RELATIVE NAVIGATION BY MEANS OF

PASSIVE RANGINGS

by

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ABSTRACT

The subject of this work is a community of vehicles trying to to navigate relatively to one another by means of mutual passive rangings.

The solution that has been adopted so far, is analyzed. Sufficient conditions for exponential stability are found, proved and discussed. The discussion is complemented with a few simulations.

An alternative concept of relative navigation by means of passive rangings is then proposed. It is shown that this concept decouples the filters of the navigating community from one another (thereby relaxing the stability conditions) and allows every member of the community to use all the rangings that are taken.

Finally, a method that allows to reconstruct the centralized optimal estimate in a decentralized way is proposed. This method can be applied to the problem of relative navigation aided by passive rangings, yielding another alternative concept, which is discussed and compared with the others.

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TABLE OF CONTENTS

Page

CHAPTER 1

INTRODUCTION

1.1 The concept of relative navigation by means of passive rangings.

A passive ranging is a measurement of the distance between the user and some other object, which the user obtains without sending any signal. Such a measurement can be obtained if the object spontaneously emits a signal, and if the user knows the time at which the signal was emitted. If this is the case, all the user has to do to obtain the distance is record the signal's time of arrival, subtract the time of broadcast, and multiply by the signal propagation speed.

A community of vehicles can perform relative navigation by means of passive rangings, if it has a Time Division Multiple Access (TDMA) communication facility and some computational capability. The operation of a TDMA network requires all members of the network to have a clock, and the clocks to be at least roughly synchronized. Let it be agreed that every member send a special message, each in its turn, at prearranged times. Then all other members can use the time of arrival of this message to find their distance from the source of the message, according to the simple algorithm described in the previous paragraph. The value found for the distance, however, will be corrupted by a systematic error due to the phase difference between the source's and the receiver's clocks. This systematic error can be modeled. The raw value of distance (henceforth called simply the "ranging") can be used to estimate both the distance between source and receiver and the phase difference between their clocks. The latter will be used to ensure a better synchronization. The former will be used to establish the position of the members in a relative coordinate grid.

What is meant here by a relative coordinate grid, is a coordinate grid that is insensitive to rigid translations and rotations of the whole community. Such movements leave all mutual distances unchanged, and cannot, therefore, be observed by means of mutual rangings alone. Similarly, one must define community time, that is, time measured with respect to an origin that is insensitive to an equal phase shift of all clocks of the community. The problem of defining such a relative position and time coordinate system will be touched on in Section 2.6.

1.2 Integration of rangings with other kinds of measurement.

Rangings can also be used in conjunction with measurements of a different kind, in particular with measurements of absolute (i.e., either Earth-fixed or inertial) coordinates. In this case the definition of "relative" coordinates may be changed appropriately, since rigid motions of the community become observable. Suppose all members of the community have dead reckoners, and rangings are used to estimate the dead reckoners' errors (a typical application). Then the relative coordinate grid may be one that is insensitive to a rigid translation or rotation of all the dead reckoner errors of the community. Such a grid will be slowly moving, due to the dead reckoners' drift.

The use of a relative coordinate system (along with the absolute one) is advisable even in the case of mixed measurements. In absolute coordinates, the effect of rangings is partly to correlate estimation errors, rather than reduce their value. Besides, there are cases where the knowledge of relative coordinates is valuable in itself, as the next section will describe.

1.3 Usefulness of relative navigation

The concept of navigation outlined in Section 1.1, alone or integrated with other facilities, is practically useful in any situations. Namely, whenever the knowledge of a member's relative position in the community is either more valuable or more precise than the knowledge of its absolute position.

This formula includes a variety of both civil and military aeronautical scenarios; in particular

(i) crowded terminal areas or flight routes, where the danger of collision is high;

(ii) missions requiring the cooperation of several aircraft, or of aircraft and ships (rendezvous, air refueling, formation flying, etc.);

(iii) whenever high-precision absolute positioning systems (such as GPS) are available to some aircraft, but not to many others; in this case relative positioning is used to provide good absolute positioning; relative coordinates do not have to be introduced.

Item (i) is mainly civil; item (ii) mainly military. Item (iii) can be both; the airplanes, to which GPS precision is passed on, may be either small civil craft that cannot afford GPS, or military units in a jammed zone.

There may even exist a few situations where relative positioning is considered only a by-product, and the desirable effect is the good synchronization of the clocks, which would otherwise require more expensive apparatus.

1.4 Past and present applications

The inherent ability of a TDMA network to perform both relative navigation and synchronization by means of passive rangings was recognized early; however, only navigation was attempted at first, with synchronization ensured by other means. Passive synchronization was tried only later.

The earliest application of passive rangings for position finding was probably a collision avoidance system, developed by McDonnel Douglas in 1965 for the flight testing of high performance aircraft. In this system, time of arrival was used only to infer range and range rate. Clock synchronization was achieved by means of an interrogaterespond procedure (usually called round-trip timing). The estimates of range and range rate were used to detect collision danger situations.

In the early seventies the Navy sponsored Singer, the Kearfott Division, to develop an experimental Integrated Tactical Navigation System for which see Stow (1) and Danik (2). ITNS used both rangings and inertial dead reckoners. Its main objective was to establish the feasibility of navigation in purely relative coordinates, with no surveyed reference point. One or more master members established the relative grid. All members reported their position in relative coordinates; range was calculated from times of arrival and compared with its inertially predicted value; inertial corrections were then applied. Synchronization, however, was still achieved by means of round-trip timings. ITNS flight tests gave good results. Early trials were marred by an oversight: the relative grid's origin had been defined but its azimuth had not; this resulted, predictably, in rotational instability. The problem was detected and corrected.

At about the same time, the Mitre Corporation, sponsored by the Air Force, was trying for the first time the concept described in Section 1.1: fully passive synchronization, as well as navigation. The results of the of the Position Location Reporting And Control of Tactical Aircraft program are summarized by Westbrook and Snodgrass (3). PLRACTA included simulations and flight tests, with two aircraft and a few ground-fixed surveyed stations as members of the community. The inclusion of these surveyed stations eliminated the problem of defining a relative coordinate grid. Each member estimated its own relevant variables (including position and clock phase) and exchanged position messages with the others. A disappointing result, predicted by simulations and confirmed by flight tests, was that unrestricted ranging between members produced instability. Further simulation work led to recommending the use of a source selection rule, by which rangings could be accepted or refused. The rule was based on the comparison of source's and receiver's computed error covariances (a case of what will be called "covariance-based hierarchy" later in this work). Also recommended was the introduction of infrequent round-trip timings for synchronization. After these changes, the simulation results were judged satisfactory.

The experience gained with experimental programs PLRACTA and ITNS is being used now in the JTIDS-RelNav program. JTIDS, short for Joint Tactical Information Distribution System, is a spread-spectrum TDMA communication facility, considered jamproof and meant to provide a data link between military units in tactical situations. An introduction to the

JTIDS concept and a description of a terminal's hardware are given by Dell'Imagine (4). A few years ago it was decided JTIDS should include a Rel(ative) Nav(igation) capability, and JTIDS-RelNav terminals are being built at the present time.

The JTIDS-RelNav concept, as described for instance by Fried (5), relies heavily on PLRACTA and ITNS experience. Like ITNS, it integrates rangings with inertial data and uses a relative coordinate system (as well as an absolute one). Like PLRACTA, clock errors are modeled too, and synchronization can be achieved passively, in principle. However, as recommended by PLRACTA reports, also round-trip timings are used by a few members whenever the computed covariance of clock error crosses a threshold. Members are divided into "primary users", which perform round-trip timings, and "secondary users", which usually do not. Also in view of PLRACTA findings, a covariance-based hierarchy is enforced within each class of members. Each member estimates its own relevant variables (the errors of its inertial navigator and of its clock) in absolute coordinates, and the origin and azimuth of the relative grid. Relative position is found by difference. An attempt to keep into account the errors of other members is made: the ranging innovations are weighed with the reported position and time accuracy of the ranging's source.

1.5 Review of the state of the art

A state of the art in the subject of relative navigation by means of passive rangings has been reached, as a result of PLRACTA, ITNS, and several papers that appeared in the last decade. Its main features are reviewed in this section.

It is recognized that a relative spatial grid and relative time can be defined consistently. Early problems were satisfactorily solved ana several alternative ways of defining the relative coordinates are now known; see Section 2.6 for a survey of them.

It is also recognized that passive rangings give the members of a TDMA network the ability of both finding their relative positions and synchronizing their clocks on a community time. Some uncertainty exists about the way to use the data as to achieve a stable estimation process. The present trend is to evade the problem, by resorting to active synchronization (round-trip timings) whenever instability threatens.

It has always been implicitly assumed that each member of the community should model its own relevant variables, not those of the other members. Another implicit assumption is that every member should use the rangings it takes, without sharing their value to the community at large. A centralized filter, modeling the relevant variables of all members, and using all the measurements is considered impractical, because of the large size of the state space. Intermediate solutions have not been proposed. The accepted way for a member's filter to keep into account the effect of other members' errors consists on weighing the innovations with the reported accuracy of the source of the ranging.

It was found that source selection logic (that is, a logic deciding which rangings are to be used and which are to be rejected) plays an important role in community performance. Several alternatives were described and evaluated by Rome (6), Rome and Stambaugh (7), Greenberg and Rome (8). There is a consensus that unrestricted ranging between any two members leads to instability. Beside the PLRACTA results, simulations of this effect are reported in Refs. (7) and (8). On the other hand, if a covariance-based hierarchy is enforced, opinions are mixed. References (3) and (5) report stability, while Ref. (8) reports instability in some cases.

All the previous results were reached in an empirical way, using simulations. There is pessimism about the possibility of an analytical investigation. Reference (8) pronounces "pencil and paper analysis" and "optimization in the mathematical sense" to be "virtually impossible" (p. 336).

An exception to this pessimism has appeared lately. Kerr (9) recognized that our problem is a particular case of decentralized estimation, and examined the extant body of literature on that subject to find a solution that could fit our case. A few theoretical contributions on decentralized estimation that the present writer thinks noteworthy, will be briefly mentioned in the next section.

1.6 Our problem as a case of decentralized estimation. Review of some known results

Decentralized estimation differs from its centralized counterpart in the fact that the total body of information (the incoming measurements and the memory of past measurements and estimates) is not generally available. When an estimate is to be taken, only a certain subset of the total information can be used towards its computation.

Our problem is a case of decentralized estimation for two different reasons:

(i) rangings are supposed to be known only to the member of the community that takes them;

(ii) members are supposed to model only a part of the state space;

therefore, they do not store all the nonredundant information contained in the rangings.

The first difficulty can be removed if the members agree to share their rangings to the whole community; the penalty is an increase in the size of messages. The second difficulty could be removed, with a filter modeling all state variables; this is not considered feasible, at the present state of the art in electronics.

A complete and satisfactory theory of decentralized estimation does not exist. What solutions are known are either suboptimal or concerning very particular cases.

Kerr (9) reviewed some of these solutions and found two of them to be applicable to our problem. They are the Surely Locally Unbiased filter proposed by Sanders et al (10) and the Sequentially Partitioned Algorithm, proposed by Shah (11). The application of both methods to our problem (for which see Ref. (9)) consists in the state-of-the art filter of Section 1.5, with attribution of a weight to the accuracy of the sources of rangings. All these methods add is a way of calculating these weights.

Speyer (12) and Willsky et al. (13) propose decentralized methods that reconstruct the centralized optimum estimate.

Speyer's method requires all members to have a fullsize filter, modeling all the relevant sate variables. They must also have an auxiliary vector, of the same size as the complete state vector; it is data-dependent and must be updated on line. Each member knows only a part of the measurements, but it can reconstruct the optimal centralized estimate by linearly combining the estimates and the auxiliary vectors of all members (which must be shared to the community).

Willsky's method, which is an elaboration of Speyer's, allows every member to have an incomplete or aggregate state vector, provided certain conditions are satisfied; in particular, a member's state vector must not drop any state variable that is used to complete the expected value of that member's measurements. There is only one auxiliary vector, which has the same size as the complete state vector, is not data-dependent, and must be updated by a central processor provided with knowledge of all members' covariance matrices. The central processor must also know all the local estimates, and obtains the centralized optimal estimate by linearly combining them and the auxiliary vector.

Neither Speyer's nor Willsky's method is applicable to our case. Speyer's does not reduce the amount of computations (increases it, in effect, since every member must update a full-sized state vector and matrix and an equally large auxiliary vector and matrix). Willsky's requires a central processor with large computational capability. Both methods require a large amount of information to be shared by the local estimators to the community: all local estimates, all local covariance matrices (which, in our problem, depend on the community's geometry and must be computed on line), and, in Speyer's case, auxiliary vectors as well.

A more detailed description of Speyer's and Willsky's methods as they could by applied to our problem will be provided in Chapter 7. These methods have provided some inspiration for the method that will be described in Chapter 7.

1.7 Scope and plan of this work

The present work has two aims. The first is to analyze and discuss the state-of-the art solution that was described in Section 1.5 and is being applied to JTIDS-RelNav. The second is to propose alternative solutions. The state-of-the art solution consists in having each member model only its own relevant state variables. There is uncertitude as to whether a community so arranged can perform a stable estimation process without requiring the help of occasional (or perhaps frequent) active clock synchronication; also, as to the best way of organizing the community in matter of source selection. Such communities will be called "ownstate" communities in the remainder of this work. They will be analytically described in Chapter 2. In Chapter 3 it will be proved that one class of them is, indeed, stable under certain observability conditions, and without the help of round-trip timings. In Chapter 5 simulations will be presented, using models derived in Chapter 4.

An alternative solution will be presented in Chapters 6. Chapter 7 will present a method by the use of which the community can cooperatively reconstruct the optimal centralized estimates, without some of the inconveniences of Speyer's and Willsky's methods. Chapter 8 contains the conclusions.

CHAPTER 2

OWNSTATE FILTER COMMUNITIES

2.1 Introduction

The problem of navigation by means of rangings involves a large number of relevant state variables neatly divided into uncoupled subsets. The physical attributes of one given member of the network (its position, velocity, the phase of its clock, etc.) evolve independently from similar variables of all other members. On the other hand, every observation, i.e., ranging, strongly couples variables belonging to two different members, the source and the receiver of the signal.

Under these conditions, an optimal estimator would have to include the relevant variables of at least all active members of the network in its state vector and could use all rangings taken by all members. This is impractical at the present (1981) state-of-the-art electronics. The relevant variables of each member are at least fifteen; even a small community of four active stations would need a 60-state filter, with a 60 x 60 error covariance matrix to update at every ranging (every 2 or 3 sec. at most, typically oftener than that). Therefore a suboptimal solution is necessary.

The simplest idea that may come to one's mind is to have each member estimate only its own states and incorporate only its own rangings. This is what this writer chose to call an <u>ownstate</u> filter community (the name is not standard). The remainder of this chapter will be used to set up a mathematical model for such a process, and describe a few alternative ways of organizing it. A motivation for doing so is that almost only ownstate filters have been proposed or considered so far for applications; the system that is being implemented now for JTIDS-RelNav is also, substantially, an ownstate filter community.

2.2 A mathematical model of passive rangings

Consider a three-dimensional Cartesian coordinate grid x,y,z; let t be the time, in units so chosen that the speed of light is 1. Let

$$\underline{x}^i \triangleq [x^i, y^i, z^i, t]' \tag{2.1}$$

be a vector containing the coordinates of member i and the time. Let

$$\hat{\underline{x}}^i \triangleq [\hat{x}^i, \hat{y}^i, \hat{z}^i, \hat{\tau}^i]'$$

be the best estimate of $\underline{x}^i$ available to member i; τ^i is the phase of member i's clock. Let

$$\tilde{\underline{x}}^i \triangleq \underline{x}^i - \hat{\underline{x}}^i = [\tilde{x}^i, \tilde{y}^i, \tilde{z}^i, \tilde{\tau}^i]'$$

be the estimation error (this will always be the sign convention).

Say an active member, labelled j and called "source," is supposed to send a p(osition)-message at a scheduled time t_{sch}. Member j will send it when it believes the scheduled time to have come; the estimated time of broadcast will be according to schedule:

$$\hat{\tau}^j_{ob} = t_{sch}$$

but the actual t.o.b. will be instead:

$$t^j_{ob} = t_{sch} + \tilde{\tau}^j$$

Consider now another member, labelled i and called "receiver;" let d^{ij} be the distance between receiver and source. The p-message will reach member i at the actual time of arrival:

$$t^i_{oa} = t^j_{ob} + d^{ij} = t_{sch} + d^{ij} + \tilde{\tau}^j$$

but, at that time, the phase of the receiver's clock will be:

$$\hat{\tau}_{oa} = t^i_{oa} - \tilde{\tau}^i = t_{sch} + d^{ij} + \tilde{\tau}^j - \tilde{\tau}^i$$

One must also consider that the arrival of the p-message will be recorded by member i with a certain stochastic error v^i; the recorded t.o.a. will be:

$$t_{sch} + d^{ij} + \tilde{\tau}^j - \tilde{\tau}^i + v^i$$

The ranging ρ^{ij} is found by subtracting the scheduled t.o.b.:

$$\rho^{ij} = d^{ij} + \tilde{\tau}^j - \tilde{\tau}^i + v^i \qquad (2.2)$$

One sees that a passive ranging contains not only a stochastic error v^i, but also a systematic error due to the phase difference between the clocks.

The expected value of the ranging is:

$$\hat{\rho}^{ij} = \hat{d}^{ij} = (\hat{x}^i-\hat{x}^j)^2 + (\hat{y}^i-\hat{y}^j)^2 + (\hat{z}^i-\hat{z}^j)^2 \; ^{1/2} \qquad (2.3)$$

The innovation, defined as:

$$\delta\rho^{ij} = \rho^{ij} - \hat{\rho}^{ij}$$

must be a linear function of errors and noise, in order to fit into a

Linear Quadratic Gaussian estimation process. The common practice (extended Kalman filter) is to use a first order approximation:

$$\delta\rho^{ij} = \frac{\hat{x}^i-\hat{x}^j}{\hat{d}^{ij}}(\tilde{x}^i-\tilde{x}^j) + \frac{\hat{y}^i-\hat{y}^j}{\hat{d}^{ij}}(\tilde{y}^i-\tilde{y}^j) + \frac{\hat{z}^i-\hat{z}^j}{\hat{d}^{ij}}(\tilde{z}^i-\tilde{z}^j) - (\tilde{\tau}^i-\tilde{\tau}^j) \tag{2.4}$$

Defining the geometry vector:

$$\underline{h}^{ij} \triangleq \left(\frac{\hat{x}^i-\hat{x}^j}{\hat{d}^{ij}}, \frac{\hat{y}^i-\hat{y}^j}{\hat{d}^{ij}}, \frac{\hat{z}^i-\hat{z}^j}{\hat{d}^{ij}}, -1 \right)' \tag{2.5}$$

one can rewrite Eq. (2.4) simply as:

$$\delta\rho^{ij} = \underline{h}^{ij'}(\underline{\tilde{x}}^i - \underline{\tilde{x}}^j) + v^i \tag{2.6}$$

which shows that the errors of receiver and source have an antisymmetrical effect on the innovation.

It is useful to notice that $\underline{h}^{ij}$ and $\underline{h}^{ji}$ are linearly independent from each other; two opposite passive rangings, taken at the same time, are two independent measurements. If a processor knows them both, it can separate the distance and the clock phase difference:

$$(\rho^{ij} + \rho^{ji})/2 = d^{ij} + (v^i + v^j)/2 \tag{2.7}$$

$$(\rho^{ij} - \rho^{ji})/2 = \tilde{\tau}^j - \tilde{\tau}^i + (v^i-v^j)/2$$

So, two opposite passive rangings are equivalent to one active (or round-trip) ranging and one round-trip timing.

Another remark that will be useful later is that $|\underline{h}^{ij}| = \sqrt{2}$. This result is due to the spatial components of $\underline{h}$ being the direction cosines of the line of sight, while the time component always equals -1.

Lemma. Three coplanar (but not coincident) lines of sight give rise to three independent rangings. However, adding other coplanar lines of sight does not increase the number of independent rangings.

Proof. Say $\underline{e}_1$, $\underline{e}_2$ are two line-of-sight cosine vectors, and $\underline{e}_3 = \alpha\underline{e}_1 + \beta\underline{e}_2$ is a third one on the same plane. Obviously:
$|\underline{e}_1|^2 = 1$, $|\underline{e}_2|^2 = 1$ and:

$$1 = |\underline{e}_3|^2 = \alpha^2 + \beta^2 + 2\,\alpha\beta\,\underline{e}_1'\,\underline{e}_2 \quad ;$$

$\underline{e}_3$ not coinciding with $\underline{e}_2$ implies $\alpha \neq 0$; $\underline{e}_3$ not coinciding with $\underline{e}_1$ implies $\beta \neq 0$; $\underline{e}_1$ not coinciding with $\underline{e}_2$ implies $\underline{e}_1'\underline{e}_2 - 1 \neq 0$.
Together they imply:

$$1 - (\alpha+\beta)^2 = 2\alpha\beta(\underline{e}_1'\underline{e}_2 - 1) \neq 0$$

that is: $\alpha+\beta \neq \pm 1$.

The three geometry vectors, arranged in a matrix, are:

$$\begin{matrix} \underline{e}_1, & -1 \\ \underline{e}_2, & -1 \\ \alpha\underline{e}_1+\beta\underline{e}_2, & -1 \end{matrix} \qquad \text{equivalent to:} \qquad \begin{matrix} \underline{e}_1, & -1 \\ \underline{e}_2, & -1 \\ \underline{0}\,, & (\alpha+\beta-1) \end{matrix}$$

and the matrix has rank 3, since $\alpha+\beta\neq 1$.

Now add another coplanar line-of-sight vector:

$$\underline{e}_4 = \gamma\underline{e}_1 + \delta\underline{e}_2$$

The four geometry vectors, arranged in a matrix, are:

$$\begin{bmatrix} \underline{e}_1, & -1 \\ \underline{e}_2, & -1 \\ \alpha\underline{e}_1+\beta\underline{e}_2, & -1 \\ \gamma\underline{e}_1+\delta\underline{e}_2, & -1 \end{bmatrix} \quad \text{equivalent to:} \quad \begin{bmatrix} \underline{e}_1, & -1 \\ \underline{e}_2, & -1 \\ \underline{0}, & (\alpha+\beta-1) \\ \underline{0}, & (\gamma+\delta-1) \end{bmatrix}$$

Obviously, the rank is still 3, Q.E.D.

2.3 Generalities about ownstate filters

Now that passive rangings have been described, ownstate filters can be defined more precisely. One can see from Eq. (2.6) what the trouble with them is. The ranging ρ^{ij}, taken by member i, should have an innovation depending on member i's errors. Instead, $\delta\rho^{ij}$ also depends on $\underline{\tilde{x}}^j$, which are errors of member j's state variables; member i does not check or update the estimate of such variables. Unless one is willing to do additional calculations to keep track of them, there is no way to handle the term $-\underline{h}^{ij'}\underline{\tilde{x}}^j$ except as an additional uncertainty in the ranging, like the additive measurement noise v^i.

Suppose the covariance of measurement noise is:

$$r^i \triangleq E\,((v^i)^2)$$

Then the receiver i will model in its filter a measurement noise covariance $r^i+\bar{r}^i$, where $\bar{r}^i$ is the weight attributed to the uncertainty of the source j. In an ownstate community, this weight is calculated only on the basis of information received from the source. If member j has an estimate of the covariance of $\underline{\tilde{x}}^j$:

$$P^j \triangleq E\,(\underline{\tilde{x}}^j \underline{\tilde{x}}^{j'}) \tag{2.8}$$

and includes P^j (or some truncated version of it) in its p-message, then a receiving member i can calculate the covariance of the term $-\underline{h}^{ij'}\underline{\tilde{x}}^j$ in Eq. (2.6) and let:

$$\bar{r}^i = \underline{h}^{ij'} P^j \underline{h}^{ij} \tag{2.9}$$

The estimated position coordinates of the source $\hat{x}^j$, $\hat{y}^j$, $\hat{z}^j$, must be included in the p-message as well, because receivers need them for the computation of the expected ranging, Eq. (2.3), and the geometry vector, Eq. (2.5). Again, in an ownstate community, receivers have to accept the source's estimates of those variables without question.

The definition of an ownstate filter community can now be given. Its features are:

(i) each member estimates only the variables that physically belong to itself, and performs no other recursive computation;

(ii) each active member sends p-messages containing (a) its estimated position and (b) the estimated error covariance P^j of a vector that includes its position and its clock phase (or abbreviated information to that effect);

(iii) each member incorporates only the rangings it takes, and does not share them to the community;

(iv) a term $\bar{r}^i$, being some function of P^j, is added to the measurement noise variance, in order to weigh the uncertainty of the source.

Item (iv) is very questionable from a conceptual standpoint. The variance of the innovation, from Eq. (2.6), is

$$E((\delta\rho^{ij})^2) = \underline{h}^{ij'} \; E(\tilde{\underline{x}}^i\tilde{\underline{x}}^{i'}) \; \underline{h}^{ij} + \underline{h}^{ij'} \; E(\tilde{\underline{x}}^j\tilde{\underline{x}}^{j'})\underline{h}^{ij} +$$

$$+ \; r^i - \underline{h}^{ij'} \; E \; (\tilde{\underline{x}}^i\tilde{\underline{x}}^{j'} + \tilde{\underline{x}}^j\tilde{\underline{x}}^{i'}) \; \underline{h}^{ij} \qquad (2.10)$$

The filter gains may well be chosen in such a way as to handle the first and third terms; if $\bar{r}^i$, from Eq. (2.9), is added to the noise variance, then the second term is accounted for as well (at least if P^j evaluates $E(\tilde{\underline{x}}^j\tilde{\underline{x}}^{j'})$ correctly); but the fourth term, the cross-covariance of source's and receiver's errors, cannot be recovered.

Yet, there is no reason to think that this cross-covariance will be negligible. On the contrary, the collaborative nature of an ownstate community will keep it high. Suppose for an instant that, when member j broadcasts, its errors are uncorrelated with those of all the other members. After the receivers have incorporated their ranging to j, their errors will be correlated with those of member j, and, consequently, with one another. If any of the receivers is an active member, it will, when its turn comes, be the source of a p-message, and it will be ranged to by other former receivers of member j's message, possibly by j itself. Besides, when member j broadcasts again in the next round, its new errors will be correlated with its old errors, and, therefore, with the errors of the receivers which have used its old message. In conclusion, the incorporation of a ranging is done correctly only the first time.

2.4 Mathematical model of an ownstate filter

Consider a community, whose members are trying to estimate their relative position. Suppose no rangings other than passive ones are used.

However, any or all members are allowed to have a dead reckoning facility. Suppose the state vectors estimated by each member are not coupled, and evolve in time linearly. A superscript identifies the member, a subscript the discrete time. Assume the state vector of member i obeys:

$$\underline{x}_n^i = \Psi_n^i \underline{x}_{n-1}^i + \underline{w}_n^i \tag{2.11}$$

The input is stochastic, Gaussian, unbiased, and has a covariance

$$E(\underline{w}_n^i \underline{w}_m^{k'}) = Q_n^i \, \delta_{nm} \, \delta_{ik} \tag{2.12}$$

A passive ranging of i to j at time n is given by:

$$\rho_n^{ij} = f\,(\underline{x}_n^i - \underline{x}_n^j) + v_n^{ij} \tag{2.13}$$

where f(.) is the distance-and-clock-phase function. It is assumed that the coordinate grid of $\underline{x}^i$ and $\underline{x}^j$ is cartesian. The input v^{ij} is stochastic, Gaussian, unbiased, and:

$$E(v_n^{ij} \, v_m^{k\ell}) = r_n^{ij} \, \delta_{nm} \, \delta_{ik} \tag{2.14}$$

The second superscript (typically j) identifies the source of the ranging, and can be dispensed with, since it is a function of the discrete time (i.e., of the subscript):

$$j = s(n) \quad , \quad \ell = s(m) \tag{2.15}$$

where s(.) is the (integer scalar) schedule function, saying which member broadcasts at which time. The innovation is, to the first order:

$$\delta\rho_n^{ij} \cong \underline{h}_n^{ij'} (\underline{x}_{n-}^i - \underline{x}_{n-}^j) + v_n^{ij} \tag{2.16}$$

with

$$\underline{h}_n^{ij} \triangleq \frac{df}{d\underline{x}^i} \Bigg|_{\substack{\underline{x}^i = \hat{\underline{x}}_{n-}^i \\ \underline{x}^j = \hat{\underline{x}}_{n-}^j}} \tag{2.17}$$

A minus sign after the subscript indicates the estimate or the error before the incorporation of the ranging taken at that particular time.

One possible case is when the members have no dead reckoner. In this case, $\underline{x}^i$ may include relative position and clock time only; if it is so, then the displacement and clock shift from t_{n-1} to t_n must be entirely attributed to the stochastic input $\underline{w}_n$; consequently: $\Psi_n^i = I$ for all n and i; $\underline{h}$ will be given by Eq. (2.5).

A case that shall often be discussed supposes the presence of a dead reckoner, but also supposes its output to be corrupted only by a random-walk error. The state may include only the position errors and the clock phase error (the latter, too, is modeled as a random walk); again, $\Psi_n^i = I$, and $\underline{h}$ is given by Eq. (2.5).

In a more typical case, all members will have a dead reckoner, say an INS. The state $\underline{x}^i$ will include all variables that are necessary to make the dead reckoner's, the clock's and the altimeter's errors Markovian. The one-step transition matrix Ψ_n^i will have a complicated form, and the geometrical vector $\underline{h}$ will be an extension of the form given by Eq. (2.5).

The filter of typical member i is:

$$\hat{\underline{x}}_{n-} = \Psi_n^i \, \hat{\underline{x}}_{n-1}^i \tag{2.18}$$

$$\hat{\underline{x}}_n^i = \hat{\underline{x}}_{n-}^i + \underline{k}_n^{ij} \, \delta\rho_n^{ij} \tag{2.19}$$

Either no assumptions will be made on how the gains $\underline{k}$ are computed, or else they will be supposed to be, as in a Kalman filter, the solution of a recursive Riccati equation:

$$P_{n-}^i = \Psi_n^i \, P_{n-1}^i \, \Psi_n^{i'} + Q_n^i \tag{2.20}$$

$$\underline{k}_n^{ij} = P_n^i \underline{h}_n^{ij} \,/\, (\underline{h}_n^{ij'} P_n^i \underline{h}_n^{ij} + \bar{r}_n^{ij} + r_n^{ij}) \tag{2.21}$$

$$P_n^i = (I - \underline{k}_n^{ij} \underline{h}_n^{ij'}) \, P_{n-}^i \tag{2.22}$$

If one remembers that, generally:

$$E((\delta\rho_n^{ij})^2) \neq \underline{h}_n^{ij} P_{n-}^i \, \underline{h}_n^{ij} + r_n^{ij} + \bar{r}_n^{ij} \tag{2.23}$$

(see Eq. (2.10) for the correct expression), one cannot expect these Kalman gains to be optimal in any sense, nor P^i to be equal to the estimation error covariance. Some of the results that follow are independent from the choice of the gains, others require them to be Kalman.

The difference equations for the estimation error will now be derived. They will be necessary in the next two chapters. No assumption will be made on the gains, but linearity will be supposed; that is, approximate Eq. (2.16) will be considered correct and the dependency of $\underline{h}$ on the estimates (Eq. (2.17)) will not be considered. Plugging it into Eq. (2.19) one gets:

$$\hat{\underline{x}}_n^i = \hat{\underline{x}}_{n-}^i + \underline{k}_n^{ij}\underline{h}_n^{ij'}\ (\tilde{\underline{x}}_{n-}^i - \tilde{\underline{x}}_n^j) + \underline{k}_n^{ij}\ v_n^{ij} \qquad (2.24)$$

and subtracting this from $\underline{x}_n^i$ gives the error variation when a ranging is incorporated:

$$\tilde{\underline{x}}_n^i = (I-A_n^{ij})\ \tilde{\underline{x}}_{n-}^i + A_n^{ij}\tilde{\underline{x}}_{n-}^j - \underline{k}_n^{ij}v_n^{ij} \qquad (2.25)$$

For brevity's sake, it was defined:

$$A_n^{ij} \triangleq \underline{k}_n^{ij}\underline{h}_n^{ij'} \qquad (2.26)$$

The propagation of the estimation error between rangings is found subtracting Eq. (2.18) from Eq. (2.11):

$$\tilde{\underline{x}}_{n-}^i = \Psi_n^i\ \tilde{\underline{x}}_{n-1}^i + \underline{w}_n^i \qquad (2.27)$$

Combining eqs. (2.25) and (2.27), and noticing that

$$\tilde{\underline{x}}_{n-}^j = \tilde{\underline{x}}_n^j$$

(the broadcaster makes no measurement incorporation) one finds:

$$\tilde{\underline{x}}_n^i = (I-A_n^{ij})\ \Psi_n^i\ \tilde{\underline{x}}_{n-1}^i + A_n^{ij}\ \tilde{\underline{x}}_n^j + (I-A_n^{ij})\ \underline{w}_n^i - \underline{k}_n^{ij}v_n^{ij} \qquad (2.28)$$

This is the difference equation for the estimation error. It provides a starting point for next chapter's analysis of stability.

2.5 Community organization: (a) coordinate setting.

The organization of a community of ownstate filters must provide for the definition of the relative coordinate grid. Most of the choices have no effect on the analysis that will follow. This section will only explain briefly what is necessary to make this work self-contained.

It is obvious from Eq. (2.2) that only differences of coordinates and of clock phases are measured. Absolute time and geodesic position are unobservable, if only relative rangings are used. The following quantities must be defined in some way:

(i) the origin of community time, in absolute time;

(ii) the origin of the relative spatial coordinate grid, in absolute coordinates;

(iii) the orientation angle(s) of the relative coordinate grid, in absolute coordinates.

It is convenient to fix the third relative axis coincident with the local vertical of the origin, so that only one azimuth angle is needed for (iii).

Each of these three items can, separately, be defined either: (a) by one single member, or (b) by the combined effort of the community. The sufficient conditions for stability proved in the next chapter include explicitly the requirement that items (i) and (ii) be fixed by one single member.

The member that fixes the time origin is called time master; the one that fixes the spatial grid origin is called navigation controller. The spatial grid azimuth can be fixed by the orientation of the inertial system of the navigation controller; or else, it can be fixed by a position comparison between the navigation controller and another member, which is called the end-of-baseline.

Further options concern how the chosen member will fix its allotted indeterminacy. The options are two; the indeterminacy can be fixed:

(1) by the physical position of the chosen member; or (2) by the instantaneous zero of its auxiliary dead reckoner. For instance, take the navigation controller. Under option (1) its relative position is always defined as (0, 0, 0), with deterministic certainty. Under option (2) its relative position is whatever its auxiliary dead reckoner indicates, and that value is defined as deterministically correct. In both cases, the essential thing is that the error of the chosen member is defined to be zero. Therefore error and stability analysis are insensitive to whether option (1) or (2) is used. In all what follows, this problem is ignored completely, but, of course, the choice has a great practical effect. If no auxiliary dead reckoner is present, only option (1) is available, and that is quite bad, in the aircraft case, because the relative grid will be moving rapidly. If dead reckoners are provided, then it is opportune to choose option (2), under which the relative grid will only be slowly drifting.

2.6 Community organization: (b) source selection

It will be shown in the next chapters that source selection is a most important feature that sets different organizations apart. The problem is to decide who is allowed to range to whom. A great many variations on this theme are possible; here they are grouped in three categories. The reader must be warned that most of the terms introduced by Rome and several co-authors in Refs. (6), (7), (8) are not used here; the adjective "democratic" is used, but in a different sense.

1. Democratic organization

In this case there is no source selection at all. Everybody, including masters and controllers, is allowed to range to everybody else.

2. Fixed-rank hierarchy

In this case each member has a rank, which is attributed beforehand and never changed during the mission. Every member ranges to those which have a rank superior to its own. It does not range to other members of equal or inferior rank. The lowest-rank members are passive; they need not send p-messages because nobody would accept them anyway.

3. Covariance-based hierarchy

In this case members do not have a fixed rank. Every time a p-message is received, the receiver compares the supposed accuracy of its own estimates (in terms of position, orientation, clock time or whatever) with the reported accuracy of the source's estimates. If the latter are better than the former, the p-message is accepted, the ranging finalized and incorporated. If otherwise, nothing happens.

Many variations of this kind of hierarchy are possible, depending on precisely what quantity's accuracy is compared. However, the only evaluation a member has of its error covariance is contained in its P matrix. What will be compared is always some function of P^i with the same function of P^j (which is included in the p-message).

A combination of fixed-rank and covariance-based hierarchy is being implemented now for JTIDS-RelNav. Members are divided into masters, "primary users" and "secondary users", and these ranks are fixed. Within primary and secondary users a covariance-based hierarchy is enforced.

This outline of possible organizations concludes the description of ownstate communities.

CHAPTER 3

STABILITY THEORY OF AN OWNSTATE COMMUNITY

3.1 Introduction

This chapter will address the problem of the stability of an ownstate community in an analytical way, without relying on simulation results.

A recursive estimation algorithm may be considered as a dynamic system whose state is the estimation error, and whose input is the noise. Leaving precise definitions to the next section, it is obvious that some kind of stability is a necessary requirement, since the very purpose of the process is to bring and keep the estimation error down to acceptable levels, independent of its initial value, which may be arbitrarily large. The history of JTIDS-RelNav studies has been largely influenced by concerns about stability, but most insights have been obtained by means of simulations; analytical results are rare.

There is a widespread opinion among JTIDS-RelNav authors that the democratic organization is unstable, see e.g. Westbrook and Snodgrass (3), because of simulation and flight test results to this effect for PLRACTA. This opinion was instrumental in causing the present implementation of JTIDS-RelNav to have a covariance-based hierarchy, instead of a democratic organization. However, there is uneasiness about this choice as well. The results of Rome and Stambaugh (7) and Greenberg and Rome (8) raised doubts about a covariance-based hierarchy, although Fried (5) had contrary opinion. More recently Kerr and Chin (14) echoed the uneasiness and proposed algorithms that could give more assurance of stability.

Although many of these opinions are well grounded, yet, this work will prove that there is one organization of ownstate filters that is stable. It is the one that was called a fixed-rank hierarchy, in Chapter 2. Most of the remainder of this chapter is devoted to the proof of its stability under sufficient conditions, and to an analysis of these conditions.

3.2 Definitions of Stability

This section contains a reminder of the accepted definitions of stability that are relevant to our case. They stem mostly from Lyapunov, and are summarized in Kalman and Bertram (15).

Consider a free, or unforced, discrete-time dynamic system, whose state $\underline{x}$ obeys the difference equation

$$\underline{x}_n = \underline{f}_n (\underline{x}_{n-1}) \tag{3-1}$$

The presence of a subscript indicates functional dependence on discrete time, e.g., $\underline{x}_n$ means $\underline{x}(t_n)$. Given an initial state $\underline{a}$ at discrete time m, Eq. (3-1) will have a solution:

$$\underline{x}_n = \phi(n; \underline{a}, m) \tag{3-2}$$

ϕ is called a trajectory of the free system. It always exists, since it can be found from Eq. (3-1) by induction; it is unique if $\underline{f}$ (.) is single-valued, as we shall suppose; it satisfies Eq. (3-1) and the initial condition, that is:

$$\phi(n; \underline{a}, m) = \underline{f}_n(\phi(n-1; \underline{a}, m))$$

$$\phi(m; \underline{a}, m) = \underline{a}$$

An equilibrium state $\underline{e}$ is defined by:

$$\underline{e} = \underline{f}_n(\underline{e}) \qquad \text{or:} \qquad \underline{e} = \phi(n;\ \underline{e},\ m)$$

for all n and m. One at least must be supposed to exist, since what is going to be addressed is precisely the stability of an equilibrium state.

Uniformity may be present or not. Definitions and lemmas will usually come in pairs, one for the uniform and one for the nonuniform case. The remarks between brackets will refer to the uniform case.

In the following analysis, the operator "norm" will be left undefined. The Euclidean norm operator will be used in proofs and specific calculations; the proofs, however, will have general validity, whenever only topological properties are concerned. All normings of a finite-dimensional vector space define equivalent topologies (see, e.g., Loomis and Sternberg (16) p.208).

Definition 1. An equilibrium state $\underline{e}$ of a free discrete-time dynamical system is called [uniformly] stable in the sense of Lyapunov if, for any given $\delta > 0$ and any given time m [for any $\delta>0$ and all $m>M$] there exists a $\varepsilon(\delta, m) > 0$ [$\varepsilon(\delta)>0$] such that $|\underline{a} - \underline{e}| < \varepsilon$ implies $|\phi(n;\ \underline{a},\ m) - \underline{e}| < \delta$ for all $n > m$.

The intuitive meaning of Def. 1 is that, if the state is initially close enough to the equilibrium value, it will stay as close to it as one wishes. This may be a satisfactory kind of stability in some cases, but usually one will wish the state to converge to equilibrium independently of the initial value. This more restrictive condition is defined as follows.

Definition 2. An equilibrium state $\underline{e}$ of a free discrete-time dynamical system is [uniformly] asymptotically stable if:

(i) it is [uniformly] stable, according to Def. 1;

(ii) there exists at least one $\varepsilon > 0$ with this property: for any given $\delta > 0$ and time m [for any $\delta>0$ and all $m>M$] there exists an integer $N(\delta,m)>0$ [$N(\delta)>0$] such that $|\underline{a} - \underline{e}| < \varepsilon$ implies $|\phi(n; \underline{a}, m) - \underline{e}| < \delta$ for all $n > m + N$.

Definition 2 states that there is a (generally finite) region of stability. This restriction is removed in the next definition, which concerns not one particular equilibrium state, but the system at large.

Definition 3. A free discrete-time dynamical system is [uniformly] asymptotically stable in the large if:

(i) it has an equilibrium state $\underline{e}$, which is [uniformly] stable according to Def. 1;

(ii) for any given $\varepsilon > 0$, $\delta > 0$ and time m there exists an integer $N(\varepsilon,\delta,m) > 0$ [$N(\varepsilon, \delta) > 0$] such that $|\underline{a} - \underline{e}| < \varepsilon$ implies $|\phi(n; \underline{a}, m) - \underline{e}| < \delta$ for all $n > m + N$ [and for all m greater than some M];

[(iii) all trajectories are uniformly bounded; that is, for any given $\tau>0$ there exists a $\beta(\tau)>0$ such that $|\underline{a} - \underline{e}| < \tau$ implies $|\phi(n; \underline{a}, m)-\underline{e}| < \beta$ for all $n > m$].

If uniformity is not desired, Def. 3 (iii) can be dispensed with. The arbitrariness of ε is what makes Def. 3 different from Def. 2. As a consequence of Def. 3, all trajectories will converge to $\underline{e}$ and there cannot be another equilibrium state.

Let us now go back to Def. 2 and specialize it in a different direction, namely, impose a condition on how fast the state converges to equilibrium.

Definition 4. An equilibrium state $\underline{e}$ of a free discrete-time dynamical system is exponentially stable if there exists a constant $\beta > 0$ and, for some values of ε, a $\delta(\varepsilon)>0$, such that $|\underline{a} - \underline{e}| < \varepsilon$ implies $|\phi(n; \underline{a}, m)-\underline{e}| < \delta \exp(-\beta(n-m))$ for all $n > m$, and for at least one m (therefore, as it would be easy to see, for all m greater than that: there is a built-in uniformity).

Of course, one can also define exponential stability in the large, by saying that Def. 4 must hold for all values of ε.

All the previous definitions concern a free system. Instead, if an input $\underline{u}$ is present, the system will be described by an equation of this type:

$$\underline{x}_n = f_n (\underline{x}_{n-1}, \underline{u}_n) \tag{3.4}$$

The solutions of Eq. (3-4), called forced trajectories, will depend on the whole time-history of $\underline{u}$. Given the input as a certain function of time:

$$\underline{u}_n = \underline{g}(n) \tag{3.5}$$

and an initial state $\underline{a}$ at discrete time m, one can find the forced trajectories by induction:

$$\underline{x}_n = \phi_g(n; \underline{a}, m) \tag{3.6}$$

but the form of function $\underline{\phi}_g(.)$ depends on the form of $\underline{g}(.)$.

If one keeps $\underline{g}(.)$ constant, i.e., considers only trajectories forced by one particular input history, one can then generalize all the previous definitions, with the following changes. Instead of considering the difference between the free trajectory starting from an arbitrary point $\underline{a}$, and the equilibrium state, one will consider the difference between two forced trajectories starting from any two arbitrary points $\underline{a}$ and $\underline{b}$. In Defs. 1 to 4 one has to change:

$$|\underline{a} - \underline{e}| \quad \text{into} \quad |\underline{a} - \underline{b}|$$
$$|\phi(n; \underline{a}, m) - \underline{e}| \quad \text{into} \quad |\phi_g(n; \underline{a}, m) - \phi_g(n; \underline{b}, m)| \tag{3.7}$$

and appropriately reword the statements. This generalization is useful in case the system is linear (as it will be shown).

3.3 Stability of linear systems.

This section contains the statements and proofs of a few lemmas about the stability of linear systems. They build up a basis for next section's theorem. The lemmas range from most elementary to possibly original. The fact that a linear estimation process is being considered, provides the motivation for analyzing linear systems.

A free linear system is described by

$$\underline{x}_n = \Psi_n \underline{x}_{n-1} \tag{3.8}$$

The free trajectory starting from $\underline{x}_m = \underline{a}$ is:

$$\phi(n; \underline{a}, m) = \Phi(n, m)\, \underline{a} \tag{3.9}$$

with

$$\Phi(n, m) = \prod_{k=m+1}^{n} \Psi_n \quad , \quad \text{for } n>m \tag{3.10}$$
$$\Phi(n,n) = I$$

The case $n < m$ is of no interest here, and may be left undefined. There always exists an equilibrium state $\underline{e} = 0$, (in most cases this is the only equilibrium state).

The same system, forced by an input $\underline{u}_n$, is described by an expression of this type:

$$\underline{x}_n = \Psi_n \underline{x}_{n-1} + \underline{u}_n \tag{3.11}$$

The forced trajectories are given by

$$\phi_u(n; \underline{a}, m) = \Phi(n, m) \underline{a} + \sum_{k=m+1}^{n} \Phi(n, k)\underline{u}_k \tag{3.12}$$

as it can be easily seen by induction from Eq. (3.11).

Lemma 1: A linear system has the same stability properties when free and when forced by any input.

Proof. Consider two trajectories starting from $\underline{a}$ and $\underline{b}$ at the same time m and forced by the same input $\underline{u}_n$; from Eq. (3.12):

$$\phi_u(n; \underline{a}, m) - \phi_u(n; \underline{b}, m) = \Phi(n, m)(\underline{a} - \underline{b}) \tag{3.13}$$

The difference between a free trajectory starting from $\underline{a}$ and the null equilibrium state $\underline{e} = 0$ is, from Eq. (3.9):

$$\underline{\phi}(n; \underline{a}, m) - \underline{e} = \Phi(n, m)(\underline{a} - \underline{e}) \tag{3.14}$$

Comparison of (3.13) with (3.14) shows that the substitutions (3.7) do not alter the truth of a statement, Q.E.D.

Lemma 2. If a linear system is [uniformly] stable in the sense of Lyapunov, the norms of its transition matrices $| \Phi (n,m) |$ are bounded uniformly in n [and uniformly in m].

Lemma 2-a. If a linear system is [uniformly] asymptotically stable, the norms of its transition matrices $|\Phi(n,m)|$ converge to zero [uniformly in m]. That is, for any given $\tau>0$ and any given m, there exists $N(\tau,m)$ [for any given $\tau>0$ and all $m>M$, there exists $N(\tau)$] such that $n> m + N$ implies $|\Phi(n,m)| < \tau$.

Proof. For lemma 2: take any $\delta > 0$, find $\varepsilon(\delta)$ according to Def. 1. Take all $\underline{a}$ such that $|\underline{a}| = \varepsilon$; then all $\underline{b} \triangleq \underline{a}/\varepsilon$ will cover the unit circle. From Def. 1 and e.q. (3.14), for $n > m$: $|\Phi(n,m)\underline{a}| < \delta$ or, equivalently:

$$\underline{a}'\Phi'(n,m)\,\Phi(n,m)\,\underline{a} < \delta^2$$

$$\underline{b}'\Phi'(n,m)\,\Phi(n,m)\,\underline{b} < \delta^2/\varepsilon^2$$

Remember that the Euclidean norm of a matrix A has the property:

$$|A|^2 = \max_{\underline{x}\varepsilon U} \{\underline{x}'A'A\underline{x}\}$$

where $U \triangleq \{\underline{x} \,|\, \underline{x}'\underline{x} = 1\}$ is the unit circle. Therefore, $|\Phi(n,m)|^2 < \delta^2/\varepsilon^2$; $\delta/\varepsilon(\delta)$, with any δ, provides an upper bound; it is uniform in m if ε does not depend on m, Q.E.D.

For lemma 2-a: take any ε and any δ, find $N(\varepsilon,\delta)$ according to Def. 2. Then take all $\underline{a}$ with $|\underline{a}| = \varepsilon$ and repeat the same argument used above. It will be found that $|\Phi(n,m)| < \delta/\varepsilon$ for $n > m + N$, [and for all m]. Let $\delta/\varepsilon \triangleq \tau$, $N(\varepsilon,\delta) \triangleq N(\tau)$, Q.E.D.

Lemma 3. If the equilibrium state $\underline{e} = 0$ of a linear system is [uniformly] asymptotically stable, then the system is [uniformly] asymptotically stable in the large.

Proof. Def. 2(i) translates into Def. 3(i). From (3.14) one sees that $|\phi(n;\underline{a},m) - \underline{e}|$ and $|\underline{a} - \underline{e}|$ are proportional. If: $|\underline{a} - \underline{e}| < \varepsilon$

implies $|\phi(n;\underline{a},m)-e| < \delta$ under suitable conditions, then: $|\underline{a} - \underline{e}| < c\varepsilon$ implies $|\phi(n;\underline{a},m)-\underline{e}| < c\delta$ under the same conditions, for all $c > 0$. Therefore Def. 2(ii) becomes Def. 3(ii). Def. 3(iii) remains to be proved. Take $\varepsilon > 0$ and $\delta > 0$ arbitrarily; find $N(\delta)$, according to Def. 2. Then, for $n > m + N$:

$$|\phi(n;\underline{a},m) - \underline{e}| < \delta$$

whereas, for $m < n < m + N$:

$$|\phi(n;\underline{a},m) - \underline{e}| = |\Phi(n,m)\,\underline{a}| < |\Phi(n,m)| \,.\, |\underline{a}| < s\varepsilon$$

where s is an upper bound for $|\phi(n,m)|$, according to lemma 2. Therefore $\max\{s\varepsilon, \delta\}$ is an upper bound for the trajectory for all $n > m$, and it is a uniform bound if s does not depend on m, Q.E.D.

Lemmas 1 to 3 are very well known. The next one is also known, but it is seldom stated explicitly. It goes as follows.

Lemma 4. For a linear system, uniform asymptotic stability and exponential stability are equivalent.

Proof. That exponential stability implies uniform asymptotic stability is obvious enough. I shall prove only the converse. Using lemma 2-a find N (1/2), such that $|\Phi(n,m)| < 1/2$ for $n > m + N$ and for all m greater than a certain value M. Then find:

$$\alpha \triangleq \max_{m<i<m+N} |\Phi(i,m)| \,.\, 2^{(i-m)/N}$$

α is finite because $|\Phi(i,m)|$ is bounded above by lemma 2, and $2^{(i-m)/N} < 2$. Take now n and m arbitrarily (but with $n > m > M$); find the integer k such that : $k < \frac{n-m}{N} < k+1$; therefore: $m < n-kN < m+N$. Then:

$$|\phi(n;\underline{a},m) - \underline{e}| = |\Phi(n,m).\underline{a}| =$$

$$= |\Phi(n,n-N).\ \Phi(n-N,n-2N) \ldots\ \Phi(n-(k-1)N,n-kN).$$

$$.\ \Phi(n-kN,m).\underline{a}| <$$

$$< |\Phi(n,n-N)|\ .\ |\Phi(n-N,n-2N)| \ \ldots\ |\Phi(n-(k-1)N,n-kN)|\ .$$

$$.\ |\Phi(n-kN,m)|\ .\ |\underline{a}| <$$

$$< 2^{-k}\ |\Phi(n-kN,m)|\ .\ |\underline{a}| =$$

$$= 2^{-(n-m)/N}\ .\ 2^{(n-kN-m)/N}\ .\ |\Phi(n-kN,m)|\ .\ |\underline{a}| <$$

$$< 2^{-(n-m)/N}\ .\ \alpha\ .\ |\underline{a}|$$

Define $\delta(\varepsilon) = \alpha\cdot\varepsilon$; define $\beta = (\ln 2)/N$; β is independent of ε , as required. Then $|\underline{a}| < \varepsilon$ will imply:

$$|\phi(n;\underline{a},m) - \underline{e}| < 2^{-(n-m)/N}\ \alpha\ \varepsilon = \delta e^{-\beta(n-m)} \qquad \text{Q.E.D.}$$

As a corollary to lemma 4, one can add that $|\Phi(n,m)|$ is exponentially convergent to zero. The proof is contained in the proof of lemma 4.

It is apparent that the analysis of stability is particularly easy in the case of linear systems. There always is an equilibrium state $\underline{e} = 0$; if that is asymptotically stable, then the system is asymptotically stable in the large, both when free and when forced by any input; moreover, if there is uniformity, then the stability is exponential. One more basic lemma is known about linear systems; namely, when they are uniformly asymptotically stable, a bounded input gives rise to a bounded state trajectory (and therefore to a bounded output, if there is any). This lemma will not be proved here (see Kalman and Bertram, loc. cit.). Instead, a stricter statement will be proved; it is the key to next section's theorem, and it is possibly original, or, at least, independently found.

Lemma 5. In a uniformly asymptotically stable linear system, an input that exponentially converges to zero gives rise to a state trajectory that exponentially converges to zero.

Proof. Suppose $|\underline{u}_k| < c\, e^{-\tau(k-m)}$ for all $k > m$. From Eq. (3.12):

$$|\underline{x}_n| = |\phi_u(n;\underline{a},m)| =$$

$$= |\Phi(n,m)\underline{a} + \sum_{k=m+1}^{n} \Phi(n,k)\underline{u}_k| \leqslant$$

$$\leqslant |\Phi(n,m)| \cdot |\underline{a}| + \sum_{k=m+1}^{n} |\Phi(n,k)| \cdot |\underline{u}_k| <$$

$$< \alpha\, e^{-\beta(n-m)} |\underline{a}| + \sum_{k=m+1}^{n} \alpha\, e^{-\beta(n-k)}\, c e^{-\tau(k-m)}$$

because of the hypothesis and of lemma 4 (corollary). Take $\gamma = \min\{\beta,\tau\}$; then:

$$|\phi_u(n;\underline{a},m)| < \alpha\, e^{-\gamma(n-m)} |\underline{a}| + \sum_{k=m+1}^{n} \alpha\, c\, e^{-\gamma(n-m)} =$$

$$= \alpha\, e^{-\gamma(n-m)} (|\underline{a}| + (n-m)c).$$

The proof will be complete if one shows that this last expression is less than $\delta e^{-p(n-m)}$ with δ possibly a function of $|\underline{a}|$, but p independent from it.

And, indeed, take any p with $0<p<\gamma$ and take

$$\delta > \frac{\alpha c}{\gamma - p} \exp(-1+|\underline{a}|(\gamma - p)/c);$$

the function:

$$y(n) \triangleq (|\underline{a}| + cn) \exp((p-\gamma)n)$$

has, elementarily, a maximum $y^\circ = \frac{c}{\gamma - p} \exp(-1 + |\underline{a}|(\gamma - p)/c)$

Therefore:

$$|\phi_u(n;\underline{a},m)| < \alpha (|\underline{a}| + (n-m)c) \exp(-\gamma(n-m)) =$$

$$= \alpha \exp(-p(n-m)).y(n-m) <$$

$$< \alpha \exp(-p(n-m)).y^\circ < \delta \exp(-p(n-m)) \text{ Q.E.D.}$$

3.4 Proof of the Stability of a fixed-rank hierarchical organization.

This section states and proves sufficient conditions for the stability of an ownstate filter community.

Consider a community like the one defined in sections 2-3 and 2-4; the state of typical member i propagates linearly:

$$\underline{x}_n^i = \Psi_n^i \underline{x}_{n-1}^i + \underline{w}_n^i \tag{3.15}$$

and the estimator is linear, so that the estimation error follows equation (2.28):

$$\tilde{\underline{x}}_n^i = (I-A_n^{ij}) \Psi_n^i \tilde{\underline{x}}_{n-1}^i + A_n^{ij} \tilde{\underline{x}}_n^j - \underline{k}_n^{ij} v_n^{ij} + (I-A_n^{ij}) \underline{w}_n^i \tag{3.16}$$

with the matrix A defined as

$$A_n^{ij} = \underline{k}_n^{ij} \underline{h}_n^{ij'} \tag{3.17}$$

and the symbols have the same meaning as in Chapter 2. No hypotheses are made now on how the gains $\underline{k}$ are computed. They may be solutions of the recursive Riccati equation (2.20-22) or they may be computed in any other way.

The difference $\delta\underline{\tilde{x}}^i$ between two trajectories of $\underline{\tilde{x}}^i$ subject to the same sample of noise, but starting from different initial conditions, follows the equation:

$$\delta\underline{\tilde{x}}_n^i = (I-A_n^{ij})\,\Psi_n\,\delta\underline{\tilde{x}}_{n-1}^i + A_n^{ij}\,\delta\underline{\tilde{x}}_n^j \qquad (3.18)$$

All stability concepts refer to this difference. According to Lemma 1, Eq. (3.18) should be the homogeneous part of Eq. (3.16). The input $\delta\underline{\tilde{x}}^j$ does not vanish, however. One must aggregate all $\underline{\tilde{x}}^i$ into one vector $\underline{\tilde{x}}$. Then the equation for $\delta\underline{\tilde{x}}$ will be homogeneous. In other words, the evolution of the community error cannot be split up among the single members.

Theorem 1. A community, like the one just described, performs an exponentially stable estimation process if:

(i) each member is attributed a constant rank, and allowed to range only to other members whose rank is superior to its own;

(ii) there is only one first-rank member, which is also time master and navigation controller;

(iii) the filtering process of each member, considered separately, is exponentially stable, provided all other members have zero errors (this property will be called one-by-one stability);

(iv) the matrices A^{ij} are bounded from above in the norm.

Hypothesis (i) defines what was called a fixed-rank hierarchy. Hypothesis (ii) requires the only first-rank member (which will be labelled 1) to have zero relative position and clock errors by definition. However the state vectors $\underline{x}_i$ are composed, a passive ranging is sensitive only to relative position and clock errors of the source; thus, when member 1 is the source, it will be:

$$\underline{h}_n^{i1'} \tilde{\underline{x}}_n^1 = 0 \qquad \text{for all } n \text{ and all } i \neq 1.$$

Hypothesis (iii) requires a careful definition. If all other members member i ranges to happen to have no relative position or clock errors, then

$$\underline{h}_n^{ij'} \tilde{\underline{x}}_n^j = 0$$

and Eqs. (3.16), (3.18) become:

$$\tilde{\underline{x}}_n^i = (I-A_n^{ij})\Psi_n^i \tilde{\underline{x}}_{n-1}^i + (I-A_n^{ij})\underline{w}_n^i - \underline{k}_n^{ij} v_n^{ij} \tag{3.19}$$

$$\delta\tilde{\underline{x}}_n^i = (I-A_n^{ij})\Psi_n^i \delta\tilde{\underline{x}}_{n-1}^i \tag{3.20}$$

Hypothesis (iii) requires Eqs. (3.19), (3.20) to describe an exponentially stable process (forced and free, respectively).

Hypothesis (iv) is no additional burden, in practice, since every reasonable estimator that satisfies hypothesis (iii) will have bounded gains as well. However, if the matrices Ψ^i have favorable properties, one can invent pathological non-Kalman gains such that $|(I-A^{ij})\Psi^i|$ decays exponentially, and yet $|A^{ij}|$ grows unboundedly. That is why hypothesis (iv) has to be claimed separately.

Proof. Consider Eq. (3.18) separately for each member i; the second term on the r.h.s. is then considered an input to an exponentially stable system, by hypothesis (iii). Forget the master, which does not even need a filter if only relative navigation is performed. Consider a second-rank member, labelled 2. Because of hypothesis (i), it ranges only to the master; because of hypothesis (ii) its Eq. (3.18) becomes:

$$\delta \underline{\tilde{x}}_n^2 = (I - A_n^{21}) \Psi_n^2 \, \delta \underline{\tilde{x}}_{n-1}^2 \tag{3.21}$$

Because of hypothesis (iii), $|\delta \underline{\tilde{x}}^2|$ decays exponentially.

Consider now a third-rank member, labelled 3, which ranges to all second-rank members and to the master. When the master is ranged to, the equation is:

$$\delta \underline{\tilde{x}}_n^3 = (I - A_n^{31}) \Psi_n^3 \, \delta \underline{\tilde{x}}_{n-1}^3$$

with no input, by hypothesis (ii). When a generic second-rank member, labelled 2, is ranged to, the equation is

$$\delta \underline{\tilde{x}}_n^3 = (I - A_n^{32}) \Psi_n^3 \, \delta \underline{\tilde{x}}_{n-1}^3 + A_n^{32} \, \delta \underline{\tilde{x}}_n^2 \tag{3.22}$$

Since $|A^{32}|$ is bounded by hypothesis (iv), and $|\delta \underline{\tilde{x}}^2|$ has just been proved to be exponentially decaying, then also $|A^{32} \delta \underline{\tilde{x}}^2| < |A^{32}| \cdot |\delta \tilde{x}^2|$ is exponentially decaying. The input to member 3 is an interspersion of zeroes (from the master) and exponentially decaying functions (one from each second-rank member). Then the input is globally exponentially decaying, being bounded above by the least steep rate of decay among the component functions; it has been tacitly assumed that the members of second rank, and of any rank, and in total, are a finite number. Now Lemma 5 applies; since the input decays exponentially, the state $\delta \underline{\tilde{x}}^3$ also does.

Suppose now all members down to the (k-1)th rank have been proved to have exponentially decaying $\delta\underline{\tilde{x}}^i$. The input to any k-th-rank member is made of interspersed exponentially decaying functions and zeroes (from the master). By the same argument used for the third ranks, we conclude that $\delta\underline{\tilde{x}}^k$ decays exponentially. By induction, this is valid for all members, Q.E.D.

One can see that the proof is quite simple and nothing but an application of Lemma 5. However, there is a subtle point that might escape. Namely, one should resist the temptation of using the following argument. Since every member ranges only to those other members that do not range back to it, then in Eq. (3.16) $\underline{\tilde{x}}^j$ does not depend on $\underline{\tilde{x}}^i$; therefore two trajectories of $\underline{\tilde{x}}^i$, starting at different values, but with all things else equal, will also have the same $\underline{\tilde{x}}^j$; their difference obeys the homogeneous equation:

$$\delta\underline{\tilde{x}}_n^i = (I-A_n^{ij})\Psi_n^i \, \delta\underline{\tilde{x}}_{n-1}^i$$

and $\delta\underline{\tilde{x}}^i$ decays exponentially by hypothesis (iii).

This argument has an appealing simplicity and uses only hypotheses (i) and (iii), but it is not a sufficient proof. It only tells what happens to two trajectories of $\underline{\tilde{x}}^i$ initiated by two different values of $\underline{\tilde{x}}^i$ itself; it does not tell what happens to two trajectories of $\underline{\tilde{x}}^i$ initiated by two different values of $\underline{\tilde{x}}^j$, when j has a rank superior to i. In order to complete the proof one needs hypotheses (ii) and (iv), and Lemma 5.

3.5 Stability properties of a linear filter with Kalman gains

It was shown in the previous section that a fixed-rank community with a master is exponentially stable if the members' filters are exponentially stable one by one. The next task is to show that one-by-one stability exists. One good way to ensure that is to have all members use Kalman gains; but this is not sufficient unless other conditions are met. Consequently, this section presents a reminder of known theorems about the stability of Kalman filters, and elaborates on them what will come useful later.

Consider a discrete-time linear system on which linear observations are taken:

$$\underline{x}_n = \Psi_n \underline{x}_{n-1} + \underline{w}_n$$
$$\underline{y}_n = M_n \underline{x}_n + \underline{v}_n \qquad (3.23)$$

Suppose the inputs $\underline{w}$, $\underline{v}$ are stochastic, Gaussian, uncorrelated with each other and with the following statistics:

$$E(\underline{w}_n) = 0 \; ; \; E(\underline{w}_n \underline{w}_m') = Q_n \delta_{mn}$$
$$E(\underline{v}_n) = 0 \; ; \; E(v_n v_m') = R_n \delta_{mn} \qquad (3.24)$$

Suppose the state is reconstructed by a minimum variance filter:

$$\hat{\underline{x}}_{n-} = \Psi_n \hat{\underline{x}}_{n-1}$$
$$\hat{\underline{x}}_n = (I-K_n M_n)\hat{\underline{x}}_{n-} + K_n \underline{y}_n$$
$$P_{n-} = \Psi_n P_{n-1} \Psi_n' + Q_n \qquad (3.25)$$
$$K_n = P_{n-} M_n' (M_n P_{n-} M_n' + R_n)^{-1}$$
$$P_n = (I-K_n M_n) P_{n-}$$

Then, the estimation error obeys the equation:

$$\tilde{\underline{x}}_n = (I-K_nM_n)\Psi_n\tilde{\underline{x}}_{n-1} + (I-K_nM_n)\underline{w}_n - K_n\underline{v}_n \tag{3.26}$$

and the meaning of P is:

$$P_n = E(\tilde{\underline{x}}_n\tilde{\underline{x}}_n') \tag{3.27}$$

Following Kalman (17) the information matrix from discrete time m to discrete time n > m is defined as follows:

$$\mathcal{I}\ (n,m) \triangleq \sum_{k=m+1}^{n} \Phi^{-1'}(n,k)M_k'R_k^{-1}\ M_k\ \Phi^{-1}(n,k) \tag{3.28}$$

where Φ is the transition matrix of system (3.23) and is given by Eq. (3.10). Likewise, the controllability matrix from m to n > m is defined as:

$$\mathcal{C}\ (n,m) \triangleq \sum_{k=m+1}^{n} \Phi(n,k)Q_k\Phi'(n,k) \tag{3.29}$$

The system (3.23), (3.24) is called completely observable [controllable] from m to n if $\mathcal{I}$ (n,m)[$\mathcal{C}$ (n,m)] is positive definite. It is called uniformly completely observable [controllable] if, for some values of the positive integer N, there exist real numbers $\alpha(N)$,$\beta(N)$ with $\beta>\alpha>0$, such that

$$\alpha I < \mathcal{I}\ (n+N,m) < \beta I \quad [\alpha I < \mathcal{C}\ (m+N,m) < \beta I] \tag{3.30}$$

for all m>0.

The intuitive meaning of the controllability matrix can be seen by imagining the system (3.23), (3.24) without any observation (or with totally inconclusive observations: $R = \infty$). Then the error covariance is, by induction:

$$P_n = \Phi(n,m)P_m\Phi'(n,m) + \mathcal{C}(n,m) \qquad (3.31)$$

If the state $\underline{x}_m$ is completely known (i.e., $P_m = 0$), then $P_n = \mathcal{C}(n,m)$. The controllability matrix quantifies the ability of the stochastic input to bring the error from zero to a finite variance, in absence of observations; in other words, $\mathcal{C}$ measures the ability of $\underline{w}$ to control the state in a stochastic sense.

Similarly, if one supposes that the system (3.23), (3.24) has no input ($Q = 0$) then the inverse of the error covariance is

$$P_n^{-1} = \Phi^{-1'}(n,m)P_m^{-1}\,\Phi^{-1}(n,m) + \mathcal{I}(n,m) \qquad (3.32)$$

This result is reached by substituting the last of Eqs. (3.25) with the equivalent equation:

$$P_n^{-1} = P_{n-}^{-1} + M_n'\,R_n^{-1}\,M_n$$

then proceeding by induction. If the state is totally unknown at time m, i.e., $P_m^{-1} = 0$, then $P_n^{-1} = \mathcal{I}(n,m)$. The information matrix quantifies the ability of the observations to reduce the error covariance from infinity to a finite value, that is, to reconstruct the state, in a stochastic sense.

Sufficient conditions for the uniform asymptotic (exponential, by Lemma 4) stability of the filter (3.25) are given by the following proposition.

Theorem 2. The process (3.26) is uniformly asymptotically stable if the system (3.23), (3.24) is uniformly completely observable and uniformly completely controllable (Deyst and Price (18)).

For a proof, see the original authors, but notice also the corrections introduced by McGarty (19). The first step of the proof consists in showing that P_n is bounded uniformly from above and below:

$$aI < P_n < bI \quad , \quad \text{with } 0<a<b \tag{3.33}$$

This intermediate result is then used to show that $\underline{\tilde{x}}_n{}' P_n^{-1} \underline{\tilde{x}}_n$ is a Lyapunov function, and thus complete the proof. Here it will be used instead, to prove the following statement.

Theorem 3. Under the same hypotheses of theorem 2, and if M_n has full rank, the matrix norm $|K_n M_n|$ is uniformly bounded from above.

Proof. Please remember the following definitions, where $\underline{y}$ is an arbitrary nonzero vector, H an arbitrary matrix, A and B are symmetric matrices:

(i) $$|\underline{y}| \triangleq (\underline{y}'\underline{y})^{1/2}$$

(ii) $$|H| \triangleq \sup_{\underline{y}} \frac{|H\underline{y}|}{|\underline{y}|}$$

(iii) $A > B$ is equivalent to $\underline{y}'A\underline{y} > \underline{y}'B\underline{y}$

in particular: $A \gtrless aI$ is equivalent to $\underline{y}'A\underline{y} \gtrless a|\underline{y}|^2$

Since M has full rank, $\underline{y} = 0$ is equivalent to $M'\underline{y} = 0$; therefore $M'\underline{y}$ can be used, instead of $\underline{y}$, as the arbitrary vector in def. (iii). From $P > aI$

one gets

$$\underline{y}'MPM'\underline{y} > a\,|M'\underline{y}|^2$$

Since R>0 by definition, then $\underline{y}'R\underline{y} > 0$, and:

$$\underline{y}'(MPM'+R)\underline{y} > a\,|M'\underline{y}|^2 \text{ . Then:}$$

$$|M|^2 a = \sup_{\underline{y}} \frac{|M'y|^2}{|\underline{y}|^2} a < \sup_{\underline{y}} \frac{\underline{y}'(MPM'+R)\underline{y}}{|\underline{y}|^2} <$$

$$< \sup \frac{|y| \cdot |(MPM'+R)y|}{|\underline{y}|^2} = |MPM'+R|$$

From $0 < P < bI$ one gets: $P^2 < b^2 I$. Using def. (iii) with $M'\underline{y}$ as the arbitrary vector:

$\underline{y}'MP^2M'\underline{y} < |M'y|^2 b^2$; by def. (i), this is equivalent to: $|PM'\underline{y}| <$ $< |M'\underline{y}|\,b$. Then, because of def. (ii):

$$|PM'| = \sup_{\underline{y}} \frac{|PM'\underline{y}|}{|\underline{y}|} < \sup_{\underline{y}} \frac{|M'\underline{y}|}{|\underline{y}|} b = |M|\,b$$

All the previous results are then used to establish the following chain of inequalities:

$$|K_n M_n| = |P_{n-} M_n' (M_n P_{n-} M_n + R_n)^{-1} M_n| <$$

$$< |P_{n-} M_n| \cdot |M_n P_{n-} M_n + R_n|^{-1} \cdot |M_n| <$$

$$< |M_n| \cdot b \cdot (|M_n|^2 \cdot a)^{-1} \cdot |M_n|$$

$|K_n M_n| < b/a$, Q.E.D.

Theorem 3 substantiates what was claimed in the previous section, namely, that hypotheses (iii) and (iv) would usually be satisfied together.

Let us now reword the Deyst-Price theorem in the following way:

Theorem 2-a. Consider the system defined by Eqs. (3.23) but with no hypotheses about $\underline{w}$ and $\underline{v}$. Take sequences Q_n, R_n that satisfy Eq. (3.30) but are otherwise arbitrary. Build the estimator (3.25) with them. Then the estimation error (3.26) is uniformly asymptotically stable.

This rewording excludes Eqs. (3.24), i.e., makes the actual statistics of the input independent by those used in the model. In this case, the estimator (3.25) is not optimal; the matrix P computed by Eqs. (3.25) is not the error covariance (for the actual error covariance see Battin (20) p. 334 ff. and Jazwinski (21) p. 244 ff.) and the actual error covariance is not minimum.

Proof of the equivalence of theorems 2 and 2-a. Take the error (3.26) under the hypotheses of theorem 2. Change the statistics of $\underline{w}_n$ and $\underline{v}_n$ so that Eqs. (3.24) are satisfied by some previously arbitrarily chosen Q_n, R_n . What is changed in Eq. (3.26) is simply the statistics of the input $(I - K_n M_n)\underline{w}_n - K_n \underline{v}_n$. By Lemma 1, the stability of a linear system does not depend on the input's time history, much less on its statistics. Therefore the conclusion of theorem 2 is still true, Q.E.D.

This equivalence justifies the common practice of overestimating the noise variance in order to get better filter properties. The argument, however, cannot be extended to system models that differ from reality in any feature other than noise statistics.

3.6 Sufficient conditions for one-by-one stability: observability analysis and flight path strategy

The results of the previous section will now be applied to our problem. If the filter of each member has Kalman gains, Eqs. (2.20) to (2.22), then the problem of one-by-one exponential stability is equivalent to that of the exponential stability of a Kalman filter. If one defines the one-by-one controllability and information matrices of the typical member i as:

$$\mathcal{C}^i(n,m) = \sum_{k=m+1}^{n} \Phi^i(n,k)Q_k\ \Phi^{i'}(n,k) \tag{3.34}$$

$$\mathcal{I}^i(n,m) = \sum_{k=m+1}^{n} (\Phi^i(n,k))^{-1'} \underline{h}_k^{ij}\ \underline{h}_k^{ij'}\ (\Phi^i(n,k))^{-1}/(r_k^{ij} + \bar{r}_k^{ij}) \tag{3.35}$$

then the conditions:

$$\alpha_i I < \mathcal{C}^i(m+N,m) < \beta_i I\ ;\ \gamma_i I < \mathcal{I}^i(m+N,m) < \delta_i I \tag{3.36}$$

are sufficient to insure hypotheses (iii) and (iv) of theorem 1, by virtue of theorems 2-a and 3, respectively.

Further discussion requires hypotheses on the matrices ^{i}. As said in Section 2.4, if one knows no details about the auxiliary dead reckoners and the clocks, and just supposes that their errors are random walks, then one may set $^i = I$. The same is true if there are no dead reckoners at all. Since it is not opportune to be bound to detailed hypotheses about the dead reckoners (or the clocks) this approximation shall be adopted in the following discussion.

The one-by-one controllability matrix becomes:

$$\mathcal{C}^i(m+N,m) = \sum_{k=m+1}^{m+N} Q_k^i \tag{3.37}$$

Since Q^i is largely arbitrary (remember theorem 2-a) there is no difficulty in satisfying the uniform controllability condition. It is sufficient to make Q^i uniformly bounded from above and below: ${}_{i}I < Q_n^i < {}_{i}I$, for all n.

The information matrix becomes:

$$\mathcal{I}^i(m+N,m) = \sum_{k=m+1}^{m+N} \underline{h}_k^{ij}\underline{h}_k^{ij'}/(r_k^{ij}+\bar{r}_k^{ij}) \tag{3.38}$$

There is no problem in bounding the measurement noise covariance and the source uncertainty weight; the value of numerators, instead, can be fixed at will only if the members agree beforehand on a flight path strategy. The information matrix appears to be a sum of N dyadic matrices:

$$\mathcal{I}^i = \sum_{k=1}^{N} \underline{y}_k\underline{y}_k' \tag{3.39}$$

with:

$$\underline{y}_k \triangleq \underline{h}_{m+k}^{ij}/(r_{m+k}^{ij} + \bar{r}_{m+k}^{ij})^{1/2} \tag{3.40}$$

It is known that such a sum is always positive semidefinite, but positive definite if and only if the set $\{\underline{y}_1, y_2, \ldots, \underline{y}_N\}$ spans the linear space $\mathbb{R}^s$ in which it is defined. The <u>proof</u> is simple. If the set spans the space, it is never possible to find a nonzero vector $\underline{z}$ that is orthogonal to all $\underline{y}_k$. Therefore $\underline{z}'\underline{y}_k \neq 0$ for all $\underline{z} \neq 0$, $\in \mathbb{R}^s$ and at least one k and:

$$\underline{z}' \, \mathcal{I}^i \underline{z} = \sum_{k=1}^{N} (\underline{z}'\underline{y}_k)^2$$

is always > 0. Viceversa, if the set does not span $\mathbb{R}^s$, it is always possible to find a nonzero $\underline{z} \in \mathbb{R}^s$ such that $\underline{z}'\underline{y}_k = 0$ for all k. Therefore $\underline{z}' \; I^i \underline{z} = 0$, Q.E.D.

Thus, complete observability is assured if an integer N exists such that in every N consecutive rangings there are s independent ones. In order to have uniformity, one must bound I^i from above and below. The upper bound is no problem, since $|\underline{h}| = \sqrt{2}$. The lower bound is ensured by making the linear independence uniform.

In our case s = 4, and one must remember the lemma of Sec. 2.2 about the independence of geometry vectors. So one reaches the following:

Flight Path Condition. There must exist an integer N such that every set of N consecutive rangings contains four different lines of sight that are not on the same plane, nor become in time indefinitely close to the same plane.

One consequence is that a member which goes unboundedly far away from the rest of the community has no uniform observability, because all its lines of sight grow closer and closer together. Another consequence is that observability of the vertical coordinate will always be poor, since all members are typically close to the same horizontal plane.

If one takes that for granted and only wants to insure observability of the horizontal position and time (for altitude there will be altimeters, anyway), then s = 3, and the observability condition is as follows.

Two-dimensional Flight Path Condition. There must exist an integer N such that every set of N consecutive rangings contains three different lines of sight, which must not become in time indefinitely close.

A third consequence bears only on a fixed-rank hierarchical community with a master (the kind of community where one-by-one stability of all members has been proved sufficient for general stability). Since there is only one first rank member, and the second ranks range only to it, they will get just one independent line of sight, unless they move around with respect to the master (or the master moves around with respect to them). A static configuration has no observability, and, therefore, no proven stability.

All the previous conclusions are valid if the errors of the dead reckoners and the clocks are featureless random walks. If this restriction is removed, then the ownstate vectors $\underline{x}^i$ must also include sufficient states to make the errors Markovian, and the Ψ^i matrices will not equal the identity. This change increases the number s of necessary independent observations; but not necessarily the number of independent geometry vectors, as it will now be shown.

The information matrix, Eq. (3.35), is still a sum of dyadic matrices, and can be expressed by Eq. (3.39) if one lets:

$$\underline{y}_k = (\Phi^i(m+N,m+k))^{-1'}\underline{h}^{ij}_{m+k}/(r^{ij}_{m+k} + \bar{r}^{ij}_{m+k})^{1/2} \qquad (3.41)$$

Even if two successive rangings are not independent:

$$\underline{h}^{ij}_{m+k+1} = \underline{h}^{ij}_{m+k}$$

still, $\underline{y}_{k+1}$ may be not proportional to $\underline{y}_k$:

$$\underline{y}_{k+1} = \Psi_{m+k+1}\, \underline{y}_k\, (r^{ij}_{m+k} + \bar{r}^{ij}_{m+k})^{1/2} \;/(r^{ij}_{m+k+1} + \bar{r}^{ij}_{m+k+1})^{1/2}$$

since $\Psi_{m+k+1} \neq I$. This happens because the state variables are correlated in time. In theory, if the transition matrix had the opportune properties, it is even possible that any set of s consecutive rangings could give observability. This will not happen in our case, however. In practice, one will have to rely on the flight path condition to achieve observability of position coordinates and clock time. After that is achieved, one can count on time correlation to achieve observability of all other state variables, since they are, typically, derivatives (or functions of derivatives) of position and clock time.

3.7 Summary of results

It has been shown in this chapter that a fixed-rank hierarchical community of ownstate filters, with a master, is exponentially stable if the members' filters are stable one by one and have bounded gains.

It is known that one-by-one stability and gain boundedness are insured by uniform complete observability and controllability. These properties, in turn, depend, substantially, only on the satisfaction of the flight path condition (apart from the usual precautions in modeling the noise covariance).

It can be concluded that a fixed-rank community, whose flight paths satisfy the condition of Section 3.5, will certainly be exponentially stable. Fixed rank hierarchy can be obtained from the present implementation of JTIDS-RelNav by just modifying (simplifying, actually) the source selection logic. The flight path condition requires, essentially, that at least the master, or at least the second-rank members, must be moving around.

CHAPTER 4

TIME-DOMAIN COVARIANCE AND STABILITY ANALYSES

4.1 Introduction

In this chapter difference equations are derived, that are useful for the time-domain analysses of error covariance and error stability in an ownstate community. They will be used in next chapter's simulations.

First, the difference for the error covariance will be derived. Although derived specifically for the errors of an ownstate community, these equations have a somewhat more general validity. With an appropriate generalization of the symbols, they could apply to any set of linearly connected linear systems driven by a stocchastic input. Then, there will be a discussion of how to make stability properties evident in the time domain; it will be shown that the time-domain test for stability is formally equivalent to a covariance simulation. This property is valid for all linear systems.

4.2 Equations for covariance analysis

The estimation error of the i-th member of an ownstate community was found, in Chapter 2, to obey these difference equations:

$$\underline{\tilde{x}}^{i}_{n-} = \Psi^{i}_{n} \underline{\tilde{x}}^{i}_{n-1} + \underline{w}^{i}_{n} \tag{4.1}$$

$$\underline{\tilde{x}}^{i}_{n} = (I-A^{ij}_{n}) \underline{\tilde{x}}^{i}_{n-} + A^{ij} \underline{\tilde{x}}^{j}_{n-} - \underline{k}^{ij}_{n} v^{ij}_{n} \tag{4.2}$$

with:

$$A^{ij}_{n} = \underline{k}^{ij}_{n} \underline{h}^{ij'}_{n} \tag{4.3}$$

Equation (4.2) shows the mutual connection of the errors. These equations are valid to the first order only; they derive from Eq. (2.15), which truncates the Taylor expansion of the innovation at the first order; this linearization is made everywhere in this work.

Define the covariance of the estimation error of member i:

$$U_n^{ii} \triangleq E\,(\underline{\tilde{x}}_n^i\, \underline{\tilde{x}}_n^{i'}) \tag{4.4}$$

and the cross-covariance of the errors of members i and k:

$$U_n^{ik} \triangleq E\,(\underline{x}_n^i\, \underline{x}_n^{k'}) \tag{4.5}$$

and notice that $U^{ik} = U^{ki'}$. The propagation between rangings is, from Eqs. (2.27), (2.12):

$$U_{n-}^{ik} = \Psi_n^i\, U_{n-1}^{ik}\, \Psi_n^{k'} + Q_n^i\, \delta_{ik} \tag{4.6}$$

in particular, for i=k:

$$U_n^{ii} = \Psi_n^i\, U_{n-1}^{ii}\, \Psi_n^{i'} + Q_n^i \tag{4.7}$$

At time t_n, the broadcaster is member j, given by Eq. (2.15). Assuming i and k are members which do accept its p-message and incorporate, respectively, the rangings ρ_n^{ij} and ρ_n^{kj}, then from Eqs. (2.25), (2.14):

$$U_n^{ik} = (I-A_n^{ij})\, U_{n-}^{ik}\, (I-A_n^{kj})' + (I-A_n^{ij})\, U_{n-}^{ij}\, A_n^{kj'} +$$

$$+ A_n^{ij}\, U_{n-}^{jk}\, (I-A_n^{kj})' + A_n^{ij}\, U_{n-}^{jj}\, A_n^{kj'} + \underline{k}_n^{ij}\underline{k}_n^{ij'}\, r_n^{ij}\, \delta_{ik} \tag{4.8}$$

If either i or k does not incorpaorate a ranging at time t_n (because it refuses the p-message from the broadcaster, or because it happens to be the broadcaster itself), then, of course, its error remains unchanged:

$$\underline{\tilde{x}}_n^i = \underline{\tilde{x}}_{n-}^i \text{ and/or } \underline{\tilde{x}}_n^k = \underline{\tilde{x}}_{n-}^k$$

The same expression can be obtained in Eq. (2.25) by formally letting $\underline{k}^{ij} = 0$ and, consequently $A^{ij} = 0$. Therefore, Eq. (4.8) is still valid provided one attributes a zero gain to ranging incorporations that do not take place.

The system of equations (4.6), (4.8) presents the recursive calculation of the actual error convenience of an ownstate community. These equations will be used in the simulation of Chapter 5 to calculate the performance of alternate organizations. Besides, inspection of these equations provides further insights into the limitations of ownstate communities.

In the particular case i = k, Eq. (4.8) becomes

$$U_n^{ii} = (I-A_n^{ij})\ U_{n-}^{ii}\ (I-A_n^{ij})' + I-A_n^{ij})\ U_{n-}^{ij}A_n^{ij'} +$$

$$+ A_n^{ij}\ U_{n-}^{ji}\ (I-A_n^{ij})' + A_n^{ij}\ U_{n-}^{jj}\ A_n^{ij'} + \underline{k}_n^{ij}\underline{k}_n^{ij'}r_n^{ij} \tag{4.9}$$

If one supposes that both member i and the source of its ranging, member j, had correct evaluations of their error covariance before the message incorporation, that is:

$$P_{n-}^i = U_{n-}^{ii} \quad , \quad P_{n-}^j = U_{n-}^{jj}$$

that the gains are Kalman and that the weight $\overline{r}^{ij}$ is computed with Eq. (2.9), then it is easy to see that:

$$U_n^{ii} = P_n^i + (I-A_n^{ij})\, U_{n-}^{ij}\, A_n^{ij'} + A_n^{ij}\, U_{n-}^{ji}\, (I-A_n^{ij}) \qquad (4.10)$$

One can see that $U^{ii} = P^i$ if there is a nonzero cross-covariance U^{ij} between i's and j's errors. The error covariance goes wrong at every incorporation of a message from a source whose error is correlated.

Now take Eq. (4.8) again and suppose that there was no prior correlation between the errors of members i, k and the source j:

$$U_{n-}^{ik} = U_{n-}^{ij} = U_{n-}^{kj} = 0$$

Then Eq. (4.8) becomes:

$$U_n^{ik} = A_n^{ij}\, U_{n-}^{jj}\, A_n^{kj'} \qquad (4.11)$$

and $U^{ik} = 0$ if $U^{jj} = 0$. A nonzero cross-covariance arises when two uncorrelated members range to the same uncorrelated source, if the source has any errors. If, later member k becomes the broadcaster, and member i takes a ranging to it, this cros-covariance will corrupt P^i according to Eq. (4.10).

Thus the remark at the end of Section 2.3 is quantitatively explained. From Eqs. (4.10) and (4.11) one sees that the neglected cross-covariance may be of the same order of magnitude as the auto-covariance. The evaluation of U^{ii} through P^i might well, therefore, be grossly mistaken.

4.3 A method for testing stability in the time domain

It is not straightforward to infer the stability properties of a system from its time-domain simulations. The definitions of stability

(Section 3.2) concern either the difference between any two forced trajectories (with equal but arbitrary input) or the difference between any free trajectory and an equilibrium state. Neither of them is visible in a single case simulation. In general, in order to verify the stability definitions, one should

(i) simulate all possible inputs;

(ii) simulate all possible initial conditions.

Both difficulties can be circumvented, in the linear case. It was shown (lemma 1 of Section 3.3) that the difference between two forced trajectories (with equal but arbitrary input) behaves like a free trajectory. This eliminates the necessity of testing all inputs: only the unforced case is needed. Consider then the unforced linear system:

$$\underline{x}_n = \Psi_n \underline{x}_{n-1} \tag{4.12}$$

its trajectory starting at $\underline{x}_m = a$ is given by:

$$\underline{\phi}(n;\underline{a}, m) = \prod_{i=m+1}^{n} \Psi_i \underline{a}$$

If $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_\ell\}$ is a basis of the linear space where the system's state is defined, then:

$$\underline{a} = \sum_{j=1}^{\ell} a_j \underline{u}_j$$

and Eq. (4.13) becomes:

$$\underline{\phi}(n; \underline{a}, m) = \left(\sum_{j=1}^{\ell} a_j \prod_{i=m+1}^{n} \Psi_i \underline{u}_j \right) =$$

$$= \sum_{j=1}^{\ell} a_j \underline{\phi} (n; \underline{u}_j, m) \tag{4.14}$$

Therefore, one needs only to test ℓ free trajectories, starting from a basis of the state space. All other free trajectories will be linear combinations of them.

These ℓ test trajectories reveal the stability properties of the linear system. If all of them converge to zero [exponentially] then the system is [uniformly] asymptotically stable. If all are bounded, the system is stable in the sense of Lyapunov. The reader can verify these statements by comparing them with the definitions given in Section 3.2.

On the contrary, simulations with nonzero input can be misleading. A bounded output, for instance, is no guarantee of asymptotic stability. If the system is asymptotically stable, a bounded input does give a bounded trajectory, but the converse is not generally true.

4.4 Formal equivalence of covariance analysis and stability test

The ℓ time-domain tests for stability can actually be reduced to only one test, using the equations for covariance analysis (Section 4.2). Although covariance analysis supposes a stochastic ensemble of trajectories, and the stability test, instead, only a finite number of trajectories, with deterministic initial conditions, yet there is a formal similarity that will now be shown.

Consider a linear stochastic discrete-time system:

$$\underline{z}_n = B_n \underline{z}_{n-1} + \underline{w}_n \tag{4.15}$$

with the usual hypotheses on $\underline{w}_n$. The covariance of the state follows:

$$U_n = B_n U_{n-1} B_n' + Q_n \tag{4.16}$$

with

$$U_n \triangleq E\,(\underline{z}_n \underline{z}_n') \;,\; Q_n \triangleq E\,(\underline{w}_n \underline{w}_n')$$

The unforced system follows:

$$\underline{z}_n = B_n \underline{z}_{n-1} \tag{4.17}$$

Suppose $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_e\}$ is a basis of the space $\mathbb{R}^e$ where the state is defined, and call $\underline{z}_n(\underline{u}_k)$ the free trajectory started by $\underline{z}_o = \underline{u}_k$. Aggregate into matrices:

$$Z_n = \left[\underline{z}_n(\underline{u}_1), \underline{z}_n(\underline{u}_2), \ldots, \underline{z}_n(\underline{u}_e)\right]$$

$$Z_o = \left[\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_e\right]$$

From eq. (4.11) and letting $S_n \triangleq Z_n Z_n'$:

$$S_n = B_n S_{n-1} B_n' \;;\; S_o = Z_o Z_n' \tag{4.18}$$

which is identical with the covariance equation (4.16) in case the input covariance Q is zero. Call z_n^{ik} the i-th element of $\underline{z}_n(\underline{u}_k)$. Then:

$$\sum_k |\underline{z}_n(\underline{u}_k)|^2 = \sum_k \sum_i z_n^{ik} z_n^{ik} = \sum_i s_n^{ii} = \text{tr } S_n \tag{4.19}$$

Therefore, tr S_n provides an excellent test for stability. If it converges to zero [exponentially] then the system is [uniformly] asymptotically stable. Besides, S_n can be found with the same algorithm that computes the covariance U_n, by letting $Q_n = 0$ and the initial U_o equal to any symmetrical positive definite matrix.

In the case of an ownstate community the state $\underline{z}$ is the aggregate error of all members:

$$\underline{z}_n' = \left[\tilde{\underline{x}}^{1'}\ \tilde{\underline{x}}^{2'} \ldots,\ \tilde{\underline{x}}^{m'}\right] \tag{4.20}$$

and the S matrix is:

$$S_n = \begin{bmatrix} U^{11} & U^{12} & \ldots & U^{1m} \\ U^{21} & U^{22} & \ldots & U^{2m} \\ \ldots & \ldots & & \ldots \\ U^{m1} & U^{m2} & \ldots & U^{mm} \end{bmatrix}_n \tag{4.21}$$

whose partitions U^{ik} follow equation (4.8), with $Q^i{}_n=0$. The error space, however, is not exactly equal to the one where eq. (4.20) is defined, because a few errors are constrained to be zero in order to eliminate the indeterminacies of the relative coordinate grid. For the initial U_o one can choose the identity matrix with zeroes instead of ones in the approp- riate diagonal places (e.g. in the simulations of chapter 5: corresponding to all errors of member 1 and to the North error of member 2).

Plots showing either the trace of S_n or the square root of one of its diagonal elements, will be presented in the next chapter. The meaning of the former has been shown. The meaning of the latter is:

$$(s_n^{ii})^{1/2} = (\sum_k (z_n^{ik})^2)^{1/2} \tag{4.20}$$

that is, the root sum square value of a given error, summed over the set of independent unitary initial conditions.

This chapter has presented the equations for the error covariance in an ownstate community; it has also shown that the equations for the covariance of a linear system can be interpreted as a test for stability as well. Both results will be used in the simulations presented in the next chapter.

CHAPTER 5

SIMULATIONS OF OWNSTATE COMMUNITIES

5.1 Introduction

This chapter has the purpose of complementing the analysis of Chapters 2 and 3 with the presentation of a few simulations. They will be used to illustrate some of the analytical results about fixed-rank hierarchies, and to show a comparison with the two other kinds of organization (democracy and covariance-based hierarchy). The simulations are based on a simplified model and must not be interpreted as a validation of the analytical results, but rather as an illustration of them. In order to make the simulations more significant, the equations derived in Chapter 4 will be used.

At the end of the chapter, conclusions will be drawn about the feasibility of ownstate communities.

5.2 Simulation features

The truth model of the simulations featured a community of four members in a two-dimensional Cartesian coordinate grid. Earth curvature was neglected, and so was the altitude dimension (for whose estimation mutual rangings are expected to be of little effect in many applications). The members were supposed to have each a dead reckoner and a passive ranging facility, including a clock. No particular modeling hypotheses were made about this hardware; the dead reckoners' indicated positions and the clocks' indicated times were supposed to be corrupted by random-walk errors.

The typical ownstate vector $\underline{x}^i$ had three elements: the East and North errors of member i's dead reckoner and the phase error of its clock, in that order. The discrete-time difference equation was eq. (2.11) with $\Psi_n^i = I$. The driving noise covariance was given the value:

$$Q_n^i = ((10 \text{ m})^2/\text{sec}) I \; (\Delta t_n \text{ sec}) \tag{5.1}$$

where Δt_n is the interval between discrete times n-1 and n. There are exceptions to eq. (5.1) which will be discussed below. The signal propagation speed was modeled as constantly equal to the speed of light in a vacuum. Clock errors were quoted in range-equivalent units. In these units 1 m is equivalent to 3.3 nanosec.

In order to avoid unpleasant nonlinear effects, the geometry vector was computed as:

$$\underline{h}_n^{ij} = \left[\frac{x^i - x^j}{d^{ij}}, \frac{y^i - y^j}{d^{ij}}, -1 \right]_n \tag{5.2}$$

where x, y, d are <u>actual</u> coordinates and distance. In practice, this feature cannot be realized; the members will compute the geometry vectors using their own and the source's best estimates of position.

The measurement noise variance was:

$$r_n^{ij} = (10 \text{ m})^2 \tag{5.3}$$

and the source uncertainty weight $\bar{r}_n^{ij}$ was computed with eq. (2.9).

The initial error covariance was given the value:

$$P_o^i = (1 \text{ Km})^2 \, I \tag{5.4}$$

Again, see below for exceptions. In runs where a sample of error was simulated, its initial values were plus or minus 1 or 2 Km.

The four members were supposed to broadcast at intervals of $\Delta t = 3$ sec, and in order of their numbering (first member 1, then member 2, then 3, then 4, then 1 again, and so on). Consequently, the schedule function, eq. (2.15), was:

$$j = n \bmod 4 \qquad (5.5)$$

The indeterminacies of the relative coordinate grid were eliminated by choosing member 1 as both time master and navigation controller, and member 2 as end-of-baseline. For this purpose, all the errors of member 1 and the North error of member 2 were set identically to zero. That is to say, the following exceptions to eqs. (5.1) and (5.4) were made:

$$Q_n^1 = P_o^1 = 0 \qquad (5.6)$$

$$Q_n^2 = \text{diag}\ (100\ \Delta t,\ 0,\ 100\ \Delta t)\ m^2 \qquad (5.7)$$

(with Δt in seconds)

$$P_o^2 = \text{diag}\ (1,\ 0,\ 1)\ Km^2 \qquad (5.8)$$

Besides modeling the operation of the ownstate filters, the simulation also computed the actual covariance of the error. This was done, in accordance with the analysis of sect. 4.2, with eqs. (4.6), (4.8). The reader will soon see how much the actual error covariance may differ from the filter-computed one, which follows eqs. (2.20) to (2.22).

The plots that follow show:

(i) a sample of error, labelled e;

(ii) plus and minus the filter-computed standard deviation of the error, labelled σ_c;

(iii) plus and minus the actual standard deviation of the error, labelled σ_a.

Abscissae are in seconds. The ordinate unit is 1 Km for position errors and 1 Km range equivalent (about 3.3 μsec) for clock phase errors.

5.3 Simulations of a static community with a fixed-rank hierarchy

The first point to be shown is the effect of the flight path condition, sect. 3.6. In a two-dimensional problem, and in view of the lemma of sect. 2.2, that condition becomes: there must exist an integer N such that every set of N consecutive rangings contains three different lines of sight, which do not become in time indefinitely close.

An obvious way of not meeting this condition is to have a fixed-rank community with no relative motion. Such is the case presented in fig. (5.1) to (5.3). The geometrical arrangement consists of four stations sitting at the corners of a 20 Km square (like in the initial geometry of fig. (5.11)). The ranks are distributed as follows. Member 1, master and controller, has first rank (as required by hyp. (ii) of theorem 1, sect. 3.4); member 2, end-of-baseline, has second rank; both members 3 and 4 have third rank.

The figures, plotting the errors of members 2, 3 and 4 respectively, bring clearly forth the consequences of the lack of observability. Member 2 ranges only to the master. The master has no errors; therefore member 2's filter model is correct, and its computed and actual error covariances

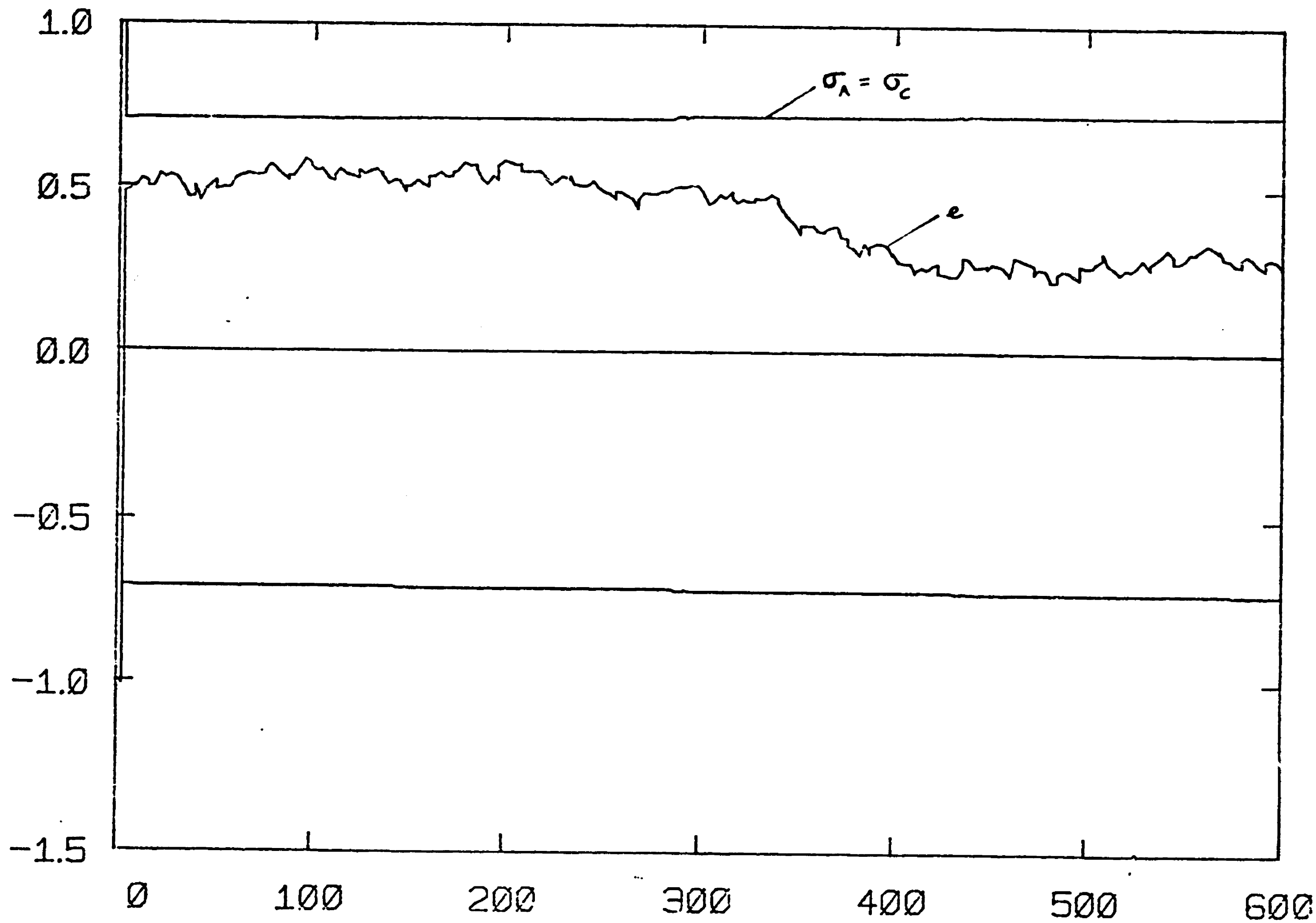


Figure 5.1(a). Static community. Fixed-rank hierarchy. East error of member 2.

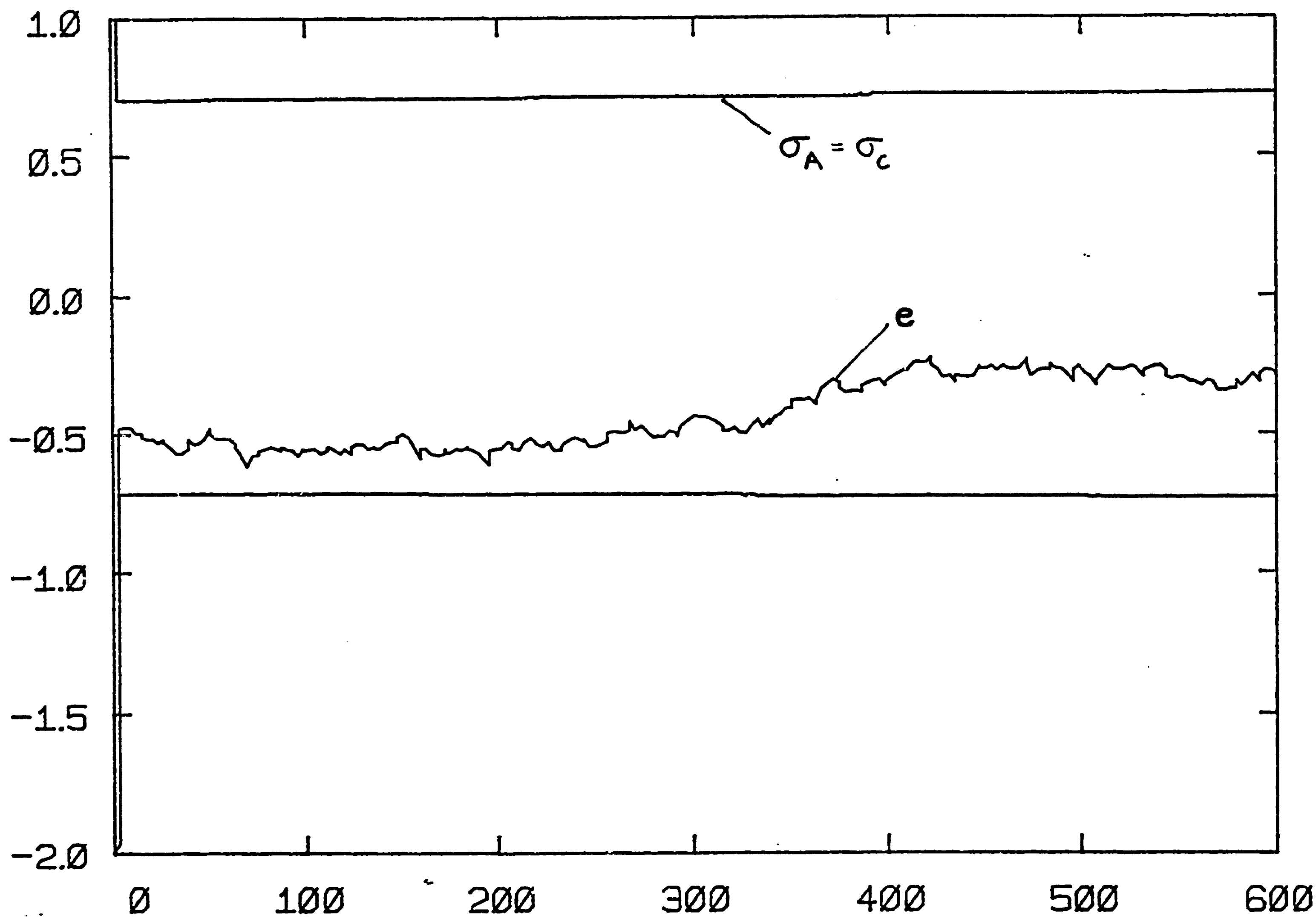


Figure 5.1(b). Static community. Fixed-rank hierarchy. Clock error of member 2.

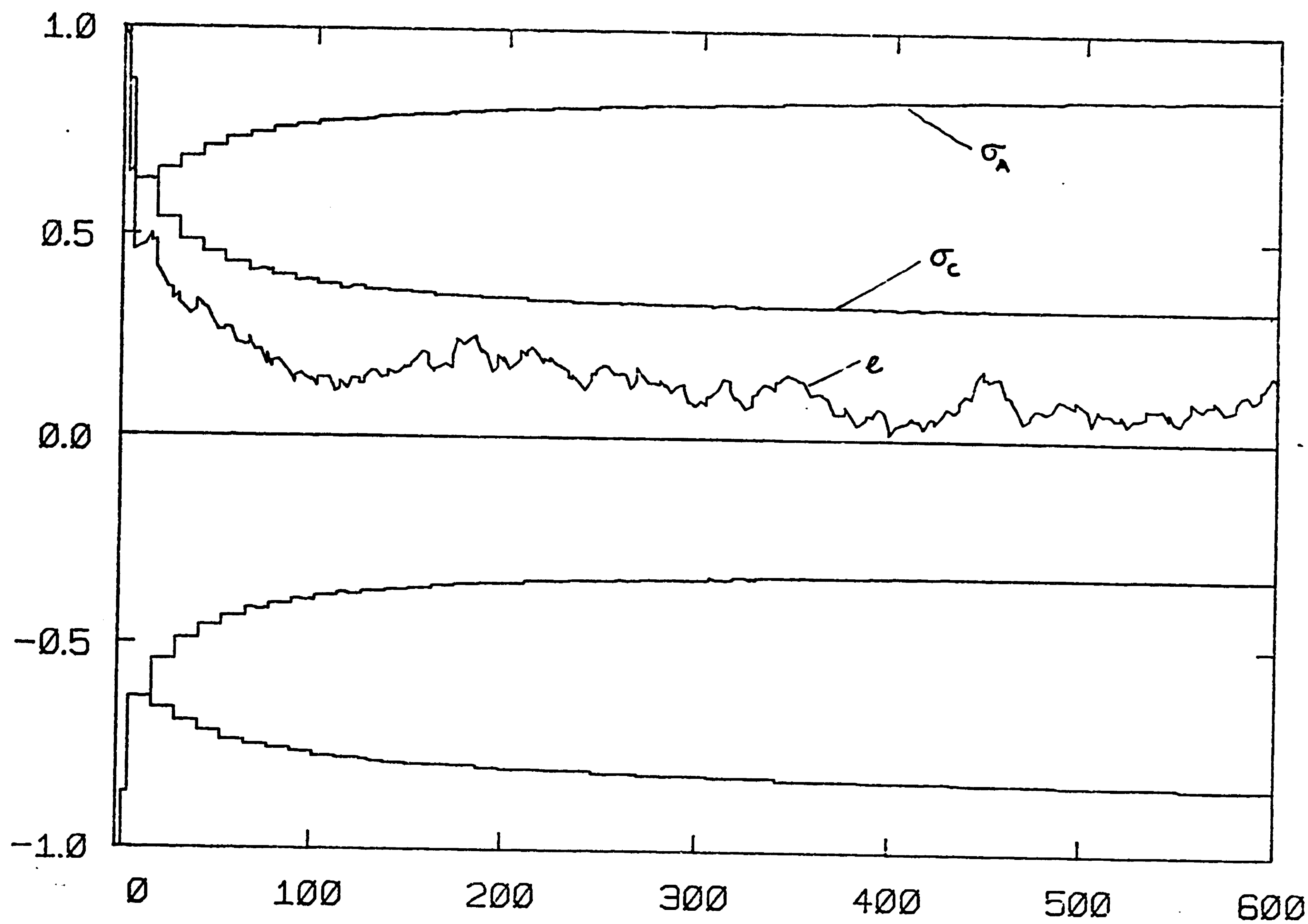


Figure 5.2(a). Static community. Fixed-rank hierarchy. East error of member 3.

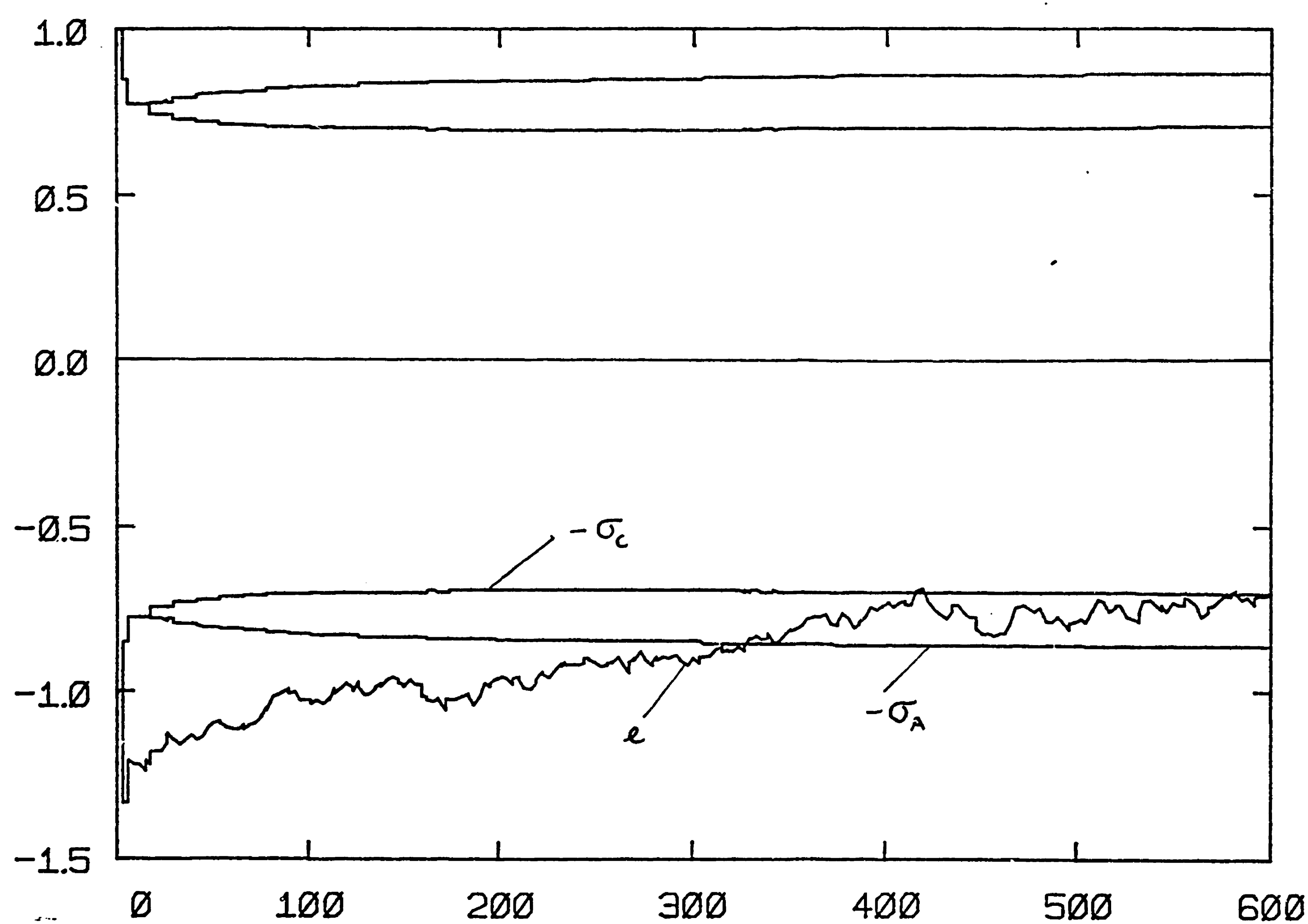


Figure 5.2(b). Static community. Fixed-rank hierarchy. North error of member 3.

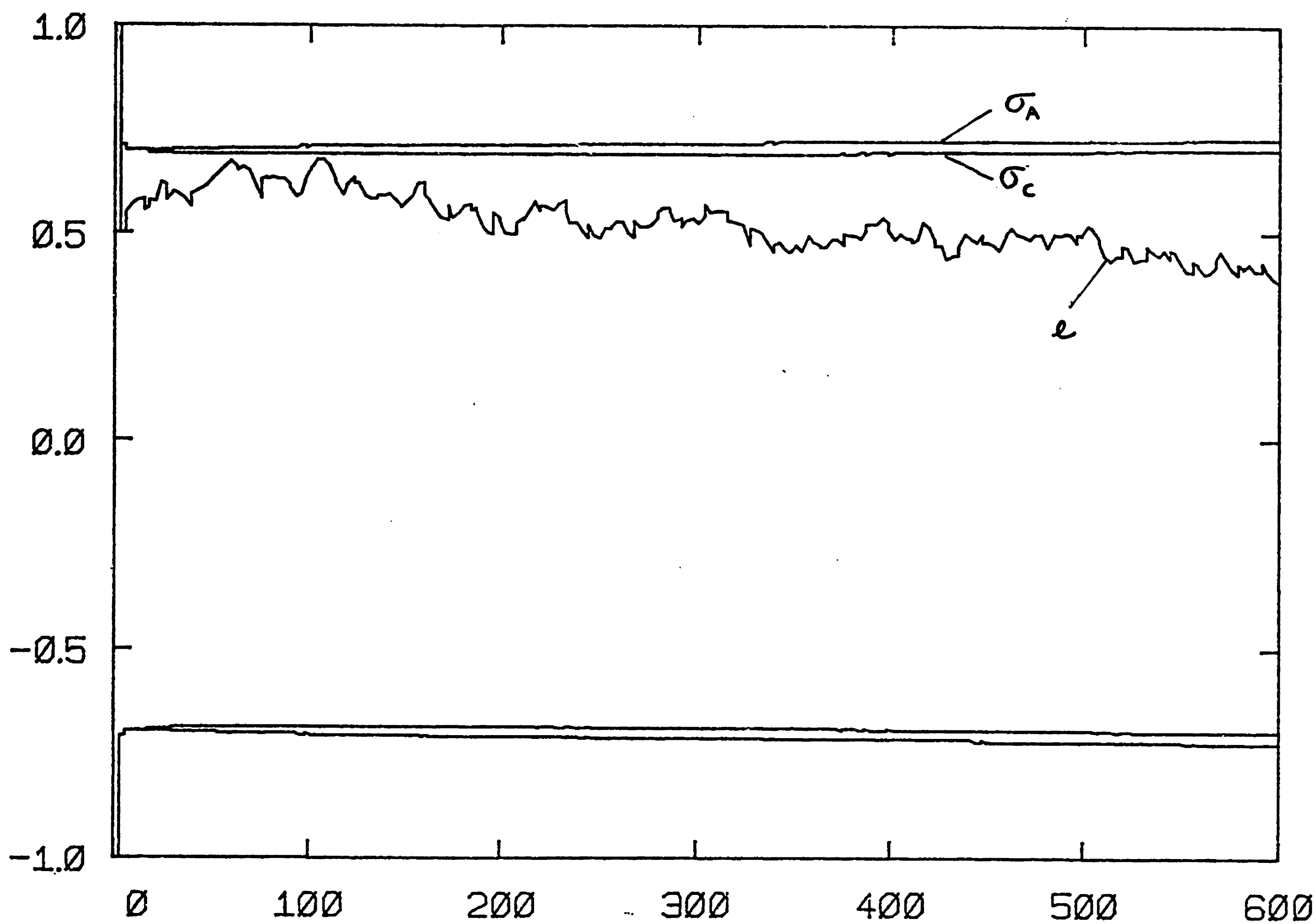


Figure 5.2(c). Static community. Fixed-rank hierarchy. Clock error of member 3.

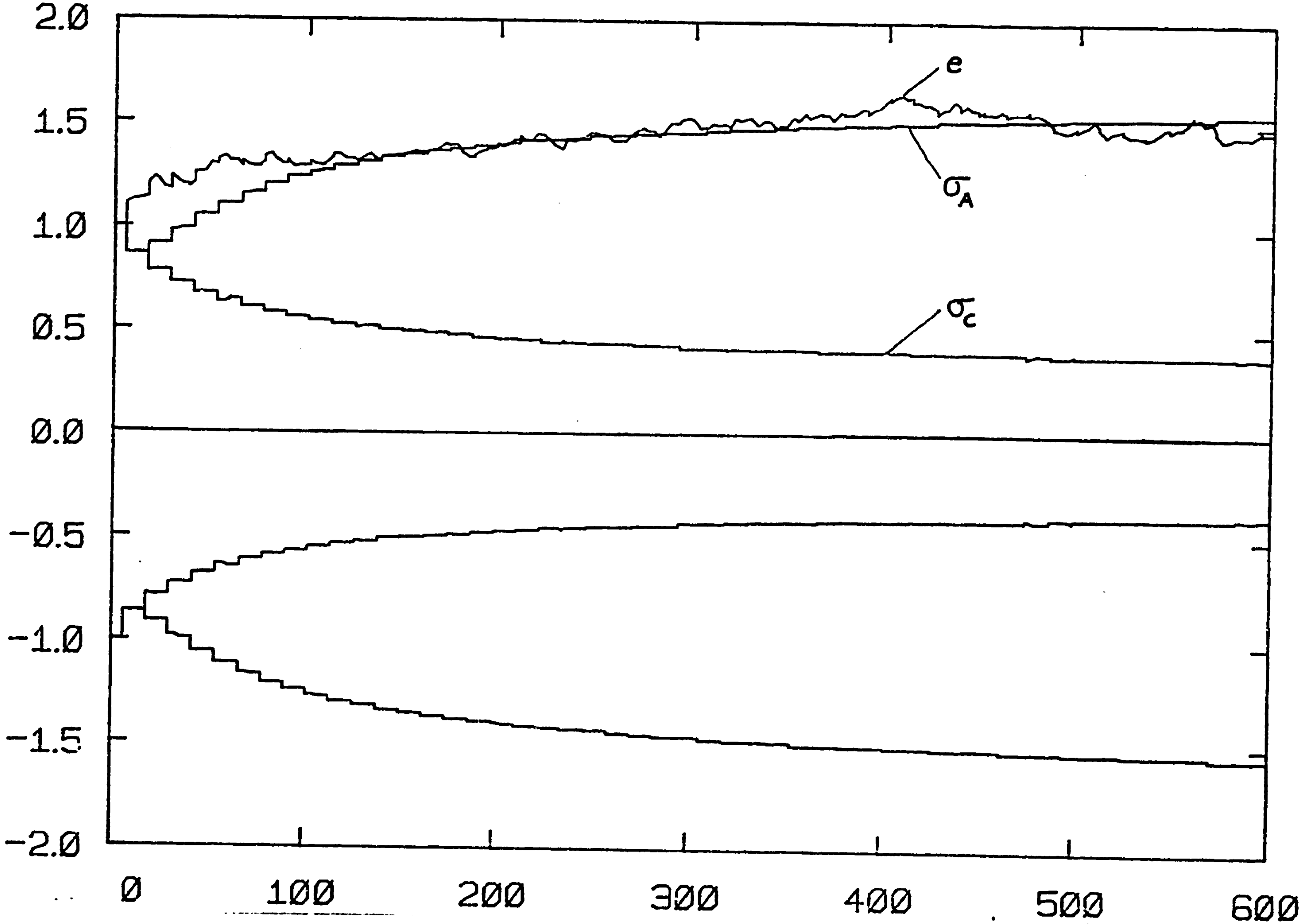


Figure 5.3(a). Static community. Fixed-rank hierarchy. East error of member 4.

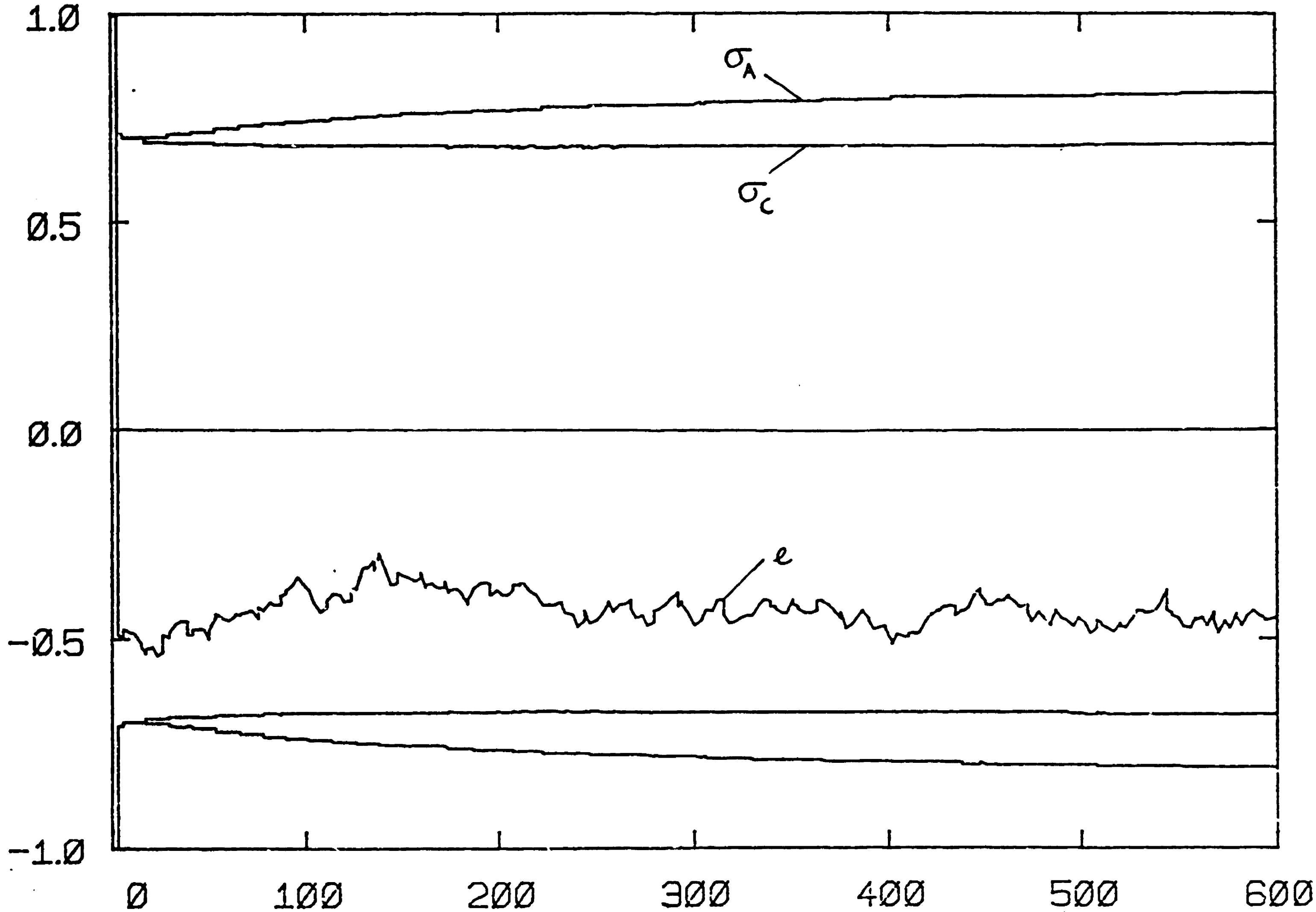


Figure 5.3(b). Static community. Fixed-rank hierarchy. North error of member 4.

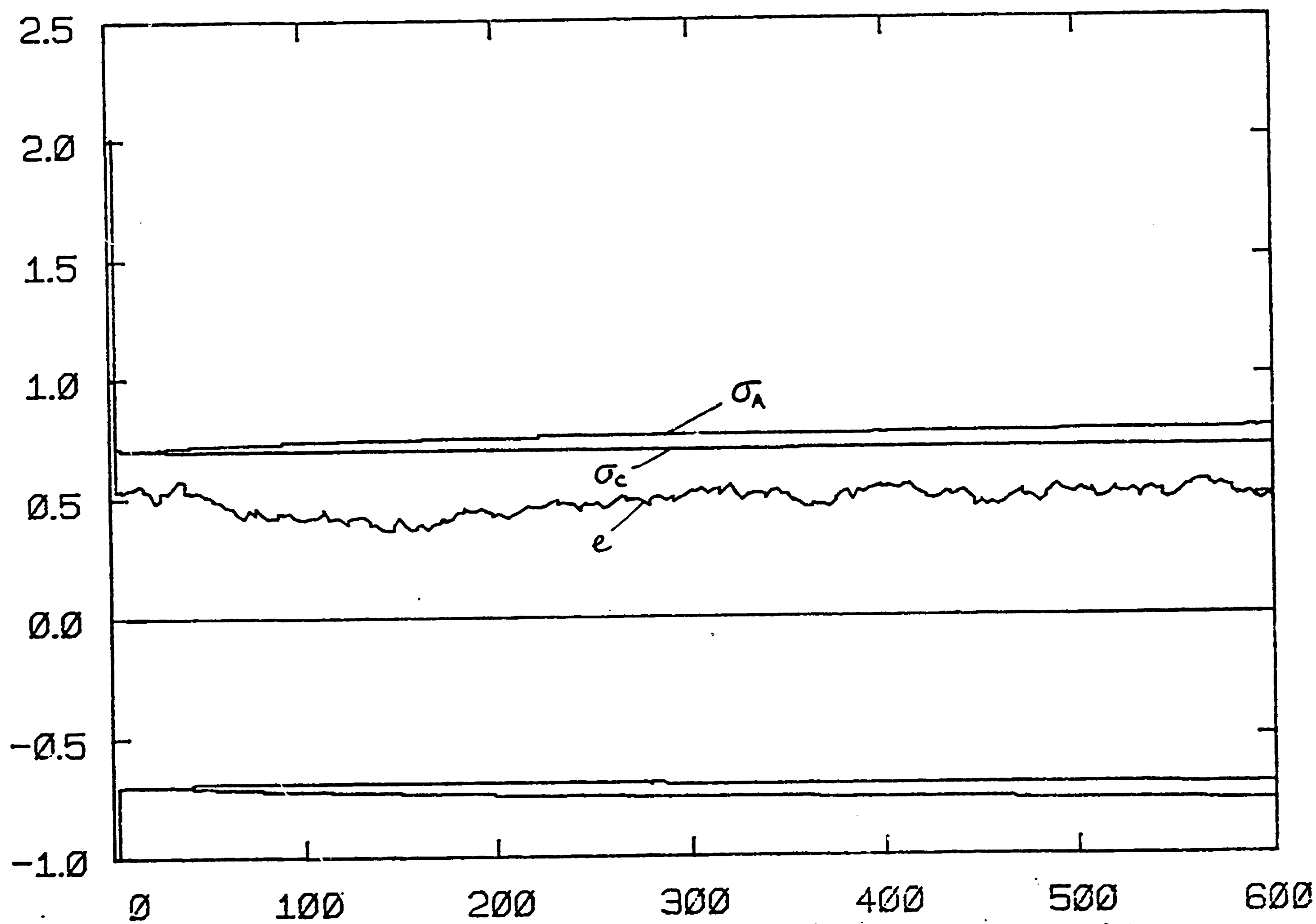


Figure 5.3(c). Static community. Fixed-rank hierarchy. Clock error of member 4.

agree. The first ranging correlates almost completely its East and clock errors. The following rangings, all taken along the same line of sight as the first, could, at most, improve this correlation, but cannot separate the East and clock errors. Actually, since the measurement noise is pretty small, there is no noticeable improvement after the first ranging.

Members 3 and 4, too, get from the master almost all the information they can get in their first ranging to it. In addition, they also range to member 2, and here an interesting effect shows. Member 2 has nonzero errors. Members 3 and 4 interpret them as a measurement noise. The filters of members 3 and 4 average this supposed noise and, therefore, their computed covariance goes down at every ranging to member 2, until a steady state is reached. What happens really is that only the first ranging of member 3 to member 2 is beneficial; the following ones increase the actual error covariance. As for member 4, not even its first ranging to member 2 is beneficial; the actual error covariance increases at every ranging.

5.4 Static communities with other organizations

A democratic community and a covariance-based hierarchical community were simulated under the same conditions as in sect. 5.3.

The democratic community, it must be remarked, does have one-by-one observability, since every member ranges to the other three but this does not guarantee community observability or stability. Figures 5.4 to 5.6, representing the error of member's 2, 3 and 4 show large oscillations, with increasing amplitude and a period of about 420 sec (or 35 rounds of broadcast); σ_a follows the oscillations of the error sample faithfully (with doubled frequency, of course). The filters are unaware of the

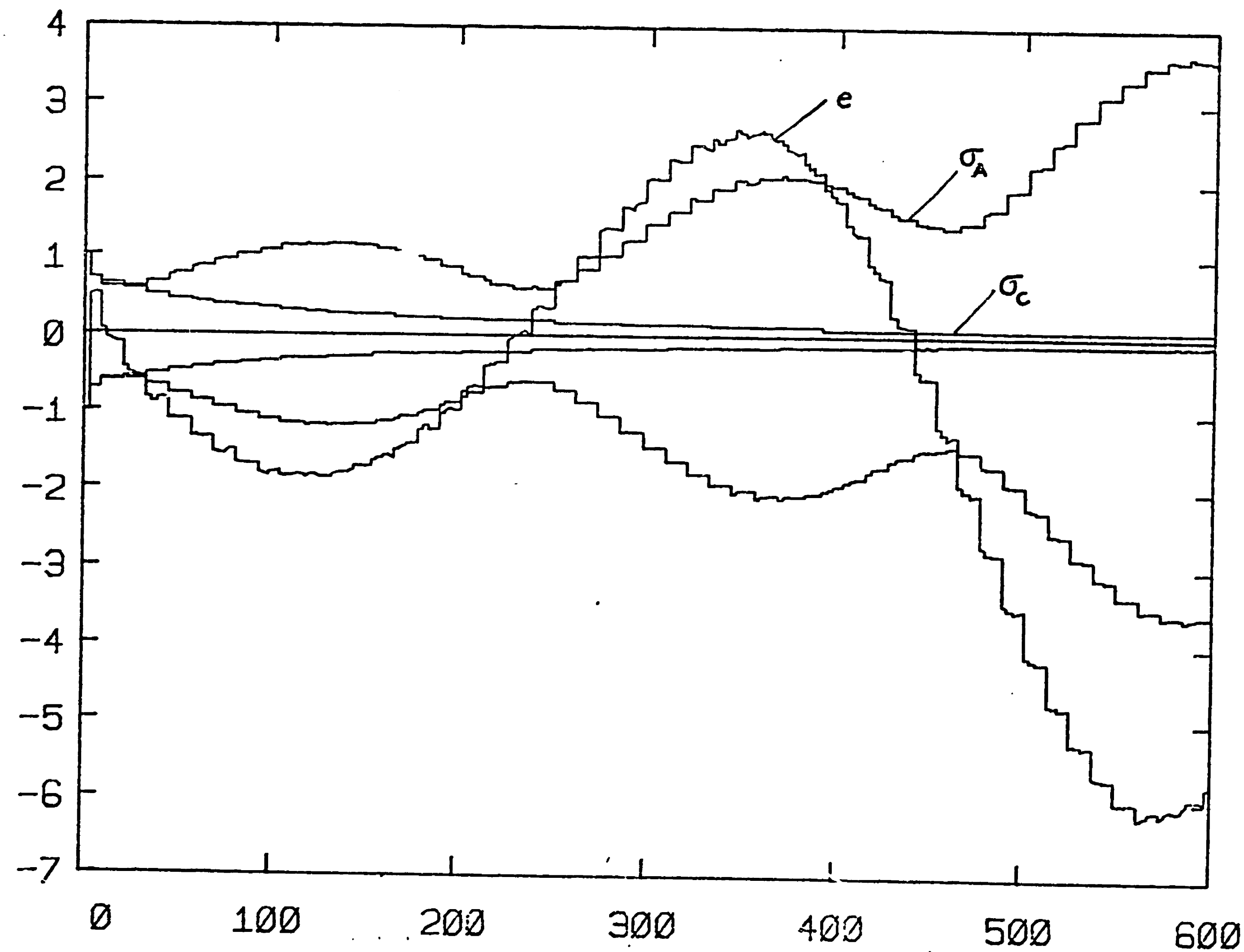


Figure 5.4(a). Static community. Democracy. East error of member 2.

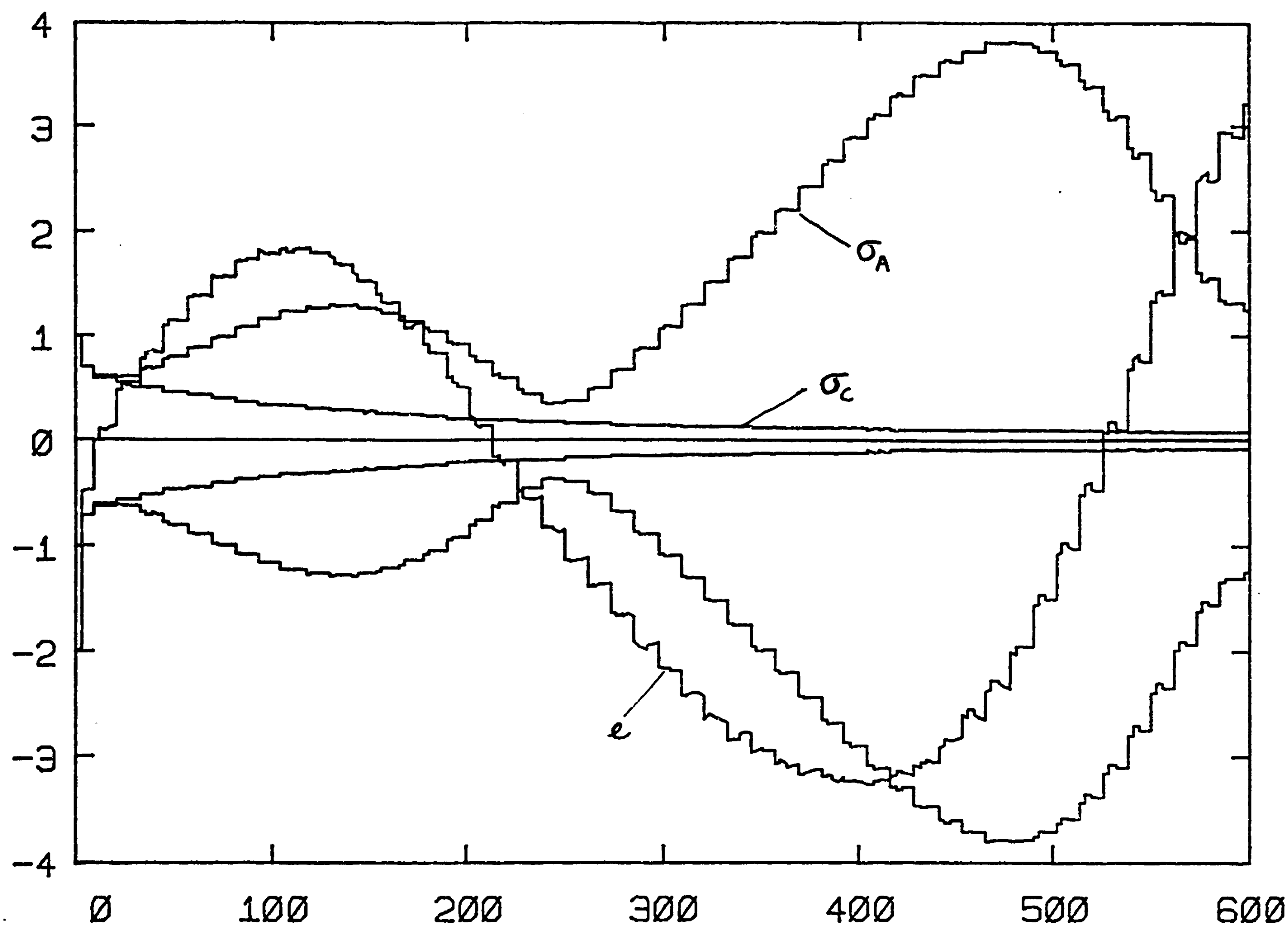


Figure 4(b). Static community. Democracy. Clock error of member 2.

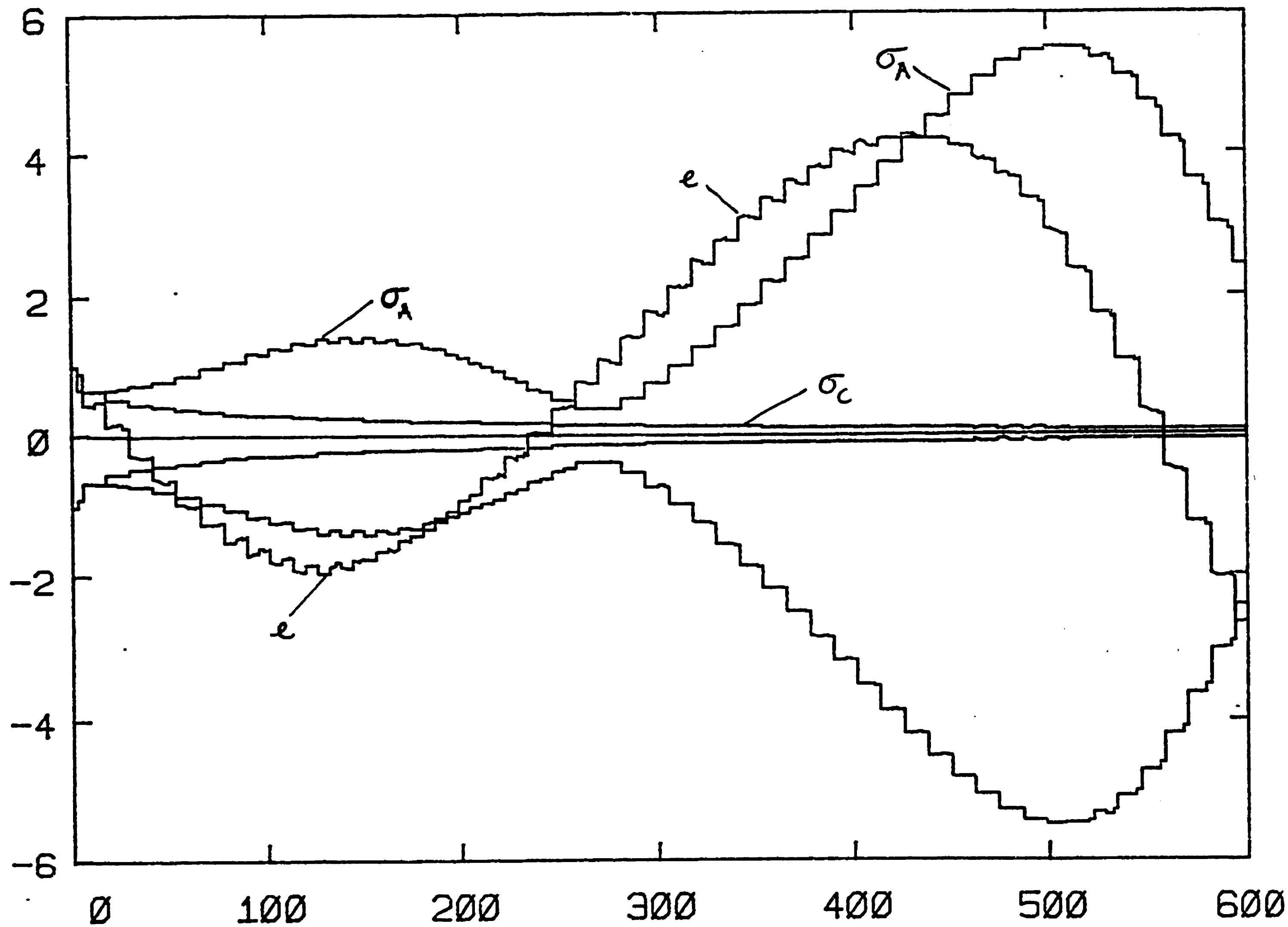


Figure 5.5(a). Static community. Democracy. East error of member 3.

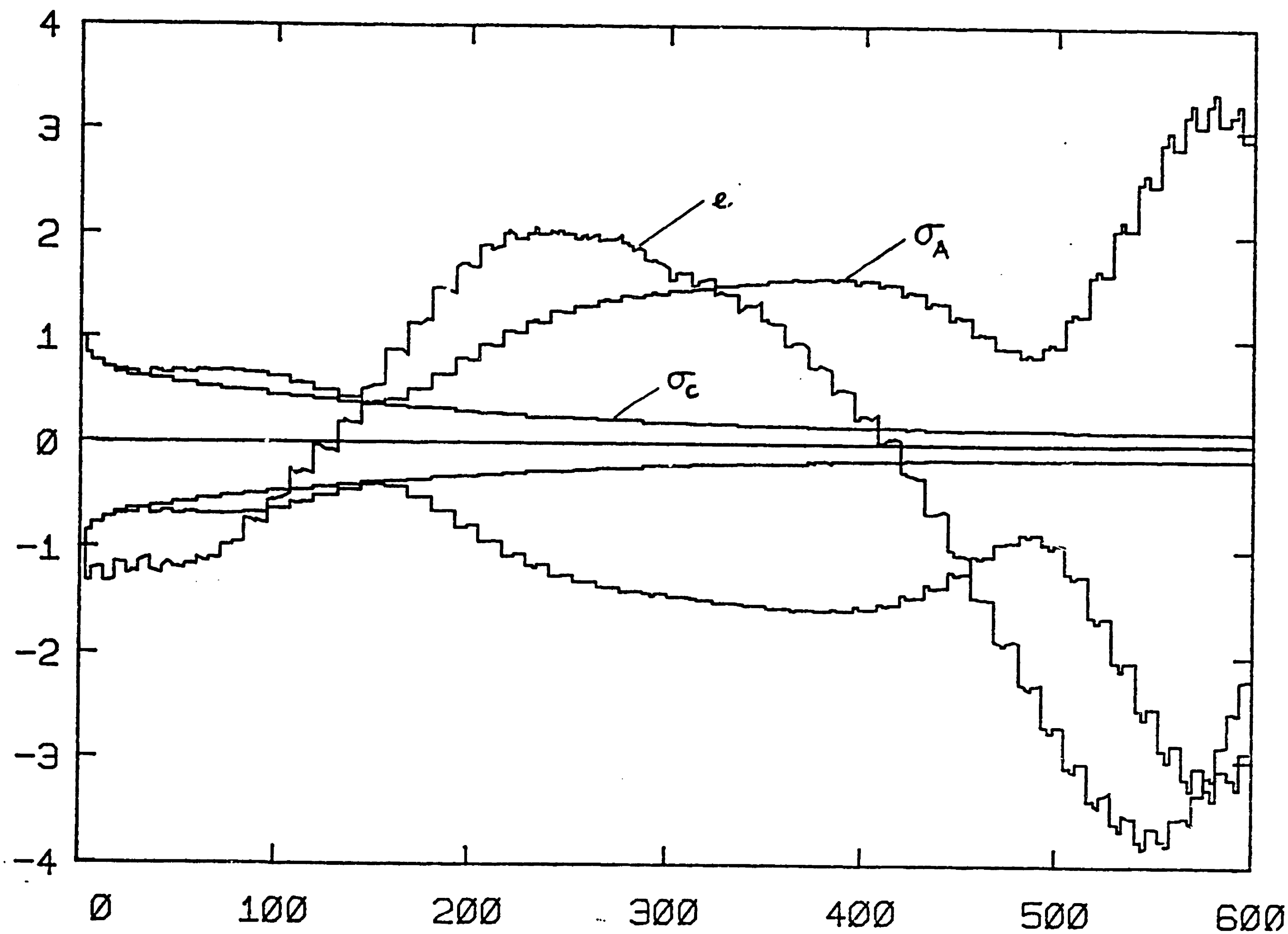


Figure 5.5(b). Static community. Democracy. North error of member 2.

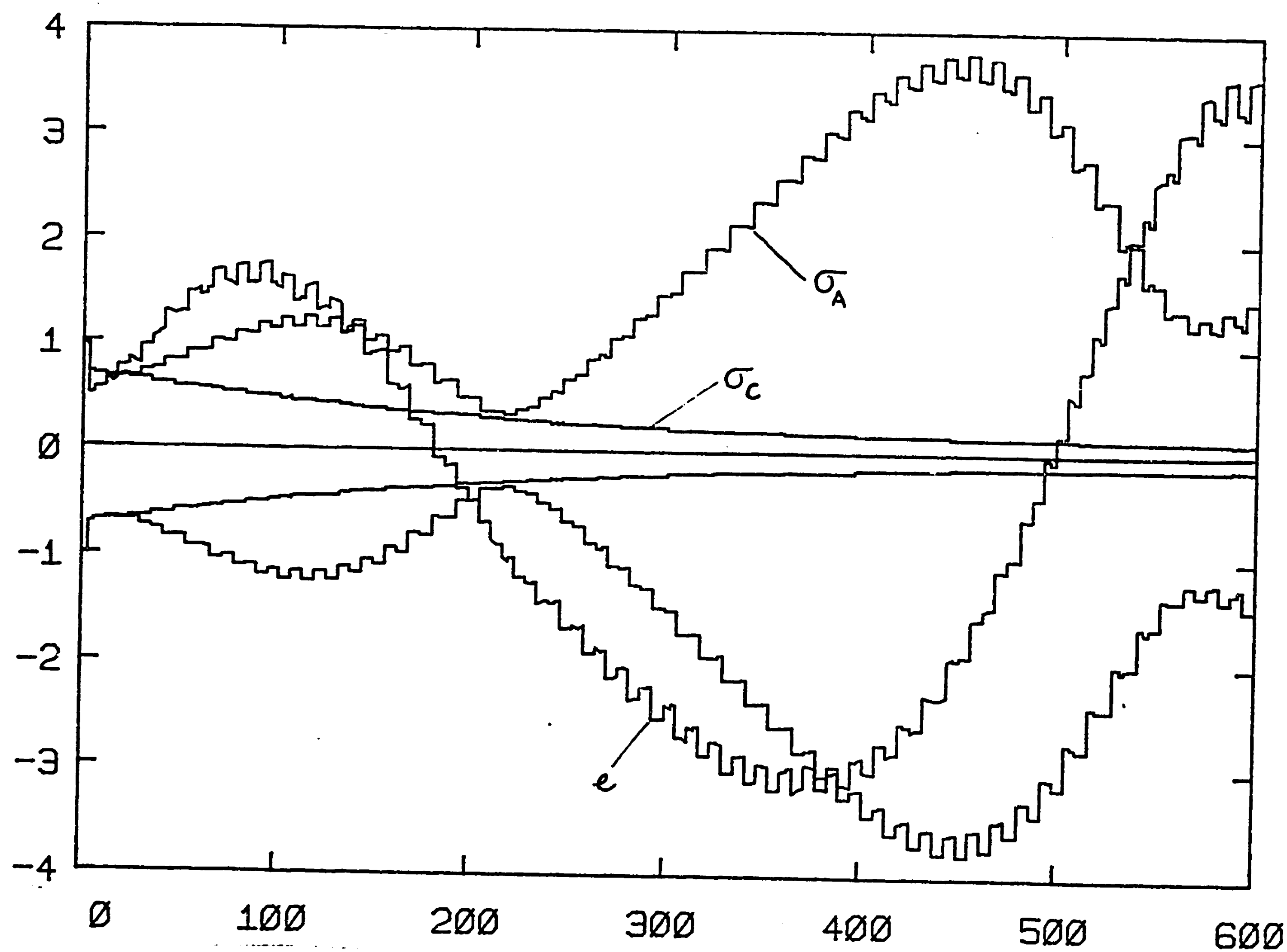


Figure 5.5(c). Static community. Democracy. Clock error of member 3.

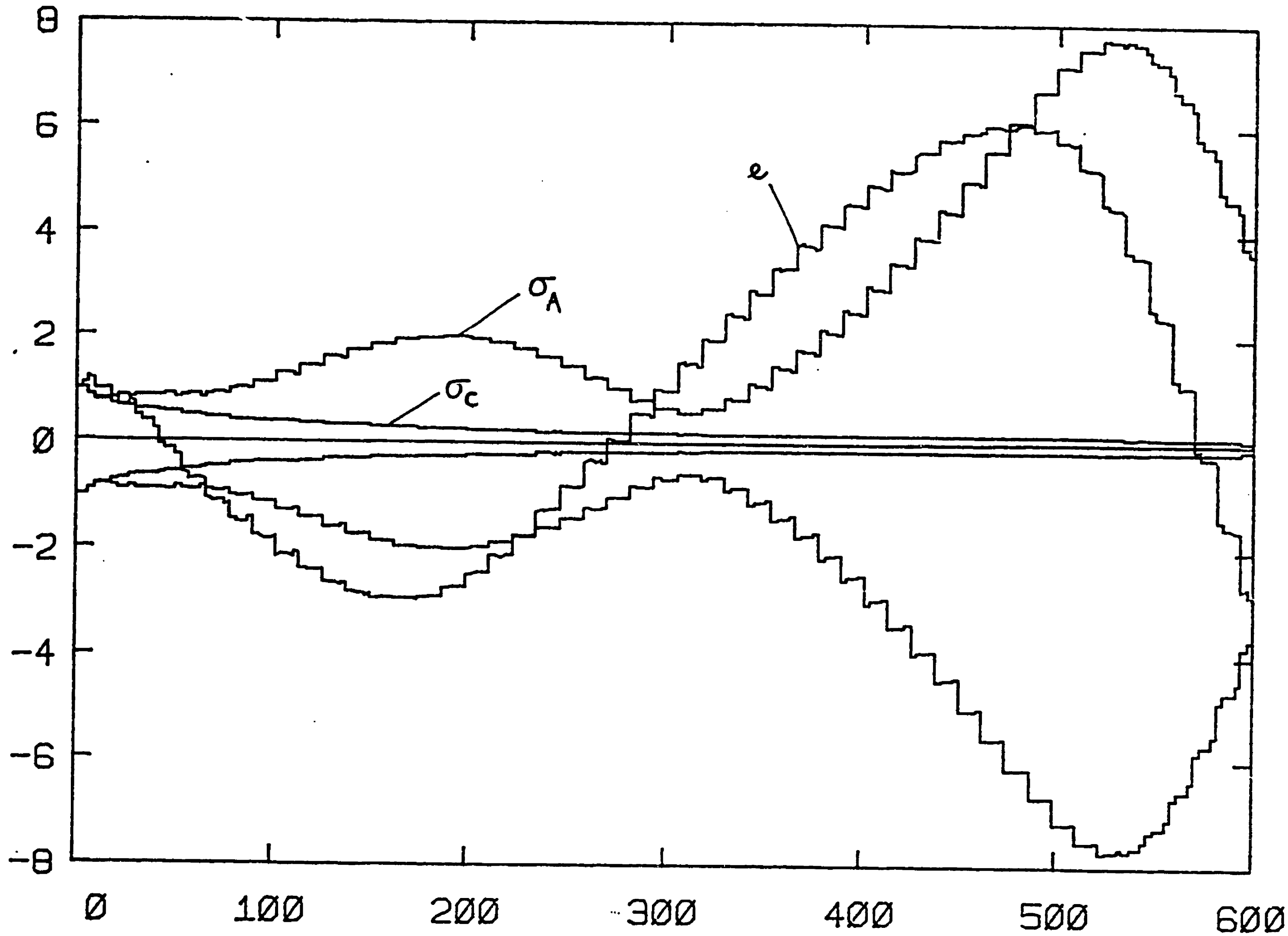


Figure 5.6(a). Static community. Democracy. East error of member 4.

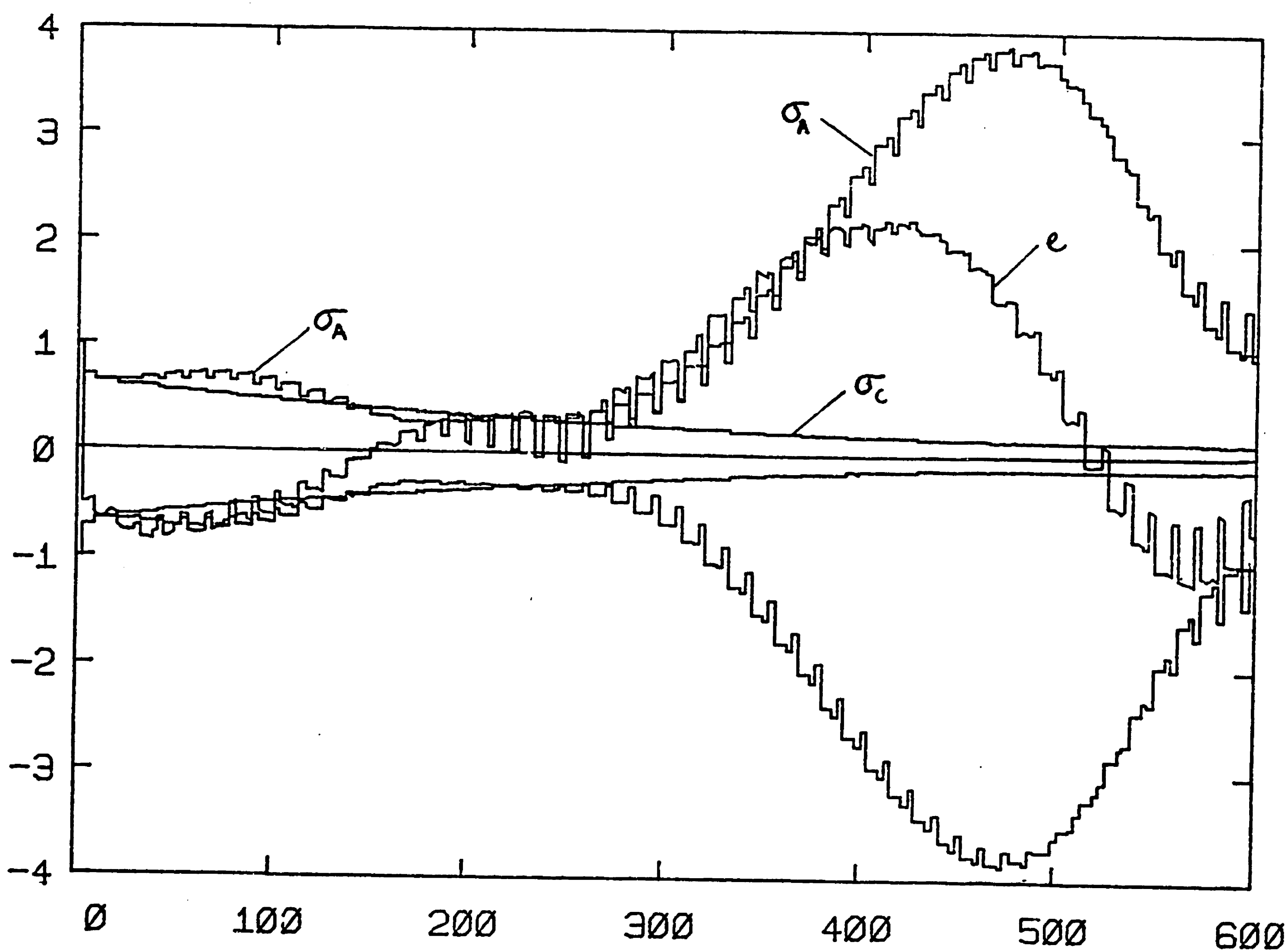


Figure 5.6(b). Static community. Democracy. North error of member 4.

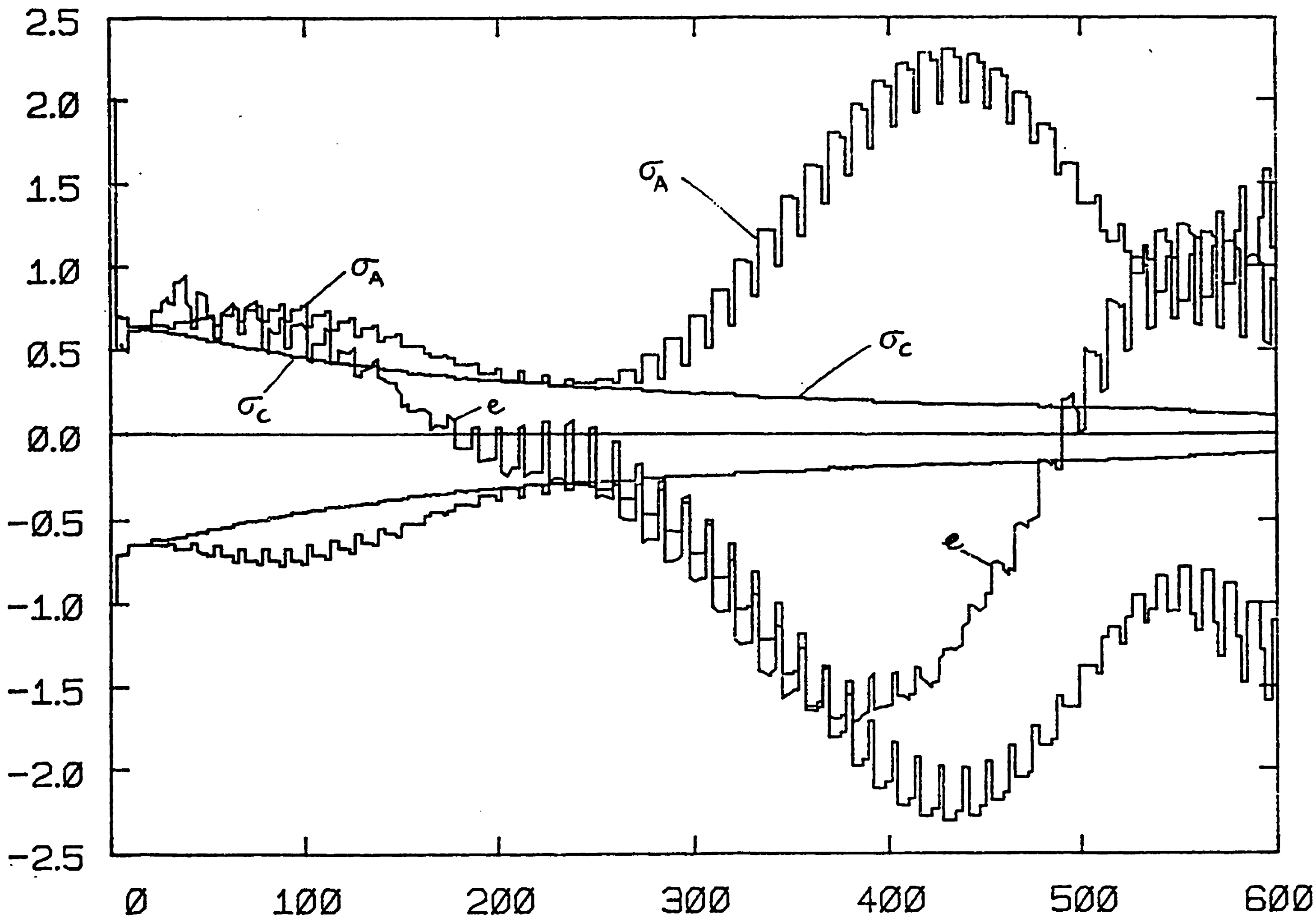


Figure 5.6(c). Static community. Democracy. Clock error of member 4.

oscillatory pattern and exhibit instead, a steadily decreasing computed covariance. This is the kind of behavior other authors have reported for the errors of a democratic organization.

A covariance-based hierarchical community is an interesting case, because one does not know, a priori, which rangings will be accepted and which will be refused, and, therefore, whether there will be one-by-one observability or not. The source selection logic used in the simulation consisted in comparing the trace of the receiver's error covariance matrix with the trace of the source's. The ranging was incorporated if:

$$\mathrm{tr}\ P_n^i > \mathrm{tr}\ P_n^j$$

i and j being the indices of receiver and source, respectively.

Figure 5.7 plots the East error of member 2 for the first 5 minutes. Member 2 usually refuses rangings other than to the master, and has, therefore, no observability. Alterations of this pattern occur occasionally at t = 45 sec, 129 sec and 225 sec, when member 2 agrees to range to member 3; the first of these rangings is beneficial, the others make the error worse. Between these occasional rank reversals, the behavior is quite similar to that of a fixed-rank hierarchy (fig. 5.1).

Now look at fig. 5.8, which plots the same error over a duration of 15 minutes. The sparse rank reversals (i.e. member 2's ranging to members 3 or 4) give rise to an unmistakably oscillatory pattern that looks like the one of fig. 5.4 (democratic organization).

Oscillatory behavior is also evident in the case of members 3 and 4, whose errors are plotted in fig. 5.9 and 5.10, respectively. Member 4 usually ranges to member 3 (and to the others); member 3 ranges to member 4, infrequently (every 40 or 50 seconds).

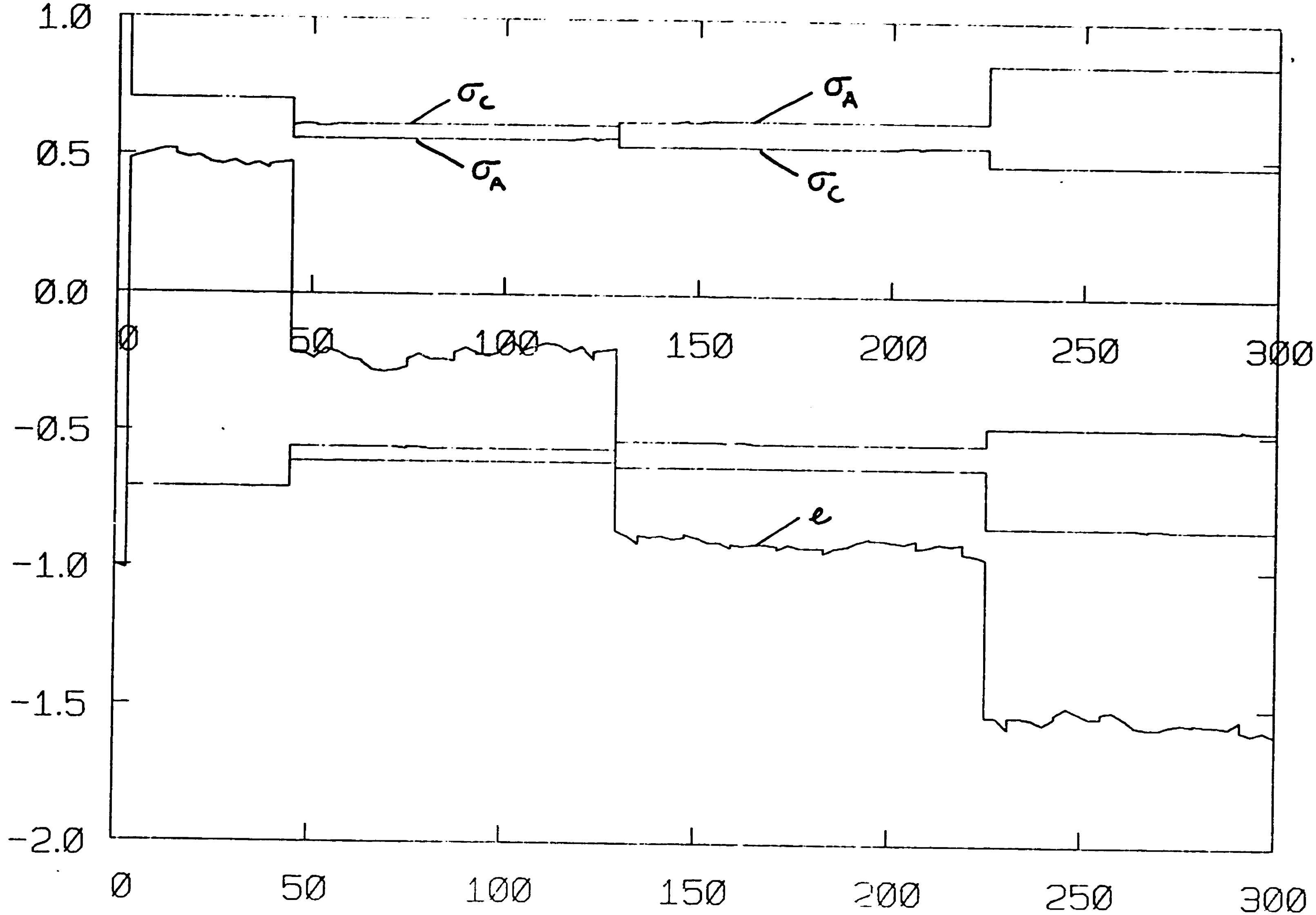


Figure 5.7. Static Community. Covariance-based hierarchy. East error of member 2.

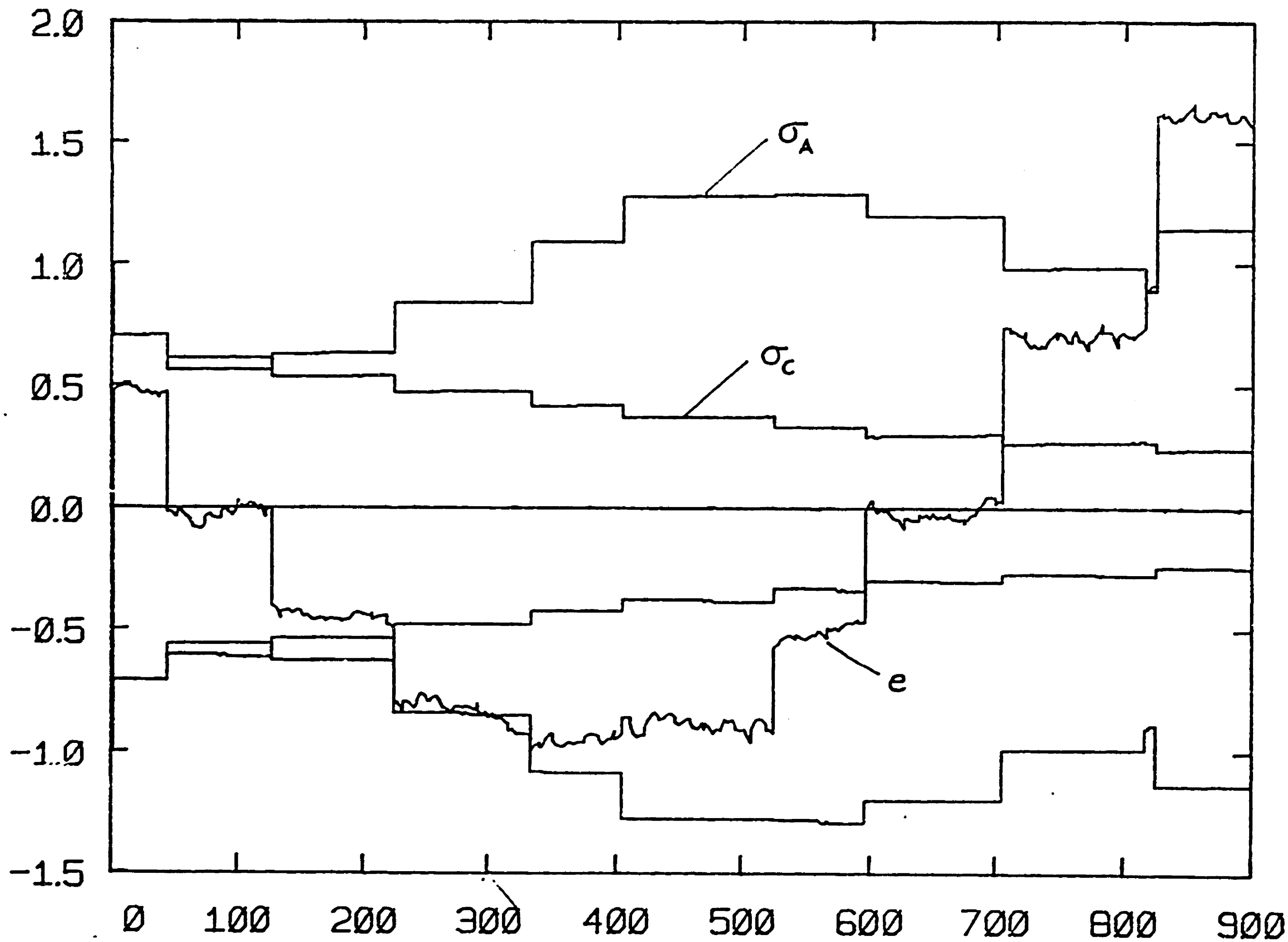


Figure 5.8(a). Static community. Covariance-based hierarchy. East error of member 2.

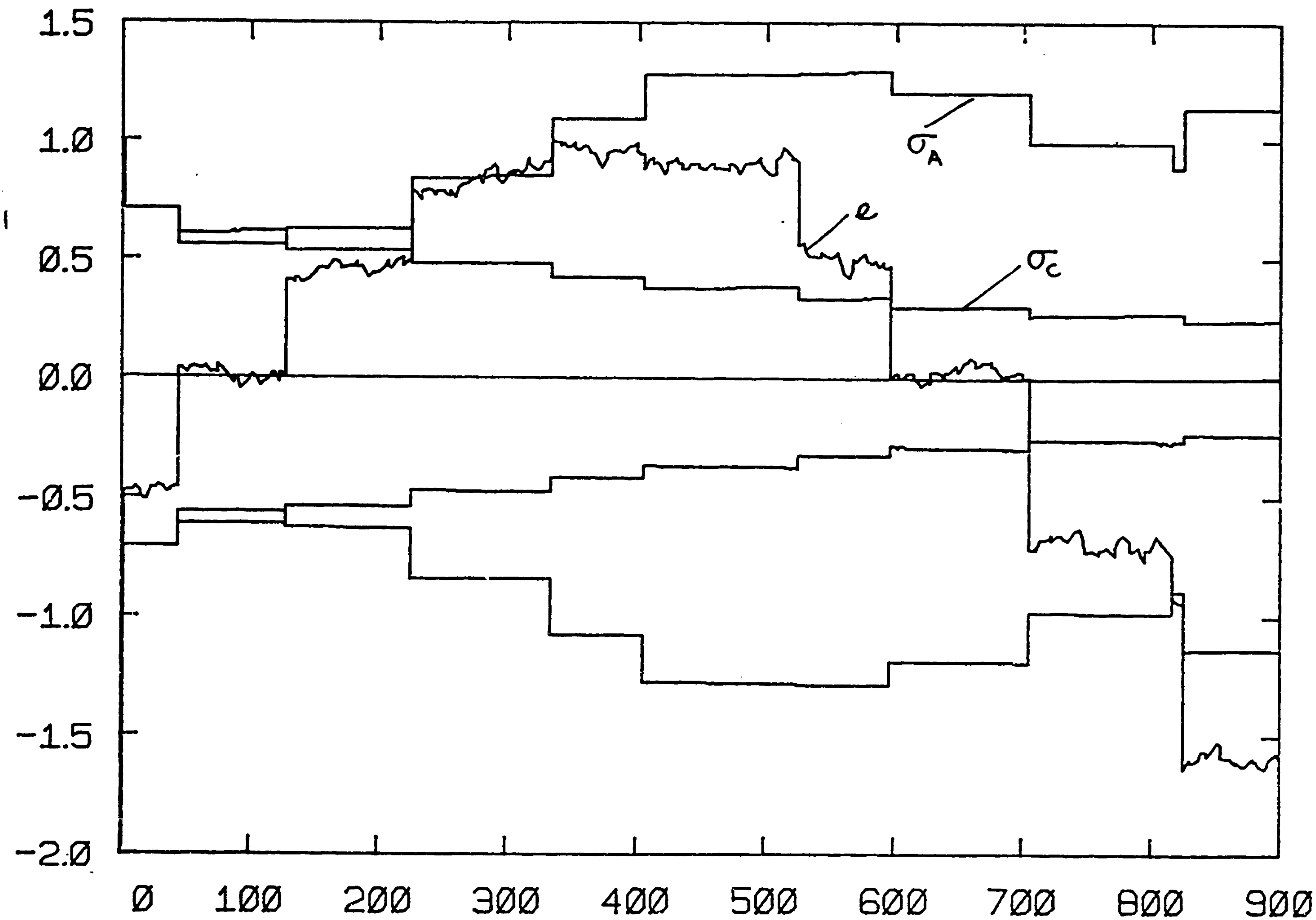


Figure 5.8(b). Static community. Covariance-based hierarchy. Clock error of member 2.

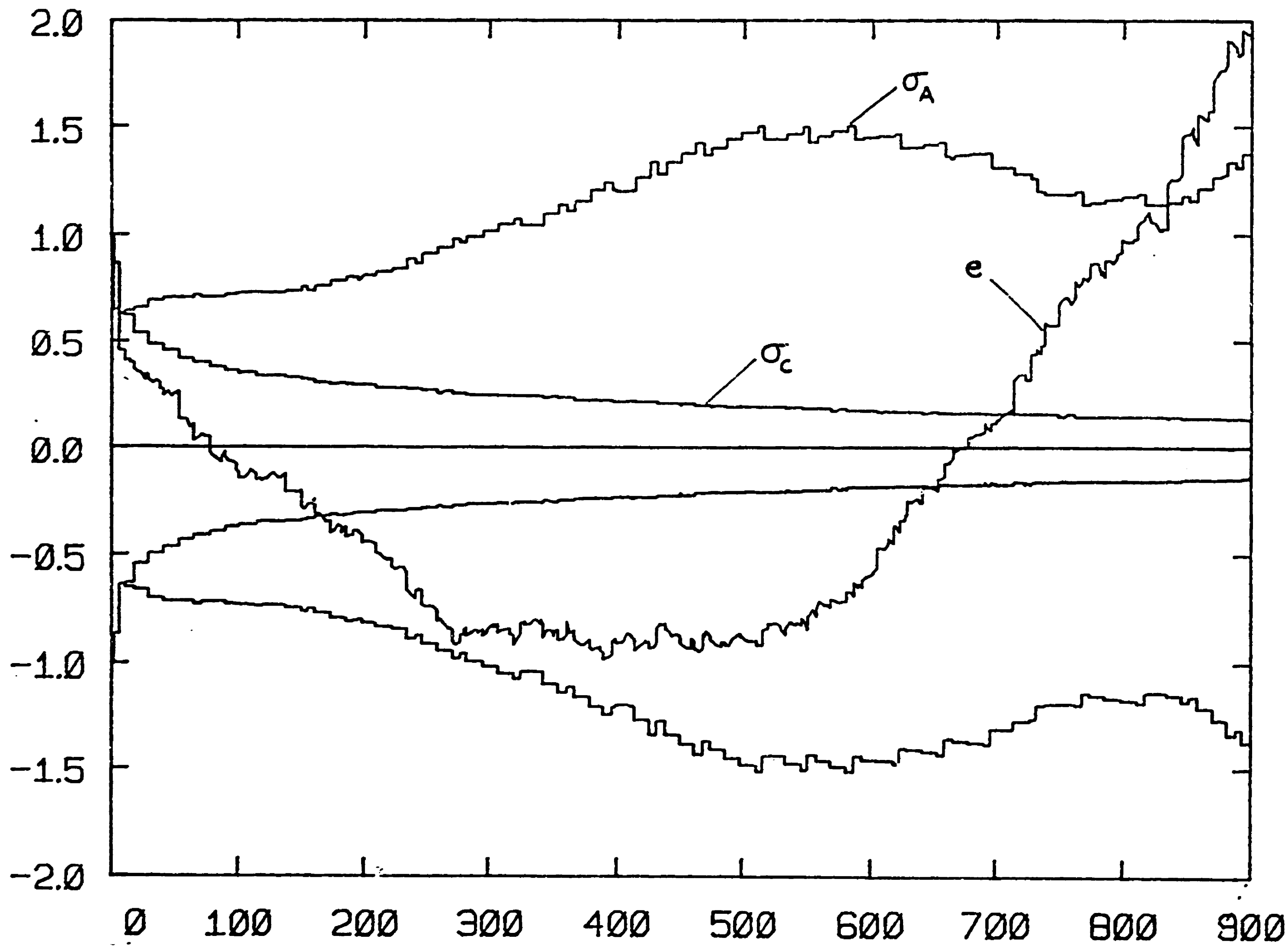


Figure 5.9(a). Static community. Covariance-based hierarchy. East error of member 3.

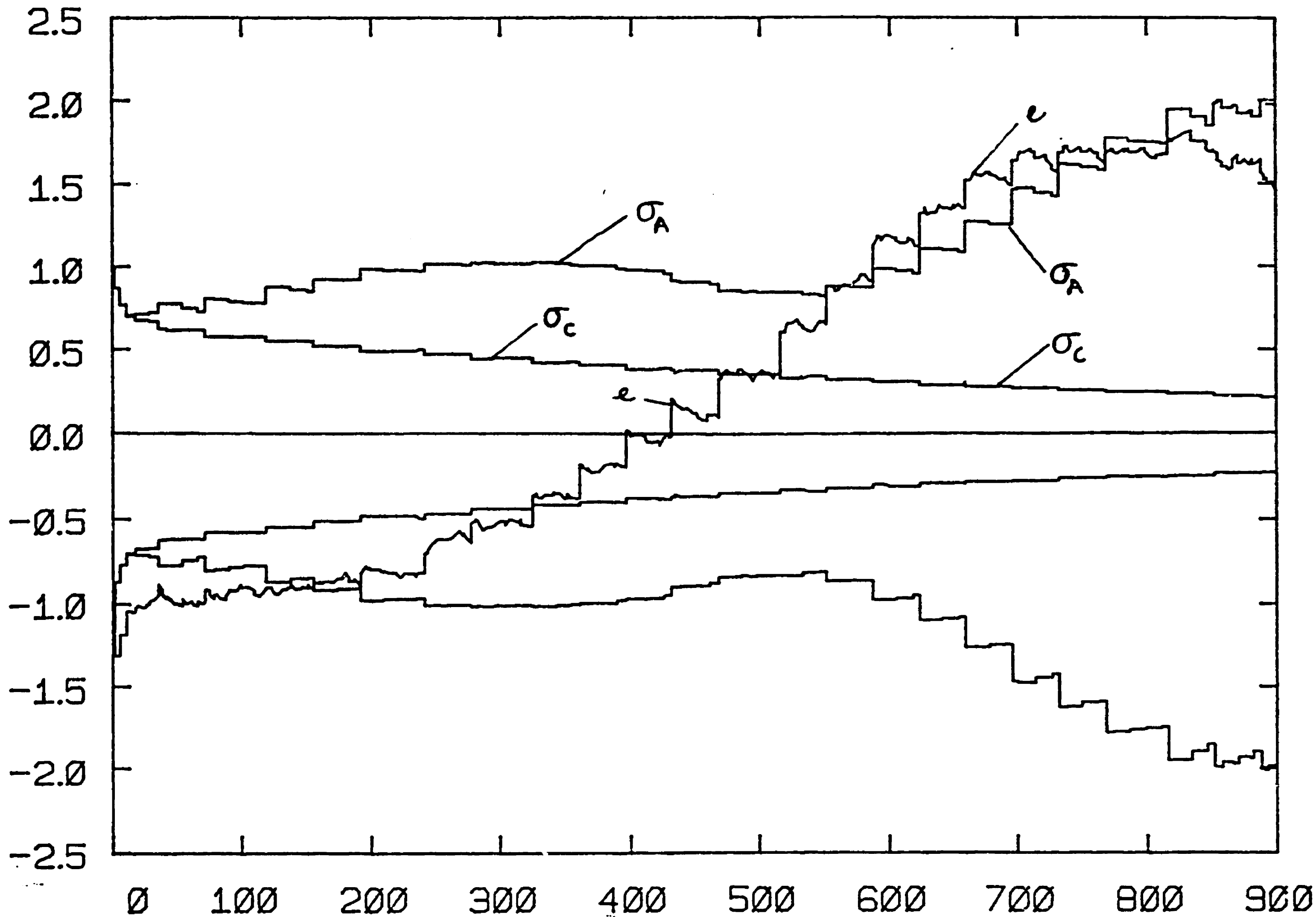


Figure 5.9(b). Static community. Covariance-based hierarchy. North error of member 3.

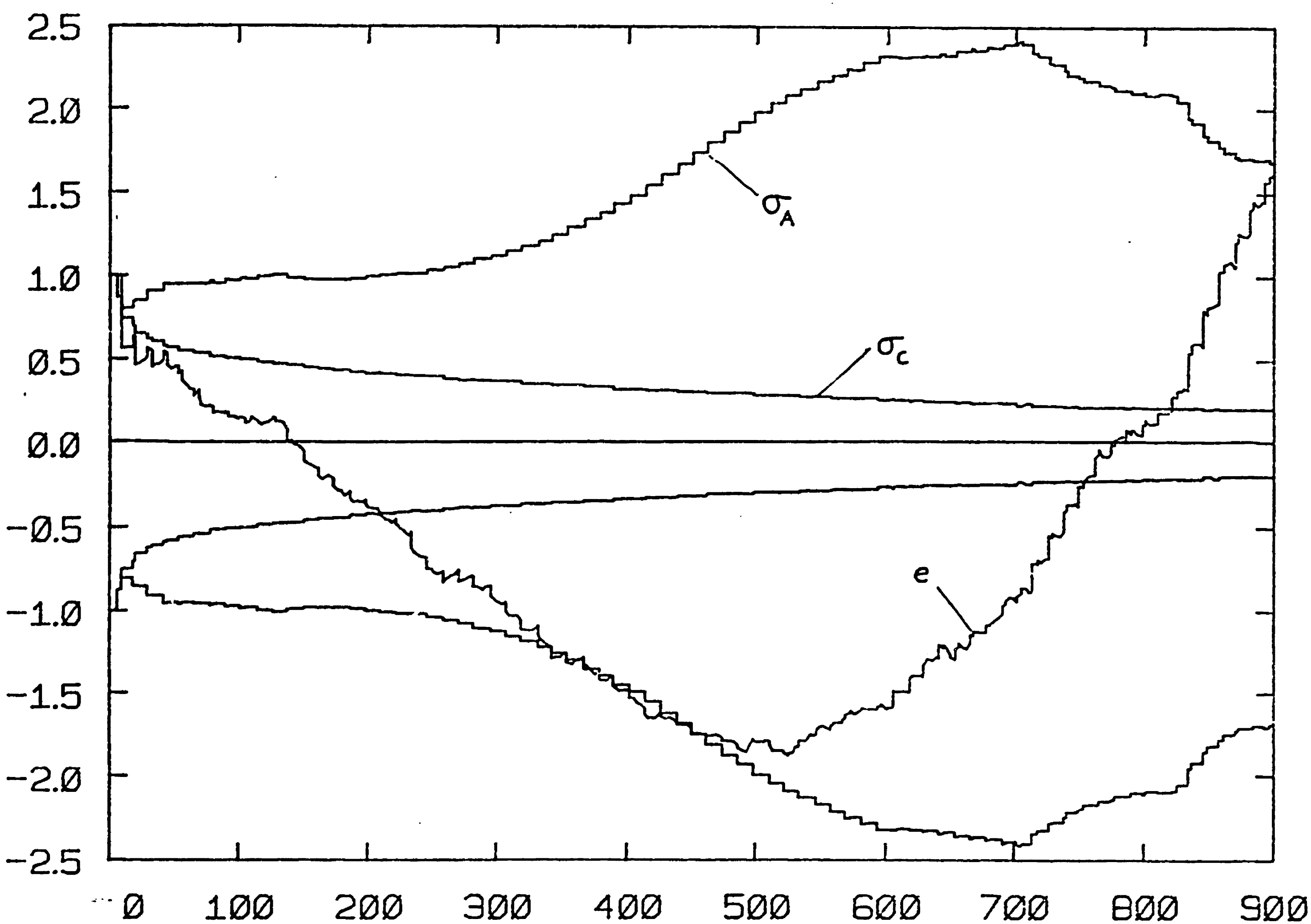


Figure 5.10(a). Static community. Covariance-based hierarchy. East error of member 4.

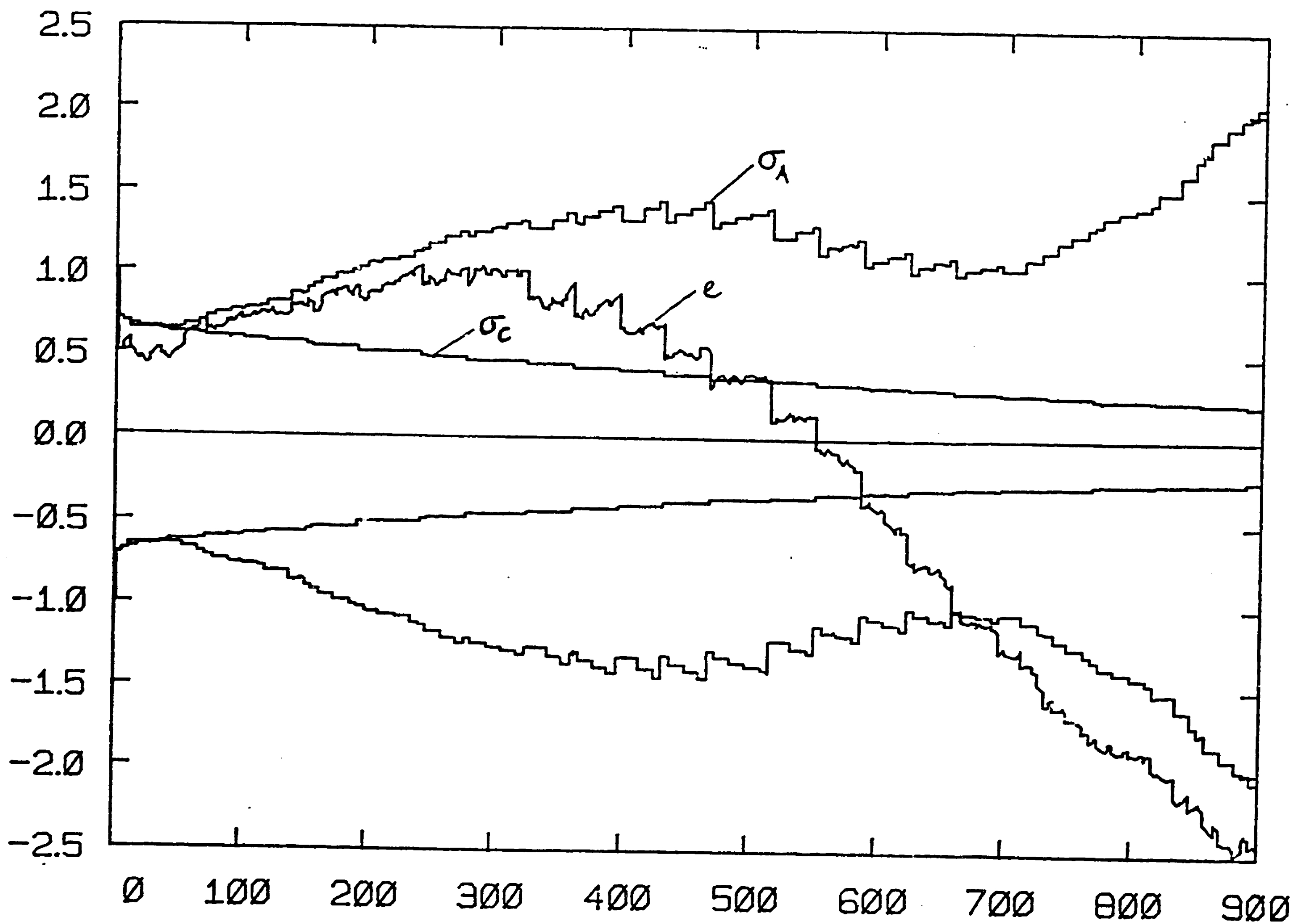


Figure 5.9(c). Static community. Covariance-based hierarchy. Clock error of member 3.

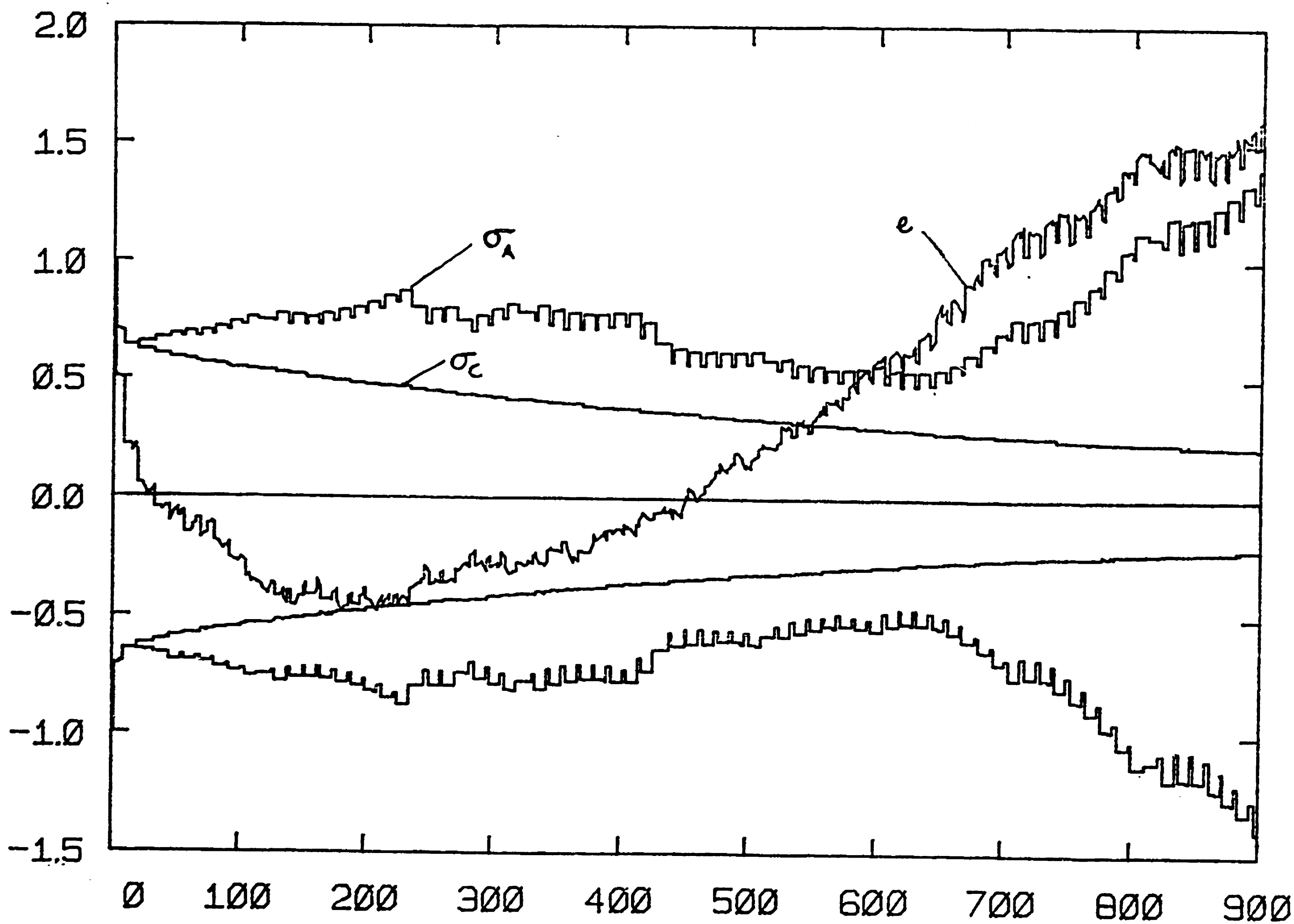


Figure 5.10(b). Static community. Covariance-based hierarchy. North error of member 4.

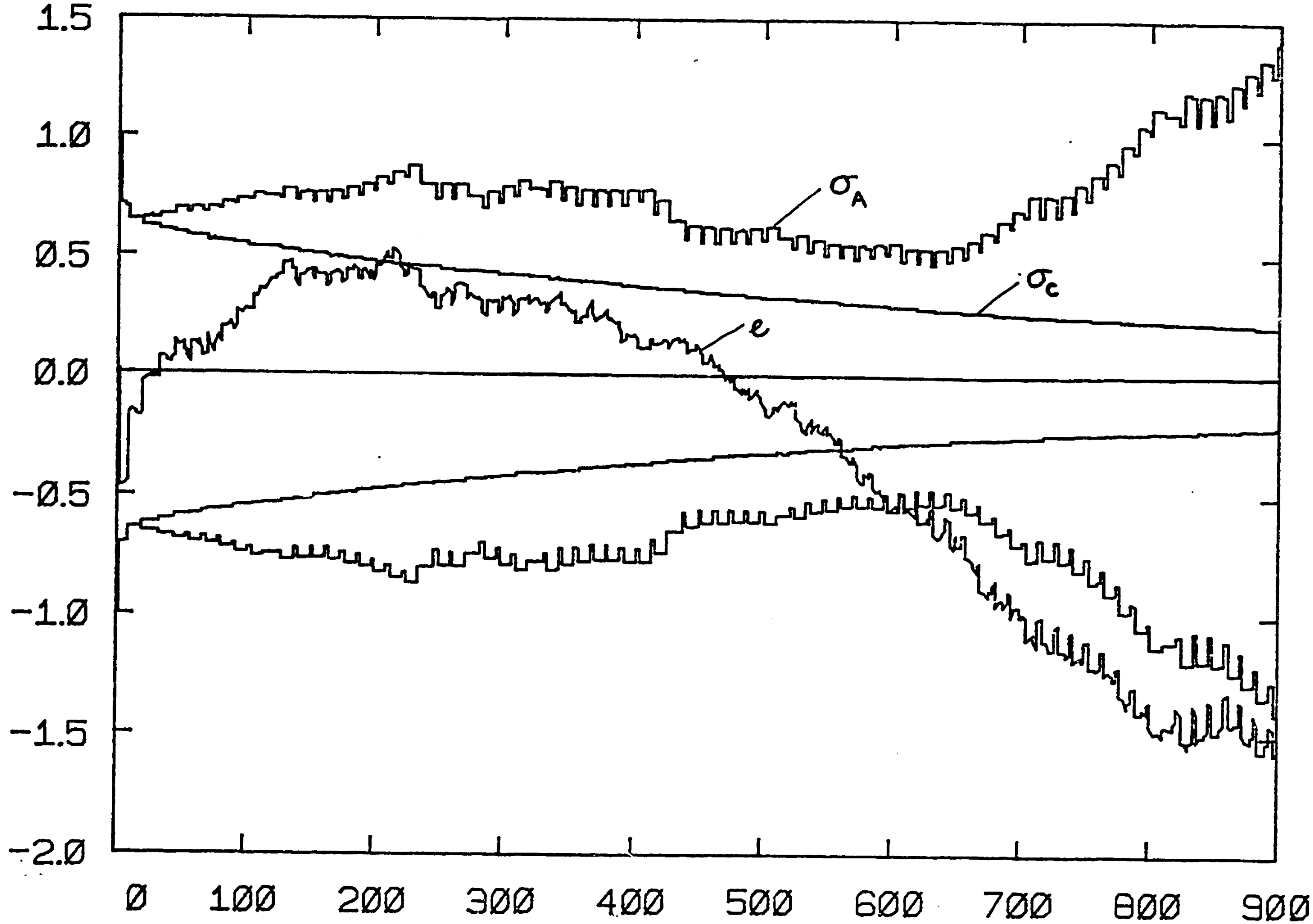


Figure 5.10(c). Static community. Covariance-based hierarchy. Clock error of member 4.

5.5 Dynamic communities

It was remarked in sect. 3.6 that motion will give one-by-one observability to a fixed-rank community. In particular, there is observability if either the master or all the second ranks move around. The simulation discussed in this section had member 2 (the only one of second rank) moving back and forth, as shown in fig. 5.11, at a little above the speed of sound. The remainder was like in sect. 5.2.

In a fixed-rank community (figures 5.12, 5.13, 5.14, showing the errors of members 2, 3 and 4) the presence of observability causes a marked improvement. Error covariance goes down to a steady state (modulated by the 300 sec back-and-forth motion of member 2). When the separation of the three components of the error occurs it is marked by the sudden decrease of the error covariance from an initial-condition dependent value to a steady-state value.

Member 2 (who only ranges to member 1) experiences this change when the angular velocity of its line of sight to member 1 is greatest (at about 180 sec). Member 4 (who ranges to members 1 and 2) soon benefits from the improved accuracy of its rangings to member 2. Member 3's accuracy also shows some improvement in this same time period, but the steady state is reached later, at about 330 sec, when the angular velocity of its line of sight to member 2 is greatest.

This case has present the sufficient conditions for community stability: fixed rank hierarchy and one-by-one observability and controllability. The simulation results do show a stable behaviour.

A democratic community, under the same conditions, shows, instead, still an unstable oscillatory behavior. The errors of members 2, 3 and 4, plotted in fig. 5.15, 5.16 and 5.17, show this pattern. The waveform is

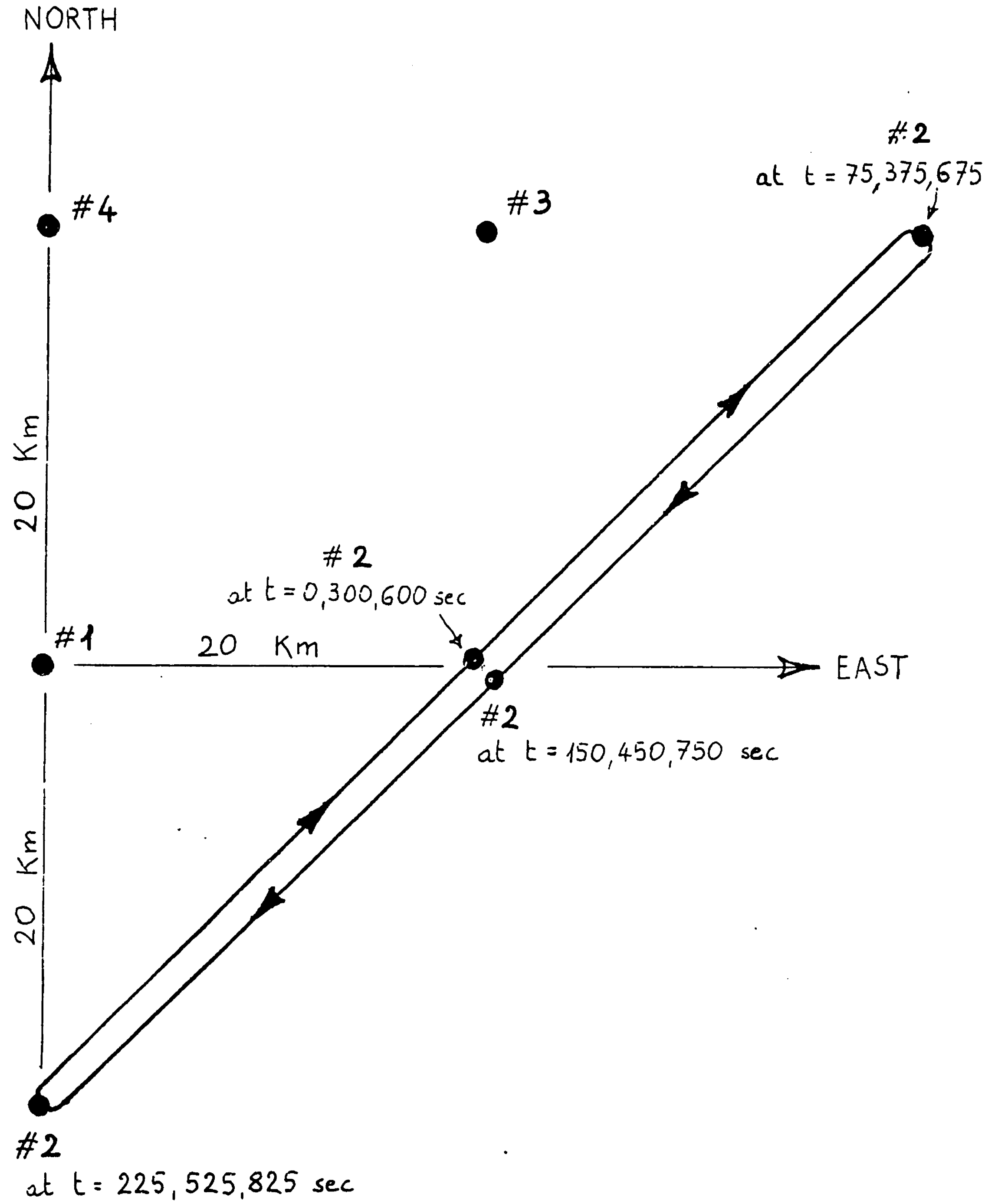


Figure 5.11. Geometry of the community

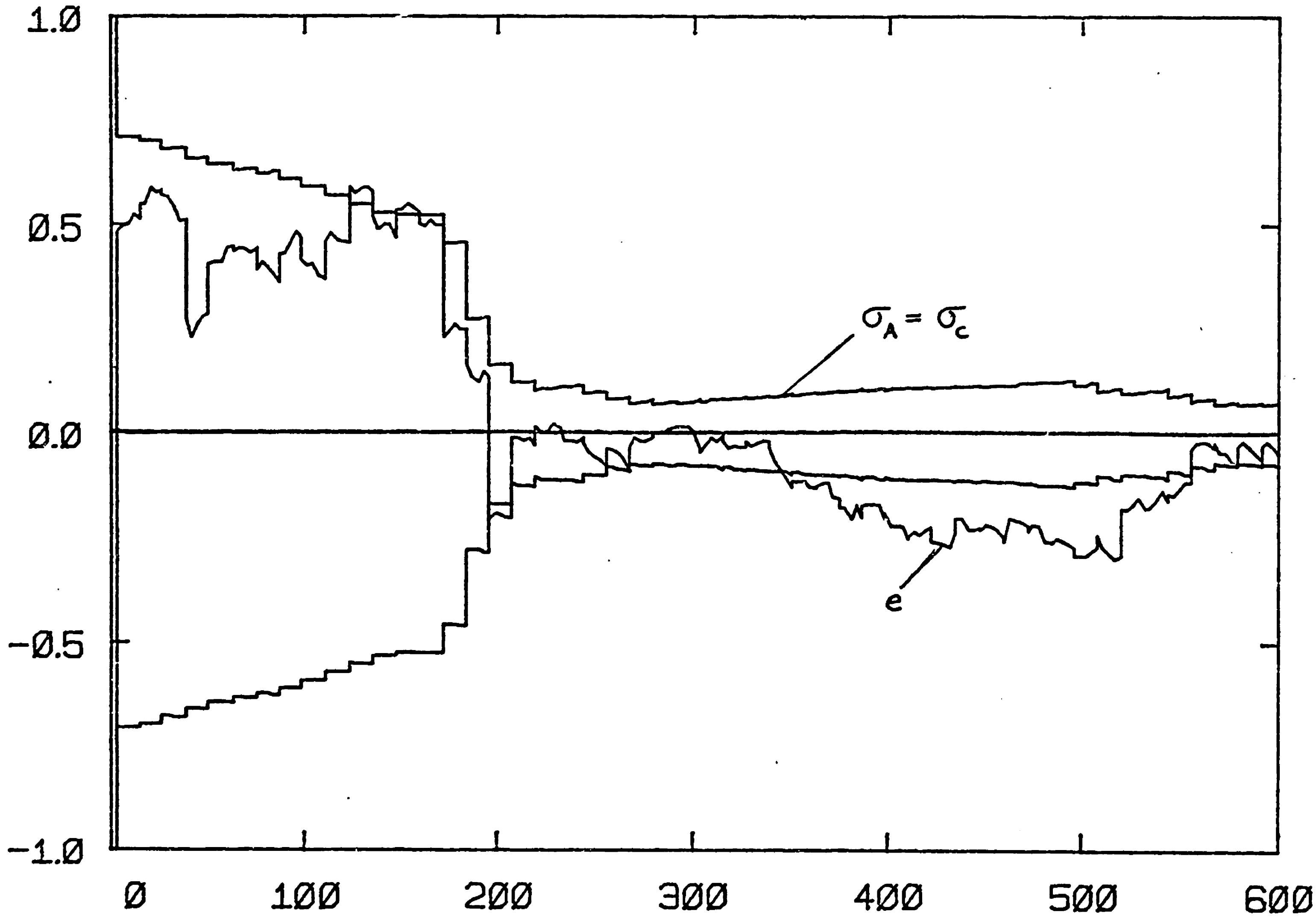


Figure 5.12(a). Dynamic community. Fixed-rank hierarchy. East error of member 2.

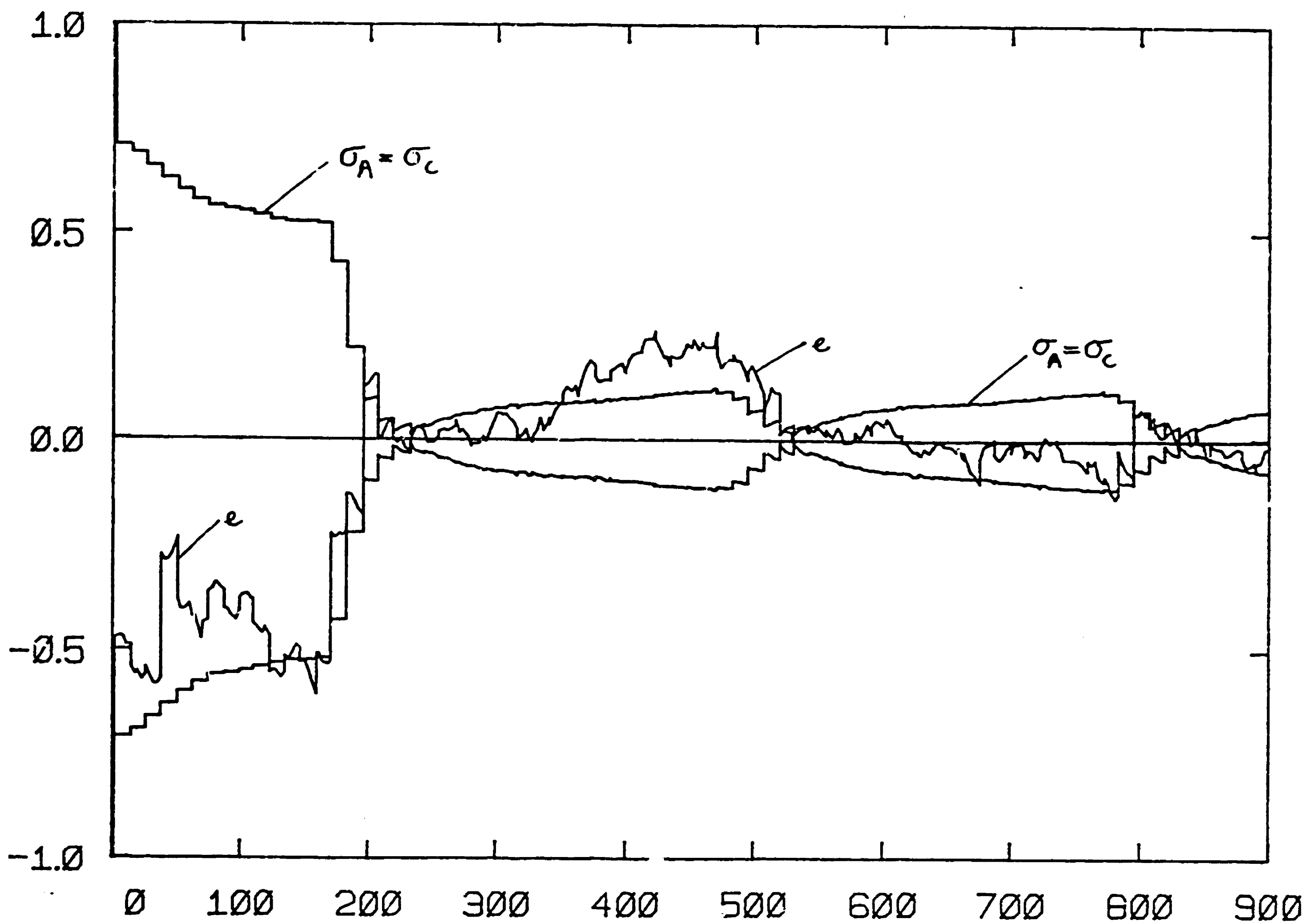


Figure 5.12(b). Dynamic community. Fixed-rank hierarchy. Clock error of member 2.

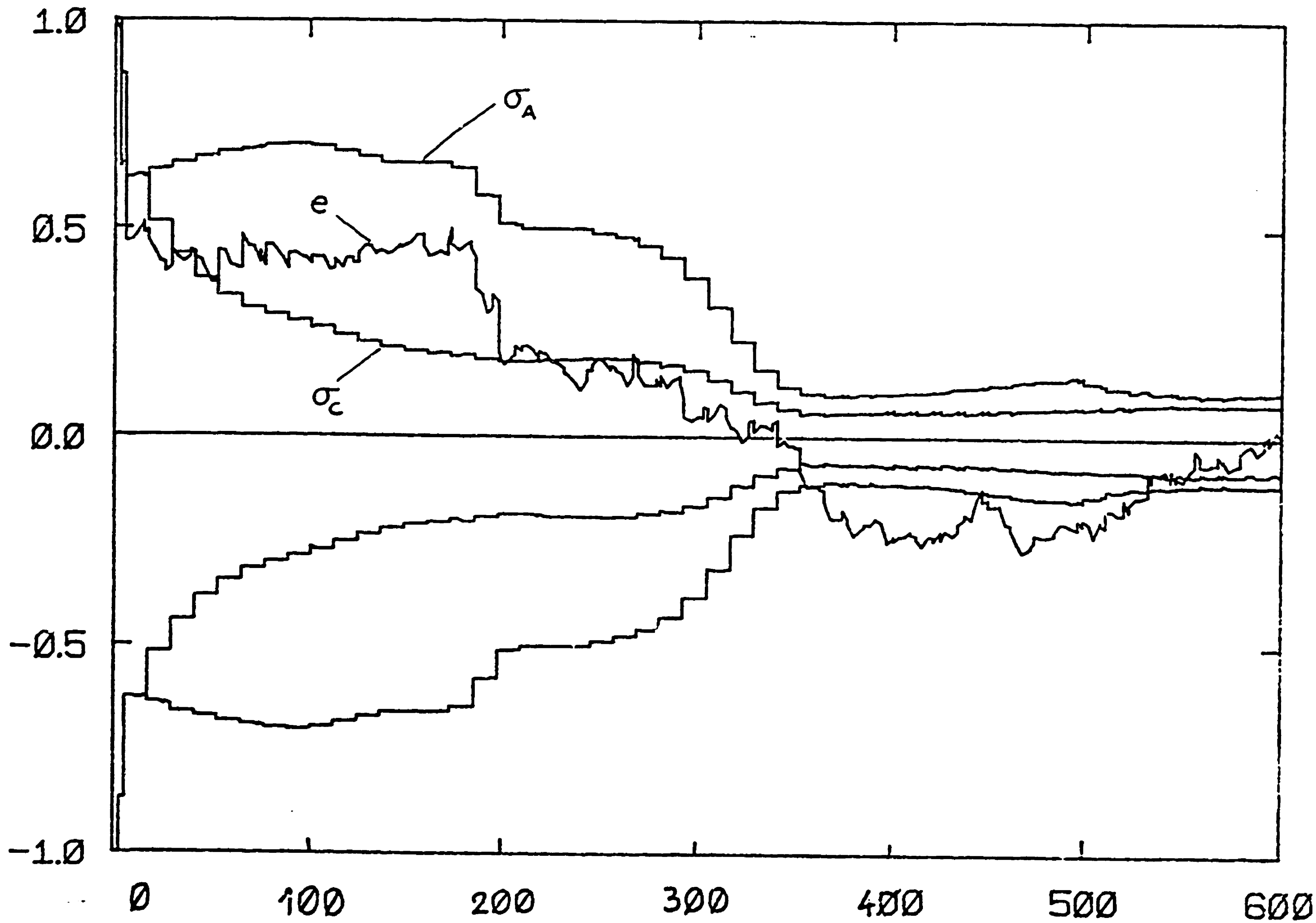


Figure 5.13(a). Dynamic community. Fixed-rank hierarchy. East error of member 3.

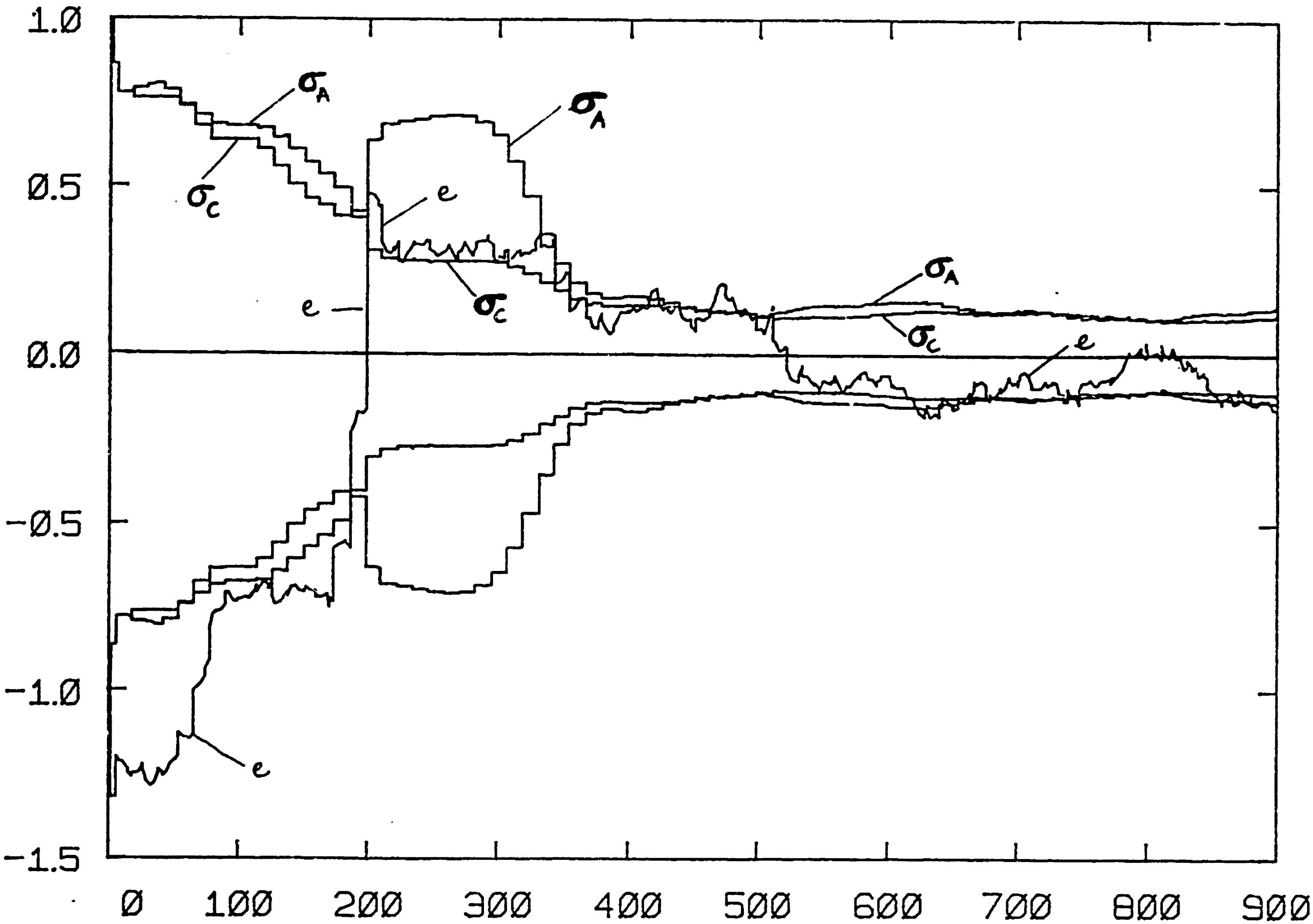


Figure 5.13(b). Dynamic community. Fixed-rank hierarchy. North error of member 3.

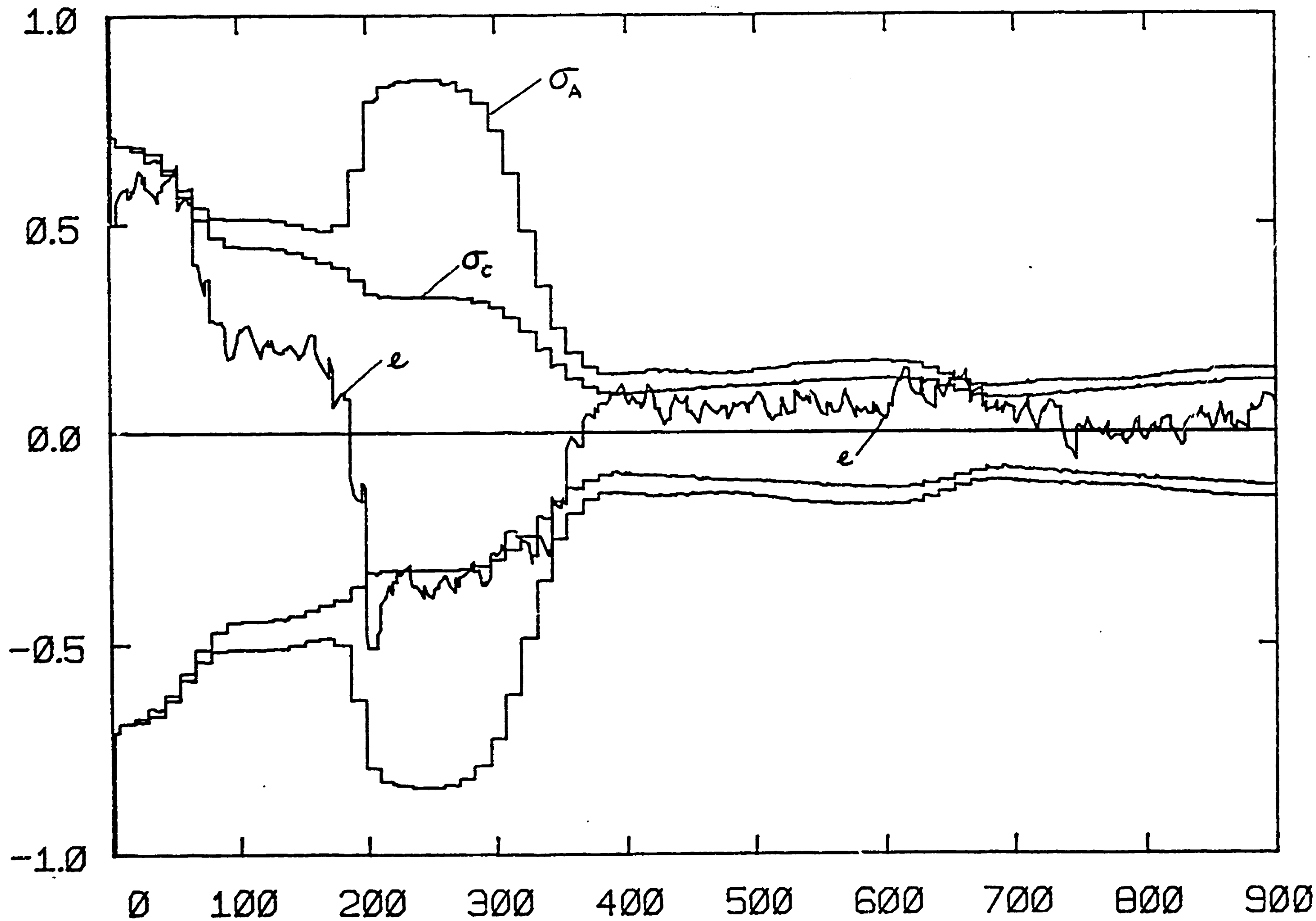


Figure 5.13(c). Dynamic community. Fixed-rank hierarchy. Clock error of member 3.

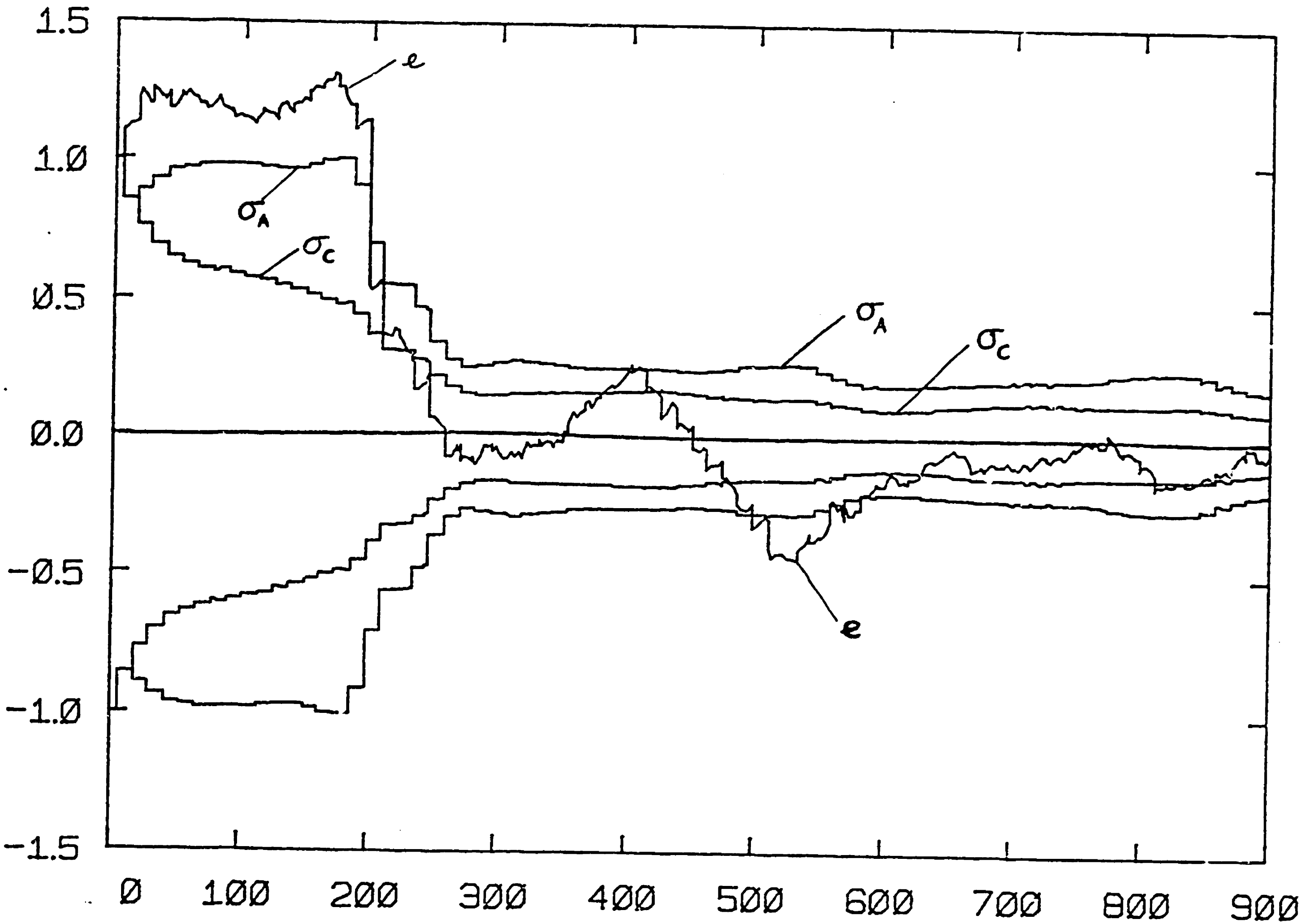


Figure 5.14(a). Dynamic community. Fixed-rank hierarchy. East error of member 4.

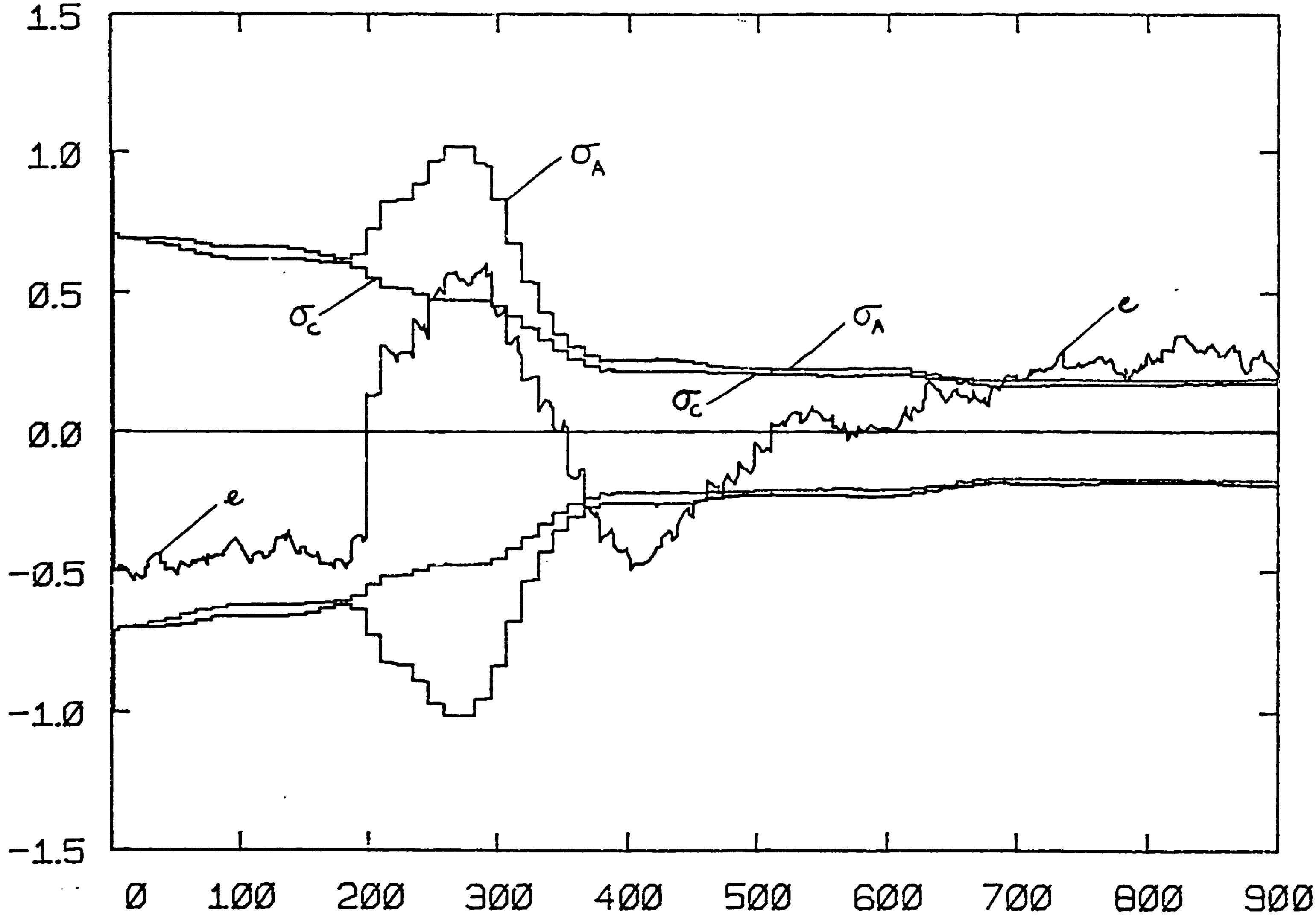


Figure 5.14(b). Dynamic community. Fixed-rank hierarchy. North error of member 4.

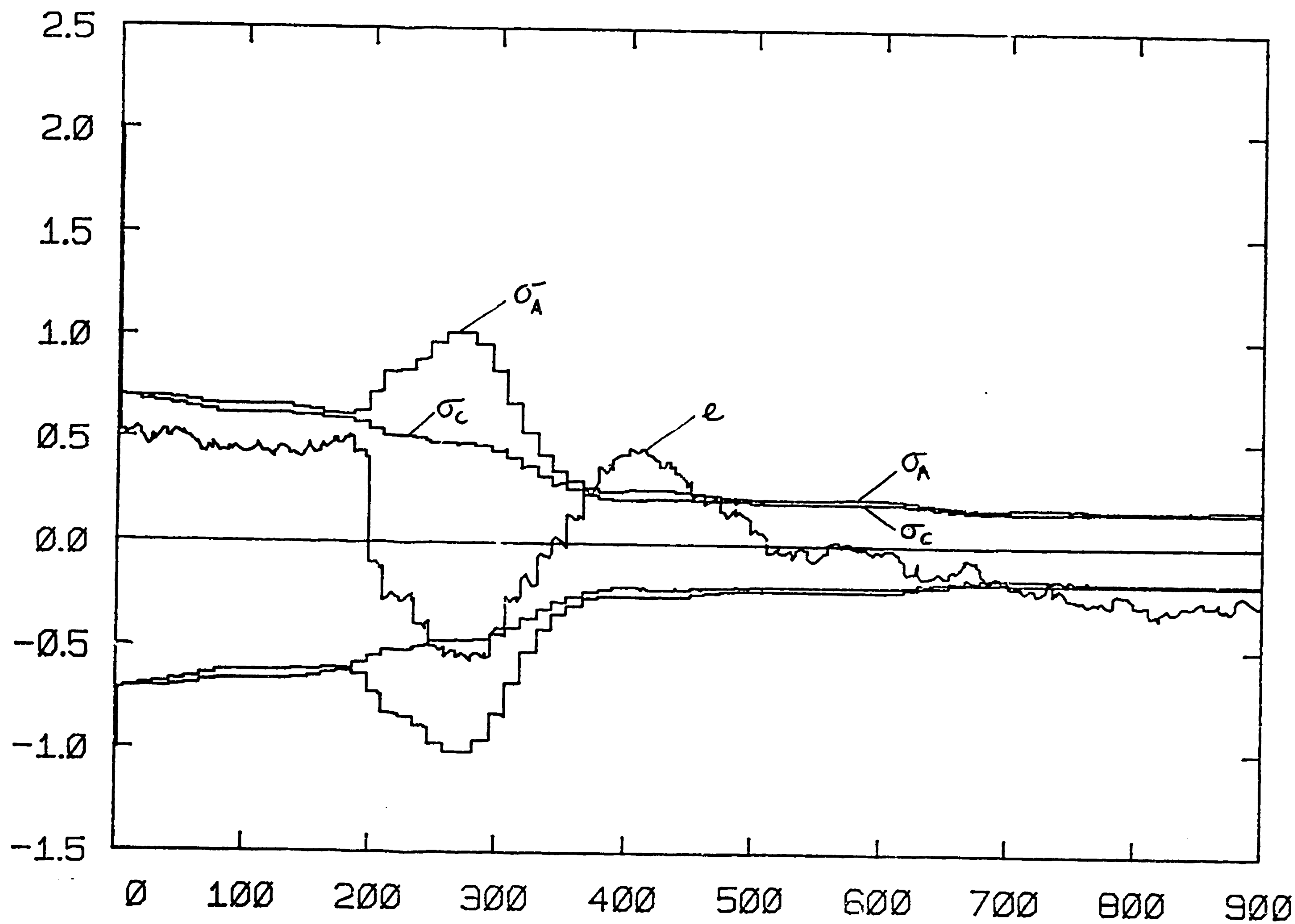


Figure 5.14(c). Dynamic community. Fixed-rank hierarchy. Clock error of member 4.

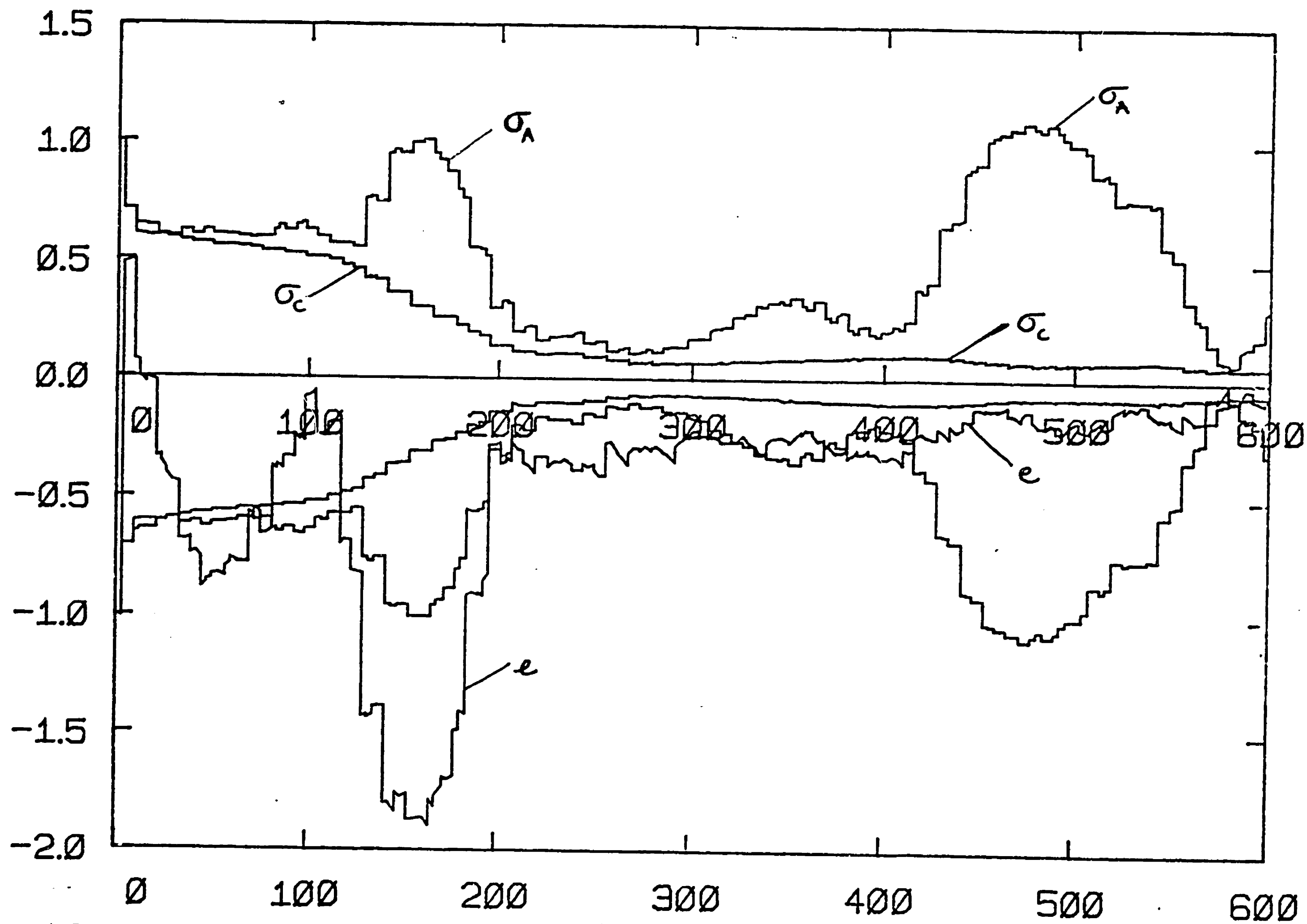


Figure 5.15(a). Dynamic community. Democracy. East error of member 2.

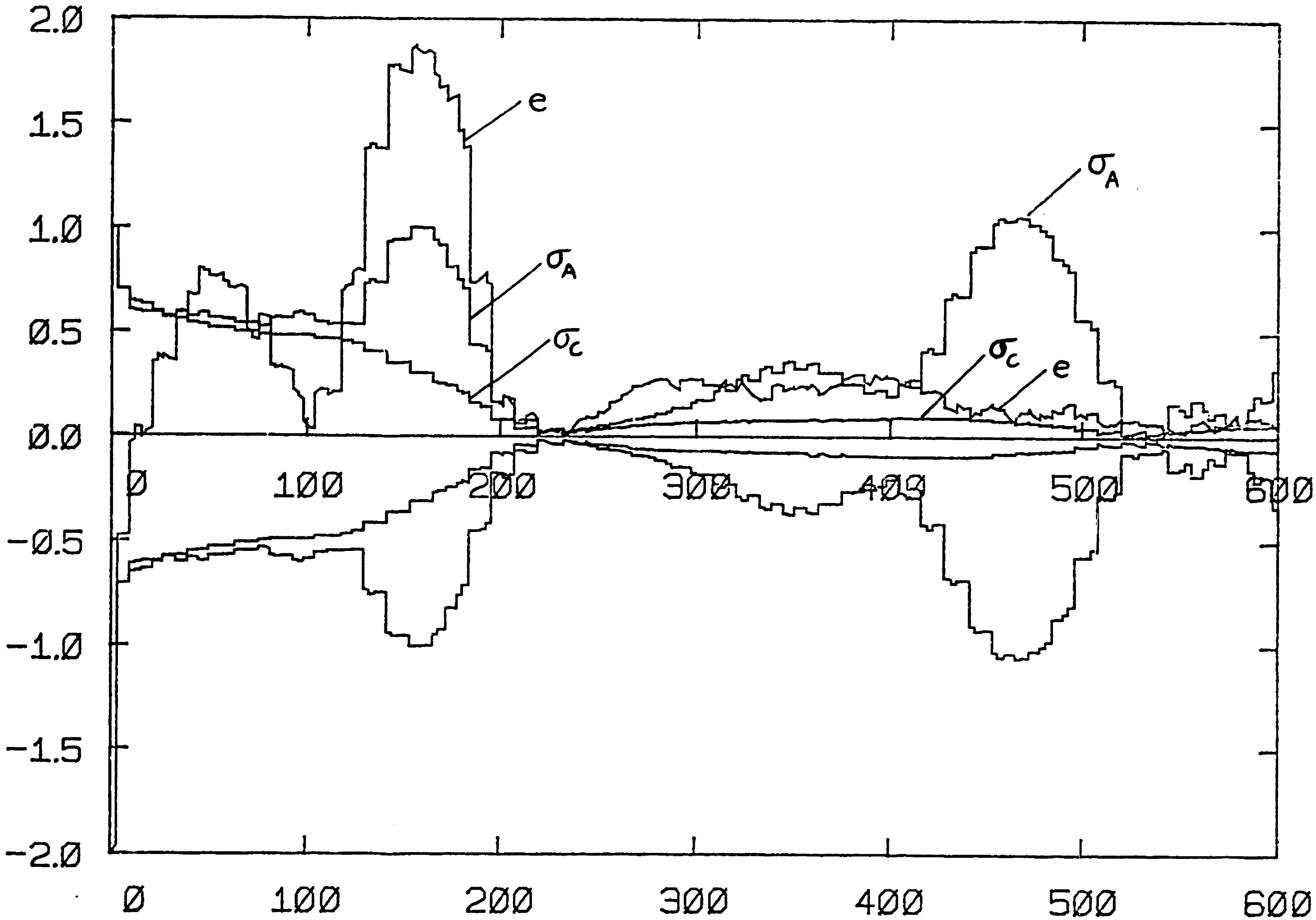


Figure 5.15(b). Dynamic community. Democracy. Clock error of member 2.

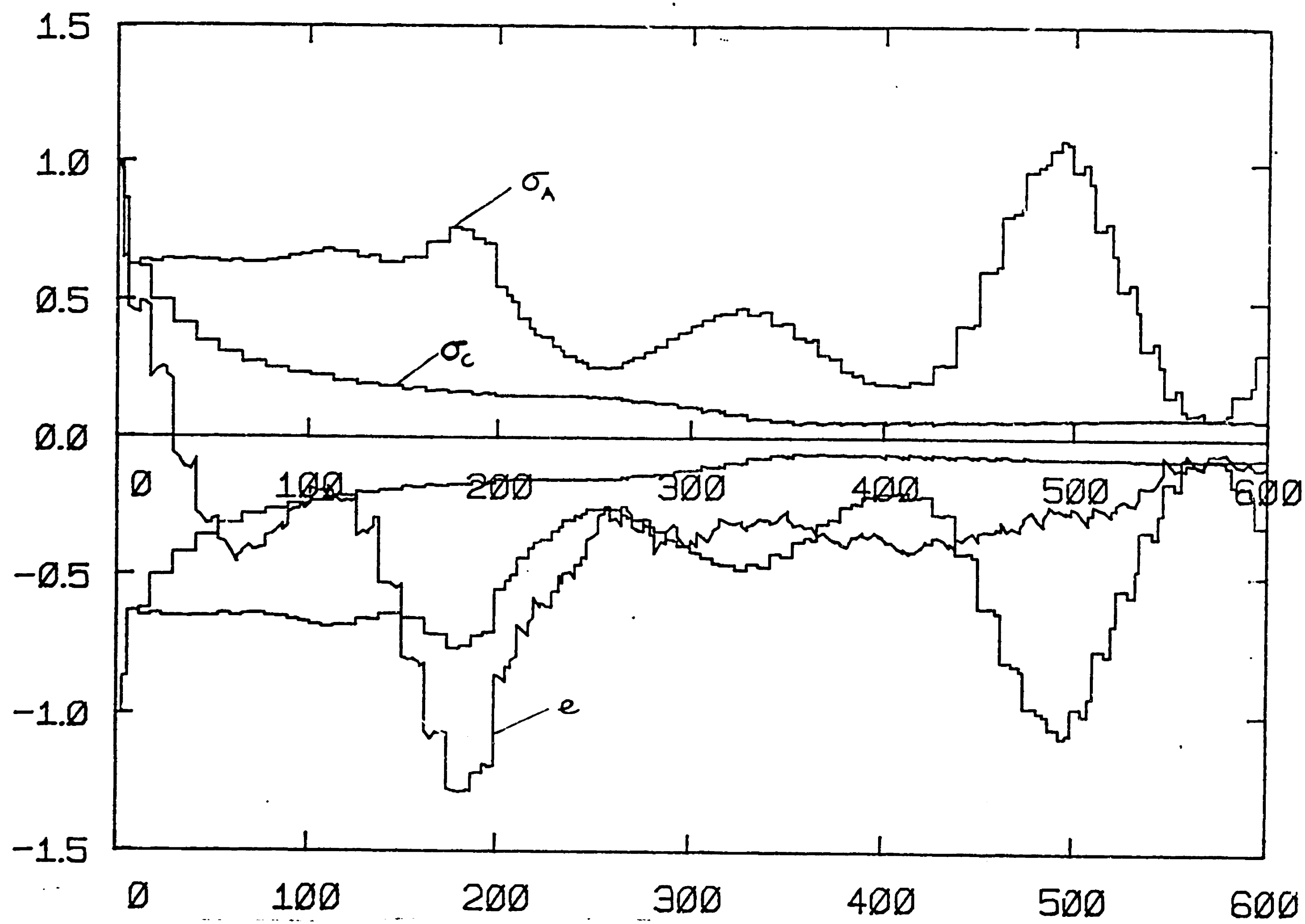


Figure 5.16(a). Dynamic community. Democracy. East error of member 3.

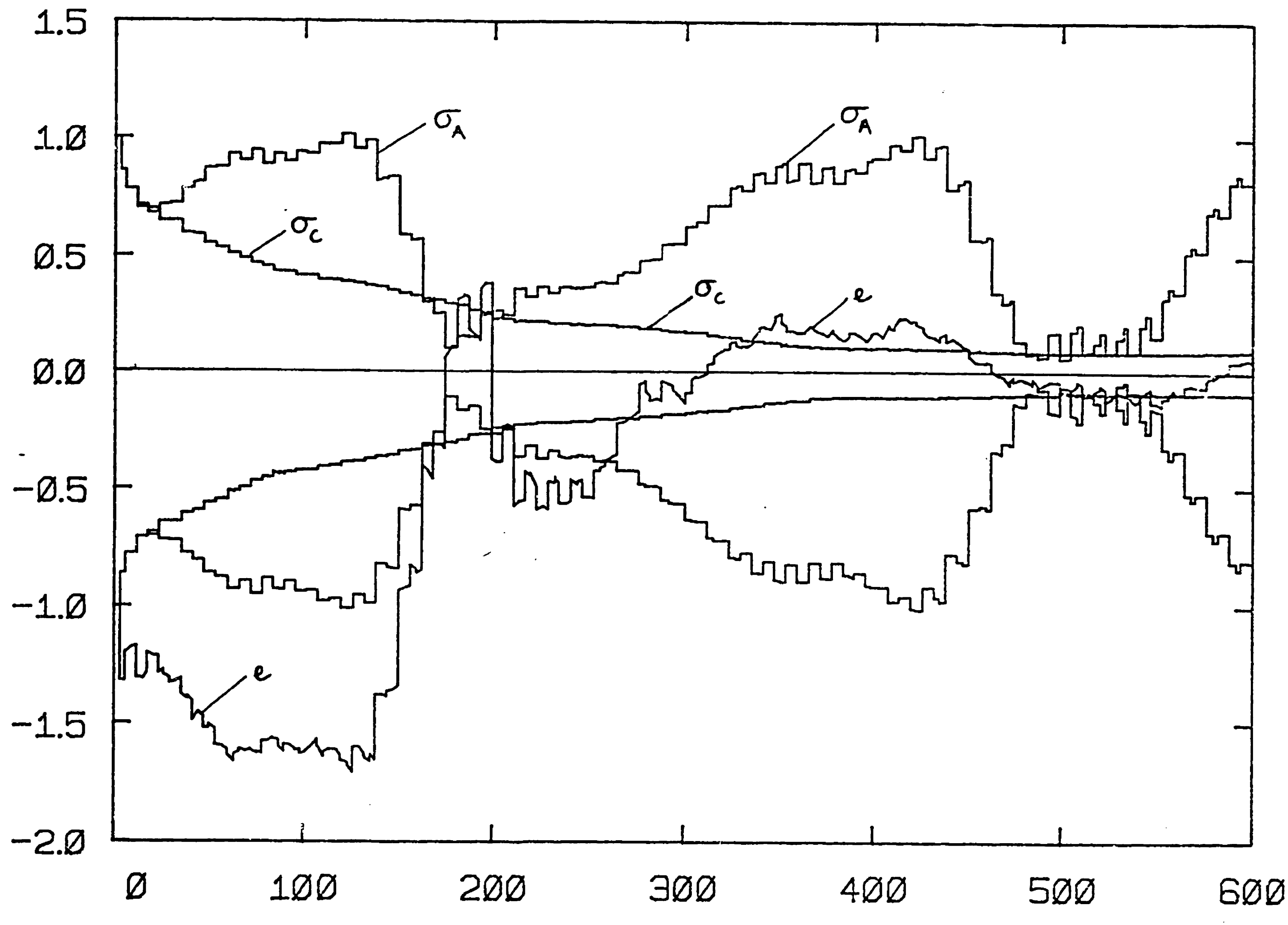


Figure 5.16(b). Dynamic community. Democracy. North error of member 3.

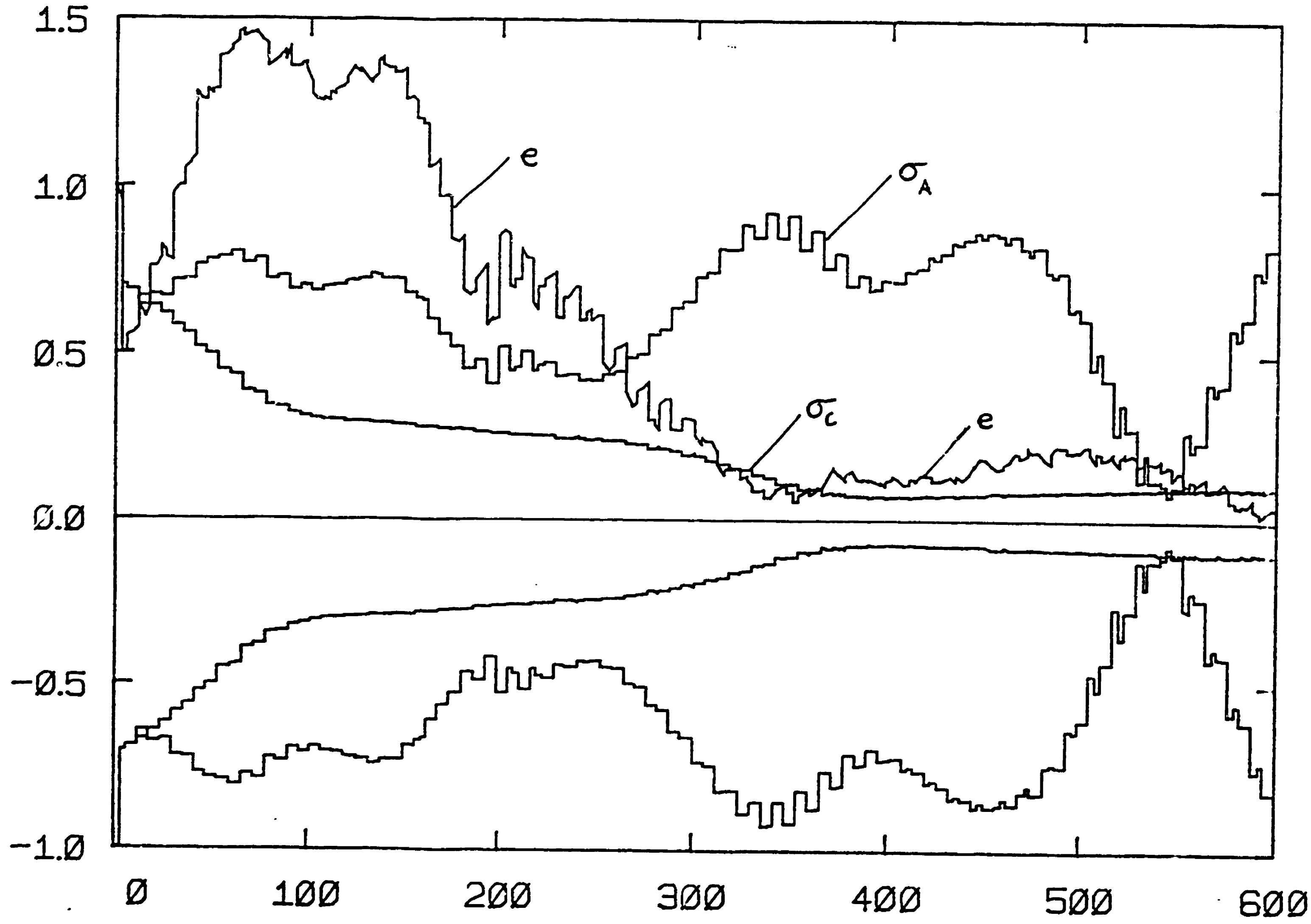


Figure 5.16(c). Dynamic community. Democracy. Clock error of member 3.

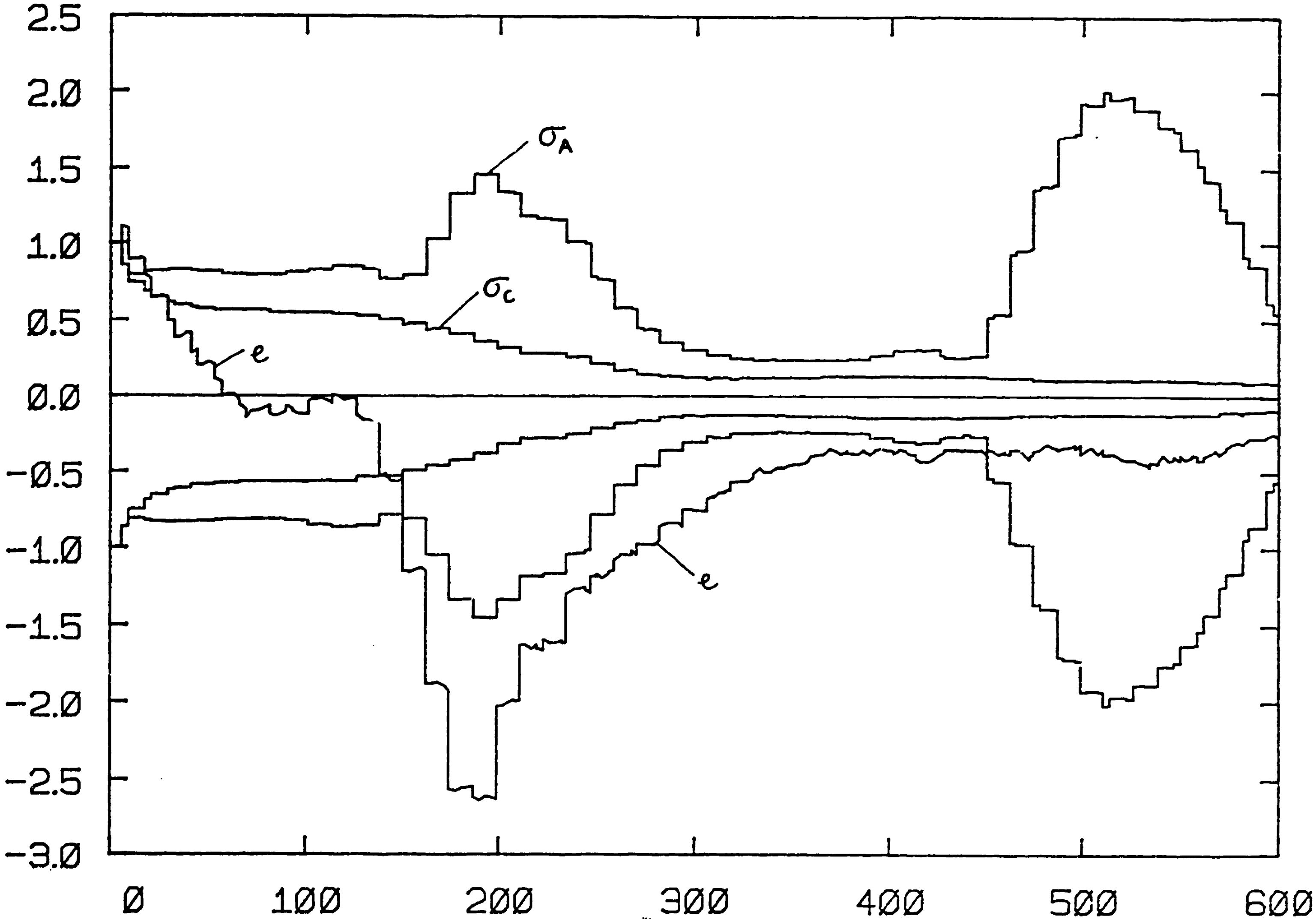


Figure 5.17(a). Dynamic community. Democracy. East error of member 4.

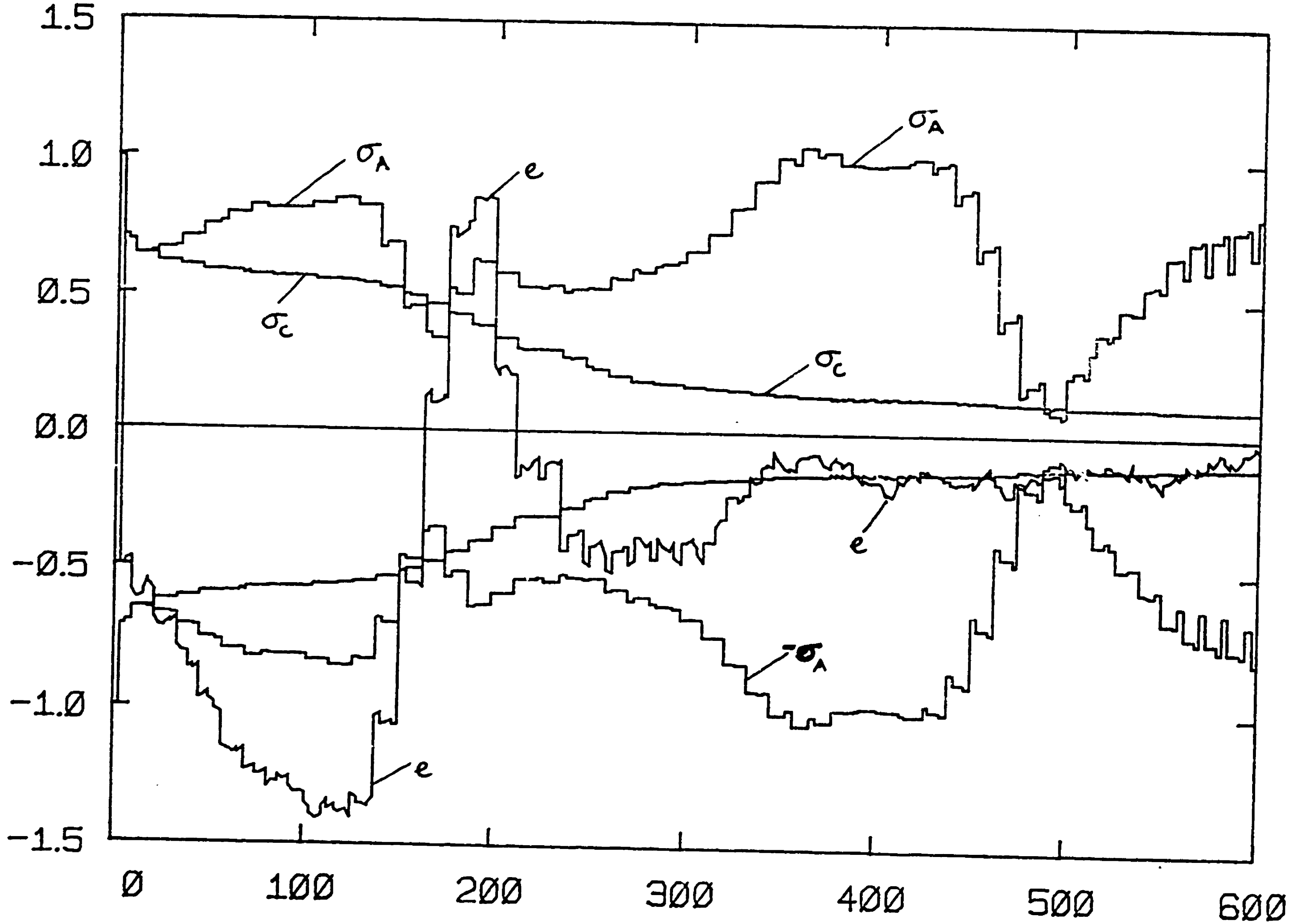


Figure 5.17(b). Dynamic community. Democracy. North error of member 4.

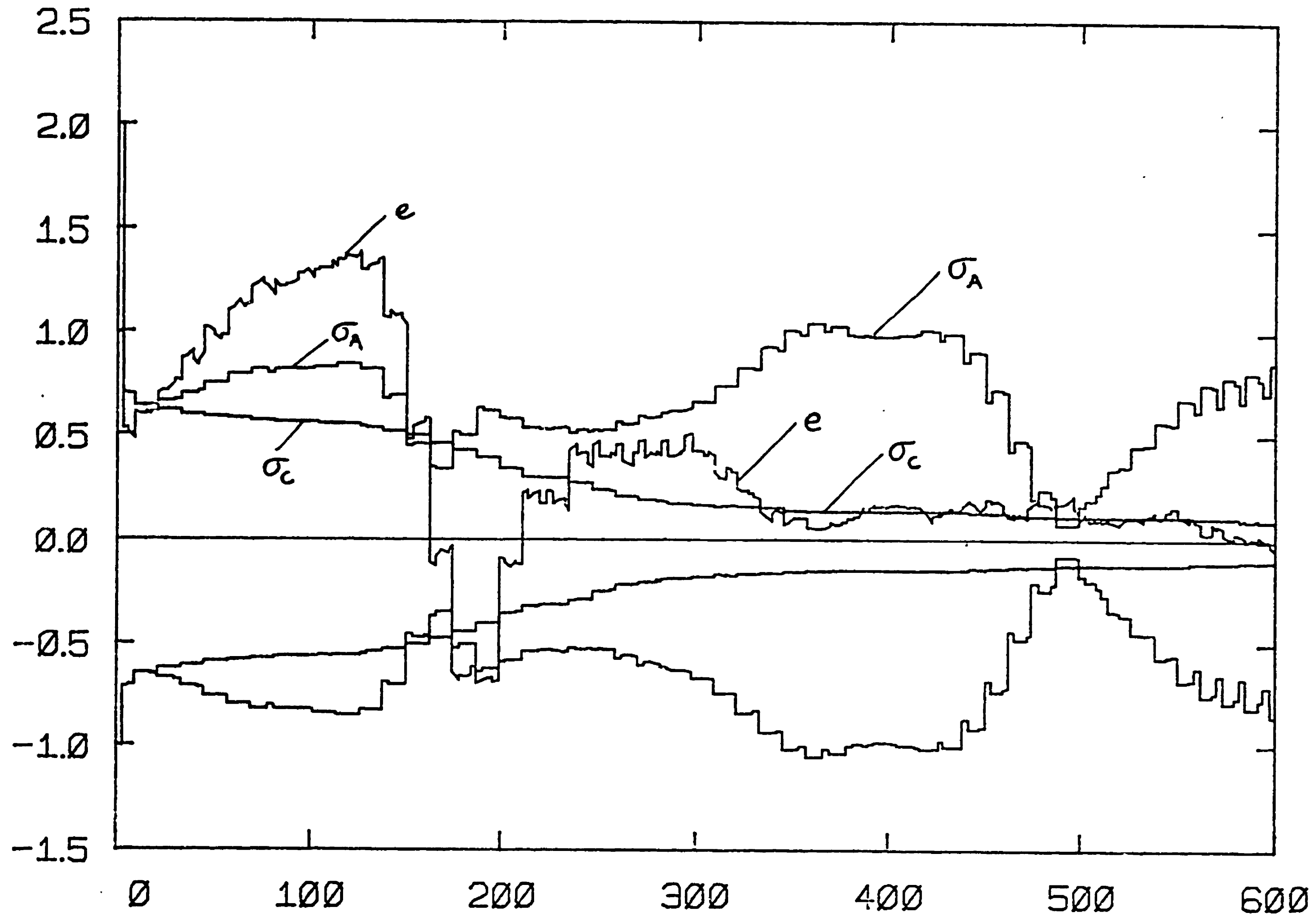


Figure 5.17(c). Dynamic community. Democracy. Clock error of member 4.

odd and may result from the modulation of a proper oscillation by the 1/300 Hertz geometry change.

A covariance-based community (errors in fig. 5.18, 5.19, 5.20) behaves much like its fixed-rank counterpart. Member 2 seldom ranges to anybody else than the master, and the disturbances aroused by these rank reversals die out quickly (unlike the static case). Their insurgence and decay is clearly visible, because rangings to a non-master cause σ_a to be different from σ_c. Member 3 usually ranges only to members 1 and 2; member 4 ranges to everybody. Thus, the community behaves almost as if it had a fixed-rank hierarchy, with one member per rank.

These plots exemplify a kind of behavior quite different from the one observed in sect. 5.4 for the same organization, and indicate that both the partisans and the opponents of covariance-based hierarchies might be right.

5.6 Time-Domain tests for stability

An illustration of the proven stability of a fixed-rank community satisfying the flight path condition, was obtained by running the same case of sect. 5.5, with member 2 flying back and forth. A different feature was introduced, however; member 4 was given the fourth rank, allowing it to range to member 3 (which still had the third rank). The test derived in section 4.4 was used.

Figure 5.21 shows tr S_n. As expected, it converges to zero faster than an exponential function. This convergence is not monotonic; in particular, there is a large increase at about t = 300 sec. The explanation of this behavior is found by looking at the three following figures.

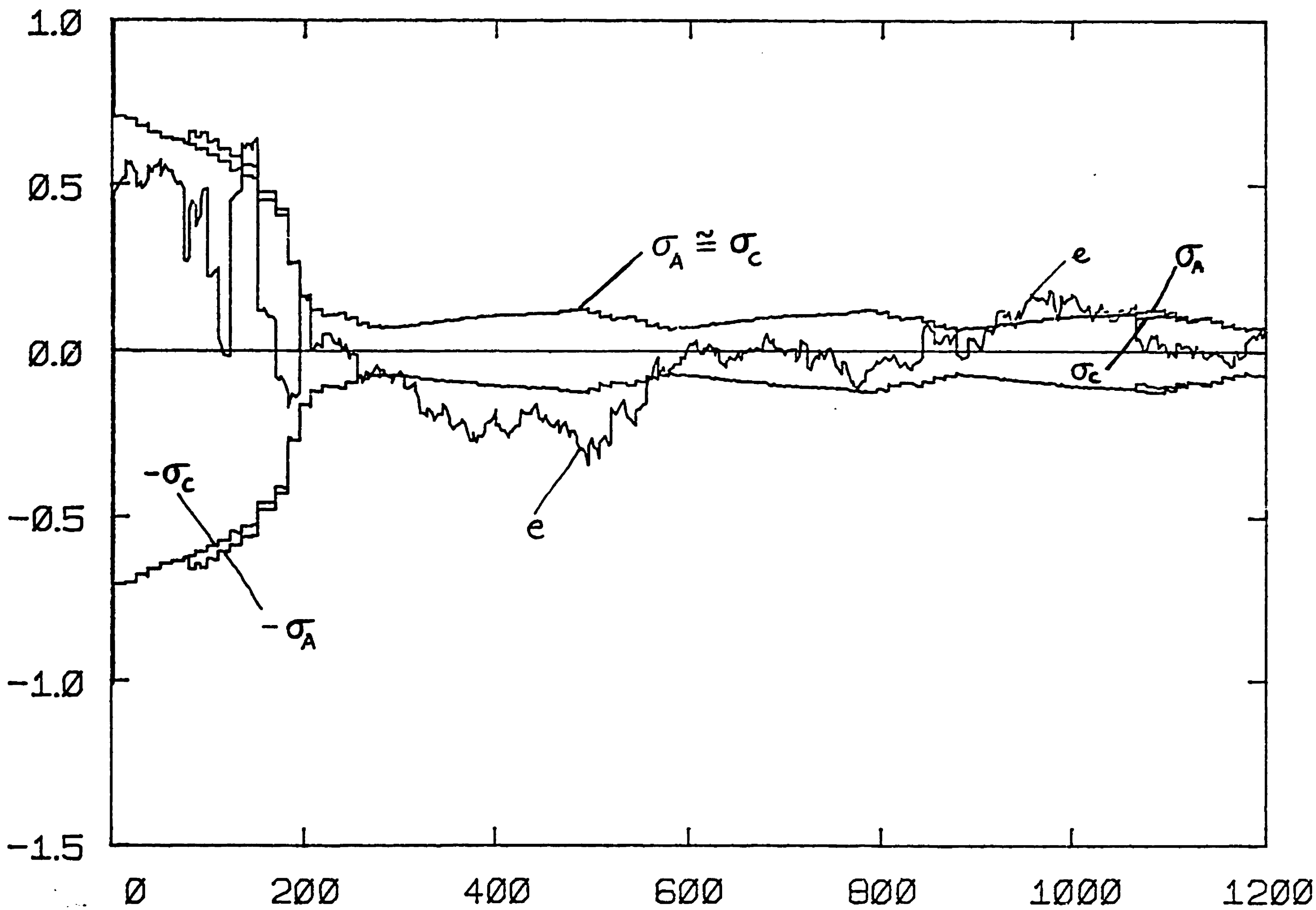


Figure 5.18(a). Dynamic community. Covariance-based hierarchy. East error of member 2.

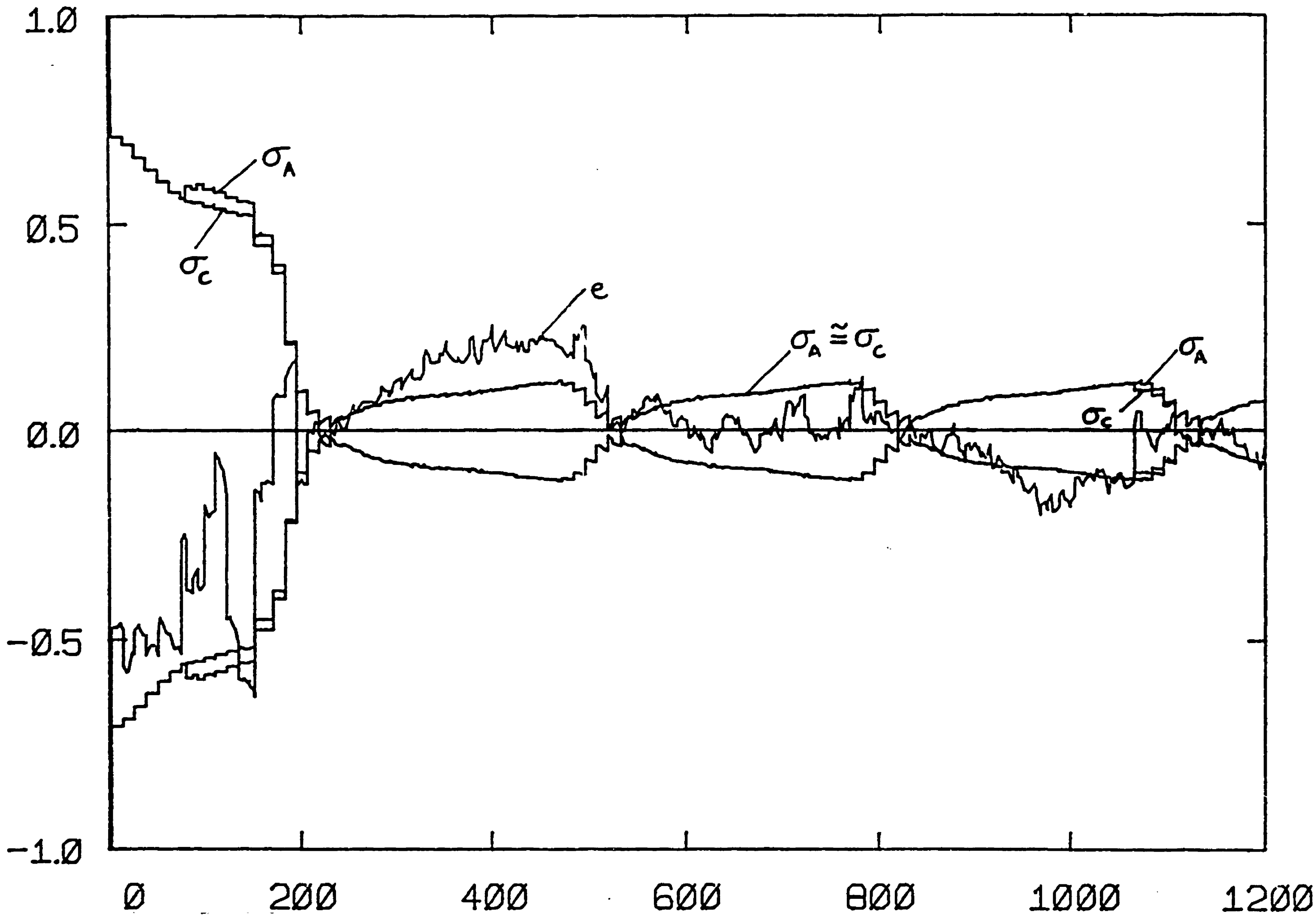


Figure 5.18(b). Dynamic community. Covariance-based hierarchy. Clock error of member 2.

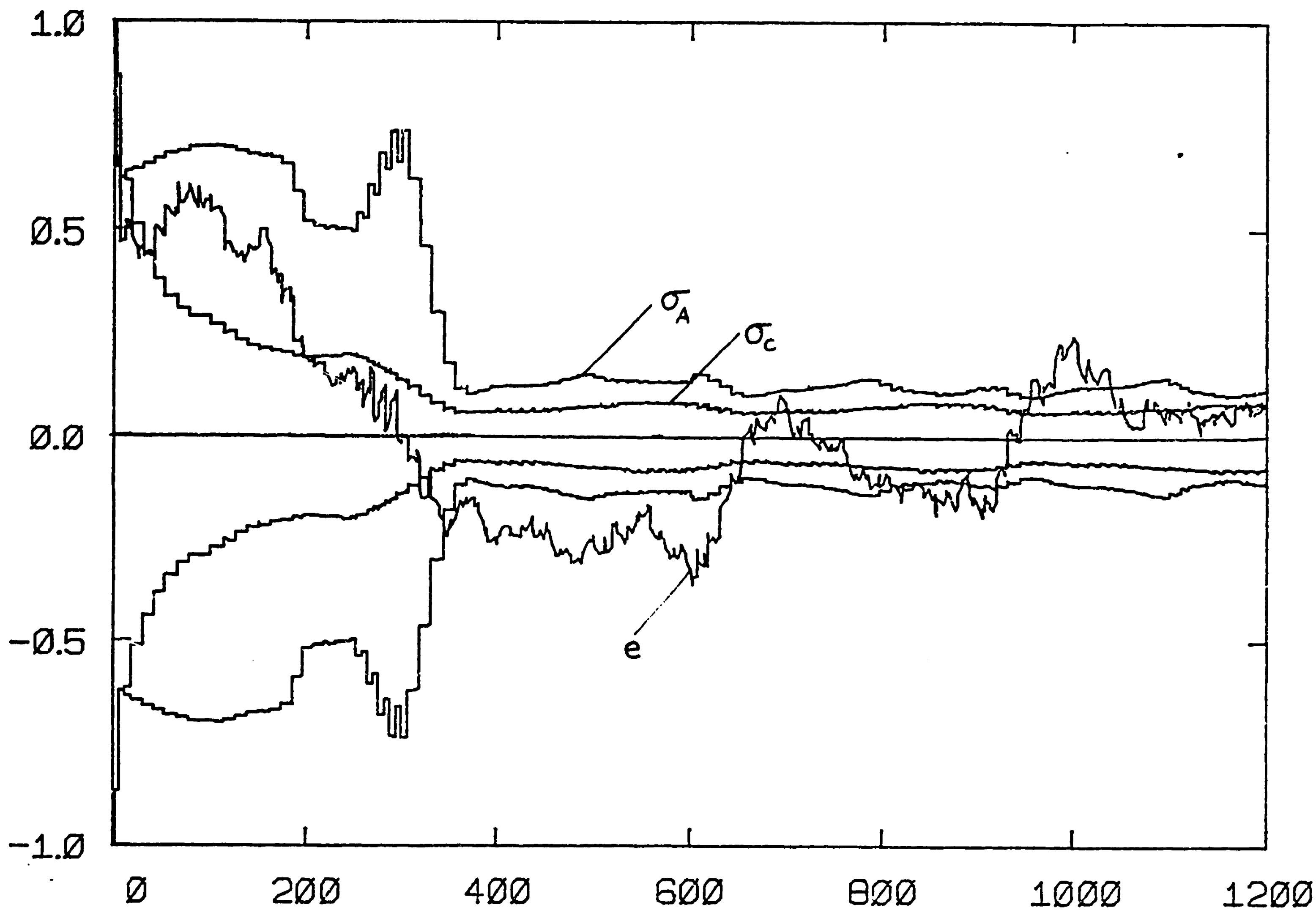


Figure 5.19(a). Dynamic community. Covariance-based hierarchy. East error of member 3.

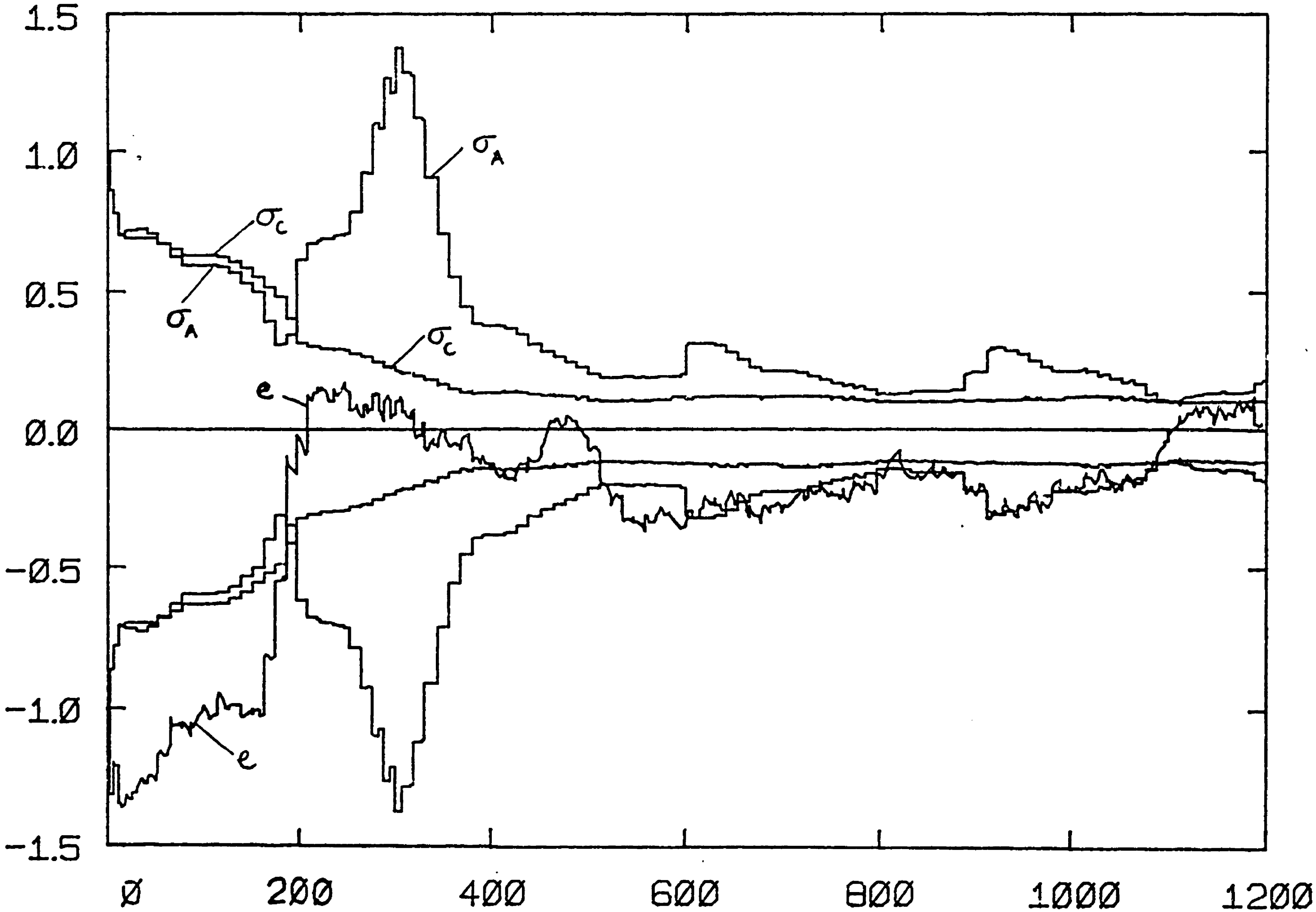


Figure 5.19(b). Dynamic community. Covariance-based hierarchy. North error of member 3.

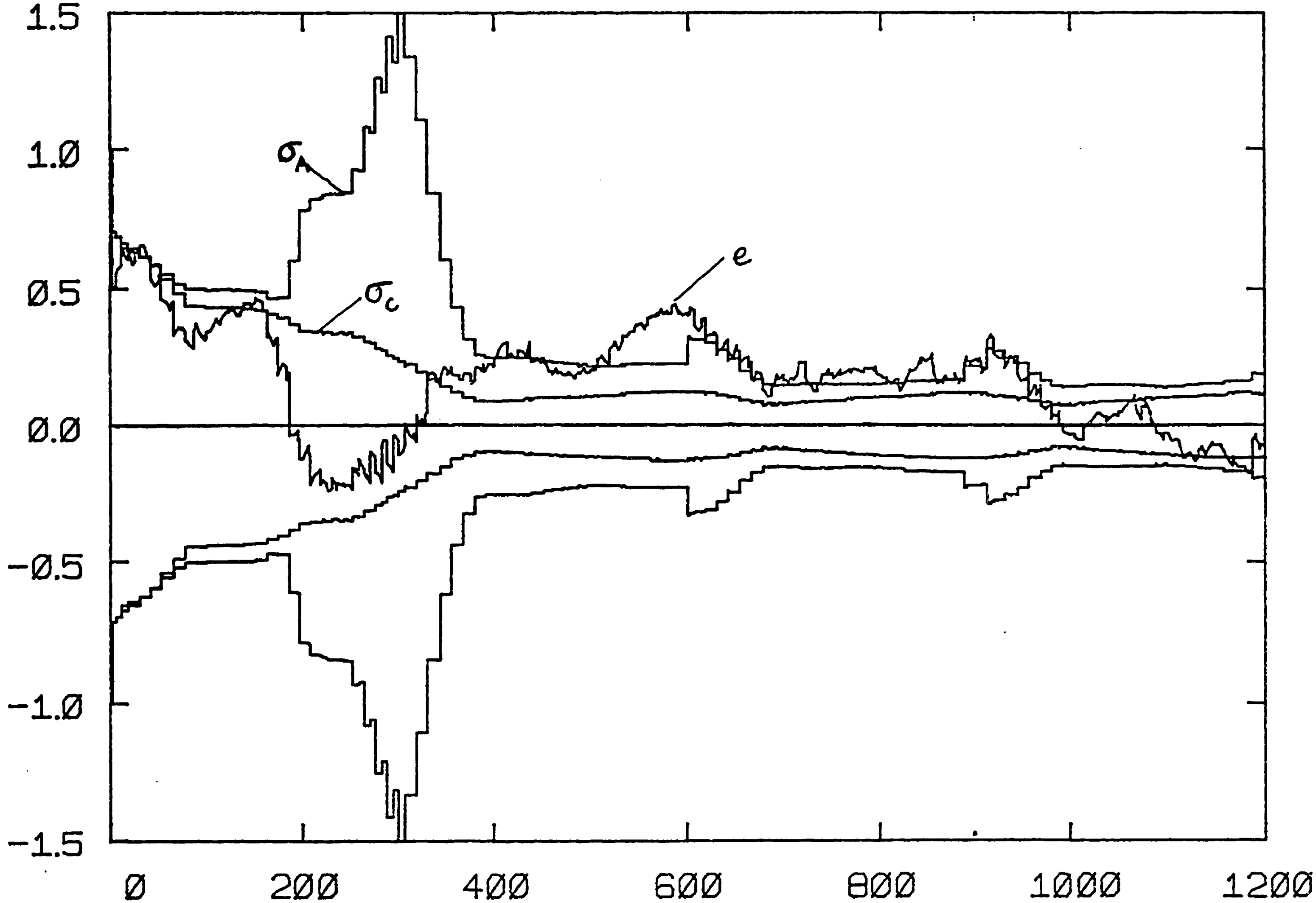


Figure 5.19(c). Dynamic community. Covariance-based hierarchy. Clock error of member 3.

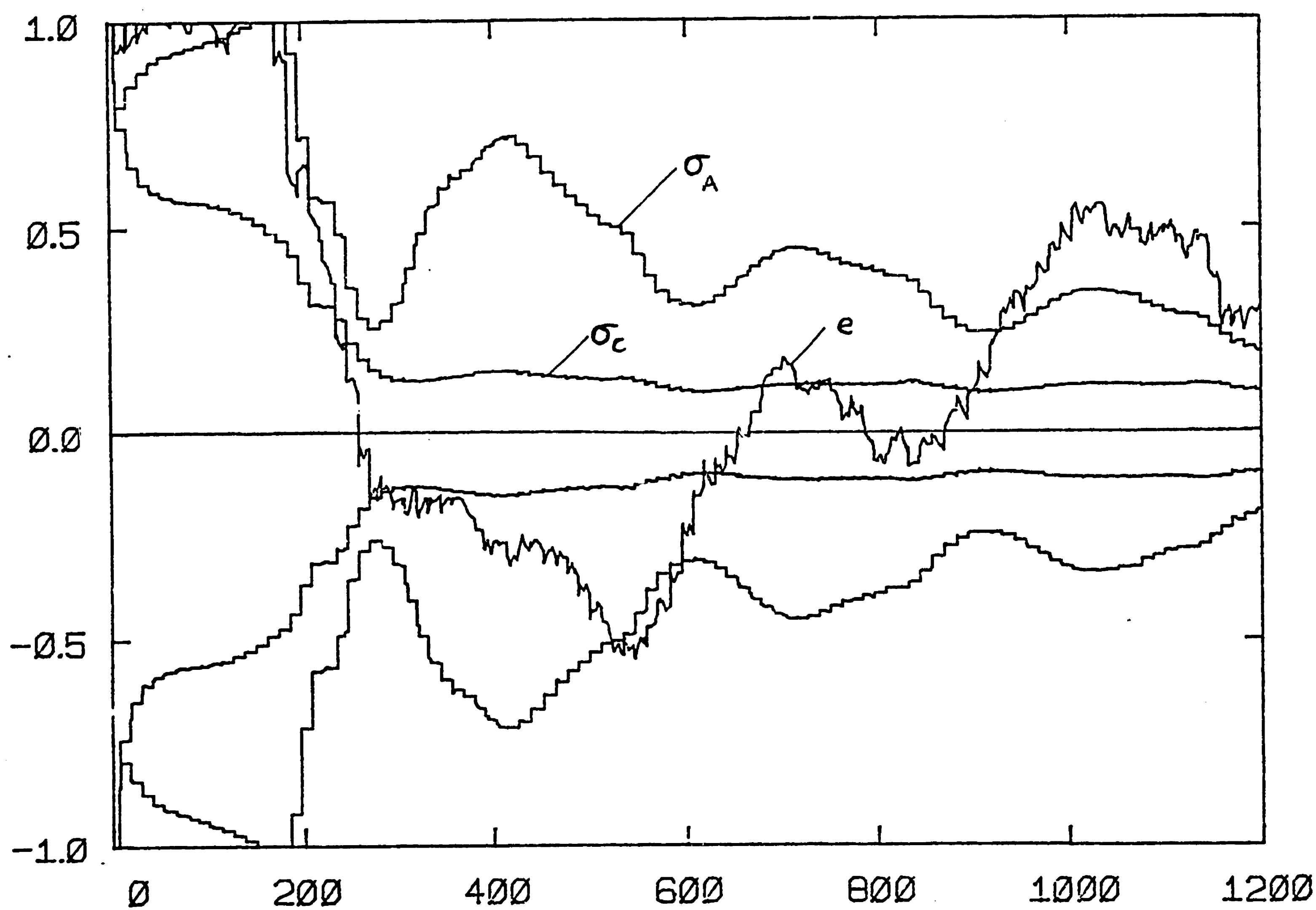


Figure 5.20(a). Dynamic community. Covariance-based hierarchy. East error of member 4.

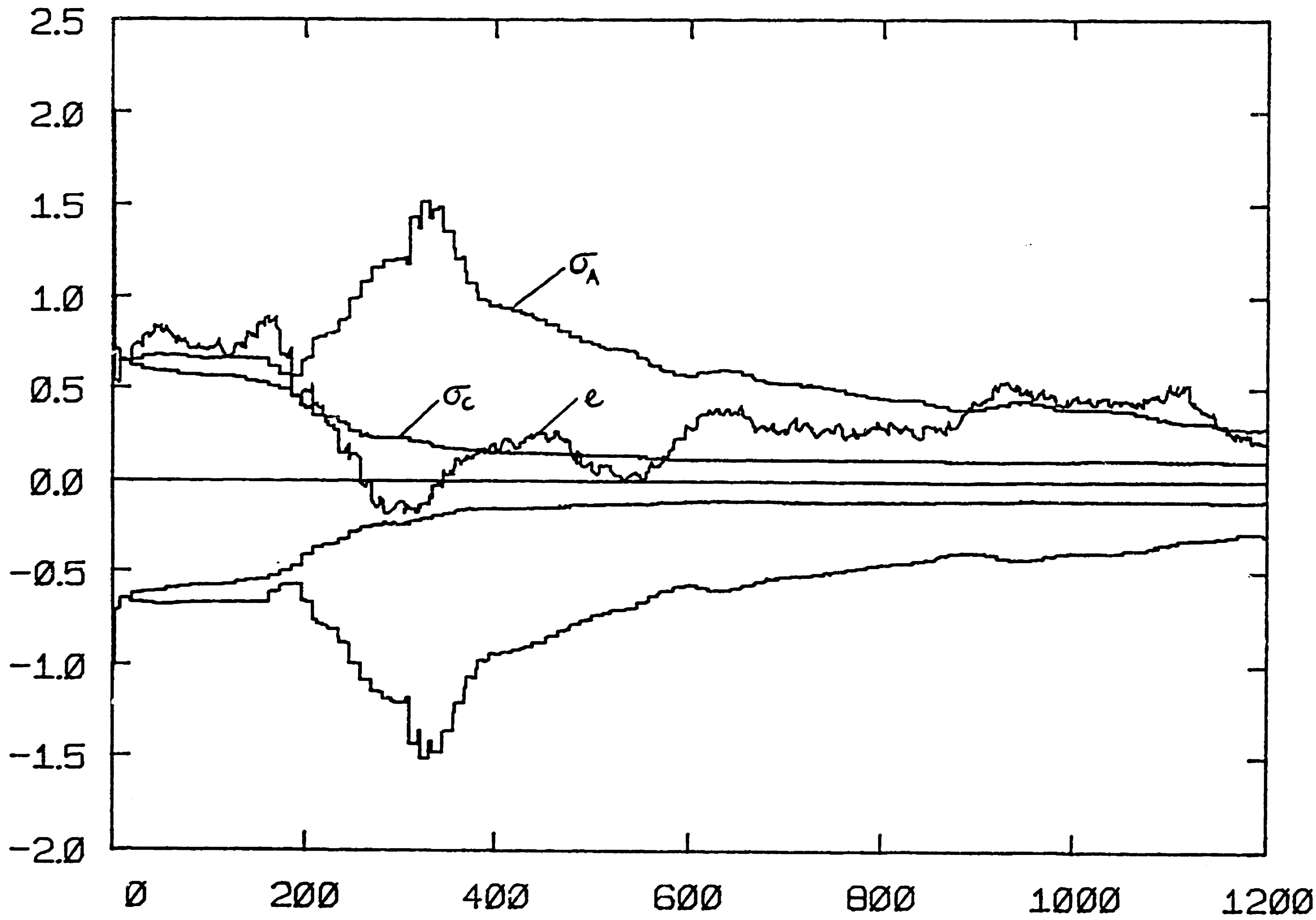


Figure 5.20(c). Dynamic community. Covariance-based hierarchy. Clock error of member 4.

Figures 5.22, 5.23, 5.24 refer, respectively, to member 2's clock error, member 3's North error and member 4's East error. These errors have been selected as particularly representative each of one member's behavior. The root sum square value of the unforced error (summed over a set of linearly independent unity initial conditions) is labelled σ_S. The filter-computed error standard deviation σ_C is also shown, so that the reader may see what the filter activity is; σ_C is not expected to go to zero.

Member 2's σ_S decreases monotonically; member 3's has a peak; member 4's has many peaks. The reader is asked to remember that a monotonically decreasing input to an exponentially stable linear system gives rise to a response with a peaked transient (see, for instance, the proof of lemma 5 in sect. 3.3). Consider, besides, that the input to a member's estimation errors are the errors of the members it ranges to (eq. (3.18)). Member 2 ranges only to the master, which has no errors; the unforced decay of member 2's errors is monotonical. Member 3 ranges to the master and to member 2; its errors are expected to be the sum of a monotonically decaying function (the unforced response) and a function with a peaked transient (the response to member 2's errors). This is exactly what fig. 5.23 shows. Member 4, finally, ranges to members 1, 2 and 3. Its transient will be complicated, with a peak as a response to member 2's monotonical error, and many secondary peaks as a response to member 3's peaked error. All this is punctually verified in fig. 5.24.

This property probably indicates the necessity of a trade-off. On one hand it is advisable to hierarchise a community with as many ranks as possible, in order to increase the number of accepted rangings, and to reduce the probability of some members having to depend only on far-away

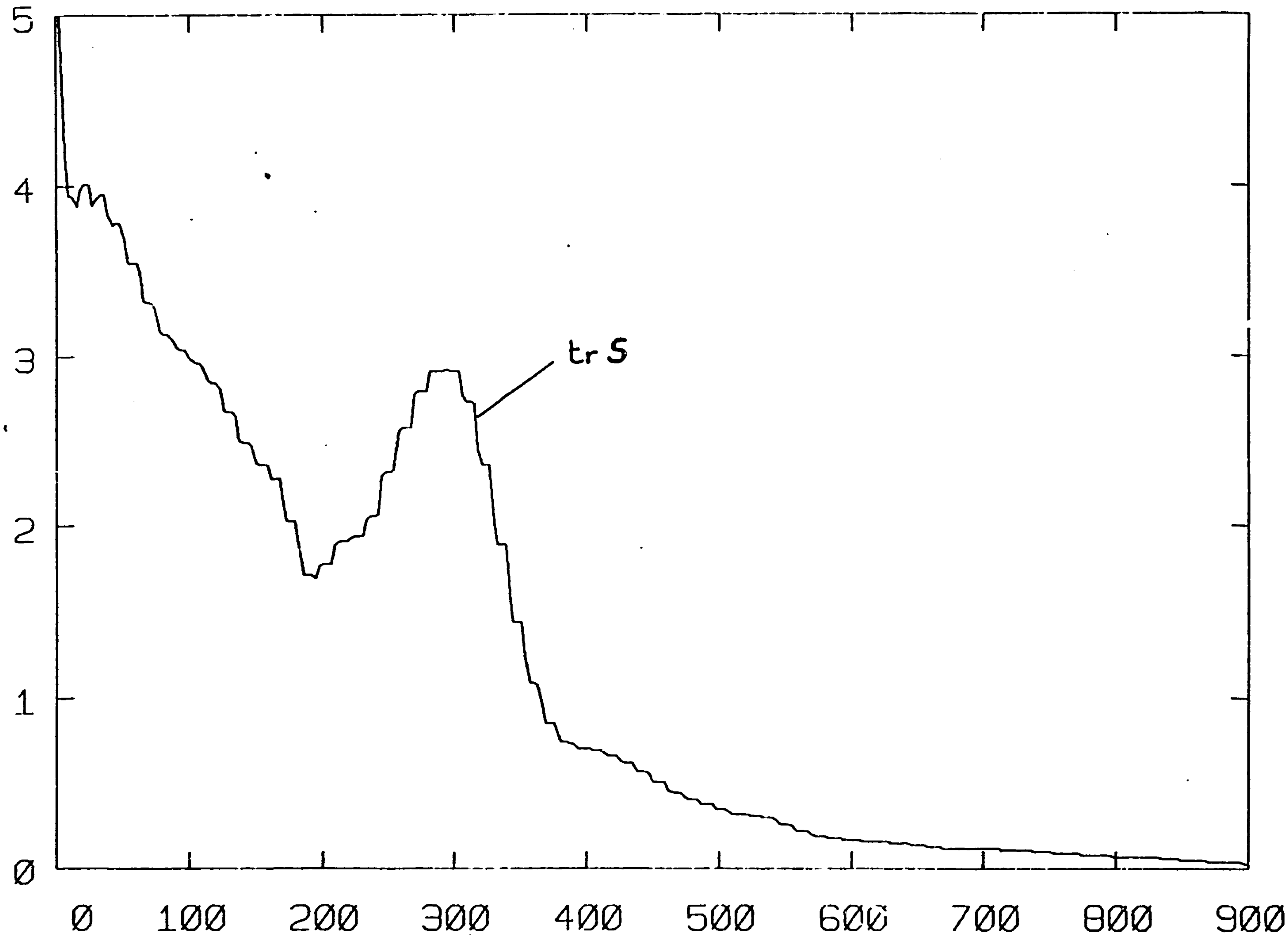


Figure 5.21. Stability test. Fixed-rank hierarchy. Square sum of all errors.

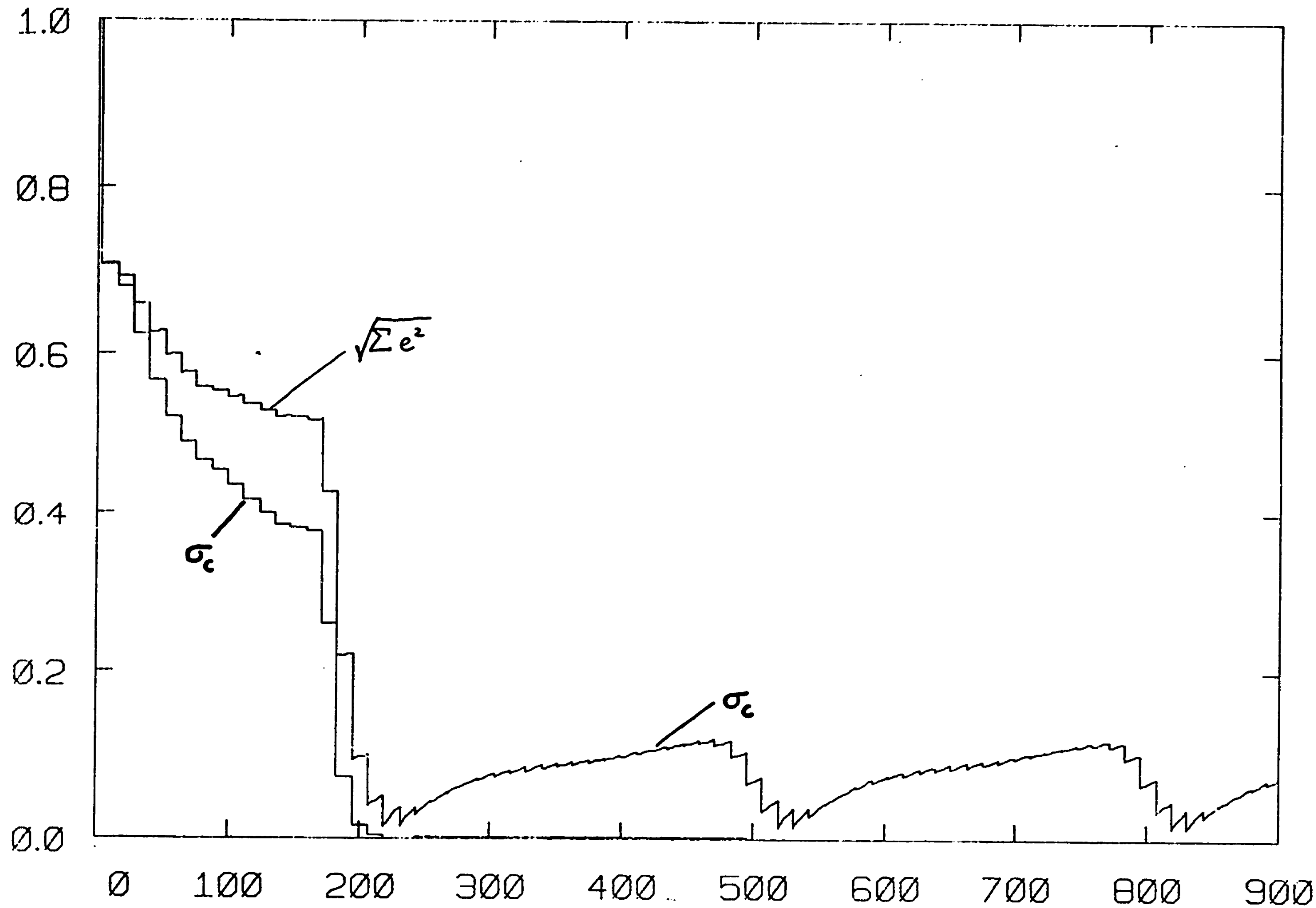


Figure 5.22. Stability test. Fixed-rank hierarchy. Root sum square clock error of member 2.

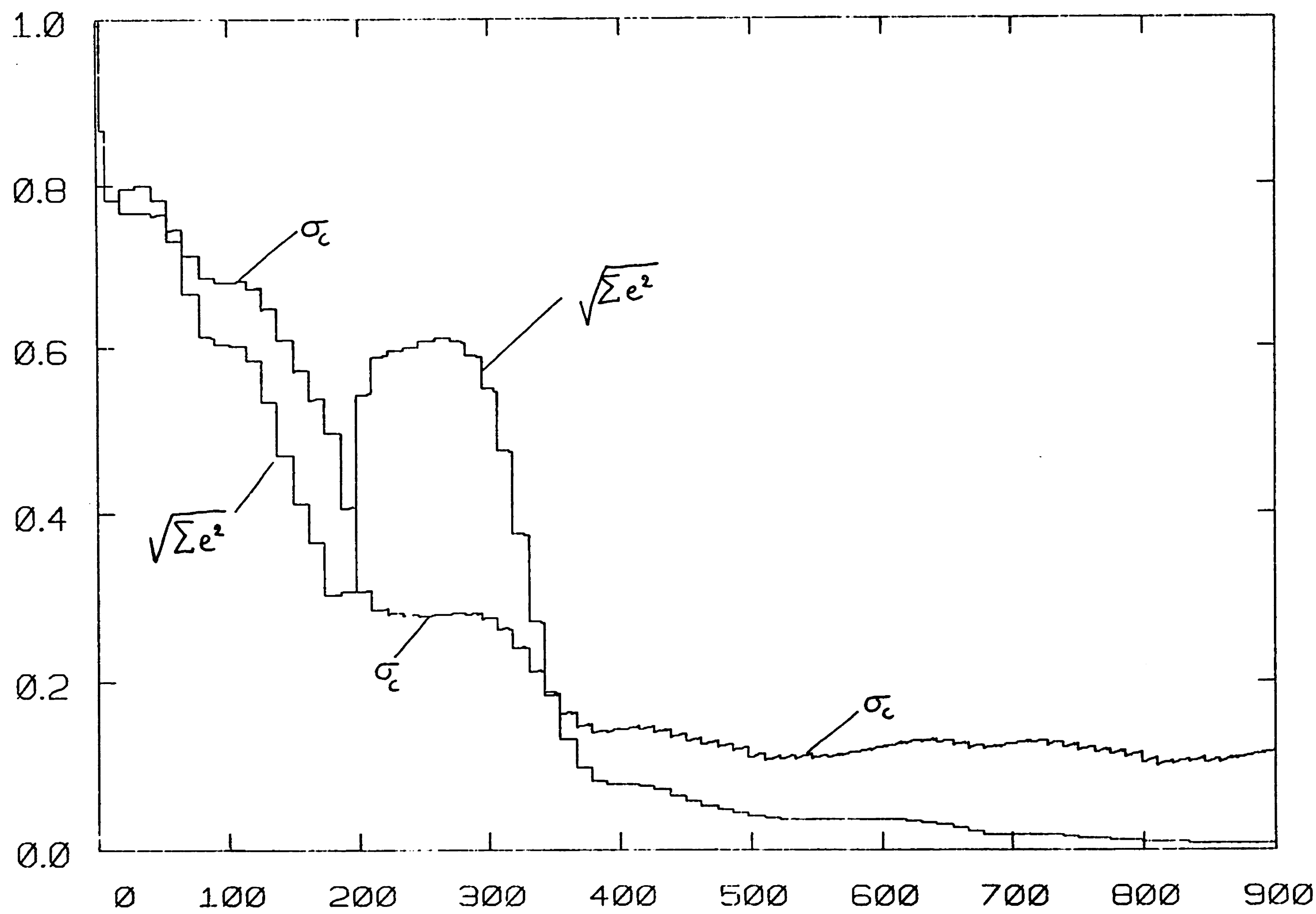


Figure 5.23. Stability test. Fixed-rank hierarchy. Root sum square north error of member 3.

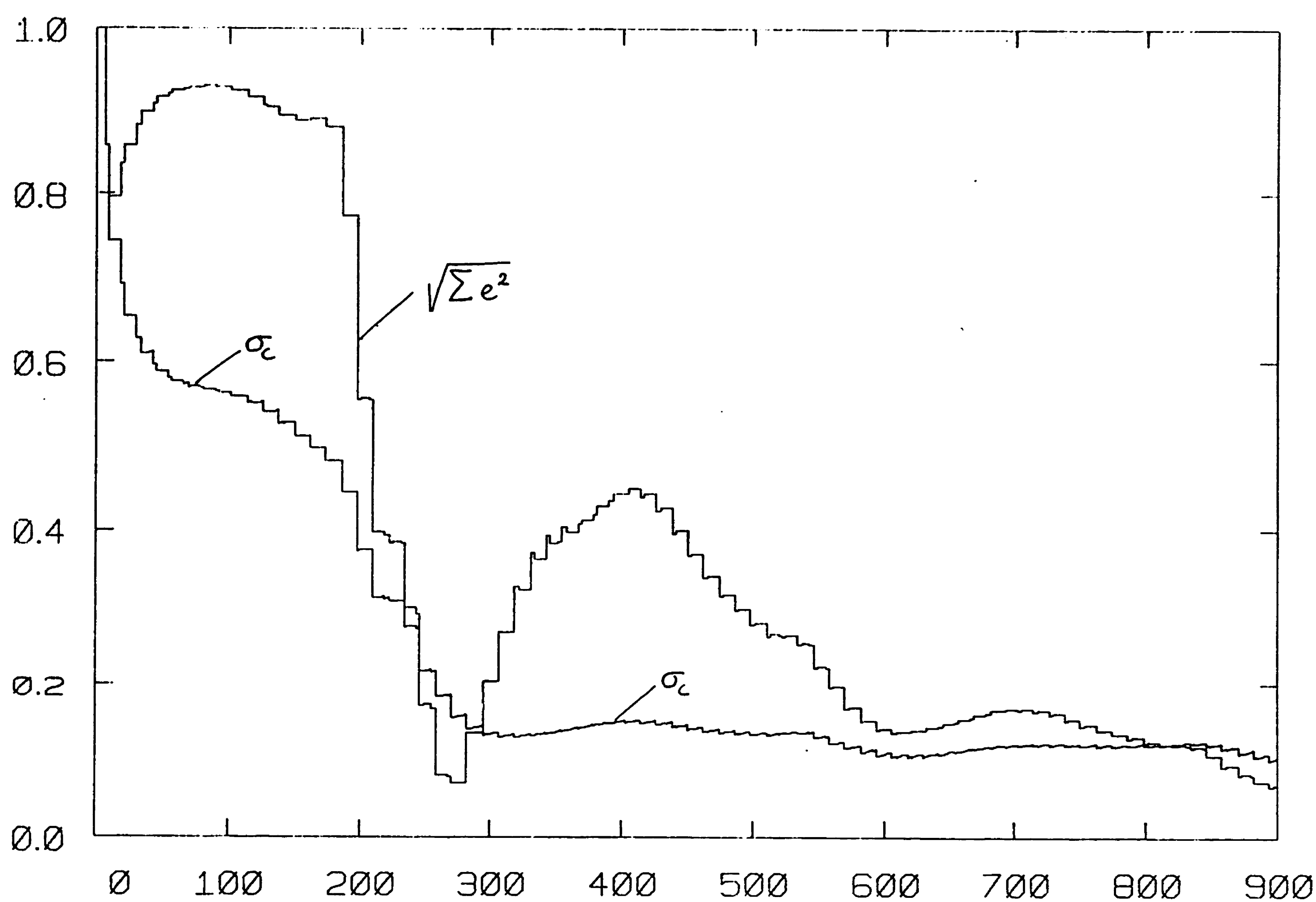


Figure 5.24. Stability test. Fixed-rank hierarchy. Root sum square east error of member 4.

sources. On the other hand the error transient gets worse and worse as one goes down the hierarchical ladder.

The same test was applied to a democratic community, with identical geometry. Figure 5.25 shows tr S_n. The pattern is unmistakably that of an unstable oscillation, with a 1/300 Hertz component (the back-and-forth motion of member 2) and a proper frequency component. No need to show the several σ_s, which only repeat the same pattern.

A covariance-based community under the same conditions proved, in sect. 5.5, to behave much like a fixed-rank hierarchy. This is confirmed by the stability test. Figure 5.26 shows tr S_n. The increase is larger and more jagged, the secondary peaks are more visible, but otherwise it resembles fig. 5.21. The same is true for the several σ_s, which are not shown.

5.7 Conclusions about ownstate filter communities

The behavior of an ownstate filter community has proved to be critically determined by its source-selection logic. Therefore, there will be separate conclusions for the three main kinds of organization.

(a) Fixed-rank hierarchy

This organization has been proved exponentially stable under mild observability and controllability conditions, which are insured by the presence of motion. The simulations shown in this chapter confirmed and illustrated the analytical results.

(b) Democracy

The general opinion is that a democratic organization is inherently unstable. Simulations and flight tests for PLRACTA, simulations for JTIDS

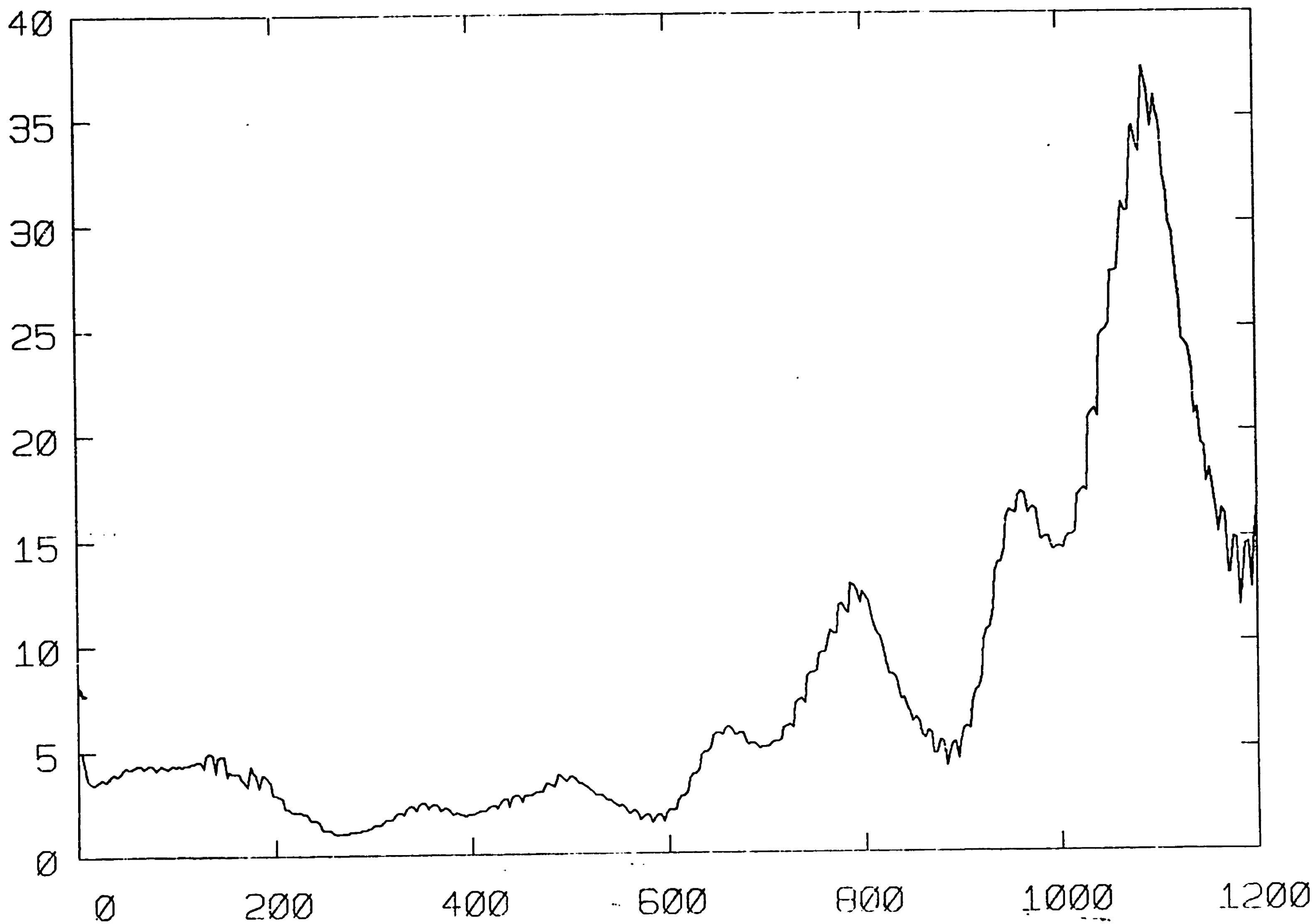


Figure 5.25. Stability test. Democracy. Square sum of all errors.

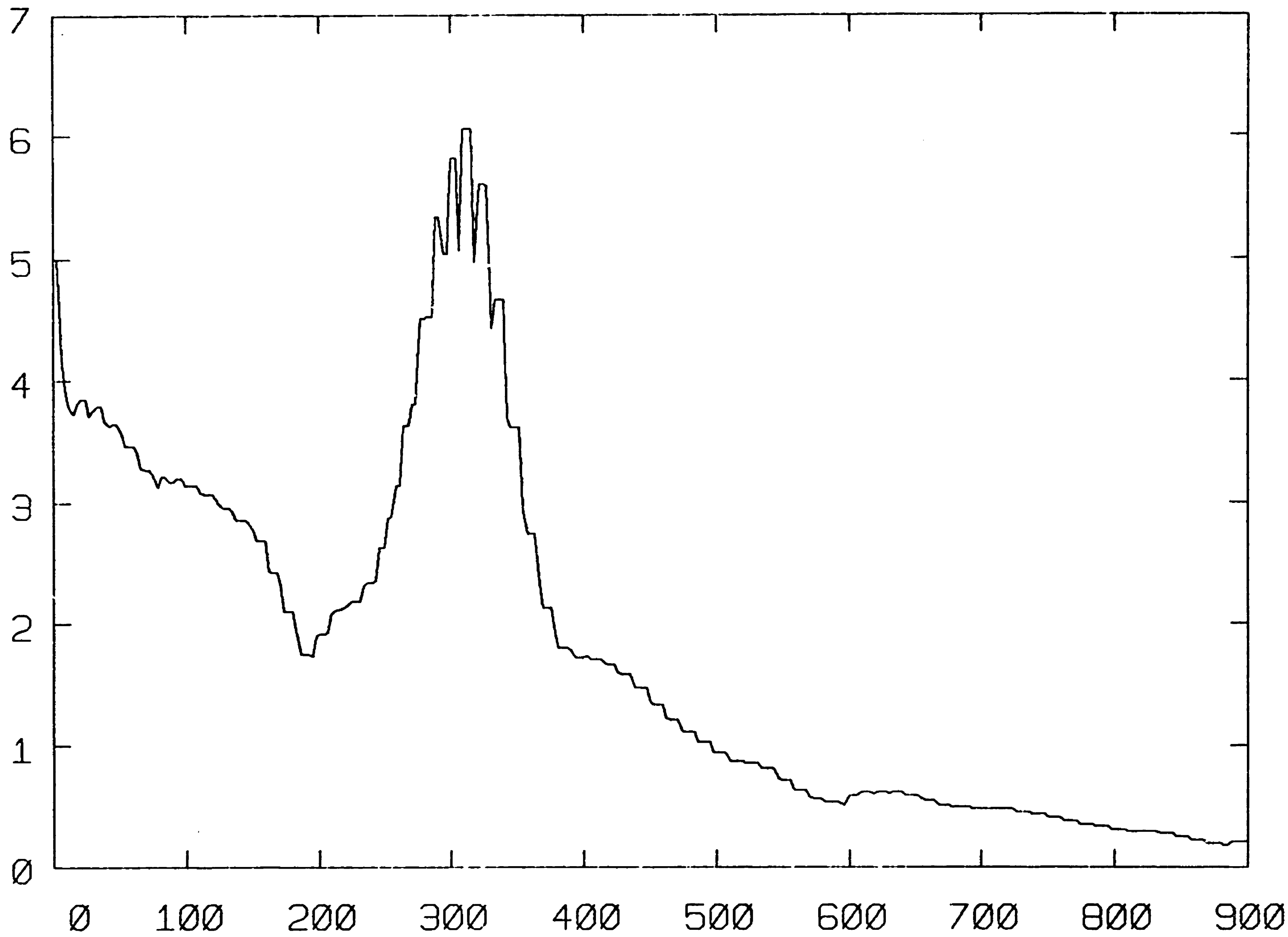


Figure 5.26. Stability test. Covariance-based hierarchy. Square sum of all errors.

by several authors, and the simulations by the present writer, all support this opinion.

The stability tests presented here have not the same cogency of an analytic proof. However, begging the question of theoretical instability, it can be said that a democratic organization has an unacceptable behavior, which looks unstable over operative durations.

(c) Covariance-based hierarchy

This organization looks particularly impervious to analysis, since one does not know what the pattern of source selections will be; it surely varies from case to case. In both examples simulated here, the members settled down spontaneously into a one-member-per-rank hierarchy, with only rare reversals of this pattern. This may be sheer chance, of course, and other cases may, instead, verify Rome and Stambaugh's conjecture.

These authors speculated in Ref. ( ) that a covariance-based hierarchy would, in the long run, behave like a democracy. They argued that the computed error covariances of most members, including masters, would in time drop down to the same level; after that, the source selection logic would no more enforce a hierarchy and rank reversals would happen continuously.

An either-or discussion can be attempted. A covariance-based organization may

i) either have frequent closed-loop rangings;

(ii) or it may approximate a fixed-rank hierarchy with only rare rank reversals.

In the first case the behavior approximates that of a democracy, and shares the same negative judgment. The second case can be analyzed as a fixed-rank hierarchy into which perturbations are introduced now and then. Consequently, two subcases are possible:

(ii-a) either the approximated fixed-rank hierarchy has a strong asymptotic stability; in this case (exemplified in the dynamic example of sect. 5.5) the perturbations will die out and the general behavior will be like a fixed-rank hierarchy's, but possibly with a longer transient;

(ii-b) or the approximated fixed-rank hierarchy will have a weak asymptotic stability, or none at all; in this case the perturbations will not die out soon enough, and their combined effect will create a general pattern like a democracy's, although on a longer time-scale (this is precisely the case of the static example of sect. 5.4).

There will be intermediate cases between these extremes. The probable conclusion is that a covariance-based hierarchy will behave, at its worst, like a democracy, and, at its best, like a fixed-rank hierarchy.

(d) <u>General conclusions</u>

The only kind of organization that provenly satisfies the basic requirement for a filter's feasibility (i.e. is asymptotically stable), is a fixed-rank hierarchy with a master.

As for covariance-based hierarchy, it is likely that it will behave quite similarly to a fixed-rank hierarchy, if there is strong one-by-one observability. When this is the case, covariance-based hierarchy has the advantage of being able to adapt automatically the ranks to the situation. However, all rank reversals will introduce perturbations, which, if they do not die out soon, may add up and produce instability. Choosing covariance-based over fixed-rank hierarchy involves a risk and requires a value judgement.

CHAPTER 6

MEASUREMENT-SHARING COMMUNITIES WITH UNCOUPLED FILTERS

6.1 Introduction

It was announced in Chapter 1 that, besides analyzing ownstate communities, this work would present alternative solutions. One such alternative solution is described in this chapter.

The main features of the proposed solution are:

(i) every member shares its measurements to the community;

(ii) the members' filters are uncoupled with one another;

(iii) every member's filter models a reduced state vector, which includes not only its own states, but also some of the other members' states; namely, their positions and clock phases.

Sections 6.2, 6.3 and 6.4 present reasons for adopting, respectively, these three features. Section 6.5 presents the filter equations. The remaining sections go into a few practical details, giving examples of state vector composition and message contents, scheduling and length.

6.2 Advantages of measurement sharing

Sharing measurements is costly in terms of communication requirements, but it involves remarkable advantages.

A filter which knows all the rangings that are taken in the community has observability even in a static geometry (apart from a few exceptions). Consider that, if m is the number of members, m(m-1) rangings are taken during every round of broadcasts. The unknown variables the rangings are supposed to determine are four per member (clock phase and three-dimensional position) minus the seven that are unobservable (rigid translations and rigid rotations of the whole community, an equal phase shift of all clocks). It is easy to verify that:

$$m\ (m-1) > 4m-7 \qquad (6.1)$$

for all m. If one considers a planar case, which implies three unknowns per member and four unobservables, one finds, instead:

$$m\ (m-1) > 3m-4 \qquad (6.2)$$

the equality being valid for m = 2. In both cases the number of rangings equals or exceeds the number of unknowns. Apart from a few exceptional geometries (e.g., all members on the same plane, for the three-dimensional case; all members on a line, for the planar case) the rangings taken during a round of broadcasts will be sufficient to make positions and clock phases observable, whether there is motion or not.

An illustration of the effects of increased observability is provided by Fig. 6.1 to 6.4. They result from the simulation of an optimal centralized filter, which knows all the measurements as soon as they are taken, and models all state variables. The other features of the simulation are like in the static case of Chapter 5 (see Sections 5.2 and 5.3). Figure 6.1 shows the errors of member 2, and is to be compared with Fig. 5.1, 5.4, 5.8 (illustrating the same errors in the three kinds of ownstate community organizations). A low-valued steady state is reached immediately after the first 12 sec round. The other figures, 6.2 to 6.4, show the errors of members 2, 3 and 4 on a more detailed scale. The dynamic case of Chapter 5 was also simulated; the results are almost identical. Figure 6.5 shows the errors of member 2 and should be compared with figures 5.12, 5.15, and 5.18.

6.3 The use of uncoupled filters

The subjects of this section are: under what conditions ranging-aided navigation can be performed by uncoupled filters, and what the advantages are.

Calling $\underline{\tilde{\xi}}^i$ the error state vector of the i-th filter of a set, its being uncoupled from the others is expressed by

$$\underline{\tilde{\xi}}_n^i = (\underline{\tilde{\xi}}_{n-1}^i, \underline{w}_n^i, \underline{v}_n^i) \tag{6.3}$$

where $\underline{w}^i$ and $\underline{v}^i$ have the usual meanings (driving noise and measurement noise, respectively). Equation (6.3) states that there must be no dependency of $\underline{\tilde{\xi}}^i$ on $\underline{\tilde{\xi}}^j$, for $j \neq i$. Since the estimates are functions of the measurement innovations, this means that the innovations must be functions of $\underline{\tilde{\xi}}^i$ and the noise, but not of $\underline{\tilde{\xi}}^j$. Ranging innovations (see Eq. (2.4)) are functions of position and clock phase of both source and receiver. The consequence is that, if a member's filter has to be uncoupled from the others, it must model positions and clock phases of all possible sources and receivers, that is, of all members.

The possible advantages of having uncoupled filters come from the fact that they can be analyzed one by one. The theory of an isolated filter is better known than that of set of coupled filters. Sufficient conditions for stability are given by the Deyst and Price theorem; other properties are also well known.

The ownstate community is an example of the consequences of filter coupling. Equation (2.28) showed that the error of a ranging source is an input to the difference equation for the error of the receiver. This

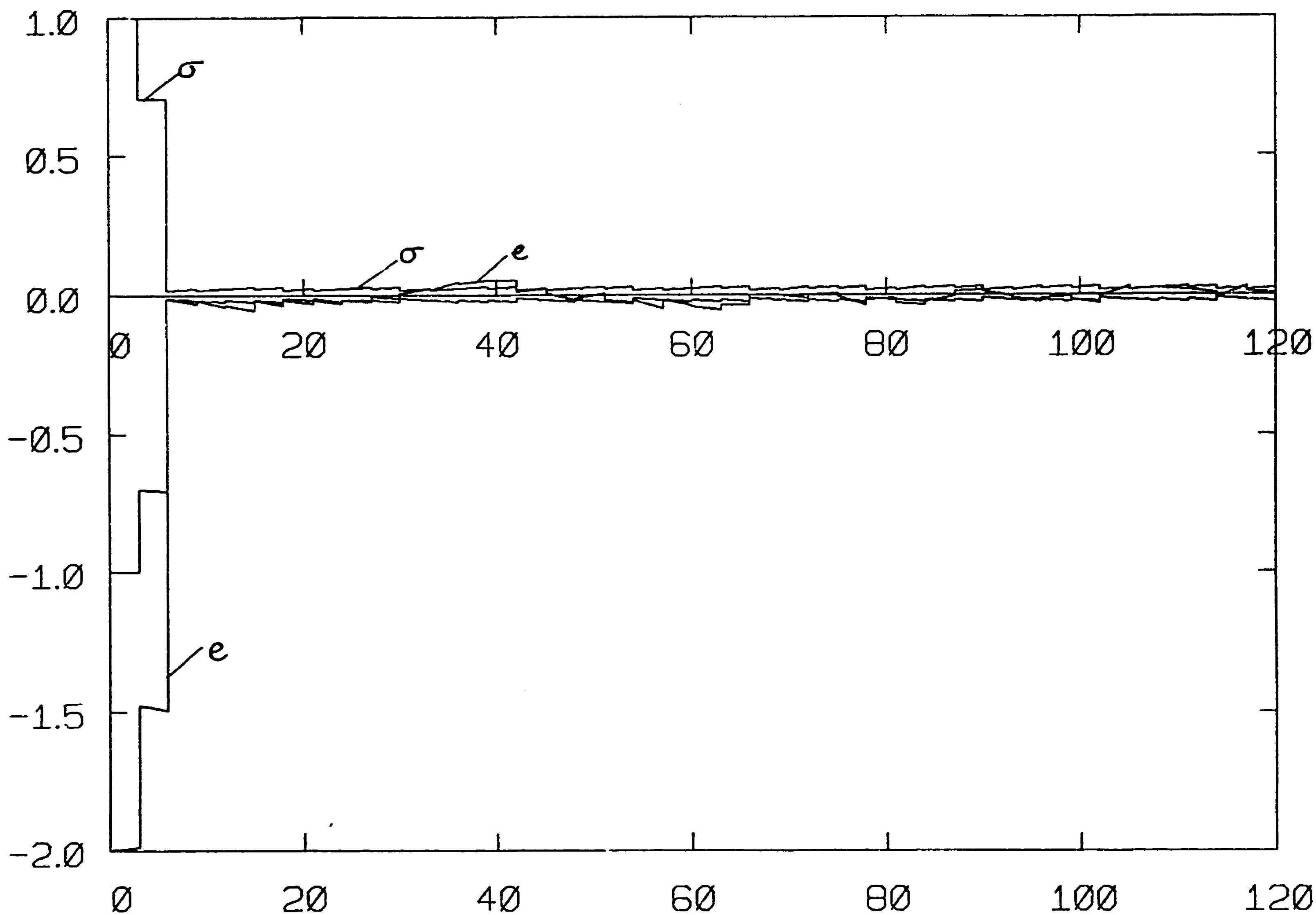


Figure 6.1(a). Static community. Centralized optimal filter. East error of member 2.

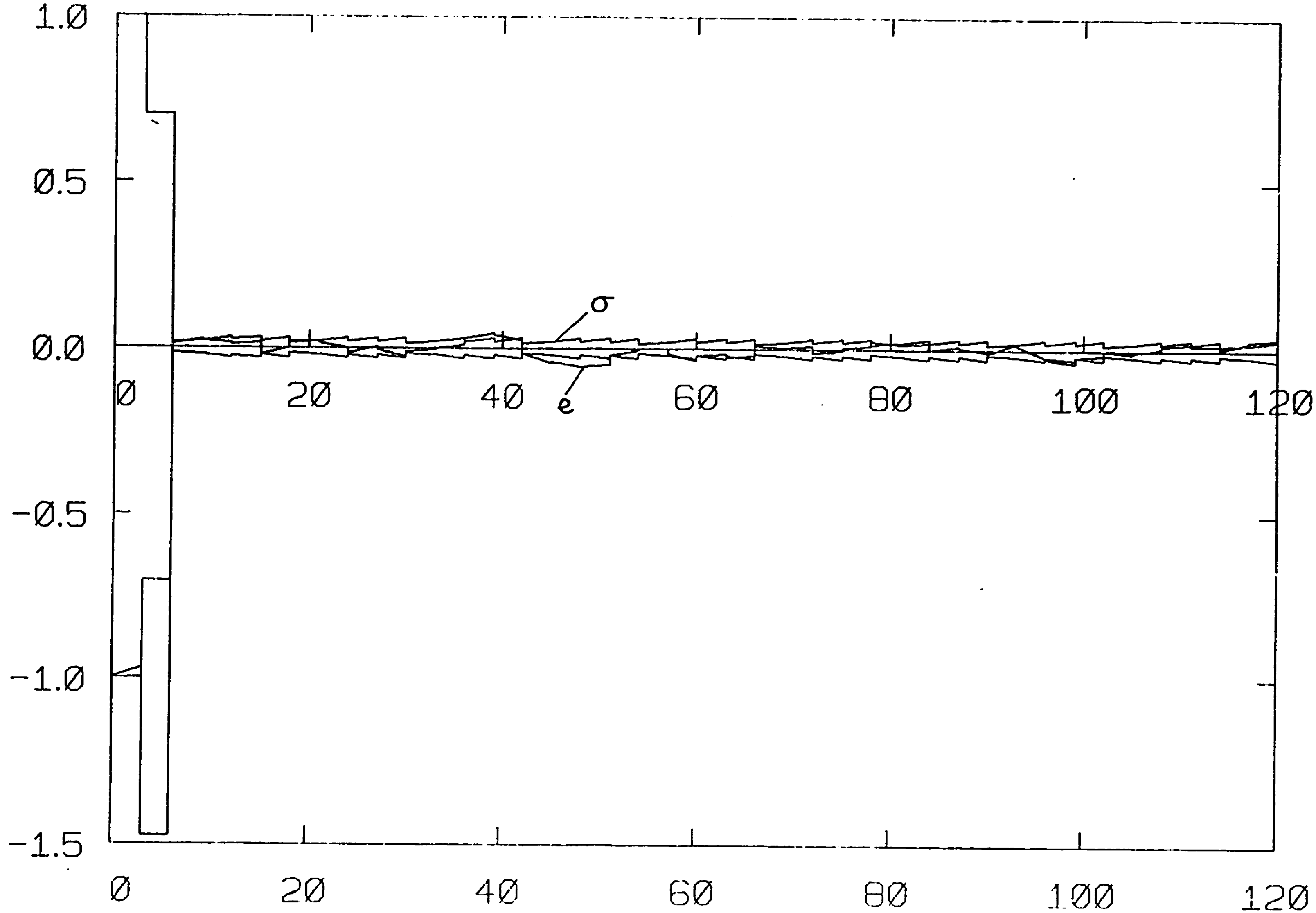


Figure 6.1(b). Static community. Centralized optimal filter. Clock error of member 2.

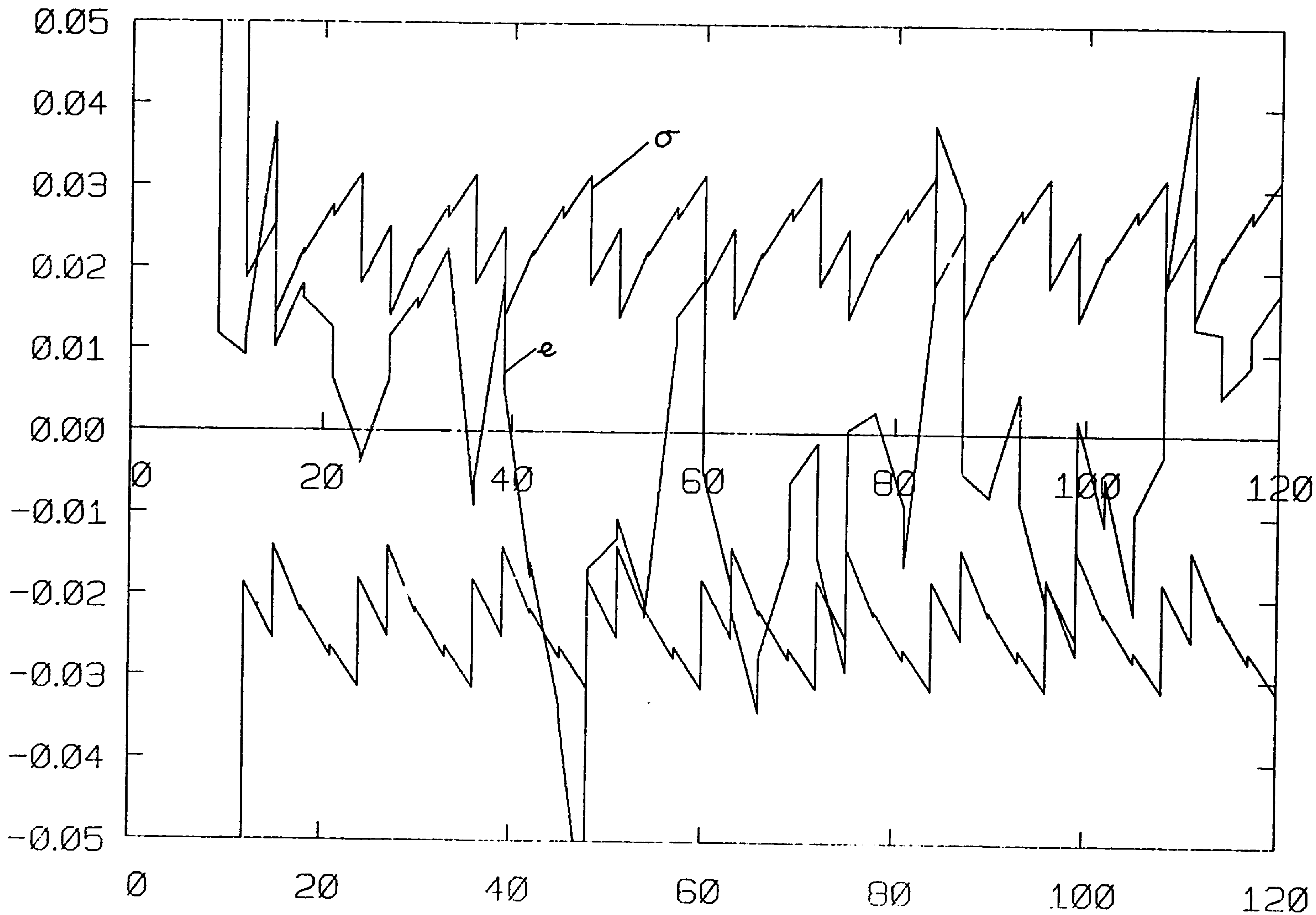


Figure 6.2(a). Static community. Centralized optimal filter. East error of member 2.

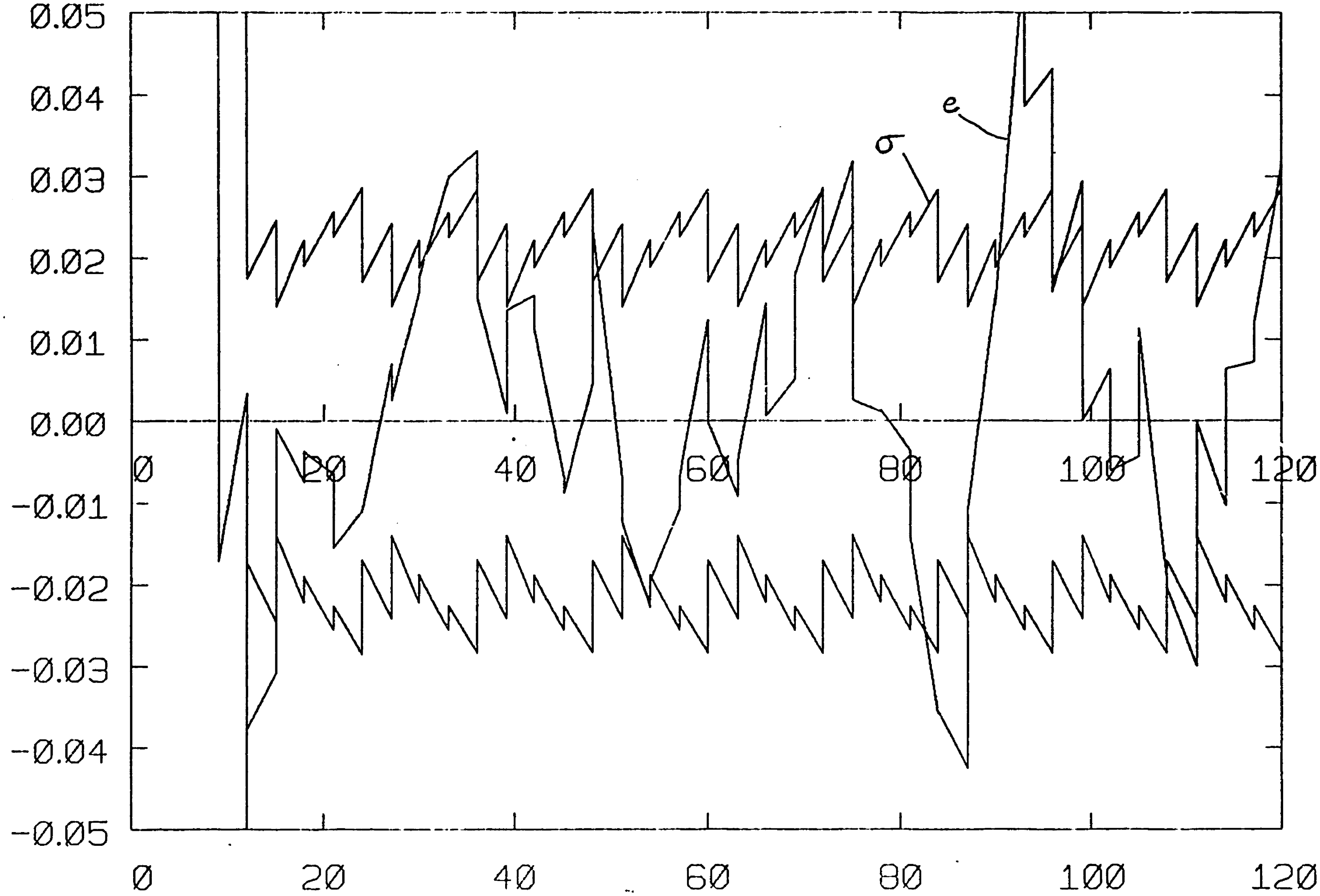


Figure 6.2(b). Static community. Centralized optimal filter. Clock error of member 2.

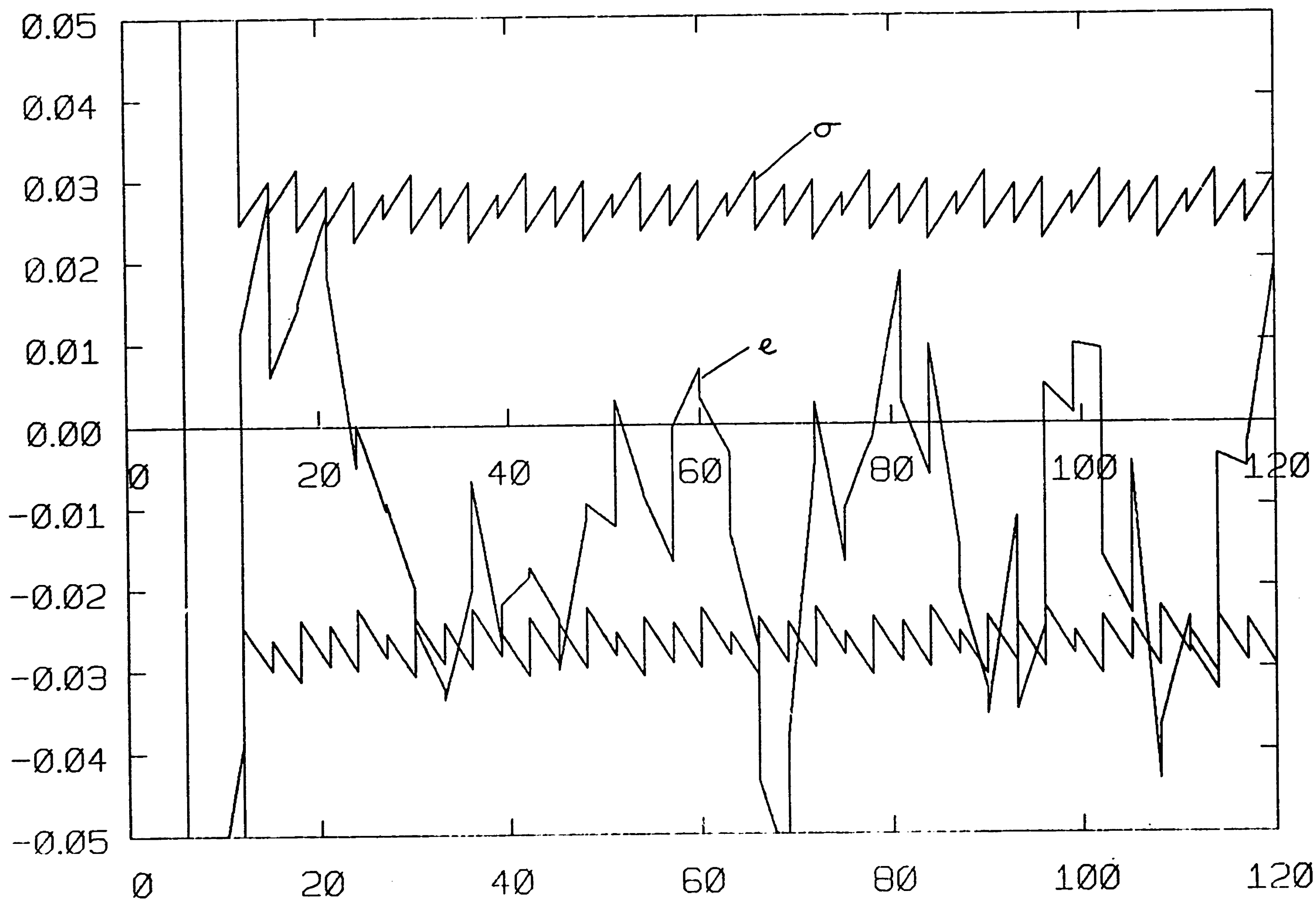


Figure 6.3(a). Static community. Centralized optimal filter. East error of member 3.

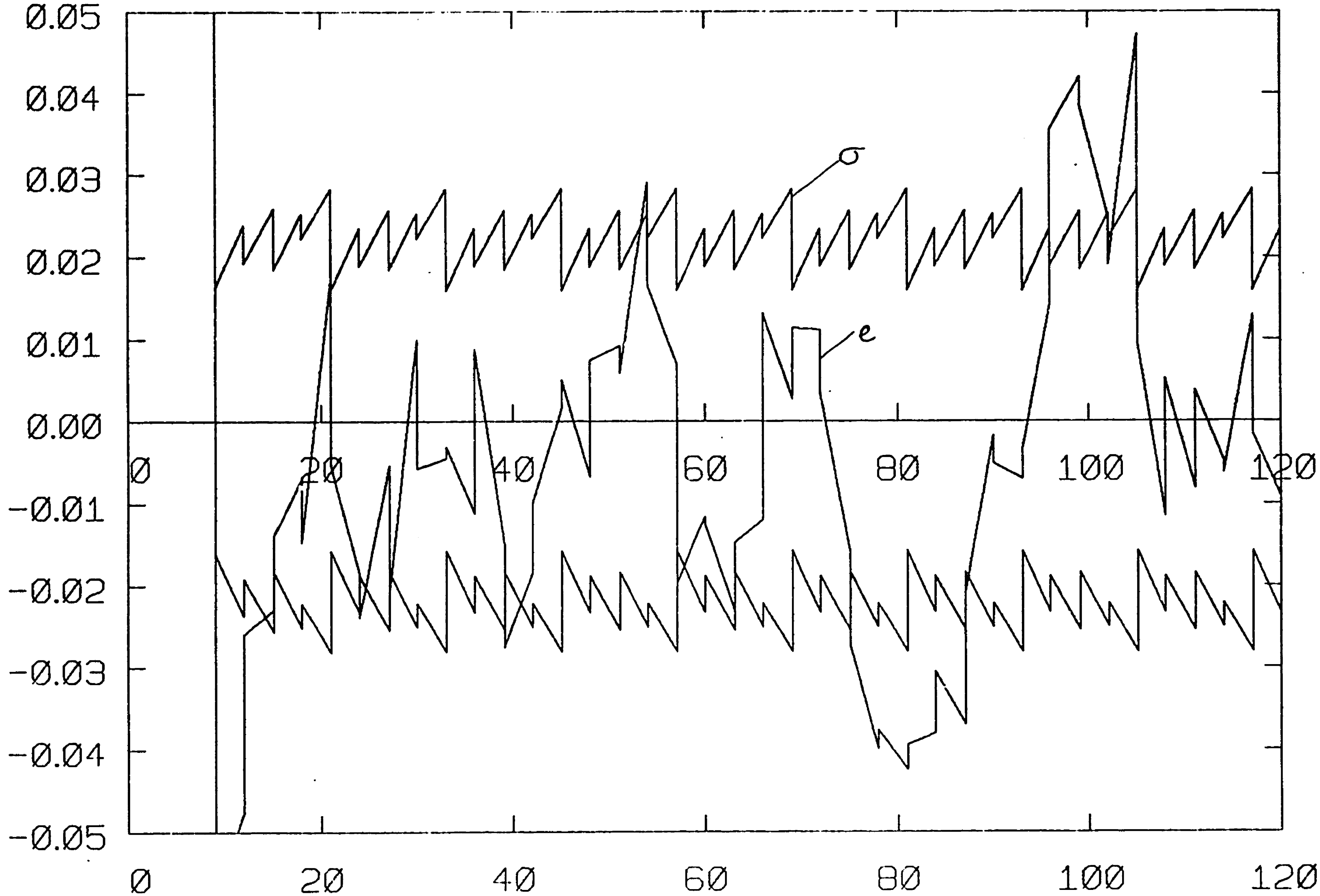


Figure 6.3(b). Static community. Centralized optimal filter. North error of member 3.

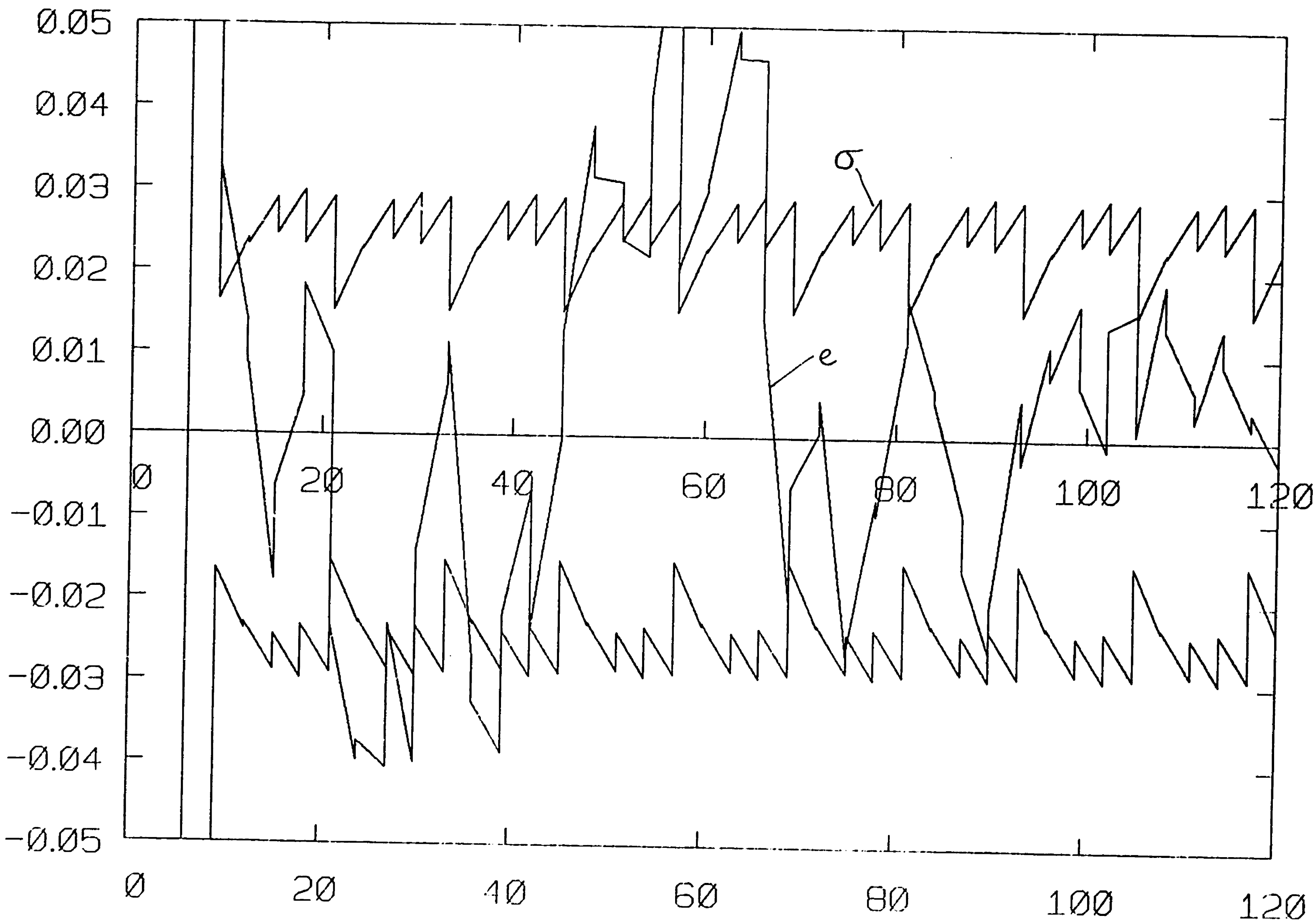


Figure 6.3(c). Static community. Centralized optimal filter. Clock error of member 3.

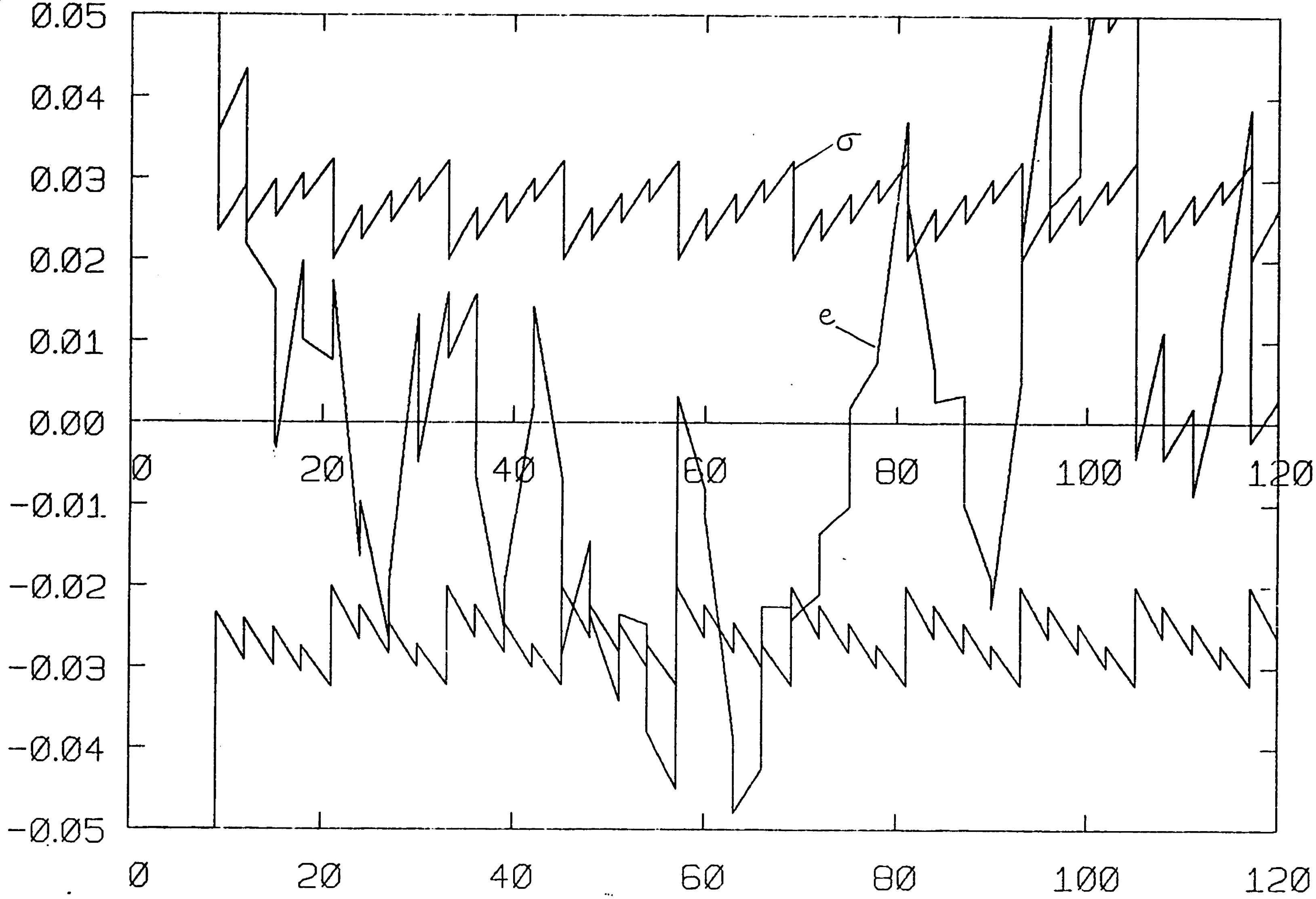


Figure 6.4(a). Static community. Centralized optimal filter. East error of member 4.

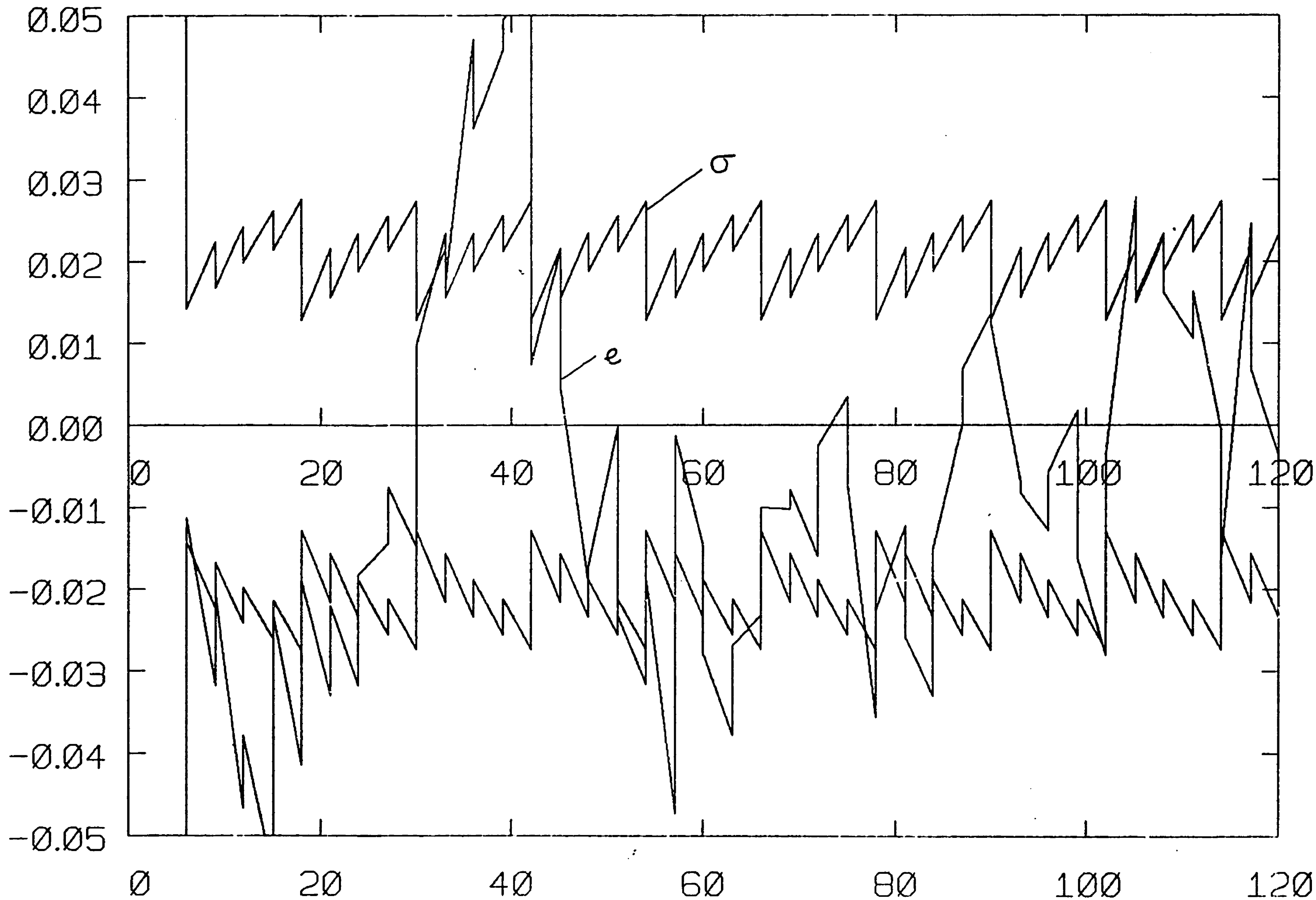


Figure 6.4(b). Static community. Centralized optimal filter. North error of member 4.

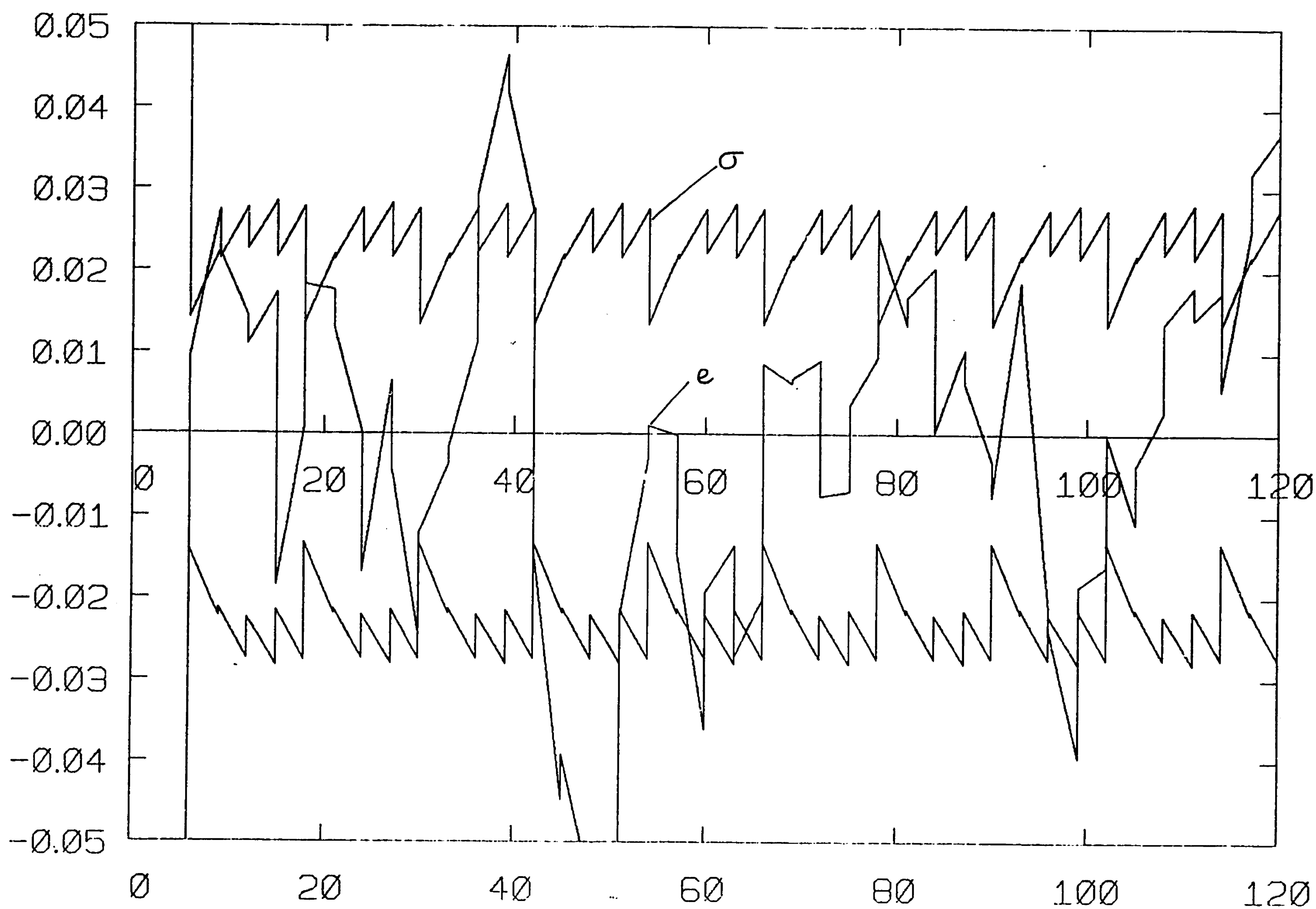


Figure 6.4(c). Static community. Centralized optimal filter. Clock error of member 4.

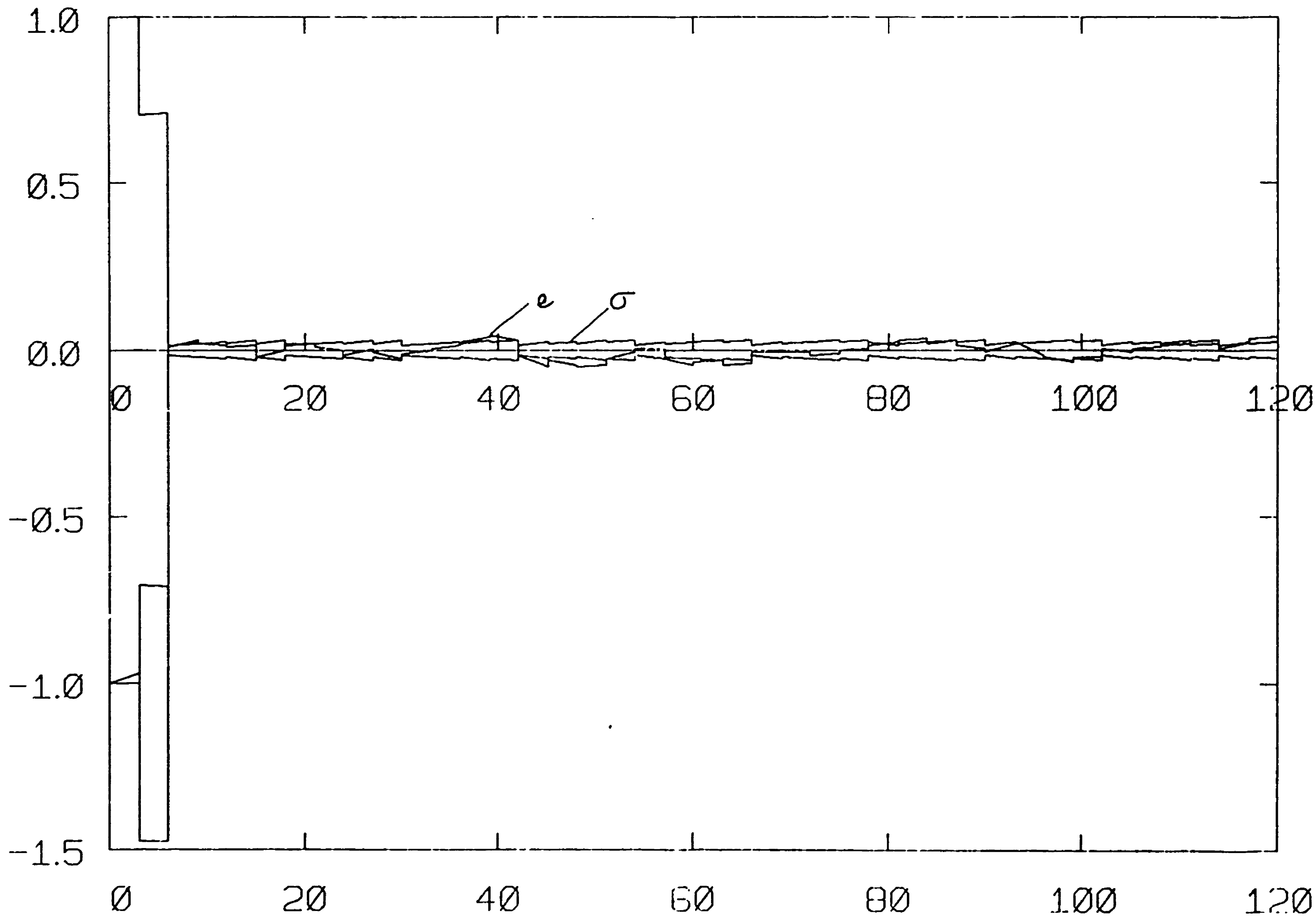


Figure 6.5(a). Dynamic community. Centralized optimal filter. East error of member 2.

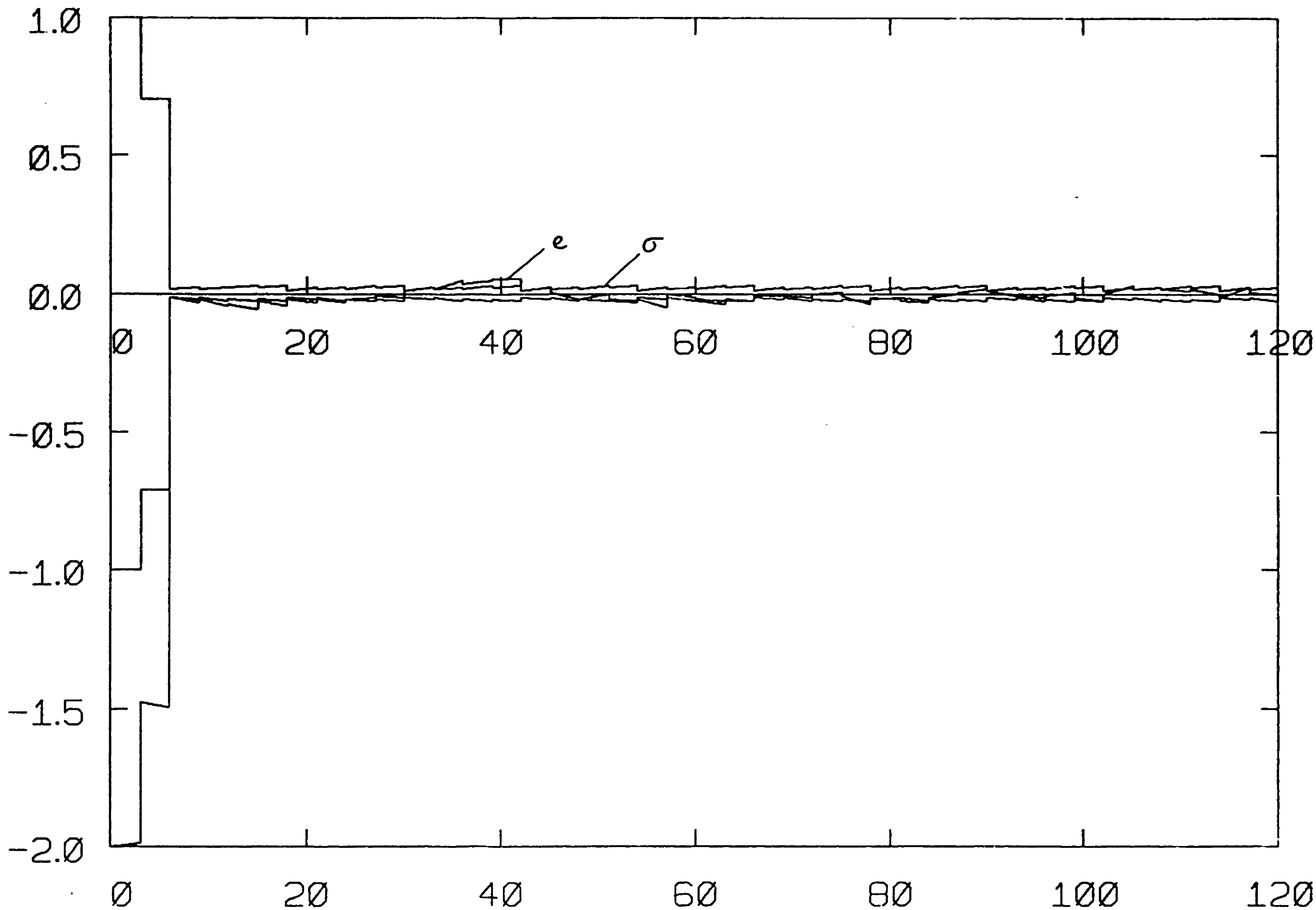


Figure 6.5(b). Dynamic community. Centralized optimal filter. Clock error of member 2.

coupling causes the long transient of hierarchical communities and (very likely) the instability of democratic communities. In a community whose filters are uncoupled, instead, hierarchy is not necessary for stability.

6.4 Advantages of retaining all measured variables

The filters proposed in this chapter drop all the state variables of other members, except those that are directly measured by the rangings. One advantage of retaining those variables appeared in the last section; namely, it allows every member's filter to be uncoupled from the others. But there is another advantage, which will be shown in this section.

A Kalman filter whose state vector (otherwise shortened with respect to reality) retains all the measured variables, is equivalent to a filter with a complete model, as far as measurement incorporation is concerned. Its only fall from optimality consists in incorrect propagation between measurements. This proposition will be better explained and proved presently.

Consider a linear system with linear or linearized measurements:

$$\underline{x}_n = \Psi_n \underline{x}_{n-1} + \underline{w}_n \quad ; \quad E\,(\underline{w}_n \underline{w}_m{}') = Q_n \delta_{nm} \tag{6.4}$$

$$\delta\rho_n = M_n \tilde{\underline{x}}_{n-} + \underline{v}_n \quad ; \quad E\,(\underline{v}_n \underline{v}_m{}') = R_n \delta_{nm} \tag{6.5}$$

and suppose a filter estimates only the shortened state vector:

$$\underline{\xi} = S\underline{x} \tag{6.6}$$

The projection matrix S has either the form

$$S = [I \;\; 0] \tag{6.7}$$

or can be derived from this expression by permuting columns. If one supposes that all measured variables are retained, the filter must be able to describe the innovations as functions of $\tilde{\underline{\xi}}$:

$$\delta\underline{\rho}_n = H_n \tilde{\underline{\xi}}_{n-1} + \underline{v}_n \tag{6.8}$$

Comparison of Eqs. (6.5), (6.6) and (6.8) gives:

$$M_n = H_n S \tag{6.9}$$

Call P and X the computed covariance matrices of the optimal and shortened filters, respectively.

Lemma. If the shortened filter's estimate and covariance matrix are optimal before a measurement incorporation, they are optimal after it; that is,

$$\text{if:} \quad \hat{\underline{\xi}}_{n-} = S\hat{\underline{x}}_{n-} \quad , \quad X_{n-} = SP_{n-}S' \tag{6.10}$$

$$\text{then:} \quad \hat{\underline{\xi}}_{n} = S\hat{\underline{x}}_{n} \quad , \quad X_{n} = SP_{n}S' \tag{6.11}$$

Proof. Plug Eqs. (6.9) and (6.10) into the measurement incorporation equations of a Kalman filter:

$$\hat{\underline{\xi}}_n = \hat{\underline{\xi}}_{n-} + X_{n-} H_n' (H_n X_{n-} H_n' + R_n)^{-1} \sigma\underline{\rho}_n =$$

$$= S\hat{\underline{x}}_{n-} + SP_{n-}S'H_n'(H_n SP_{n-}S'H_n' + R_n)^{-1} \, \delta\rho_n =$$

$$=S\hat{\underline{x}}_{n-} + P_{n-}M_n'(M_nP_{n-}M_n'+R_n)^{-1}\,\delta\rho_n =$$

$$=S\hat{\underline{x}}_n$$

$$X_n = X_{n-} - X_{n-}H_n'(H_nX_{n-}H_n'+R_n)^{-1}H_nX_{n-} =$$

$$=SP_{n-}S' - SP_{n-}S'H_n'(H_nSP_{n-}S'H_n'+R_n)^{-1}H_nSP_{n-}S' =$$

$$=S(P_{n-} - P_{n-}M_n'(M_nP_{n-}M_n'+R_n)^{-1}M_nP_{n-})S' =$$

$$=SP_nS'$$

Q.E.D.

The estimate and covariance matrix of the shortened filter fall away from their optimal values in their propagation between measurements. The shortened filter uses $S\Psi S'$ as transition matrix. Supposing that $\hat{\underline{\xi}}$ and X were optimal before the propagation:

$$\hat{\underline{\xi}}_{n-1} = S\hat{\underline{x}}_{n-1} \qquad ; \qquad X_{n-1} = SP_{n-1}S'$$

they become:

$$\hat{\underline{\xi}}_{n-} = S\Psi_nS'\hat{\underline{\xi}}_{n-1} = S\Psi_nS'S\hat{\underline{x}}_{n-1} \neq S\Psi_n\hat{\underline{x}}_{n-1} = S\hat{\underline{x}}_{n-}$$

$$X_{n-} = S\Psi_nS'X_{n-1}S\Psi_n'S' + SQ_nS' =$$

$$= S\Psi_n S'SP_{n-1}S'S\Psi_n'S' + SQ_nS' \neq$$

$$\neq S\Psi_n P_{n-1}\Psi_n'S' + SQ_nS' = SP_{n-}S$$

The deviation from optimality comes from the presence of the matrix

$$S'S = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{post-multiplying} \quad \Psi .$$

In order to reduce this effect, an established engineering practice is to propagate the error covariance matrix with the following equation:

$$X_{n-} = \Phi_n X_{n-1} \Phi_n' + SQ_nS' + \overline{Q}_n \tag{6.12}$$

where $\overline{Q}_n$ is an equivalent noise covariance meant to compensate for the lost terms and Φ is an appropriate approximation of $S\Psi S'$.

6.5 Equations for the shortened filter

The features of a filter with shortened model have already been explained and motivated in Sections 6.1 to 6.4. It remains to summarize its equations for our case.

The necessary symbols will now be introduced. Call $\underline{x}^i$ the ownstate of member i. Partition it as follows:

$$\underline{x}^{i'} = [\underline{z}^{i'} , \underline{y}^{i'}] \tag{6.13}$$

where the partition $\underline{z}^i$ is defined in such a way that a ranging of i to j is a function only of $\underline{z}^i$ and $\underline{z}^j$; the innovation is:

$$\delta\rho_n^{ij} = \underline{h}_n^{ij'} \tilde{\underline{z}}_{n-}^i + \underline{g}_n^{ij'} \tilde{\underline{z}}_{n-}^j + v_n^{ij} \tag{6.14}$$

The ownstate difference equation is:

$$\underline{x}_n^i = \Psi_n^i \underline{x}_{n-1} + \underline{w}_n^i \tag{6.15}$$

which can be partitioned into:

$$\begin{bmatrix} \underline{z}^i \\ \underline{y}^i \end{bmatrix}_n = \begin{bmatrix} A^i & B^i \\ C^i & D^i \end{bmatrix}_n \begin{bmatrix} \underline{z}^i \\ \underline{y}^i \end{bmatrix}_{n-1} + \underline{W}_n^{i\,'} \tag{6.16}$$

The driving noise covariance and the measurement noise covariance are:

$$E\,(\underline{w}_n^i \underline{w}_m^{j\,'}) = Q_n^{ij} \delta_{nm} \tag{6.17}$$

$$E\,(v_n^{ij} v_m^{kl}) = r_n^{ij} \delta_{nm} \delta_{ik} \tag{6.18}$$

A noise covariance matrix for the whole community can be defined as:

$$Q_n \triangleq \begin{bmatrix} Q^{11} & \cdots\cdot & Q^{1m} \\ \cdots & & \cdots \\ Q^{m1} & \cdots\cdot & Q^{mm} \end{bmatrix}_n \tag{6.19}$$

Likewise, a global measurement noise covariance matrix is:

$$R_n = \text{diag}\,(r_n^{1j}, \ldots, r_n^{j-1,j}, 0, r_n^{j+1,j}, \ldots, r_n^{mj}) \tag{6.20}$$

j being the member that broadcasts at time t_n.

Now the filter equations can be given. The state of member i's filter is:

$$\underline{\xi}^{i\,'} \triangleq \left[\underline{x}^{i\,'}, \underline{z}^{1\,'}, \ldots, \underline{z}^{i-1\,'}, \underline{z}^{i+1\,'}, \ldots, \underline{z}^{m\,'}\right] \tag{6.21}$$

The propagation between measurements follows the equations:

$$\hat{\xi}^i_{n-} = \Phi^i_n \xi^i_{n-1} \tag{6.22}$$

$$X^i_{n-} = \Phi^i_n X^i_{n-1} \Phi^{i'}_n + SQ_n S' + \overline{Q}^i_n \tag{6.23}$$

where S is the projection matrix, defined like in Section 6.4, $\overline{Q}^i_n$ is the equivalent noise covariance, and:

$$\Phi^i_n \triangleq \text{diag}\ (\Psi^i_n, I, I, \ldots , I) \tag{6.24}$$

The incorporation of all rangings taken at time t_n, whose innovations can be aggregated into one vector:

$$\delta\underline{\rho}_n = [\delta\rho^{1j}, \ldots , \delta\rho^{j-1,j}, \delta\rho^{j+1,j}, \ldots , \delta\rho^{mj}]_n \tag{6.25}$$

follows the usual extended Kalman filter equations:

$$\underline{\hat{\xi}}^i_n = \underline{\hat{\xi}}^i_{n-} + K^i_n \delta\underline{\rho}_n \tag{6.26}$$

$$K^i_n = X_{n-} H^{i'}_n \ (H^i_n X_{n-} H^{i'}_n + R_n)^{-1} \tag{6.27}$$

$$X^i_n = (I - K^i_n H^i_n) X^i{}_{n-} \tag{6.28}$$

The measurement matrix H^i is defined implicitly by:

$$\delta\underline{\rho}_n = H^i_n \underline{\tilde{\xi}}_{n-} + [v^{1j}_n, \ldots , v^{mj}_n]' \tag{6.29}$$

and is composed by the geometry vectors $\underline{h}$ and $\underline{g}$ of Eq. (6.14) as well as by null partitions.

6.6 Example of performance of a shortened filter

The practice of dropping state variables from the filter model, and compensating for their absense with an equivalent driving noise covariance, is well established. It is known that, in many cases, if the equivalent noise covariance is opportunely chosen, with regard to the dynamics of the dropped variables and to the duration of the mission, the shortened-state filter performs almost as well as an optimal one.

Two simulations are plotted in Fig. 6.6 to 6.9 which illustrate the success of this approach in our application.

The truth model of these simulations included four members with the same geometry as in the static case of Chapter 5; horizontal position errors and clock phases were modeled, and their first derivatives (horizontal velocity and clock frequency errors) were modeled as well. The indicated velocities and frequencies (from dead reckoners and clocks, respectively) were corrupted by a random-walk noise, with incremental variance:

$$\sigma_v^2 = (3 \cdot 10^{-5} \text{ Km/sec})^2$$

over the $\Delta t = 3$ sec interval between measurements. This level of noise is meant to provide a crude model of the effect of the Schuler oscillation on the velocity errors of an inertial system; clock frequency noise was chosen at the same level for simplicity. Positions and clock phases were modeled to be the correct integrals of indicated velocities and clock frequencies, with no additional noise. The relative coordinates were defined like in the simulations of Chapter 5. As a consequence, there was no driving noise for the rates of the states which are zero by definition.

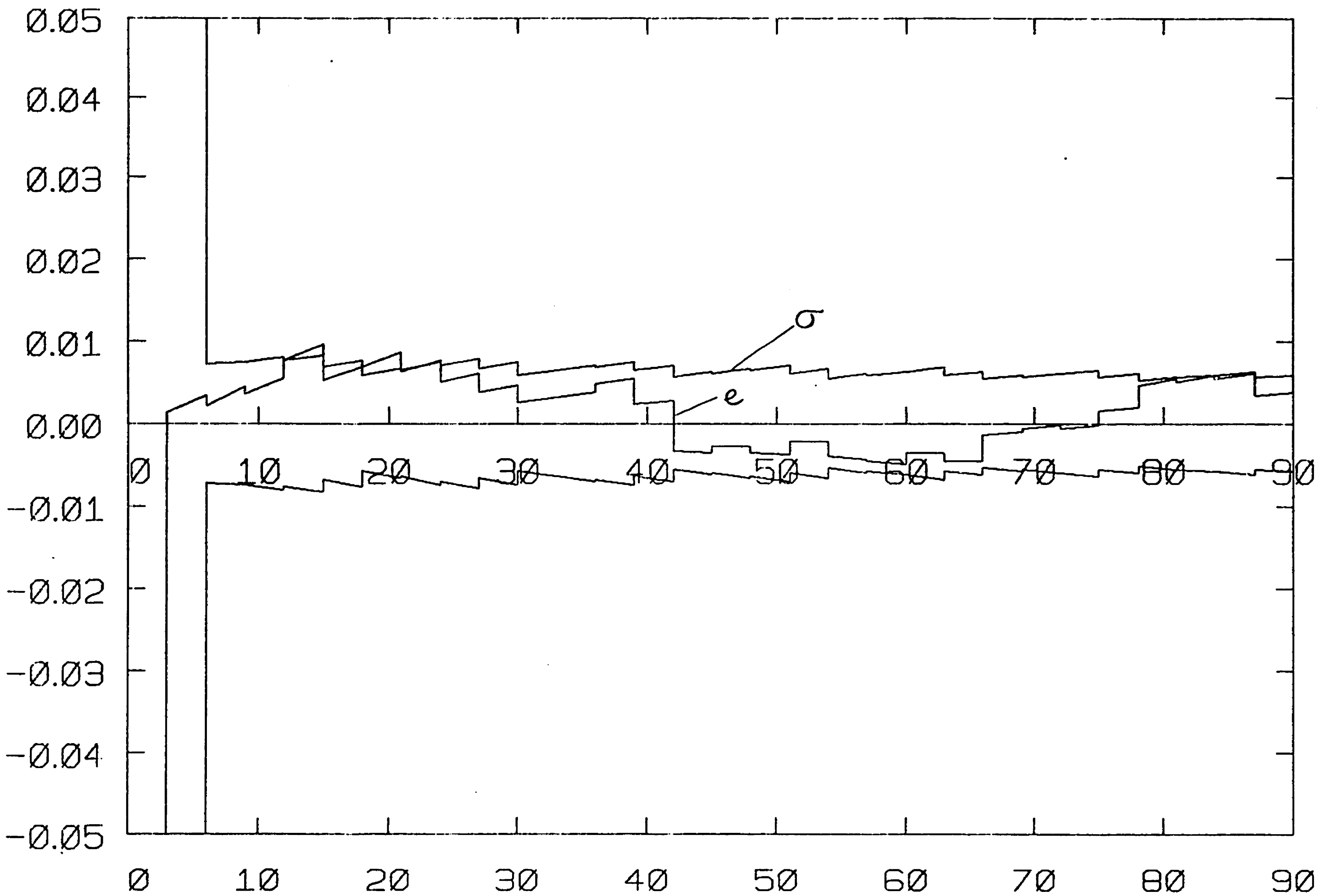


Figure 6.6(a). Centralized optimal filter including rate states. East error of member 2.

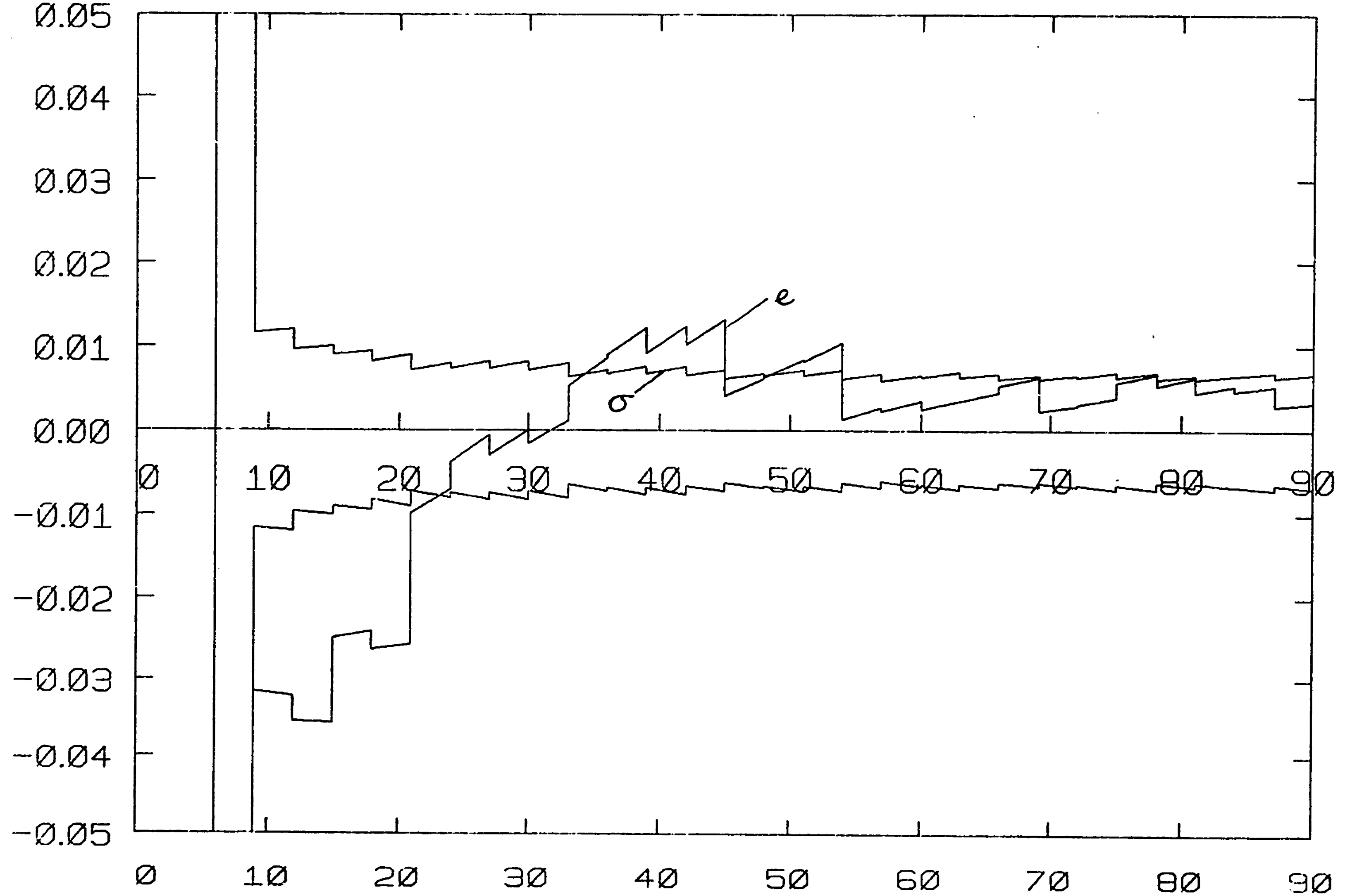


Figure 6.6(b). Centralized optimal filter including rate states. East error of member 3.

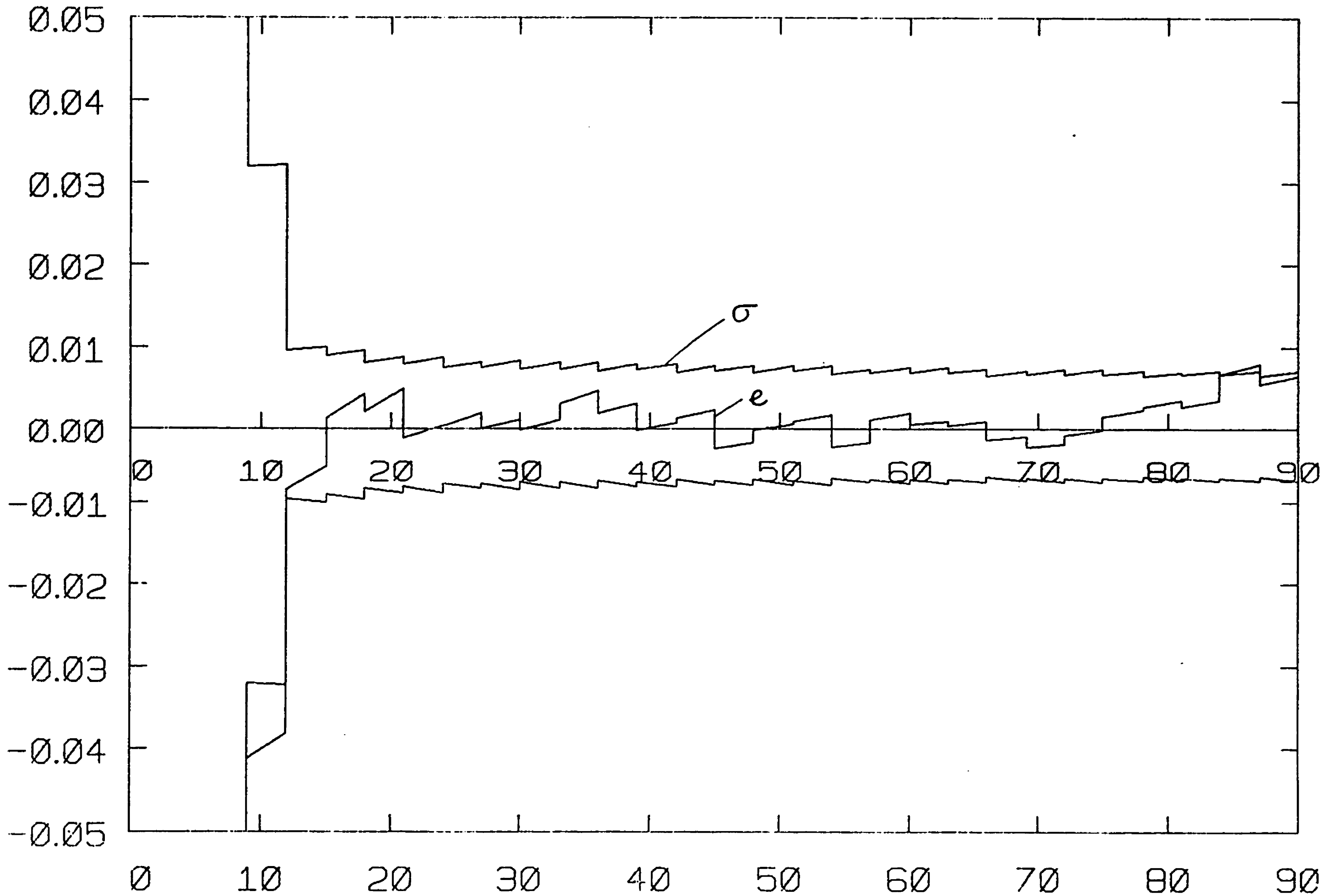


Figure 6.6(c). Centralized optimal filter including rate states. East error of member 4.

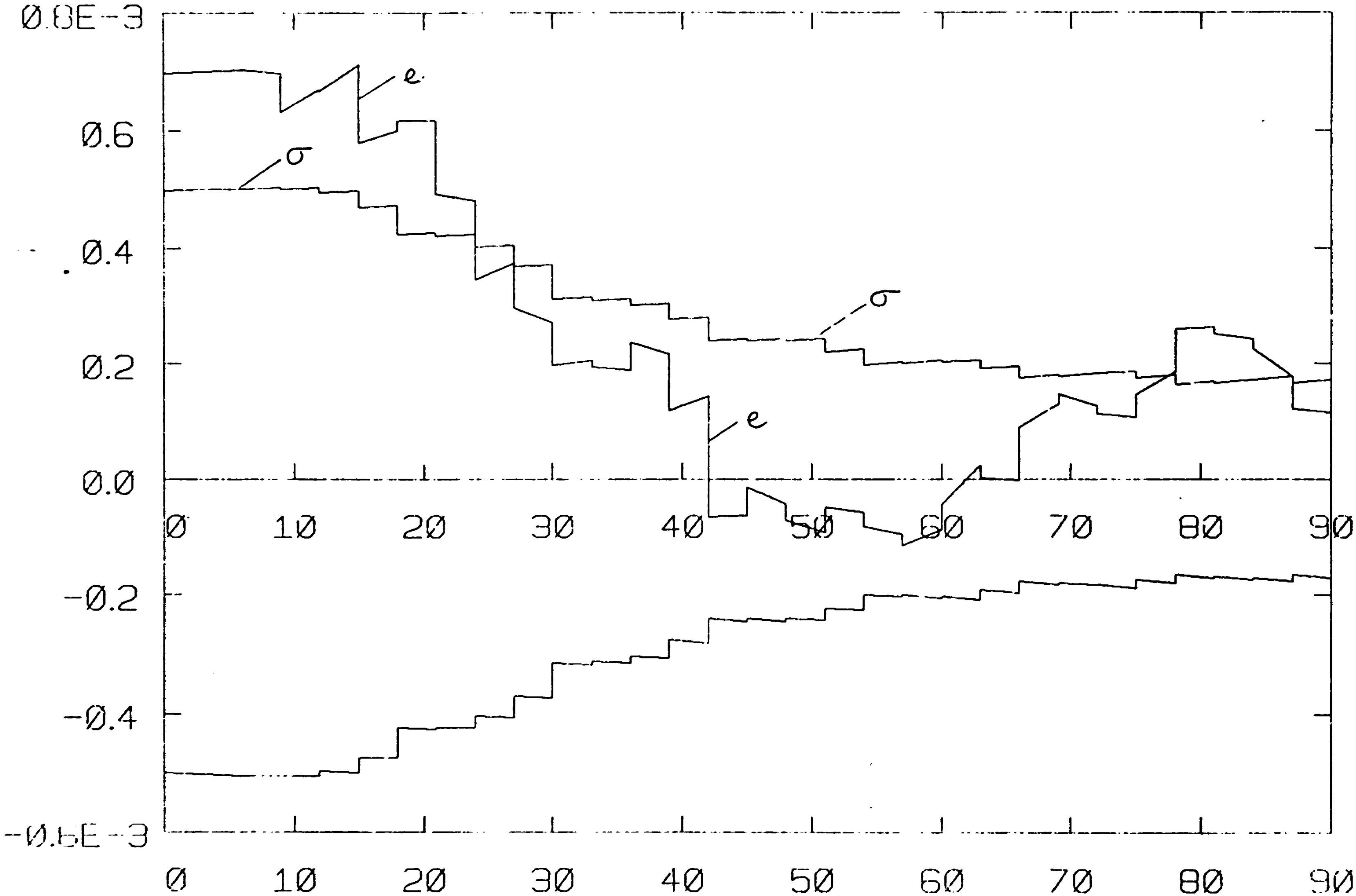


Figure 6.7(a). Centralized optimal filter including rate states. East velocity error of member 2.

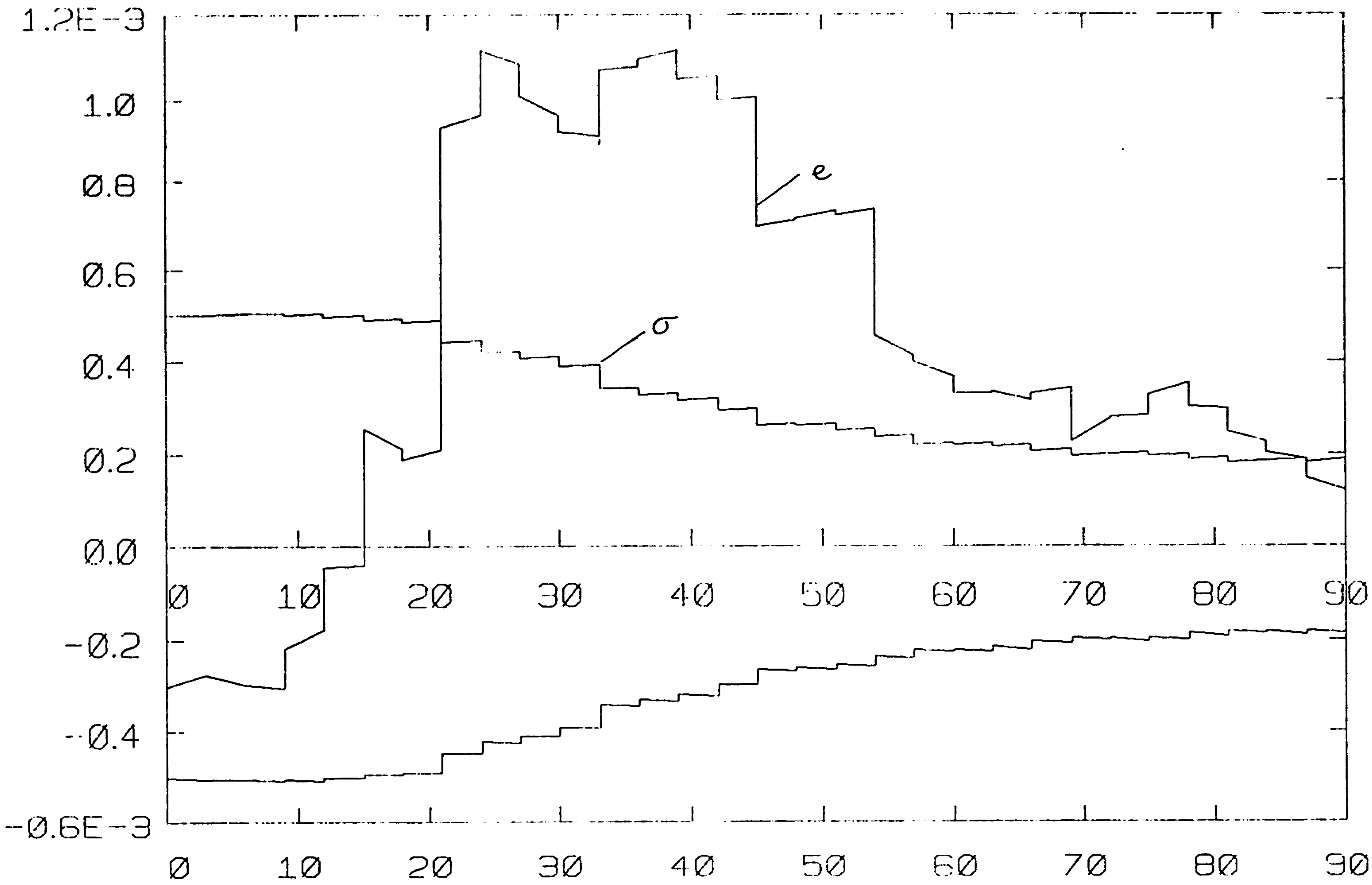


Figure 6.7(b). Centralized optimal filter including rate states. East velocity error of member 3.

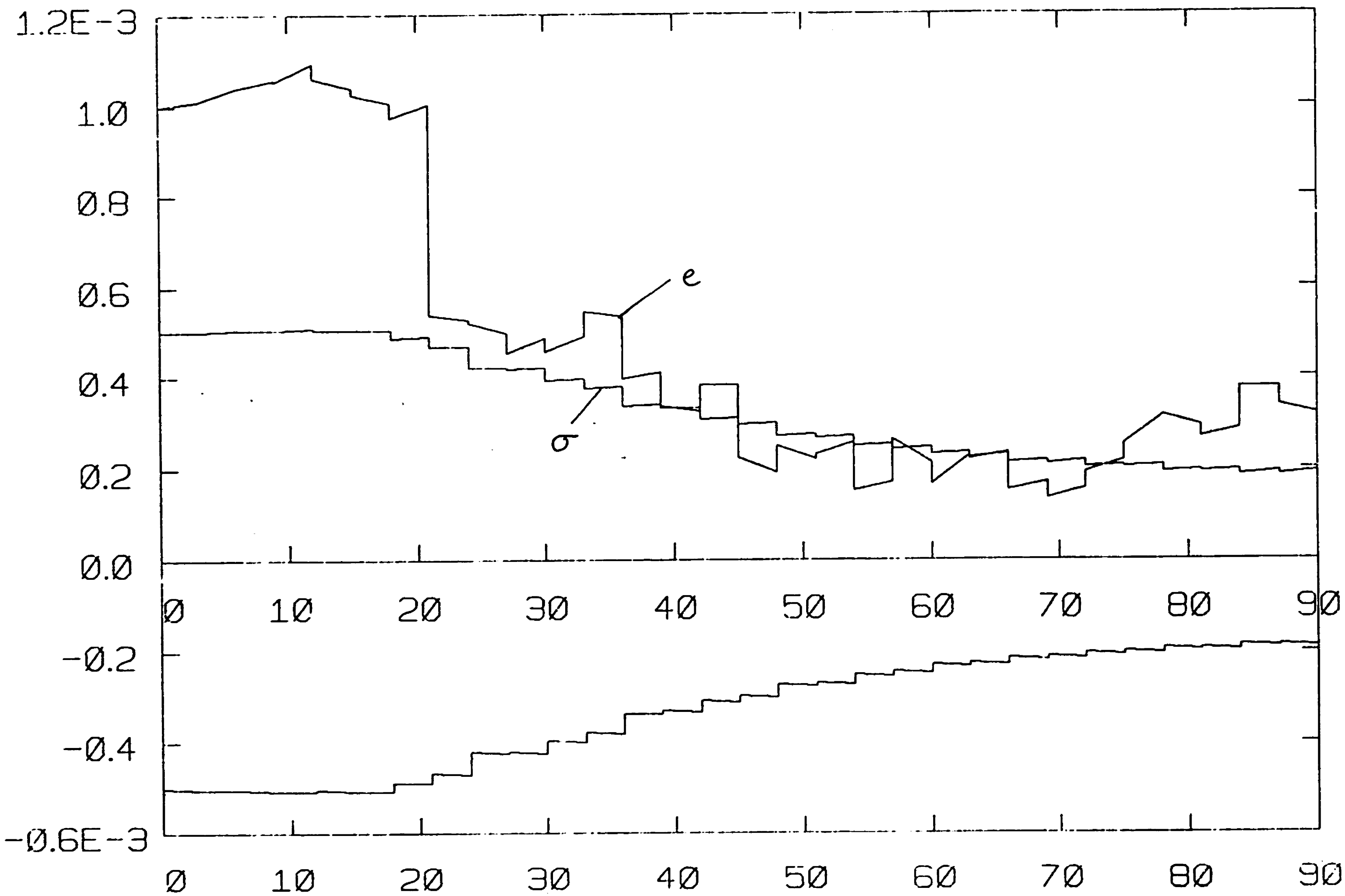


Figure 6.7(c). Centralized optimal filter including rate states. East velocity error of member 4.

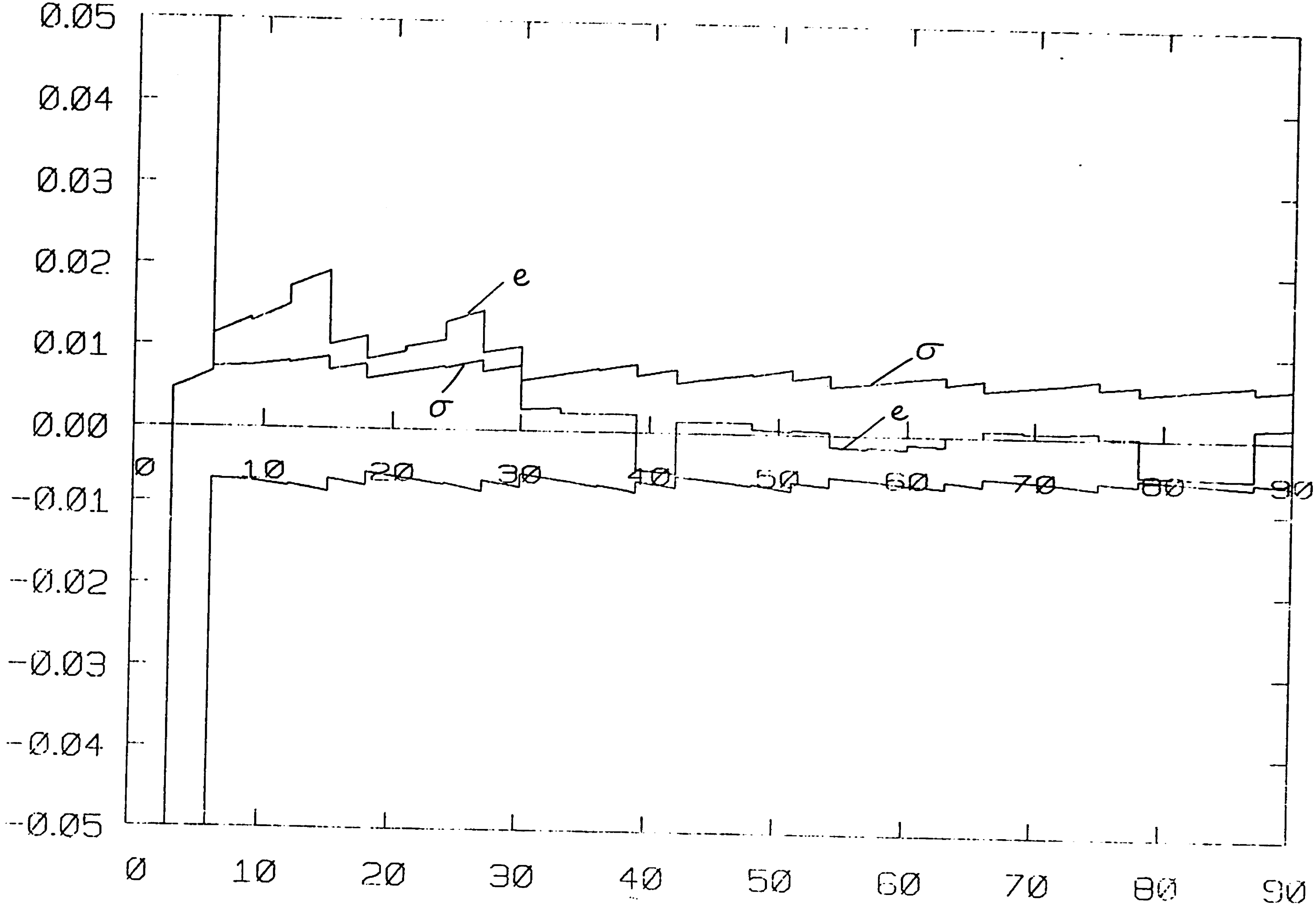


Figure 6.8(a). Suboptimal filter of member 2. East error of member 2.

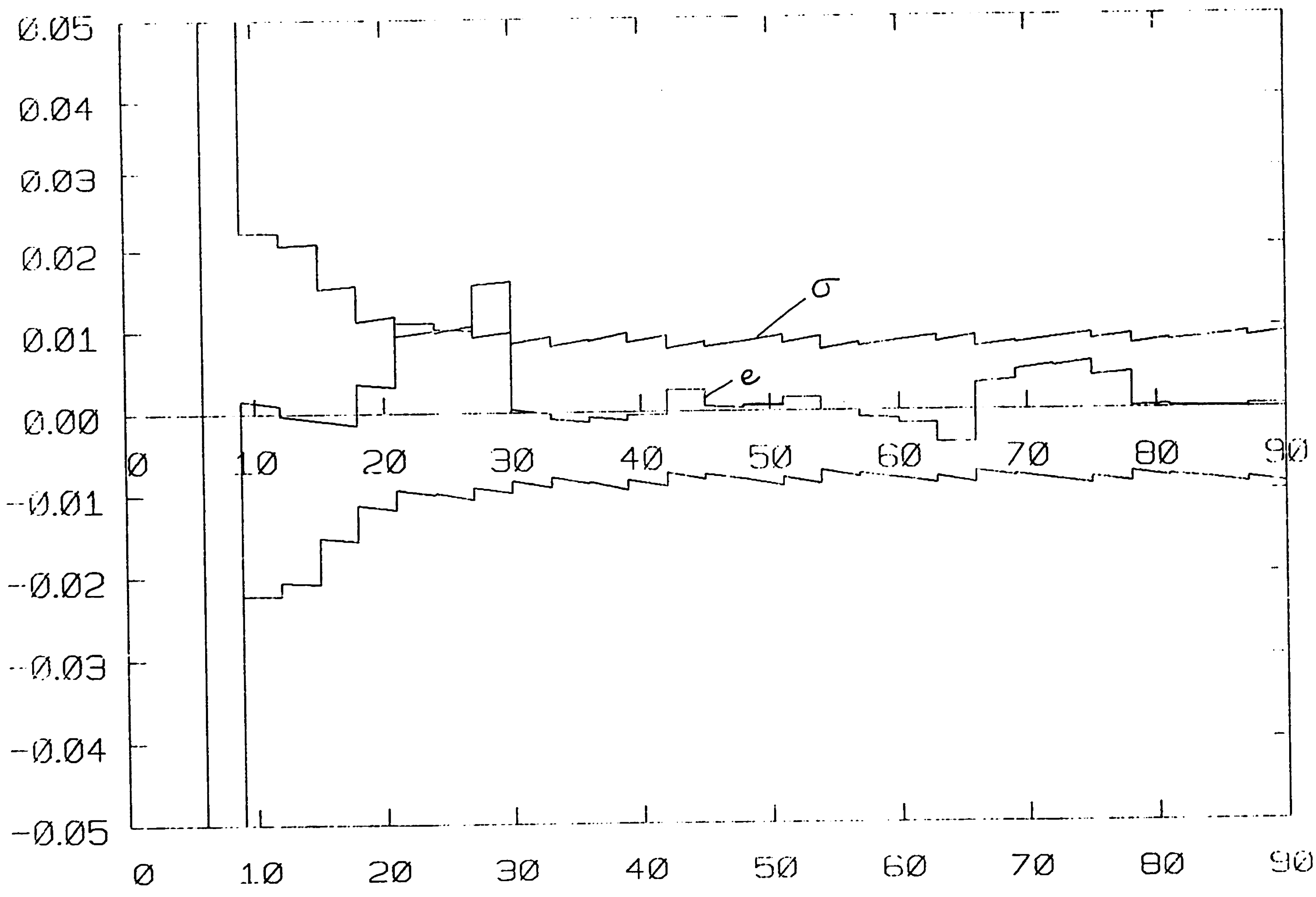


Figure 6.8(b). Suboptimal filter of member 3. East error of member 3.

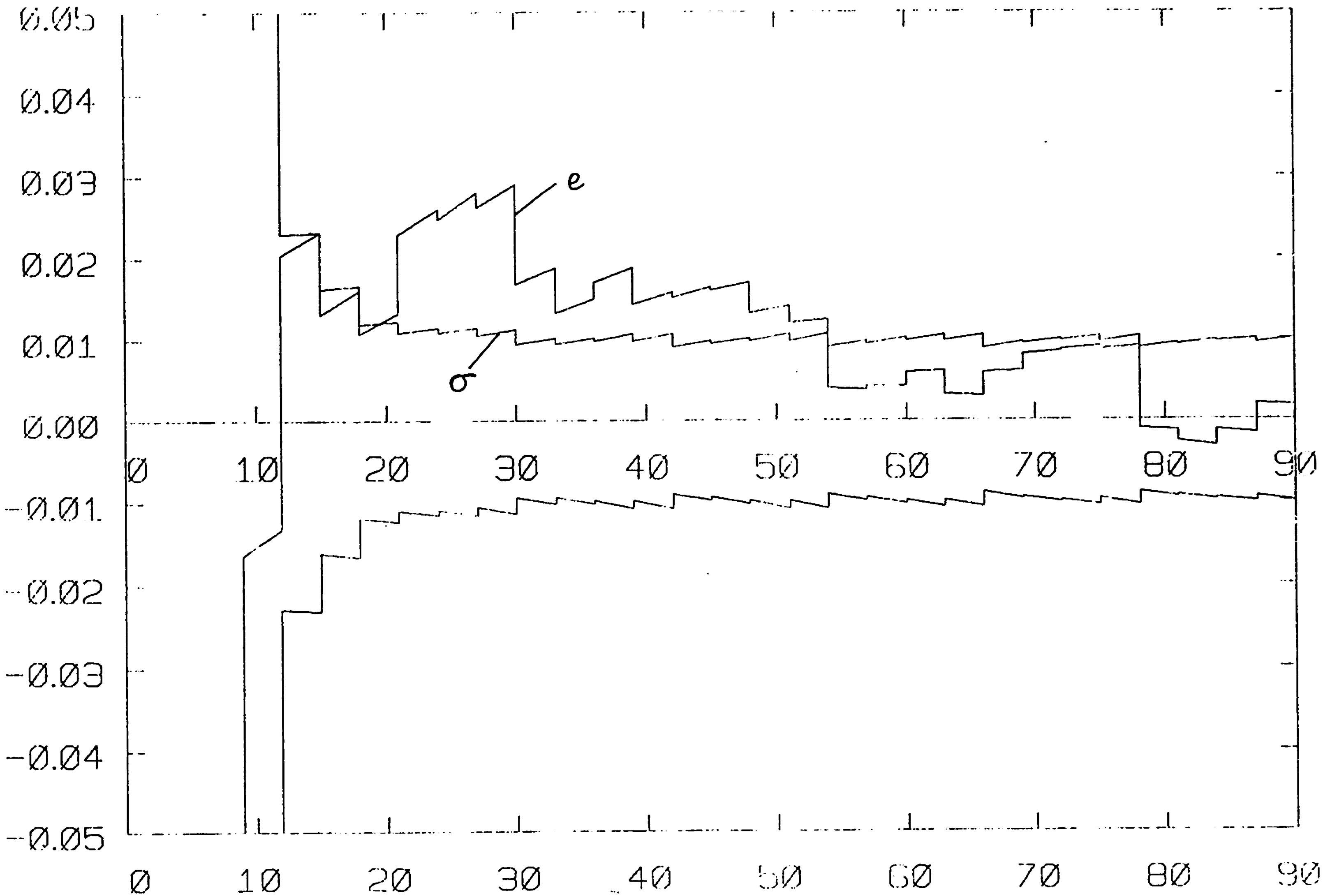


Figure 6.8(c). Suboptimal filter of member 4. East error of member 4.

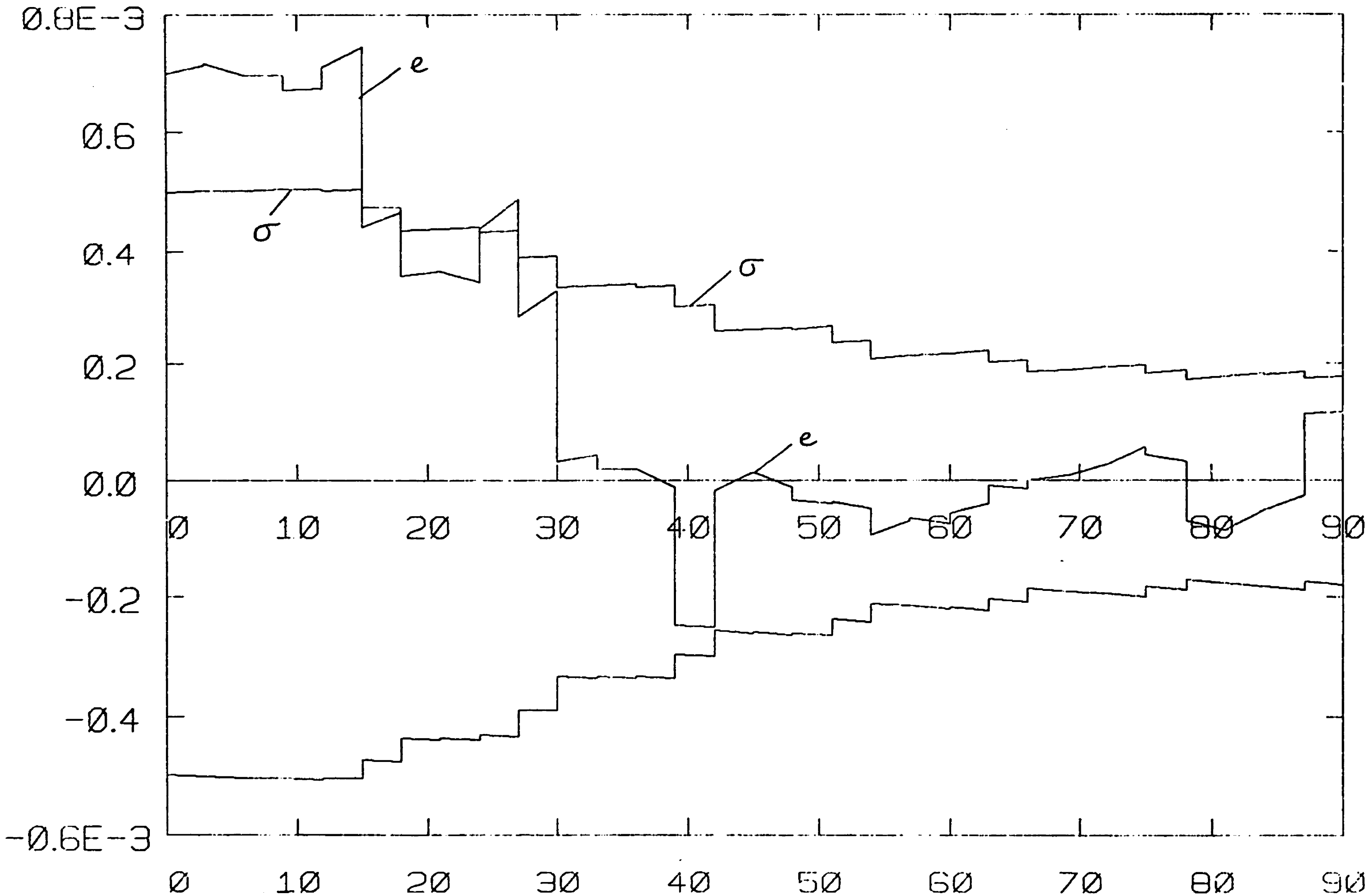


Figure 6.9(a). Suboptimal filter of member 2. East velocity error of member 2.

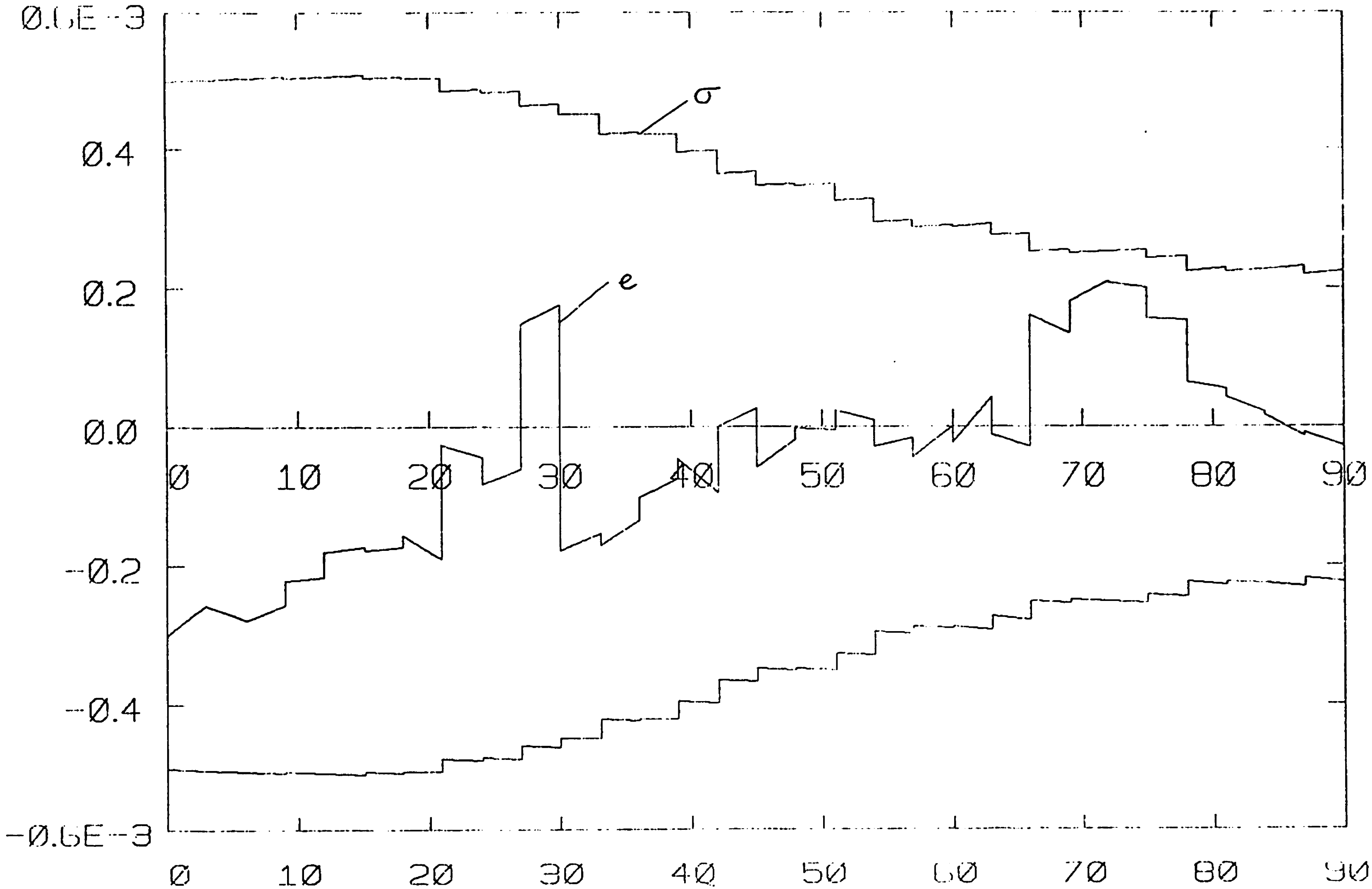


Figure 6.9(b). Suboptimal filter of member 3. East velocity error of member 3.

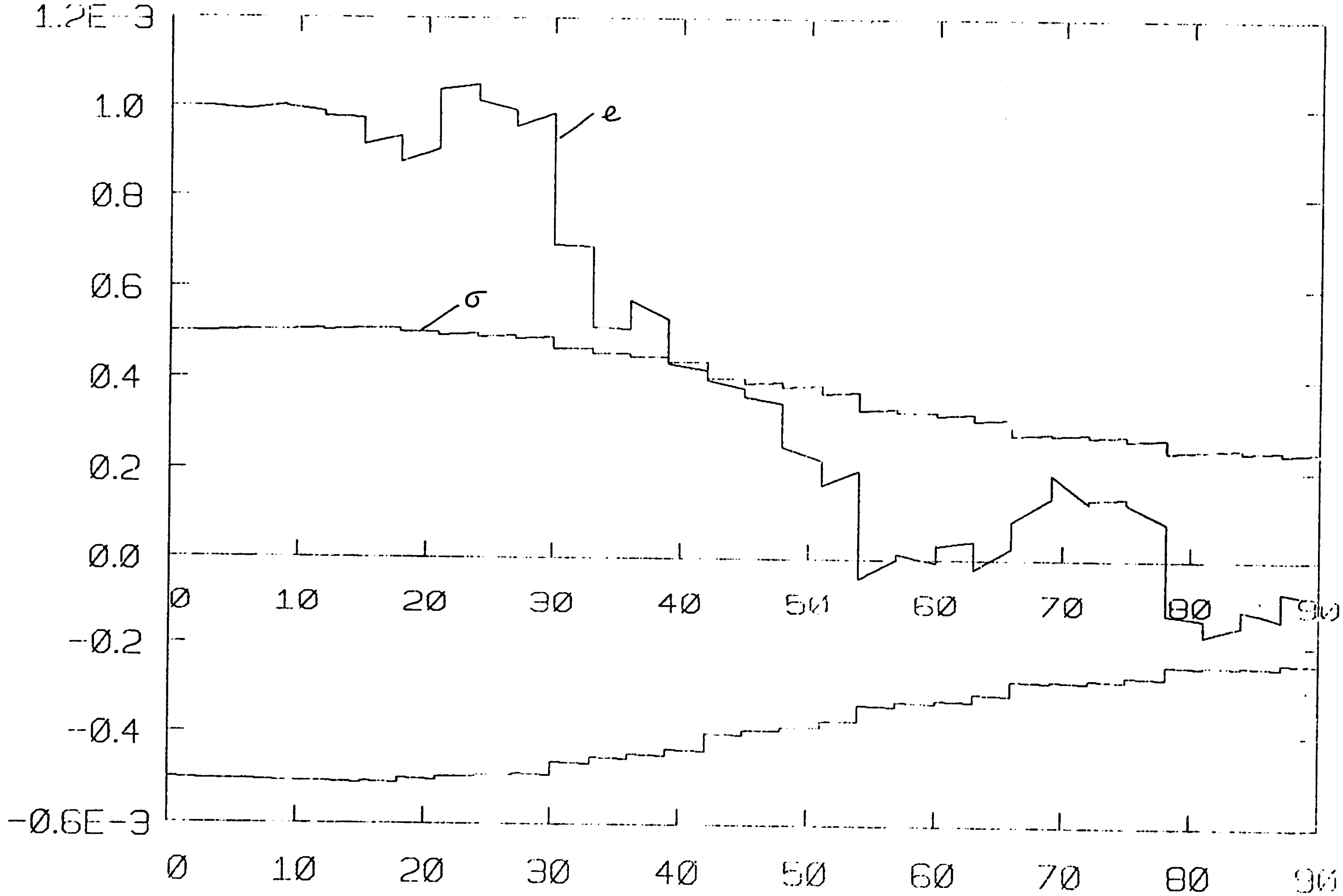


Figure 6.9(c). Suboptimal filter of member 4. East velocity error of member 4.

Rangings were corrupted by an additive noise with variance:

$$r = (10^{-2}\text{Km})^2$$

Two filter models were simulated. The first was a centralized filter, modeling all member's state variables and provided with immediate knowledge of all rangings. The filter noise model matched the truth model. Initial errors had variances of $(1 \text{ km})^2$ for positions and clock phases, $(0.5 \text{ m/sec})^2$ for velocities and clock frequencies. The values of the initial errors were chosen pseudorandomly, according to the variances given above.

The second filter model included four different filters, one per member. Each member modeled all its own variables and the position errors and clock phases of the others, but not their velocity or clock frequency errors. The unmodeled increase of position errors and clock phases, due to the dropped velocity and frequency errors, was compensated by attributing them a random-walk error with incremental variance:

$$\sigma^2 = (10^{-4}\text{Km}^2/\text{sec})\ (\Delta t \text{ sec})$$

The standard deviation of the position error due to a random walk of such intensity matches the one produced by the initial velocity error at t = 400 sec. These shortened-model filters also had knowledge of all rangings as soon as they were taken.

The performance of the shortened-model filters was very similar to that of the optimal filter. Figures 6.6 and 6.7 show the performance of the optimal filter. Figure 6.6 contains the East errors of members 2,3 and 4; Figure 6.7 contains the East velocity errors of the same members. Figures 6.8 and 6.9 show the performance of the shortened-model filters; they contain, respectively, the East errors and the East velocity errors of members 2, 3 and 4. In these plots the estimator is the same member whose variable is estimated. The plots also show plus and minus the computed standard deviations; they are in good agreement with the errors.

This example is not meant as a validation of the estimation concept proposed in this chapter. A validation would require a more complicated truth model, including all the relevant variables; besides, the delay with which measurements are received and incorporated would have to be modeled as well. Such simulations would be necessary, if the proposed concept were adopted for some application, but they are beyond the scope of this work.

6.7 Example of state vector composition

This section contains an example of how the ownstate vector $\underline{x}^i$ and its partitions $\underline{z}^i$ and $\underline{y}^i$ might be composed. This question must be touched on, because it is essential for the feasibility of the solution proposed in this chapter, that ξ^i may have not too large a size. This implies, in turn, that $\underline{z}^i$ must have a much smaller size than $\underline{y}^i$.

The vector $\underline{x}^i$, for the application that has been discussed, would typically include the errors of member i's dead reckoner, clock and altimeter. Under each of these categories, enough variables must be included to model adequately the dynamics of all major sources of error. For instance, an inertial dead reckoner may require the modeling of position errors, velocity errors, attitude ("platform") errors, and gyro drifts; three of each kind gives a total of twelve variables. A clock model requires at least two variables (phase and frequency errors). One variable is usually enough for the altimeter error.

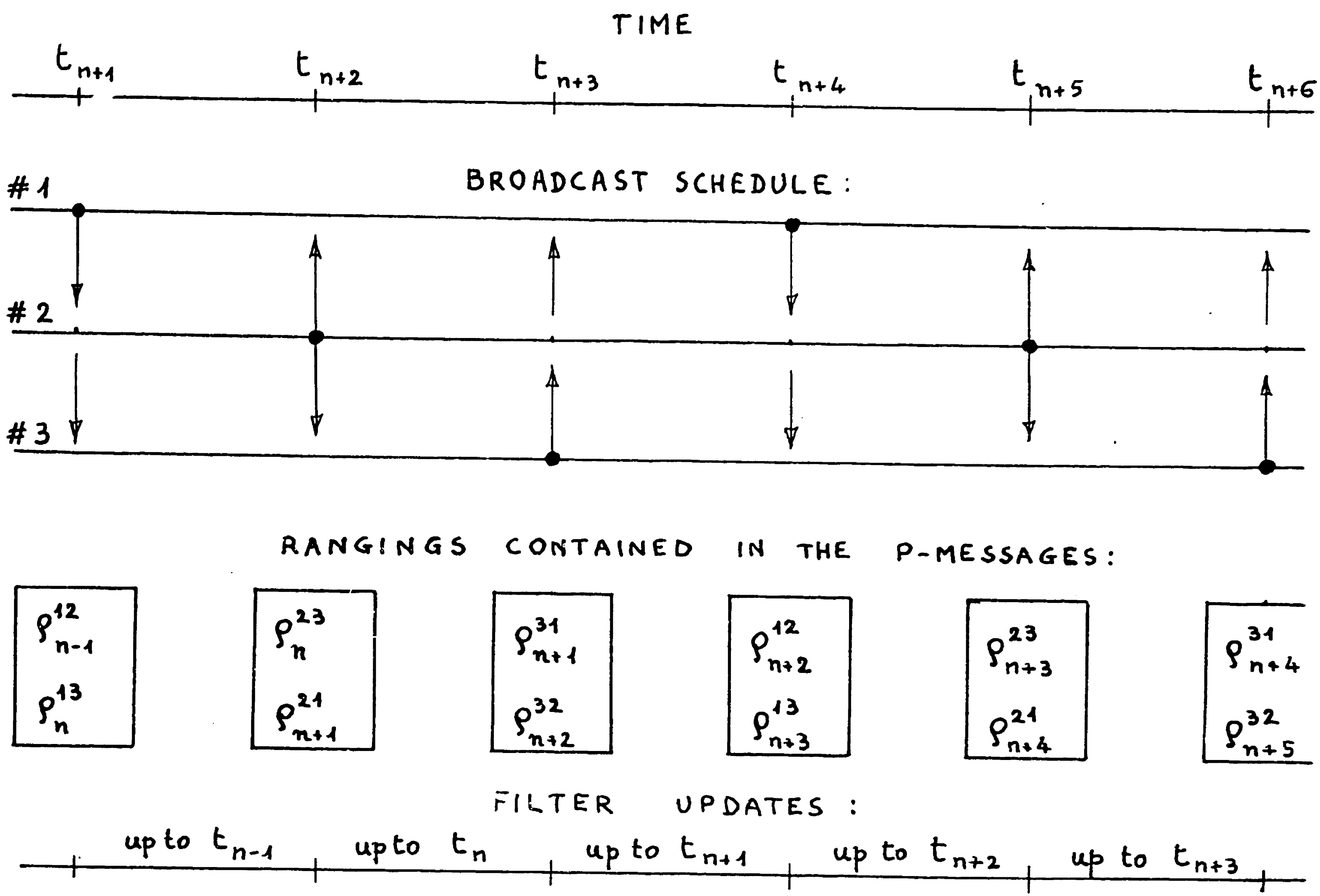


Figure 6.10. Time schedule of a measurement-sharing community of three members.

There is a delay of one round of broadcasts between real time and filter updates. That length of time is usually a constant of the TDMA network, and does not depend on the number of members. For JTIDS, for example, it is of the order of magnitude of 12 sec.

6.9 Contents of the p-messages

This section examines in detail what the messages must contain. The reader will recall that the aim of the arrangement proposed in this chapter is to enable every member to incorporate all the rangings that are taken, and to do so without coupling its filter with the other filters. Knowledge of all rangings is ensured by having each member include in its message all those it took since its last broadcast. In order to give every member the ability of computing the expected values of the rangings, it may be necessary to include other information as well. In order not to have any member use other members' estimates (i.e., in order to avoid filter coupling), one must be careful about choosing this additional information. These are the subjects on which this section elaborates.

In case navigation is performed only by means of rangings, without dead reckoners, no in formation needs to be exchanged other than the rangings' values. Every member estimates the position and clock phase of every other member. Therefore, every member can compute the expected value of all rangings using nothing but its own estimates.

In the more typical case in which rangings are integrated with dead reckonings, the situation is different. What every member estimates is not the other members' positions, but the position errors of their dead reckoners. A member can compute the expected value of rangings only if it knows some of the output of the other members' dead reckoners.

A detailed discussion is made easier by the introduction of a few symbols. Call $\underline{\zeta}^i$ a vector including member i's position, in relative coordinates, and community time. Then define:

$\underline{\zeta}^i$ the actual value of the vector just defined;

$\underline{\bar{\zeta}}^i$ the value indicated by member i's dead reckoner and clock;

$\underline{z}^i \triangleq \underline{\zeta}^i - \underline{\bar{\zeta}}^i$ the dead reckoner and clock errors; it is the same $\underline{z}^i$ defined in Section 6.5;

$\underline{\hat{z}}^{i,k}$ member k's estimate of $\underline{z}^i$; it is a portion of $\underline{\hat{\xi}}^k$;

$\underline{\tilde{z}}^{i,k}$ member k's error in estimating z^i; it is a portion of $\underline{\tilde{\xi}}^k$;

$\underline{\hat{\zeta}}^{i,k} \triangleq \underline{\bar{\zeta}}^i + \underline{\hat{z}}^{i,k}$ the best estimate of $\underline{\zeta}^i$ available to member k.

One must remember that, usually, $\underline{\hat{z}}^{i,i}$ is zero. The updates are added directly to $\underline{\bar{\zeta}}^i$, by appropriately resetting one's dead reckoner and clock.

The expected value of the ranging ρ_n^{ij}, as computed by member k, is a function of $\underline{\hat{\zeta}}_{n-}^{i,k}$ and $\underline{\hat{\zeta}}_{n-}^{j,k}$. These two must be functions only of member k's estimates. Combining the two expressions:

$$\underline{\hat{\zeta}}_{n-}^{i,k} = \underline{\bar{\zeta}}_{n-}^{i} + \underline{\hat{z}}_{n-}^{i,k} \quad ; \quad \underline{\hat{\zeta}}_{n-1}^{i,k} = \underline{\bar{\zeta}}_{n-1}^{i} + \underline{\hat{z}}_{n-1}^{i,k}$$

one finds:

$$\underline{\hat{\zeta}}_{n-}^{i,k} = \underline{\hat{\zeta}}_{n-1}^{i,k} + (\underline{\bar{\zeta}}_{n-}^{i} - \underline{\bar{\zeta}}_{n-1}^{i}) + (\underline{\hat{z}}_{n-}^{i,k} - \underline{\hat{z}}_{n-1}^{i,k}) \tag{6.30}$$

The third term on the right-hand side can be calculated by propagating the filter states $\underline{\tilde{z}}^{i,k}$. The second term

$$\Delta\bar{\underline{\zeta}}_n^i \triangleq \bar{\underline{\zeta}}_{n-}^i - \bar{\underline{\zeta}}_{n-1}^i \tag{6.31}$$

must be provided by member i. Notice that, although both terms on the r.h.s. of Eq. (6.31) depend on member i's estimation error, their difference does not; $\bar{\underline{\zeta}}_{n-}^i$ is the dead reckoner's indication before the new correction is applied. Filter coupling is thus avoided.

Consequently, the contents of member i's message must be:

(i) the values of the rangings it took since its last broadcast;

(ii) for each time interval between events, the displacement $\Delta\bar{\underline{\zeta}}_n^i$ indicated by the dead reckoner before corrections were applied,

The $\Delta\bar{\underline{\zeta}}_n^i$ defined above includes a clock component as well, but in this case it is more convenient to broadcast the resetting of the clock, rather than the elapsed clock time. The elapsed clock time equals the scheduled interval between events (which is known to all members) minus the latest clock resetting.

The presence of a practical inconvenience could require a change in the nature of item (ii). If member k should miss a message from member i, it would lose not only the latest rangings taken by member i but also the dead-reckoned value of its displacement. Member k's filter would then be stuck with a large estimation error, from which it might be unable to recover. The inconvenience is obviated if member i broadcasts not $\Delta\bar{\underline{\zeta}}_n^i$ but its running sum:

$$(\bar{\underline{\zeta}}_n^i)_{\text{uncor}} \triangleq \sum_{\ell=1}^{n} \Delta\bar{\underline{\zeta}}_\ell^i \tag{6.32}$$

The "uncorrected" dead-reckoned position defined by Eq. (6.32) is what the dead reckoner would indicate if it had never been reset. Upon receiving it, member k can find the difference between its present value and the most recent value known to itself. In this way, no large estimation error is introduced, even if one or more messages in a row are missed.

6.10 Evaluation of the size of a p-message and comparison with the ownstate case.

This section will try to evaluate the length in bits of a typical p-message, both for the community described in this chapter and for an ownstate community.

The numbers of bits necessary for the transmission of a given datum are symbolized as follows:

c for the clock resetting;

p for the vector $(\underline{\bar{\zeta}}_n^i)_{uncor}$ or $\Delta\underline{\bar{\zeta}}_n^i$ (without clock component);

r for a ranging;

q for the position-and-clock-phase partition of a covariance matrix.

With these symbols, the length of a p-message, according to the analyses of this section and of Section 2.3, is, for an ownstate community:

$$\ell_o = p + q \qquad (6.33)$$

and, for the community of uncoupled filters defined in this chapter:

$$\ell_u = (p+c)m + r\,(m-1) \qquad (6.34)$$

where m is the number of members.

The values of p, q, r will be now estimated. For a ranging, supposing a LSB (Least Significant Bit) value of 1 m and a MR (Maximum Range) of 100 Km, one obtains r = 17.

For the clock resetting, assuming a LSB of 1/512 mile (range equavalent) and a MR of 1 mile, we obtain c - 10.

For horizontal relative position, taking the LSB to be 1/512 of a mile and the MR to be 1024 miles, one needs 20 bits per each coordinate Altitude, assuming a LSB of 25 feet and a MR of 50,000 feet, requires 11 bits. The total is p = 51.

The covariance matrix partition has size 4x4, with 10 independent elements. It may be assumed that a half byte per element is enough. This gives q = 40.

The following table gives the message lengths for the solution proposed in this chapter, in function of the number of members:

m:	3	4	5	6	7	8	9
ℓ_u:	217	295	373	451	529	607	695

For comparison, an ownstate community has:

$\ell_0 = 91$.

The comparison shows that this chapter's proposal involves considerably longer p-messages than an ownstate community, but still their length is not forbidding if the community is not a large one.

Consider, for instance, the JTIDS application. According to ref. (4), every round of JTIDS broadcasts contains 1536 time slots, whose capacity is 545 bits each, for a total of 837, 120 bits. Of course, only a fraction of this channel capacity may be used for navigation purposes. Still, a nine-member community would require only 6165 bits, or less than 1% of the total capacity. This seems quite feasible.

6.11 Conclusions

A measurement-sharing community of uncoupled filters promises quite a better performance than an ownstate community. The expectation of a better performance is justified by its advantages in terms of observability and filter decoupling.

The fact that every member can use all the rangings gives observability even in a static geometry. Motion is not necessery, and a satisfactory estimate is reached sooner than in an ownstate community.

Filter decoupling also affords advantages. Community stability is no more an issue. No hierarchical source selection has to be introduced. The members' filters are only required to be stable one by one, and, if any of them are not stable, the others are not affected.

The performance of each filter seems to be close to optimality, in spite of the incompleteness of the model. However, this result must be confirmed (or qualified) with more realistic simulations than the one shown in Section 6.6.

The price to be paid is a larger size of the filters' state vectors and of the p-messages. Both sizes increase linearly with the size of the community, whereas they are constant in the ownstate case. It is possible that, for many applications, there may be a certain critical community size, beyond which the concept proposed in this chapter becomes unfeasible.

CHAPTER 7

THE COOPERATIVE OPTIMAL FILTER

7.1 Introduction

The "cooperative optimal filter" introduced in this chapter is a decentralized estimation algorithm whose results are identical to those of a centralized optimal filter. The centralized optimal estimates are reconstructed by several processors working cooperatively and exchanging information.

This method can be applied to the problem of navigation aided by passive rangings. The decentralized processors are identified each with a member of the community. The required exchange of data is performed partly by means of the usual p-messages, which also serve for ranging, partly by means of additional messages.

The plan of this chapter is as follows. Section 7.2 contains a review of other methods that allow to decentralizedly reconstruct the centralized optimal estimates. It will be shown that their application field and that of the method proposed here are complementary. Sections 7.3 to 7.5 describe the cooperative optimal filter. Sections 7.6 and 7.7 describe how it could be applied to our problem; in particular, a possible time schedule is proposed, and the amount of data to be exchanged is evaluated. Conclusions are drawn in the last section.

7.2 Other methods of decentralized reconstruction of the centralized optimal estimates

The methods described in this section are those proposed by Speyer (12) and by Willsky et al. (13). They will be taken out of the context for which they were proposed, and described in a way suitable for application to an estimation problem similar to ours. However, most of the original notation will be retained.

Speyer's method is described first. Consider a linear system:

$$\underline{x}_n = A_n \underline{x}_{n-1} + \underline{w}_n \quad ; \quad E(\underline{w}_n \underline{w}_m') = Q_n \delta_{nm}$$

with measurements partitioned among several local processors; processor i knows:

$$\underline{z}_n^i = H_n^i \underline{x}_n + \underline{v}_n^i \quad ; \quad E(\underline{v}_n^i \underline{v}_m^j) = R_n^i \delta_{nm} \delta_{ij}$$

Each processor is supposed to have a complete model of the state vector $\underline{x}$; the Kalman filter based on this model is updated with the measurement subset $\underline{z}^i$, obtaining a suboptimal estimate $\underline{x}^i$. Besides, each processor must also have a data-dependent auxiliary vector $\underline{h}^i$, of the same size as $\underline{x}$, which is updated with the following equations:

$$\underline{h}_n^i = F_n h_{n-1}^i + G_n^i (\underline{z}_n^i - H_n^i A_n \hat{\underline{x}}_{n-1}^i)$$

$$F_n = P_n (A_n P_{n-1} A_n' + Q_n)^{-1} A_n$$

$$G_n^i = P_n (A_n P_{n-1} A_n' + Q_n)^{-1} A_n P_{n-1} (P_{n-1}^i)^{-1} A_n^{-1} -$$

$$-P_n (A_n P_{n-1}^i A_n' + Q_n)^{-1}$$

where P and P^i are respectively the centralized error covariance matrix (supposing the incorporation of all measurements) and the error covariance matrix of local processor i (supposing the incorporation of $\underline{z}^i$ only); they known to all processors. Supposing the suboptimal estimates $\hat{\underline{x}}^i$ and the auxiliary vectors $\underline{h}^i$ are known for all i, Speyer shows that the centralized optimal estimate can be obtained from:

$$\hat{\underline{x}}_n = \sum_i [P_n \ (P_n^i)^{-1} \ \hat{\underline{x}}_n^i + \underline{h}_n^i]$$

Speyer's method is not suited for our case because the amount of computations every processor must perform is probably than it would be with a centralized optimal filter; both $\hat{\underline{x}}^i$ and $\underline{h}^i$ must be updated with matrix operations, and they are of the same size as the centralized optimal estimate $\hat{\underline{x}}$. Besides, the amount of data exchange is also large; in our case, P and P^i must be computed on line, and, therefore, processor i must share P^i to the community, along with $\hat{\underline{x}}^i$ and $\underline{h}^i$. In our case, it makes more sense to share the measurements and let each member compute the centralized optimal solution by itself.

Willsky's method is an extension of Speyer's, but a few new features are introduced. It will be described for the continuous-time case, as it is done in Ref. (13). The local processors are allowed to have shortened or aggregate models of the system. The true model is:

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + d\underline{\beta}/dt \quad ; \quad E(d\underline{\beta}(t) \ d\underline{\beta}'(\tau)) = Q(t) \ dt \ \delta(t-\tau)$$

and the model used by processor i is:

$$\dot{\underline{x}}^i(t) = A^i(t)\underline{x}^i(t) + d\underline{\beta}^i/dt \quad ; \quad E \ (d\underline{\beta}^i(t)d\underline{\beta}^{i\prime}(\tau)) = Q^i(t)dt \ \delta(t-\tau)$$

The measurement subset available to processor i can be described as a function of the true model:

$$\underline{y}^i(t) = C^i(t)\underline{x}(t) + d\underline{\gamma}^i/dt$$

or as a function of the local model:

$$\underline{y}^i(t) = H^i(t)\underline{x}^i(t) + d\underline{\gamma}^i/dt$$

with

$$E\,(d\gamma^i(t)d\gamma^{j\prime}(\tau)) = R^i(t)\ dt\ \delta(t-\tau)\ \delta_{ij}$$

Each processor has a Kalman filter patterned after its local model. The centralized optimal estimate can be recovered under the following condition. It must be possible to find matrices M^i such that:

$$C^i = H^i M^i$$

(Notice the similarity of this condition with the one given in Section 6.4, Eq. (6.9), for the decoupling of the members' filters.) If the condition is satisfied, any member that wishes to do so can reconstruct the optimal estimates, by computing on-line the auxiliary vector $\underline{\xi}$ (which has the same size as the true-model state $\underline{x}$) with these equations:

$$\dot{\underline{\xi}} = F\underline{\xi} + \sum_i K^i\hat{\underline{x}}^i$$

$$F = A - \sum_i PC^{i\prime}(R^i)^{-1}C^i$$

$$K^i = PM^i(P^i)^{-1}Q^i(P^i)^{-1} - QM^{i\prime}(P^i)^{-1} + PM^{i\prime}A^{i\prime}A^{i\prime}(P^i)^{-1} -$$

$$- PA'M^{i\prime}(P^i)^{-1} - PM^{i\prime}(P^i)^{-1}$$

P and P^i have the same meaning as in Speyer's method. If $\underline{\xi}$ is computed and all the local estimates $\hat{\underline{x}}^i$ are shared to the community, the optimal centralized estimate can be found from:

$$\hat{\underline{x}} = \underline{\xi} + \sum_i G^i \hat{\underline{x}}^i \quad , \; G^i \triangleq PM^{i'} (P^i)^{-1}$$

If applied to our case, Willsky's method would be less burdensome than Speyer's. The local model state vector $\underline{x}^i$ may be considerably shorter than the complete state vector $\underline{x}$; but still, a vector $\underline{\xi}$ of the same size of $\underline{x}$ must be updated and the error covariance matrices must be computed and shared. The conclusions are the same as for Speyer's method.

The unsuitability of these methods to our problem is no disparagement of their value. They were conceived with different applications in mind. Namely, cases where computational ability is unlimited, but the measurements taken by each local processor cannot be shared to the community. Our case has opposite features. Measurement sharing, although costly, can be done, whereas the recursive updating of the complete state vector (or of an equally large auxiliary vector) is not considered feasible.

The method that will be described in the next three sections assumes unlimited ability to exchange data, but seeks to reduce the amount of computations each local processor has to perform. Its applications should be complementary to those of Speyer's and Willsky's methods.

7.3 Cooperative optimal filter: (a) generalities

This section and the following two will describe an original method of reconstructing the centralized optimal (i.e., Kalman) estimate by several processors working cooperatively. Each processor has full knowledge of the latest measurements, but limited memory and computational ability. In particular, no processor has a full-sized state vector and error covariance matrix, nor any full-sized auxiliary vector or matrix. Unlike the methods quoted in the previous section, this method accepts an increased amount of communication among the processors for the sake of reduced computations in each of them. However, unlike Speyer's and Willsky's methods, it cannot be applied to all systems.

The kind of system to which the method does apply will now be described. Consider a linear discrete-time system driven by a stochastic Gaussian unbiased input:

$$\underline{x}_n = \Psi_n \underline{x}_{n-1} + \underline{w}_n \;;\; E(\underline{w}_n \underline{w}_\ell') = Q_n \delta_{n\ell} \tag{7.1}$$

Suppose the single-step transition matrices have a block-diagonal structure:

$$\Psi_n = \text{diag}\,(\Psi_n^1, \Psi_n^2, \ldots, \Psi_n^m) \tag{7.2}$$

so that the state vector can be split up into m subvectors that evolve without coupling:

$$\underline{x}' = [\underline{x}^{1'}, \underline{x}^{2'}, \ldots., \underline{x}^{m'}] \tag{7.3}$$

$$\underline{x}_n^i = \Psi_n^i \underline{x}_{n-1}^i + \underline{w}_n^i \quad , \quad i = 1,2, \ldots, m \tag{7.4}$$

The $\underline{x}^i$ will be called ownstate vectors, for consistency with Chapters 2 to 4. Partition each ownstate vector further as follows:

$$\underline{x}_n^i = \begin{bmatrix} \underline{z}^i \\ \underline{y}^i \end{bmatrix}_n = \begin{bmatrix} A^i & B^i \\ C^i & D^i \end{bmatrix} \begin{bmatrix} \underline{z}^i \\ \underline{y}^i \end{bmatrix}_{n-1} + \underline{w}_n^i \qquad (7.5)$$

call the $\underline{z}^i$ coupled subsets, the $\underline{y}^i$ uncoupled subsets, and suppose each scalar measurement is a function only of the coupled subsets of two different ownstate vectors:

$$\delta\rho_n^{ij} = f\,(\underline{z}_n^i, \underline{z}_n^j) + v_n^{ij} \qquad (7.6)$$

so that the innovation is, to the first order:

$$\delta\rho_n^{ij} = \underline{h}_n^{ij'}\ \tilde{\underline{z}}_n^i + \underline{g}_n^{ij'}\ \tilde{\underline{z}}_n^j + v_n^{ij} \qquad (7.7)$$

with

$$\underline{h}_n^{ij} = \left.\frac{\partial f}{\partial \underline{z}_n^i}\right|_{\substack{z_n^j = \hat{\underline{z}}_n^j \\ \underline{z}_n^i = \hat{\underline{z}}_n^i}} , \qquad \underline{g}_n^{ij} = \left.\frac{\partial f}{\partial \underline{z}_n^j}\right|_{\substack{\underline{z}_n^j = \hat{\underline{z}}_n^j \\ \underline{z}_n^i = \hat{\underline{z}}_n^i}} \qquad (7.8)$$

The term v^{ij} is stochastic, with the usual hypotheses and notations. This completes the description.

This kind of system fits well the case of dead-reckoned navigation aided by rangings. The ownstates are made up of the errors of the dead reckoner, the clock, the altimeter, etc. of each aircraft. They evolve uncoupledly, and only a subset of them (position and clock phase errors) are coupled by the measurements (rangings).

Now other symbols will be introduced. The Error Covariance Matrix (ECM) of a supposed centralized optimum estimator:

$$P_n \triangleq E(\underline{\tilde{x}}_n \, \underline{\tilde{x}}_n') \qquad (7.9)$$

is partitioned as follows:

$$P_n = \begin{bmatrix} P^{11} & P^{12} & \dots & P^{1m} \\ P^{21} & P^{22} & \dots & P^{2m} \\ \dots & \dots & \dots & \dots \\ P^{m1} & P^{m2} & \dots & P^{mm} \end{bmatrix} \qquad (7.10)$$

with $P_n^{ik} \triangleq E\,(\underline{\tilde{x}}_n^i \, \underline{\tilde{x}}_n^{k'})$ and $P_n^{ki} = P_n^{ik'}$.

Each block is further partitioned as follows:

$$P_n^{ik} = \begin{bmatrix} Z^{ik} & V^{ik} \\ V^{ki'} & Y^{ik} \end{bmatrix}_n \qquad (7.11)$$

with

$$Z_n^{ik} \triangleq E\,(\underline{\tilde{z}}_n^i \, \tilde{z}_n^{k'}) \quad ; \quad Y_n^{ik} \triangleq E\,(\underline{\tilde{y}}_n^i \, \underline{\tilde{y}}_n^{k'})$$

$$V_n^{ik} \triangleq E\,(\underline{\tilde{z}}_n^i \, \underline{\tilde{y}}_n^{k'}) \quad ; \quad V^{ki'} \triangleq E\,(\underline{\tilde{y}}_n^i \, \underline{\tilde{z}}_n^{k'})$$

Besides, another partitioning will be used:

$$P_n^{ik} = [\bar{P}^{ik}, \hat{P}^{ik}]_n \qquad (7.12)$$

obviously with:

$$\bar{P}_n^{ik} \triangleq \begin{bmatrix} Z^{ik} \\ \\ V^{ki'} \end{bmatrix}_n \quad ; \quad \hat{P}^{ik} \triangleq \begin{bmatrix} V^{ik} \\ \\ Y^{ik} \end{bmatrix}_n \qquad (7.13)$$

The general structure of the cooperative filter will now be described. There are supposed to be m decentralized estimators; each of them knows:

(i) sufficient data to compute all partitions of the transition matrix Ψ;

(ii) all the latest measurements and sufficient data to compute their expected values; or, equivalently, all the latest innovations.

Here "latest" means "not incorporated yet." Although all processors have full access to the incoming information, they are decentralized, because each remembers a different portion of it. Namely, each processor updates a different state vector. The typical processor i has the following reduced state:

$$\underline{\xi}_n^{i'} \triangleq \left[\underline{x}^{i'}, \underline{z}^{1'}, \ldots, \underline{z}^{i-1'}, \underline{z}^{i+1'}, \ldots, \underline{z}^{m'}\right] \qquad (7.14)$$

which includes all the i-th ownstate, but only the coupled subsets of the other ownstates. Processor i records and updates a <u>main ECM</u>:

$$X_n^i \triangleq E\,(\tilde{\underline{\xi}}_n^i\, \tilde{\underline{\xi}}_n^{i'}) = \begin{bmatrix} P^i & \overline{P}^{i1} & \ldots & \overline{P}^{i,i-1} & \overline{P}^{i,i+1} & \ldots & \overline{P}^{im} \\ \overline{P}^{i1'} & Z^{11} & \ldots & Z^{1,i-1} & Z^{1,i+1} & \ldots & Z^{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \overline{P}^{i,i-1'} & Z^{i-1,1} & \ldots & Z^{i-1,i-1} & Z^{i-1,i+1} & \ldots & Z^{i-1,m} \\ \overline{P}^{i,i+1'} & Z^{i+1,1} & \ldots & Z^{i+1,i-1} & Z^{i+1,i+1} & \ldots & Z^{i+1,m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \overline{P}^{im'} & Z^{m1} & \ldots & Z^{m,i-1} & Z^{m,i+1} & \ldots & Z^{mm} \end{bmatrix}_n \qquad (7.15)$$

and, moreover, a set of <u>auxiliary ECM's</u>:

$$\hat{P}_n^{ik} = E\,(\tilde{\underline{x}}_n^i\ \tilde{\underline{y}}_n^{k'}) \qquad \text{with } k = 1,2,\ldots, \tag{7.16}$$

Figure 7.1 gives a graphic representation of the relationship of processor i's main end auxiliary ECM's to the centralized ECM. The figure also shows the ECM of an ownstate community member.

In the cooperative optimal filter, instead, the estimates and ECM's of every processor are at all times portions of their centralized optimal counterparts. In symbols:

$$\underline{\xi}_n^i = S^i\ \underline{x}_n \quad ; \quad X_n^i = S^i\ P_n\ S^{i'} \tag{7.17}$$

It is left to the reader to find the expression of S^i and verify that

$$S^{i'}\ S^i = \text{diag}\ (I,\ 0,\ \ldots.,\ I,\ 0,\ I,\ I,\ I,\ 0,\ \ldots.,\ I,\ 0) \tag{7.18}$$

where the diagonal blocks of Eq. (5.18) correspond to a partition of $\underline{x}$ into:

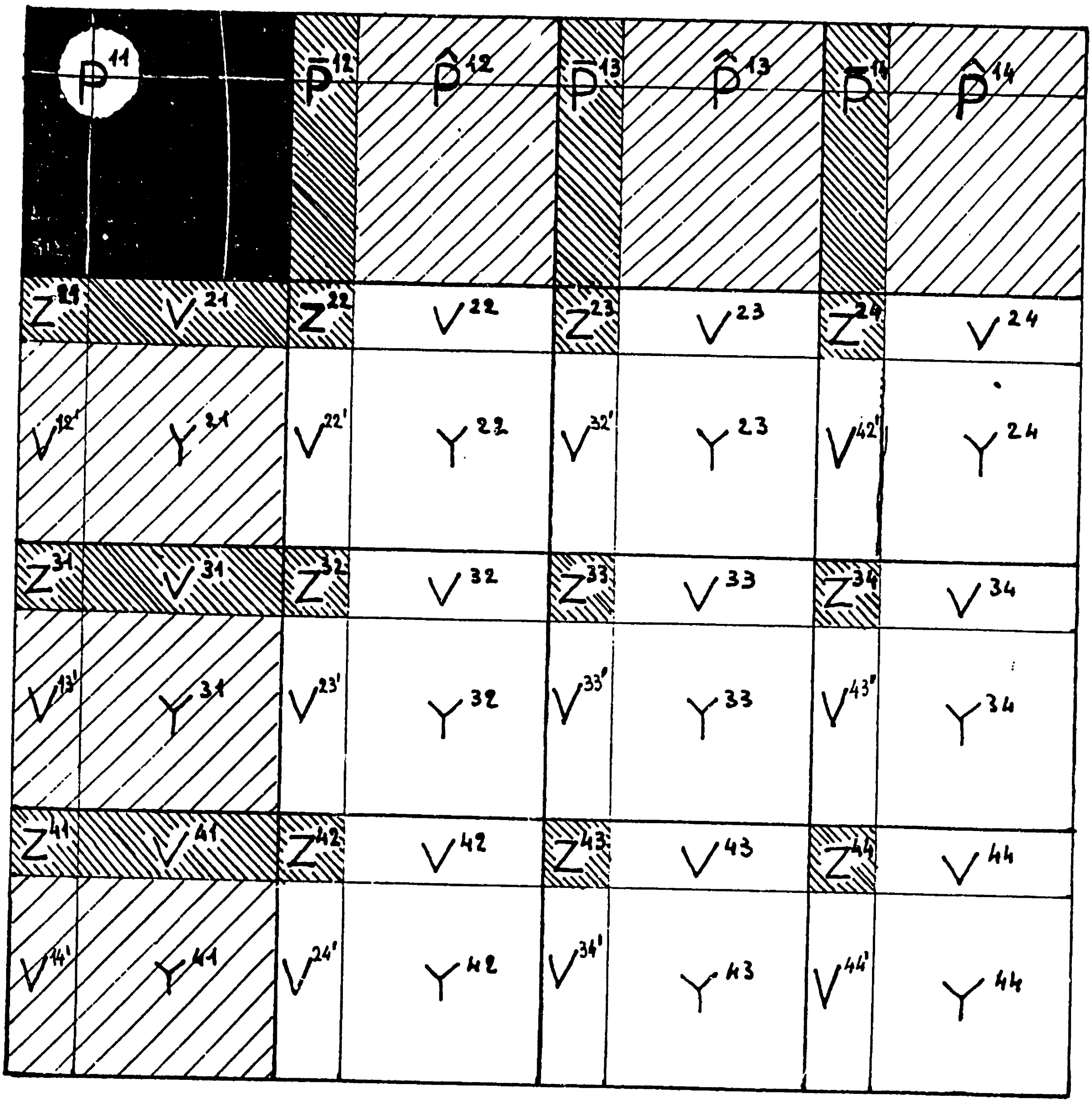
$$\underline{x}_n' = [\underline{z}^{1'},\ \underline{y}^{1'},\ \ldots.,\ \underline{z}^{i-1'},\ \underline{y}^{i-1'},\ \underline{z}^{i'},\ \underline{y}^{i'},\ \underline{z}^{i+1'},\ \underline{y}^{i+1'},\ \ldots,\ \underline{z}^{m'},\ \underline{y}^{m'}]_n \tag{7.19}$$

Equation (7.18) will be used later.

<u>7.4 Cooperative optimal filter: (b) propagation between measurements</u>

It will be shown in this section how it is possible to propagate the estimates and the main auxiliary ECM's of every processor in such a way that they still are, after the propagation, subsets of their centralized

Figure 7.1. Error Covariance Matrices of member 1 in a community of four.



Ownstate ECM.

Main ECM in a cooperative optimal filter.

Auxiliary ECM's in a cooperative optimal filter.

counterparts. It will be found that no processor can do that by itself, and a certain amount of information must be exchanged.

The centralized filter propagation between discrete times n - 1 and n is:

$$\hat{\underline{x}}_{n-} = \Psi_n \hat{\underline{x}}_{n-1} \tag{7.20}$$

$$P_{n-} = \Psi_n P_{n-1} \Psi_n' + Q_n \tag{7.21}$$

Partitioning $\underline{x}$ into ownstate vectors, and P and Q accordingly, one obtains, for i,k = 1,2 ..., m:

$$\hat{\underline{x}}^i_{n-} = \Psi^i_n \hat{\underline{x}}^i_{n-1} \tag{7.22}$$

$$P^{ik}_{n-} = \Psi^i_n P^{ik}_{n-1} \Psi^{k'}_n + Q^{ik}_n \tag{7.23}$$

Further partioning into coupled and uncoupled subsets gives:

$$\hat{\underline{z}}^i_{n-} = A^i_n \hat{\underline{z}}^i_{n-1} + B^i_n \hat{\underline{y}}^i_{n-1} \tag{7.24}$$

$$\underline{y}^i_{n-} = C^i_n \hat{\underline{z}}^i_{n-1} + D^i_n \hat{\underline{y}}^i_{n-1} \tag{7.25}$$

$$Z^{ik}_{n-} = A^i_n Z^{ik}_{n-1} A^{k'}_n + A^i_n V^{ik}_{n-1} B^{k'}_n + B^i_n V^{ki'}_{n-1} A^{k'}_n + B^i_n Y^{ik}_{n-1} B^{k'}_n + Q^{ik}_{zn} \tag{7.26}$$

$$Y^{ik}_{n-} = C^i_n Z^{ik}_{n-1} C^{k'}_n + C^i_n V^{ik}_{n-1} D^{k'}_n + D^i_n V^{ki'}_{n-1} C^{k'}_n + D^i_n Y^{ik}_{n-1} D^{k'}_n + Q^{ik}_{yn} \tag{7.27}$$

$$V^{ik}_{n-} = A^{i}_{n}\, Z^{ik}_{n-1}\, C^{k'}_{n} + A^{i}_{n}\, V^{ik}_{n-1}\, D^{k'}_{n} + B^{i}_{n}\, V^{ki'}_{n-1}\, C^{k'}_{n} + B^{i}_{n}\, Y^{ik}_{n-1} D^{k'}_{n} + Q^{ik}_{vn} \tag{7.28}$$

$$V^{ki'}_{n-} = C^{i}_{n}\, Z^{ik}_{n-1} A^{k'}_{n} + C^{i}_{n}\, V^{ik}_{n-1} B^{k'}_{n} + D^{i}_{n}\, V^{ki'}_{n-1}\, A^{k'}_{n} + D^{i}_{n}\, Y^{ik}_{n-1}\, B^{k'}_{n} + Q^{ki'}_{vn} \tag{7.29}$$

having partitioned Q^{ik} as follows:

$$Q^{ik}_{n} \overset{\Delta}{=} \begin{bmatrix} Q^{ik}_{z} & Q^{ik}_{v} \\ Q^{ki'}_{v} & Q^{ik}_{y} \end{bmatrix}_{n}$$

Rearranging Eqs. (7.26) to (7.29) in accordance with Eq. (7.13):

$$\bar{P}^{ik}_{n-} = \Psi^{i}_{n}\, P^{ik}_{n-1} [A^{k}_{n}\ ,\ B^{k}_{n}\ ]' + [Q^{ik'}_{zn}\ ,\ Q^{ik}_{vn}]' \tag{7.30}$$

$$\beta^{ik}_{n-} = \Psi^{i}_{n}\, P^{ik}_{n-1}\ [C^{k}_{n}\ ,\ D^{k}_{n}]' + [Q^{ik}_{vn},\ Q^{ik}_{yn}]' \tag{7.31}$$

Consider now the propagation algorithm of member i; if one neglects the cross-correlation of $\underline{\xi}^{i}$ with the omitted states, one might try:

$$\hat{\underline{\xi}}^{i}_{n-} = \Phi^{i}_{n}\, \hat{\underline{\xi}}^{i}_{n-1} \tag{7.32}$$

$$X^{i}_{n-} = \Phi^{i}_{n}\, X^{i}_{n-1}\, \Phi^{i'}_{n} + S^{i} Q_{n} S^{i'} \tag{7.33}$$

where Φ^{i} is the projection of Ψ on the shortened space:

$$\Phi^{i}_{n} = S^{i}\, \Psi_{n}\, S^{i'} = \text{diag}\ (\Psi^{i}_{n},\ A^{1}_{n},\ \ldots,\ A^{i-1}_{n},\ A^{i+1}_{n},\ \ldots,\ A^{m}_{n})$$

This algorithm is insufficient. The partitions of $\hat{\underline{\xi}}^i$ and X^i are propagated by it in the following way (with k, ℓ ≠ i):

$$\hat{\underline{x}}^i_{n-} = \Psi^i_n \hat{\underline{x}}^i_{n-1}$$

$$\hat{\underline{z}}^k_{n-} = A^k_n \hat{\underline{z}}^k_{n-1}$$

$$P^{ii}_{n-} = \Psi^i_n P^{ii}_{n-1} \Psi^{i'}_n + Q^{ii}_n$$

$$\overline{P}^{ik}_{n-} = \Psi^i_n \overline{P}^{ik}_{n-1} A^{k'}_n + [Q^{ik'}_z, Q^{k\ell}_v]'_n$$

$$Z^{k\ell}_{n-} = A^k_n Z^{k\ell}_{n-1} A^{\ell'}_n + Q^{k\ell}_{zn}$$

Comparison with Eqs. (7.22) to (7.31) shows that only $\hat{\underline{x}}^i$ and P^{ii}, the ownstate and its autocovariance, are propagated correctly.

Now the advantage of updating the auxiliary ECM's appears. If one uses this algorithm:

$$\hat{P}^{ik}_{n-} = \Psi_n \overline{P}^{ik}_{n-1} C^{k'}_n + \Psi^i_n \hat{P}^{ik}_{n-1} D^{k'}_n + [Q^{ik'}_v, Q^{ik'}_y]'_n \tag{7.34}$$

$$X^i_{n-} = \phi^i_n X^i_{n-1} \phi^{i'}_n + S^i Q_n S^{i'} + \begin{bmatrix} 0 & \Psi^i_n \hat{P}^{i1}_{n-1} B^1 \ldots \Psi^i_n \hat{P}^{im}_{n-1} B^m_n \\ B^{1'}_n \hat{P}^{i1'}_{n-1} \Psi^{i'}_n & \\ \cdots & O \\ B^{m'}_n \hat{P}^{im'}_{n-1} \Psi^{i'}_n & \end{bmatrix} \tag{7.35}$$

comparison with Eqs. (5.23) to (5.31) shows that the auxiliary ECM's P^{ik} are propagated correctly, and the main ECM X^i is propagated correctly except in the lower right hand corner (partitions $Z^{k\ell}$, with k, ℓ = i).

The correctly propagated value of these partitions (as well as the correct value of $\hat{\underline{z}}^k$, if it is desired) must be supplied by the other members. Member k will broadcast $Z^{k\ell}_{n-}$ with ℓ = 1, 2, ..., m. Member i must help the others likewise. The third transmission requirement is then for member i to include in its p messages:

(iii-a) the latest propagated values of $Z^{i\ell}$, with ℓ = 1, 2, ..., m.

See the previous section for requirements (i) and (ii).

7.5 Cooperative optimal filter: (c) measurement incorporation.

It has been supposed that the measurements have the form given by Eq. (7.6). In our case $\rho^{\ell j}$ is the ranging of member ℓ to member j, and it is supposed that, after some delay, member i (possibly i ≠ ℓ) will come to know the innovation $\delta\rho^{\ell j}$. It will now be shown how $\delta\rho^{\ell j}$ is incorporated into member i's estimates and ECM's. For simplicity of notation's sake, the values before the incorporation will have the subscript n- and those after the incorporation the subscript n; one must remember, though, that more than one scalar measurement is taken at time t_n; so, the values with subscript n must then take the subscript n- and be the subject of another incorporation, and so on, down to exhaustion of the measurements.

The innovation equals, to the first order:

$$\delta\rho_n^{\ell j} = \underline{c}_n^{\ell j'} \, \tilde{\underline{x}}_{n-} + v_n^{\ell j} \qquad (7.36)$$

with:

$$\underline{c}_n^{\ell j'} \triangleq [\underline{0}', \ldots, \underline{0}', \underline{h}_n^{\ell j'}, 0', \ldots, \underline{0}, g_n^{\ell j'}, \underline{0}', \ldots \underline{0}'] \tag{7.37}$$

$\underline{h}^{\ell j}$ and $\underline{g}^{\ell j}$ being in the places occupied by $\underline{z}^{\ell}$ and $\underline{z}^{j}$ in the partition of $\underline{x}$ according to Eq. (7.19). The optimum centralized incorporation would give:

$$\delta\hat{\underline{x}} \triangleq \hat{\underline{x}}_n - \hat{\underline{x}}_{n-} = P_{n-} \underline{c}_n^{\ell j} (\underline{c}_n^{\ell j'} P_{n-} \underline{c}_n^{\ell j} + r_n^{\ell j})^{-1} \delta\rho_n^{\ell j} \tag{7.38}$$

$$P_n = P_{n-} - P_{n-} \underline{c}_n^{\ell j} (\underline{c}_n^{\ell j'} P_{n-} \underline{c}_n^{\ell j} + r_n^{\ell j})^{-1} \underline{c}^{\ell j'} P_{n-} \tag{7.39}$$

Likewise, for processor i:

$$\delta\rho_n^{\ell j} = \underline{c}_n^{i,\ell j'} \tilde{\underline{\xi}}_{n-}^{i} \tag{7.40}$$

with:

$$\underline{c}_n^{i,\ell j'} \triangleq [\underline{0}', \ldots, \underline{0}', \underline{h}_n^{\ell j'}, \underline{0}', \ldots, \underline{0}' \; \underline{g}_n^{\ell j'}, \underline{0}', \ldots, \underline{0}'] \tag{7.41}$$

$\underline{h}^{\ell j}$ and $\underline{g}^{\ell j}$ being in the places occupied by $\underline{z}^{\ell}$ and $\underline{z}^{j}$ in the partition of $\underline{\xi}^{i}$ according to Eq. (7.14). It is obvious that:

$$\underline{c}_n^{i,\ell j} = S^i \underline{c}_n^{\ell j} \tag{7.42}$$

The incorporation by processor i, using Kalman gains, is:

$$\delta\hat{\underline{\xi}}_n^{i} = X_{n-} \underline{c}_n^{i,\ell j} (\underline{c}_n^{i,\ell j'} X_{n-} \underline{c}_n^{i,\ell j} + r_n^{\ell j})^{-1} \delta\rho_n^{\ell j} \tag{7.43}$$

$$X_n^i = X_{n-}^i - X_{n-}^i \underline{c}_n^{i,\ell j} (\underline{c}_n^{i,\ell j'} X_{n-}^i \underline{c}_n^{i,\ell j} + r_n^{\ell j})^{-1} \underline{c}_n^{i,\ell j'} X_{n-}^i \qquad (7.44)$$

It will now be shown that this incorporation is correct. That is, if $\hat{\xi}^i$ and X^i were the appropriate projections of $\hat{\underline{x}}$ and P before the incorporation, they remain such after the incorporation. This is a particular case of a lemma that has been proved in Chapter 6, but it is just as simple to prove it directly.

Proof: Suppose: $\hat{\underline{\xi}}_{n-}^i = S^i \hat{\underline{x}}_{n-}$, $X_{n-}^i = S^i P_{n-} S^{i'}$

and introduce these expressions into Eqs. (7.43), (7.44), together with Eq. (7.42):

$$\delta\hat{\underline{\xi}}_n^i = S^i P_{n-} S^{i'} S^i \underline{c}_n^{\ell j} \left(\underline{c}_n^{\ell j'} S^{i'} S^i P_{n-} S^{i'} S^i \underline{c}_n^{\ell j} + r_n^{\ell j}\right)^{-1} \delta\rho^{\ell j}$$

$$X_n = S^i P_{n-} S^{i'} - S^i P_{n-} S^{i'} S^i \underline{c}_n^{\ell j} \left(\underline{c}_n^{\ell j'} S^{i'} S^i P_{n-} S^{i'} S^i \underline{c}_n^{\ell j} + r_n^{\ell j}\right)^{-1} \cdot$$

$$\cdot \underline{c}_n^{\ell j'} S^{i'} S^i P_{n-} S^{i'}$$

From Eqs. (7.18) and (7.37) one sees that:

$$S^{i'} S^i \underline{c}_n^{\ell j} = \underline{c}_n^{\ell j}$$

This result, and comparison with Eqs. (7.38), (7.39) give:

$$\delta\hat{\underline{\xi}}_n^i = S^i \delta\hat{\underline{x}}_n \quad ; \quad X_n = S^i P_n S^{i'}$$

Q.E.D.

It remains to see how the auxiliary ECM's can incorporate a measurement. Introducing Eq. (7.38) into Eq. (7.39), one finds:

$$P_n = P_{n-} - \delta\hat{\underline{x}}_n^i \, \delta\hat{\underline{x}}_n^{i'} \; . \; p \; . \; (\delta\rho_n^{\ell i})^{-2} \tag{7.45}$$

having let:

$$p \triangleq \underline{c}_n^{\ell j'} \; P_{n-} \; \underline{c}_n^{\ell j} + r_n^{\ell j} \tag{7.46}$$

Then, by taking the appropriate partition of Eq. (7.45):

$$\hat{P}_n^{ik} = \hat{P}_{n-}^{ik} - \delta\hat{\underline{x}}_n^i \; \delta\hat{y}_n^k \; . \; p \; . \; (\delta\rho^{\ell j})^{-2} \tag{7.47}$$

Member i can compute p, because Eq. (7.46) is equivalent to:

$$p = \underline{c}_n^{i,\ell j} \; X_{n-} \; c_n^{i,\ell j'} + r_n^{\ell j} \tag{7.48}$$

(the proof is contained in what was proved above); $\delta\hat{\underline{x}}^i$ is also known correctly (as just proved) and the innovation is known by hypothesis. The only missing term in Eq. (7.47) is $\delta\hat{\underline{y}}^k$.

The term $\delta\hat{\underline{y}}^k$ (the updates of the uncoupled subset of member k's ownstate) must be supplied by member k. Likewise, member i must supply the other members with its own $\delta\hat{\underline{y}}^i$. This is possible, because $\delta\hat{\underline{y}}^i$ is a projection of $\delta\hat{\underline{\xi}}^i$, which can be computed correctly by member i (as just proved) by means of Eq. (7.43) without help from the others.

Consequently, the last transmission requirement is for the p-message of member i to contain:

(iii-b) the updates $\delta\hat{\underline{y}}^i$ of its uncoupled subset $\underline{y}^i$ made at all the "latest" measurement incorporations.

Numerical problems may make it advisable to broadcast some equivalent information. For instance, $\delta\hat{\underline{y}}^i/\delta\rho$ might be broadcast instead of $\delta\hat{\underline{y}}^i$

Here "latest" means the measurements that have been already incorporated into the main ECM, but not yet into the auxiliary ECM's. Item (iii-b) alternates with item (iii-a), for which see Section 7.4. The latter is required during the propagation phase, between measurement incorporations; the former is required during the measurement incorporation phase. Items (i) and (ii) (see Section 7.3) are required all the time.

The phasing of all this information transfer may be slightly different in different problems. As far as our problem is concerned, a possible time schedule is described in the next section.

7.6 Phasing of the cooperative optimal filter in the case of relative navigation

This section and the next one discuss the application of the cooperative optimal filter to our case. In particular, this section deals with the phasing of the required information transfer.

The broadcast schedule and content definition requires more attention for our problem than it might for different problems, because of the dual role of p-messages. Namely, the same p-message can both convey information from the source to all receivers, and serve as physical support for the rangings of all receivers to the source. If the cooperative optimal filter is adopted, this dual role cannot be retained by all the messages. Some of them have to serve only for information transfer, and their time of arrival cannot be used.

The operation of the cooperative optimal filter may be summarized as follows. It was seen in Section 7.3 that every member has a main ECM X^i, partitioned by Eq. (7.16) into submatrices of the form P^{ii}, $\overline{P}^{ik}$, $Z^{k\ell}$) and a set of auxiliary matrices $\hat{P}^{ik}$. It was shown in Section 7.4 that the submatrices $Z^{k\ell}$ cannot be propagated correctly by member i; they must be supplied by other members. As for measurement incorporation, Section 7.5 showed that the main ECM X^i can incorporate rangings correctly, but the auxiliary ECM's require help from other members.

Figures 7.2 and 7.3 will help to describe the time schedule of a three member community, and can be generalized to larger numbers of members. Figure 7.2 shows one round of broadcasts. Black circles symbolize broadcasts whose time of arrival is taken by the receivers and used for rangings. The contents of their messages are written in the boxes. Open circles indicate broadcasts made only for information exchange. Apart from the increased number of broadcasts, the schedule is similar to that of Fig. 6.10. There is a one-round lag between real time and filter update. The messages used for ranging contain all rangings taken lately and enough data to compute the single-step transition matrices (items (i) and (ii) of Section 7.3).

Figure 7.3 shows the interval between t_{n+1} and t_{n+2} in more detail. The messages used only for information exchange have their contents spelled out in the boxes; they contain, alternatively, item (iii-a) or item (iii-b), for which see Sections 7.4 and 7.5. At time t_{n+1} all the rangings taken up to time t_{n-1} have been shared to the community. Propagation of the matrices from t_{n-2} to t_{n-1} may begin. Every member i propagates all submatrices of its ECM except $Z^{k\ell}$ (with $k,\ell \neq i$) using Eqs. (7.34), (7.35). Then the missing submatrices are shared, and propagation of X^i is completed. Then measurement incorporation begins. Every member i incorporates the two rangings taken at time t_{n-1} into its main ECM using Eq. (7.44) repeatedly,

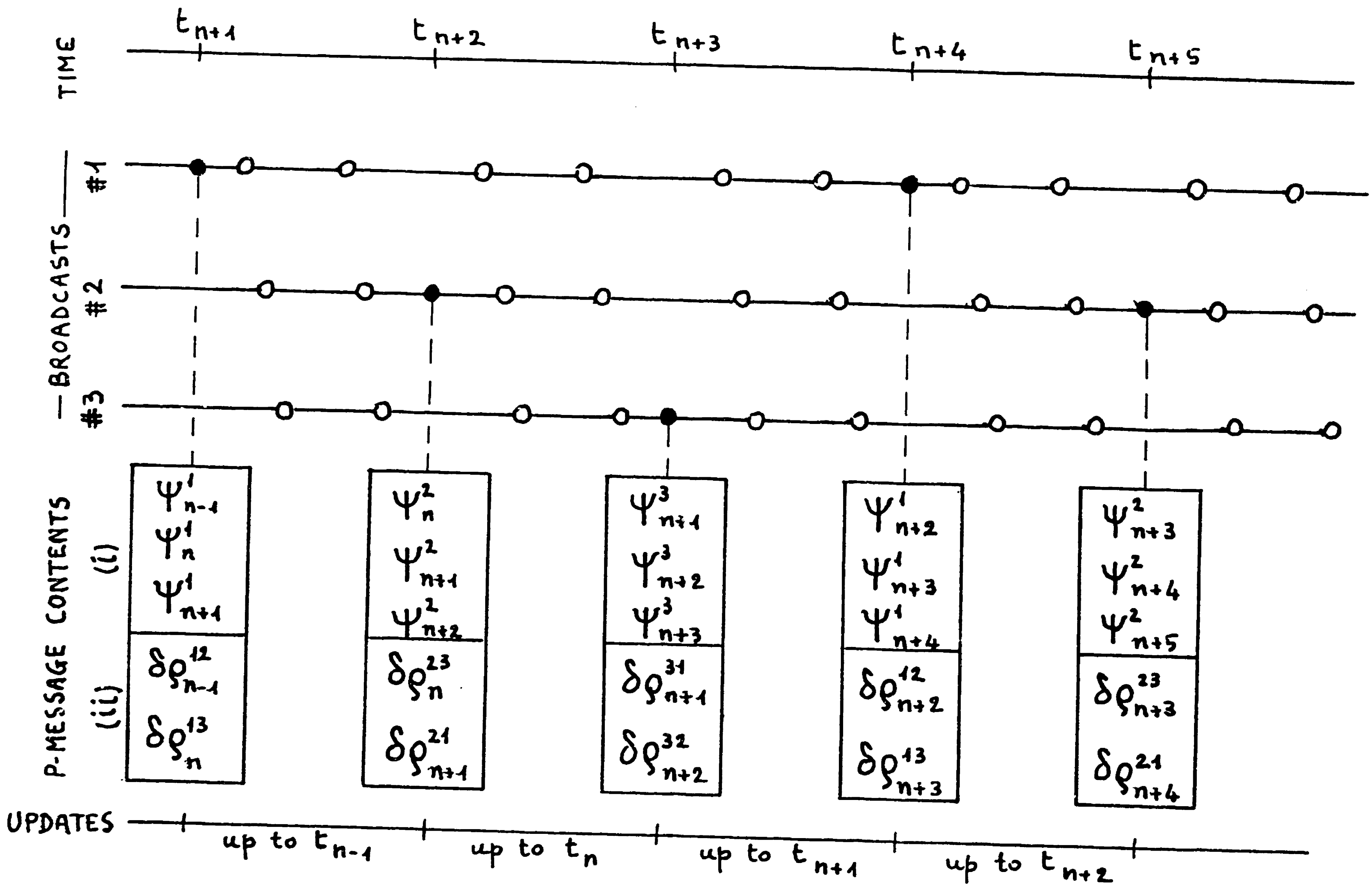


Figure 7.2. Time schedule of a three-member cooperative optimal filter.

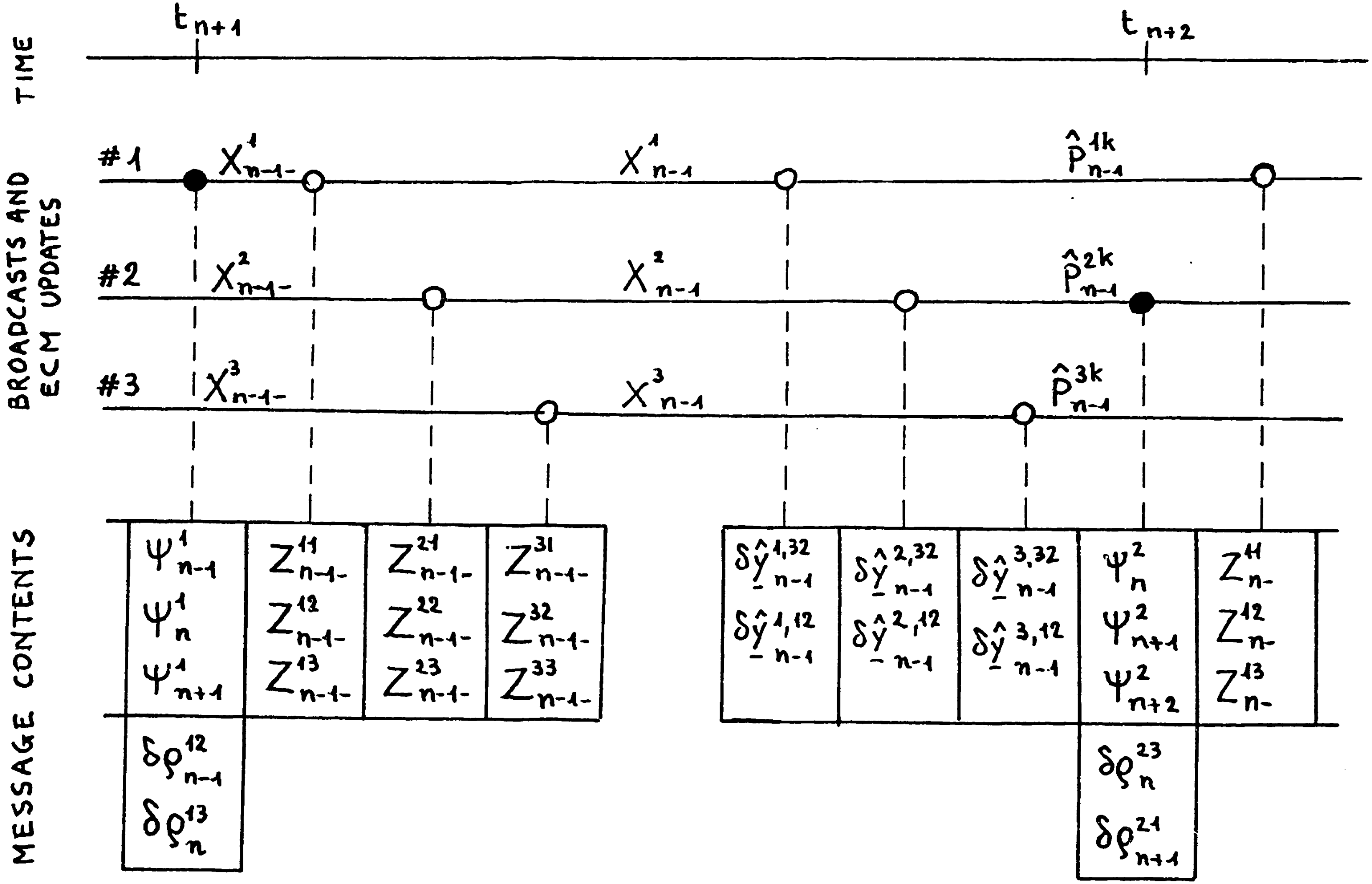


Figure 7.3. Time schedule of a three-member cooperative optimal filter.
Expanded view of the interval between two p-messages.

and finds two sets of estimate updates using Eq. (7.43). The $\delta\hat{\underline{y}}^i$ portions of these updates are shared to the community. After that, every member can use them to update its auxiliary ECM's, using Eq. (7.47) repeatedly. At time t_{n+2}, when the next broadcast for ranging purposes in scheduled, the filter is updated up to t_{n-1}.

In general, if the community has m members, the contents of p-messages are as follows. Item (i) includes data for m single-step transition matrices; item (ii) includes m-1 rangings; item (iii-a) includes m sub-matrices and item (iii-b) includes m-1 update vectors.

7.7 Evaluation of the size of p-messages

This section contains an evaluation of the number of bits that must be transmitted by each member during a round of broadcasts. It parallels what was done in Section 6.10, and uses the symbols q, r defined there.

Additional symbols are:

t   number of bits for a transition matrix;

u   number of bits for an update vector.

During a round of broadcasts, the message containing items (i) and (ii) is broadcast once by every member. The messages containing item (iii-a) are broadcast m times by every member, and so are those containing item (iii-b). Item (i) requires mt bits; item (ii) requires (m-1)r bits; item (iii-a) requires mq bits and item (iii-b) requires (m-1)u bits.

Therefore, the number of bits per round to be sent by any member is:

$$\ell_c = m^2 (q+u) + m(t+r-u) - r$$

Notice that ℓ_c is proportional to the square of the number of members; compare with ℓ_u, which is linear in m, and ℓ_o, which is constant (see Section 6.10).

In order to attribute likely values to the symbols q, t, r and u, several considerations have been made, which will not be reported in detail. The reader can substitute more appropriate values, if he wishes. For q and r the values of Section 6.10 have been adopted: 40 bits and 17 bits, respectively. For the update vector, 11 elements of 13 bits each, giving a total of 143 bits for u. As for the transition matrix Ψ^i, only the elements due to the dead reckoner have to be computed on line. The data necessary for such computation (see Widnall and Grundy (22) p. 26 ff) and the bits attributed to them are as follows: latitude and longitude (24 bits each), altitude (11 bits), three components of the accelerometer output (10 bits each), three components of platform angular velocity (8 bits each). The total is t = 113. Notice that latitude, longitude and altitude are also necessary for the computation of the expected time of arrival and of the geometry vector.

Using these values for q, r, t, u, the value of ℓ_c for several sizes of the community is:

m:	3	4	5	6	7	8	9
ℓ_c:	1591	2859	4493	6493	8859	11591	14689

Comparison with Section 6.10 shows that ℓ_c is at least one order of magnitude greater than ℓ_u. It also shows that a nine-member community would require a total of 132, 201 bits per round, or 17% of the whole JTIDS capicity.

7.8 Conclusions

This chapter has presented, under the name of "cooperative optimal filter," a method of decentralized reconstruction of the centralized optimal estimate. Although this method may find a variety of other applications, it was conceived specially for the problem of ranging-aided navigation, and some details of its application to this problem were discussed in Sections 7.6 and 7.7.

The output of a cooperative optimal filter is identical to that of a centralized optimal filter, apart from a time lag. Therefore, its advantages over a community of ownstate filters, in terms of stability and accuracy, cannot be neglected. Its advantages over the solution proposed in Chapter 6 are, instead, probably not great. It is reasonable to expect that the performance of the community of uncoupled filters proposed in Chapter 6 will not be too far from optimality.

A disadvantage of the cooperative optimal filter is the large amount of data sharing required from the members. Section 7.7 tried to give a quantitative assessment, and it was found that the number of bits each member must send per round of broadcasts grows with the square of the number of members.

Another disadvantage is the vulnerability, resulting from the rigidly collaborative nature of the process. The successful performance of the cooperative optimal filter depends on the uninterrupted exchange of data. Temporary or permanent disabling of a single member's radio link would put a serious strain on the community.

The relative importance of advantages and disadvantages is a value judgment, and must be left to prospective users. However, the writer's opinion is that the concept proposed in Chapter 6 has most of the advantages of and less disadvantages than the cooperative optimal filter, and should be preferred to it in most cases.

CHAPTER 8

CONCLUSIONS

8.1 Overview of the results

The purposes of this work were two: to analyze the concept that has so far been adopted for relative navigation aided by passive rangings, and to propose alternative concepts. A few positive results were obtained in both areas of investigation.

A community performing ranging-aided navigation in the accepted fashion (for which the name of "ownstate" community was coined) was the subject of Chapters 2 to 5. Equations describing its performance were derived (Chapters 2 and 4); it was proved analytically that the community estimation process is expontially stable, if the community is organized in an appropriate way and certain observability and controllability conditions are met (Chapter 3). Simulations (Chapter 5) confirmed this result and gave further insights.

An alternative concept was proposed in Chapter 6. This concept requires greater amounts of computation and data sharing but it promises a better performance. It should be evaluated with realistic simulations.

An original method of decentralized reconstruction of the centralized optimal estimates was proposed in Chapter 7. It is applicable to a certain class of problems, wider than and including the one investigated in this work. If a community performs relative navigation by means of this method, everyone of its members will obtain a portion of the same estimates that a centralized optimal processor would obtain. This solution is costly in terms of amount of data to share, and rather vulnerable to structural perturbations.

All these results, which are described in greater detail in the following sections, suppose the use of purely passive synchronization. The help of round-trip timings has never been invoked.

8.2 Conclusions about ownstate communities.

The word "ownstate" was coined to describe the arrangement by which every member of the community estimates only its own physical attributes, to the exclusion of those of other members. This is the main feature of the concept adopted so far for relative naviagation aided by passive rangings.

It was known that source selection logic is crucial in determining the performance of an ownstate community. The terms used in this work for communities with different kinds of source selection are: democracy (no source selection, every member ranges to everyone else), covariance-based hierarchy (a member ranges to those members which report a better accuracy than its own), fixed-rank hierarchy (a member ranges to those which have a higher prearranged rank, regardless of the accuracy of their estimates). Democratic communities were known to show an unstable behavior; it was thought that the introduction of a covariance-based hierarchy was sufficient to make the community stable (on this point the opinions were not unanimous; see Section 3.1). The results of this work partly support and partly qualify these opinions. Namely, stability could be proved for a fixed-rank hierarchy, not for a covariance-based one.

The mechanism that couples together the filters of an ownstate community was identified (Eq. (2.27)). With that insight, it was possible to prove analytically that a fixed-rank hierarchy can be exponentially stable. The same conditions that are sufficient for one-by-one stability of each filter of the community (supposing it received only absolutely correct information from the others) were proved to be sufficient also for the stability of the community as a whole. Such conditions are given, for each filter, by the Deyst and Price theorem; they are secured by the presence of relative motion in the community, (according to the requirements discussed in Section 3.6).

The widespread opinion, that a covariance-based hierarchy is stable, could be confirmed only with qualifications. It seems that a covariance-based hierarchy behaves in many cases almost like a fixed-rank hierarchy. That is, source selection only rarely allows a closed-loop pattern of rangings (e.g., member a ranges to member b, which ranges to c, which ranges to a). Supposing this to happen covariance-based hierarchy can be as stable as fixed-rank hierarchy, and has the advantage of having adaptive ranks. However, it is uncertain how often this is the case. Choosing a covariance-based hierarchy over a fixed-rank one involves an element of risk and requires a value judgment.

Democratic communities received only brief attention. Simulations of their behavior tend to confirm the report that they are unstable. The coupling mechanism identified by Eq. (2.27), which results, for a democratic community, in multiple feedback, could explain this effect.

8.3 An alternative concept: measurement-sharing community with uncoupled filters.

An alternative concept was proposed in Chapter 6. It was suggested that the filter state vector of every member should include not only that member's significant variables, but also a few of the other members' variables. Namely, it should include all other members' positions (or position errors) and clock phases. Furthermore, it was suggested that measurements should be shared. It was shown that this arrangement presents two advantages over the "ownstate" concept.

One advantage is that the filters will be decoupled from one another. Positions and clock phases of source and receiver are the only arguments of a ranging. If a member estimates all other members' positions and clock phases (as well as its own) it can compute the expected value of all rangings using only its own estimates; its estimation error, then, will not depend on anybody else's estimation error. Filter decoupling allows separate analysis and independent performance of each filter of the community; source selection is no longer necessary for stability.

The other advantage is that any member can compute the expected value of another member's rangings. Therefore, it can incorporate all rangings that are taken by the community, if it knows their values. In order to exploit this advantage, it was suggested that every member should share its rangings to the community. In this way, every member will have a centralized filter, which is provided with knowledge of all measurements, and is suboptimal only because some of the lower-order derivatives are not modeled. It is known that the performance of such a filter is, in many cases, not far from optimality, if the noise model is well chosen. A simulation indicated that this is likely to happen for our case as well.

The use of this concept involves greater computational burden and greater data exchange than an ownstate community does. Neither of these inconveniences seems forbidding, if the size of the community is small.

The advantages, in terms of better performance, are probably remarkable; but the expectation of a performance close to optimality should be verified with more realistic simulations than this work could offer.

In any case, it is important to have stated that a viable alternative to the ownstate concept exists.

8.4 Another alternative concept: the cooperative optimal filter.

Under the name of "cooperative optimal filter" Chapter 7 presented a method of reconstructing the centralized optimal estimates by the cooperative work of several decentralized processors. The method applies to all systems which can be partitioned into a set of subsystems that are dynamically uncoupled, but pairwise coupled by measurements. This class includes a ranging-aided navigation community.

When applied to our problem, this method involves some more computations than the concept of Section 8.3, and a much larger amount of data to share; this amount grows with the cube of the size of the community. The filter models of each member include the same variables as those of a measurement-sharing community of uncoupled suboptimal filters; in addition, the recursive computation of a few auxiliary matrices is required. Measurements must be shared, and so must many other data, including, at every step, portions of the estimate updates and of the propagated error covariance matrix.

Weaknesses of this method are the large amount of data sharing and the vulnerability to structural perturbations. The latter results from the fact that every update of the error covariance matrix must be accomplished with the help of all members. A member cannot be left alone, nor can the community easily do without it.

The advantage of this method consists in producing the centralized optimal estimates, without any member having the computational burden of a centralized optimal filter.

A comparison of the two concepts proposed as alternatives to the one adopted so far one (uncoupled suboptimal filters, Chapter 6, and cooperative optimal filter, Chapter 7) seems rather in favor of the former. The former has less data exchange, uncoupled filters that allow any member to operate independently from the others, and, probably, an output not too far from optimality.

It must be pointed out, however, that the method of Chapter 7 may find other applications. Other available methods of decentralized reconstruction of the centralized optimal estimates assume unlimited computational capability, but no possibility of sharing measurements. This method, instead, assumes limited computational capability and the possibility of sharing measurement and other on-line data. It could, therefore, find applications without being in competition with the other methods.

8.5 Suggestions for further research.

Since this work was mainly analytical, a good amount of simulation work remains to be done. Some of it has been already pointed out. Namely, it was suggested that the approach proposed in Chapter 6 (measurement-sharing uncoupled filters) should be validated with realistic simulations; it was also suggested that simulations could tell more about the cases in which it is safe to use a covariance-based hierarchical ownstate community, rather than one with fixed ranks. It can be added now that one could try to retain the advantage of covariance-based hierarchy (that is, adaptive ranks) while minimizing the occurrence of rank reversals. For instance, it could be established that a member can range to another only if the other reports a better accuracy for several times in a row, or if the difference between the computed error covariances exceeds a certain threshold. Such source selection logics can be conveniently evaluated through simulations.

Besides, it is possible that the results of this work suggest analytical developments in a seemingly unrelated area. An essential feature of the problem that was investigated here is the presence of a set of subsystems that are dynamically uncoupled, but can be measured only relative to one another. Such a feature may not be unique to our problem. The respective advantages and disadvantages of the three approaches considered in this work (ownstate filters, uncoupled suboptimal filters, cooperative optimal filter) would probably be different, when applied to a different problem. One of them could turn out to be particularly advantageous and to deserve an appropriate extension or generalization.

The reader may also remember that the proof of the exponential stability of a suitably hierarchized ownstate community was obtained in an unconventional way. Exponential stability was supposed in each isolated subsystem (i.e., in each member's filter when ranging to errorless sources); then the previously proved property, that an exponentially decaying input gives an exponentially decaying output, was applied. The same device could be useful in establishing the stability of hierarchical layer-structured estimation and control systems, which have an important role in large-scale systems theory.

If any of these suggestions is carried on, it will surely be welcomed by the present writer, who chooses the expression of such a wish to be the conclusion of this work.

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