RENEWAL PROCESS AND DIFFUSION MODELS OF 1/f NOISE

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February, 1975

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May, 1979

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JUL 15 1980
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Submitted to the Department of Electrical Engineering and Computer Science on May 18, 1979 in partial fulfillment of the requirements for the Degree of Doctor of Science

ABSTRACT

A non-stationary autocorrelation function for 1/f noise is derived and shown to depend logarithmically on absolute time. For observation times that are short when compared to the elapsed time since the process began, the non-stationary autocorrelation function is approximated and shown to be almost stationary, with only its steady or D.C. value dependent on absolute time. The almost stationary autocorrelation function is then used in conjunction with linear system models that yield 1/f noise to characterize the memory of 1/f random processes.

Techniques for computing the power spectral density of renewal processes that yield 1/f noise are developed and two illustrative examples are considered that demonstrate the connection between stable distributions and 1/f noise.

Finally, in an effort to support the conclusion of Voss and Clarke that the 1/f noise in thin-film metal resistors results from temperature fluctuations, the power spectral densities of temperature fluctuations arising from the diffusion of heat are computed for 1, 2, and 3 dimensions, both for the case of very small and very large resistor lengths. None of the models yields 1/f noise.

Thesis Co-supervisors: W. M. Siebert and R. B. Adler
Professors of Electrical Engineering
ACKNOWLEDGEMENTS

The relationships that I have enjoyed with my thesis co-supervisors, R. B. Adler and W. M. Siebert, have extended well outside the usual bounds. Their encouragement and intense interest in my work as well as their friendship has made my years at M.I.T. very rewarding. I would also like to acknowledge the continued support and friendship of R. G. Mark and S. K. Burns, who did not contribute to this work, but with whom I have been long acquainted. Finally, I would like to thank Gloria McAvenia, who may never again volunteer to type an equation, for painstakingly typing this document.
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INTRODUCTION

In 1968, Mandelbrot and Wallis published a paper entitled, "Noah, Joseph and Operational Hydrology", in which they raised an interesting question. What is the likelihood of 40 days and 40 nights of continuous rain or of 7 years of adequate rainfall and abundant crops followed by 7 years of inadequate rainfall and famine? If the random fluctuations in rainfall had statistics like white noise, the likelihood would be vanishingly small, like that of a group of monkeys, banging on typewriters, accidently typing one of Shakespeare's sonnets. White noise has an equal concentration of power at all frequencies and large events lasting for a long time are very rare. But, Mandelbrot and others have shown that the statistics of rainfall are more like $1/f$ noise. $1/f$ noise is characterized by a power density that increases at lower frequencies. In fact, its power density spectrum is inversely proportional to frequency; hence the name, $1/f$ noise. The dominance of energy at low frequencies leads one to expect long-lived events, like 40 days and 40 nights of rain.

Physical measurements whose values fluctuate as a function of time are often described with random process theory. A random process is a function of time that can be described exactly by giving the joint probability density function for the values of the process at all times. However, random processes are often partially characterized by their autocorrelation functions or their power spectral density (PSD), both of which are only second moment characterizations and do not completely and uniquely specify the random process.
In other words, many different random processes may have in common the same
PSD without being identical to each other.

To be precise, we shall define "1/f-type noise" or a "1/f-type pro-
cess" to be a random process whose PSD has an amplitude versus frequency
relationship of the form:

\[ S(f) \propto \frac{1}{|f|^{2-\alpha}} \quad 0 < \alpha < 2 \]

The parameter \( \alpha \) usually lies close to 1.

Interest in 1/f-type processes results, in part, from their almost
ubiquitous occurrence in descriptions of fluctuations in physical measure-
ments. Besides characterizing fluctuations in rainfall, for low enough
frequencies, they also characterize resistance fluctuations in resistive
materials of all types, including: metal film, semiconductor, carbon and
aqueous solution (Hoog, 1976). 1/f noise is also commonly observed in the
voltages or currents in MOS and bipolar transistors. (Other processes such
as Johnson Noise or shot noise are also frequently present. But, their
PSD's are white and at low enough frequencies, the PSD of the 1/f-type
process will dominate.) 1/f-type processes have also been used to charac-
terize, in part, the fluctuations in:

- seasonal temperature (Brophy, 1970)
- traffic flow (Musha, 1976)
- the frequency of quartz crystal oscillators (Wainwright, et. al.,
  1974)(Atkinson, et. al., 1963)
- the rate of insulin uptake by diabetics (Campbell and Jones, 1972)
- the voltage across nerve membranes (Verveen and Dirksen, 1965)
  (Poussart, 1971).
the voltage across synthetic membranes (Michalides, Wallaart and DeFelice, 1973)
economic data and the rate of computer errors (Mandelbrot, 1977)
the amplitude and frequencies of music (Voss and Clarke, 1975)

A few models have been proposed that successfully explain the presence of $1/f$ type noise in certain measurements, but a general theory is still lacking.

The purpose of this work is to provide an analytic framework for describing $1/f$ noise and for discussing its important and distinguishing characteristics. Although a general theory will not be presented, hopefully, with the understanding to be gained from this work, the ubiquitous occurrence of $1/f$-type noise in physical measurements may seem a little less surprising.

One of the unusual properties of $1/f$ noise and one that has attracted considerable attention to the topic was the so-called 'infrared catastrophe'. If the PSD of a random process continued down to zero frequency while remaining proportional to:

$$\frac{1}{|f|^{2-\alpha}}$$

where: $\alpha < 1$ only

then the integral of the PSD diverges and the variance, thus computed, is infinite. Of course, infinite variance represents infinite power which cannot be a property of a finite physical system that exhibits $1/f$ noise.

One class of attempts at avoiding this difficulty was to assume that at some low-frequency limit, the PSD would become flat like white noise. (A high-frequency limit is also assumed above which the PSD decreases at least as fast as $1/f^2$, exactly as is assumed for white noise to assure
integrability.) With low and high frequency limits, the integral of the PSD may be large, but it, and therefore, the variance is finite.

Almost all experimentally observed $1/f$ noise fits the definition of $1/f$-type noise, with $\alpha \leq 1$. Therefore, according to the above discussion, there must exist a low-frequency limit for the variance to be finite. However, after intensive experimental effort, low-frequency limits have not been detected. For example, in MOS transistors, Mansour, et. al., (1968) have observed a $1/f$ PSD to $10^{\ast(-5)}$ Hz, and Brophy (1970) has observed $1/f$-type seasonal temperature fluctuations to $10^{\ast(-10)}$ Hz, or about 1 cycle in 300 years. In those cases and almost all others no change in slope was observed at low frequencies. In only two cases has a low frequency flattening been observed: in thin films of tin at the temperature of the superconducting transition (Clarke and Hsiang, 1975,1976) and in nerve membranes (Verveen and Derkson, 1965).

Mandelbrot (1967) settled the apparent contradiction posed on one hand, by a finite physical system generating a random process with infinite variance and, on the other, by the lack of a low-frequency limit. He noted that a random process with a PSD that is not integrable does not satisfy the conditions of the Weiner-Khinchine Theory and that the random process cannot be stationary. Hence, its variance would be a function of time as is the case for a Weiner-Levy process (Brownian motion). Rather than postulate a low-frequency limit to be lower than can ever be measured, Mandelbrot considered only processes for which the starting state is known and computed conditional PSD's that depend on the exact starting state of the process. In this way, observing the process for a finite amount of time will always yield a finite variance.
Unfortunately, for a complicated system, Mandelbrot's approach is difficult to use. Only rarely would an experimenter have complete knowledge of the initial state of a system that manifests 1/f noise in one of its parameters. For example, the complete starting state of a conduction process in a solid would require the exact position and momentum of each lattice atom and each conduction electron, as well as knowledge of any 'external' influences. This knowledge is almost impossible to obtain with adequate precision. Only for the cases where an aggregate initial state is sufficient would Mandelbrot's approach be convenient.

In chapter III, I develop a method for describing a 1/f-type process, assuming that, before some time, the system producing it was at rest, but without knowing exactly when that time was. A linear system model that yields 1/f noise exactly is used to compute a non-stationary autocorrelation function, conditioned on knowing that at time, \( t = 0 \), the system was at rest. Approximating the function for the case where the time of observation \( T \) is much shorter than the time \( (t_2) \) elapsed since \( t = 0 \), yields an autocorrelation function with two terms. One term depends only on relative times \( \tau \) between values of the process and is independent of \( t_2 \), while the other term depends only on \( t_2 \) and not on \( \tau \). Except for the term with \( t_2 \), the autocorrelation function would be stationary. Furthermore, as we shall see in chapter IV, the term with \( t_2 \) has only a small effect on the corresponding PSD, leading to the concept of an almost stationary PSD. The derivation is generalized to include all 1/f-type noises, with similar results.

In Chapter IV, I examine the PSD's that are the transforms of the approximate non-stationary autocorrelation functions derived in Chapter III. Only the D.C. value of the PSD is influenced by the total elapsed time, \( (t_2) \).
Therefore, we conclude that the PSD's are almost stationary.

In chapter V is discussed the influence of past values of a 1/f-type process on present values. While the PSD represents the distribution of power as a function of frequency, its Fourier Transform, the autocorrelation function represents the effect of past events on the present and, in an informational context, can be used to characterize the "memory" of a random process. The derivation of chapter III demonstrated that the shape of the autocorrelation function is independent of the elapsed time, \( t_2 \), since the system ceased to be at rest. For values of \( \alpha \) close to 1, the autocorrelation function is quite flat. Current values of the process are almost equally correlated with values at all past times. A 1/f-type process is strongly influenced by its history - almost all of it.

Having established in chapter IV that 1/f-type noises are strongly influenced by their history, we also pursue in Chapter V the question of how much information they remember. A state-variable description of two linear systems that yield 1/f noise is developed to estimate the number of initial conditions required to specify the initial state of a system that manifests 1/f noise. The minimum number is rather low, approximately one initial condition for each decade of frequency over which the 1/f PSD extends.

The combined result of chapters III, IV, and V is the following description of 1/f noise. 1/f noise is an almost stationary random process (logarithmically non-stationary) whose autocorrelation function is extremely flat. Therefore, current values of the process are strongly influenced by all past values. However, the influence of the past for a linear system that exhibits 1/f noise can be condensed into the values of a surprisingly
small number of state variables, one for each decade of frequency over which the 1/f PSD extends. In short, a summary of the history of the process requires one number that averages the values of the last 1 second, one for the past 10 seconds, 100 seconds and so on.

The above description might well constitute a sensible approach to history. Events of the preceding week are often summarized day-by-day, those of a year ago, week-by-week, and those of thousands of years ago, century-by-century. The trends established thousands of years ago that still influence the present are relatively small in number when compared with the number of possible short term trends established yesterday. In an informational context, 1/f noise represents a process that evolves in a curious and characteristic way. New data is constantly added, while older data is condensed into summaries on which the added data has a diminishing effect as it gets older.

Although no description has been offered for what causes 1/f noise, it no longer seems surprising to me that information retaining processes like the evolution of the species, the development of culture, of music and art, and of economics and government might be aptly described as 1/f-type processes. For physical systems, whose measured parameters exhibit 1/f noise and to which no new information is presently being added, the observed 1/f noise may represent the gradual forgetting of an initial state: one number after 1 second, another after 10 seconds and so on.

In chapter VI, two families of renewal process models for 1/f-type processes are developed. Their PSD's are computed using an operational definition of a PSD developed by W. M. Siebert that is computed on the basis of a finite-time observation. This definition yields consistent
results for the value of the exponent \( \alpha \) of a \( 1/f \)-type PSD, but allows for widely varying results for the magnitude of the PSD in complete agreement with experimental observations of \( 1/f \)-type noise in physical systems.

The two families of renewal process models are used to analyze two aspects of \( 1/f \)-type noise. The first family uses simple events (step functions) whose interarrival times are stably distributed. The result is the entire range of \( 1/f \)-type noises plus one case where the interarrival times are Normally distributed that yields white noise. In the limit of large numbers, sums of identical, independent random variables approach a Normal distribution, but only if their variance is finite. If their variance is infinite, the sum will approach one of the other stable distributions, each of which has its particular domain of attraction (see Feller, volume II). For example, the Cauchy distribution is stable and in the first family of renewal process yields a \( 1/f \) PSD exactly.

The second family uses more complicated events (ramp functions) and for a \( 1/f \) PSD requires an interarrival time density function that for long times approaches zero as \( t^{-4} \). Both the mean and variance of this interarrival distribution are finite and the distribution meets the conditions of the law of large numbers. Therefore, a large sum of identical, independent distributions would yield a Normal distribution. However, choosing the interarrival times according to a Normal distribution is shown to yield white not \( 1/f \) noise in the second family as well as in the first. Hence, applying the law of large numbers somehow eliminates whatever is responsible for the \( 1/f \) noise.

The Normal distribution can provide a good fit to the 1st and 2nd moments, but not the third and higher which are infinite. Calculating the
PSD for each independent process and then adding the PSD's would yield, overall, 1/f noise; but first computing the overall interarrival distribution function by applying the law of large numbers to obtain a Normal distribution would yield only white noise. Ignoring the tails of the distribution or equivalently the third and high moments eliminates the 1/f noise.

Finally, in chapters VII - IX, we consider the Langevin approach for modelling 1/f noise, and apply it to the description of thermal fluctuations in a diffusing medium, which have been suggested by Voss and Clarke (1976a) to account for the 1/f noise in thin-film metal resistors.
CHAPTER II

BACKGROUND

The first theoretical model for 1/f noise was proposed by Bernamont (1937) to explain the flicker effect in vacuum tubes. Van der Ziel (1950) noted its general form and McWhorter (1955) applied it to 1/f noise in semiconductors. Independent shot noise processes with PSD's of the form:

\[ S(f) = \frac{\tau}{1 + (2\pi f \tau)^2} \]

can be superimposed to give a 1/f PSD. The processes must have a uniform distribution of time constants (\( \tau \)) and are weighted by \( \left( \frac{1}{\tau} \right) \) as shown below:

\[ S_N(f) = \sum_{m=1}^{N} \left( \frac{1}{m \tau} \right) \frac{m \tau}{1 + (2\pi f m \tau)^2} \]

computing the sum yields:

\[ S_N(f) \propto \frac{1}{|f|} \quad \text{for} \quad \frac{1}{N \tau} \ll |f| \ll \frac{1}{\tau} \]

As \( N \) increases, longer time constants are required and the 1/f PSD will extend to lower frequencies. This can easily be demonstrated by proceeding to the continuous limit.

\[ S(f) = \int_{\tau_{\text{min}}}^{\tau_{\text{max}}} \left( \frac{1}{\tau} \right) \frac{\tau}{1 + (2\pi f \tau)^2} \]

\[ = \frac{1}{2\pi|f|} \left[ \tan^{-1}(2\pi f |\tau_{\text{max}}|) - \tan^{-1}(2\pi f |\tau_{\text{min}}|) \right] \]

which is proportional to 1/f for:

\[ \frac{1}{\tau_{\text{max}}} \ll |f| \ll \frac{1}{\tau_{\text{min}}} \]

McWhorter proposed that 1/f noise in semiconductors was a surface ef-
fect caused by the emptying and filling of trapping states in the oxide layer at the surface. Each trapping process could be regarded as an independent shot-noise process that effectively modulated the carrier density and, hence, the resistance of the semiconductor. He assumed that the carriers moved between the bulk and the traps via a tunnelling process. Thus, the time constants of the shot-noise processes would depended exponentially on the distance the carrier was required to tunnel. Thus:

\[ t = t_{\text{min}} e^{-x/x_0} \]

where \((x)\) is the tunnelling distance from within the surface oxide layer to the bulk and is assumed to be a random variable with a uniform density. The derived probability density function (pdf) for \((\tau)\) is:

\[ \text{pdf} (\tau) \propto \frac{1}{\tau} \]

Thus a 1/f PSD results that is valid over a large range of frequencies.

Another theoretical model, which forms the basis for a commercial 1/f noise generator, (Barnes and Jones, 1971) derives from the continuous RC transmission line. The driving-point impedance of an infinite, continuous RC line is:

\[ Z(f) = \left( \frac{R}{j2\pi fC} \right)^{1/2} \]

Exciting the line with a white noise current source at the input yields an input voltage whose PSD is exactly 1/f. A line of finite length will give a 1/f PSD down to a lowest frequency, inversely proportional to the square of the length.
Many other theoretical models of 1/f-type noise have been proposed, but none is generally accepted. Some have been shown to be in error by subsequent publications. Some do not allow a sensible physical embodiment (Mandelbrot, 1967). Some apply only to a specific physical situation (e.g., Tunaley, 1972c). Rather than attempt to discuss them all, I have chosen a few which have not been disproven and that apply to a broad range of systems in which 1/f-type noise is observed.

Mandelbrot (1967) correctly discussed the system theoretic difficulties inherent in 1/f-type noise. He developed a few mathematical descriptions based on conditional PSD's. Though correct, these descriptions are difficult to apply to the physical systems in which 1/f noise is observed and do not enable one to readily predict in which systems 1/f noise will be present.

Halford (1968) proposed a renewal process model in which events are generated by a Poisson process with amplitudes and durations that depend on their interarrival time. To obtain 1/f-type noise, the amplitudes must be irrational functions of the interarrival time.

Hoog (1969) has suggested that at least part of the 1/f noise in resistors is not caused by surface traps or by any other surface effect and cannot be explained with McWhorter's model. Strictly on the basis of experimental data, he proposes that 1/f noise in resistive materials is not a surface effect and that its magnitude is independent of the choice of resistive material or geometry and is inversely proportional to the total
number of current carrying particles in the resistor.

\[ S(f) = \frac{C}{N} \frac{1}{|f|} \]

where: \( C = 2 \times 10^{-3} \)

\( N = \text{total number of mobile charge carriers} \).

The data presented by Hoog and his colleagues, Vandamme and Kleinpenning (see Bibliography) exhibits the enormous scatter characteristic of 1/f noise and loosely fits their theory except for the data from resistors made of aqueous solutions that is contradictory. In aqueous solutions (Hoog and Gaal, 1971), the magnitude of 1/f noise is dependent on the volume of the resistor instead of on the total number of carriers. Voss and Clarke (1976a) also found that the magnitude of the 1/f noise in the semimetal Bismuth was inversely proportional to the volume rather than the total number of carriers. The rest of the data is fit as well by an inverse dependence on the total number of mobile charge carriers as it is by an inverse dependence on the volume of the resistor. In addition to data taken on continuous materials, Hoog and his colleagues also present data on spreading resistances formed by the contact of two metallic bars. Again the data loosely fits their theory. However, the 1/f noise in these resistors may not be of the same origin as in the continuous metal film resistors. Hoog also maintains that the magnitude of the 1/f PSD is independent of material and geometry, once the total number of carriers has been considered. This is incorrect for aqueous solutions (Hoog and Gaal, 1971) and is also incorrect for copper. Dutta, Eberhard and Horn (1977) measured the 1/f noise in single crystal filaments and in thin films of copper. Their data on thin films agreed with Hoog's, but on the filaments it was 300 times too large. Finally, Van der Ziel (1973) has shown that
Hoog's data can be interpreted as a surface effect.

I will take the position that 1/f noise occurring in similar experimental situations may be of distinctly different physical origins and that a single model like Hoog's, broadly applied, is inadequate and incorrect. Even among experiments with metal film resistors, there is evidence for different physical processes. For example, Hoog observed a small magnitude versus temperature dependence in gold films. But Eberhard and Horn (1977) found such a strong temperature dependence in silver and copper films that they modeled it as an activation energy process. Furthermore, Vandamme (private communication to Hoog) observed 1/f noise in manganin contact resistance, but Voss and Clarke (1976) found none in thin films of manganin.

Tietler and Osborne (1970) proposed that frequency dependent non-linear coupling among normal modes can produce a PSD with equal energy per octave, i.e., 1/f noise. However, they did not indicate the form of the non-linear coupling required or how it might be accomplished.

Tunaley (1972a, 1972b) adopted an operational definition of the PSD that enables one to consider random processes for which the Wiener-Khinchine Theory is not appropriate. He then proceeded to discuss renewal processes whose interarrival times are stably distributed with infinite variances and obtains (1/f)**(2-a) PSD's. A comparison of his approach to one developed by W.M. Siebert will be discussed at length later. Tunaley (1972c) applied his model to thin-film resistors by assuming islands of conductivity between which carriers must hop. If the variance of the hopping time is unbounded, a 1/f-type noise results. Hoog (1976) has noted that this model most likely does not apply to homogeneous thin-films for reasonable film
thickness, because the films are not likely to consist of discontinuous islands.

Handel (1975) suggested that 1/f noise in a current carrying system derives from the electron scattering process. He proposed that each scattering event produces bremsstrahlung radiation with a high probability of low photon frequencies, and that the bremsstrahlung interacts with the wave properties of the charge carrier to produce "quantum beats" at very low frequencies with a 1/f PSD. Although this paper was mentioned by Hoog (1976) in his recent review article, it was not critically discussed by him or by anyone else. Handel's paper seems to be in error for at least two reasons. In bulk, homogeneous material, the length of time during which an electron is influenced by a scattering potential is at most the time that the electron is in the solid and not hitting the boundaries. But, this time is very short, microseconds or less. Therefore, the bremsstrahlung spectrum will not extend to f=0, but will truncate for frequencies below 1 mHz. Furthermore, the interaction between the bremsstrahlung radiation and the charge carriers can continue and be coherent in phase, again, only while the electron is in the solid and not hitting the boundaries. Again, times are on the order of microseconds. But 1/f noise is observed at frequencies much lower than 1 Hz. Handel's model cannot produce such low frequencies.

Musha (1975) proposed that the resistance of an electrical conductor in thermodynamic equilibrium fluctuates because of fluctuations in carrier density that have a 1/f PSD. This idea may appear to be in conflict with Nyquist's derivation of thermal noise from a resistance, but it is not. Specifically, Nyquist's theorem states that a resistance in thermodynamic
equilibrium will exhibit only white noise in its terminal current or voltage. It implicitly assumes that the resistance does not vary with time; but this assumption is unnecessary. Consider the system in thermodynamic equilibrium illustrated in Figure 2a.

![Diagram of R(t) and PARTICLE RESERVOIR]

**Figure 2a**

Charged particles are freely exchanged between the resistance, R(t), and the reservoir. Since R(t) is inversely proportional to the number of carriers within it, the value of R(t) will fluctuate about an average \( <R(t)> \). Of course, the PSD of the fluctuation remains to be determined. McWhorter's model for 1/f noise fits exactly into the above description. The resistance is the interior volume of the semiconductor and the reservoir is the oxide layer on the surface. Communication between the two takes place in or out of equilibrium and is relatively unaffected by the presence of a D.C. current through R(t); a 1/f PSD that does not extend to f=0 results. If the 1/f process were to extend to f=0, the average \( <R(t)> \), would not exist and the system would not have a well-defined equilibrium.

Musha solved the Boltzman Transport Equation in the one dimensional limit to obtain the PSD for carrier density fluctuations within a resistor, as a function of position and time, \( \Delta n(x,t) \). He uses this result to calculate the PSD of the fluctuating resistance. For small fluctuations about an average level:
\[ \Delta R(t) \propto \int \Delta n(x,t) \, dx \]

For the purpose of examining physical situations to which Musha's model might apply, it is helpful to consider an analogue for his equations. His equations for \( \Delta n(x,t) \) reduce to the one dimensional diffusion equation; therefore, his model is analogous to an RC line, with distributed noise sources that are uncorrelated in both space and time. Current noise sources are used to represent the random generation and recombination of carriers in a small region of the sample and the voltage on the RC line represents fluctuations in the carrier density as a function of position and time, \( \Delta n(x,t) \). Musha considers noise sources in a finite length, but does not consider the effects of finite boundaries on the diffusion equation. Therefore, in the analogue, the piece of an RC line containing the distributed noise sources and representing the resistor must be terminated at each end with a noiseless matched impedance, i.e., an infinite RC line with an identical characteristic impedance but containing no noise sources. Thus we have the analogue illustrated in Figure 2b.

![Diagram](image)

**Figure 2b**

If the length of the resistor is much greater than a diffusion length, Musha obtains a \((1/f)^2\) PSD. Only if the length is much shorter does he obtain a \(1/f\) PSD. When the length of the RC line representing the re-
sistor becomes much shorter than a diffusion length, the voltages at all nodes become completely correlated. Therefore, the RC line can be treated as a single node with a single voltage and driven by a single noise source. This situation is illustrated in Figure 2c.

Termination | Resistor | Termination

![Figure 2C](image)

The single noise source excites the input impedance of the terminating structures which is proportional to $1/\sqrt{f}$ resulting in a carrier density with a $1/f$ PSD. This model is well known and does not represent a new contribution. It will be discussed again in the chapters on diffusion models.

The well known RC line model illustrated in Figure 2c is difficult to apply to the description of experimental $1/f$ noise. The noise source would be accompanied by a resistor in accordance with the Fluctuation - Dissipation Theorem and the terminations that are analogous to a diffusion, an entropy producing process, would not be noiseless. If the resistor noise dominated, the overall noise would be white; and if the termination noise dominated the result would be $f^{-1/2}$. This is discussed in the chapters on diffusion models.

Furthermore, carrier density fluctuations cannot occur in metals. A non-uniform charge density would give rise to a large electric field that would restore uniformity in a very short time. It can only apply to
materials in which both positive and negative charge carriers are present and charge neutrality need not be violated by fluctuations in density. Such is the case for ions in aqueous solutions and for the minority carriers in a semiconductor. Each ion has an oppositely charged counter ion and the charge of each minority carrier can be balanced by an oppositely charged majority carrier. Finally, in practice, any structure will be of finite length and therefore the PSD will truncate at some lowest frequency. For example, in silicon the minority carrier diffusion coefficient is:

\[ D = 40 \text{ cm}^2/\text{sec} \]

Taking the length \( L \) of the "terminations" to be 1 cm., yields:

\[ f_{\text{lower}} = \frac{D}{\pi L^2} \approx 10 \text{ Hz} \]

This value is not low enough to be consistent with experimental data.

Musha also derives a one dimensional diffusion model for temperature fluctuations which leads to a 1/f PSD for sufficiently small size and low frequency. In a system capable of both carrier density and temperature fluctuations, he notes that the former will dominate because it has many fewer degrees of freedom. Many of the above criticisms also apply here.

Voos and Clarke (1976a) have suggested that 1/f noise in thin-film resistors is caused by temperature fluctuations modulating the value of the resistance and have demonstrated that the resistance fluctuations occur in equilibrium and are not dependent on the presence of a D.C. current (1976b). They have obtained a better fit to experimental data than Hoog by assuming that the magnitude of the PSD is proportional to the square of the thermal coefficient of resistance and have noted that for the material, manganin, with a nearly zero coefficient, no 1/f noise can be observed.
Furthermore, they have simultaneously measured the $1/f$ noise at different points along the length of the resistor and found the $1/f$ noise to be spatially correlated with a characteristic distance equal to a thermal diffusion length.

These results strongly suggest that $1/f$ noise in thin-film resistors is caused by temperature fluctuations. To provide a theoretical basis for temperature fluctuations exhibiting a $1/f$ PSD, Voss and Clarke have revived the idea that diffusion processes would lead to $1/f$ noise, first proposed by Richardson (1950). His argument was dismissed by Van Vliet and Van der Ziel (1958), who emphasized that the PSD would become flat for low enough frequencies. Therefore, in accord with Van Vliet, Voss and Clarke have predicted a low frequency cutoff for the $1/f$ noise in thin films that depends on the dimensions of the resistor. This prediction has been tested (Dutta, Eberhard and Horn, 1977) and was shown to be incorrect. In addition, Voss and Clarke were unable to obtain the correct form of the PSD.

Recently, Sato (1978) has shown that diffusion of heat in two dimensions with distributed noise sources, analogous to generation-recombination noise in a semiconductor, does yield a $1/f$ PSD. His result is correct in its $1/f$ dependence but its magnitude depends on the size of the substrate acting as a heat bath, which cannot be correct. The size should influence the value of the lowest frequency at which the PSD is still proportional to $1/f$, but not the magnitude. At frequencies well above the low-frequency limit, the diffusion length is very short compared to the overall size of the bath. Therefore, the boundaries cannot have an important effect.

Two-dimensional diffusion of heat in a thermal bath is considered in
Chapter VII of this paper. In thermal equilibrium, fluctuations in temperature arise from two sources, of which, Sato considered only one. The one he considered does, in fact, have a PSD proportional to $1/f$, but it is insignificant in magnitude to the result of the other noise source whose PSD is proportional to $\ln 1/f$. 
CHAPTER III

NON-STATIONARY AUTOCORRELATION FUNCTIONS FOR 1/f PROCESSES

Mandelbrot's correct conclusion (1967) that a random process whose PSD is proportional to:

\[
\frac{1}{f^{2-\alpha}} \quad 0<\alpha<1 \quad \text{only}
\]

cannot be stationary, may appear to be in conflict with numerous experimental observations of 1/f noise. If the process is non-stationary, both the autocorrelation function and the PSD must also be non-stationary. However, experimenters commonly observe 1/f-type noise with \(0 < \alpha \leq 1\) for which at least, the value of the parameter (\(\alpha\)) appears to be stationary. Among identical experiments performed at different times, the observed value of (\(\alpha\)) varies by only a few percent. Under the same conditions, the observed magnitude of the PSD may vary by multiple factors of ten, but for a few experimental situations, its variation is also small.

A resolution of this apparent conflict is the ultimate goal of this and the next chapters. We will derive a non-stationary autocorrelation for 1/f noise from a linear system model by assuming that at some time (\(t=0\)) the system was constructed so that all initial conditions had the value: zero. Using the non-stationary autocorrelation function, we can then discuss the effect of observing the process for a time much shorter than the time elapsed since the process began and without any knowledge of exactly how long it has been continuing. The result will be that the process appears to be almost stationary.

By considering in detail a simple system that gives 1/f noise exactly, we shall derive the non-stationary autocorrelation function for 1/f noise.
The procedure will then be generalized to derive the non-stationary auto-correlation functions for all $1/f$-type noises. The simple system to be considered is illustrated in Figure 3a, below. It consists of a current

![Figure 3a](image)

noise source with a white PSD driving the input impedance of a 1-dimensional RC line that is infinite in length. The impedance of an infinite RC line is:

$$Z(f) = \left[ \frac{R}{j2\pi fC} \right]^{1/2}$$

where:

$R$ = resistance of line/unit length

$C$ = capacitance of line/unit length

Therefore, the PSD of the voltage produced by a constant (white)spectral density of current, $I_N$, is:

$$S(f) = I_N^2 \frac{R}{2\pi fC}$$

Provided that the line is infinite in length, the PSD is exactly proportional to $1/f$ down to zero frequency. However, if the line is finite (and terminated in a finite resistance), the PSD will have a low-frequency limit to the $1/f$ behavior determined by the length of the line:
\[ f_{\phi} = \frac{1}{2\pi R C x^2} \]

Well below the frequency, \( f_{\phi} \), the PSD will be white. To derive the non-stationary autocorrelation function, we will construct the system at time \( t=0 \), such that the voltage on the line is zero, everywhere.

The white noise source will be constructed in a way that greatly facilitates calculations, but actually is a quite general form of shot noise with a D.C. value of zero. Consider a process that once during the interval of observation \((T)\) produces a unit impulse of current that is either positive or negative with equal probability. The time at which the impulse occurs will be uniformly distributed throughout the interval of observation.

\[
I = \pm u_o(t - t_o)
\]

\[
\text{pdf}(t_o) = \frac{1}{T} \quad 0 < t_o < T
\]

\[
= 0 \quad \text{otherwise}
\]

Summing a large number \((N)\) of independent, identical processes would create a shot noise current, without an accompanying D.C. current, whose PSD would be white with magnitude:

\[
S(f) = \frac{N}{T} \equiv \frac{I_o^2}{N}
\]

For our computation, an increase in the time of observation must be accompanied by a corresponding increase in the number of independent processes. Actually, this is equivalent to requiring a constant arrival rate of a single Poisson process, but is computationally simpler.

A current impulse at \( t = t_o \) excites the RC line to give a voltage response of:
\[ v(t) = \pm \left( \frac{R}{\pi C} \right)^{1/2} \frac{1}{\sqrt{t - t_0}} u_{-1}(t - t_0) \]

The autocorrelation function of the voltage resulting from a single impulse is:

\[ R(t_2, t_1) = E[v(t_1)v(t_2)] \]

where \( E[\ ] \) denotes expected value

\[
= \frac{R}{\pi C} \left\{ \left[ \frac{1}{\sqrt{t_2 - t_0}} u_{-1}(t_2 - t_0) \right] \left[ \frac{1}{\sqrt{t_1 - t_0}} u_{-1}(t_1 - t_0) \right] \right\}
\]

Noting that both terms are non-zero only if the impulse occurs before \( t_1 \) and \( t_2 \), and assuming \( t_2 > t_1 \), we obtain:

\[ R(t_2, t_1) = \int_0^{t_2} \frac{R}{\pi C} \frac{1}{\sqrt{t_2 - t_0}} \frac{1}{\sqrt{t_1 - t_0}} \, dt_0 \]

For \( N \) identical and independent processes, the final result is just \( N \) times the above, which is:

\[ R(t_2, t_1) = \frac{N}{\pi T} \frac{R}{C} \cosh^{-1} \left( \frac{t_2 + t_1}{t_2 - t_1} \right) \]

For the case where the total observation time (\( T \)) is much less than the time since the process began we have:

\[ t_2 - t_1 < T << t_1 \text{ or } t_2 \]

Thus we obtain:

\[ R(t_2, t_1) = \frac{NR}{\pi TC} \cosh^{-1} \left( \frac{1 + t_1/t_2}{1 - t_1/t_2} \right) \]

\[ - \frac{NR}{\pi TC} \ln 2 \left( \frac{1 + t_1/t_2}{1 - t_1/t_2} \right) \]
\[ -\frac{NR}{\pi TC} \ln \left( \frac{4}{1 - t_1/t_2} \right) \]
\[ -\frac{NR}{\pi TC} \ln 4t_2 - \frac{NR}{\pi TC} \ln (t_2 - t_1) \]

For the purpose of comparing this with the 'stationary' autocorrelation function that will be derived in the next chapter, we rewrite this in terms of the difference \( \tau = t_2 - t_1 \):

\[ R(t_2, \tau) = \frac{NR}{\pi TC} \ln 4t_2 - \frac{NR}{\pi TC} \ln \tau \]

One of the puzzles of 1/f noise has been associated with its 'infinite variance' or more correctly, the wildly varying magnitude of its PSD. With the above non-stationary autocorrelation function, I would like to speculate that a careful experimenter observing the noise of this RC line model would observe an almost stationary PSD.

Assuming that the system is initially at rest, a careful experimenter would start it up, allow ample time for it to warm up and become 'stable', and then take his data. In repeating his experiment, he would most likely wait about the same amount of time for each trial. The result would be that \( t_2 \) would be roughly the same. Even if it varied, the logarithm of \( t_2 \) would only vary by a small amount. Therefore, the PSD would appear to be stationary and well behaved.

The key to the above argument is the assumption that at the beginning of the experiment, the system is at rest. Without this assumption, the process could have begun at any time in the past and there would be no way to determine whether the value of \( t_2 \) was minutes, hours, decades or centuries. Nevertheless, even observing the process without knowing when it began would still produce a constant shape PSD, independent of the value
of $t_2$. The first term of the autocorrelation function for small ($\tau$) depends only on $t_2$ and its Fourier transform with respect to ($\tau$) is an impulse at the origin whose magnitude depends on $t_2$. The second term does depend on ($\tau$), but not on $t_2$. Therefore, it gives rise to a PSD that must be independent of $t_2$. Combining both terms results in a PSD whose magnitude and shape are independent of $t_2$ except at low frequencies for which ($\tau$) would no longer be much smaller than $t_2$.

These results can be generalized to $1/f$-type noises with PSD's of the form:

$$S(f) = \frac{1}{|f|^{2-\alpha}} \quad 0 < \alpha < 2$$

Those processes for which $1 < \alpha < 2$ do not violate the conditions of the Weiner-Khinchine Theory at low frequencies and when truncated at high frequencies are stationary with an autocorrelation function of:

$$R(\tau) \rightarrow |\tau|^{1-\alpha}$$

For $0 < \alpha < 1$, the processes do not meet the conditions of the Weiner-Khinchine Theory and are not stationary. To derive a non-stationary autocorrelation function, postulate a generalized RC line with a frequency dependent input impedance of:

$$Z(f) = \frac{1}{f^{1-\alpha/2}} \quad 0 < \alpha < 1$$

Driving it with a white noise current source would result in a voltage whose PSD would be proportional to:

$$S(f) = \frac{1}{|f|^{2-\alpha}} \quad 0 < \alpha < 1$$

The impulse response of this RC line would be (Erdelyi, page 137):
\[ z(t) = \frac{1}{t^{\alpha/2}} \mu_{-1}(t) \]

Therefore, the autocorrelation function would be:

\[ R(t_2, t_1) = \int_{0}^{t_1} \frac{1}{(t_2 - t_0)^{\alpha/2}} \frac{1}{(t_1 - t_0)^{\alpha/2}} \, dt_0 \]

Substituting \( \mu = t_2 - t_0 \), this becomes:

\[ R(t_2, \tau) = \int_{\tau}^{t_2} \frac{1}{\mu^{\alpha/2}} \frac{1}{(\mu - \tau)^{\alpha/2}} \, d\mu \]

This is equivalent to the following:

\[ R(t_2, \tau) = \int_{\tau}^{\infty} \frac{1}{\mu^{\alpha/2}} \frac{1}{(\mu - \tau)^{\alpha/2}} \, d\mu - \int_{t_2}^{\infty} \frac{1}{\mu^{\alpha/2}} \frac{1}{(\mu - \tau)^{\alpha/2}} \, d\mu \]

The first term can be evaluated with the help of an integral table (Gradshteyn and Ryzhik, p.284, equation 3.191-2). The second term is easily evaluated after again assuming that the maximum value of \( \tau \), which would be equal to the time of observation is much smaller than \( t_2 \). Therefore, \( (\mu - \tau) \) can be replaced by \( (\mu) \). The final result is:

\[ R(t_2, \tau) = \frac{t_2^{1-\alpha}}{1 - \alpha} + \frac{\Gamma(\alpha - 1)\Gamma(1 - \frac{\alpha}{2})}{\Gamma\left(\frac{\alpha}{2}\right)} \tau^{1-\alpha} \]

Noting that \( \Gamma(\alpha - 1) < 0 \) for \( 0 < \alpha < 1 \), we have:

\[ R(t_2, \tau) = \frac{t_2^{1-\alpha}}{1 - \alpha} - \text{(constant)} \tau^{1-\alpha} \]

The results for 1/f-type noises are similar in form to the autocorrelation functions for 1/f noise. For observation times that are much shorter than the total time since the process began, we obtain two terms. One is a function only of \( t_2 \) and the other is a function only of \( \tau \). Again, the
shape of the PSD will be unaffected by the value of $t_2$, except for its apparent steady or D.C. value.
CHAPTER IV

ALMOST STATIONARY AUTOCORRELATION FUNCTIONS FOR 1/f-TYPE PROCESSES

In the previous chapter, we derived the non-stationary autocorrelation functions for 1/f-type noise. These functions were then approximated for the case where the time of observation (T) was much shorter than the total time (t_2) since the process began. The result was that each approximate autocorrelation function could be written as the sum of two terms, one dependent only on t_2 and one dependent only on \( \tau \).

\[ R(t_2, \tau) \sim f(t_2) + f(\tau) \]

In effect, the non-stationary autocorrelation functions are approximately stationary. The non-stationary behavior is entirely contained in the added term that depends only on t_2. By taking the real part of the Fourier transform with respect to the variable, \( \tau \), we will compute an approximately stationary PSD and evaluate the effect of the added term. Since there is no point in considering correlations over times larger than the total observation time, T, we will limit the value of \( \tau \) to the range:

\[ 0 < |\tau| \leq T \]

For 1/f noise exactly, by dividing the arguments of both terms of T, we have:

\[ R(t_2, \tau) = \frac{2}{N} \frac{R}{\pi C} \left( \ln \frac{4t_2}{T} - \ln \frac{|\tau|}{T} \right) \quad 0 < |\tau| \leq T \]

\[ = \frac{2}{N} \frac{R}{\pi C} \ln \frac{4t_2}{T} \quad T < |\tau| \]

Using transform pair (1) from Erdelyi (p.17) and the scaling properties of Fourier transforms, we obtain for the second term the PSD given below:
\[ S(f) = I_n^2 \frac{R}{\pi} \frac{1}{2\pi f C} \int_0^{2\pi f T} \frac{\sin \mu}{\mu} \, d\mu \]

The transform of the first term with respect to \( \tau \) is just the transform of a constant.

\[ S(f) = \left( I_n^2 \frac{R}{\pi C} \ln \frac{4t_2}{T} \right) \nu_0(f) \]

The overall result is:

\[ S(f) = I_n^2 \frac{R}{\pi C} \left( \ln \frac{4t_2}{T} \nu_0(f) + \frac{1}{4|f|} \right) |f| \gg \frac{1}{T} \]

\[ = I_n^2 \frac{R}{\pi C} \left( \ln \frac{4t_2}{T} \nu_0(f) + T \right) |f| \ll \frac{1}{T} \]

This PSD, illustrated in Figure 4a, is proportional to \( 1/f \) down to the lowest frequency allowed by the limited observation time. Both the exponent and the magnitude of the PSD for frequencies above this lowest frequency are independent of the value of both \( t_2 \) and \( T \) and are therefore stationary.

The apparent steady value of the process over the interval, \( T \), would be the integral of the PSD from \( f = -\frac{1}{T} \) to \( f \sim \frac{1}{T} \) and would equal:

\[ \frac{1}{T} \int \frac{1}{T} S(f) \, df = I_n^2 \frac{R}{\pi C} \left( \ln \frac{4t_2}{T} + 2 \right) \]

Thus, we conclude that the values of \( t_2 \) and \( T \) only affect the apparent steady value of the process.
Figure 4a

The above derivation shows that even in observing a 1/f process without knowing when it started (the value of $t_2$), an experimenter would conclude that over the frequency range allowed by the limited time of observation ($f \gg \frac{1}{T}$), the PSD was stationary. However, the apparent steady or D.C. value of the process, unless it was dominated by an independent D.C. term, would depend logarithmically on the value of $t_2$. It is common practice in the literature to normalize the PSD by dividing by the square of the steady value of the process. For the cases where an independent D.C. term is not dominant, the magnitude of the PSD would depend logarithmically on $t_2$. This is consistent with the experimental observations that the value of the exponent varies slightly and that usually the value of the magnitude varies enormously.

For 1/f-type processes with $1 < \alpha < 2$, again the autocorrelation functions are stationary. For $0 < \alpha < 1$, we have:
\[ R(t_2, \tau) \approx t_2^{1-\alpha} - |\tau|^{1-\alpha} \quad 0 < |\tau| < T \]
\[ - t_2^{1-\alpha} \quad T < |\tau| \]

Using the transform pair (3) from Erdelyi (p.137) and evaluating the result, we obtain:

\[ S(f) \approx t_2^{1-\alpha} \nu_0(f) + \left| \frac{1}{2\pi f} \right|^{2-\alpha} \Gamma(2-\alpha) \quad |f| >> \frac{1}{T} \]
\[ - t_2^{1-\alpha} \nu_0(f) + \frac{T^{2-\alpha}}{2-\alpha} \quad |f| << \frac{1}{T} \]

These PSD's are illustrated in Figure 4b.

---

Figure 4b

Again, above the lowest frequency allowed by the observation time, the PSD is independent of both \( t_2 \) and \( T \) and is, therefore, stationary. The apparent steady value depends on both \( t_2 \) and \( T \) and would be:

\[ \text{D.C. value} \approx \int_{-\frac{1}{T}}^{\frac{1}{T}} S(f) df = t_2^{1-\alpha} + \frac{2T^{1-\alpha}}{2 - \alpha} \]
CHAPTER V

THE MEMORY OF 1/f-TYPE PROCESSES

White noise has no memory of the past, that is, current values of the process are independent of past values. But, 1/f-type processes do have memory. The question is how much and for how long are these processes influenced by their past. In this chapter, we will explore these questions by first examining the autocorrelation functions from Chapter III to illustrate how long the process remembers and then by constructing two linear systems that yield 1/f noise exactly to illustrate how many initial conditions the process remembers.

The approximate autocorrelation functions derived in Chapter III are listed below and illustrated in Figure 5a.

\[ R(t_2, \tau) = t_2^{1-\alpha} \tau^{1-\alpha} \]

\[ \ln 4t_2 - \ln \tau \]

\[ t^{1-\alpha} \]

\( 0 < \alpha < 1 \)
\( \alpha \equiv 1 \)
\( 1 < \alpha < 2 \)

Figure 5a
They all decay as a power of time, slower than any exponential, except for
the case of (α) exactly equal to 1, for which the decay is logarithmic and
slower than any power of time. Thus, we can see that 1/f is a process
with a very long memory. The closer (α) is to 1, the greater the influence
of the distant past when compared with the influence of the recent past.
The autocorrelation function is quite flat, so that present events are
approximately equally correlated with events from the recent past and with
those from the very distant past.

Since it is clear from the autocorrelation functions that 1/f-type
process are strongly influenced by the past, especially for (α) nearly 1,
the next question is how many initial conditions does the process remember.
If the process is modelled with a linear system, the question becomes how
many state variables or how many poles are required. Two linear systems
will be considered. First will be the RC line driven by a white noise
source, the same model that was used in Chapter III. It yields 1/f noise
exactly, and as we shall see, its analysis will place an upper limit on
the required number of state variables. The analysis of a second system,
which will be described later, will place a lower limit on the required
number of state variables and will also be used to illustrate the similari-
ty among 1/f-type processes with slightly different values of the para-
meter (α).

The continuous RC line, used in Chapter III to construct a simple
linear system that produces 1/f noise exactly can be analyzed by consider-
ing it to be the limit of a lumped, finite length RC line and allowing the
length to approach infinity. Lumped lines have a high-frequency limit a-
bove which the PSD will no longer be 1/f. A finite length line will have
a low-frequency limit below which its PSD will not be proportional to \(1/f\). Allowing the length to approach infinity causes the low-frequency limit to approach zero.

The finite-length, lumped RC line has been analyzed in detail (Weber). With \(N\) sections, containing \(N\) capacitors, the input impedance will have exactly \(N\) poles and \(N\) zeros, all lying on the negative real axis of the complex plane, as given below:

Poles at: \(S = -\frac{1}{RC} \left( 1 - \cos \frac{(n - \frac{1}{2})\pi}{N} \right)\)

Zeros at: \(S = -\frac{1}{RC} \left( 1 - \cos \frac{n\pi}{N} \right)\)

where \(n = 1, 2, 3, 4, \ldots\)

The \(1/f\) frequency response of the input impedance and thus the \(1/f\) PSD, results from the alternating pattern of poles and zeros.

To approximate the \(1/f\) PSD of a continuous and infinite RC line over a frequency range of:

\[ f < |f| < f_h \]

would require a finite length, lumped line whose lumping was fine enough to accommodate frequencies as high as \(f_h\) and whose length was long enough to give a \(1/f\) PSD down to frequencies as low as \(f_\lambda\). Thus, the length of the continuous line \(l_h\) corresponding to each lumped section must be approximately:

\[ l_h \sim \left( \frac{1}{2\pi f_{h,RC}} \right)^{1/2} \]

The total length must be:

\[ l_T \sim \left( \frac{1}{2\pi f_{\lambda,RC}} \right)^{1/2} \]
Therefore, the total number of sections and hence the total number of poles (N) is just:

\[ N = \left( \frac{f_h}{f_L} \right)^{1/2} \]

The RC line model gives an approximate fit to an exact 1/f PSD that has equal magnitude of deviation independent of frequency. Thus, the fit is rather poor at high frequencies where the magnitude of the PSD is low and the error relatively large, but is almost perfect for middle frequencies. A smaller number of poles can be used if we are willing to accept a fit that has a uniform percentage deviation, i.e., whose deviation is proportional to frequency.

A simple linear system that can be constructed to have a PSD with a constant percentage error for any of the 1/f-type PSD's is illustrated in Figure 5b. It bears little relation to the previous model except that it is also a linear system driven by a white noise source and that the PSD of some variable, \( V_o(f) \), will be 1/f. It consists of resistor-capacitor sections separated and isolated by unity-gain buffer amplifiers.

![Figure 5b](image)

Each section contributes one pole and one zero to the overall response.

Of course, the required number of poles and zeros depends on the exactness of the fit. An exact fit would require an infinite number.
However, it is interesting to determine how many are required for say a ± 3db fit, and then to use that value to discuss the number of state-variables and, hence, the number of initial conditions needed by the process.

A construction based on the foregoing concept is presented in Figure 5c. The continuous lines are the PSD for various 1/f-type noises plotted on a log-log scale. Superimposed on each of these lines are dotted lines representing the Bode plots of an approximating linear system. The positions of the poles and zeros were chosen for a maximum error of 3db.

The results may be somewhat surprising. Approximately 1 pole (that is 1 state variable) per decade of frequency is required to fit a 1/f PSD. (This result was obtained in a different context by J. K. Roberge (p.604)).

\[
\frac{1}{2} \quad \frac{3}{2}
\]

Slightly fewer are required for 1/f\(^{1/2}\) and 1/f\(^{3/2}\), with the value decreasing towards zero as the PSD approaches either white noise or brownian motion (1/f\(^2\)). Again, the number of poles and zeros depends critically on how closely the linear system must fit the 1/f-type PSD. However, the point is that the number of poles is fairly small for a fit corresponding to the usual quality of experimental data and that the distribution of poles is exactly uniform as a function of frequency, regardless of the exact exponent of the 1/f type process.

For a 1/f PSD extending from 10\(^{-50}\) Hz to 10\(^{50}\) Hz, only a hundred poles are required for a better than 3db fit. For a 0.5 db fit, the number is probably less than 500. This means that the system would have only hundreds of state variables and could remember only hundreds of initial conditions. Some of the initial conditions would be associated with capacitors whose sections produced high-frequency poles and, therefore, the effect of these initial conditions would decay quickly. Other initial conditions would be
associated with very low-frequency poles; their effect would persist for a very long time.

If we demand a perfect fit to a 1/f-type PSD, but only observe the process for a finite time, the number of state variables whose initial conditions must be treated individually is not infinite. The effect of a finite observation time (T) is to smooth the frequency response by averaging over a bandwidth ($f_0$) of approximately the reciprocal of that time.

$$f_0 = \frac{1}{2\pi T}$$

Therefore, at most, we could have a pole and zero at every multiple of ($f_0$). Thus, the maximum number of distinguishable state variables is the ratio of the high-frequency limit to the reciprocal of the observation time.

$$\text{Maximum} = \frac{f_h}{f_0}$$

For purposes of comparison with the minimum number, choose the low-frequency limit to coincide with the lowest frequency allowed by the observation time. (The effect of all state variables whose time constants are much longer than the observation time can be grouped into one initial condition: the apparent D.C. value.) The result is a range for the number of state variables (N) given by:

$$\log_{10} \left( \frac{f_h}{f_0} \right) < N < \frac{f_h}{f_0}$$

The construction of Figure 5c supports the view that all 1/f-type noises are fundamentally similar. They all have a uniform distribution of poles per decade of frequency and over the range:
\[ \frac{1}{2} < \alpha < \frac{3}{2} \]

have approximately the same number of poles per decade.

Combining the result for the minimum number of state-variables required with the characterization of the autocorrelation functions developed earlier, we arrive at the following description of the memory of \(1/f\)-type noise for \(\alpha\) very near 1. The correlation of present values with those from a time long ago \(\tau\) would result from the voltage on all capacitors whose discharge rate was too slow to allow an appreciable change in the time, \(\tau\). All capacitors whose poles were at low enough frequencies but above the lowest frequency, \(f_0\), would be included.

\[ f_0 < f < \frac{1}{2\pi \tau} \]

the magnitude of the correlation is just proportional to total number of capacitors. Hence,

\[ R(\tau) \sim \log_{10} \left( \frac{1}{2\pi \tau} \div f_0 \right) \sim \text{constant} - \log \tau \]

Of course, this is the same result as before.

A physical system that was started with particular initial conditions, would gradually over time cease to be influenced by the starting values of its state variables. The variable corresponding to a pole at 1 Hz would cease to reflect its initial state after a few seconds. The variable corresponding to 0.1 Hz, would cease to reflect its initial state after tens of seconds, and another after hundreds of seconds and so on.
For the case of an informational system, for which the inputs have meaning and are not considered to be noise, but whose statistics, nevertheless, have a $1/f$ PSD, we have the converse description. The system constantly adds new data to the old. The new data has a strong influence on current events, but its influence gradually diminishes as it becomes averaged with other data. If it was ten seconds old, it might be averaged with other data from 9 to 11 seconds. The average would be the value of the state variable with a time constant of ten seconds, and would represent a trend in the data over a few seconds. The same piece of data also would be averaged with the data of the last 100 seconds to arrive at the value of the 100-second state variable which would represent a trend in the data over many tens of seconds. Hence, a $1/f$ process can be though of as viewing its history in terms of averages or trends. Its current behavior is influenced equally by those trends whose duration has been 0.1 seconds, 1, 10, 100 or 1000 seconds, and so on.

The notions presented above are descriptive of an evolutionary process, in which new data is constantly being added to the old. For the case of a constant rate of new information, the process either increases its store of information linearly, requiring the maximum number of state variables or logarithmically, requiring the minimum. In the first case, no information is ever lost, although the influence of a single state variable diminishes logarithmically with time. In the second case, the information is summarized so that only averages or trends (on all time scales) remain.
CHAPTER VI

RENEWAL PROCESS MODELS OF 1/f NOISE

This chapter begins a discussion of specific models for physical systems that exhibit 1/f-type noise. After a brief discussion comparing the relative convenience of using either renewal process models or the Langevin Equation for describing fluctuations in physical systems, we proceed to develop mathematical techniques for computing the PSD's of renewal processes. These techniques are then used to discuss two families of renewal processes. Choosing the interarrival times of the first to be stably distributed, with infinite variance, results in the entire group of 1/f-type processes. The Normal distribution is stable, but with finite variance. In the limit of large numbers, sums of independent random variables with finite variance (and other necessary conditions) will converge to a Normal random variable. The same is true for all stable distributions, and each has its own domain of attraction from among random variables whose variance is infinite.

The second family requires an interval distribution that is proportional to $t^{-4}$ for long times in order to obtain 1/f noise. The $t^{-4}$ distribution is not stable and has a finite mean value and variance. In fact, in the limit of large numbers, sums of independent, identically distributed random variables, each of which had a $t^{-4}$ long-time tail, would converge to a Normal distribution. But, as is shown in the Appendix, the Normal distribution in this second family of renewal processes results in white noise. The third moment of a $t^{-4}$ random variable is infinite, but all moments of a Normal random variable are finite. The Normal provides a good fit, in the limit of large numbers, only to the first two moments,
not to the higher moments. The limit is insensitive to the tail behavior. Hence, relying on the law of large numbers to compute the distributions for random variables, in this case and probably for many others, would neglect the tail behavior and eliminate $1/f$ as a possible result.

Fluctuations or noise in physical systems is often modelled using the Langevin Equation in which the dynamical equations of the system are driven by fluctuating source terms. In thermodynamic equilibrium, each source term represents the behavior of a dissipative subsystem in accordance with the Fluctuation-Dissipation Theorem. For convenience, the sub-systems are usually chosen such that their fluctuations are correlated over much shorter distances and times than the correlations introduced by the overall system. Therefore, the source terms are taken to be both spatially and temporally uncorrelated. The dynamics act as a passive filter that attenuates the noise at some frequencies to reshape the PSD. The overall effect is system noise that is not, in general, white.

The semi-infinite RC line, driven by a white noise current source, is an excellent example of the form of the Langevin approach. In that case the resulting input voltage is $1/f$ noise. However, it is not an equilibrium model because the noise source is not associated with a dissipative sub-system. (Placing an appropriately valued resistor in parallel with the noise source would satisfy the Fluctuation-Dissipation Theorem but would also cause the PSD of the input voltage to be white.) In chapters VII – IX, the Langevin Equation will be used to model temperature fluctuations in a heat bath governed by the diffusion equation.

The Langevin approach ceases to be convenient when the dynamics of the system are exceedingly complex and difficult to model with a group of
equations. The process by which electrons in a solid collide with lattice atoms and scatter provides an excellent example. In applying the Langevin Equations, one would first write the Hamiltonian description of the electron movement given the exact positions and momenta of every lattice atom, and then drive the equations by allowing the positions of the lattice atoms to fluctuate. A more convenient approach would be to adopt a probabilistic description of the entire process that characterizes the behavior of the electrons.

A convenient description of the scattering process, and of other random processes, that lends itself easily to the calculation of a PSD, is the general form called a renewal process. A renewal process consists of events arriving in time, where both the events and the interarrival times are random variables. The interarrival times must be independent, identically distributed random variables, and the structure of the event may be correlated only with the interarrival time. Beginning with any event, the description of the process is the same, i.e., it starts over again or is renewed. (See, for example, Feller.)

Unlike the RC line model, the processes that will be considered in this chapter do not have interarrival times with a Poisson distribution and, for the most part, will have simply structured events such as step functions or ramps. With a simple event like a step function, we must rely on the distribution of intervals to achieve a $1/f$ PSD. From the other models, we know that the process has an excess of energy at low frequencies and requires very long time constants. Therefore, the distribution of intervals must favor long intervals over short. In fact, as we shall see, the number of long intervals is so great, that for some $1/f$ renewal
processes, the expected value of the length of an interval is infinite.

Since the 1/f-type PSD may extend to f=0, the Wiener-Khinchine Theory may not apply (see Introduction) and other techniques must be used to compute the PSD from the time functions. Mandelbrot (1967) and Tunaley (1972a) have proposed alternate techniques. Both obtain PSD's that depend conditionally on the time of observation and both use ensemble rather than time averages to compute the PSD's. Mandelbrot's technique, though correct, is extremely difficult to apply to any physical situation. Tunaley proceeds by computing a sample spectrum:

\[ S_T(f) \triangleq \frac{1}{T} \left| \int_0^T x(t) e^{-j2\pi ft} \, dt \right|^2 \]

and then taking its expected value.

\[ S(f|T) \triangleq E\left( S_T(f) \right) \]

Of course, this procedure depends on knowing the underlying probability distributions.

In general, S(f|T) is strongly dependent on (T), the time of observation. Unfortunately, as we shall see, for some renewal process generating a (1/f)**(2-a) PSD, observing for a finite time, (T), one is likely to witness very few events. The interarrival times may have infinite variances and, for (a) less than or equal to 1, infinite mean values. The expected waiting time for the next event would be infinite and the expected number of events in any finite interval of time would be zero. Thus, the expected value of the sample spectrum will not be ergodic.
An alternate technique has been proposed by W. M. Siebert (unpublished) that is especially applicable to renewal processes. Rather than observe for a fixed time, observe until a fixed number of events \( N \) have occurred. Again, compute a sample spectrum:

\[
S_N(f) \triangleq \frac{1}{N} \left| \int_0^{T_N} x(t) e^{-j2\pi ft} \, dt \right|^2
\]

then take the expected value

\[
S(f \mid N) \triangleq E \left( S_N(f) \right)
\]

This technique constitutes an operational definition of a PSD, in which the number of events \( N \) replaces the length of the interval \( T \) as the normalization factor. This change gives a PSD that converges uniformly to a well-defined function as \( N \) approaches infinity. The normal definition of a PSD does not give convergent behavior as \( T \) approaches infinity because for \( 1/f \)-type noises, the statistics of \( T \) are erratic. Siebert and I both believe that this operational definition also corresponds with the experimental procedure used to observe \( 1/f \) noise. Experimenters observe the process for a 'reasonable' length of time \( T \) and then compute the usual PSD, normalizing by \( T \). 'Reasonable' means long enough to observe a significant number of events. The observation is repeated many times and a number of PSD's are computed, each corresponding to only one observation. The result is a group of \( 1/f \)-type PSD's with different values of the exponent (2-\( a \)) and of the magnitude. The exponent is always found to lie close to a particular value (usually within 5%), but the magnitude is excessively variable. To convey some notion of a typical value for
the magnitude, the experimenter may choose to consider very large and very small values atypical and ignore them, or he may attempt to average the magnitudes at a particular frequency. With Siebert's technique, the experimental value of the exponent is given by $S_N(f)$; the value of the magnitude is $(N/T)S_N(f)$. The exponent will be well behaved but the magnitude will not.

One example, which yields an $1/f$ PSD exactly, is the renewal process illustrated in Figure 6a.

\[
\begin{align*}
E[a_i] &= 0 \\
E[a_ia_j] &= 0 \quad i \neq j \\
&= 1 \quad i = j
\end{align*}
\]

\[
pdf[T_{i+1} - T_i] \triangleq pdf(\Delta T) = \frac{2}{\pi} \frac{1}{1 + (\Delta T)^2} \quad \Delta T \geq 0
\]

**Figure 6a**

The interval between events is distributed according to a one-sided Cauchy pdf, and the events are uncorrelated with each other. The sample spectrum can be computed directly. However, it is simpler to recognize in advance that the expected value of the cross products will be zero, and leave them out.
\[ S(f|N) = E \left[ \sum_{i=1}^{N} \frac{2 - e^{-j2\pi f \Delta T} - e^{+j2\pi f \Delta T}}{N(2\pi f)^2} \right] \]

\[ = \frac{2 - \phi(f) - \phi(-f)}{(2\pi f)^2} \]

where:
\[ \phi(f) = E \left[ e^{-j2\pi f \Delta T} \right] \]

Please note that for the probability density function of the inter-arrival times I have been using a one-sided Cauchy pdf, whose characteristic function \( \phi(f) \) is not the same as that of a symmetrical Cauchy pdf. However, a one-sided pdf can always be created from a symmetrical pdf by adding a function that is both real and odd valued. The transform of an odd, real function is odd and will vanish in the expression for \( S(f|N) \). Therefore, in this discussion and in those that follow, I will use characteristic functions corresponding to symmetrical pdf's. The characteristic function for a symmetrical Cauchy pdf is:

\[ \phi(f) = e^{-|2\pi f|} \]

Thus:
\[ \lim_{f \to 0} S(f|N) = \frac{2}{|2\pi f|} \]

The same derivation applies with any other stable pdf substituted for the Cauchy pdf. Stable distributions other than the Cauchy have characteristic functions of the form: (Gnedenko and Kolmgorov)

\[ \ln \phi(f) = j\gamma f - c|f|^\alpha(1 + j\beta \frac{|f|}{f} \tan \frac{\pi}{2} \alpha) \]

where:
\[ -1 \leq \beta \leq 1 \]
\[ 0 < \alpha \leq 2 \]
\( \gamma \) is real
\( c > 0 \)
\( \alpha \neq 1 \)
For these pdf's:

\[ \lim_{f \to 0} S(f|N) = \frac{2a}{|2\pi f|^2 - a} \quad 0 < a \leq 2 \]

With the Cauchy pdf \((a=1)\), we obtain a PSD of \(1/f\). The Normal pdf is stable with \(a=2\) and give white noise. As \(a\) approaches zero, we obtain the PSD of a Wiener Process or Brownian Motion. In between are all the \(1/f\)-type noises.

These models correspond rather closely with the behavior of a molecule undergoing diffusion. The steps of uncorrelated amplitude, represent the projection of the molecule's velocity vector in the overall direction of diffusion, and the interarrival times correspond to the times between scattering. Therefore, in this example, to obtain \(1/f\)-type noise, the scattering times must be stably distributed.

Another family of renewal processes is illustrated below. Here the events are ramps, beginning with zero amplitude at each arrival and growing linearly with time until the next arrival. Since the events of these processes are roughly the integral of the events from the previous case, one might expect that the PSD's for the new family would be \(1/f^2\) times the old. More interesting is the possibility that the long-time behavior of the interarrival distributions required to obtain \(1/f\)-type noise would be different.
Since the amplitudes of the events in these processes are not uncorrelated random variables, one must consider all the cross products when computing the PSD. The PSD has been derived in terms of the characteristic function of the interarrival pdf and is presented in the appendix. Also derived in the appendix is the resulting PSD for a Normal distribution of arrival times and for a distribution that for large times is proportional to:

$$\lim_{t \to \infty} \text{pdf}(t) \propto t^{-4}$$

The Normal distribution yields white noise, and the \( t^{-4} \) distribution yields 1/f noise.

This family of renewal processes and, in particular, the result that a \( t^{-4} \) long-time tail yields a 1/f PSD, can be used to illustrate a few of the critical issues in the modelling of 1/f noise. In contrast to the Cauchy pdf with infinite mean and variance, both the mean value and variance of a pdf with a \( t^{-4} \) tail are finite. Therefore, the conventional definition of a PSD could have been used. As was the case with the RC line driven by white noise, both the shape and magnitude of the PSD
will show little variation from sample to sample provided that one does not normalize by the apparent D.C. value.

Unlike the Cauchy pdf, a pdf with a \((t^{-4})\) tail is not a stable distribution. Because the mean and variance are finite, the sum of many independent processes, in the limit of large numbers, would converge to a Normal pdf. But using a Normal pdf for the interarrival times yields only white noise. Only the first two central moments of the \((t^{-4})\) pdf are finite; the skewness and higher moments are all infinite and account for the \(1/f\) PSD. However, the central moments of the Normal pdf are all finite.

Computing separately the PSD of many independent renewal processes of this family with an interarrival pdf that has a \((t^{-4})\) long-time tail and then adding together the results, yields a total PSD that is \(1/f\). However, first adding together the pdf's and then taking the limit for large numbers of independent processes, yields a PSD that is white. The normal distribution is the correct limiting distribution for the sum only in that it minimizes the mean and the mean squared errors, i.e., it provides a good first and second moment fit. However, it does not necessarily provide a good fit to the higher moments which are highly dependent on the tails of the pdf and are responsible for the \(1/f\) noise.

In the limit of large numbers, the Normal distribution provides a good fit to the 'bulk' of the pdf resulting from a sum of pdf's with \((t^{-4})\) long-time tails. But, the 'bulk' is influenced mostly by the values of the pdf's at short times and not by the tails. To preserve the \(1/f\) character of the noise, one must be careful not to incorrectly alter the tail dependence by appealing to the law of large numbers and deducing that the pdf must be
Normal, as is common practice in physics.

The occurrence in physical systems of interarrival time tails proportional to \( t^{-2} \) or \( t^{-4} \) is not as unlikely as one may think. For example, consider the scattering of particles from impurities uniformly distributed in a solid. If we assume a Maxwell-Boltzmann velocity distribution for the particles, we have:

\[
pdf(v) = \left( \frac{2}{\pi} \right)^{1/2} \frac{a^{3/2} v^2}{2} e^{-\frac{a^2 v^2}{2}}
\]  

(velocity)

\[
pdf(l) = \frac{1}{l_0} e^{-l/l_0}
\]  

(path length)

Since \( t = \frac{v}{l} \), we obtain

\[
pdf(t|l) = \left( \frac{2}{\pi} \right)^{1/2} \frac{a^{3/2}}{t^4} e^{-\frac{a^2 t^2}{2t^2}}
\]

limit \( t \to \infty \) \[
\lim_{t \to \infty} pdf(t) \sim \left( \frac{2}{\pi} \right)^{1/2} \frac{6(a l_0)^3}{t^4}
\]

The result is a \( t^{-4} \) long-time tail, yielding a \( 1/f \) PSD.

Stable distributions such as the Cauchy whose long-time tail is \( t^{-2} \) are probably even more common. For example, the distance travelled in the y-direction before a two dimensional random walk crosses the line x = a has a Cauchy distribution (Feller, p.175). The time it takes for a 1-dimensional random walk to return to the origin has a distribution that is stable with parameter \( \alpha = \frac{1}{2} \), and would yield a \( 1/f^{3/2} \) PSD in the family of renewal processes with step-function events. Furthermore, sums of identical, independent, stably-distributed random variables are stable with
the same parameter (α), and products of stable random variables are often stable with a different parameter (α) (Feller, p. 176).

The renewal process models exhibited in this section closely correspond to the behavior of particles in a scattering process. The first family, those with step-function events, would apply to the case where no external force acts on the particles, while the superposition of the first and the second families would describe the behavior of electrons with a small applied field, i.e., ohmic conduction.

Unfortunately, in a solid, for sufficiently long-interarrival times the distribution must truncate. The maximum time allowed would be the time it takes for a particle to hit the walls. If a small accelerating field is present, this time is quite short. For example, in a 1 cm length of metal with an applied field of 1 volt, the transit time would be approximately 1 μsec. Therefore, the longest time between scattering intervals would be 1 μsec and the resulting 1/f PSD would have a low-frequency limit of at least 1 MHz.

Thus, no model that focuses on the details of individual electron behavior in the presence of an accelerating field can yield a 1/f PSD extending to frequencies as low as 1 Hz. Therefore, one must consider slow fluctuations in the average behavior of the electrons, either in their average number (e.g. the McWharter Model) or their average mobility. The suggestion by Voss and Clark that temperature fluctuations modulate the conductivity of thin-film, metal resistors fits into the second category and will be explored in detail in subsequent chapters.
DIFFUSION MODELS

The idea that 1/f noise might be the result of diffusion is not new. However, because of the experimental work of Voss and Clarke (1976a and 1976b), diffusion models are of renewed interest. Voss and Clarke have suggested that the diffusion of heat results in temperature fluctuations with a 1/f PSD that modulate the resistivity of conducting materials and are responsible for the 1/f noise observed in metal-film resistors. However, they have been unable to show that temperature fluctuations can have a 1/f PSD.

In this chapter and those that follow, we will carefully model the dynamics of heat transport in a thermal reservoir and compute the PSD of the thermal fluctuations. Prior theoretical treatments of generalized diffusions can be found by Van Vliet and Van der Ziel (1958) and by Van Vliet and Chenette (1965). Their results will be rederived here (case 1) along with new results.

The diffusion models to be considered are general and apply to any diffusing specie, including heat. If one considers the flow of heat to be the movement of particles of heat (phonons), the application is immediate. More generally, entropy flow and inverse temperature

\[ \dot{S} \text{ and } \frac{1}{T} \]

are a legitimate force-flow pair in the Onsager formalism. For small fluctuations around room temperature \( T_0 \), we have:

\[ \dot{S} = \frac{Q}{T_0} \text{ and } V\left(\frac{1}{T}\right) = -\frac{\nabla T}{T_0^2} \]

Therefore, we can consider a flow of heat \( Q \) in response to a gradient
in temperature normalized by a constant \( T_0 \). A more complete discussion of the irreversible flow of heat can be found in texts on thermodynamics such as Reif and on solid-state physics such as Smith, Janak and Adler.

The derivation of diffusion models of temperature fluctuations will be based on the Langevin approach, in which the dynamical equations of the system, in our case the diffusion equation, are driven by fluctuating source terms associated with dissipative sub-systems in accordance with the Fluctuation-Dissipation Theorem. For convenience, the dynamics of the sub-systems are assumed to be substantially faster than those of the overall system and the correlations between sub-systems are assumed to be insignificant when compared with the correlations introduced by the dynamical equations of the overall system. Therefore, the source terms driving the dynamical equation are taken to be uncorrelated in both space and time, i.e., spatially and temporally white noise.

Since the models that follow are based on the Fluctuation-Dissipation Theorem, which can be applied properly only to systems in thermodynamic equilibrium, we must discuss whether \( 1/f \) noise can be present in a system that is in equilibrium. In terms of linear system theory, the dynamics of a system act as a passive filter that attenuates some frequencies and, thereby, alters the shape of the PSD from that of white noise. Provided that the effective filtering operation is causal and passive, the behavior of the system is consistent with the laws of thermodynamics. However, one of the requirements for equilibrium to hold is that the behavior of the system must be stationary in time. Of course, to determine whether the behavior is stationary, the system must be observed for a time much longer than its longest relaxation time, much longer than the time required for
the slowest process to reach steady-state.

As demonstrated in the previous chapters, a system that exhibits $1/f$ noise over a frequency range that includes zero frequency has no longest relaxation time. The longest time is infinite. Furthermore, even if a low-frequency limit exists, unless the observation time is sufficiently long to allow an observation of the PSD becoming flat, it is not longer than the longest relaxation time. Hence, in order to be certain that a system exhibiting $1/f$ noise is in equilibrium, one must have observed the PSD to become white at low frequencies. But since the low-frequency limits have almost never been observed, there can be no assurance that the system is in equilibrium.

For the discussion of diffusion models, thermal equilibrium will be assumed only for the small dissipative sub-systems and not for the system as a whole. The sub-systems will be assumed to be in equilibrium with the thermal environment and to be unaffected by the state of the entire system and, in particular, those processes that are too slow to have reached steady state. These assumptions will allow the proper application of the Fluctuation-Dissipation Theorem without requiring that the system as a whole be in thermodynamic equilibrium.

In the following discussions, extensive use will be made of resistor-capacitor transmission lines (RC lines) as exact analogues for the diffusion equation in 1, 2 or 3-dimensions. The correspondence between electrical and thermal variables is given below:

\[
\begin{align*}
I & \leftrightarrow Q \\
V & \leftrightarrow T/T_0 \\
G & \leftrightarrow T_0 \cdot \text{(Thermal Conductance)}
\end{align*}
\]
C \leftrightarrow T_0 \cdot (\text{Thermal Capacitance})

The fluctuations of the small sub-systems will be represented as current sources whose PSD is given below:

\[ |I_N(f)|^2 = 4kT_0 G = 4kT_0^2 \] (Thermal Conductance)

The experimental situation that will be modelled consists of a region of conducting material placed on a thermally conductive but electrically insulating substrate. The substrate will be modelled as a homogeneous thermal bath whose temperature fluctuations cause the temperature of the resistor to fluctuate. In turn, the value of the resistance fluctuates in proportion to the temperature coefficient of the material. The placement of the resistor in the bath is assumed not to affect the bath in any way. The bath will be modelled as first 1-dimensional and then 2 and 3-dimensional with distributed sources of thermal noise.

Two groups of models will be considered, one in which the conductive area is small compared to a thermal diffusion length in all its dimensions and a second in which the length of the conductive area is long but the width and thickness are both short compared to a diffusion length.

Focusing on the thermal bath and appealing to the RC line analogues we can see that for a given dimensionality only two configurations of distributed sources are possible. In case 1, the sources are placed across the resistors and in case 2, they are placed across the capacitors. These are illustrated with a lumped 1-dimensional RC line in Figures 7a and 7b below. Because the noise sources must be associated with dissipative sub-systems, resistors must be placed across the capacitors. However, the resistors will be assumed to be large enough to be neglected when compared
with the impedance of the capacitors over the frequency range of interest. This sets a low-frequency limit above which the diffusion equation applies to case 2.

\[ f >> \frac{1}{2\pi R_p C} \equiv f_c \]

Below this limit, the system is entirely resistive and the noise must be white.

Case 1

Figure 7a

Case 2

Figure 7b

In accordance with the Fluctuation-Dissipation Theorem, the amplitude of the white noise current sources is:
\[ I_N(f) = \left( \frac{4kT}{R_Q} \right)^{1/2} \quad \text{Case 1} \]

\[ I_N(f) = \left( \frac{4kT}{R_P} \right)^{1/2} \quad \text{Case 2} \]

For the first group of models, the dimensions of the conductive area are very small compared to a diffusion length. Therefore, it will be modelled as a small region around the origin without any important structure. We need to compute the temperature fluctuations resulting from all the noise sources distributed throughout the infinite bath. (Of course, finite dimensions would impose an additional low-frequency limit on the PSD.)

The PSD of the temperature fluctuations can be derived by computing the PSD resulting from each independent current source and then integrating over all sources to compute the total PSD. This method will be used for Case 2 and also in the next chapter. However, a simpler technique is available. Since the noise of each section of the analogue obeys the Fluctuation-Dissipation Theorem, it also obeys Nyquist's Theorem for electrical systems. Furthermore, systems that obey Nyquist at every point also obey Nyquist at the input terminals. Therefore, the PSD of the voltage fluctuations at the input terminals is proportional to the real part of the frequency dependent input impedance as follows:

\[ |V(f)|^2 = 4kT \text{ Re}[Z(f)] \]

This result is general and applies to non-electrical systems as well. To compute the PSD of the temperature fluctuations at the origin, we need only compute the impedance of the analogue at that point and take its real part. Assuming that the bath is geometrically 1-dimensional, we need to
compute the impedance of a doubly infinite RC line. From transmission line theory it is:

\[ Z(f) = \frac{1}{2} \left( \frac{R_o}{j2\pi f C_o + \frac{1}{R_p}} \right)^{1/2} \]

Note that the effects of \( R_p \) are included explicitly. Therefore, this represents both cases 1 and 2 for 1-dimension, simultaneously. For \( f >> f_\lambda \), the PSD is:

\[ S(f) = kT \left( \frac{R_o}{\pi f C} \right)^{1/2} \]

The noise is proportional to \( f^{-1/2} \) for frequencies above \( (f_\lambda) \) and is white for frequencies below. By including the noise source associated with \( R_p \) but, at the same time violating the Fluctuation-Dissipation Theorem by not including the effects of the resistor, some authors have obtained an \( f^{-3/2} \) PSD. This will be derived later. However, when including \( R_p \), the \( f^{-1/2} \) term would be much larger than the \( f^{-3/2} \) term for frequencies above \( (f_\lambda) \).

Assuming that the bath is geometrically 2-dimensional, we have the situation illustrated in Figure 7c.
Again, the conductive region is located at $r = 0$. The amplitude of the temperature fluctuation at $r = 0$ resulting from a noise source a distance ($r$) away depends only on the magnitude of the distance and not on its angular direction. Consider a thin, annular region, a distance ($r$) from the origin with thickness ($dr$).

![Figure 7d](image)

For the purpose of deriving an analogue, associate with that region a capacitance ($C$) and parallel conductance ($G_p$) of:

\[
C = C_0 \ 2\pi r \ dr \\
G_p = G_{po} \ 2\pi r \ dr
\]

and between adjacent annular regions a resistance of:

\[
R = \frac{R_0}{2\pi r} \ dr \\
R_0 = \text{Resistance/square}
\]

Although the geometry is 2-dimensional, the analogue is functionally 1-dimensional and can be represented by an RC line for which ($r$) is the only spatial parameter. However, unlike the result for a 1-dimensional bath, the $R$'s and $C$'s are not constant, but are functions of ($r$). The effective diffusion coefficient and the low-frequency limit of the model are still independent of ($r$):
\[ D = \frac{1}{RC} (dr)^2 = \frac{1}{R_o C_o} \]

\[ f_\chi = \frac{G_o}{2\pi C_o} \]

Assuming that the bath is geometrically 3-dimensional, again the fluctuations at \( r = 0 \) from a source located a distance \( r \) away depend only on the magnitude of the distance. Therefore, consider a thin-spherical shell and associate with it a capacitance and parallel conductance:

\[ C = C_o \ 4\pi r^2 \ dr \quad C_o = \text{capacitance/units of volume} \]

\[ G_p = G_o \ 4\pi r^2 \ dr \quad G_o = \text{conductance/units of volume} \]

and between adjacent shells, a resistance:

\[ R = \frac{R_o}{4\pi r^2} \ dr \quad R = \text{Resistance - unit length} \]

For 1-dimensional model of the bath, the impedance could be obtained immediately. However, for 2 and 3-dimensions, because the \( R \)'s and \( C \)'s are not constants the derivation is more complicated. The voltage of the analogues, as a function of time, \( v(t) \) at the origin in response to a unit impulse of current at \( t = 0 \) at a distance \( r \) away from the origin is:

\[ v(t) = \frac{1}{C_o} \left( \frac{1}{4\pi Dt} \right) 1/2 e^{-\frac{r^2}{4Dt}} \quad 1\text{-Dimension} \]

\[ v(t) = \frac{1}{C_o} \left( \frac{1}{4\pi Dt} \right) e^{-\frac{r^2}{4Dt}} \quad 2\text{-Dimensions} \]

\[ v(t) = \frac{1}{C_o} \left( \frac{1}{4\pi Dt} \right)^{3/2} e^{-\frac{r^2}{4Dt}} \quad 3\text{-Dimensions} \]

These responses can be verified by direct substitution into the diffusion equation. The voltage at the origin as a function of frequency can be found by taking the transform of the above results (Erdelyi, p.146):
\[ V(f) = \frac{1}{2C_0} \left( \frac{1}{j2\pi Df} \right)^{1/2} e^{-r \left[ \frac{12\pi f}{D} \right]^{1/2}} \]

1-Dimension

\[ V(f) = \frac{1}{2C_0} \left( \frac{1}{\pi D} \right) K_0 \left[ r \left( \frac{12\pi f}{D} \right)^{1/2} \right] \]

2-Dimensions

\[ V(f) = \frac{1}{2C_0} \left( \frac{1}{2\pi Dr} \right) e^{-r \left[ \frac{12\pi f}{D} \right]^{1/2}} \]

3-Dimensions

As we shall see, the impedance of the bath may become infinite for conductive areas of zero size. Therefore, to compute the impedance assume a small but non-zero radius \( r_0 \) much smaller than a diffusion length for frequencies low enough to be of interest.

\[ r_0 \ll \left( \frac{D}{2\pi f} \right)^{1/2} \]

Excite the line with a unit amplitude sinusoidal current source at \( r = 0 \) and compute the resulting voltage at \( r_0 \). The impedance is then:

\[ Z_{r_0}(f) = \frac{V_{r_0}(f)}{I_{r_0}(f)} \]

Computing the current, \( I(f) \), proceeds as follows:

\[ I(f) = \int \frac{1}{R_0} \hat{E} \cdot dS \quad \text{(Surface Integral)} \]

and \[ \hat{E} = -\frac{d}{dr} [V(f)] \]

Thus:

\[ I(f) = -\frac{1}{R_0} \frac{d}{dr} [V(f)] \cdot dS \]

For a 1-dimensional bath, we obtain:

\[ I_{r_0}(f) = e^{-r \left[ \frac{12\pi f}{D} \right]^{1/2}} \]
\[ Z_{r_0}(f) = \frac{1}{2C_0} \left( \frac{1}{j2\pi f} \right)^{1/2} = \frac{1}{2} \left( \frac{R_o}{j2\pi f C_0} \right)^{1/2} \]

This is the same result as previously obtained. For a 2-dimensional bath, we obtain:

\[ I_{r_0}(f) = \frac{1}{\pi R_0 C_0} \left( \frac{12\pi f}{D} \right)^{1/2} K_1 \left[ \frac{r_0 \left( \frac{12\pi f}{D} \right)^{1/2}}{K_0} \left( \frac{12\pi f}{D} \right)^{1/2} \right] \cdot 2\pi r_0 \]

\[ Z_{r_0}(f) = \frac{R_o}{2\pi r_0} \left( \frac{D}{j2\pi f} \right)^{1/2} K_0 \left[ \frac{r_0 \left( \frac{12\pi f}{D} \right)^{1/2}}{K_0} \left( \frac{12\pi f}{D} \right)^{1/2} \right] \]

Taking the limit of \( r_0 \ll \left( \frac{D}{2\pi f} \right)^{1/2} \), yields:

\[ Z_{r_0}(f) \sim \frac{R_o}{2\pi r_0} \left( \frac{D}{j2\pi f} \right)^{1/2} \frac{1}{r_0} \frac{\ln \left( r_0 \frac{2\pi f}{D} \right)}{\frac{1}{2}} \]

\[ \sim \frac{R_o}{4\pi} \ln \left( \frac{r_0 \frac{2\pi f}{D}}{\frac{1}{2}} \right) \]

Therefore, if the bath is geometrically 2-dimensional, the PSD of temperature fluctuations for case 1 is:

\[ S(f) = \frac{k T R}{\pi} \ln \left| r_0 \frac{2\pi f}{D} \right| \]

If the bath is geometrically 3-dimensional, we obtain:

\[ I_{r_0}(f) = \frac{4\pi R_0}{R_0} \frac{1}{4\pi D C_0} \frac{1}{r_0^2} \left( \frac{12\pi f}{D} \right)^{1/2} e^{-r_0 \left( \frac{12\pi f}{D} \right)^{1/2}} \]

\[ Z_{r_0}(f) = \frac{R_o}{4\pi r_0} \frac{1}{1 + r_0 \left( \frac{12\pi f}{D} \right)^{1/2}} \]

In the limit of \( r_0 \ll \left( \frac{D}{2\pi f} \right)^{1/2} \), we have the result:
\[ S(f) = \frac{kT R_0}{\pi r_0} \text{ (white noise)} \]

Nyquist's Theorem cannot be applied for the topology of case 2 unless the parallel current sources are accompanied by parallel resistors. The presence of the parallel resistors would modify the diffusion equation and require new solutions for \( V(f) \). It is easier to compute the PSD of the thermal fluctuations by considering each independent source and then integrating. In addition, this procedure would also apply to non-equilibrium situations that are not constrained to be consistent with the Fluctuation-Dissipation Theorem. Therefore, for frequencies above \( f_c \), we will compute the PSD resulting from current sources across the capacitors but without the accompanying resistors, \( R_p \), and show that the temperature fluctuations from case 2 are much smaller than those from case 1.

The following procedure will be used to compute the fluctuations in case 2. First, compute the voltage as a function of frequency at the origin resulting from a source located a distance \( r \) away. These functions were listed in the previous section. Next, square the magnitude of the voltage to find the PSD resulting from a single source. Finally, since the sources are uncorrelated, sum the PSD's of all sources, i.e., integrate over \( r \).

For a geometrically 1-dimensional bath, we have:

\[
S(r, f) = \frac{4kT G_p}{4C_0} \frac{1}{2\pi D} e^{-2r \left( \frac{\pi f}{D} \right)^{1/2}}
\]

\[
S(f) = \int_0^\infty S(r, f) \, dr
\]

\[
= \frac{1}{2} kT \left( \frac{f_c}{f} \right) \left( \frac{R}{\pi f C} \right)^{1/2}
\]

Case 2
This result is proportional to $f^{-3/2}$, but is insignificant when compared with the result of case 1 unless $f \ll f \omega$, contrary to assumption:

$$S(f) = K T \left( \frac{R}{\pi f C} \right)^{1/2}$$

Case 1

For a bath that is geometrically 2-dimensional, we have:

$$S(r, f) = 2 \pi r \left( 4 K T G \rho_0 \frac{1}{2 \pi C_0 D} \right)^2 K_0 \left[ r \left( \frac{4 \pi f}{D} \right)^{1/2} \right]^2$$

Since the modified Bessel function is real rather than a complex function of its argument, this becomes:

$$S(r, f) = \frac{2 K T G \rho_0 R_o}{\pi} \frac{R^2}{\pi} K_0 \left[ r \left( \frac{4 \pi f}{D} \right)^{1/2} \right] K_0 \left[ r \left( \frac{-4 \pi f}{D} \right)^{1/2} \right]$$

Integrating over $r$ from $r_o$ to $\infty$ with the help of a formula (Gradshteyn and Ryzhik, p.634, equation 5.54), and taking the limit as $r_o \to 0$

$$S(f) = \frac{2 K T G \rho_0 R_o}{\pi} \left[ \frac{D}{4 \pi |f|} \right]$$

Case 2

$$S(f) = \frac{K T \rho_0}{\pi} \ln \left| r_o \left( \frac{2 \pi f}{D} \right) \right|$$

This is exactly proportional to $1/f$, but again the result from case 1, repeated below, will dominate.

$$S(f) = \frac{K T \rho_0}{\pi} \ln \left| r_o \left( \frac{2 \pi f}{D} \right) \right|$$

Finally, for a 3-dimensional bath, we obtain:

$$S(r, f) = (4 K T G \rho_0) (4 \pi r^2) \frac{1}{C_0} \left( \frac{1}{4 \pi D r} \right)^2 e^{-2 r \left( \frac{\pi f}{D} \right)^{1/2}}$$

$$S(f) = \frac{K T \rho_0}{\pi} \left( \frac{1}{\pi D} \right)^{1/2} \pi f$$

Case 2
The result for case 1 was:

\[ S(f) = \frac{KTR_o}{\pi} \frac{1}{r_o} \]

The sum of the two results is:

\[ S(f) = \frac{KTR_o}{\pi} \left[ \frac{1}{r_o} + \left( \frac{f \lambda}{f} \right)^{1/2} \left( \frac{\pi f \lambda}{D} \right)^{1/2} \right] \]

If \( f \lambda > f \) and \( r_o \ll \left( \frac{D}{\pi f} \right)^{1/2} \), the second term, the one for case 2, is negligible when compared with the first.

The results of this section are summarized in the table below. In all dimensions the case 1 term dominates the PSD of the temperature fluctuations. Note that the term for case 2 in 2-dimensions gives exactly \( 1/f \) noise. Unfortunately, its magnitude is insignificant when compared with the term from case 1. Sato (1978) correctly derived a PSD proportional to \( 1/f \) for 2-dimensions with the topology of case 2. However, he incorrectly computed its magnitude and failed to compare it with case 1.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-D</td>
<td>[ \frac{KTR_o}{\pi} \frac{R}{\pi f c} ]^{1/2}</td>
</tr>
<tr>
<td>2-D</td>
<td>[ \frac{KTR_o}{4\pi} \ln \left( \frac{r_o^2}{2\pi f D} \right) ]</td>
</tr>
<tr>
<td>3-D</td>
<td>[ \frac{KTR_o}{\pi r_o} ]</td>
</tr>
</tbody>
</table>
CHAPTER VIII

MORE DIFFUSION MODELS

The structures used by Voss and Clarke and others for their experimental measurements of 1/f noise in thin metallic films have one dimension that is much greater than a diffusion length at frequencies for which the observed PSD is 1/f. The results of the previous chapter were derived from the assumption that all dimension were much smaller than a diffusion length and cannot be applied here. In this chapter, we will consider 1, 2, and 3-dimensional models where the length of the resistor exceeds a thermal diffusion length. Again, we will use uniformly distributed noise sources, but only in the topology of case 1, across the resistors. Since the noise from case 2 is always much smaller than the noise from case 1, case 2 will not be considered in this context.

We will begin with a 1-dimensional resistor, placed on a geometrically 1-dimensional thermal bath such that the resistor does not disturb the thermal homogeneity of the bath. This is illustrated in Figure 8a. To calculate the PSD of the resistance fluctuations, consider the resistor

![Figure 8a](image-url)
to consist of very small resistors in series, with the resistance of each small resistor proportional to its local temperature.

\[ R(T_r)dr \quad R(T_r)dr \quad R(T_r)dr \quad \text{etc.} \]

**Figure 8b**

The local temperature is determined by the thermal bath, whose analogue is a 1-dimensional RC line with distributed noise sources. The voltage on the line is analogous to the local temperature of the small resistors.

\[ \begin{align*}
I_N & \\
+ & \\
\Delta T_r & \\
- & \\
\end{align*} \]

\[ C_0 \]

**Figure 8c**

It is important to recognize that the temperature fluctuations resulting from a single noise source are correlated along the length of the resistor. Therefore, to compute the PSD of resistance fluctuations arising from a single source, the amplitude of the fluctuations rather than the magnitude squared must be summed. After first integrating alone the resistor and then computing the magnitude squared of the resistance fluctuations from a single noise source, compute the total PSD by then integrating over all sources.

We will use the following procedure for the computations in all dimensions:
1. Replace each of the current sources located across the resistors in the RC line with an equivalent current source located across the capacitors.

2. Using the previous results, compute the temperature at a single point on the resistor resulting from a single noise source in the bath.

3. Assuming that the resistance is linearly proportional to temperature for small fluctuations with constant of proportionality ($\beta$), integrate the temperature along the length of the resistor, and thereby compute the fluctuation in resistance resulting from a single noise source.

4. Compute the magnitude squared of the resistance fluctuation from one source and then integrate over the entire bath to compute the overall PSD.

For a 1-dimensional bath, a current source located across the resistor of an RC line can be replaced by two equal but opposite sources located across the adjacent capacitors as illustrated in Figure 8d. Since the response of the RC line to a source across the capacitor is known, it is advantageous to modify the topology to make use of the former result. The sources across the resistors were uncorrelated in space (impulses in space), but the two sources across the capacitors act as a doublet.
Figure 5d

The response of the line to the doublet will be the spatial derivative of the former response. A factor of \( \sqrt{2} \) arises because, across each capacitor, the current source consists of the difference between two uncorrelated currents that add as their magnitude squared rather than as their amplitudes.

The voltage (temperature) on the line at a point \( R \) resulting from a single doublet source at a point \( r \) is:

\[
V(\hat{r}) = \frac{I}{\sqrt{2}} \frac{d}{d(r-R)} \left[ \frac{1}{2C_o} \left( \frac{1}{4\pi D\hat{r}} \right)^{1/2} - |r-R| \left( \frac{12\pi \hat{t}}{D} \right)^{1/2} \right]
\]
\[
= \left( \frac{KT}{R_o^2} \right)^{1/2} \frac{|r-R|}{(r-R)} e^{-|r-R| \left( \frac{12\pi \hat{t}}{D} \right)^{1/2}}
\]

Note that the voltage is an odd function of the distance \( (r-R) \). The temperature on one side of the doublet goes up and on the other side, goes down. Because the response decays in a distance of a few thermal diffusion lengths, sources far from the ends of the resistor would contribute two equal but opposite responses. Therefore, the sum over the length of the resistor would vanish. Only those sources near the ends will provide a net contribution. The resistance fluctuation from a single source is:
\[
\beta \int_{-\ell}^{\ell} V(f) \, dR = \frac{\beta}{2} \left( \frac{KT}{2R} \right)^{1/2} \left( \frac{R_o}{12\pi fC_o} \right)^{1/2} e^{-|r-R| \left( \frac{12\pi f}{D} \right)^{1/2}} \left|_{-\ell}^{\ell} \right.
\]

Assuming that the length of the resistor (2\ell) is much greater than a diffusion length, a source located near one end will have an insignificant response at the other end. Therefore, in computing the magnitude squared the cross terms will vanish. Thus the PSD of the resistance fluctuations from a single source located at (r) is:

\[
S(r,f) = \frac{\beta^2 KT}{8} \left( \frac{1}{2\pi f C_o} \right) \left[ e^{-2|r-\ell| \left( \frac{\pi f}{D} \right)^{1/2}} + e^{-2|r+\ell| \left( \frac{\pi f}{D} \right)^{1/2}} \right]
\]

Integrating over all sources, yields:

\[
S(f) = \frac{\beta^2 KT}{4} \frac{1}{2\pi f C_o} \left( \frac{D}{\pi f} \right)^{1/2}
\]

It is customary to normalize the PSD by the square of the resistance to obtain:

\[
\frac{S(f)}{R^2} = \frac{\beta^2 KT}{4(2\ell)^2} \frac{1}{2\pi f C_o} \left( \frac{D}{\pi f} \right)^{1/2}
\]

Factors such as the width, thickness and resistivity would enter both in the computation of the PSD of resistance fluctuations and in the resistance squared. Therefore, in the above expression these terms would cancel.

For thermal baths that are effectively 2 and 3-dimensional, the problem is more complicated. The integrations cannot be computed in closed
form without some simplifying assumptions, and, as before, the impedance of a small region tends toward infinity as its size decreases. We will begin by assuming a minimum size, much smaller than a diffusion length at the highest frequency of interest to construct a lumped model, and then proceed to the continuous limit.

In figure 8e is illustrated a long, but very narrow resistor placed on a 2-dimensional thermal bath. The bath is divided into small regions with lumped parameters. Again, the resistor is assumed not to disturb the thermal homogeneity of the bath.

![Diagram](image)

Figure 8e

In terms of a 2-dimensional RC analogue, the heat bath is modelled as pictured in Figure 8f.
Exactly as we did for 1-dimension, a single noise source can be replaced by two sources of equal but opposite amplitudes across the capacitors.
The source may be oriented either in parallel with (x-direction) or perpendicular to the resistor (y-direction). First, consider the response at a point \( x_0 \) on the resistor to a source located at \((x,y)\) and oriented parallel to the resistor.

![Figure 8h](image)

The response to each source depends only on the distance between the source and the point on the resistor and will be represented as \( F(r) \). Thus we have for the response to both sources:

Net response \( = (I)F(r_2) - (I)F(r_1) \)

\[
= (I)F \left[ \left( y^2 + (x_0 - x - dx)^2 \right)^{1/2} \right] - (I)F \left[ y^2 + (x_0 - x)^2 \right]^{1/2}
\]

\[
= (I) \frac{dF(r_2)}{dr_2} \left( \frac{x_0 - x}{r_2} \right) dx
\]

Proceeding in a similar manner, the net response for a source oriented perpendicular to the resistor would be:
Each capacitor will have a current source that is the result of two sources in the y-direction and two in the x-direction. Because their responses are different we will continue to separate the x and y-directed currents. Adding the two x or two y-directed currents together results in a factor of $\sqrt{2}$. Thus, we have:

\[
I_x = \left( \frac{2KT}{R_0} \right)^{1/2} \left( \frac{x_0 - x}{r} \right) \frac{d}{dr} \left[ \right]
\]

\[
I_y = \left( \frac{2KT}{R_0} \right)^{1/2} \left( \frac{y}{r} \right) \frac{d}{dr} \left[ \right]
\]

For a 2-dimensional bath, we have:

\[
F(r) = \frac{1}{2\pi D c} K_0 \left[ r \left( \frac{j2\pi f}{D} \right)^{1/2} \right]
\]

Therefore, the temperature fluctuations at a point $x_0$ on the resistor from a source at $(x,y)$ is:

\[
V(f) = \frac{1}{\pi} \left( \frac{KTR_0}{2} \right)^{1/2} \left( \frac{j2\pi f}{D} \right)^{1/2} K_1 \left[ r \left( \frac{j2\pi f}{D} \right)^{1/2} \right] \left[ \frac{(x_0 - x)}{r} + \frac{y}{r} \right]
\]

The sum of response along the resistor is:

\[
\frac{1}{\pi} \left( \frac{KTR_0}{2} \right)^{1/2} \int_{-\ell}^{\ell} \alpha K_1(\alpha r) \left[ \frac{x_0 - x}{r} + \frac{y}{r} \right] dx_0
\]

where $\alpha = \left( \frac{j2\pi f}{D} \right)^{1/2}$

\[
r = \left[ y^2 + (x_0 - x)^2 \right]^{1/2}
\]

Eventually, we will compute the magnitude squared of the above integral, remembering that the $\frac{x_0 - x}{r}$ and the $\frac{y}{r}$ terms are uncorrelated, and then
integrate the result over the entire bath. The \( \frac{x_0 - x}{r} \) term is easy; it is an exact derivative and equals:

\[
\frac{1}{\pi} \left( \frac{K T R_0}{2} \right) \left[ K_o \left[ \alpha \left( y^2 + (\xi - x)^2 \right)^{1/2} \right] \right. - \left. K_o \left[ \alpha \left( y^2 + (\xi + x)^2 \right)^{1/2} \right] \right]
\]

This is the integral of an odd function. Only for sources near the ends of the resistor will it not vanish. Furthermore, since the length of the resistor, \( \xi \) is much greater than a diffusion length \( \frac{1}{\alpha} \), only one term at a time will be non-zero, i.e., when computing the magnitude squared, the cross terms vanish. Thus the PSD from a single noise source is:

\[
S_{x,y}(f) = \frac{1}{\pi} \left( \frac{K T R_0}{2} \right) \left[ K_o \left[ \alpha \left( y^2 + (\xi - x)^2 \right)^{1/2} \right] \right. \left. K_0 \left[ \alpha \left( y^2 + (\xi + x)^2 \right)^{1/2} \right] \right]
\]

\[
+ \frac{1}{\pi} \left( \frac{K T R_0}{2} \right) \left[ K_o \left[ \alpha \left( y^2 + (\xi + x)^2 \right)^{1/2} \right] \right. \left. K_0 \left[ \alpha \left( y^2 + (\xi - x)^2 \right)^{1/2} \right] \right]
\]

Integrating over \((x)\) and \((y)\) by first substituting

\[ x' = \xi - x \text{ or } x' = \xi + x \]

and then switching to polar coordinates, yields:

\[
\frac{S(f)}{R^2} = \frac{\beta^2 K T}{16 \xi^2 \pi |f| C_0} \quad [\text{1/f Noise}]
\]

The result of summing the responses for all the \( x \)-directed doublet sources is \( 1/f \) noise. Unfortunately, this term was an odd function and most of the \( x \)-directed sources contribute zero. Therefore, the response resulting from the \( y \)-directed sources whose response is an even function will dominate, as we shall see below.
The \( y \)-directed term is not an exact differential and, to the author's knowledge, cannot be solved in closed form over finite limits. But, because the length of the resistor is long when compared to a diffusion length, those sources away from the ends and toward the middle are unaffected by the ends. One makes no error in integrating their response from \(-\infty\) to \(\infty\). Within a few diffusion lengths of the ends, the boundaries matter and, of course, the contribution of the sources that are far away from the resistor is zero. Provided that \( \xi > \frac{1}{\alpha} \), the sources within a few diffusion lengths are a small fraction of the total. Therefore, we can approximate the integral by integrating from \(-\infty\) to \(\infty\) all sources that lie within the region \(-\xi < x < \xi\) and neglecting entirely all sources that lie outside that region. Thus, we have for the \( y \)-directed term:

\[
S(f) = \frac{1}{\pi} \left( \frac{KTR_0}{2} \right)^2 \int_{-\infty}^{\infty} \int_{-\xi}^{\xi} \int_{-\infty}^{\infty} \alpha \frac{K_1(\sigma r)}{r} \frac{Y}{x_0} \, dx \, dx_0 \, dy
\]

The integral over \( dx_0 \) has been evaluated in closed form (Gradshteyn and Ryzhik, p.706, Equation 6.596-3):

\[
= \frac{1}{\pi} \left( \frac{KTR_0}{2} \right)^2 \int_{-\infty}^{\infty} \int_{-\xi}^{\xi} \left[ \frac{\pi}{2ay} \right]^{1/2} K_\frac{1}{2}(ay) \, dx \, dy
\]

which equals:

\[
= \frac{1}{\pi} \left( \frac{KTR_0}{2} \right)^2 \int_{-\infty}^{\infty} \int_{-\xi}^{\xi} \left( \frac{\pi}{2a} \right)^{1/2} e^{-ay} \, dx \, dy
\]

Finally we obtain:

\[
S(f) = \beta^2(2\xi) \frac{KTR_0}{2\pi^2} \frac{\pi^2}{4} \frac{1}{\alpha + \alpha^*}
\]
\[
\frac{S(f)}{R^2} = \frac{\beta^2 K T R^2}{8} \frac{1}{2} \left( \frac{D}{\pi f} \right)^{1/2}
\]

Adding this to the term for the x-directed sources, we have:

\[
\frac{S(f)}{R^2} = \frac{\beta^2 K T R^2}{32 \pi} \frac{D}{\pi f} \left( \frac{D}{\pi f} \right)^{1/2} \left[ 1 + \frac{2}{\pi} \left( \frac{D}{\pi f} \right)^{1/2} \right]
\]

since \( \ell >> \left( \frac{D}{2\pi f} \right)^{1/2} \) by assumption, the overall result is proportional to \( f^{-1/2} \).

The final case to be considered is a long, thin resistor in a 3-dimensional heat bath. The structure has cylindrical symmetry and therefore is functionally 2-dimensional. However, the response to a source in 3-dimensions is different from the response in 2-dimensions. Otherwise, with the exception of a factor of \( (2\pi r) \), the problem is identical.

The response to a single source located at \((x_0, y_0)\), where \((x_0)\) is along the z-axis and \((y_0)\) is in the radial direction is:

\[
\left( \frac{2 K T}{R_0} \right)^{1/2} \int_{-\ell}^{\ell} \left[ \frac{(R-x_0) + y_0}{r} \right] \frac{d}{dr} \left[ \frac{1}{4\pi D C_0 r} \right] e^{-ar} \frac{dR}{\alpha}
\]

where \( \alpha = \left( \frac{j 2\pi f}{D} \right)^{1/2} \)

\[ r^2 = (R - x_0)^2 + y_0^2 \]

Again, for \( \ell >> \frac{1}{|\alpha|} \), we will evaluate this integral over infinite limits.

Thus we obtain:

\[
\frac{1}{4\pi} \left( \frac{2 K T R_0}{2} \right)^{1/2} \int_{-\infty}^{\infty} \left( \frac{1 + (ar)}{r^2} \right) e^{-ar} \left( \frac{R + y_0}{r} \right) dR
\]

The term proportional to \( R \) is an odd function and yields zero when integrated over symmetric limits. Thus, we have:
\[
\frac{(2KTR_o)^{1/2}}{4\pi y_o} y_o \int_{-\infty}^{\infty} \left( \frac{-\alpha r}{r^3} + \frac{\alpha e^{-\alpha r}}{r^2} \right) \, dr
\]

where \( r = (R^2 + y_o^2)^{1/2} \)

Integrating the first term by parts, we obtain:

\[
\frac{(2KTR_o)^{1/2}}{4\pi y_o} y_o \left[ \frac{R/y_o^2}{(R^2 + y_o^2)^{1/2}} e^{-\alpha(R^2 + y_o^2)^{1/2}} \right]_{-\infty}^{\infty} = 0
\]

\[
+ \int_{-\infty}^{\infty} \left[ \frac{R/y_o^2}{(R^2 + y_o^2)^{1/2}} \right] \left[ \frac{\alpha R}{(R^2 + y_o^2)^{1/2}} \right] e^{-\alpha(R^2 + y_o^2)^{1/2}} \, dR
\]

Combining this with the second term yields:

\[
\frac{(2KTR_o)^{1/2}}{4\pi y_o} \int_{-\infty}^{\infty} \frac{R^2 + y_o^2}{R^2 + y_o^2} e^{-\alpha(R^2 + y_o^2)^{1/2}} \, dR
\]

Substituting \( u^2 = R^2 + y_o^2 \), obtains:

\[
\frac{u \, du}{(u^2 - y_o^2)^{1/2}} = dR
\]

\[
\frac{(2KTR_o)^{1/2}}{4\pi y_o} \alpha \int_{-\infty}^{\infty} \frac{u}{(u^2 - y_o^2)^{1/2}} e^{-\alpha u} \, du
\]

The integral has been solved in closed form and equals (Gradshteyn and Ryzhik, p.316, Equation 3.365-2):
\[
\frac{(2K\mathcal{R}_o)^{1/2}}{4\pi} \frac{\alpha}{y_o} \left( y_o K_1(\alpha y_o) \right)
\]

Finally computing the magnitude squared and integrating over all sources, we obtain:

\[
S(f) = \frac{2K\mathcal{R}_o}{16\pi^2} a^2(2\ell) 2\pi \int \int_\infty y_o K_1(\alpha y_o) K_1(\alpha^2 y_o) \, dy_o
\]

This integral can be evaluated in closed form and equals (Gradshteyn and Ryzhik, p. 634, Equation 5.54-1):

\[
\frac{S(f)}{R^2} = \frac{K\mathcal{R}_o}{16\ell} \quad \text{(white noise)}
\]
CHAPTER IX

DISCUSSION OF DIFFUSION MODELS

The idea that the $1/f$ noise in thin, but continuous, metal films results from temperature fluctuations is supported by three pieces of experimental evidence from a paper by Voss and Clarke (1976a).

1. They obtain a good fit to the magnitude of the PSD versus the inverse volume of the sample by including as a constant of proportionality the square of the resistive temperature coefficient.

2. The material manganin with a temperature coefficient of nearly zero exhibited unmeasurably small $1/f$ noise.

3. They observed a spatial correlation of the noise along the resistor with a spatial constant that was approximately a thermal diffusion length.

In the same paper, Voss and Clarke attempted a theoretical derivation to demonstrate that equilibrium temperature fluctuations would have a $1/f$ PSD, but were unable to do so. They appealed mostly to the work of Van Vliet, who derived the fluctuations in the situations corresponding to case 1 of this paper, none of which have a $1/f$ PSD. Since then, Sato (1978) has derived a $1/f$ PSD for a 2-dimensional bath in the topology of case 2. But, he computed the magnitude of the PSD incorrectly. When this is done correctly and compared with the 2-dimensional case 1 result, the case 2 result is found to be insignificantly small.

A third group of models was considered in this paper that closely corresponds to the geometries of the resistors and baths in the experi-
ments. None of these models yields a PSD that fits the data. The results for the PSD's computed in the prior chapters are summarized below.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Long Resistor Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1-D$</td>
<td>$KT \left( \frac{R_o}{\pi f C_o} \right)^{1/2}$</td>
<td>$KT \left( \frac{R_o}{\pi f C_o} \right)^{1/2} \frac{\ell}{f}$</td>
</tr>
<tr>
<td>$2-D$</td>
<td>$\frac{KTR_o}{4\pi} \ln \left( \frac{r_o}{D} \frac{2\pi f}{\ell} \right)$</td>
<td>$\frac{KTR_o}{\pi} \left( \frac{f\ell}{f} \right)$</td>
</tr>
<tr>
<td>$3-D$</td>
<td>$\frac{KTR_o}{\pi r_o}$</td>
<td>$\frac{KTR_o}{\pi} \left( \frac{1}{\pi D f} \right)^{1/2} \pi f \ell$</td>
</tr>
</tbody>
</table>

In the light of these results, one must conclude that $1/f$ noise in thin metallic films is not the result of equilibrium temperature fluctuations, unless the effects of thermal inhomogeneities are substantial, which seems unlikely to the author.

Voss and Clarke (1976b) claim to have measured equilibrium $1/f$ noise in $L_{N_b}$, a semiconductor, and in discontinuous films of $N_b$; but could not detect any spatial correlations and concluded that thermal diffusion was not responsible for this noise. I am unaware of any measurements of equilibrium PSD of resistance fluctuations for continuous metal films that show a $1/f$ PSD, and would be surprised if the PSD was not proportional to $f^{-1/2}$ as predicted by the last group of models for a 2-dimensional heat bath. This should be tested.

Numerous non-equilibrium models have been suggested in the literature. Any non-equilibrium effect that would increase the noise of case 2 compared with case 1, in a 2-dimensional bath, would yield a $1/f$ PSD. One specula-
tion might be some form of shot noise associated with a redistribution of energy among various modes.

Finally, Voss and Clarke's experimental results may be the result of an entirely different kind of process. For example, consider a schematic of their experimental geometry illustrated below. A steady (D.C.) current

\[ I \rightarrow + \quad v(t) \rightarrow - \rightarrow I \]

Figure 9a

is passed through the outer contacts and the fluctuations of the voltage are measured across the inner contacts. Because of discontinuities, the presence of tunnelling or other phenomena, contacts are notorious for exhibiting much 1/f noise. Of course, the voltage, \( v(t) \), is measured across a pair of contacts without appreciable current flow. Hence, \( v(t) \) is thought not to be influenced by the contact 1/f noise.

However, the power dissipated at the contact is \( I(\text{D.C.}) \times v_{\text{contact}} \). Since this voltage has a 1/f PSD, the dissipated power and, therefore, the heat flowing into the substrate, must have fluctuations proportional to \( f^{-1/2} \). The temperature of the surrounding area will be caused to fluctuate by the heat and the PSD of the temperature fluctuations will be 1/f. These fluctuations will be spatially correlated, will decay with a thermal diffusion length and will modulate the value of the resistance between the inner terminals in proportion to the magnitude of the resistive temperature
coefficient. This is exactly consistent with Voss and Clarke's data. However, the PSD of the resistive fluctuations would be proportional to the D.C. current squared which is inconsistent with the data.

The above model is rather ad hoc and unappealing, but one like it might be correct. Without any theoretical basis for \(1/f\) equilibrium temperature fluctuations, it is likely that some ad hoc explanation is correct. The model above could be tested by varying the distance between the outer and the inner contacts.
CHAPTER X

SUMMARY AND DISCUSSION

In this paper, a non-stationary autocorrelation function for 1/f noise has been derived by analyzing a linear system that gives 1/f noise exactly. The dependence of the autocorrelation function on absolute time is found to be logarithmic and since logarithms are slowly varying functions, the autocorrelation function can be considered to be almost stationary.

Furthermore, by computing the Fourier transform of the autocorrelation function with respect to the relative time between values and not their absolute times of occurrence, a PSD was derived that was stationary except for its apparent steady or D.C. value which depended on absolute time.

The almost stationary autocorrelation function was used to characterize how long the past influenced present values of the process. Combining it with estimates of the number of state variables required for a linear system that manifests 1/f noise resulted in a description of the information storage inherent in a 1/f-type process. With the lowest estimate of the required number of state variables, a 1/f process was described as evolutionary. Its history was condensed into the values of the state variables of the system; one that represented an average of the values of the process over 1 second, one over 10 seconds, 100 seconds and so on.

The above description suggests that 1/f-type processes might be useful for modelling informational systems that are strongly influenced by their history, systems whose response to current inputs depends on everything that has ever happened to them. Likely candidates might be personality
development, the development of cultures, social systems and governments, and the development of descriptive sciences such as economics, psychology and physics. In addition, works of art that are intended to communicate the development of an idea are likely to have the statistics of a 1/f-type process.

Statistical tools for handling 1/f-type processes are lacking and will be needed before it will be possible to describe the behavior of a system like the economic system of the United States as a 1/f process. For example, based on the concept that economic variables can be modeled as stationary random processes, regression analysis is used to compute parameters in a model that predicts the future behavior of the economy. However, because the behavior of businessmen is strongly influenced by economic history, the economic relationships upon which the models are based will evolve. In this case, an alternative style of regression analysis based on the statistics of a 1/f-type process might be appropriate. But, what would this alternative be? Also, what types of predictive estimators would give the best results for a 1/f process?

Another interesting question would be the extent to which systems that manifest 1/f noise could be controlled, and at what cost. Since systems that manifest 1/f-type processes must be capable of storing information, it would probably be useful to develop a notion of how much, i.e., the information capacity of a 1/f rather than a white noise channel. Also, it would be useful to compare the advantages or disadvantages of various coding strategies for use in a 1/f noise channel.

In another part of this paper, renewal processes whose PSD was 1/f were used to illustrate two important points. First, 1/f noise is inti-
ately related to the tail behavior of probability distributions. Neglecting the tail behavior by appealing to the law of large numbers to obtain a Normal distribution will throw out the 1/f noise. Second, 1/f noise can also be generated with stable distributions other than the Normal distribution.

In the final part of this paper, the idea that the diffusive transport of heat in a substrate might give rise to temperature fluctuations that modulate the resistance of a thin metal film is explored. None of the models yield a 1/f PSD suggesting that a model based on either thermal inhomogeneities, far from equilibrium heat flow, or some ad hoc explanation should be used to fit the Voss and Clarke experiments.
APPENDIX

DERIVATION OF THE PSD FOR THE SECOND FAMILY OF RENEWAL PROCESSES

The PSD's of the second family of renewal processes will be derived in this section. As is the case for any renewal process, the interarrival times are identically distributed. We will refer to their density function and its transform as:

$$\text{pdf}(T_{i+1} - T_i) = \text{pdf}(\Delta T) \leftrightarrow \Omega(f)$$

The events of the process are ramps, starting at zero at the beginning of each interval, growing linearly with time and then returning again to zero at the beginning of the next interval. This process is illustrated below.

The process, \( I(t) \), is represented by the expression below:

$$I(t) = \sum_{i=0}^{N} \int_0^t \left[ \mu_1(t-T_i) - \mu_1(t-T_{i+1}) - (T_{i+1} - T_i) \mu_0(t-T_{i+1}) \right] dt$$

Using the procedure described in the section on renewal processes, we have:

$$S_N(f) = \frac{1}{N} \left| \int_0^{T_{N+1}} I(t) e^{-j2\pi ft} dt \right|^2 = \left| \frac{I(f)}{N} \right|^2$$
and

\[ S(f|N) \triangleq E[S_N(f)] \]

Taking the transform of \( I(t) \), we obtain:

\[
I(f) = \frac{1}{j2\pi f} \text{pv} \left[ \sum_{i=0}^{N} e^{-j2\pi f T_{i+1}} - e^{-j2\pi f T_{i+1}} e^{-j2\pi f T_{i+1}} \right]
\]

where \( \text{pv} \left[ \cdot \right] \) denotes principal value.

Recognizing that the first two terms in the sum exactly cancel except for their first and last members we have:

\[
I(f) = \frac{1}{(j2\pi f)^2} \text{pv} \left[ 1 - e^{-j2\pi f T_{N+1}} \sum_{i=0}^{N} j2\pi f (T_{i+1} - T_i) e^{-j2\pi f T_{i+1}} \right]
\]

Finally, we have:

\[
|I(f)|^2 = I(f) I^*(f)
\]

\[
= \frac{1}{(2\pi f)^4} \text{pv} \left[ 2 - e^{-j2\pi f T_{N+1}} + e^{j2\pi f T_{N+1}} \right] \quad \text{term 1}
\]

\[
+ \frac{1}{(2\pi f)^4} \text{pv} \left[ \left( 1 - e^{-j2\pi f T_{N+1}} \right) \sum_{i=0}^{N} j2\pi f (T_{i+1} - T_i) e^{j2\pi f T_{i+1}} \right] \quad \text{term 2}
\]

\[
- \frac{1}{(2\pi f)^4} \text{pv} \left[ \left( 1 - e^{+j2\pi f T_{N+1}} \right) \sum_{i=0}^{N} j2\pi f (T_{i+1} - T_i) e^{-j2\pi f T_{i+1}} \right] \quad \text{term 3}
\]

\[
- \frac{1}{(2\pi f)^4} \text{pv} \left[ \sum_{i=0}^{N} (T_{i+1} - T_i) e^{-j2\pi f T_{i+1}} \right] \left( \sum_{\ell=0}^{N} (T_{\ell+1} - T_{\ell}) e^{+j2\pi f T_{\ell+1}} \right) \quad \text{term 4}
\]
To compute the PSD, we must divide by \((N+1)\) and then take the expected value.

\[
S(f/N) = \left( \frac{I(f)I^*(f)}{N + 1} \right)
\]

In order to keep track of the algebra, this will be done term by term. The expected value of the first term can be evaluated by recognizing that the interval from 0 to \(T_{N+1}\) is the sum of \((N+1)\) independent, identical random variables. Therefore, the characteristic function of the sum is:

\[
\left( \Omega(f) \right)^{N+1}
\]

Therefore, for term 1, we have:

\[
\frac{1}{(2\pi f)^4} \frac{1}{N + 1} \left[ 2 - \Omega^{N+1}(f) - \Omega^{N+1}(-f) \right]
\]

Terms 2 and 3 are similar and will be discussed together. Consider the result of term 2 for a single value of \(i\).

\[
\frac{1}{(2\pi f)^4} \left[ 1 - e^{-j2\pi fT_{N+1}} \right] j2\pi f \left( T_{i+1} - T_i \right) e^{+j2\pi fT_{i+1}}
\]

\[
= \frac{1}{(2\pi f)^3} \left[ \left( \Delta T_i \right) e^{j2\pi f\Delta T_i} e^{j2\pi fT_i} - \left( \Delta T_i \right) e^{j2\pi f\Delta T_i} e^{j2\pi f(T_i - T_{N+1})} \right]
\]

where \(\Delta T_i = T_{i+1} - T_i\)

Taking the expected value yields

\[
\frac{1}{(2\pi f)^3} \frac{1}{(N+1)} \left( \frac{d\Omega(f)}{d(12\pi f)} \right) \cdot \left( \Omega^1(f) - \Omega^{-1}(f) \right)
\]

Summing over \(i\) and recognizing a geometric series in \(\Omega(f)\) with the assumption that \(|\Omega(f)| < 1\), yields for term 2:

\[
\frac{1}{(2\pi f)^3} \frac{1}{(N+1)} \frac{d\Omega(f)}{df} \left( \frac{1 - \Omega^{N+2}(f)}{1 - \Omega(f)} - \frac{1 - \Omega^{N+2}(-f)}{1 - \Omega(-f)} \right)
\]

For term 3, we have a similar result:
\[- \frac{1}{(2\pi)^4 f^3} \cdot \frac{1}{(N+1)} \frac{d\Omega(-f)}{df} \left( \frac{1 - \Omega^{N+2}(-f)}{1 - \Omega(-f)} - \frac{1 - \Omega^{N+2}(f)}{1 - \Omega(f)} \right)\]

The fourth and final term can be written as:

\[- \frac{1}{(2\pi)^2} \cdot \frac{1}{(N+1)} \cdot \mathcal{E} \left[ \sum_{\lambda=0}^{N+1} \sum_{l=0}^{N+1} \Delta T_{\lambda} \Delta T_{l} e^{j2\pi f(T_{\lambda+1} - T_{l+1})} \right] \]

\[= \frac{1}{(2\pi)^2} \cdot \frac{1}{N+1} \left( \sum_{i=0}^{N+1} \sum_{\lambda=0}^{i-1} \Delta T_{\lambda} \Delta T_{l} e^{-j2\pi f\Delta T_{l}} e^{-j2\pi f(T_{l} - T_{\lambda+1})} \right. \]

\[+ \mathcal{E} \left( \Delta T_{l} \right)^2 \]

\[+ \left. \sum_{\lambda=L}^{N+1} \Delta T_{\lambda} \Delta T_{l} e^{j2\pi f\Delta T_{l}} e^{j2\pi f(T_{\lambda} - T_{l+1})} \right) \]

Recognizing a geometric series, we have:

\[= \frac{-1}{(2\pi)^2} \cdot \frac{1}{(N+1)} \sum_{L=0}^{N+1} \left[ \mathcal{E}(\Delta T_{l}) \frac{d\Omega(+f)}{d(j2\pi f)} \frac{1 - \Omega^{i+1}(f)}{1 - \Omega(f)} \right] \]

\[+ \mathcal{E}(\Delta T)^2 \]

\[+ \mathcal{E}(\Delta T) \frac{d\Omega(-f)}{d(j2\pi f)} \frac{1 - \Omega^{N-i+1}(-f)}{1 - \Omega(-f)} \right) \]

Recognizing another geometric series, we have:

\[= \frac{1}{(2\pi)^4 f^2 (N+1)} \left[ (N+1) \mathcal{E}(\Delta T^2) + \mathcal{E}(\Delta T) \frac{d\Omega(f)}{df} \left\{ \frac{(N+2)}{1 - \Omega(f)} + \frac{1 + \Omega^{N+2}(f)}{[1 - \Omega(f)]^2} \right\} \right] \]

\[+ \mathcal{E}(\Delta T) \frac{d\Omega(-f)}{df} \left\{ \frac{N+2}{1 - \Omega(-f)} + \frac{1 + \Omega^{N+2}(-f)}{[1 - \Omega(-f)]^2} \right\} \]

Combining all four terms, we obtain:

\[S(f|N) = \frac{1}{(2\pi)^4} \cdot \frac{1}{N+1} \left[ 2 - \Omega^{N+1}(f) - \Omega^{N+1}(-f) \right] \]

\[+ \frac{1}{(2\pi)^4 f^3 (N+1)} \frac{d\Omega(f)}{df} \left\{ \frac{1 - \Omega^{N+2}(f)}{1 - \Omega(f)} - \frac{1 - \Omega^{N+2}(-f)}{1 - \Omega(-f)} \right\} \]
\[ + \frac{1}{(2\pi)^3 f^2 (N+1)} \frac{d\Omega(-f)}{df} \left( \frac{1 - \Omega_N^{N+2}(-f)}{1 - \Omega(-f)} - \frac{1 - \Omega_N^{N+2}(f)}{1 - \Omega(f)} \right) \]

\[ + \frac{E(\Delta T^2)}{(2\pi)^4 f^2} \]

\[ + \frac{E(\Delta T)}{(2\pi)^4 f^2 (N+1)} \frac{d\Omega(f)}{df} \left( \frac{N+2}{1 - \Omega(f)} - \frac{1 - \Omega_N^{N+2}(f)}{[1 - \Omega(f)]^2} \right) \]

\[ + \frac{E(\Delta T)}{(2\pi)^4 f^2 (N+1)} \frac{d\Omega(-f)}{df} \left( \frac{N+2}{1 - \Omega(-f)} - \frac{1 - \Omega_N^{N+2}(-f)}{[1 - \Omega(-f)]^2} \right) \]

To obtain a \(1/f\) PSD, choose the interval density function to be:

\[
\text{pdf (\Delta T)} = \frac{3\Delta T_0^3}{(\Delta T)^4} \mu_1 (\Delta T - \Delta T_0)
\]

Actually any pdf with a \(t^{-4}\) long-time tail will do as well. The characteristic function of the above pdf is:

\[
\Omega(s) = e^{-sT_0} \left( 1 - \frac{sT_0}{2} + \frac{(sT_0)^2}{2} \right) + \frac{(sT_0)^3}{2} \ln sT_0 + \ldots
\]

where: \(s = j2\pi f\)

Substituting this expression into the equation for the PSD and performing some laborious but straightforward algebra, yields a PSD proportional to \(\frac{1}{|f|}\).

Choosing instead a Normal distribution for the interarrival times, with the characteristic function:

\[
\Omega(s) = e^{-sE(\Delta T) - \frac{s^2}{2} E(\Delta T^2)}
\]

yields a PSD that is white.
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