4 and 5.

---

![Diagram of Generalized Tau-Dither Tracking System](image)

Figure 7.2 Generalized tau-dither tracking system.
7.2 Derivation of Equivalent Nonlinearity for Rapidly Dithered Systems

The results in Chapter 3 indicate that the output of the correlator in Figure 7.1 can be modelled as the sum of two terms. The first term is correlation between the input code and feedback code. The second term is a noise. Recall that the correlation between two codes is a function of the time shift between them. This correlation function is plotted in Figure 7.3.

![Figure 7.3 Correlation function for random number code.](image)

This familiar model for correlation process leads to the system block diagrammed in Figure 7.4 as a model for the physical system in Figure 7.1. In the model of Figure 7.4 a function $d(t)$ represents the switch shown in Figure 7.1. The switch of Figure 7.1 which controls whether the correlator output is multiplied by plus or minus one has been replaced by a single multiplier multiplying the output of the nonlinearity by $sgn[d(t)]$. The noise $n(t)$ in Figure 7.4 is the baseband equivalent of whatever additive noise is present at the input at Figure 7.1. For the loop in Figure 7.4 to model the loop in Figure 7.1, the dither function $d(t)$ must take values plus or minus one. To model the generalized tau-dither loop, the dither function takes integer
Figure 7.4 A baseband version of the T-dither loop.

values, and some weighting other than plus or minus one must be applied. Even for the generalized loop, only weightings of plus and minus one will be considered. It will be seen that there is enough flexibility in the function \( d(t) \) to obtain a broad class of error responses and the more general case will not be required. The goal of this section is to find a nonlinearity which models the dither function and the \( \text{sgn}[d(t)] \) multiplier (the elements enclosed in dashed boxes in Figure 7.4). It is desired that this model permit use of the design procedures offered earlier. Recall that in the case of parallel correlators, the procedure was to model the weighted correlator outputs by an equivalent forward loop nonlinearity related to the correlator weightings. For the tau-dither system, a forward loop nonlinearity will be found which models the tau-dither system and which is related to the dither function \( d(t) \).

It seems intuitively obvious that if \( d(t) \) is a periodic function with period \( T \), then the loop output using \( d(\alpha t) \) as a dither function should approach a limit in some sense as \( \alpha \to \infty \). That is, as the period
of \( d(\alpha t) \) shrinks, the loop output approaches the output of a loop using the average input. After some definitions, a somewhat more general proposition will be stated precisely.

Consider a sequence of tracking loops of the form shown in Figure 7.4, the difference between the loops being that they use different dither functions. Suppose that the \( n^{th} \) loop uses a dither function \( d_n(t) \). Consider also a function \( h(i) > 0 \) defined on the integers which has

\[
\sum_{i=-\infty}^{\infty} h(i) = 1 \quad (7.1)
\]

Roughly, \( h(i) \) is the fraction of time which the dither \( d(t) \) spends at the integer \( i \). Let \( T_n > 0 \) be called the fundamental period of each function \( d_n(t) \), and suppose that \( T_n \to 0 \) as \( n \to \infty \). Specify the function \( d_n(t) \) only insofar as that for all integers \( n > 0 \) and \( j \geq 0 \)

\[
\lambda \left\{ t : d_n(t) = i \cap [jT_n, (j+1)T_n) \right\} = h(i) \cdot T_n \quad (7.2)
\]

where \( \lambda \) is Lebesgue measure. This says that inside each interval of the form \([jT_n, (j+1)T_n)\) the dither function \( d_n(t) \) spends 100 \( h(i) \) percent of the time at the integer \( i \). Requirement (2) is the only one that needs to be placed on the dither function. It does not have to be periodic. It will be shown that Eqn. (7.2) is sufficient to obtain a limiting loop output as \( T_n \to 0 \). The function \( h(i) \) is called the percent-time function and for deterministic \( d(t) \) is the primary determinant of the loop performance.
Define a function \( \tilde{h}(e) \) by

\[
\tilde{h}(i) = \text{sgn}(i)h(i) \quad \text{for } i \text{ an integer}
\]

(7.3)

and

\[
\tilde{h}(e) = \tilde{h}([e])(1-([e]-e)) + ([e]-e)\tilde{h}([e]+1)
\]

(7.4)

for \( e \) not an integer. The symbol \([e]\) means the greatest integer in \( e \).

The graph of \( h(e) \) is constructed by graphing \( \text{sgn}(i)h(i) \) and connecting the dots with straight lines.

The function \( \tilde{h}(e) \) is the forward loop nonlinearity which is used to obtain a time invariant loop that is equivalent to the original tau-dither loop. The nature of this equivalence can be stated more concretely once the input dynamics and measurement noise are specified.

Suppose that the input phase is a Brownian motion \( \beta_t \) and that the measurement noise \( n(t) \) is the formal derivative of a Brownian motion \( \gamma_t \).

It is sufficient for present purposes to take \( G(s) = K/s \). Then the differential equation for the \( n^{th} \) \( \tau \)-dither loop error is

\[
de_n = -K[\text{sgn}(e_n(t))d_n(t)]dt + d\gamma_t \text{sgn}(d_n(t)) + dB_t
\]

(7.5)

The autonomous equivalent of the loop in Figure 7.3 is block diagrammed in Figure 7.4.
For the input signal, dynamics, and loop filter used to obtain Eqn. (7.5) the error of the loop in Figure 7.4 satisfies

\[ de = -K[h(e)dt + dy_t] + d\beta_t \quad (7.6) \]

Now the equivalence of the solution to Eqn. (7.5) as \( n \to \infty \) and that described by Eqn. (7.6) can be stated precisely. The result is that as \( n \to \infty \) (which means that the dither rate gets high) the statistics of the error process generated by the dither loop, \( \epsilon_n(t) \), approach those of the error in the autonomous loop, \( \epsilon(t) \). This is true of all statistics of these processes including, for example, mean square values, autocorrelations, and power spectra. This result is proven in full detail in Appendix E.

### 7.3 Failure of Fokker-Planck Approach

Since the error probabilities for the loop described by Eqn. (7.6) are the same as for the limiting \( \tau \)-dither loop, then questions involving error statistics for the \( \tau \)-dither loop may be resolved by considering the equivalent loop.

Now all steady-state statistics of \( \epsilon \), the solution to Eqn. (7.6) may be computed once the steady-state probability density \( p(\epsilon) \) is known. The density \( p(\epsilon) \) may be easily computed by solving the steady state Fokker-Planck equation as done in Chapter 4. The steady-state density \( p(\epsilon) \) which solves the steady state Fokker-Planck equation is
\[ p(e) = p_0 \exp \left\{ - \frac{K}{K^2 + 1} \int_0^e \bar{h}(s) \, ds \right\} \quad (7.7) \]

For \( p(e) \) to be a probability density necessitates

\[ \lim_{e \to \infty} \int_0^e \bar{h}(s) \, ds = \infty \quad (7.8) \]

Otherwise \( p(e) \) approaches a constant for large \( e \) and thus has infinite area. The function \( h(i) \) used to define \( \bar{h}(e) \) has the property that

\[ \sum_{i=-\infty}^{\infty} h(i) = 1 \quad (7.9) \]

From the definition of \( \bar{h} \) in Eqns. (7.3) and (7.4) it follows that there is no percent time function \( h(i) \) which will make both Eqns. (7.8) and (7.9) true. It is concluded that the Fokker-Planck equation does not have a steady-state solution which is a probability density, and conjectured that

\[ \lim_{t \to \infty} \mathbb{P}\{e^2(t)\} = \infty \quad (7.10) \]

In the infinite correlator case considered in Chapter 4 there is a condition analogous to Eqn. (7.8). In the infinite correlator case, however, instead of the condition of Eqn. (7.9) there is the condition that \( \mathbb{E}h^2(i) = 1 \). As determined in Chapter 4, there is a narrow band of functions satisfying Eqn. (7.8) and \( \mathbb{E}h^2(i) = 1 \) simultaneously. This means that with an infinite number of correlators there is only a narrow
band of nonlinearities which can be built from these correlators and which yield a finite steady state variance. In light of the care required to choose a nonlinearity when an infinite number of detectors can be used, it is perhaps not surprising that no suitable nonlinearity can be found when only one correlator is available.

Something different will have to be done before the steady-state techniques used in the multiple correlator case can be applied here. Several possibilities arise.

One possibility is a change in the performance index. Since it seems that the τ-dither loop of Eqn. (7.6) will eventually violate any fixed bounds, perhaps the mean time required to do so could be maximized. This is a problem of the same magnitude as the optimum parallel correlator solution. [Stratonovich (1)] and [Darling and Siegert (1)] give differential equations for the mean time to first exit. These depend on the coefficient function \( \tilde{h}(e) \) in a fashion similar to the dependence of the steady-state Fokker-Planck equation on the coefficient function \( \tilde{h}(e) \). The obvious procedure would be to pick fixed bounds and determine \( \tilde{h}(e) \) so as to maximize the mean time to first exit from those bounds.

Another approach might be to pick the input process and the noise process so that the loop is stable no matter what \( \tilde{h}(e) \) is chosen. This approach is similar to that taken by [Kushner(1)]. This approach is a very reasonable one. The Fokker-Planck equation fails to have a steady-state solution partly because the input process is a Brownian motion. This same input process, if input to a traditional early-late tracking
loop, would result in the traditional tracking loop's inevitable loss of lock. This is true no matter how small (nonzero) the measurement noise and input process variance parameters are. In some sense this indicates that the Brownian motion is too violent an input process to model physical motions, because many real tracking loops operate without losing lock. It may be that the correlated input process is a better model of physical input processes. This failure to stabilize with a Brownian input is nonetheless a manifestation of the basic inferiority of the tau-dither loop with respect to the multiple correlator loops considered earlier.

The treatment of the correlated input does not seem substantially different from the analogous case with multiple correlators and will not be pursued further here.

There is still another possibility for improving tau-dither performance. In the schemes considered so far the detector shape has not been data-dependent. That is, given the noise statistics, the shape could be precomputed and did not depend on the particular realization of the measurement and state processes. It may be that, by making the detector depend on the data properly, an improvement in performance can be achieved. A procedure will now be investigated which will give a data-dependent tau-dither scheme.

7.4 Exact Nonlinear Filtering Approach in the Absence of Target Motion

In Chapter 5 the optimal detector shape was determined by approximating the nonlinear filter solution (solution to Kushner's equations)
for arbitrary detector shape, then resolving the detector shape by optimizing the filter performance. In this section a similar program is followed. The target is assumed to be stationary, which makes it possible to solve for the nonlinear filter explicitly. The filter is determined for an arbitrary dither history. The dither history is then determined to yield optimum filter performance. In the tau-dither case the optimum dither policy at a given point in time depends on the measurement history up to that time.

The effect of target motions will be considered later. For the present, assume that the target is stationary. That is, suppose that the target state $\theta$ satisfies

$$d\theta = 0$$

(7.11)

and that the measurement $z_t$ satisfies

$$dz_t = h[\theta(t) - d(t)]dt + d\beta_t$$

where $h$ is the usual PRN code autocorrelation function discussed in Chapter 3.

$$h(x) = 1 - |x| \quad |x| \leq 1$$

$$= 0 \quad |x| > 1$$
and \( d(t) \) is a \( Z_0^t \) measurable function. The process \( \beta_t \) is a Brownian motion. As a thought experiment, suppose that the measurement strategy \( d(t) = 0 \) is chosen and that the initial target state \( \theta \) is distributed as drawn in Figure 7.5(a). If, in a particular trial, the real initial state lies well away from zero then after measurements have been collected for some time, the posterior density will be depressed in the middle as shown in Figure 7.5(b). It seems more or less reasonable to move the measurement device (shown centered at zero in Figure 7.5(c)).

![Figure 7.5 Thought experiment for sensor placement problem.](image-url)
to a new position where there is a higher probability of finding the signal. Instead of leaving the measurement device centered at zero, it should be moved to where the signal most probably lies. For example, the centering \( d^*(t) \) might be chosen by

\[
\int_{-\infty}^{\infty} h(\theta - d^*(t)) \cdot p_\theta(\theta) \, d\theta = \max_d \int_{-\infty}^{\infty} h(\theta - d) \cdot p_\theta(\theta) \, d\theta \quad (7.12)
\]

where \( p_\theta(\theta) \) is the a-posteriori probability density for \( \theta \), the target state, given measurements up to time \( t \). This criterion for selecting \( d^*(t) \) results in a random search; that is, one which depends on the measurements taken and which will be different for different realizations of the measurement noise. This strategy can be justified by considering the following idealized problem.

As an idealization of the acquisition problem, suppose that the following situation exists. The a-priori probability for \( \theta \) is concentrated on the integers. The state is constant in \( \mathbb{Z} \) and the measurement is characterized by

\[
dz = \delta_{\theta, d} \, dt + d\beta_t \quad (7.13)
\]

where \( \delta_{\theta, d} \) is a Kronecker delta function, \( \theta \) is the state being estimated and \( d \) is the "centering" used to take the measurement. If the centering is held constant in the interval [s,t] then for \( t \in [s,T] \) the conditional probability given measurements up to time \( t \) is given by
If \( d \) is constant over \([s,t]\) then the expression simplifies to

\[
p_t(i) = \frac{p_s(i)}{1 - p_s(d) + \exp\left\{z - z_s - \frac{t-s}{2}\right\} p_s(d)} \quad i \neq d
\]  

(7.15)

\[
p_t(d) = \frac{p_s(d) \exp\left\{z - z_s - \frac{t-s}{2}\right\}}{1 - p_s(d) + \exp\left\{z - z_s - \frac{t-s}{2}\right\} p_s(d)}
\]  

(7.16)

This probability mass function has two possible behaviors depending on whether the state \( \theta \) is \( \theta = d \) or \( \theta \neq d \). Specifically

\[
p_t(d) = \frac{\exp\left\{\frac{t-s}{2} + \beta - \beta_s\right\} p_s(d)}{1 - p_s(d) + p_s(d) \exp\left\{\frac{t-s}{2} + \beta - \beta_s\right\}}
\]  

(7.17)

\[
p_t(i) = \frac{p_s(i)}{1 - p_s(d) + p_s(d) \exp\left\{\frac{t-s}{2} + \beta - \beta_s\right\}} \quad i \neq d
\]  

(7.18)

if \( \theta = d \). On the other hand,
\[ p_t(d) = \frac{\exp\left(\beta_t - \beta_s - \frac{t-s}{2}\right) p_s(d)}{[1-p_s(d)] + p_s(d) \exp\left(\beta_t - \beta_s - \frac{t-s}{2}\right)} \] (7.19)

\[ p_t(i) = \frac{p_s(i)}{[1-p_s(d)] + p_s(d) \exp\left(\beta_t - \beta_s - \frac{t-s}{2}\right)} \quad i \neq d \] (7.20)

if \( \theta \neq d \).

The problem of choosing the centering for \( d \) will be tackled by considering the conditional density as the state of a control system, and considering the centering to be the control. Generally the state will be an element in an infinite dimensional vector space, since the a-priori density will generally have a nonzero mass at each of the integers. Since the evolution of the state can be described explicitly (as indicated above) this is not as serious a problem as it might normally seem to be. Several types of solutions are available. First suppose that measurement strategy is chosen to maximize

\[ \lim_{t \to \infty} E \left[ \sup_i p_t(i) \right] \] (7.21)

The quantity \( \sup_i p_t(i) \) is the conditional probability at time \( t \) of the most probable point. Maximizing this performance index then yields a measurement policy which maximizes the expected steady-state probability of the most likely point. Suppose that a feedback control law is to be
found assuming that future measurements will be taken using constant centering. This is like seeking the open loop optimal control. The control is open loop in the sense that the control is computed without taking into account the fact that the control can be changed in the future. The expressions given above indicate that

$$E_{p_s} \lim_{t \to \infty} \sup_i p_t(i) = 1 \cdot p_s(d) + \left[1 - p_s(d)\right] \sup_{i \neq d} \frac{p_s(i)}{1 - p_s(d)} \quad (7.22)$$

$$= p_s(d) + \sup_{i \neq d} p_s(i) \quad (7.23)$$

This is maximized by choosing $d^*$ (the optimal centering) to yield

$$p_s(d^*) = \sup_i p_s(i) \quad (7.24)$$

or by choosing $d^*$ corresponding to the second largest probability. The feedback control law is then to use the formula above to propagate the conditional probability mass function, and to pick the centering function $d^*$ corresponding to the maximum probability. This is an altogether reasonable policy.

A different performance yielding this same measurement policy will now be considered. Suppose that it is desired to maximize the amount of information available for estimating the phase $\theta$ at some final time $T$. The Shannon information measure for this problem is
\[ E \int_0^T \left( \delta_{\theta,d} - \hat{\delta}_{\theta,d} \right)^2 \, dt = E \int_0^T E_{F_t} \left( \delta_{\theta,d} - \hat{\delta}_{\theta,d} \right)^2 \, dt \]  

(7.25)

where \( E_{F_t} \) is the conditional expectation given measurements up to time \( t \).

Eqn. (7.25) can be simplified by recognizing that

\[ \left( \delta_{\theta,d} - \hat{\delta}_{\theta,d} \right)^2 = \left( 1 - p_t(d) \right)^2 \quad \text{if} \quad \theta = d \]

\[ = \left( -p_t(d) \right)^2 \quad \text{if} \quad \theta \neq d \]  

(7.26)

and thus computing

\[ E_{F_t} \left( \delta_{\theta,d} - \hat{\delta}_{\theta,d} \right)^2 = \left( 1 - p_t(d) \right)^2 p_t(d) + p_t^2(d) \left( 1 - p_t(d) \right) \]

\[ = \left( 1 - p_t(d) \right) p_t(d) \]  

(7.27)

Substituting this expression into Eqn. (7.25) then yields

\[ E \int_0^T \left( \delta_{\theta,d} - \hat{\delta}_{\theta,d} \right)^2 \, dt = E \int_0^T \left( 1 - p_t(d) \right) p_t(d) \, dt \]  

(7.28)

At each point in time, \( t \), \( d(t) \) can be chosen to maximize the integrand.

It will be shown that the point \( d^* \) satisfying

\[ p_t(d^*) = \max_i p_t(i) \]  

(7.29)
also yields

\[
\left(1 - p_t(d^*)\right)p_t(d^*) = \max_i \left(1 - p_t(i)\right)p_t(i)
\]

(7.30)

This means that the policy which maximizes the information available is the same as the policy derived earlier to yield maximum probability. The equivalence of the policy indicated by Eqn. (7.29) and that indicated by Eqn. (7.30) can be established with reference to Figure 7.6. This figure shows how \(1-p)p\ behaves as a function of \(p\). This function is symmetric about 0.5. If \(p(i)\) is uniformly less than 0.5 then the equivalence of Eqns. (7.29) and (7.30) is clear. If one of the \(p(i) > 0.5\) then there are two possibilities. First, if there are only two points of nonzero mass, then these two probabilities are symmetrically placed about 0.5 and it makes no difference which point is chosen for centering the measurement. In the second case, if there are more than two points of nonzero mass, one of which has probability greater than 0.5, then those of probability less than 0.5 must be smaller than the reflection of the largest one about 0.5.
The policy suggested here is the same as one derived by A. N. Shiryaev [Shiryaev (2),(3),(4)] in another way. Shiryaev attaches costs to making mistakes and to taking measurements. The policy which minimizes the expected cost is then composed of two rules: one rule for stopping the process and another for deciding where to look if the decision is made to continue. The rule for where to look is the same as the one derived here, namely, look in the most probable location.

The acquisition procedure which this strategy suggests was the object of a Monte Carlo simulation whose description follows. The acquisition begins from an initial condition whose probability is concentrated on the integers one through 40. The initial probabilities are given by

\[ p(i) = \exp \left\{ -\frac{(i-20.5)^2}{20} \right\} /C \]  

(7.31)

where \( C \) is a normalizing factor

\[ C = \sum_{i=1}^{40} \exp \left\{ -\frac{(i-20.5)^2}{20} \right\} \]  

(7.32)

The initial conditions are generated by sampling Gaussian random numbers with mean 21 and variance 10, taking the greatest integer in the samples and discarding any samples greater than 40 or less than one. The centering is selected to correspond to the point of maximum probability and a measurement is taken for some small predetermined length of time; for the run plotted in Figure 7.7 this time is 0.5 sec. The measurement
given is as described above (Eqn. (7.13)) where the variance parameter of the additive Brownian motion is 1.0. Figure 7.7 shows one realization for the centering using this policy. The procedure is stopped when

$$\max_{i} p(i) > .9$$  \hspace{1cm} (7.33)

The realization shown generally starts at the center of the a-priori density and works its way outward by switching from one side of the density to the other. The procedure does return to inspect points which have been once discarded.

<table>
<thead>
<tr>
<th>$P_{\text{acc}}$</th>
<th>$P_{\text{succ}}$</th>
<th>$T_{\text{avg}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.9</td>
<td>.93</td>
<td>76.7</td>
</tr>
<tr>
<td>.95</td>
<td>.97</td>
<td>74.2</td>
</tr>
<tr>
<td>.99</td>
<td>.98</td>
<td>81.7</td>
</tr>
<tr>
<td>.999</td>
<td>.99</td>
<td>98.6</td>
</tr>
</tbody>
</table>

The table above shows how the procedure behaves based on the average of 100 runs. For each set of 100 runs the search was stopped when $\max_{i} p(i) > P_{\text{acc}}$ where $P_{\text{acc}}$ is entered in the first column of Table 7.1. The second and third columns indicate the fraction correct detections and average stopping time achieved with the given stopping rule.

As a point of comparison the dual threshold sequential search, which is normally used for GPS code acquisition, is considered. This procedure, as indicated in [Shiryayev (1)], is to consider each possible
Figure 7.7 Sample of optimal centering policy.
position in its turn (according to a preassigned pattern) and to inspect that position until it is relatively certain that the target is either present or absent. For this test the probability that the signal is in the \(i^{th}\) cell is given by

\[
p_t(i) = \frac{p_0(i) \exp\left(\frac{z_i - t/2}{\tau}\right)}{[1 - p_0(i)] + p_0(i) \exp\left(\frac{z_i - t/2}{\tau}\right)}
\]  

(7.34)

where \(t\) is the time since the inspection of the \(i^{th}\) cell began and \(p_0(i)\) is the a-priori probability of the \(i^{th}\) cell. One obvious deficiency of this procedure is that the prior probability is used to compute \(p_t(i)\). It would be more reasonable to use the conditional probability given the measurements up to the time when the inspection of the \(i^{th}\) cell begins. The procedure in current use then dismisses much of the information it has collected each time it begins to consider a new position. When \(p_t(i)\) moves outside an integral \([p_{\text{rej}}, p_{\text{acc}}]\) the \(i^{th}\) location is declared to contain or not contain the target depending on whether \(p_t(i)\) crosses \(p_{\text{acc}}\) or \(p_{\text{rej}}\). As long as \(p_t\) stays between \(p_{\text{acc}}\) and \(p_{\text{rej}}\) the inspection of the \(i^{th}\) cell continues. These two thresholds then characterize the procedure. Figure 7.8 shows the result of using this procedure and how the result depends on these thresholds. In Figure 7.8 the abcissa is the acceptance threshold \(p_{\text{acc}}\) and the ordinate is the rejection threshold \(p_{\text{rej}}\). The numbers at a point \((p_{\text{acc}}, p_{\text{rej}})\) are the percentage of successful acquisitions and the average stopping time respectively. For example, at the point \((.99, .001)\) 84/113 is written. This means that
(NUMBER OF SUCCESSES
PER 100 TRIALS)/(AVERAGE TIME TO DECISION)

0.01  -  76/43  63/
0.001 -  83/100  84/113  80/
0.0001 -  80/136  80/136  81/131  75/136
0.00001 -  76/208

ACCEPTANCE THRESHOLD

0.000001 -  0.9999  0.999  0.99  0.9

Figure 7.8 Monte Carlo simulation results for standard acquisition procedure.

when $p_{\text{acc}} = .99$ and $p_{\text{rej}} = .001$ then out of 100 Monte Carlo runs with the initial conditions used earlier, 84 trials were successful and the average stopping time was 113 seconds. This procedure does not work well at all and is clearly inferior to the maximum information procedure suggested earlier.

Several features need to be added to this acquisition scheme before it can be used for acquiring and tracking a random number code. One feature which is missing is a realistic model of the correlator output as a function of the error. The simulation was based on a Kronecker delta measurement nonlinearity and an initial error taking only integer values. In reality the measurement nonlinearity is the triangular function drawn in Figure 7.5(c) and the error can take on a continuum
of values. A reasonable approximation will be achieved using a probability mass function, but more points at a finer spacing will have to be employed. The nonlinearity can be approximated by several Kronecker delta functions which are spaced as closely as the mass points and whose envelope is the triangular correlation function in Figure 7.5(c). By making the mass points close together an arbitrary degree of accuracy may be achieved at the expense of greater complexity and computational burden. The traditional search procedure cannot be made to reflect better the true nature of the nonlinearity and error distribution. In practice, an average value of the correlation weight (1/2) and a 0.5 chip spacing would be chosen for the traditional search procedure.

Another feature which needs to be added is target motion. The algorithm suggested here was derived under the supposition that the target did not move. It does appear that this algorithm will function with a moving target, however. For example, suppose that the target remains in one location for a time and then jumps to a new location where it remains. The problem of finding the target just after it has jumped looks, in a sense, like a new acquisition. If the algorithm has found the target just before it jumps (that is, the algorithm has assigned high probability to the correct location and is centering the correlator there) then after the target jumps the algorithm loses confidence in the old location, hunts around, and finds the new location.

A second indication that the algorithm will function in the presence of target motion is that the nonlinear filter without plant
noise appears to be approximately the correct filter when the plant noise is small. That is, the filter with zero noise looks to be in error by a constant times the plant noise magnitude.

There are some problems with using this algorithm to track moving targets. If the algorithm is very confident of a given location, then when the target moves from that location the algorithm will probably be slow to respond. When it does respond it will return to jumping around as it did during the initial acquisition. This algorithm does not necessarily look first at those points close to where it last saw the target. More work is needed to decide how severe these problems are and to discover ways around them.
CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

In this dissertation a problem called the correlation tracking problem has been identified and stated abstractly. Many problems which fit the abstract definition of a correlation tracking problem have been addressed, but to this author's knowledge these problems have not been visualized as feedback estimation problems. The viewpoint provided here appears to offer some advantages over those normally taken. The principal advantage is that both the shape of the measurement nonlinearity and the filtering algorithm are outputs of the feedback estimation design procedure. The principal disadvantage is that the feedback estimation problem is difficult to solve exactly.

It has been shown that correlation tracking problems and, more generally, feedback estimation problems can be thought of as two problems combined: a nonlinear filtering problem and an optimization problem. Several different approaches to the correlation tracking problem have exhibited this two-problem behavior. Fokker-Planck approaches have been used to solve the nonlinear filtering part of the problem by considering a restricted class of estimators, computing the resulting steady-state probability density, and then finding the optimal estimator
by a variational calculus procedure. The designs resulting from this procedure have been shown to track successfully in conditions which render traditional tracking loops inoperative. The Fokker-Planck designs distinguish themselves over the extended Kalman filter by their global treatment of the system nonlinearities as opposed to the local treatment used by the extended Kalman filter.

A second procedure used in this problem was to solve the nonlinear filtering problem by a density approximation method. When this was done the optimization step could be carried out by a simple Schwartz inequality method. This procedure again treated the nonlinearity in a global manner. Using Gaussian approximation was shown to be equivalent to linearizing with a describing function and using a Kalman filter on the resulting linearized system. It was also seen that in the small error limit this approximation reduced to the extended Kalman filter. This Gaussian approximation was seen to offer the advantage of being easily extensible to arbitrary order linear input processes and of covering the important transient operation (acquisition in particular). In the particular case of a Brownian input delay process, the Gaussian approximation was observed to be over-optimistic in predicting its performance. Both the Fokker-Planck and density approximation methods were justified for use in noncoherent GPS code tracking.

The problem of tracking using the output of a single correlator was addressed. A model for a dithered loop was derived and was used to show that the Fokker-Planck approach would not work in the single correlator problem. An approach which was effective, at least for a non-
moving input, was to solve the associated class of nonlinear filtering problems exactly and to approximate the solution to the optimization problem which resulted. This gave an acquisition which was shown to outperform the procedure in current GPS use and which seems to have potential for use in tracking a moving input as well.

A GPS receiver designer will find two of the results here of immediate interest. One is the use of Gaussian approximation for computing tracking loop bandwidths and changing these bandwidths in response to changing noise and vehicle dynamics. It should be noted that the Gaussian approximation may be used without range extension to change the gains in existing tracking loops. The second item of interest is the acquisition scheme presented in Chapter 7. This scheme needs some modification before it is ready to use, but the modifications need not be involved. These two, in particular, are of immediate interest because they may be implemented in current receivers as a software change.

Several things which appeared indicate areas where further work would be useful. Perturbation approximations for probability densities seem to hold potentially useful results. The approximation given here could be extended in several ways as was indicated earlier. Another approach to the same problem would be to obtain a perturbation solution to the stochastic differential equation first and then use these solutions to obtain the probability densities. There should be a connection between these, but it is not clear how to determine it.
Acquisition procedures for the Gaussian approximation could profit from some attention. The procedure was observed to accurately predict the errors for most of the sample paths (about 80%), but to fail miserably on the others. To fix this requires an adaptive procedure which has a direct link between the system errors and the covariance. Such a procedure would result, for example, if two nonlinearities, one symmetric and one antisymmetric, were used instead of a single, antisymmetric nonlinearity. This idea was elaborated upon at the end of Chapter 7.

The approach used to obtain a fast tau-dither acquisition seems extensible in two interesting and practically important ways. It would be useful to have an algorithm which would work for a two-dimensional (phase and frequency) process. It would also be useful to have a solution for the case where there is plant noise. The addition of plant noise fundamentally complicates the problem, but it seems as though the no-noise algorithm would track some motion as it stands. It also seems that a small motion perturbation might be very useful in this context.
Appendix A

A Representation Theorem for Feedback Measurements

Several estimation problems of interest here are most conveniently modeled by

\[
\text{(state)} \quad x(t, \omega) = x(0, \omega) + \int_0^t f(x(s, \omega), s) \, ds + \int_0^t \sigma(x(s, \omega), s) \, d\alpha(s, \omega)
\]

\[
\text{(meas.)} \quad z(t, \omega) = z(0, \omega) + \int_0^t h(z(\cdot, \omega)x(s, \omega), s) \, ds + \int_0^t (z(\cdot, \omega), s) \, d\eta(s, \omega)
\]

That is, the state \(x(t, \omega)\) (or signal) satisfies a usual stochastic differential equation and the measurement \(z(t, \omega)\) satisfies a stochastic functional differential equation. These equations arise in several time-of-arrival tracking problems where the measurement is modeled by

\[
h(z(\cdot, s), x(s, \omega), s) = g(x(s, \omega) - \hat{x}(s, \omega), s). \quad \text{This measurement is a function of the difference between the current state and the current estimate of the state. Another place where this situation arises is with a deformable measurement nonlinearity. A deformable nonlinearity is one which can be made to assume one of a variety of shapes. The object here is to derive an equation for the conditional density of the \(x(t)\) given measurements \(z(t)\). The equation turns out to be an unsurprising modification to the}
representation theorem. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Always consider $\omega$ to be an element of $\Omega$. Let $\alpha(s, \omega)$ and $\eta(s, \omega)$ be $p$ dimensional and $r$ dimensional standard Brownian motions for $0 \leq s \leq T$ and which are independent. Let $\sigma(x(s, \omega), s)$ and $\rho(z(\cdot, s), s)$ be respectively $p \times p$ and $r \times r$ dimensional invertible matrices. Let $f(x(s, \omega), s)$ be a $p$-vector and $h(z(\cdot, s), x(s, \omega), s)$ be an $r$-vector.

Consider the following two sets of differential equations.

\begin{align*}
  x(t, \omega) &= x(0, \omega) + \int_0^t f(x(s, \omega), s) \, ds + \int_0^t \sigma(x(s, \omega), s) \, d\alpha(s, \omega) \\
  z(t, \omega) &= z(0, \omega) + \int_0^t h(z(\cdot, s), x(s, \omega), s) \, ds + \int_0^t \rho(z(\cdot, \omega), s) \, d\eta(s, \omega) \\
\end{align*}

\begin{align*}
  (A.1) \\

  \text{and}

  x(t, \omega) &= x(0, \omega) + \int_0^t f(x(s, \omega), s) \, ds + \int_0^t \sigma(x(s, \omega), s) \, d\alpha(s, \omega) \\
  z(t, \omega) &= z(0, \omega) + \int_0^t \rho(z(\cdot, \omega), s) \, d\eta(s, \omega) \\
  (A.2)
\end{align*}

Suppose that the conditions on these equations (i.e., $f$, $\sigma$, $\rho$, $h$) are such that there exist solutions which have continuous versions and which are unique in probability law. (Sufficient conditions for these things will be given later.) Suppose also that $x(0, \omega)$ and $z(0, \omega)$ are independent of $\alpha$ and $\eta$. Then the solutions to (A.1) and (A.2) induce probability measures on the space of continuous functions $C^{p+r}[0,T]$ - the space of continuous $\mathbb{R}^{p+r}$ valued functions on $[0,T]$. Denote by $\mu$ the probability measure on $C^{p+r}[p,T]$ induced by the solution to (A.2)
and by $\tilde{\mu}$ that induced by the solution to (A.1). It is desired for $\mu$ and $\tilde{\mu}$ to be equivalent measures and to have a formula for the Radon-Nicodym derivation $d\mu/d\tilde{\mu}$. It will be shown later that, with the restriction that $\sigma$ and $\rho$ are invertible, $\mu$ and $\tilde{\mu}$ are equivalent and that an extension of Girsanov's formula gives

$$
\frac{d\tilde{\mu}}{d\mu} = \exp \left\{ \int_0^T \rho^{-1}(z(\cdot,\omega),s)h(z(\cdot,s),x(s,\omega),s)dz(s,\omega) - \frac{1}{2} \int_0^T h(z(\cdot,\omega),x(s,\omega),s)\left[\rho^{-1}(z(\cdot,\omega),s)\right]^T \rho^{-1}(z(\cdot,\omega),s)h(z(\cdot,\omega),x(s,\omega),s)ds \right\}
$$

(A.3)

Given this formula and the independence of $x$ and $z$ in Equation (A.2), the derivation of the representation theorem given by [Zakai (1)] may be used without change.

Let $u \in \mathbb{R}^p$ and define

$$Q(u,t) \equiv P \left\{ \omega: x(t,\omega) \leq u \right\}, \quad t \in [0,T]$$

Denote by $E(\cdot)$ the expectation operator with respect to $\mu$ and by $\tilde{E}(\cdot)$ that with respect to $\tilde{\mu}$. These are integrals defined for functions which are measurable with respect to $\mathcal{M}_T$ (the $\sigma$-algebra generated by sets $x(\cdot): x(\cdot) \in \mathbb{P}[0,T], x(t) < u \in \mathbb{R}^p$, $t \in [0,T]$ and $\mathcal{P}^+[0,T]$). Let $B(x_0^S, z_0^S)$ be the sub $\sigma$-algebra generated by the families of random variables
\[ x(t, \omega); 0 \leq t \leq s \text{ and } z(t, \omega); 0 \leq t \leq s; \text{ similar for } B(x^s_0) \]
or \( B(x^s_s, y^s_0) \). Denote the conditional expectation with respect to the
sub \( \sigma \)-algebra above by \( E \cdot B(x^s_0, y^s_0) \) (similar for \( \tilde{E} \)). Let \( \Lambda(u, z^t_0) \) be the
value of some version of
\[
E \left\{ E \left| \frac{d\tilde{u}}{du} \mid B(x^t_0, z^t_0) \right| B(x^t_0, z^t_0) \right\}
\]
evaluated at the point \( x(t, \omega) = u \) and along the trajectory \( z^t_0 \). Then
for any real-valued Borel function \( f(x) \), \( x \in \mathbb{R}^p \) the conditional expectation
is given by
\[
\tilde{E} \left\{ f(x(t, \omega)) \mid B(z^t_0) \right\} = \frac{\int_{\mathbb{R}^p} \Lambda(u, z^t_0) f(u) dQ(u, t)}{\int_{\mathbb{R}^p} \Lambda(u, z^t_0) dQ(u, t)}
\]
In addition, if \( Q(u, t) \) is absolutely continuous with respect to Lebesgue
measure in \( \mathbb{R}^p \) and has probability density \( p(u, t) \) then the conditional
probability density \( p(u, t \mid B(z^t_0)) \) exists and is given by
\[
p(u, t \mid B(z^t_0)) = \frac{\Lambda(u, z^t_0) p(u, t)}{\int_{\mathbb{R}^p} \Lambda(u, z^t_0) p(u, t) du}
\]
As already mentioned, the derivation, starting from Girsanov's
formula, follows [Zakai (1)] closely, so details are not included here.

To obtain Girsanov's formula as given in (A.3), the derivation in
[Friedman (1)] may be used with some minor alterations. To establish this
this formula, consider a new probability space \((Ω, F, P)\) and two
\(p\)-vector differential equations

\[
dξ(t,ω) = σ(ξ(·,ω),t)dα(t,ω) + b(ξ(·,ω),t)dt \tag{A.4}
\]

\[
d\tilde{ξ}(t,ω) = σ(\tilde{ξ}(·,ω),t)dα(t,ω)+\tilde{b}(\tilde{ξ}(·,ω),t)dt \tag{A.5}
\]

where \(α(t,ω)\) is a standard \(p\)-dimensional Brownian motion and \(σ(x(·),t)\)
is a non-singular matrix \(V x(·)t \in C^p[0,T]\) and \(t∈[0,T]\). Suppose that
solutions to (A.4) and (A.5) exist and have continuous versions. Suppose
also that these solutions are unique in the sense of the probability law
that they induce on the space \((C^p[0,T], M_T)\), where \(M_T\) is the \(σ\)-algebra
generated by sets of the form \(\{x(·) : x(·) ∈ C^p[0,T], x(t) < u, t ∈ [0,T], u ∈ \mathbb{R}^p\}\).
Assume the solutions \(ξ\) and \(\tilde{ξ}\) are such that

\[
E \int_0^T f^2(ξ(·,ω),t)dt < \infty
\]

for \(f = σ_{ij}, f = b_i\), or \(f = \tilde{b}_i, i, j = 1, \ldots, p\). Define

\[
ψ(x,t) = σ^{-1}(x,t)[\tilde{b}(x,t) - b(x,t)]
\]

and

\[
ϕ(t,ω) = ψ(ξ(·,ω),t)
\]
Then, applying the results in [Friedman (1), Chapter 7], the following is obtained.

**Theorem 0.** With the conditions and definitions above, the measures \( \mu^{-\zeta} \) and \( \mu^{+\zeta} \) induced on \((C^p[0,T],M_T)\) by the solutions to (A.4) and (A.5) respectively are equivalent and

\[
\frac{d\mu^{+\zeta}}{d\mu^{-\zeta}} = \exp \left\{ \zeta^+_T(\phi) \right\}
\]

where \( \zeta^+_s(\phi) = \int_s^t \phi(u,\omega) d\alpha(u,\omega) - \frac{1}{2} \int_s^t |\phi(u,\omega)|^2 du. \)

Now it only remains to establish sufficient conditions for the solutions to equations (A.4) and (A.5), and the coefficients evaluated along the solutions, to satisfy the conditions imposed for Theorem 0. Let \( \Omega = C[0,T] \) \( \omega \in \Omega, \ P = \) Wiener measure and \( M_T = (\sigma\text{-algebra generated by sets } \{x(\cdot) \in C[0,T] : x(t) < a\} \ a \in \mathbb{R}, \ t \in [0,T]\}). \) The underlying probability space is then \((\Omega, M_T, P)\). Let

\[
x(t,\omega) = x(0,\omega) + \int_0^t \sigma(x(\cdot,\omega),s) dw(s,\omega) + \int_0^t b(x(\cdot,\omega),s) ds \quad (A.6a)
\]

where \( w(s,\omega) \) is a standard Brownian motion over \((\Omega, M_T, P)\) \( x(0,\omega) \in L^2(\Omega) \)

and \( \sigma \) and \( b \) are functionals defined on \( C[0,T] \times [0,T] \) satisfying

(let \( f = b \) or \( \sigma \))
\[ |f(x(^*), s)| \leq k(1+|x_1|_T) \]

and

\[ |f(x_1(^*), s) - f(x_2(^*), s)| \leq K(|x_1 - x_2|_T) \]  \hspace{1cm} (A.6b)

where \( x_1, x_2 \in C[0, T] \) and \(|x|_T = \sup_{0 \leq s \leq T} |x(s)| \). In addition, \( b(^*, s) \) and \( \sigma(^*, s) \) are \( M_T \) measurable. \( M_T \) is the \( \sigma \)-algebra generated by sets \( \{x_s \in A \} \) \( 0 \leq s \leq t \) and \( A \) a Borel set is \( \mathbb{R} \) and \( x(^*) \in C[0, T] \).

First, it will be shown that the measureability condition on \( b \) and \( \sigma \) gives a stronger bound than hypothesized.

**Lemma 1** for \( b \) and \( \sigma \) as defined above and \( x_1, x_2 \in C[0, T] \)

\[ |b(x_1(^*), s)| \leq K(1+|x_1|_s) \]

and

\[ |b(x_1(^*), s) - b(x_2(^*), s)| \leq K(|x_1 - x_2|_s) \]

and similar conditions hold for \( \sigma \).

**Proof.** Since \( b(^*, s) \) is the measureable, then it can be approximated point-wise by a functional \( b_n(^*, s) = \sum_{i=1}^{N} a_i n^{-1} E_{i,n} \) where the \( E_{i,n} \) are sets in the algebra generated by finite unions and intersections of finite cylinders. \( x(t) \in A \) \( 0 \leq t \leq s \). Since \( b \) does not take on the value \( \pm \infty \), then the \( b_n \)s can be chosen such that \( \forall \ x(^*) \in C[0, T] \ \epsilon < 0 \exists N_0 \in \mathbb{N} \ s.t. \)

\[ |b_n(x(^*), s) - b(x(^*), s)| < \epsilon \ \forall \ n > N_0. \] For a given continuous function \( x(^*) \)
define $x^s = \{ y \in C[0,T] : y(t) = x(t) \mid 0 \leq t \leq s \}$

By definition, if $y \in x^s$ then

$$b_n(x^s, s) = b_n(y^s, s) = b_n(x^s, s)$$

In particular, for $x'(s) \in x^s$ defined by

$$x'(t) \equiv x(t) \mid 0 \leq t \leq s \equiv x(s) \mid s < t$$

then $b_n(x'(s), s) = b_n(x(s), s)$ for every $n$.

Since $b(x(s), s) = \lim_{n \to \infty} b_n(x(s), s) = \lim_{n \to \infty} b_n(x'(s), s) = b(x'(s), s)$

then it follows that

$$|b(x(s), s)| = |b(x'(s), s)| \leq K(1+|x'|_T$$

$$= K(1+|x|_s)$$

Analogously define $x_1^s(s)$ and $x_2^s(s)$ for two arbitrary functions $x_1(s), x_2(s) \in C[0,T]$ by
\[ x_i'(t) \equiv x_i(t) \quad 0 \leq t \leq s \]

\[ \equiv x_i(s) \quad s < t \]

for \( i = 1, 2 \). Then

\[ |b(x_1(\cdot), s) - b(x_2(\cdot), s)| = |b(x_1'(\cdot), s) - b(x_2'(\cdot), s)| \]

\[ \leq K(||x'_1 - x'_2||_T) = K(||x_1 - x_2||_s) \]

Replace the bs with cs and the identical proof works for c. Q.E.D.

Now successive approximations can be used to establish existence of solutions to Eqns. (A.4) and (A.5). Define a sequence of stochastic processes \( \{x_n(t, \omega)\}_{n \geq 0} \) by

\[ x_0(t, \omega) = x(0, \omega) \]

\[ x_{m+1}(t, \omega) = x_0(t, \omega) + \int_0^t \sigma(x_m(\cdot, \omega), s)dw(s, \omega) + \int_0^t b(x_m(\cdot, \omega), s)ds \]

Then

**Lemma 2.** Under conditions (A.6) with definitions (A.7), then

\[ E||x_m(\cdot, \omega)||_T^2 < \infty \]
Proof. Obviously true for $m = 0$. For $m > 0$ assume (induction hypothesis) true for $m - 1$.

\[
E \left| \int_0^T \sigma(\xi_{m-1}(\cdot, \omega), s) dw(s, \omega) + \int_0^T b(\xi_{m-1}(\cdot, \omega), s) ds \right|^2 \\
\leq 4E \left| \int_0^T \sigma_0(\cdot, \omega) \right|_T^2 + 4E \left| \int_0^T \sigma(\xi_{m-1}(\cdot, \omega), s) dw(s, \omega) + \int_0^T b(\xi_{m-1}(\cdot, \omega), s) ds \right|^2 \\
\leq 4E \left| \int_0^T \sigma_0(\cdot, \omega) \right|_T^2 + 16E \left| \int_0^T \sigma(\xi_{m-1}(\cdot, \omega), s) dw(s, \cdot) \right|_T^2 + \\
+ 16E \left| \int_0^T b(\xi_{m-1}(\cdot, \omega), s) ds \right|^2
\]  
(A.8)

The first term on the right is bounded by hypothesis. The second term may be bounded by proving that (given the induction hypothesis) the time function $\int_0^T \sigma(\xi_{m-1}(\cdot, \omega), s) dw(s, \omega)$ is a separable martingale. It is sufficient [Friedman (1), Theorems 3.1 and 3.2, pg. 67] to show that

\[
E \int_0^T \sigma^2(\xi_{m-1}(\cdot, \omega), s) ds < \infty
\]

By hypothesis (A.6)

\[
E \int_0^T \sigma^2(\xi_{m-1}(\cdot, \omega), s) ds \leq E \int_0^T k^2(1 + \int_0^T |\xi_{m-1}(\cdot, \omega)|_T^2) ds \\
\leq k^2 E \left\{ 4 + 4 \left( \int_0^T |\xi_{m-1}(\cdot, \omega)|_T^2 \right) \right\}
\]  
(A.9)
which is finite by the induction hypothesis. Then the result
(Friedman [16] Theorem 3.7, pg. 71) that if $X(t)$ is a separable martingale, then for any $\alpha > 1$,

$$E \left\{ \sup_{0 \leq t \leq T} |X(t)|^\alpha \right\} \leq \left( \frac{\alpha}{\alpha - 1} \right)^{\alpha} E |X(T)|^\alpha$$

may be used to bound the second term in (A.8).

$$E \left| \int_0^T \sigma(\xi_{m-1}(\cdot, \omega), s) dw(s, \omega) \right|^2 \leq 2^2 E \left| \int_0^T \sigma(\xi_{m-1}(\cdot, \omega), s) dw(s, \omega) \right|^2$$

$$= 4E \int_0^T \sigma^2(\xi_{m-1}(\cdot, \omega), s) ds$$

which was shown to be bounded in (A.4). The last term in (A.8) may be bounded by

$$E \left| \int_0^T b(\xi_{m-1}(\cdot, \omega), s) ds \right|^2 \leq E \left| \int_0^T b^2(\xi_{m-1}(\cdot, \omega), s) ds \right| T$$

$$\leq E \int_0^T K^2 (1 + |\xi_{m-1}(\cdot, \omega)|_T)^2 ds$$

which was also shown to be bounded in (A.9). Finally,

$$E \left| \xi_m^2(\cdot, \omega) \right| T \leq 4E \left| \xi_0^2(\cdot, \omega) \right| T + 80K^2 E(4 + 4 |\xi_{m-1}(\cdot, \omega)|_T^2) T$$

By induction Q.E.D.
Lemma 3. The sequence \( \{\xi_n(t, \omega)\}_{n \geq 0} \) defined by (A.7) is a Cauchy sequence in the norm \( E\left|\left|\xi_m(\cdot, \omega) - \xi_n(\cdot, \omega)\right|\right|_T^2. \)

**Proof.** Assume (induction hypothesis) that \( M > 0 \) with

\[
E\left|\left|\xi_m(\cdot, \omega) - \xi_{m-1}(\cdot, \omega)\right|\right|_T^2 \leq \frac{(Mt)^m}{m!}
\]

This is true for \( m = 1 \) since

\[
E\left|\left|\xi_1(\cdot, \omega) - \xi_0(\cdot, \omega)\right|\right|_T^2 = E\left|\left|\int_0^t \sigma(\xi_0(\cdot, \omega), s) \, dw(s, \omega) + \int_0^t b(\xi_0(\cdot, \omega), s) \, ds\right|\right|_T^2
\]

\[
\leq 4E\left|\left|\int_0^t \sigma(\xi_0(\cdot, \omega), s) \, dw(s, \omega)\right|\right|_T^2 + 4E\left|\left|\int_0^t b(\xi_0(\cdot, \omega), s) \, ds\right|\right|_T^2
\]

\[
\leq 16\int_0^t \sigma^2(\xi_0(\cdot, \omega), s) \, ds + 4E\int_0^t b^2(\xi_0(\cdot, \omega), s) \, ds
\]

\[
\leq 20K^2(2 + 2E\left|\left|\xi_0(\cdot, \omega)\right|\right|_T^2) T
\]

The inequality (A.10) for martingales was used again, as was Lemma 2, to establish that \( \int_0^t \sigma \, dw \) is a martingale. For \( m > 1 \)
\[ E \left| \xi_m(\cdot, \omega) - \xi_{m-1}(\cdot, \omega) \right|_T \leq 4E \int_0^t \left[ \sigma(\xi_{m-1}(\cdot, \omega), s) - \sigma(\xi_{m-2}(\cdot, \omega), s) \right] \, dw(s, \omega) \, ds + \]
\[ + 4E \int_0^t \left[ b(\xi_{m-1}(\cdot, \omega), s) - b(\xi_{m-2}(\cdot, \omega), s) \right] \, ds \]
\[ \leq 16E \int_0^t \left[ \sigma(\xi_{m-1}(\cdot, \omega), s) - \sigma(\xi_{m-2}(\cdot, \omega), s) \right]^2 \, ds + \]
\[ + 4E \int_0^t \left[ b(\xi_{m-1}(\cdot, \omega), s) - b(\xi_{m-2}(\cdot, \omega), s) \right]^2 \, ds \]
\[ \leq 20K^2 \int_0^t \left| \xi_{m-1}(\cdot, \omega) - \xi_{m-2}(\cdot, \omega) \right|^2 \, ds \]
\[ \leq 20K^2 \int_0^t \frac{(Ms)^{m-1}}{(m-1)!!} \, ds = 20K^2 M^{m-1} \frac{t^m}{m!} \]

Then if \( M > 20K^2(2 + 2E \left| \xi_0(\cdot, \omega) \right|_T^2) \) the desired result at stage \( m \) follows from the induction hypothesis. By induction, then, it is established that

\[ E \left| \xi_m(\cdot, \omega) - \xi_{m-1}(\cdot, \omega) \right|_T^2 \leq \frac{(Mt)^m}{m!} \]

It is clear by the triangle inequality that for any integer \( N \) and \( n, m > N \)

\[ E \left| \xi_n(\cdot, \omega) - \xi_m(\cdot, \omega) \right|_T^2 \leq \sum_{i=N}^{\infty} \frac{(Mt)^i}{i!} < \frac{(Mt)^N}{N!} e^{MT} \]
This can be made arbitrarily small by choosing N sufficiently large so Q.E.D.

Now the limit function from the sequence $\xi_n(t,\omega)_{n \geq 0}$ is the solution to the differential equation (A.6). To obtain this, first show

**Lemma 4.** There exists a sample continuous process $\xi(t,\omega)$ with $E|\xi(\cdot,\omega)|_T^2 < \infty$ and $E|\xi(\cdot,\omega) - \xi_n(\cdot,\omega)|_T^2 \to 0$ as $n \to \infty$.

**Proof.** Since $E|\xi_n(\cdot,\omega) - \xi_m(\cdot,\omega)|_T \to 0$ in $L^2(\Omega)$ as $n,m \to \infty$ then a subsequence $\xi_{nk}(t,\omega)$ which has $E|\xi_{nk}(t,\omega) - \xi_{nk}(t,\omega)|_T \to 0$ almost everywhere in $\Omega$ as $k,i \to \infty$. Then let $\xi(t,\omega) = \lim_{k \to \infty} \xi_{nk}(t,\omega)$ for those $\omega$ for which the limit exists and let $\xi(t,\omega) = 0$ elsewhere. If at $\omega_0 |\xi_{nk}(t,\omega_0) - \xi_{ni}(t,\omega_0)|_T \to 0$ as $k,i \to \infty$, then $\xi(t,\omega_0) = \lim_{k \to \infty} \xi_{nk}(t,\omega_0)$ is well defined for each $t \in [0,T]$ since at such $\omega_0$ the sequence of time functions is uniformly Cauchy in $t$. Since each $\xi_n(t,\omega)$ has a continuous version and since $\xi(t,\omega)$ is either the uniform limit (in $t$) of $\xi_{nk}(t,\omega)$ or is identically zero, then it follows that $\xi(t,\omega)$ has a continuous version. At those $\omega_0 \in \Omega$ for which $\xi_{nk}$ is Cauchy in sup norm

$$
|\xi(t,\omega_0)|_T^2 = |\xi(t,\omega_0) - \xi_n(t,\omega_0) + \xi_n(t,\omega_0)|_T^2
$$

$$
\leq 2|\xi(t,\omega_0) - \xi_{nk}(t,\omega_0)|_T^2 + 2|\xi_{nk}(t,\omega_0)|_T^2
$$

$$
\leq 2 \sup_{i \geq k} |\xi_{ni}(t,\omega_0) - \xi_{nk}(t,\omega_0)|_T^2 + 2|\xi_{nk}(t,\omega_0)|_T^2
$$

$$
+ 2|\xi_{nk}(t,\omega_0)|_T^2
$$
and since such $\omega_0$ have probability one, then

$$E \left| \xi(t, \omega_0) \right|^2_T \leq 2 \sup_{i > nk} E \left| \xi_i(t, \omega) - \xi_{nk}(t, \omega) \right|^2_T + 2E \left| \xi_{nk}(t, \omega) \right|^2_T$$

$$\leq 2 \sum_{i > nk} E \left| \xi_{i+1}(t, \omega) - \xi_i(t, \omega) \right|^2_T + 2E \left| \xi_{nk}(t, \omega) \right|^2_T$$

The right hand side has already been shown to be finite. Since $\xi_{nk}$ uniformly in $t$ almost everywhere in $\Omega$ and $\xi_n$ is Cauchy in $E \left| \cdot \right|^2_T$ then $E \left| \xi(t, \omega) - \xi_n(t, \omega) \right|^2_T \to 0$. Q.E.D.

These results may be combined to obtain

**Theorem 1.** Under conditions (A.6b), Equation (A.6a) has a solution $x(t, \omega)$ which has the following properties.

(i) $E \left| x(t, \omega) \right|^2_T < \infty$

(ii) $x(t, \omega)$ is sample continuous

(iii) $x(t, \omega)$ is unique in that if $y(t, \omega)$ also satisfies (A.6a) and condition (i), then $E \left| x(t, \omega) - y(t, \omega) \right|^2_T = 0$ which implies that for almost all $\omega$ $x(t, \omega) = y(t, \omega)$.

**Proof.** Let $x(t, \omega) = \xi(t, \omega)$ as defined above. Then
\[ E \left| x(\cdot, \omega) - x(0, \omega) - \int_0^T \sigma(x(\cdot, \omega), s) dw(s, \omega) - \int_0^T b(x(\cdot, \omega), s) ds \right|^2_T \]

\[ = \lim_{n \to \infty} E \left| \xi_n(\cdot, \omega) - \int_0^T (\cdot, \omega) - \int_0^T \sigma(\xi_n(\cdot, \omega), s) dw(\cdot, \omega) - \int_0^T b(\xi_n(\cdot, \omega), s) ds \right|^2_T \]

\[ = \lim_{n \to \infty} E \left| \xi_n(\cdot, \omega) - \xi_{n+1}(\cdot, \omega) \right|^2_T = 0 \]

so \( x(t, \omega) \) satisfies the differential equation for almost all \( \omega \in \Omega \).

Assume that \( y(t, \omega) \) also satisfies Eqn. (A.6a) and has \( E \left| y(\cdot, \omega) \right|^2_T < \infty \)

\[ E \left| x(\cdot, \omega) - y(\cdot, \omega) \right|^2_T \leq 4E \left| \int_0^T [\sigma(x(\cdot, \omega), s) - \sigma(y(\cdot, \omega), s)] dw(s, \omega) \right|^2_T + \]

\[ + 4E \left| \int_0^T [b(x(\cdot, \omega), s) - b(y(\cdot, \omega), s)] ds \right|^2_T \]

\[ \leq 20K^2 \int_0^T E \left| x(\cdot, \omega) - y(\cdot, \omega) \right|^2_s ds \]

since \( x(0, \omega) - y(0, \omega) = 0 \), then \( E \left| x(\cdot, \omega) \right|^2_T = 0 \) so Q. E. D.

Finally, the necessary result can be stated and proven.

Let \( \xi \) and \( \tilde{\xi} \) satisfy

\[ \xi(t, \omega) = \xi(0, \omega) + \int_0^T \sigma[\xi(\cdot, \omega), s] dw(s, \omega) + \int_0^T b[\xi(\cdot, \omega), s] ds \quad (A.13) \]

and
\[ \tilde{\xi}(t, \omega) = \xi(0, \omega) + \int_0^t \sigma(\tilde{\xi}(s, \omega), s) d\tilde{w}(s, \omega) + \int_0^t b(\tilde{\xi}(s, \omega), s) ds \quad (A.14) \]

where \( \omega \) and \( \tilde{\omega} \) are two Wiener processes.

**Theorem 2.** \( \xi \) and \( \tilde{\xi} \) (the solutions to Eqns. (A.14) and (A.14)) induce the same measures on the space of continuous functions.

**Proof:** Consider \( \phi[\xi(t_1), \ldots, \xi(t_k)] \). From the recursion of Eqn. (A.17) it is clear that the process \( \xi(t) \) may be approximated by functions \( F_m \) of the form

\[ F_m[t, \xi(0, \omega), w(u_{m, 1}), \ldots, w(u_{m, l_m})] \]

where \( 0 < u_{m, i} < t \) and \( F_m(t, x_0, x_1, \ldots, x_{l_m}) \) are Borel measurable functions depending only on \( b(x, t) \) and \( \sigma(x, s), b(x, s) \) for \( 0 \leq s \leq t \).

Then,

\[ \mathbb{E}[\phi[\xi(t_1), \ldots, \xi(t_k)]] = \]

\[ = \lim_{m \to \infty} \left[ \mathbb{E}[F_m[t_1, \xi(0, \omega), w(u_{m, 1}), \ldots, w(u_{m, l_m})], \ldots, \right. \]

\[ \left. F_m[t_k, \xi(0, \omega), \tilde{w}(u_{m, 1}), \ldots, \tilde{w}(u_{m, l_m})]} = \right) \]

\[ = \lim_{m \to \infty} \mathbb{E}\left[ F_m[t_1, \xi(0, \omega), w(u_{m, 1}), \ldots, w(u_{m, l_m})], \ldots, \right. \]

\[ \left. F_m[t_k, \xi(0, \omega), w(u_{m, 1}), \ldots, w(u_{m, l_m})] \right] = \mathbb{E}[\phi(\tilde{\xi}(t_1), \ldots, \tilde{\xi}(t_k)] \]

Q.E.D.
This last result shows that the solutions Eqn. (A.1) and (A.2) are unique in probability law. The results of Theorems (1) and (2) establish that, with the restrictions (A.6a), the conditions of Theorem (0) are met. Theorem (0) is Girsanov's theorem for this feedback estimation problem, and with it the proof of the representation theorem which is given in [Zakai (1)] may be used without modification.
APPENDIX B

ONE PARAMETER NATURE OF THE
NORMALIZED THRESHOLD CROSSING PROBABILITY FOR A
FIRST ORDER EXTENDED KALMAN FILTER

In Chapter 3 a Monte Carlo simulation is used to demonstrate how an extended Kalman filter and traditional early-late correlators fail as measurement noise and dynamics become bad enough. The evidence for this failure is the one sigma error that the extended Kalman filter exhibits after running for six time constraints under a variety of conditions.

It was asserted that only the noise-dynamics product (qr), as a single parameter, affected the performance in Chapter 3 of the extended Kalman filter considered. The one sigma error after si. time constants was asserted to depend on q and r only through the product qr and not in some more complicated way. This appendix proves that the threshold crossing time normalized by the loop time constant does indeed depend upon only the qr product. Figure B.1 below depicts the extended Kalman filter in question.

![Block diagram of extended Kalman filter](image)

Figure B.1 Block diagram of extended Kalman filter.
Suppose that the filter is in steady state. Then the error variance \( p \) is given by

\[
p = \sqrt{2qr}
\]

and the Kalman gain \( K \) by

\[
K = \frac{p}{2r} = \sqrt{\frac{q}{2r}}
\]

This means that the differential equation for the loop error "e" is

\[
de = -\sqrt{\frac{q}{2r}} f(e) dt - \sqrt{q} \ dn + \sqrt{q} \ dB
\]

where \( f(e) \) is the forward loop nonlinearity. The Fokker-Planck equation for this diffusion is

\[
\frac{\partial p(e,t)}{\partial t} = \sqrt{\frac{q}{2r}} \frac{\partial}{\partial e} (f(e)p(e,t)) + q \frac{\partial^2 p(e,t)}{\partial e^2}
\]

Suppose that \( m(e_0) \) is the mean first passage time for the error to grow past \( \pm 1 \) given that at time zero \( e(0) = e_0 \epsilon(-1,1) \). [Stratonovich (1)] shows that this function \( m(e_0) \) satisfies

\[
-1 = -\sqrt{\frac{q}{2r}} e_0 \frac{\partial m}{\partial e_0} + q \frac{\partial^2 m}{\partial e_0^2}
\]
with the boundary conditions

\[ m(-1) = m(1) = 0 \]  \hspace{1cm} (B-5)

Notice that considering first passage from the linear range has simplified the problem somewhat since in the linear range \( f(e) = e \). The equation for the mean first passage time can be converted to a first order matrix differential equation of the form

\[ \frac{d}{de_0} \begin{bmatrix} m \\ m' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \sqrt{\frac{1}{2qr} e_0} \end{bmatrix} \begin{bmatrix} m \\ m' \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{q} \end{bmatrix} \]  \hspace{1cm} (B-6)

The fundamental solution to this problem is a 2 x 2 matrix function satisfying

\[ \phi(e_0, e) = \begin{bmatrix} 0 & 1 \\ 0 & \sqrt{\frac{1}{2qr} e_0} \end{bmatrix} \phi(e_0, e) \quad \forall e, e_0 \]  \hspace{1cm} (B-7)

and \( \phi(e, e) = I \). This fundamental matrix is given by

\[ \phi(e_0, e) = \begin{bmatrix} 1 & \int_{e_0}^{e} \exp \left\{ \frac{1}{2qr} \frac{\sigma^2 - e_0^2}{2} \right\} d\sigma \\ 0 & \exp \left\{ \frac{1}{2qr} \frac{e_0^2 - e^2}{2} \right\} \end{bmatrix} \]  \hspace{1cm} (B-8)
The usual variation-of-parameters formula for the complete solution and the boundary condition $m(-1) = 0$ combine to yield

$$m(e_0) = \int_{-1}^{e_0} \exp \left\{ \sqrt{\frac{1}{2qr}} \frac{\sigma^2 - 1}{2} \right\} d\sigma m'(-1) -$$

$$- \frac{1}{q} \int_{-1}^{e_0} \int_{s}^{e_0} \exp \left\{ \sqrt{\frac{1}{2qr}} \frac{\sigma^2 - s^2}{2} \right\} d\sigma ds \quad (B-9)$$

Since $m(l) = 0$, $m'(-1)$ can be eliminated from this equation and the final expression for $m(e_0)$ is

$$m(e_0) = \frac{1}{q} \frac{\int_{-1}^{1} \int_{s}^{1} \exp \left\{ \sqrt{\frac{1}{2qr}} \frac{\sigma^2 - s^2}{2} \right\} d\sigma ds}{\int_{-1}^{1} \exp \left\{ \sqrt{\frac{1}{2qr}} \frac{\sigma^2 - 1}{2} \right\} d\sigma} \quad (B-10)$$

$$\cdot \int_{-1}^{e_0} \exp \left\{ \sqrt{\frac{1}{2qr}} \frac{\sigma^2 - 1}{2} \right\} d\sigma - \int_{-1}^{e_0} \int_{s}^{e_0} \exp \left\{ \sqrt{\frac{1}{2qr}} \frac{\sigma^2 - s^2}{2} \right\} d\sigma ds$$

The mean exit time does not depend solely on the product $qr$. But different loop bandwidths ($K = \sqrt{\frac{\sigma}{2r}}$) may be associated with the same $qr$ product. If the escape time is normalized by the loop bandwidth then the resulting random variable is the number of loop time constants to exit. Suppose that $t(\omega)$ is the exit time as a function on the underlying
probability space. Let $\tau(\omega) \overset{\Delta}{=} K t(\omega)$. Then since $K$ is one over the loop time constant, $\tau(\omega)$ is the normalized exit time. Since

$$E_t(\omega) = K E_t(\omega) \quad (B-11)$$

the normalized mean exit time may be computed using $E_t(\omega)$ which is just $\omega(e_0)$ given above. That is, substituting $K = \sqrt{\frac{a}{2r}}$

$$E(\tau(\omega)) = \sqrt{\frac{1}{2qr}} \left[ \int_{-1}^{1} \int_{s}^{1} \exp \left\{ \frac{1}{2qr} \frac{c^2-s^2}{2} \right\} ds \, d\sigma \right] \cdot \int_{-1}^{e_0} \exp \left\{ \frac{1}{2qr} \frac{c^2-1}{2} \right\} d\sigma - \int_{-1}^{e_0} \int_{s}^{e_0} \exp \left\{ \frac{1}{2qr} \frac{c^2-s^2}{2} \right\} ds \, d\sigma \right] \quad (B-12)$$

This expression clearly depends on the qr product alone.

Since the plot given in Chapter 3, comparing the predicted and actual loop errors, shows the errors at a fixed, normalized time (i.e., six loop time constants), then it has to be concluded that the apparent failure of the prediction for large predicted errors is due to the influence of the nonlinearity and the qr product, and is not an artifice of the ending times chosen for the various simulation runs.

This assertion can be strengthened by considering the exit time function $W(t, e_0)$. $W(t, e_0)$ is the probability that the state trajectory
does not reach the boundary by time \( t \) starting from \( e_0 \) at time \( t_0 = 0 \) (the starting time may be assumed to be zero with no loss of generality here). \( W(t, e_0) \) is defined for all \( t \geq 0 \) and \( e_0 \in (-1, 1) \) and \( W(0, e_0) = 1 \). The exit time distribution, \( P(t, e_0) \), is the probability that a trajectory exits in the interval \([0, t]\) and is related to \( W(\cdot, \cdot) \) by

\[
P(t, e_0) = W(0, e_0) - W(t, e_0) = 1 - W(t, e_0) \tag{B-13}
\]

This distribution and thus \( W(t, e_0) \) characterizes all the exit time statistics. [Stratonovich (1)] shows that \( W(t, e_0) \) solves

\[
\frac{\partial W}{\partial t} = -\sqrt{\frac{a}{2r}} f(e_0) \frac{\partial W}{\partial e_0} + a \frac{\partial^2 W}{\partial e_0^2} \tag{B-14}
\]

Suppose that the distribution \( \tilde{P}(\tau, e_0) \) of \( \tau = Kt \) is desired. Then

\[
\tilde{P}(\tau, e_0) = P\left(\frac{\tau}{K}, e_0\right) = 1 - W\left(\frac{\tau}{K}, e_0\right) \tag{B-15}
\]

Define \( \tilde{W}(\tau, e_0) \) by

\[
\tilde{W}(\tau, e_0) = W\left(\frac{\tau}{K}, e_0\right) \tag{B-16}
\]

then

\[
\tilde{P}(\tau, e_0) = 1 - \tilde{W}(\tau, e_0) \tag{B-17}
\]
By the chain rule

\[ \frac{\partial \tilde{W}(\tau, e_0)}{\partial \tau} = \frac{\partial W(u, e_0)}{\partial u} \frac{\partial \tilde{L}}{\partial \tau} \quad \mid u = \frac{\tau}{K} \]  

(B-18)

Substitution of Eqn. (B-18) and substitution of \( K = \sqrt{\frac{q}{2r}} \) yields

\[ \sqrt{\frac{q}{2r}} \frac{\partial \tilde{W}}{\partial \tau} = -\sqrt{\frac{q}{2r}} f(e_0) \frac{\partial \tilde{W}}{\partial e_0} + q \frac{\partial^2 \tilde{W}}{\partial e_0^2} \]  

(B-19)

Division of both sides of Eqn. (B-19) by \( \sqrt{\frac{q}{2r}} \) gives finally that

\[ \frac{\partial \tilde{W}}{\partial \tau} = f(e_0) \frac{\partial \tilde{W}}{\partial e_0} + \sqrt{2qr} \frac{\partial^2 \tilde{W}}{\partial e_0^2} \]  

(B-20)

This shows that \( \tilde{F}(\tau, e_0) \), which is the distribution of the number of time constants required for a trajectory to exit a given bounded region, is a function only of the \( qr \) product. Notice that the boundary conditions are not included in this derivation. This means that the same scaling works for any bounded region. This result may be summarized as follows.

For any constants \( q_1, r_1, q_2, r_2 \) for which

\[ q_1 r_1 = q_2 r_2 \]  

(B-21)
and for any \( e_1 \leq e_2 \leq e_3 \) let \( x_1(t,\omega) \) be a trajectory with the noise conditions \( q_1, x_1 \) and \( x_2(t,\omega) \) a trajectory with conditions \( q_2, x_2 \).

Suppose that

\[
x_1(0,\omega) = e_2 = x_2(0,\omega)
\]  

(B-22)

Define

\[
T_1(\omega) = \inf_{t>0} \left\{ t : x_2(t,\omega) \notin (e_1, e_3) \right\}
\]

(B-23)

and

\[
T_2(\omega) = \inf_{t>0} \left\{ t : x_2(t,\omega) \notin (e_1, e_3) \right\}
\]

(B-24)

Then \( \sqrt{\frac{q_1}{2x_1}} T_1(\omega) \) and \( \sqrt{\frac{q_2}{2x_2}} T_2(\omega) \)

have the same probability distributions.

This means that not only are the means of the normalized exit times the same, but all moments are the same. In addition, this is true for the exit from any region, not just \((-1,1)\), since the partial differential equations are the same.
APPENDIX C

CORRELATION MODEL FOR PSEUDO RANDOM
NUMBER CODE MULTIPLIER

The purpose of this section is to prove that the autocorrelation model of the output of the code multiplier is accurate and to derive the relation between the noise present on the signal before code multiplication and that present after code multiplication.

The final result on the multiplier model can be summarized as follows. Let $S(\cdot)$ be a pseudo random code function and let $S[\omega t + \hat{\theta}(t)]$ be a received code and $S[\omega t + \hat{\theta}(t)]$ be the code generated using the estimated delay $\hat{\theta}$. Here $\omega$ is the chipping rate in chips/s, $\theta(t)$ is the line-of-sight delay in chips and $\hat{\theta}(t)$ is an estimate of the line-of-sight delay. The autocorrelated model for the multiplier is accurate when

$$\frac{d}{dt} [\omega t + \hat{\theta}(t)] = \omega$$

and

$$\frac{d}{dt} [\omega t + \theta(t)] = \omega \quad (C.1)$$

or equivalently $\frac{\hat{\theta}(t)}{\omega}$ and $\frac{\theta(t)}{\omega}$ are slowly varying. These conditions
amount to a sample rate criterion on the processes $\theta(t)$ and $\hat{\theta}(t)$. These processes can be visualized as being sampled every $1/\omega$ seconds and this sampling rate must be faster than the changes in $\theta(t)$ and $\hat{\theta}(t)$. In the GPS case $\omega$ is $10^7$ chips/s. Then if the change in $\theta(t)$ and $\hat{\theta}(t)$ are small compared to $10^9$ ft/s, the autocorrelation model is valid. This result will now be established rigorously.

In what sense (if any) does

$$S[\omega t + \theta(t)]S[\omega t + \hat{\theta}(t)] \overset{?}{=} R_{SS}[\theta(t) - \hat{\theta}(t)] \quad (C.2)$$

It will be shown that as $\varepsilon \to 0$ and $\omega \to \infty$

$$E\left[\frac{1}{T} \int_0^T S[\omega t + \theta_{\varepsilon}(t)]S[\omega t + \hat{\theta}_{\varepsilon}(t)] - R_{SS}[\theta_{\varepsilon}(t) - \hat{\theta}_{\varepsilon}(t)] dt\right] \to 0 \quad (C.3)$$

where

$$d\theta_{\varepsilon} = \varepsilon a_1 \theta_{\varepsilon} dt + \varepsilon d\gamma_1$$

$$d\hat{\theta}_{\varepsilon} = \varepsilon a_2 \hat{\theta}_{\varepsilon} dt + \varepsilon d\gamma_2 \quad (C.4)$$

Notice that when $|\theta(t) - \hat{\theta}(t)| > 1$, the product on the LHS of Eqn. (C.2) can be either plus one or minus one, but not identically zero. For example, if $\theta_{\varepsilon} = 1$ and $\hat{\theta}_{\varepsilon} = 2$ then the indicated product is another
±1 random code taking values ±1 and is never equal to 0. Consider Figure C.1 wherein two codes and their products are drawn.

\[ S(\omega t + \theta_\varepsilon) \]
\[ S(\omega t + \hat{\theta}_\varepsilon) \]
\[ S(\omega t + \theta_\varepsilon) \cdot S(\omega t + \hat{\theta}_\varepsilon) \]

Figure C.1 Product of two PN codes.

The product \( S(\omega t + \theta_\varepsilon) \cdot S(\omega t + \hat{\theta}_\varepsilon) \) has value 1 whenever

\[ [\omega t + \theta_\varepsilon] - [\omega t + \hat{\theta}_\varepsilon] = 0 \] (C.5)

where \([\cdot]\) in Eqn. (C.5) is the greatest integer function. When the condition in Eqn. (C.5) is not satisfied then the product is either plus one or minus one in a more or less random fashion. Denote by \( Z[\theta(t), \hat{\theta}(t), t] \) the function

\[ Z[\theta_\varepsilon(t), \hat{\theta}_\varepsilon(t), t] = 1 \quad \text{If condition in Eqn. (C.5) is satisfied;} \]
\[ = 0 \quad \text{otherwise.} \] (C.6)

\( Z(\cdot, \cdot, \cdot) \) looks something like the product but can take the value zero.
For the code shown in Figure C.1, $Z(\cdot,\cdot,\cdot,\cdot)$ is plotted in Figure C.2 along with the product $S(\omega t + \theta_\varepsilon)S(\omega t + \hat{\theta}_\varepsilon)$

![Diagram](image)

Figure C.2 Product of two PN codes and $Z(\theta,\hat{\theta},t)$. 

Now it will be shown that

$$
\begin{align*}
E \left[ Z(\theta_\varepsilon,\hat{\theta}_\varepsilon,t) - R_{ss}(\theta_\varepsilon - \hat{\theta}_\varepsilon) \right]^2 dt & \to 0 \\
& \text{ as } \varepsilon \to 0 \text{ and } \omega \to \infty \\
\end{align*}
$$

(C.7)

Any $T$.

Figure C.3 shows the arguments of two code functions $S(\omega t + \theta_\varepsilon(t))$ and $S(\omega t + \hat{\theta}_\varepsilon(t))$ for $\theta_\varepsilon$, $\hat{\theta}_\varepsilon$ constants and shows how the function $Z$ behaves for this particular choice of $\theta_\varepsilon$ and $\hat{\theta}_\varepsilon$. Some elementary geometric manipulations indicate that $Z(\theta_\varepsilon,\hat{\theta}_\varepsilon,t)$ is equal to 1 for a length of time equal to $t_2 - t_1 = (1/\omega)(1 - |\theta_\varepsilon - \hat{\theta}_\varepsilon|)$ in this case. A similar construction shows that this expression holds true for other choices of $\theta_\varepsilon$ and $\hat{\theta}_\varepsilon$. It is also true that $Z(\theta_\varepsilon,\hat{\theta}_\varepsilon,t)$ is identically zero if
Figure C.3 Arguments of the code functions for some particular constant values of $\theta_\varepsilon$ and $\hat{\theta}_\varepsilon$, and resulting $Z(\theta_\varepsilon, \hat{\theta}_\varepsilon, t)$.

$|\theta - \hat{\theta}|$ is greater than one. Then if $\theta_\varepsilon, \hat{\theta}_\varepsilon$ are constant it has been demonstrated that

$$\int_0^{1/\omega} Z(\theta_\varepsilon, \hat{\theta}_\varepsilon, t) dt = \int_0^{1/\omega} R_{ss}(\theta_\varepsilon - \hat{\theta}_\varepsilon) dt \quad (C.8)$$

Suppose now that $\theta_\varepsilon$ and $\hat{\theta}_\varepsilon$ are not constants. There may then be some difference between the right and left sides of Eqn. (C.2).
Figure C.4 Arguments for time dependent delays.

For the phase functions shown in Figure C.4, and in general if $\theta_\varepsilon$ and $\hat{\theta}_\varepsilon$ are continuous,

$$\int_{0}^{1/\omega} Z(\theta_\varepsilon, \hat{\theta}_\varepsilon, t) dt = \frac{1}{\omega} R_{ss}[\theta_\varepsilon(t_2) - \hat{\theta}_\varepsilon(t_1)]$$

(C.9)

for some $t_2, t_1 \in [0, \frac{1}{\omega}]$ ($t_2$ not necessarily greater than $t_1$). The accuracy of the approximation is then given by

$$E \int_{0}^{1} \left( R_{ss}[\theta_\varepsilon(t_2) - \hat{\theta}_\varepsilon(t_1)] - R_{ss}[\theta_\varepsilon(t) - \hat{\theta}_\varepsilon(t)] \right)^2 dt \quad (C.10)$$

since $R_{ss}$ is Lipschitz with Lipschitz constant $1$. If $\omega$ is large then
\[
E \left\{ \int_0^T \left[ Z(\theta_\varepsilon, \hat{\theta}_\varepsilon, t) - R_{ss}(\theta_\varepsilon - \hat{\theta}_\varepsilon) \right]^2 dt \right\} = \\
= E \sum_{i=1}^{[\omega T]} \int_{\frac{i-1}{\omega}}^{\frac{i}{\omega}} \left[ R_{ss}(\theta_\varepsilon(t_{2,i}) - \theta_\varepsilon(t_{1,i}) - R_{ss}(\theta_\varepsilon(t) - (t)) \right]^2 dt \\
\text{for some choice of } t_{1,i}, t_{2,i} \in \left[ \frac{i-1}{\omega}, \frac{i}{\omega} \right]. \text{ Since } R_{ss} \text{ is Lipschitz}
\]
\[
\leq 4E \sum_{i=1}^{[\omega T]} \int_{\frac{i-1}{\omega}}^{\frac{i}{\omega}} \left\{ \left[ \theta_\varepsilon(t_2) - \hat{\theta}_\varepsilon(t) \right]^2 + \left[ \theta_\varepsilon(t_1) - \hat{\theta}_\varepsilon(t) \right]^2 \right\} dt \\
\leq 8E \sum_{i=1}^{[\omega T]} \frac{(i-1)}{\omega} \sup_{s,u} \frac{i}{\omega} \left[ \theta_\varepsilon(u) - \theta_\varepsilon(s) \right]^2 + \\
\frac{i-1}{\omega} \sup_{s,u} \frac{i}{\omega} \left[ \hat{\theta}_\varepsilon(u) - \hat{\theta}_\varepsilon(s) \right]^2 
\] (C.11)

This last expression may be bounded by referring to the differential equation for \( \theta_\varepsilon(t) \); for example,
\[ \theta_\varepsilon(t) - \theta_\varepsilon(0) = \varepsilon a_1 \int_0^t \theta_\varepsilon(s) \, ds + \varepsilon \int_0^t dY_s \quad \text{(C.12)} \]

then

\[ \sup_t \theta_\varepsilon(t) - \theta_\varepsilon(0)^2 \leq 16\varepsilon^2 a_1^2 \int_0^t \sup_s \theta_\varepsilon(s) - \theta_\varepsilon(0)^2 \, ds \]

\[ + 4\varepsilon^2 \sup_t \left( \int_0^t dY_s \right)^2 \]

\[ + 16\varepsilon^2 a_1^2 \int_0^t \theta_\varepsilon^2(0) \, ds \quad \text{(C.13)} \]

[Friedman (1) Thm. 3.6] gives that

\[ E \sup_t \left( \int_0^t dY_s \right)^2 \leq 4t \quad \text{(C.14)} \]

It follows that
\[
E \sup_t [\theta_\varepsilon (t) - \theta_\varepsilon (0)]^2 \leq 16\varepsilon^2 a_1^2 \int_0^t E \sup_s [\theta_\varepsilon (s) - \theta_\varepsilon (0)]^2 \, ds
\]

\[
+ 16\varepsilon^2 \int_0^t [1 + a_1^2 E\theta^2(0)] \, ds
\tag{C.15}
\]

or

\[
E \sup_t [\theta_\varepsilon (t) - \theta_\varepsilon (0)]^2 \leq \left[ \frac{16\varepsilon^2 a_1^2 t}{e} - 1 \right] \left[ \frac{1}{a_1^2} + E\theta^2(0) \right]
\]

\[
\leq 16\varepsilon^2 a_1^2 t \left[ \frac{1}{a_1} + E\theta^2(0) \right]
\tag{C.16}
\]

Notice that if $\varepsilon^{1/2}$ had premultiplied $\int dy$ in Eqn. (C.12) then the expression Eqn. (C.16) would be

\[
16\varepsilon^2 a_1^2 t \left[ \frac{1}{a_1^2} + E\theta^2(0) \right]
\tag{C.17}
\]
plugging this into Eqn. (C.11) gives

\[
E \left[ \int_0^T \left( Z(\theta_\varepsilon, \theta_\varepsilon t) - R_{ss}(\theta_\varepsilon - \theta_\varepsilon) \right)^2 dt \right]
\]

\[
\leq 128 \frac{T \varepsilon^2}{\omega} \left[ a + a^2 \left. \sup_{0 \leq s \leq 1} E \hat{Y}_{\varepsilon}^2(s) \right] \right. + a^2 \left. \sup_{0 \leq s \leq 1} E \hat{\varepsilon}^2(s) \right] \right]
\]

(C.18)

Now it has to be shown that as \( \varepsilon \to 0 \) and \( \omega \to \infty \)

\[
E \int_0^1 \left[ S(\omega t + \theta_\varepsilon(t)) S(\omega t + \hat{\theta}_\varepsilon(t)) - Z(\theta_\varepsilon(t), \hat{\theta}_\varepsilon(t), t) \right] dt \to 0 \quad (C.19)
\]

Figure C.5 shows what the integrand in Eqn. (C.19) might look like.

![Integrand from Eqn. (C.19)](image)
The sequence of \( t_i \)s which are the heights of the pulses shown in Figure C.5 form another binary code. This code is random if \( S(\cdot) \) is random, pseudo-random if \( S(\cdot) \) is pseudo-random. The basic idea then is that the average of the signal shown in Figure C.5 approaches zero as the averaging time gets long. If the code is either random or pseudo-random then for the constant \( \theta \) and \( \hat{\theta} \) case this averaging to zero is obtained, but if the delays \( \theta, \hat{\theta} \) are functions of time then the situation is complicated slightly. The problem is that if \( \theta \) and \( \hat{\theta} \) are functions of time then the widths of the pulses in Figure C.5 vary somewhat. The previous argument showed that they could not vary by much over a short time interval. Suppose that over a time interval \([0,t]\)

\[
t_1 \leq \sup_{s \leq t} [\theta(s) - \hat{\theta}(s)] - [\theta(0) - \hat{\theta}(0)] = \Delta t_1^2 \tag{C.20}
\]

Then the error to be introduced in the interval

\[
E \int_{t_1}^{t_2} \left[ S[\omega t + \theta_\epsilon(t)] S[\omega t + \hat{\theta}_\epsilon(t)] - Z[\theta_\epsilon(t), \hat{\theta}_\epsilon(t), t] \right] dt \leq
\]

\[
\leq E \left[ (t_2 - t_1) \Delta t_1^2 + \frac{1}{\omega} \sum_{t_1}^{t_2} \sum_{\omega} \hat{s}^2 (i) \right] \tag{C.21}
\]
The first term is arrived at by assuming that the delay motion over the
interval $s \in [0,t]$ is such that $[\theta(s) - \theta(0) - \hat{\theta}(s) + \hat{\theta}(0)]$ has magnitude
$t_2$ in each pulse and that its sign is positive for positive pulses and
negative for negative pulses. In the interval $[t_1,t_2]$, there are $\omega [t_2-t_1]$
pulses, and this phase difference makes the integral over one pulse
different by at most $\Delta \Delta^2 / \omega$. So the maximum error introduced by time
variations in $\theta(t)$ and $\hat{\theta}(t)$ is $(t_2-t_1) \Delta^2$. The second term upper
bounds the integral of these pulses assuming that the phases are con-
stant. $\tilde{S}(\cdot)$ is a new code obtained from multiplying a code by a version
of itself which is shifted by an integral number of chips. Then $\tilde{S}(\cdot)$ is
a code which has many of the properties of the code $S(\cdot)$.

Recognize that

$$\left[ \sup_{0 \leq s \leq t} |[\theta(s) - \hat{\theta}(s)] - [\theta(0) - \hat{\theta}(0)]|^2 \right]^{1/2}$$

$$= \sup_{0 \leq s \leq t} |[\theta(s) - \hat{\theta}(s)] - [\theta(0) - \hat{\theta}(0)]|$$

(C.22)

Furthermore, if $f \leq 0$ then

$$E(f)^{1/2} \leq [Ef]^{1/2}$$

(C.23)
Then

\[ E \Delta_{t_1}^2 \leq \left( E \left( \Delta_{t_1}^2 \right) \right)^{1/2} \]  

(C.24)

From the earlier derivation it then follows that

\[ E \Delta_{t_1}^2 \leq 4E(t_2 - t_1)^{1/2} \left[ \alpha_1^2 + \alpha_2^2 E \delta^2(t_1) + \alpha_2^2 E \delta^2(t_1) \right]^{1/2} \]  

(C.25)

It is shown in [Gold (2)] that for sufficiently large \( \omega \),

\[ E \left( \frac{1}{\omega} \sum_{t_1, \omega} \tilde{S}(i) \right) \leq K_2 \left( \frac{t_2 - t_1}{\omega} \right)^{1/2} \]  

(C.26)

If \( \tilde{S}(i) \) were from a coin toss sequence then this is an obvious conclusion. [Gold (2)] proves that the same is true for PRN codes. A bound on the total error may be obtained by adding the errors on any collection of subintervals which exhausts the interval \([0,1]\). Suppose, for example, that \([0,1]\) is partitioned into \( N \) equal-length subintervals. Then
\[ \mathbb{E} \int_0^1 S[\omega t + \hat{\theta}_\varepsilon(t)]S[\omega t + \hat{\theta}_\varepsilon(t)] - Z[\theta_\varepsilon(t), \hat{\theta}_\varepsilon(t), t] \, dt \]

\[ \leq N \varepsilon K_1 \left( \frac{1}{N} \right)^{1/2} + N \frac{K_2}{\omega^{1/2}} \left( \frac{1}{N} \right)^{1/2} \]  \hspace{1cm} \text{(C.27)}

\[ = 4\varepsilon K_1 + \frac{K_2}{\omega^{1/2}} N^{1/2} \]  \hspace{1cm} \text{(C.28)}

It follows that the error approaches zero as \( \varepsilon \to 0 \) and \( \omega \to \infty \). This means that the autocorrelation model for the detector output is valid in this limiting sense. The type of convergence is the so-called weak convergence in \( L^1 \). That is, the detector output approaches the output of the autocorrelation model weakly in \( L^1 \) where the measure space is \( \Omega \times [0,T] \), the product of the probability space and the time interval \([0,T]\). The \( L^1 \) functions are those \( f(\cdot) \) with \( \mathbb{E} \int_0^T |f| \, dt < \infty \).

The nature of the noise \( n(t) \) will now be determined. The received signal and noise is assumed to be of the form

\[ r(t) = S[\omega t + \hat{\theta}(t)] \sin[\omega_c t + \hat{\theta}_c(t)] + n(t) \]  \hspace{1cm} \text{(C.29)}

where \( n(t) \) is white and Gaussian. Two multiplications are performed on this signal. The first is a multiplication by \( 2 \sin [\omega_c t + \hat{\theta}_c(t)] \).
to remove the carrier. In [Van Trees (2)] it is shown that after this multiplication and a wide band low pass filter to remove the double frequency terms, the signal can be assumed to be

\[
\tilde{r}(t) = S[\omega t + \theta(t)] + \tilde{n}(t) \tag{C.30}
\]

where \(\tilde{n}(t)\) can be considered white (in the carrier frequency bandwidth) and

\[
E[\tilde{n}(t)\tilde{n}(\tau)] = 2E[n(t)n(\tau)] = N \tag{C.31}
\]

The second multiplication is of \(\tilde{r}(t)\) times a code \(S(\omega t + \hat{\theta})\). The nature of the product \(S(\omega t + \hat{\theta})S(\omega t + \hat{\theta})\) was shown in the first part of this Appendix. The nature of the product \(S(\omega t + \hat{\theta})\tilde{n}(t)\) will be shown here. Suppose that

\[
\int_0^t \tilde{n}(s)ds = \sqrt{N} \beta_t \tag{C.32}
\]

where \(\beta_t\) is a standard Brownian motion. It will be shown that

\[
\int_0^t S[\omega s + \hat{\theta}(s)]n(s)ds \overset{\Delta}{=} \sqrt{N} \int_0^t S[\omega s + \hat{\theta}(s)]d\beta_s = \sqrt{N} \tilde{\beta}_t \tag{C.33}
\]
where \( \tilde{\beta}_t \) is another standard Brownian motion. To that end, suppose that \( \hat{S}(t, \gamma) \) is a stochastic process which is non-anticipative with respect to an increasing family of \( \mathcal{G} \)-algebras \( F_t \). This means that \( F_{t_2} \subseteq F_{t_1} \) for \( t_2 > t_1 \), and

(i) \( \hat{S}(t, \gamma) \) is a separable process

(ii) \( \hat{S} \) is a measurable process

(iii) for each \( t \), \( \hat{S}(t, \cdot) \) is \( F_t \)-measurable (this definition is that used in [Friedman (1)]). Suppose also that \( F(\beta_{t, \lambda}; \gamma^{-3}_t; \lambda > 0) \) is independent of \( F_t \). Now \( \hat{S} \) is going to be the feedback code process so suppose \( \hat{S}(t, \gamma) = \pm 1 \). Then

\[
E \left( \int_0^t \hat{S}(s, \gamma) d\beta_s \right)^2 = \int_0^t E \hat{S}^2(s, \gamma) \, ds = t \quad \text{(C.34)}
\]

\[
E \int_0^t \hat{S}(s, \gamma) \, d\beta_s \int_t^\infty \hat{S}(s, \gamma) \, d\beta_s = 0 \quad \text{(C.35)}
\]

since \( \int_0^t \hat{S}(s, \gamma) \, d\beta_s \) is a square integrable martingale then the Levy theorem [Liptser and Shiryaev (1)] applies and the noise \( \tilde{n} \) is then the formal derivation of a Brownian motion.

Several correlations can be formed at once by multiplying the input by each of several shifted versions of the feedback signal. Having several parallel correlations available will turn out to be useful later. It is important to know how the noises on these correlations are
related. Suppose that \( \hat{S}_i(s, \gamma) \) and \( \hat{S}_j(s, \gamma) \) are distinct feedback codes separated by an integral number of chips. Then the corresponding noises will be independent if

\[
E \int_0^t \hat{S}_i(s, \gamma) d\beta s \int_0^t \hat{S}_j(s, \gamma) d\beta s = 0 \tag{C.36}
\]

But

\[
E \int_0^t \hat{S}_i(s, \gamma) d\beta s \int_0^t \hat{S}_j(s, \gamma) d\beta s = E \int_0^t \hat{S}_i(s, \gamma) s_j(s, \gamma) ds \tag{C.37}
\]

so independence is shown if

\[
\int_0^t \hat{S}_i(s, \gamma) \hat{S}_j(s, \gamma) ds = 0 \text{ a.s.} \tag{C.38}
\]

Since \( \hat{S}_i(s, \gamma) \hat{S}_j(s, \gamma) \) is, for fixed \( \gamma \), just another code [Gold (2), Golomb (1)] say \( \hat{S}_K(s, \gamma) \), the same argument as used in showing that the autocorrelation model of the multiplier was valid shows that as \( \varepsilon \to 0 \) and \( \omega \to \infty \) the last expression approaches zero a.s.
The following things have been learned about the PRN code tracking problem:

(a) After multiplication by a feedback code the PRN tracking problem can be viewed as a correlation tracking problem.

(b) If the additive noise on the input is white Gaussian, the noise on the multiplier output is also white Gaussian.

(c) If several multiplications are performed, each with a different shift of the feedback code, then the corresponding noises are independent.
APPENDIX D

REGULAR PERTURBATION EXPANSION FOR PROBABILITY DENSITY UNDER SMALL RANDOM FREQUENCY FLUCTUATIONS

In this Appendix a regular perturbation approximation for a weakly second order system is derived. An exponential approximation is given for this problem in Section 4.6. Taking a power series solution of the exponential approximation is shown to lead to the regular approximation given here. The regular approximation is interesting because of the eigen function expansion technique required to obtain it and because its form is used to obtain the exponential approximation given in Chapter 4. At several points in the derivation of the exponential expansion solution forms, which are shown to be correct, appear without intuitive justification. In reality it is the form of the regular expansion which provides guidance.

The system under consideration is

\[
\begin{bmatrix}
\dot{\theta}_e \\
\dot{\omega}_e
\end{bmatrix} = \begin{bmatrix}
0 & \varepsilon \\
-a & 0
\end{bmatrix} \begin{bmatrix}
\theta_e \\
\omega_e
\end{bmatrix} + \begin{bmatrix}
d\delta t \\
d\alpha t
\end{bmatrix} - \begin{bmatrix}
K_1 f(\theta_e) \\
\varepsilon K_2 f(\theta_e)
\end{bmatrix} dt - \begin{bmatrix}
K_1 K \varepsilon f \\
\varepsilon K_2 K \varepsilon f
\end{bmatrix} d\gamma_t \tag{D.1}
\]
The steady state Fokker Planck equation for the probability density $p(\theta_e, \omega_e)$ is [Jazwinski (1)]

$$
0 = -\frac{\partial}{\partial \theta_e} [(-K_1 f + \varepsilon \omega_e) P] - \frac{\partial}{\partial \omega_e} [(-\omega_e - \varepsilon K_2 f) P] + \frac{1}{2} \left( q_1 + K_2 K_{2r} \right) \frac{\partial^2 P}{\partial \theta_e^2}
$$

$$
+ \varepsilon K_2 K_{2r} \frac{\partial^2 P}{\partial \theta_e \partial \omega_e} + \frac{1}{2} \left( q_2 + \varepsilon^2 K_2^2 K_{2r} \right) \frac{\partial^2 P}{\partial \omega_e^2}
$$

(D.2)

For a regular perturbation expansion a solution of the form

$$
p(\theta_e, \omega_e) = p_0 + \varepsilon p_1
$$

(C.3)

is assumed. Substituting this into the Fokker-Planck Equation (D.2) and separating according to powers of $\varepsilon$ yields
\[
\varepsilon^0: - \frac{\partial}{\partial \theta_1} (\mathbf{K}_f \mathbf{p}_0) - \frac{\partial}{\partial \omega} (-\mathbf{a}_e \mathbf{p}_0) + \frac{1}{2} (q_1 + \mathbf{K}_f^2 \mathbf{r}) \frac{\partial^2 \mathbf{p}_0}{\partial \theta_1^2} + \\
+ \frac{1}{2} q_2 \frac{\partial^2 \mathbf{p}_0}{\partial \omega^2} = 0 \quad (D.4)
\]

\[
\varepsilon^1: - \frac{\partial}{\partial \theta_1} (\mathbf{K}_f \mathbf{p}_1) - \frac{\partial}{\partial \omega} (-\mathbf{a}_e \mathbf{p}_1) + \frac{1}{2} (q_1 + \mathbf{K}_f^2 \mathbf{r}) \frac{\partial^2 \mathbf{p}_1}{\partial \theta_1^2} + \\
+ \frac{1}{2} q_2 \frac{\partial^2 \mathbf{p}_1}{\partial \omega^2} - \frac{\partial}{\partial \omega} (\mathbf{a}_e \mathbf{p}_0) - \frac{\partial}{\partial \omega} (-\mathbf{K}_2^2 \mathbf{p}_0) + \mathbf{K}_f \mathbf{K}_2^2 \mathbf{r} \frac{\partial^2 \mathbf{p}_0}{\partial \theta_1 \partial \omega} = 0 \quad (D.5)
\]

Define operators \(L_1(\cdot), L_2(\cdot)\) by

\[
L_1(p) = - \frac{\partial}{\partial \theta_1} (\mathbf{K}_f p) + \frac{1}{2} (q_1 + \mathbf{K}_f^2 \mathbf{r}) \frac{\partial^2 p}{\partial \theta_1^2} \quad (D.6)
\]

\[
L_2(p) = \frac{\partial}{\partial \omega} (-\mathbf{a}_e p) + \frac{1}{2} q_2 \frac{\partial^2 p}{\partial \omega^2} \quad (D.7)
\]

Now Eqn. (D.4) and Eqn. (D.5) can be rewritten as

\[
\varepsilon^0: L_1(p_0) + L_2(p_0) = 0 \quad (D.8)
\]
\[ \varepsilon^1: \quad L_1(p_1) + L_2(p_1) - \frac{\partial}{\partial \theta_e} (w_e p_0) - \frac{\partial}{\partial \omega_e} (-K_2 fp_0) + K_1 K_2 K_r^2 \frac{\partial^2 p_0}{\partial \theta_e \partial \omega_e} = 0 \quad (D.9) \]

The \( \varepsilon^0 \) equation, Eqn. (D.8), is solved by a product solution

\[ p_0(\theta_e, \omega_e) = \Theta(\theta_e) \Omega(\omega_e) \quad (D.10) \]

and

\[ L_1(\Theta(\theta_e)) = 0 \quad (D.11) \]

\[ L_2(\Omega(\omega_e)) = 0 \quad (D.12) \]

\( \Theta \) and \( \Omega \) are the probability densities corresponding to the uncoupled differential equations. That is:

\[ \Theta(\theta_e) = C_1 \exp \left\{ -\frac{2K_1}{q_1 + K_1 K_r^2} \int_0^e f(u) du \right\} \quad (D.13) \]

and

\[ \Omega(\omega_e) = \frac{1}{\sqrt{\frac{q_2}{a}}} \exp \left\{ -\frac{\omega_e^2}{q_2/a} \right\} \quad (D.14) \]
\[ 1/C_1 = \int_{-\infty}^{\infty} \Theta(\theta_e) d\theta_e \quad (D.15) \]

To zeroth order then the probability of \( \theta_e \) events and \( \omega_e \) events are independent.

To obtain an \( \epsilon^1 \) correction suppose that there exist functions \( \Omega^n(\omega_e) \) and constants \( \lambda^n_\omega \) satisfying

\[ L_2(\Omega^n(\omega_e)) + \lambda^n_\omega \Omega^n(\omega_e) = 0 \quad (D.16) \]

The eigenfunctions and eigenvalues for this differential equation can be found from [Hildebrand (1)] and [Rektorys (1)]

\[ \frac{d^2}{dx^n} \left( \frac{d^n}{dx^n} e^{-x^2} \right) + 2 \frac{d}{dx} x \left( \frac{d^n}{dx^n} e^{-x^2} \right) + 2n \left( \frac{d^n}{dx^n} e^{-x^2} \right) = 0 \quad (D.17) \]

Let \( \frac{\omega_e}{\sqrt{q_2/a}} = x \), then

\[ \frac{d^j}{dx^j} = \left( \frac{\sqrt{q_2}}{a} \right)^j \frac{d^j}{d\omega^j} \quad (D.18) \]
Substituting this into Eqn. (D.17) gives
\[
\frac{q_2}{a} \frac{d}{d\omega_e} \left( \frac{d^n}{d\omega_e^n} \exp \left\{ -\frac{\omega^2 a}{q_2} \right\} \right) + 2 \frac{d}{d\omega_e} \omega e \left( \frac{d^n}{d\omega_e^n} \exp \left\{ -\frac{\omega^2 a}{q_2} \right\} \right) +
\]
\[+ 2n \left( \frac{d^n}{d\omega_e^n} \exp \left\{ -\frac{\omega^2 a}{q_2} \right\} \right) = 0 \quad (D.19)
\]
Rearranging Eqn. (D.19) gives
\[
\frac{q_2}{2} \frac{d}{d\omega_e^2} \left( \frac{d^n}{d\omega_e^n} \exp \left\{ -\frac{\omega^2 a}{q_2} \right\} \right) + a \frac{d}{d\omega_e} \omega e \left( \frac{d^n}{d\omega_e^n} \exp \left\{ -\frac{\omega^2 a}{q_2} \right\} \right) +
\]
\[+ na \left( \frac{d^n}{d\omega_e^n} \exp \left\{ -\frac{\omega^2 a}{q_2} \right\} \right) = 0 \quad (D.20)
\]
Comparison of Eqn. (D.20) with Eqn. (D.16) and Eqn. (D.17) gives that
\[
\Omega^n(\omega_e) = \frac{d^n}{d\omega_e^n} \exp \left\{ -\frac{\omega^2 a}{q_2} \right\} \quad (D.21)
\]
and that
\[
\lambda^n_\omega = na \quad (D.22)
\]
The $\varepsilon^1$ equation, Eqn. (D.9), can be rewritten as

$$L_1(p_1) + L_2(p_1) + \frac{\partial \Omega}{\partial \omega_e} \left( \frac{q_2}{2a} \frac{\partial \Theta}{\partial \theta_e} + K_2 f\Theta + K_1 K_2 K_f^2 r \frac{\partial \Theta}{\partial \theta_e} \right) = 0$$  \hspace{1cm} (D.23)

Assume that the $\varepsilon^1$ equation, Eqn. (D.23), has a solution in the form

$$p_1(\theta_e, \omega_e) = \Theta_1(\theta) \Omega^1(\omega_e)$$  \hspace{1cm} (D.24)

Substitute this into Eqn. (D.23) to give

$$\Omega^1 L_1(\Theta_1) + \Theta_1 L_2(\Omega^1) +$$

$$+ \frac{\partial \Omega}{\partial \omega_e} \left( \frac{q_2}{2a} \frac{\partial \Theta}{\partial \theta_e} + K_2 f\Theta + K_1 K_2 K_f^2 r \frac{\partial \Theta}{\partial \theta_e} \right) = 0$$  \hspace{1cm} (D.25)

Now $\Omega^1$ satisfies $L_2(\Omega^1) = -\lambda_\omega \Omega^1$

and

$$\Omega^1 = \frac{\partial \Omega}{\partial \omega_e}$$  \hspace{1cm} (D.26)

so Eqn. (D.25) can be rewritten as

$$\Omega^1 \left[ L_1(\Theta_1) - a \Theta_1 + \left( \frac{q a}{2a} + K_1 K_2 K_f^2 r \frac{\partial \Theta}{\partial \theta_e} + K_2 f\Theta \right) \frac{\partial \Theta}{\partial \theta_e} \right] = 0$$  \hspace{1cm} (D.27)

Dividing out $\Omega^1$ gives an ordinary differential equation for $\Theta_1$. 
\[ L_1(\Theta_1) - a\Theta_1 = - \left( \frac{q_2}{2a} + K_1 K_2 K_f r \right) \frac{\partial \Theta}{\partial e} - K_2 f \Theta \]  
\[ (D.28) \]

Using Eqn. (D.6) to write out Eqn. (D.28) gives

\[ \frac{1}{2} \left( q_1 + K_1 K_2 K_f r \right) \frac{\partial^2 \Theta_1}{\partial e^2} + K_1 \frac{\partial}{\partial e} f \Theta_1 - a \Theta_1 = \]

\[ = - \left( \frac{q_2}{2a} + K_1 K_2 K_f r \right) \frac{\partial \Theta}{\partial e} - K_2 f \Theta \]  
\[ (D.29) \]

It was demonstrated earlier that

\[ \Theta = 0 \left( 1/\Theta_e^{-4} \right) \text{ as } |\Theta_e| \to \infty \]  
\[ (D.30) \]

and that an optimal detector shape for a Brownian input is

\[ f(\Theta_e) \sim \frac{1}{\Theta_e} \text{ as } |\Theta_e| \to \infty \]  
\[ (D.31) \]

Then Eqn. (D.29) has a solution which is

\[ \Theta_1 = 0(1/\Theta_e^{-5}) \text{ as } |\Theta_e| \to \infty \]  
\[ (D.32) \]

This guarantees that \( \Theta_1 \) is integrable and since \( \Omega^1 \) is obviously integrable it follows that

\[ p_1 = \Theta_1 \Omega^1 \]  
\[ (D.33) \]

is also integrable.
The regular perturbation approximation just derived and the exponential approximation are intimately related. To show this, denote the regular approximation by $p'$

$$p'(\theta_e, \omega_e) = \exp \left\{ - \frac{2K_1}{q_1 + K_1^2 K_2 r} \int_0^{\theta_e} f(x) dx \right\} \exp \left\{ - \frac{a}{q_2} \omega_e \right\} +$$

$$+ \Theta_1 \left( - \frac{2a}{q_2} \omega_e \right) \exp \left\{ - \frac{a}{q_2} \omega_e \right\}$$

(D.34)

where $\Theta_1$ satisfies Eqn. (D.29). Eqn. (D.29) can be made more explicit by substituting

$$\Theta = \exp \left\{ - \frac{2K_1}{q_1 + K_1^2 K_2 r} \int_0^{\theta_e} f(x) dx \right\}$$

(D.35)

and

$$\frac{\partial \Theta}{\partial \theta_e} = - \frac{2K_1}{q_1 + K_1^2 K_2 r} f(\theta_e) \Theta$$

(D.36)

The exponential approximations in Chapter 4 led to an approximate solution, denoted here by $p^2$, of the form

$$p^2(\theta_e, \omega_e) = \exp \left\{ - \frac{2K_1}{q_1 + K_1^2 K_2 r} \int_0^{\theta_e} f(x) dx - \frac{a}{q_2} \omega_e + \varepsilon h(\theta_e) \omega_e \right\}$$

(D.37)
Expanding $\rho^2(\theta_e, \omega_e)$ gives

$$
p^2(\theta_e, \omega_e) = \exp \left\{ - \frac{2K_1}{q_1 + K_1^2 K_f^2} \int_0^{\theta_e} f(x) \, dx - \frac{a}{q_2} \omega_e^2 \right\} + \\
+ \text{sh}(\theta_e) \exp \left\{ - \frac{2K_1}{q_1 + K_1^2 K_f^2} \int_0^{\theta_e} f(x) \, dx \right\} \omega_e \exp \left\{ - \frac{a}{q_2} \omega_e^2 \right\} + o(\varepsilon^2)
$$

(D.38)

If these expansions are both correct, then it should be that

$$
\text{h}(\theta_e) \exp \left\{ - \frac{2K_1}{q_1 + K_1^2 K_f^2} \int_0^{\theta_e} f(x) \, dx \right\} = - \frac{2a}{q_2} \Theta_1
$$

(D.39)

If this equation is solved for $\Theta_1$ and the result is substituted into Eqn. (D.29), then Eqn. (4.98) results. This shows that the above equality is correct and illustrates the relation between these two perturbation solutions.
APPENDIX E

TIME INVARIANT MODEL OF A TAU-DITHER

This Appendix proves rigorously that the statistics of a rapidly dithered loop match those of the time invariant loop which was used in Chapter 7 to analyze the behavior of a rapidly dithered loop. The basic definitions are restated here. The noise input in Figure E.1 is a white Gaussian noise.

![Diagram](image)

**Figure E.1** A baseband version of the $\tau$-dither loop.

Consider a sequence of tracking loops of the form shown in Figure E.1, the difference between them being that the $n^{th}$ loop uses a dither function $d_n(t)$. Consider also a function $h(i) > 0$ defined on the integers which has

$$\sum_{i=-\infty}^{\infty} h(i) = 1 \quad (E.1)$$
Let $T_n > 0$ be the fundamental period of each function $d_n(t)$, and suppose that $T_n \to 0$ as $n \to \infty$. Specify the function $d_n(t)$ only insofar as that for all integers $n > 0$ and $j \geq 0$

$$\lambda \left\{ t : d_n(t) = i \right\} \cap [jT_n, (j+1)T_n \right\} = h(i) \cdot T_n \quad (E.2)$$

where $\lambda$ is Lebesque measure. This says that inside each interval of the form $[jT_n, (j+1)T_n]$ the dither function $d_n(t)$ spends $100 \cdot h(i)$ percent of the time at the integer $i$. Requirement (2) is the only one that needs to be placed on the dither function. It does not have to be periodic. It will be shown that (2) is sufficient to obtain a limiting loop output as $T_n \to 0$. The function $h(i)$ is called the percent-time function and is the primary determinant of the loop performance.

Define a function $\tilde{h}(e)$ by

$$\tilde{h}(i) = \text{sgn}(i)h(i) \quad \text{for } i \text{ an integer} \quad (E.3)$$

and

$$\tilde{h}(e) = \tilde{h}([e]) \left[ 1 - ([e] - e) \right] + ([e] - e)\tilde{h}([e] + 1) \quad (E.4)$$

for $e$ not an integer. The symbol $[e]$ means the greatest integer in $e$.

The graph of $h(e)$ is constructed by graphing $\text{sgn}(i)h(i)$ and connecting the dots. $\tilde{h}(e)$ is the nonlinearity for which I am looking. I have still to specify the input phase process and the measurement noise process.

For this, assume that the input phase is a Brownian motion $\beta_t$ and that
the measurement noise \( h(t) \) is the formal derivative of a Brownian motion \( \gamma_t \). It is sufficient for present purposes to take \( G(s) = K/s \). Then the
differential equation for the \( n^{\text{th}} \) \( \tau \)-dither loop error is

\[
d e_n = -K \{ f(e_n(t) + d_n(t)) dt + d\gamma_t \} \text{sgn}(t) + d\beta_t
\]  

(E.5)

The autonomous equivalent of the loop in Figure E.1 is block-diagrammed in Figure E.2.

![Diagram](image)

**Figure E.2** An equivalent time-invariant loop.

For the input signal, dynamics, and loop filter used to obtain Eqn. (E.5) the error of the loop in Figure E.2 satisfies

\[
d e = -K[\tilde{n}(e) dt + d\gamma_t] + d\beta_t
\]  

(E.6)

Now the equivalence of the limiting loop and that described in Eqn. (E.6) is proven.

**Theorem:** Let \( P_n \) be the probability measure induced on the space of continuous functions by \( e_n \), the solution to Eqn. (E.5). Let \( P \) be the probability measure induced on the same space by the solution to Eqn. (E.6). Then \( P_n \to P \) as \( n \to \infty \) in the sense the \( P_n(A) \to P(A) \) for any measurable set \( A \). (The sigma algebra is the usual one generated by finite cylinders.)
Proof: If the initial condition for both Eqns. (E.5) and (E.6) is $e_0$, then in integrated form those equations are

$$e_n(t) = e_0 - K \int_0^t f(e_n + d_n) \text{sgn}(d_n) \, dt - K \int_0^t \text{sgn}(d_n) \, d\gamma_t + \beta_t$$  \hspace{1cm} (E.7)

and

$$e(t) = e_0 - K \int_0^t h(e) \, dt - K \gamma_t + \beta_t$$  \hspace{1cm} (E.8)

Since $\int_0^t \text{sgn}(d_n) \, d\gamma_t$ is another Brownian motion, then by [Friedman (1), Theorem 5.3.5] the solution to Eqn. (E.7) induces the same probability measure on the continuous functions as

$$\tilde{e}_n(t) = e_0 - K \int_0^t f(\tilde{e}_n + d_n) \text{sgn}(d_n) \, dt - K \gamma_t + \beta_t$$  \hspace{1cm} (E.9)

The solution to Eqn. (E.9) will be shown to approach the solution to Eqn. (E.8) in the $L_2$-space of nonanticipative square integrable processes. Define functions $\tilde{e}_{n,k}(t)$ and $e_k(t)$ recursively by

$$\tilde{e}_{n,0}(t) = e_0(t) = e_0$$  \hspace{1cm} (E.10)

$$\tilde{e}_{n,k} = e_0 - K \int_0^t f(\tilde{e}_{n,k-1} + d_n) \text{sgn}(d_n) \, dt - K \gamma_t + \beta_t$$  \hspace{1cm} (E.11)

and

$$e_k(t) = e_0 - K \int_0^t h(e_{k-1}) \, dt - K \gamma_t + \beta_t$$  \hspace{1cm} (E.12)
These functions are the successive approximants to the solutions of Eqns. (E.8) and (E.9). Consider in particular

\[ e_{n,1} = e_0 - k \int_0^t f(e_0 + d_n) \text{sgn}(d_n) \, d\tau - K \gamma + \beta_t \]  
(E.13)

and

\[ e_1 = e_0 - k \int_0^t \tilde{h}(e_0) \, d\tau - K \gamma + \beta_t \]  
(E.14)

Now

\[ e_{n,1} - e_1 = k \int_0^t [\tilde{h}(e_0) - f(\tilde{e}_0 - \tilde{d}_n)] \text{sgn}(d_n) \, d\tau \]  
(E.15)

where \( \tilde{h} \) is defined in Eqn. (E.4) and \( f(e) \) is the triangular code autocorrelation. If \( t = MT_n \) for some integer \( M \), then \( e_{n,1} - e_1 = 0 \), since \( e_0 \) is a constant (in the \( t \) direction). In any case

\[ |e_{n,1} - e_1| \leq K [\sup \tilde{h} + \sup f] t \mod T_n \leq 2KT_n \]  
(E.16)

Now consider

\[ e_{n,2} - e_2 = k \int_0^t [\tilde{h}(e_1) - f(d_n + \tilde{e}_n, l)] \text{sgn}(d_n) \, d\tau \]

\[ = k \int_0^t [(\tilde{h}(e_1) - f(d_n + \tilde{e}_n, l)] \text{sgn}(d_n) + [\tilde{h}(e_1) - \tilde{h}(\tilde{e}_n, l)] \, d\tau \]  
(E.17)
Since \( \tilde{h} \) is uniformly Lipschitz with constant 1, then

\[
|\tilde{e}_{n,2} - e_2| \leq 2K T_n (1 + Kt) \tag{E.18}
\]

Continuing in this manner we obtain

\[
|\tilde{e}_{n,j} - e_j| \leq 2K T_n \sum_{i=0}^{j-1} \frac{(Kt)^i}{i!} \tag{E.19}
\]

\[
\leq 2K T_n e^{Kt} \tag{E.20}
\]

This is true uniformly in \( j \) and uniformly in \( \omega \), the probability dimension.

Now suppose that (induction hypothesis)

\[
E \left[ \sup_{0 \leq s \leq t} |\tilde{e}_{n,j}(s) - \tilde{e}_{n,j-1}(s)|^2 \right] \leq \frac{(Kt)^j}{j!} \tag{E.21}
\]

for some positive constant \( M \). Then since

\[
\tilde{e}_{n,j+1} - \tilde{e}_{n,j} = -K \int_0^t [f(\tilde{e}_{n,j+1}^+ - \tilde{e}_{n,j+1}^-) - f(\tilde{e}_{n,j-1+1}^+ - \tilde{e}_{n,j-1+1}^-)] \text{sgn}(\tilde{d}_n) d\tau \tag{E.22}
\]

\[
|\tilde{e}_{n,j+1} - \tilde{e}_{n,j}| \leq K^2 \int_0^t |\tilde{e}_{n,j} - \tilde{e}_{n,j-1}|^2 d\tau \tag{E.23}
\]

it follows that
\[
E \left\{ \sup_{0 \leq s \leq t} \left| \tilde{e}_{n,j+1}(s) - \tilde{e}_{n,j}(s) \right|^2 \right\} \leq K^2 \int_0^t \frac{(Ms)^j}{j!} \, ds \quad (E.24)
\]

\[
= \frac{K^2 M^j t^{j+1}}{(j+1)!} \quad (E.25)
\]

And if \( M \geq K^2 \) then

\[
E \sup_{0 \leq s \leq t} \left| \tilde{e}_{n,j+1}(s) - \tilde{e}_{n,j}(s) \right|^2 \leq \frac{(Mt)^{j+1}}{(j+1)!} \quad (E.26)
\]

To establish this for all \( j \) then it only needs to be established for \( j = 0 \). That is,

\[
\left| \tilde{e}_{n,1} - \tilde{e}_{n,0} \right|^2 \leq 3K^2 \int_0^t \left| f(\tilde{e}_{n,0} + \tilde{a}_n) \text{sgn}(\tilde{d}_n) \right|^2 \, dt + 3K^2 \gamma_t^2 + 3\beta_t^2 \quad (E.27)
\]

By [Friedman (1), Theorem 4.3.6]

\[
E \sup_{0 \leq s \leq t} \left| \tilde{e}_{n,1}(s) - \tilde{e}_{n,0}(s) \right|^2 \leq (6K^2 + 3)t \quad (E.26)
\]

So letting \( M = \max K^2, 6K^2 + 3 \) will give the desired result.

Now the functions \( \tilde{e}_{ni} \) are just the successive approximants to the solution of Eqn. (E.9), \( \tilde{e}_n \). \( \tilde{e}_n \) can be defined by

\[
e_n = \lim_{i \to \infty} e_{ni}
\]
in the metric

$$E \left( \sup_{0 \leq s \leq T} | \tilde{e}_{n,j} |^2 \right) \leq \frac{(MT)^{j-1}}{(j-1)!} e^{MT}$$  \hspace{1cm} (E.29)

Doing this gives

$$E \left( \sup_{0 \leq s \leq T} | \tilde{e}_{n,j} - \tilde{e}_{n,j} |^2 \right) \leq \frac{(MT)^{j-1}}{(j-1)!} e^{MT}$$  \hspace{1cm} (E.30)

A parallel argument reveals $e_j \to e$ in the same sense. Now since

$$\tilde{e}_{n,j} - e = \tilde{e}_{n,j} + \tilde{e}_{n,j} - e_j \to e$$  \hspace{1cm} (E.31)

$$E \left( \sup_{0 \leq s \leq T} | \tilde{e}_{n,j} - e_{n,j} |^2 \right) \leq 3E \limsup_{N \to \infty} \sup_{j \geq N} \sup_{0 \leq t \leq T} | e_{n,j} - e_{n,j} |^2 + 3E \sup_{0 \leq t \leq T} 2KT e^{Kt}$$

$$+ 3E \limsup_{N \to \infty} \sup_{j \geq N} \sup_{0 \leq t \leq T} | e_{n,j} - e |^2 \leq 6KT e^{KT}$$  \hspace{1cm} (E.32)

To establish the theorem it is sufficient to establish [Dudley (1), Theorem 8.3] that

$$\beta(P_n, P) \to 0$$  \hspace{1cm} (E.33)

where $\beta$ is a metric on the space of probability measures on the continuous functions defined by
\[ \beta(P, Q) = \sup \left\{ |qd(P, Q)| : \|g\|_{BL} \leq 1 \right\} \quad (E.34) \]

Here \( P \) and \( Q \) are measures on \( C[0, T] \) and \( g \) is a bounded Lipschitz function. The norm \( \|g\|_{BL} \) is defined by

\[
\|g\|_L \overset{\Delta}{=} \sup_{x \neq y} \frac{|g(x) - g(y)|}{d(x, y)} \quad \|g\|_\infty \overset{\Delta}{=} \sup_x |g(x)|
\]

and

\[
\|g\|_{BL} \overset{\Delta}{=} \|g\|_L + \|g\|_\infty \quad (E.35)
\]

These are all for \( x, y \in C[0, T] \) and with \( d(\cdot, \cdot): C[0, T] \times C[0, T] \rightarrow \mathbb{R}^+ \) defined by

\[
d(x, y) = \sup_{0 \leq s \leq T} |x(s) - y(s)|^2
\]

Now

\[ \beta(P_n, P) = \sup \left\{ |E[g(\tilde{e}_n) - g(e)]| : \|g\|_{BL} \leq 1 \right\} \quad (E.36) \]

but

\[ |E[g(\tilde{e}_n) - g(e)]| \leq E|g(\tilde{e}_n) - g(e)| \quad (E.37) \]

If \( \|g\|_{BL} \leq 1 \) then

\[ |g(e_n) - g(e)| \leq \sup_{0 \leq s \leq T} |\tilde{e}_n - e|^2 \quad (E.38) \]
so

\[ E|g(\tilde{e}_n) - g(e)| \leq E \sup_{0 \leq s \leq T} |e_n - e|^2 \]  

(E.39)

By Equation (E.32)

\[ E|g(\tilde{e}_n) - g(e)| \leq 6KT_n e^{KT} \]  

(E.40)

so

\[ 3(P_n, P) \leq 6KT_n e^{KT} \]  

(E.41)

and

\[ 3(P_n, P) \to 0 \quad \text{as } N \to \infty, \text{ Q.E.D.} \]  

(E.42)
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BIOGRAPHY

William Michael Bowles was born in Stillwater, Oklahoma on April 2, 1949. He was raised in Bartlesville, Oklahoma. In 1967 he graduated College High School in Bartlesville.

He attended Oklahoma State University in Stillwater and received a BSME in 1972. He worked for a time as a field engineer for Brown and Root Construction Company in the Java Sea and then returned to Oklahoma State for graduate work. He worked as a research assistant on fire control and target tracking problems and received his MSME in 1974.

In September 1974 he went to Massachusetts Institute of Technology for further study. He worked as a Draper Fellow and staff engineer at The Charles Stark Draper Laboratory, Inc.

In 1978 he was married to Megan Ancker. She is presently a medical student at Tufts University in Boston.