The Production of Sound by Moving Objects

by

David Herbert Munro

B.S., California Institute of Technology
(1976)

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

June 1980

© Massachusetts Institute of Technology 1980

Signature of Author

Department of Physics, May 19, 1980

Certified by

Thesis Supervisor

Accepted by

Chairman, Departmental Committee
The Production of Sound by Moving Objects

by

David H. Munro

Submitted to the Department of Physics
on May 19, 1980 in partial fulfillment of the
requirements for the Degree of Doctor of Philosophy

Abstract

A linear aeroacoustic theory is developed which applies to situations in which
turbulence and resonance phenomena are not important. The approach is based on
a generalization of the Kirchhoff formula for the solution of the wave equation in
terms of its boundary values. The generalization involves an allowance for the
motion of the boundary surface; it is originally due to Morgans. A new acoustic
source type has been identified -- a couplet of mass sources with a timelike
separation rather than a spacial separation. This source type has been dubbed the
"chronopole", and plays an important role in the theory. Connection is made to the
popular aeroacoustic theory of Flowcs Williams and Hawkings, although the
viewpoint adopted here is not fully reconcilable with that theory.

The problem of light aircraft propeller noise is the focus of the thesis. A simple
lifting line theory is developed in order to compute the boundary values of the flow
field necessary as input to Morgans' formula. A computer program uses this
performance theory, mated with the general aeroacoustic theory, to calculate the
pressure field as a function of time at an arbitrary observer location. The only
inputs to the program are the blade geometry, and the two dimensional
characteristics of its airfoil sections.

A wind tunnel experiment has been performed, in order to measure the near field
sound pressure signatures associated with three 1:4 scale model light aircraft
propellers. The results of these measurements compare quite favorably with the
predictions made by the computer model, over a wide range of operating conditions.
and microphone locations. An unexpected phenomenon was uncovered in the measurements made with one of the propellers over a narrow range of operating conditions. Namely, at a tip Mach number of 0.7, extremely intense coherent bursts of high frequency sound were produced from the region near the tips of the blades, where the airfoil chords were roughly 2 cm. The measured (Doppler shifted) frequency within the burst was 38 kHz near the disk plane, which indicates a frequency of 13 kHz in the frame of the blade. The flow instability responsible for this sound has not been positively identified, but it may be related to the phenomenon of transonic buffeting.

Thesis Supervisor: Dr. K. Uno Ingard

Title: Professor of Physics and Professor of Aeronautics and Astronautics
Acknowledgments

I cannot unleash my pen before thanking my thesis advisor, Prof. K. Uno Ingard, who has been a continual source of technical inspiration and moral support.

My teacher, colleague, and friend, Dr. George P. Succi, deserves no less honorable mention; without his technical and administrative expertise, this work could not have been done.

The parts of this thesis involving aerodynamics and wind tunnel experimentation would not have materialized without the enthusiastic support of Prof. E. Eugene Larrabee, a True Aerodynamicist.

My comrades Jeff Zimmer and Pete Dunbeck slaved long over experimental apparatus with me (and often without me). The Anechoic Wind Tunnel and Aero Department Projects Lab are maintained by Al Shaw, Fred Merlis, and Don Wiener, who are deserving of the thanks they receive in most of the theses written in the Aero Department.

I thank my best friend, and best man, Mike Coln, for technical help, for the use of his microprocessor, and for general support. Also, Paul Schluter played a vital role in one part of a data path. The present form of this document is largely the result of the efforts of Wayne Gramlich and his pet Scribe.

Finally, I gratefully acknowledge the support of an NSF Graduate Fellowship for my first three years at M.I.T. The research itself was funded by NASA Contract NAS1-15154, "Noise and Performance of Propellers for Light Aircraft".
To my Mother and my Father
# Table of Contents

Chapter One: General Introduction

Chapter Two: Mechanisms of Aerodynamic Force

2.1 Introduction

2.2 Fluid Flow Past a Wing
   2.2.1 The Vortex Structure Associated with Lift
   2.2.2 Discontinuity in the Velocity Potential
   2.2.3 Induced Drag
   2.2.4 Profile Drag
   2.2.5 The Kutta Condition
   2.2.6 The Flow Field of a Lifting Wing

2.3 Airfoil Theory
   2.3.1 The Two Dimensional Low Mach Number Case
   2.3.2 Finite Compressibility Effects
   2.3.3 Finite Aspect Ratio Effects

2.4 A Theory of Propeller Performance
   2.4.1 Statement of the Problem
   2.4.2 The Limit of Zero Solidity
   2.4.3 Strategy for Estimating Induced Velocities
   2.4.4 The Limit of Infinitesimal Sheet Spacing
   2.4.5 A Correction for Low Advance Ratios
   2.4.6 A Correction for High Advance Ratios
   2.4.7 Solution of the Propeller Performance Equations
   2.4.8 A Rough Allowance for a Nacelle

2.5 Summary

Chapter Three: General Aeroacoustic Theory

3.1 Introductory Remarks

3.2 Lowson Sources
   3.2.1 Inhomogeneous Wave Equations
   3.2.2 Solutions to the Scalar Wave Equations
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.3 Solutions to the Vector Wave Equation</td>
<td>95</td>
</tr>
<tr>
<td>3.3 Flowcs Williams’ and Hawkings’ Theory</td>
<td>103</td>
</tr>
<tr>
<td>3.3.1 The Approach</td>
<td>103</td>
</tr>
<tr>
<td>3.3.2 The Equation</td>
<td>107</td>
</tr>
<tr>
<td>3.3.3 Logical Completion of the Theory</td>
<td>110</td>
</tr>
<tr>
<td>3.3.4 Discussion and Criticism</td>
<td>111</td>
</tr>
<tr>
<td>3.4 The Flow Field of a Moving Object, Powell’s Theory</td>
<td>114</td>
</tr>
<tr>
<td>3.5 Morgans’ Extension of Kirchhoff’s Formula</td>
<td>119</td>
</tr>
<tr>
<td>3.5.1 Tooling Up</td>
<td>119</td>
</tr>
<tr>
<td>3.5.2 Derivation of Morgans’ Equation</td>
<td>123</td>
</tr>
<tr>
<td>3.5.3 Interpretation of the Result</td>
<td>124</td>
</tr>
<tr>
<td>3.5.4 Consistent Approximations</td>
<td>134</td>
</tr>
<tr>
<td>3.6 Summary</td>
<td>137</td>
</tr>
<tr>
<td>Chapter Four: Aeroacoustic Theory of Propellers</td>
<td>140</td>
</tr>
<tr>
<td>4.1 Historical Introduction</td>
<td>140</td>
</tr>
<tr>
<td>4.2 Sound from a Lightly Loaded Propeller</td>
<td>143</td>
</tr>
<tr>
<td>4.2.1 Overview</td>
<td>144</td>
</tr>
<tr>
<td>4.2.2 Modelling the Lift Force</td>
<td>149</td>
</tr>
<tr>
<td>4.2.3 Modelling the Profile Drag Force</td>
<td>151</td>
</tr>
<tr>
<td>4.2.4 Modelling the Thickness Noise</td>
<td>152</td>
</tr>
<tr>
<td>4.2.5 Antisymmetric Part of the Vorticity Distribution</td>
<td>157</td>
</tr>
<tr>
<td>4.3 A Correction for Finite Blade Loading</td>
<td>160</td>
</tr>
<tr>
<td>4.3.1 Overview</td>
<td>160</td>
</tr>
<tr>
<td>4.3.2 Modelling the Induced Drag Force</td>
<td>161</td>
</tr>
<tr>
<td>4.4 Sources in Helical Motion</td>
<td>167</td>
</tr>
<tr>
<td>4.4.1 Geometry of Source and Observer</td>
<td>167</td>
</tr>
<tr>
<td>4.4.2 The Four Source Terms</td>
<td>170</td>
</tr>
<tr>
<td>4.5 A Computer Model of Propeller Rotational Noise</td>
<td>178</td>
</tr>
<tr>
<td>4.5.1 Implementation of the Theory</td>
<td>178</td>
</tr>
<tr>
<td>4.5.2 Chordwise Load Distributions</td>
<td>186</td>
</tr>
<tr>
<td>4.5.3 Relative Importance of Various Source Terms</td>
<td>190</td>
</tr>
<tr>
<td>4.6 Summary</td>
<td>197</td>
</tr>
<tr>
<td>Chapter Five: An Experiment with Propeller Noise</td>
<td>201</td>
</tr>
<tr>
<td>5.1 Overview</td>
<td>201</td>
</tr>
</tbody>
</table>
5.2 Apparatus
   5.2.1 Wind Tunnel
   5.2.2 Propeller Thrust Stand
   5.2.3 Microphone Holder
5.3 Instrumentation
   5.3.1 Wind Tunnel
   5.3.2 Microphone
   5.3.3 Digital Oscilloscope
5.4 Data Processing
   5.4.1 Conventions
   5.4.2 Processing Program
5.5 A Review of the Data
   5.5.1 Repeatability
   5.5.2 Trends in the Data
5.6 Comparison to Theoretical Predictions
   5.6.1 Introduction
   5.6.2 Cessna Blade
   5.6.3 Windsong Blade
   5.6.4 Bruiser Blade
5.7 Vortex Noise
5.8 A Violent Flow Instability
   5.8.1 The Phenomenon
   5.8.2 The Location of the Instability
   5.8.3 The Boundaries of the Instability
   5.8.4 The Frequency of the Instability
   5.8.5 The Nature of the Acoustic Source
5.9 Conclusions

Chapter Six: Concluding Remarks
Table of Figures

Figure 2-1: Wing, Propeller, and Airfoil Geometry  22
Figure 2-2: Vortex Structure of a Lifting Wing  26
Figure 2-3: Origin of the Induced Drag  31
Figure 2-4: Origin of the Kutta Condition  36
Figure 2-5: Complete Flow Field of a Lifting Wing  39
Figure 2-6: Adopted Airfoil Section Characteristics  41
Figure 2-7: Geometry for Transonic Similarity Rule  44
Figure 2-8: The Induced Velocity  49
Figure 2-9: Complete Flow Field of a Thrusting Propeller  52
Figure 2-10: Infinite Helical Vortex Pair  59
Figure 2-11: Semiinfinite Helical Vortex Pair  62
Figure 2-12: Solution to Propeller Performance Equations  75
Figure 3-1: Infinitesimal Vortex Rings  99
Figure 3-2: Flow Field of a Moving Point Force  101
Figure 3-3: Splicing Together Two Flow Fields  104
Figure 4-1: Profile Drag Model  145
Figure 4-2: Decomposition of Vortex Structure  147
Figure 4-3: Lift Force Model  149
Figure 4-4: Thickness Noise Model  155
Figure 4-5: Thin Airfoil Geometry  157
Figure 4-6: Decomposition of Sources for Finite Loading  161
Figure 4-7: Symmetric Component of Vorticity Distribution  161
Figure 4-8: Helical Trajectory Geometry  167
Figure 4-9: Blade Coordinate System  173
Figure 4-10: Source Point Grids  181
Figure 4-11: Flow Chart for Sound Calculation Program  181
Figure 4-12: Relative Importance of Source Types: -40°  190
Figure 4-13: Relative Importance of Source Types: 0°  190
Figure 4-14: Relative Importance of Source Types: +40°  190
Figure 4-15: Transition from Near to Far Field: -40°  190
Figure 4-16: Transition from Near to Far Field: 0°  190
Figure 4-17: Transition from Near to Far Field: +40°  190
Figure 5-1: Anechoic Wind Tunnel
Figure 5-2: Propeller Thrust Stand
Figure 5-3: Bodies Tested with Propellers
Figure 5-4: Microphone Holder
Figure 5-5: Microphone Circuit
Figure 5-6: Microphone Frequency Response
Figure 5-7: Microphone Flow Noise Spectrum
Figure 5-8: Example of Raw Data
Figure 5-9: Example of Data Processing
Figure 5-10: Experimental Repeatability: 0°
Figure 5-11: Theoretical Repeatability: 0°
Figure 5-12: Experimental Repeatability: -10°
Figure 5-13: Effect of Advance Ratio
Figure 5-14: Effect of Mach Number
Figure 5-15: Effect of a Nacelle: -30°
Figure 5-16: Effect of a Nacelle: 0°
Figure 5-17: Effect of a Nacelle: +30°
Figure 5-18: Cessna Blade: 7kRPM, 29 m/s, 0°
Figure 5-19: Cessna Blade: 7kRPM, 20 m/s, 0°
Figure 5-20: Cessna Blade: 7kRPM, 15 m/s, 0°
Figure 5-21: Cessna Blade: 8kRPM, 23 m/s, 0°
Figure 5-22: Cessna Blade: 9kRPM, 26 m/s, 0°
Figure 5-23: Cessna Blade: 10kRPM, 29 m/s, 0°
Figure 5-24: Port-Starboard Comparison
Figure 5-25: Cessna Blade: 10kRPM, 29 m/s, -40°
Figure 5-26: Cessna Blade: 10kRPM, 29 m/s, -20°
Figure 5-27: Cessna Blade: 10kRPM, 29 m/s, +20°
Figure 5-28: Cessna Blade: 10kRPM, 29 m/s, +40°
Figure 5-29: Windsong Blade: 7kRPM, 20 m/s, 0°
Figure 5-30: Windsong Blade: 10kRPM, 29 m/s, -30°
Figure 5-31: Windsong Blade: 10kRPM, 29 m/s, 0°
Figure 5-32: Windsong Blade: 10kRPM, 29 m/s, +30 deg
Figure 5-33: Bruiser Blade: 10kRPM, 29 m/s, -40°
Figure 5-34: Bruiser Blade: 10kRPM, 29 m/s, 0°
Figure 5-35: Bruiser Blade: 10kRPM, 29 m/s, +40°
Figure 5-36: Blade to Blade Asymmetry
Figure 5-37: Bruiser Blade: 7kRPM, 20 m/s, 0°
Figure 5-38: Bruiser Blade: 6kRPM, 18 m/s, 0°  
Figure 5-39: Bruiser Blade: 4kRPM, 15 m/s, 0°  
Figure 5-40: Vortex Noise  
Figure 5-41: Mach Number Dependence of Vortex Noise  
Figure 5-42: Mach Number Dependence of Vortex Noise (cont.)  
Figure 5-43: Cessna Blade: Dirty Surface  
Figure 5-44: Cessna Blade: Clean Surface  
Figure 5-45: Elimination of Reflections  
Figure 5-46: Expanded View of One Tone Burst  
Figure 5-47: Boundaries of Instability  
Figure 5-48: Instability on One Blade Only  
Figure 5-49: Measured Tone Burst: -20°  
Figure 5-50: Measured Tone Burst: -10°  
Figure 5-51: Measured Tone Burst: 0°  
Figure 5-52: Measured Tone Burst: + 10°  
Figure 5-53: Measured Tone Burst: + 20°  
Figure 5-54: Measured Tone Burst: + 30°  
Figure 5-55: Measured Tone Burst: + 40°  
Figure 5-56: Source Phase vs. Observer Time  
Figure 5-57: Microphone Trajectories in Blade Fixed Coordinates  
Figure 5-58: Source Phase vs. Source Time  
Figure 5-59: Measured and Expected Burst Envelopes: + 10°  
Figure 5-60: Measured and Expected Burst Envelopes: -10°, + 40°
Table of Tables

Table 2-1: Glossary of Aerodynamic Terminology 22
Table 2-2: Summary of Propeller Performance Equations 57
Table 2-3: Solution of Propeller Performance Equations 77
Table 3-1: Useful Formulas and Notations 91
Table 3-2: Formulas for Moving Point Sources 95
Table 3-3: The Lifting Line Source 130
Table 3-4: Equivalent Sources at a Fluid Boundary 134
Table 4-1: Formulas for Helical Source Trajectories 170
Table 4-2: Formulas for Propeller Sound 175
Table 4-3: Adopted Chordwise Load Distribution 188
Table 5-1: Wind Tunnel Characteristics 210
Table 5-2: Airstream Properties 225
Table 5-3: Microphone Location 227
Table 5-4: Microphone Calibrations 233
Table of Appendices

Appendix A: Green Functions for Vector Wave Equations 351
Appendix B: Moving Source Sheets 359
Appendix C: A Four Dimensional Analogue to Solid Angle 368
Appendix D: The Pressure Field for Creation of Boundary Surface 370
Appendix E: Propeller Blade Geometries 380
Appendix F: Optical Trigger Calibration 387
Chapter One

General Introduction

How does the motion of a solid object through a fluid generate sound? Many everyday noises are the result of the compression of air by the rapid motion of an impermeable surface. The sound of a hammer striking a nail, or the sound produced by a loudspeaker cone are obvious examples. Linear acoustics, that is, the wave equation, describes both of these phenomena, as well as a more surprising example -- light aircraft propeller noise. Indeed, linear acoustics is identical with thin airfoil theory, even when the speed of the airfoil approaches the speed of sound. In each of these situations, the motion of an impermeable boundary surface generates the sound.

Problems such as the hammer and nail or the loudspeaker cone are usually solved by the application of an approximate boundary condition. Namely, the fact that the boundary undergoes a finite displacement is ignored. In this picture, the wave equation is driven by the velocity of the boundary only, rather than by its actual motion. On the other hand, it is impossible to ignore the actual displacement of the boundary surface in the problem of an aircraft propeller. The mathematics necessary to describe a situation in which the displacement of the boundary is crucial is different from that required for the better known case of negligible boundary displacement.

The thrust of this thesis will be to extend the theory of sound production by the motion of a boundary surface to situations in which the motion of the boundary is
not negligible. The case of the sound produced by the motion of a light aircraft propeller will serve as a backdrop against which the theory will be developed. The selection of a special case is very important to this study, due to the ridiculous breadth of situations to which the general theory will apply. The choice of propeller noise dictates the particular facets of the general theory which will be developed in depth.

In addition to this theoretical development, the results of a series of experiments with model light aircraft propellers will be presented. These experiments provide a counterpoint to the theory, indicating the extent to which the abstract ideas may be applied to a real physical process. The limitations of the theory in the case of propeller sound result primarily from flow instabilities (either turbulence or more structured instabilities), and partially from the lack of a complete understanding of the steady state flow field around a propeller blade.

The discussion will open with the latter problem. Wing theory and propeller theory are themselves a rich ground for growth in the understanding of fluid mechanics. Since this thesis is addressed primarily to the physicist, who may not have a background in aerodynamics per se, and since I am a physicist without a real background in aerodynamics, the presentation of the aerodynamic theory is at a very elementary level. The flow field around a wing is the result of the instability of the simple irrotational flow corresponding to a moving airfoil shaped object. This breakdown of the irrotational flow field is a tremendously complex phenomenon which is not particularly well understood. In fact, no one can compute the flow field of a wing in three dimensions from first principles (at least, not in a manner in which most physicists would consider indicative of a complete understanding of the process). Despite this fact, a gigantic amount of knowledge about the important
properties of the flow around airfoils has accumulated in the past century.

In particular, the endpoint of the breakdown of irrotational flow is very well understood, at least qualitatively. Certain limiting cases have been identified and serve as models for more realistic situations. The most important of these limiting cases is that of the lightly loaded wing or propeller, when the vorticity shed into the fluid (the irrotational flow field is unstable and breaks down!) is very weak. The so-called lifting line theory results when first order corrections for the finite strength of the shed vorticity are included. This is a very old theory, but it is still of considerable practical importance in estimating the performance of wings and of propellers. A lifting line theory of propeller performance is developed here, which provides the estimates of the boundary conditions at the surface of the propeller necessary in the general acoustic theory. My only contribution to this lifting line theory is the particular presentation given here, which stresses the nature of the flow field, the necessity and meaning of the idealized model which will be introduced, and the nature of the approximations involved in generalizing this ideal model to the point of practical utility.

With a vivid (I hope!) picture of the type of flow field to be eventually addressed, the reader is lead through a general theory of aeroacoustics. The discussion begins with moving sources of mass and of momentum in a homogeneous, unbounded fluid. My contribution here is to include a detailed description of the flow field, not just the pressure field. This description is nontrivial in the case of momentum sources. It is well known that moving sources of momentum (i.e. body forces) deposit vorticity in a fluid; a detailed description of this phenomenon is provided here. This study is particularly relevant to the case of propeller sound, since the vorticity deposited in a fluid by moving momentum sources greatly resembles the
vorticity shed into the wake of a wing or a propeller.

In addition to sources of mass and of momentum, it is expedient to describe the flow fields associated with couplets of mass sources and sinks. Such couplets play a role in fluid mechanics analogous to the role of dipoles in electromagnetic theory. A more original contribution of this work is the explicit recognition of couplets with a temporal separation as a source type distinct from couplets with a spacial separation. I have taken the liberty of naming this novel source type a "chronopole" (with apologies to readers with weak stomachs); chronopoles will play a significant role in the theoretical developments herein.

There is a considerable gap between the study of mass and momentum sources in an unbounded fluid and the study of the boundary value problem in a source free fluid. This gap has traditionally been bridged by modelling the solid boundary surface, or more generally any control surface within the fluid, as a source of both mass and momentum. One such theory, due to Ffowcs Williams and Hawkings, is examined here in considerable detail, in order to provide a taste of this sort of analysis, and to relate this treatise to an established aeroacoustic theory. Theories of this type are rejected on the grounds that they provide a very misleading picture of the flow field being modelled here. In particular, the modelling of a moving surface as a momentum source is misleading because such sources automatically deposit vorticity in the fluid along their trajectories. In reality, vorticity is deposited only in the case that there is a net force on the moving object.

Part of the problem with treatments such as Ffowcs Williams' and Hawkings' is that they work with the pressure field only. The flow regime of interest here includes incompressible flow as an important limiting case, since the concepts of wing and propeller theory were developed for this limit. In incompressible flow, the
pressure field is nonlinear; the pressure satisfies a fundamentally different equation in incompressible flow theory than in linearized acoustics. However, the velocity potential is linear both in acoustics and in incompressible flow theory. For this reason, it makes much more sense to formulate a theory which must be applicable to both cases in terms of the velocity potential. An examination of the conditions under which the velocity potential satisfies the acoustic wave equation shows that the approximations made do not exclude the case of the flow around a typical light aircraft propeller.

A generalization of Kirchhoff's formula to the case of a moving boundary surface is necessary. The Kirchhoff equation can be used to advantage in the analysis of problems in which the displacement of the boundary surface is negligible. The required generalization to the case in which the motion of the boundary surface is not negligible was made many years ago by Morgans. Here, a powerful set of mathematical tools will be introduced in order to greatly simplify the derivation of Morgans' result. Neither the tools nor the finished product is original, but I believe that the construction technique employed here is novel. Also, Morgans' result will be extended to the case in which new boundary surface is created at an edge, in addition to the motion of the boundary after its creation. The process of boundary creation is important for propeller flow, since the vortex shedding process is most naturally viewed as the creation of a boundary surface. Namely, the vortex sheet itself must be regarded as a boundary of the region of the fluid where the velocity potential has meaning.

The remainder of the thesis shows how this general aeroacoustic theory can be used to compute the sound due to a light aircraft propeller. Theoretically, the equations of the standard treatments of propeller sound are recovered in the
limiting case of a lightly loaded propeller. However, when an attempt is made to allow for the first order effects of finite aerodynamic loading, the present theory begins to deviate from theories which insist that the propeller blades can be modelled as sources of momentum (and of mass). There are two distinct mechanisms of aerodynamic drag: profile drag and induced drag. In the present analysis, these two drag mechanisms have distinguishable effects on the sound field.

The induced drag force is here modelled as a first order correction for the small residual velocity of the shed vortex sheet. Additional first order corrections, which are quite independent of the magnitude of the induced drag force, are predicted. These new terms correspond to the radial contraction of the shed vortex sheet. Unfortunately, it will be necessary to omit these new terms from the numerical work presented here, since the aerodynamic theory will not predict the contraction rate of the vortex sheet. However, the first order corrections for finite blade loading have a minimal effect on the sound field (although they have a considerable effect on the blade loading itself).

A computer model has been developed which estimates the pressure field anywhere in the fluid surrounding a rotating propeller on the basis of the quantities provided by the simple aerodynamic theory. This computation has been made with an eye toward minimizing the cost of the calculations, although I believe that as little accuracy as possible has been sacrificed. The resulting algorithm is an extremely efficient means of predicting propeller sound.

Finally, the wind tunnel experiment with quarter scale model light aircraft propellers is described in some detail. The experimentation was motivated by the desire to prove that our research group could invent a quieter propeller for light airplanes. However, no novel noise reduction strategies were identified, and the
progress toward the goal of a significantly quieter propeller was minimal. This is hardly surprizing; propeller noise reduction strategies have been intensively studied, off and on, for the past half century. Because of the close connection between the flow field responsible for the propeller thrust and the mechanisms responsible for propeller noise, measures to reduce the noise generally have an impact on performance. The cases in which this impact is not adverse were identified long ago. By increasing the number of blades, or by reducing their tip speed, striking reductions in the sound level can be achieved. Due to economic factors, our sponsors were not particularly interested in these well known, but expensive solutions. The economics of the tradeoffs between sound and performance have no place in a thesis on the physics of the problem.

After a thorough explanation of the experimental setup and the microphone data collection and reduction procedures, the measured propeller sound is compared to the predictions of the computer model. The agreement between the two is startlingly good, considering the number of approximations which go into the theory. The good agreement extends to all of the propellers tested and to most of the operating conditions and microphone locations where measurements were made. The one case in which the rotational noise differed significantly from the theoretical predictions involved low blade Reynolds numbers. Since the aerodynamic theory breaks down in this case, the discrepancy is considered well understood.

In order to make the comparison of measurement to theory, the periodic part of the microphone signal is extracted (i.e the part with the same period as the blade passage period). This so called rotational noise is the only component of the sound field predicted by the computer model. The residual aperiodic part of the sound is
briefly discussed. This component of the sound generally has a much smaller amplitude than the periodic part. However, in one situation, the aperiodic part of the sound had an amplitude approaching that of the periodic part.

In this case, the aperiodic part of the sound took the form of coherent bursts of high frequency sound which appeared whenever the tip of a propeller blade approached the microphone most rapidly in its revolution. These bursts undoubtedly represent the effects of a rather violent flow instability, perhaps transonic buffeting, but the mechanism of the sound production has not been definitely identified. The existence of such a phenomenon underscores the reliance of the sound prediction technique employed here on a qualitative understanding of the flow field. When the aerodynamic theory used to supply the boundary conditions for Morgans' equation cannot account for the actual flow field, the sound cannot be predicted. The beautiful mathematics of the general aeroacoustic theory can thus be applied to practical problems only when another, more empirical theory exists to provide the boundary values.
Chapter Two
Mechanisms of Aerodynamic Force

2.1 Introduction

Wings, and their close relatives propellers, are among the simplest and most useful tools ever devised. In many ways their apparent simplicity is illusory, and even today there remain a few unanswered questions. As in most other areas of fluid and plasma physics, wing theory is beset by flow instabilities and haunted by the spectre of turbulence. Obviously, for many engineering applications these problems with the theory make little difference. After all, birds do quite well without any theoretical understanding of wings whatsoever.

A wing is a device for efficiently transferring momentum to a fluid continuum. In this sense, the word "efficient" means that the force produced on the wing by the fluid is substantially perpendicular to the direction of its motion through the fluid. Wings operate only in high Reynolds number situations; that is, inertial forces in the fluid are considerably more important than viscous forces. (Indeed, this must be the case, since the total viscous force on an object is invariably substantially antiparallel to its direction of motion.) For the reader who is unfamiliar with practical aerodynamic terminology, a short glossary of terms and an explanatory figure (2-1) are provided. It should be borne in mind that these terms begin to lose any precise meaning when the wing geometry deviates greatly from that of a bird's wing.

In this chapter, the flow field associated with a lifting airfoil will be described. The structure of this flow field was first understood by Lanchester and Prandtl, and
Figure 2-1: Wing, Propeller, and Airfoil Geometry
Table 2-1: Glossary of Aerodynamic Terminology

**Airfoils**

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>camber line</td>
<td>The locus of centers of all circles which can be inscribed inside an airfoil</td>
</tr>
<tr>
<td>leading edge</td>
<td>The upstream end of the camber line</td>
</tr>
<tr>
<td>trailing edge</td>
<td>The downstream end of the camber line</td>
</tr>
<tr>
<td>chord (line)</td>
<td>The line segment connecting the leading and trailing edges</td>
</tr>
<tr>
<td>camber</td>
<td>The maximum distance of any point on the camber line from the chord line</td>
</tr>
<tr>
<td>attack angle ($\alpha$)</td>
<td>The angle between the chord line and the velocity of the airfoil</td>
</tr>
<tr>
<td>through the fluid</td>
<td></td>
</tr>
<tr>
<td>zero lift angle ($\alpha_0$)</td>
<td>The attack angle for which the airfoil produces no lift</td>
</tr>
<tr>
<td>thickness ($\varepsilon$)</td>
<td>The diameter of the largest circle which can be inscribed inside an airfoil</td>
</tr>
</tbody>
</table>

**Wings and Propellers**

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>sweep (angle)</td>
<td>Roughly speaking, the angle between the leading or trailing edge of a wing</td>
</tr>
<tr>
<td>and the plane perpendicular to its direction of motion</td>
<td></td>
</tr>
<tr>
<td>aspect ratio</td>
<td>Reciprocal of the wing area expressed in units of (span)$^2$</td>
</tr>
<tr>
<td>(crudely, the ratio of span to chord)</td>
<td></td>
</tr>
<tr>
<td>solidity</td>
<td>The ratio of the projected area of the propeller blades to the area of the</td>
</tr>
<tr>
<td>disk</td>
<td></td>
</tr>
<tr>
<td>advance ratio</td>
<td>Ratio of the forward speed of a propeller to the rotational speed at its</td>
</tr>
<tr>
<td>tip</td>
<td></td>
</tr>
<tr>
<td>pitch</td>
<td>The distance a screw advances in one revolution (or sometimes in one radian)</td>
</tr>
</tbody>
</table>
has been described in numerous references on wing theory [Tietjens 1934, Karman 1935, Batchelor 1967a]. The present treatment will stress the irrotational nature of the flow field to a greater extent than in these standard references. Specifically, the vortex sheet shed behind a lifting wing will here be characterized by the jump in the velocity potential across it, rather than by the vorticity of the fluid in the sheet. The origins of the drag forces and their manifestations in the wake motion will also be described in detail.

The usual theory of thin airfoil sections will be shown to be an acoustic theory over a wide range of Mach numbers. That is, the equations of thin airfoil theory are simply the equations of linear acoustics. This airfoil theory will be used to devise a simple theory of propeller performance, which occupies the larger part of the chapter. This theory is presented here for the first time, although it is heavily based on the ideas of Prof. Larrabee [Larrabee 1979a]. The goal of the theory is to provide an estimate of the strength of the shed vortex sheet and the magnitude of the profile drag force, which will be necessary inputs to the acoustic theory developed in the following chapters. Since the experiment with which this theory is to be compared involves sound measurements made over a wide variety of propeller operating conditions, it was necessary to develop a computationally inexpensive technique for estimating the performance parameters. Thus, if it seems as if some approximations are simply the result of computational laziness, bear in mind that the technique being developed had to be applied to literally hundreds of different cases.

2.2 Fluid Flow Past a Wing
2.2.1 The Vortex Structure Associated with Lift

Joukowski recognized that lift force is intimately associated with a net circulation of fluid around an airfoil. In order for a wing or a propeller, which have finite span, to develop circulation around their airfoil sections, there must be vorticity in the bulk of the fluid. If the fluid were everywhere irrotational, there can be no circulation around any loop which can be continuously deformed to zero, as can a loop around any simply connected object. However, the vorticity in the flow field of a lifting airfoil is confined to a thin sheet containing only the fluid particles which have essentially come in contact with the surface of the airfoil.

The complete vortex structure of a lifting wing is shown in Fig. 2-2. Notice that all of the vortex lines are closed loops, as they must be. The segment of the loops which runs along the wing itself is called the "bound" vorticity. The bound vorticity is simply the circulation recognized by Joukowski, which must be present if there is to be lift. The segment of the loops which is parallel to the direction of motion of the wing is called the "shed" vorticity. The shed vorticity is associated with a discontinuity in the transverse component of the velocity in the wake of the wing. The discontinuity is purely a three dimensional phenomenon, since there is no transverse velocity in two dimensional flow. This fundamental difference between two and three dimensional flow past a lifting wing was the Rosetta stone of modern airfoil theory discovered by Lanchester and Prandtl. The final segment of the vortex loops is called the "startup" vortex. This vortex is deposited in the fluid at the place where the motion of the wing through the fluid was initiated. As the wing moves forward, the position of the startup vortex remains essentially fixed, so that the entire vortex structure continuously lengthens.

In the theory of incompressible, irrotational flow, a vortex loop is associated with
Figure 2-2: Vortex Structure of a Lifting Wing
the mechanical impulse required to initiate the flow. Specifically, a force must be applied to the fluid along the direction of the dipole moment of the vortex loop (computed in a manner analogous to the dipole moment of a current loop). The force must act on the fluid long enough to produce an impulse equal in magnitude to this dipole moment [Milne-Thompson 1968, Batchelor 1967b]. The relationship between the dipole moment of a vortex loop and the impulse necessary to initiate the fluid flow described by the loop provides a beautiful interpretation of the vortex sheet shed by a lifting wing:

As the wing advances, the vortex sheet behind it lengthens, but the circulations around individual vortex lines in this sheet do not change as long as the circulation around the wing itself remains constant. Thus, if the wing is providing a steady lift force, the dipole moment of the shed vortex sheet is directly proportional to the time which has elapsed since the motion was initiated. The total momentum of the fluid which is set in motion by the wing therefore increases linearly with time. This rate of increase of momentum must be simply the lift force on the wing. Wings obtain lift in exactly the same way that rockets obtain thrust, only in the case of a wing the reaction mass is the fluid which surrounds it.

Strictly speaking, the momentum transferred to the fluid by the lift force on the wing need not actually reside in the motion of particles of fluid. In particular, if the fluid has solid boundaries (other than the surface of the wing), then some or all of the momentum associated with the lift force may be transferred to these boundaries by the pressure of the fluid. Nevertheless, it is clear that the wing must first transfer momentum to the fluid before the fluid can exert a net force on its boundaries. In this sense, the reaction to the lift force may always be regarded as the downward momentum deposited in the fluid by the motion of the wing.
The picture of a lifting wing continuously creating vorticity in its wake will be the cornerstone of the aeroacoustic theory developed below. The downward motion of the fluid connected with this vortex structure is usually very much slower than the forward motion of the wing. Indeed, this fact is related to the high efficiency of a wing. Less energy is expended in transferring a given momentum to a large mass than to a smaller mass; the larger mass will end up with a slower velocity. The amount of fluid which takes part in the downward motion is determined by the scale of the shed vortex loops. Roughly speaking, all of the fluid contained in a cylinder whose axis is the centerline of the shed vortex sheet, and whose diameter is the wingspan will take part in this downward motion.

The reaction mass which the wing uses to generate the lift force can therefore be maximized in two ways. First, and most obvious, the wingspan can be increased. Second, the forward speed of the wing can be increased. Efficient wings therefore have very large spans, and move at high speeds. Sailplanes and albatrosses both attest to the truth of these two design goals. The fact that practical wings and propellers cause very slow residual motions in very large volumes of fluid, rather than vice versa, is very important to the present work. Even though the wing or propeller may cut through the fluid quite rapidly, the resulting motion of the fluid is as gentle as its designers could manage. For this reason, the linearized equations of motion are valid in the fluid, even though the fluid boundaries may be moving at near sonic speed.

2.2.2 Discontinuity in the Velocity Potential

In the aeroacoustic theory, it will be more convenient to describe the shed vortex sheet in terms of a property of the external irrotational flow field, instead of in terms
of the vorticity of the fluid which constitutes the sheet. There exists a single valued velocity potential $\Phi$ everywhere in space except for inside the vortex sheet itself. Since both circulation and velocity potential are defined as line integrals of the velocity field, it is clear that there must be a simple description of a vortex sheet in terms of the velocity potential. Indeed, the velocity potential must have a discontinuity across a vortex sheet, of magnitude equal to the circulation associated with any given point on the surface of the sheet. Vortex lines are simply lines of constant jump in the velocity potential in this picture.

The advantages of working with the jump in velocity potential, instead of with the vorticity, are manifold. It is somewhat surprising, therefore, that none of the standard references on wing theory adopt this point of view. The biggest advantage from my point of view is that it is easy to write down the compressible flow field associated with a moving sheet of discontinuity in the velocity potential, while it is difficult to write down the compressible flow field associated with a moving vortex line. (In fact, the latter problem has thwarted my most concerted efforts to solve it.) However, there are several advantages in ordinary aerodynamic theory. For example, it is much easier for me to imagine a jump in the velocity potential at the trailing edge of an airfoil than to understand that there is a circulation around the airfoil.

The formulation of the theory in terms of the jump in velocity potential emphasizes the irrotational flow field outside the vortex sheet. An important consequence of this point of view is that the entire shed vortex sheet becomes a boundary, rather than just the surface of the wing itself. In a very real sense, this sheet is a boundary surface; it is a part of the boundary of the region of irrotational flow. The precise location of the sheet, and the boundary conditions which should
be applied there, embody the most difficult aspects of the flow field of a wing.
Therefore, the explicit recognition of the vortex sheet as a part of the fluid boundary
automatically divides the problem into a simple part (determining the linear,
irrotational flow field outside the sheet), and an intractable part (determining the
location and strength of the vortex sheet). The linear part of the problem can be
solved by the elegant mathematics of acoustics. The more difficult part of the
problem falls into the domain of expertise of the aerodynamicist.

2.2.3 Induced Drag

In addition to the lift force, an airfoil also experiences drag forces. In particular,
the wake continuously created by a wing persists essentially indefinitely after the
wing passes. The total kinetic energy of the fluid in this wake, like its total
momentum, is directly proportional to the total length of the wake. The wing must
therefore be doing work on the fluid continuously as it advances; the work per unit
distance it moves forward represents the drag force. Independent of the details of
the rotational flow inside the vortex sheet, the large scale irrotational motion which
is the reaction to the lift force has a kinetic energy associated with it. The resulting
drag force is called the induced drag, since it is an unavoidable consequence of the
fluid motion which must be present to generate lift.

Just as the downward momentum of the fluid in the wake is associated with the
lift force, it must be possible to find the fluid momentum which is the reaction to the
induced drag force. Indeed, the vortex sheet behind the wing must drift downward
along with the fluid behind the wing, according to the Helmholtz equation for the
convection of vorticity [Batchelor 1967c]. This slow downward drift of the vortex
sheet after it has been created implies that the vortex sheet actually lies at a small
Figure 2.3: Origin of the Induced Drag

- Trajectory of wing
- Shed vortex sheet drifts downward slowly after it is created
- Downward component associated with reaction to lift force
- Dipole moment of vortex sheet
- Forward component associated with reaction to induced drag force
- Static pressure in the wake tends to be higher than ambient pressure; this also contributes to induced drag
angle below the trajectory of the wing, as shown in Fig. 2-3. The dipole moment of the sheet, and thus the momentum of the fluid, therefore has a component parallel to the flight direction of the wing.

Although it is tempting to believe that this forward momentum is the fluid reaction to the induced drag force, this is not a complete picture. The forward momentum is invariably larger than the impulse provided by the induced drag force, which at first seems to violate momentum conservation. The apparent paradox has been resolved in a lovely article by Sears [Sears 1974]. He points out that the average static pressure of the fluid involved in the irrotational downward motion behind the wing exceeds the ambient pressure of the fluid in front of the wing. This higher static pressure behind the wing results in a continual transfer of momentum to the bulk of the fluid in the direction antiparallel to the direction of flight, which partially compensates for the forward component of the momentum associated with the vortex structure of the wake.

2.2.4 Profile Drag

While the induced drag is associated with the kinetic energy of the large scale irrotational fluid motion external to the shed vortex sheet, the kinetic energy of the rotational flow inside this sheet gives rise to the so called profile drag. The thin layer of fluid which passes through the boundary layer at the surface of the wing tends to be dragged along with the wing. Unlike both the lift force and the induced drag force, the profile drag force is manifested in the wake only by the motion of the fluid in the thin layer which came into direct contact with the surface of the wing.

Profile drag has its origins in the no slip boundary condition at the surface of the wing. At the leading edge of the wing, the boundary layer is a laminar shear layer.
As the fluid accelerates away from the stagnation point at the leading edge, the shear layer soon becomes turbulent. This is actually desireable aerodynamically, since a turbulent boundary layer does not separate from the surface as quickly as a laminar one. Separation leads to a very adverse effect called pressure drag, which results from the fact that the high pressure at the leading edge stagnation point cannot be offset by a high pressure at a trailing edge stagnation point if the flow separates before reaching the trailing edge. In addition to the pressure drag mechanism, viscosity and turbulent momentum transport give rise to shear forces at the surface of the wing.

All of these mechanisms may be regarded as the diffusion of vorticity from the surface of the wing into the flow. The vorticity which diffuses away from the surface is convected downstream by the flow past the wing. The vorticity "shed" by this mechanism is perpendicular to the shed vorticity associated with the lift force. Furthermore, the vorticity which diffuses into the flow from the pressure side of the wing is of opposite sign to the vorticity which originates on the suction side. Such a distribution of vorticity gives rise to a jetlike layer of fluid behind the wing, where the direction of the jet is the same as the direction of motion of the wing. This layer is sometimes called the momentum wake. For the high Reynolds number situations in which wings normally operate, the momentum wake remains quite thin relative to the dimensions of the wing until the wing has moved forward a large distance.

The fluid in the momentum wake coincides with the fluid in the shed vortex sheet associated with the lift force. In the formulation of the flow problem advocated above, the fluid in the momentum wake therefore lies interior to the vortex sheet, which is regarded as a portion of the fluid boundary. This is perfectly natural; since the fluid in the momentum wake participates in a shear flow, it should
not be included in the irrotational flow exterior to the vortex sheet. The only complication is that the motion of the fluid inside this sheet associated with the profile drag results in an apparent violation of mass conservation in the irrotational external flow. Later, the acoustic source corresponding to the profile drag will be found to be equivalent to a mass source.

Finally, it should be noted that the profile drag of a blunt object is very much larger than the profile drag of a streamlined object such as an airfoil. A blunt object typically sheds a very thick, turbulent momentum wake, as does a stalled airfoil. Acoustically, the high degree of turbulence in such a wake can produce a considerable stochastic sound, which cannot be handled by the techniques used in this thesis. In all cases in which the theory developed here is valid, the profile must be relatively small, and the momentum wake correspondingly thin.

2.2.5 The Kutta Condition

It is worthwhile to mention that the fact that the momentum wake associated with profile drag coincides with the shed vortex sheet associated with lift is not simply a coincidence. In fact, the transverse vorticity of the momentum wake is instrumental in maintaining the circulation around a wing. The Kutta condition, which states that there can be no flow around the sharp trailing edge of an airfoil, determines the magnitude of the circulation around an airfoil section. Several excellent descriptions of the physical mechanisms underlying this boundary condition exist in the literature [Batchelor 1967d, Tietjens 1934]. However, none of these descriptions stress the relationship between the circulation and the vorticity shed in the profile drag mechanism.

It is convenient to view the airfoil in a coordinate system in which it is stationary
and the fluid streams past. The flow field is irrotational everywhere except in the boundary layer and in the thin momentum wake. According to the Helmholtz equation, the only way for the circulation around the loop shown in Fig. 2-4 to change is for a net vorticity in the spanwise direction to be convected through the part of the loop cut by the momentum wake. Conversely, if a net spanwise vorticity is convected through this loop, the circulation around the loop must change.

For one unique value of the circulation, the vorticity shed from the pressure side of the airfoil will be precisely equal and opposite to the vorticity shed from the suction side. If there were slightly more clockwise circulation than this equilibrium value, then the flow velocity over the pressure side would decrease slightly, while the flow velocity over the suction side increased slightly. This would result in less counterclockwise vorticity produced by the pressure surface and more clockwise vorticity produced by the suction surface. A net clockwise circulation would therefore be convected away from the trailing edge of the airfoil, which tends to restore the circulation around the loop to its equilibrium value. That is, the equilibrium is invariably a stable one.

The sharp trailing edge of an airfoil assures that the equilibrium value of the circulation occurs when there is no flow around this sharp edge. Even a small flow around so sharp a corner would generate vorticity very rapidly, and the convection of this vorticity downstream would quickly restore the circulation to its equilibrium value. This explains why the Kutta condition is satisfied for the flow past airfoils.

The Kutta condition is therefore equivalent to the statement that the transverse component of the vorticity shed from the upper surface of an airfoil must be equal and opposite to the vorticity shed from the lower surface. However, the latter statement of this condition for stable flow is not restricted to objects which have a
Vorticity diffuses into the boundary layer from the surface of the airfoil, and is convected downstream. In order for the circulation to be independent of time, the clockwise vorticity convected away from the suction side must equal the counterclockwise vorticity convected away from the pressure side.
well defined, sharp trailing edge, as is the usual statement of the Kutta condition. Blunt objects have a much milder "restoring force" to return them to their equilibrium value of circulation than do airfoils, due to the lack of a sharp trailing edge. Indeed, the circulation around a blunt object generally undergoes large, spontaneous oscillations about its equilibrium value. These oscillations correspond to the shedding of a Karman vortex street behind such an object. Spontaneous oscillations probably exist in the case of airfoil sections, but they are very much smaller in amplitude and are difficult to observe experimentally [Paterson 1973, Tam 1974].

2.2.6 The Flow Field of a Lifting Wing

The shape of the shed vortex sheet is determined in principle by the Helmholtz equation for the propagation of vorticity. Since vorticity is convected with the fluid, no fluid can penetrate the vortex sheet (the sheet simply moves). However, this sheet has no rigidity whatsoever. Since the sheet is nearly massless, the static pressure must be continuous across it. These conditions are probably sufficient to determine the shape the sheet must adopt, although there are significant theoretical problems at the edges of the sheet in such a model. This is a moot point, since the instability of vortex sheets is intimately related to the unsolved problem of turbulence.

Qualitatively, the vortex sheet shed by a lifting wing tends to roll up into thick line vortices called tip vortices [Donaldson 1975]. This roll up process becomes more and more rapid the stronger the vorticity in the sheet, that is, the more heavily the wing is loaded. Since efficient wings are rather lightly loaded, the roll up process is not complete until the wing has moved forward on the order of a
wingspan. This roll up process will be ignored entirely in the following theoretical development. Empirically, this omission does not seem to have a noticeable effect on the prediction of wing or propeller loads, probably because the spanwise loading distribution of a typical wing is basically a structureless bump. When you've seen one structureless bump, you've seen them all.

The complete flow field of a lifting wing which has been qualitatively described in this section is shown in Fig. 2-5. The large scale fluid motion outside the thin vortex sheet is responsible for the lift and induced drag forces. It is an irrotational flow, but the vortex sheet must be regarded as a part of the boundary surface for this flow, in addition to the solid boundary presented by the surface of the wing itself. The fluid inside this vortex sheet has the character of a jet which moves along the sheet in the direction of flight of the wing. This motion is associated with the profile drag force. The startup vortex, which forms when the motion of the wing begins, closes the large vortex loops associated with the reaction to the lift force.

2.3 Airfoil Theory

2.3.1 The Two Dimensional Low Mach Number Case

Much of modern practical aerodynamics is based on an understanding of the behavior of airfoil sections in two dimensional, low Mach number flows. At sufficiently high Reynolds numbers, this incompressible flow field depends only on the shape of the airfoil section. All pressures at the surface of an airfoil under these conditions are simply proportional to the dynamic pressure $\rho U^2/2$, where $U$ is the speed of the airfoil and $\rho$ is the density of the fluid. The constant of proportionality is called the pressure coefficient. Forces are proportional to the product of the
Figure 2.5: Complete Flow Field of a Lifting Wing

- Startup vortex
- Jetlike flow inside momentum wake is associated with profile drag
- Circulation around wing
- Irrotational flow external to vortex sheet is associated with lift and with induced drag
- Forward velocity of wing
dynamic pressure and a characteristic area, which is invariably taken to be the projected area of the airfoil, \( s_X \), where \( s \) is the span and \( X \) is the chord. The constants of proportionality are called the lift coefficient, \( C_L \), in the case of the lift force, and the drag coefficient, \( C_D \), in the case of the drag force.

These coefficients have been measured and/or computed for a large number of standard airfoil shapes. Airfoil sections are always designed as a "series" of related shapes which differ from each other in camber and in thickness. Designers can then specify an airfoil shape by giving only these two numbers, and the designation of the airfoil series to be employed. The chordlength \( X \) and the angle of attack \( \alpha \) then complete the geometrical information necessary to determine the loads on the airfoil section. The bible of airfoil section properties is the book by Abbott and von Doenhoff [Abbott 1949].

The lift and drag (that is, profile drag) coefficients for a given airfoil section depend only on the attack angle of the section, provided that the Reynolds number is sufficiently large. As a reasonable engineering approximation, the lift coefficient may be assumed to increase linearly with attack angle. This line is determined by two parameters: the lift curve slope, \( m \), and the angle of zero lift, \( \alpha_0 \). At sufficiently large or small attack angles, any airfoil section will eventually stall. This will be modelled here by simply cutting off the lift coefficient at a maximum value, \( (C_L)_X \), and at a minimum value, \( (C_L)_n \), as shown in Fig. 2-6. This treatment has the virtue of simplicity, if not that of great accuracy.

The drag coefficient is frequently given as a function of the lift coefficient instead of as a function of the attack angle. I have adopted this practice here, although I regret this choice somewhat. Specifically, in all of the calculations which will be performed here, I have assumed that the drag coefficient is a piecewise linear
Figure 2-6: Adopted Airfoil Section Characteristics
function of the lift coefficient, as indicated in Fig. II.6. The drag coefficient increases dramatically at values of the lift coefficient corresponding to stall. The treatment of stall is not intended to be particularly accurate; it is included to prevent a computer from mindlessly attributing huge lift coefficients and small drag coefficients to an airfoil section which happens to be operating at a very high angle of attack.

Although this model of airfoil characteristics is rather crude, the actual section properties are fairly sensitive to the exact geometry of the sections, and to the roughness of the surface. A more accurate treatment would not necessarily be a better model of an real airfoil for these reasons.

2.3.2 Finite Compressibility Effects

The velocity potential, \( \Phi \), external to the boundary layer in the low Mach number case satisfies Laplace's equation. When the airfoil moves at higher Mach numbers, compressibility of the fluid will come into play. This compressible flow is described by a velocity potential which satisfies the acoustic wave equation instead of Laplace's equation. In this thesis, the coordinate frame in which the fluid is at rest and the airfoil is in motion will usually be adopted. Assuming that the airfoil moves at a constant speed \( U \) in the \( x \) direction, the velocity potential must satisfy the Galilean symmetry \( \Phi = \Phi(x-Ut,y) \). Therefore, the wave equation becomes

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{1 - M^2} \frac{\partial^2 \Phi}{\partial y^2} = 0
\]

(2-1)

where \( M = U/c \) is the Mach number.

This equation may be reduced to Laplace's equation by simply contracting the \( y \)
coordinate by a factor of $\sqrt{1-M^2}$:

$$\frac{\partial^2 \Phi}{\partial \bar{x}^2} + \frac{\partial^2 \Phi}{\partial \bar{y}^2} = 0$$ (2-2)

$$(x, y) \Rightarrow (x, y) = (x, y \sqrt{1-M^2})$$ (2-3)

At first glance, it may appear that the flow past a section at high Mach number is similar to the incompressible flow past a thinner airfoil by virtue of (2-2) and (2-3). Alas, this is not the case, since the boundary conditions for the high Mach number flow do not reduce to the boundary conditions for the incompressible flow when expressed in $(\bar{x}, \bar{y})$ coordinates. Specifically, the no penetration condition requires that the lines of constant velocity potential in the coordinate frame in which the airfoil is at rest and the fluid is in motion be perpendicular to the surface of the airfoil. As indicated in Fig. 2-7, the velocity potential in the transformed coordinates $(\bar{x}, \bar{y})$ no longer satisfies this boundary condition, as would the velocity potential corresponding to the thinner airfoil section in incompressible flow.

While the preceding argument demonstrates the impossibility of an exact similarity between high Mach number flow and incompressible flow, an important special case exists. If the airfoil is exceedingly thin, then the boundary condition at its surface may be applied along the line $y = 0$ (or $\bar{y} = 0$) without appreciable loss of accuracy. This is one of the standard and more ingenious of the approximations commonly made in thin airfoil theory. In this case, it allows the transverse dimension of the airfoil itself to be scaled independently of the transverse coordinate. An examination of the geometry of the lines of constant velocity potential as they must appear near the airfoil surface in the transformed $(\bar{x}, \bar{y})$ coordinates shows that the transverse dimensions of the airfoil should be scaled up by the factor of $1/\sqrt{1-M^2}$. This reverse scaling of the airfoil dimensions relative to
\[ \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} = 0 \]

plus no penetration BC

\[ \frac{\partial \Phi}{\partial \tilde{x}} + \frac{\partial \Phi}{\partial \tilde{y}} = 0 \]

lines of constant \( \Phi \)

BUT boundary condition is NOT the no penetration BC
(lines of constant \( \Phi \) are no longer perpendicular to the boundary)

However, if boundary conditions are applied at \( y = \tilde{y} = 0 \):

\[ \tilde{x} = x \]
\[ \tilde{y} = y/\sqrt{1-M^2} \]

transformed boundary

transformed lines of constant \( \Phi \)
are perpendicular to a boundary which is everywhere thicker by \( \sqrt{1-M^2} \) than the actual boundary in \((x,y)\) coordinates

**Figure 2-7: Geometry for Transonic Similarity Rule**
the coordinate space will result in the same boundary conditions as those of the incompressible flow past an airfoil thicker than the original by a factor of $1/\sqrt{1-M^2}$.

Thus, the compressible flow past a thin airfoil at Mach number $M$ is approximately similar to the incompressible flow past a section which is everywhere $1/\sqrt{1-M^2}$ thicker than the original. In particular, the camber and the angle of attack in the similar incompressible flow are larger than those in the real, compressible flow by this factor. Since the lift coefficient is roughly bilinear in these two parameters, it, too, will be $1/\sqrt{1-M^2}$ larger at a given attack angle than it would be under conditions of incompressible flow. This scaling of the lift coefficient with Mach number is called the Prandtl-Glauert rule. Since it is the attack angle, and not the lift coefficient, which is actually being scaled, the value of the minimum and maximum lift coefficients (corresponding to stall) should be essentially unaffected by the Mach number.

In addition to the camber and attack angle, the effective thickness of the airfoil is also increased by $1/\sqrt{1-M^2}$. This should affect the drag coefficient, and possibly the minimum and maximum values of the lift coefficient at stall. However, these effects of Mach number scaling will be ignored in the present theory. Both the drag coefficient and the stall properties of an airfoil are more sensitive to unguessable factors such as the surface roughness. Moreover, the variation of the drag coefficient with thickness is parabolic, unlike the more important linear variation of the lift coefficient with attack angle or camber. Furthermore, considerable guesswork has gone into the drag coefficients which have been used in the propeller calculations reported below. Including Mach number scaling in these drag coefficients would simply add a new guess.

In summary, the only change which will be made here to allow for the effects of
finite compressibility will be to increase the lift curve slope from its incompressible flow value, \( m \), to a new value, \( m' \), given by

\[
m' = m / \sqrt{1-M^2}
\] (2-4)

The angle of zero lift, the minimum and maximum values of the lift coefficient, and the drag coefficient as a function of the lift coefficient will all be assumed to be unaffected by the Mach number at which an airfoil section operates. A more complete discussion of the Prandtl-Glauert rule and its limitations may be found in Liepmann and Roshko [Liepmann 1957].

The equation (2-1) used to describe the compressible flow past a high speed airfoil is merely a disguised version of the acoustic wave equation. Indeed, Laplace's equation for incompressible fluid flow is a limiting case of the wave equation. Aerodynamic theory is therefore based on the wave equation up to speeds approaching sonic speed, at least. This fact is one of the central premises of this thesis. The role of acoustics in thin airfoil theory is even more striking: Since all the equations of thin airfoil theory are linearized, the pressure field as well as the velocity field satisfies the acoustic wave equation.

It is interesting to contemplate the fact that the lift force on a thin airfoil and the reaction force on a loudspeaker cone have the same origins, mathematically. In particular, the pressure is given in both cases by the relation \( p = \rho \partial \Phi / \partial t \). Unfortunately, since airfoil theory is usually presented from the frame of reference in which the airfoil is stationary, this equation is usually disguised in the equivalent form \( p = -\rho U \partial \Phi / \partial x \).
2.3.3 Finite Aspect Ratio Effects

The discussion of this section began with the two dimensional, incompressible flow past an airfoil section. The restriction of incompressible flow was then relaxed; the restriction of two dimensional flow will now be relaxed. Only the restriction that the Reynolds number of the flow be so large that the exact value of the fluid viscosity is irrelevent will remain. If the aspect ratio of a wing is large, but finite, a simple engineering approximation can be used to allow for the most important differences between two and three dimensional flows. The simplest version of this approximation, which is called lifting line theory, will now be discussed [Milne-Thompson 1952, Karman 1935].

The characeristics of an airfoil section measured in a two dimensional wind tunnel, and those of a three dimensional wing of identical cross section but finite aspect ratio are considerably different. The most important difference is that the lift curve slope decreases with decreasing aspect ratio. Lifting line theory was evolved in order to explain this behavior. A second difference between the two and three dimensional flows is the presence of the induced drag in the three dimensional case.

If the aspect ratio of a wing is large, almost all of the airfoil sections along its span will operate in a nearly two dimensional flow field. However, in the three dimensional case, the large scale vortex structure of the wake will give rise to a small perturbation in the flow past each airfoil section across the span. The greater the aspect ratio, the weaker the fluid motion associated with this shed vortex sheet, and the less the flow past the individual airfoil sections will be perturbed from its two dimensional value. The first correction which can be made to the limit of infinite aspect ratio consists of assuming that the airfoil sections along the wing retain their two dimensional section properties, but that their effective velocity through the fluid
is not equal to the velocity of the wing through the fluid.

The small corrections to the geometrical velocity of each section through the fluid are equal to the velocity "induced" at the location of the section by the shed vortex sheet. Since the magnitude of the circulation associated with this sheet is proportional to the chordlength of the wing, while the vortex loops comprising the sheet are spread out by the order of the wingspan, the "induced" velocities are guaranteed to be smaller the the forward velocity of the wing by a factor of the order of the advance ratio. For the sake of consistency with the assumption that the airfoil section properties are the same as in two dimensional flow, the effective velocity of the sections should be computed only to first order in the induced velocities.

The basic geometry of the wake of a lifting airfoil indicates that the velocity field associated with the shed vortex loops will be primarily in the downward direction, just as the fluid momentum associated with the loops is downward. The primary effect of the induced velocity is therefore to make the effective angle of attack, $\alpha'$, less than the geometrical angle of attack, $\alpha$ (see Fig. 2-8). For complicated wake geometries, it is also possible that the effective speed, $U'$, of the section would differ from the speed of the wing, $U$. If the downward component of the induced velocity is $u_\perp$, and its component along the direction of flight is $u_\parallel$, then to first order,

$$\alpha' = \alpha - \frac{u_\perp}{U}$$
$$U' = U - u_\parallel \tag{2-5}$$

The reduction in the attack angle by the induced velocity is proportional to $u_\perp$, which is proportional to the strength of the shed vortex sheet. This is in turn proportional to the angle of attack, measured relative to the angle of zero lift. The first equation of (2-5) therefore represents an effective change in the attack angle, relative to the angle of zero lift, by a constant factor. That is, the effective lift curve
In the reference frame comoving with one airfoil section:

lift force is perpendicular to effective velocity; component along the direction of motion represents induced drag

\[ \alpha' = \alpha - \alpha_\perp = \text{effective attack angle} \]

\[ U - u_\parallel \]

\[ \alpha_\perp = \frac{U - u_\parallel}{U} \]

**Figure 2-8: The Induced Velocity**
slope of a wing of finite aspect ratio will be lower than the two dimensional section properties would suggest, exactly as is observed.

Furthermore, the difference between the geometrical direction of motion of the wing and the effective direction of motion leads to a small component of the two dimensional lift force of each airfoil section along its direction of motion. A more detailed analysis of the magnitude of this component shows that it can be identified with the induced drag force. Thus, the lifting line theory successfully accounts for the two major differences between two and three dimensional flow past airfoils.

2.4 A Theory of Propeller Performance

2.4.1 Statement of the Problem

A synthesis of the airfoil theory discussed in section 2.3 with the wake structure described in section 2.2 would produce a serviceable theory of wings. Instead, these tools will be used here to build a theory of propellers, which are more interesting acoustically than wings. For simplicity, only the case in which the axis of rotation of the propeller coincides with its direction of forward motion will be considered. Numerous fairly gross approximations will be introduced; however, the problem will be approached in such a way that the impact of these approximations is minimized. The eventual goal of this thesis is the computation of the sound produced by a propeller, rather than the detailed description of the flow field in the immediate vicinity of the propeller blades.

Nevertheless, a detailed, but qualitative picture of the flow field of a propeller will now be given. This description is intended to serve as a template to which the
actual quantitative methods introduced below can be compared. From the
discussion of the flow field of a lifting wing, the flow field of a thrusting propeller
can be inferred to have the general structure illustrated in Fig. 2-9. Once again, the
central feature of this flow field is the vortex sheet shed as the circulation around the
propeller blades changes. This sheet must lie approximately along the screw shaped
surface swept out by the motion of the trailing edge of the blade.

As in the case of wings the actual motion of the vortex sheet is extremely
complex and of an unstable nature. Eventually, the sheet tends to roll up into a
helical "tip vortex", and a linear vortex along the propeller axis of opposite
circulation. If the propeller is lightly loaded, the roll up process will not be too
violent; in any event, the roll up process cannot be taken into account in the simple
theory discussed below. Here, as will be the rule in this thesis, the preferred frame
of reference will be the one in which the bulk of the fluid is stationary while the
propeller is in motion. A lightly loaded propeller will slice cleanly through the air,
leaving a wake in which the residual motion is very much slower than the motion of
the propeller itself. The roll up process takes place on this slower time scale, and
will therefore be ignored.

The vortex sheet once again coincides with a thin layer of fluid which passed
through the boundary layer at the surface of the blade. The fluid in this sheet forms
a screw shaped jet which tends to be dragged along the trajectory of the propeller.
The vortex sheet may therefore be regarded as a thin, but finite layer of fluid.
Outside this sheet, the fluid is irrotational; the sheet must be regarded as a portion
of the boundary of this irrotational flow field. Inside the sheet, the vorticity of the
fluid accounts not only for the jump in velocity potential across the sheet in the
external flow field, but also for the jetlike flow along the sheet associated with the

52
Figure 2-9: Complete Flow Field of a Thrusting Propeller
profile drag.

The burden of fitting this qualitative template rests primarily on the powerful acoustic theory which will be developed in the following two chapters. However, this acoustic theory is incapable of predicting the strength of the shed vortex sheet, or the magnitude of the profile drag, or of the induced drag. These three parameters enter the acoustic theory as boundary conditions at the non-physical boundary surface: the shed vortex sheet. These parameters must be estimated by the much less mathematically perfect, but no less elegant, techniques outlined in this chapter. For example, a considerable portion of the nasty nonlinearity of fluid mechanics is contained in the residual motion of the vortex sheet, which cannot possibly be determined within the confines of linear acoustics. However, a part of the effect of this residual motion can be taken into account in the aerodynamic theory outlined below.

This aerodynamic theory of propellers is an original one, although it has been heavily influenced by the ideas of Prof. Larrabee, whose work is in turn heavily based on the classical theories of Betz [Betz 1919], Glauert [Glauert 1935], Goldstein [Goldstein 1929], and Theodorsen [Theodorsen 1948]. It falls into the category of a lifting line theory, as described above. The theory is a very simple one; the emphasis in its development was on providing a theory involving the least possible amount of computation consistent with making a reasonably accurate estimate of the parameters required by the acoustic theory. The analysis will begin with the exact solution of an oversimplified propeller, and corrections for the worst of the simplifications will be subsequently added.

In propeller theory, the terminology differs slightly from wing theory. Obviously, the term "blade" replaces the term "wing". Also, instead of speaking of
the aspect ratio of a propeller blade, it is customary to speak of its solidity. And to avoid geometrical confusion, the concept of the advance ratio of a propeller often replaces that of the attack angle of a wing. Notice that these two propeller concepts bear an inverse relationship to their wing counterparts: A low solidity means a high aspect ratio, while decreasing the advance ratio corresponds to increasing the attack angle.

2.4.2 The Limit of Zero Solidity

According to the airfoil theory of section 2.3, the only difference between the operation of an airfoil in two dimensions and its operation in three dimensions is the effect of the induced velocity, as expressed by (2-5). If the chord of a propeller blade is exceedingly small relative to its tip radius, and if the total number of blades is not too large, then the induced velocities are guaranteed to be quite small. Thus, in the limit of zero solidity the induced velocities may be ignored altogether, which allows the strength of the shed vortex sheet to be easily computed from the blade geometry and the two dimensional characteristics of its airfoil sections.

In this limit, the residual motion of the shed vortex sheet is completely negligible. It remains precisely in the screw shaped surface traced out by the trailing edges of the blades. Furthermore, the jump in velocity potential across the sheet will be constant along the helical lines traced out by each point on the trailing edge of each blade. The value of this jump in velocity potential is simply equal to the circulation, \( \Gamma(r) \), around the airfoil section located a distance \( r \) from the propeller axis. According to Joukowsky's theorem, the lift force per unit span is \( \rho U \Gamma \), where \( \rho \) is the fluid density and \( U \) is the speed of the airfoil section at radius \( r \). If the forward speed of the propeller is \( V \), and its angular speed is \( \Omega \), then \( U = \sqrt{V^2 + (\Omega r)^2} \). In
terms of the lift coefficient, \( C_L \),

\[
\Gamma = C_L \frac{U \chi}{2} \quad (2-6)
\]

The lift coefficient itself is given by

\[
C_L = \begin{cases} 
(C_L)_x, & \text{if } (C_L)_x < m'(\alpha - \alpha_0) \\
 m'(\alpha - \alpha_0), & \text{if } (C_L)_n > m'(\alpha - \alpha_0)
\end{cases} \quad (2-7)
\]

Here, the lift curve slope, \( m' \), is given by (2-4), with \( M = \frac{U}{c} \). The geometrical attack angle, \( \alpha \), is given in terms of the angle \( \beta \) of the chordline of the airfoil section at radius \( r \) from the disk plane by the formula

\[
\alpha = \beta - \tan^{-1}(V/\Omega r) \quad (2-8)
\]

The blade angle \( \beta \) and the chordlength \( \chi \) are known as functions of \( r \) from the blade geometry. The lift curve parameters \( m, \alpha_0, (C_L)_x, \) and \( (C_L)_n \) are known as functions of \( r \) from the two-dimensional properties of the airfoil sections. Equations (2-4), (2-8), (2-7), and (2-6) thus serve to completely specify the strength of the shed vortex sheet in terms of the propeller geometry.

For a typical light aircraft propeller, the blind application of the equations of this zero solidity limit will result in an overestimate of the jump in velocity potential by roughly 30%. Obviously, some allowance must be made for the finite solidity if a reasonably accurate estimate of the performance of a real propeller is to be made. Since the ratio of the induced velocities to the geometrical velocities of the airfoil sections is of the order of the solidity, only first-order corrections in the solidity can be attempted using the techniques introduced above. Only two modifications to the equations of the zero solidity limit need be introduced for this purpose. According to (2-5), the necessary modifications are
\[
C_L = \begin{cases} 
(C_L)_x, & \text{if } (C_L)_x < m'(\alpha - \alpha_0) \\
(m'(\alpha - \alpha_0) & \text{if } (C_L)_n > m'(\alpha - \alpha_0)
\end{cases} \tag{2-9}
\]

\[
U = \sqrt{V^2 + (\Omega r)^2} - u_\parallel \tag{2-10}
\]

Equation (2-9) replaces (2-7).

Equations (2-4), (2-5), (2-8), (2-9), (2-10), and (2-6) completely determine the jump in velocity potential across the shed vortex sheet in terms of the known blade geometry, and the as yet unknown induced velocities \( u_\perp \) and \( u_\parallel \). The radial component of the induced velocity gives rise to a small effective sweep angle, which has no first order effect on the characteristics of an airfoil section. Since the induced velocity itself depends on the jump in the velocity potential, \( \Gamma \), equation (2-6) in reality contains \( \Gamma \) on both its right and left hand sides. The remainder of this chapter will be devoted to the solution of this equation. The equations which must be solved are summarized in Table 2-2.

### 2.4.3 Strategy for Estimating Induced Velocities

A consistent theory can include corrections to the limit of zero solidity only to first order in the induced velocity. Although the velocity induced by the vortex sheets causes a residual motion of the sheets themselves, the resulting displacement of the sheets is proportional to the induced velocity. Therefore, to include the change in shape of the sheets in the calculation of the induced velocity would be to attempt to compute the correction to the zero solidity limit to second order in the induced velocity. For the purposes of a first order correction, it is thus sufficient to assume that the vortex sheet remains motionless after it has been created. (There is one important case in which this assumption cannot be made. This case is treated in section 2.4.5.)
Table 2.2. Summary of Propeller Performance Equations

\[ \alpha = \beta - \tan^{-1}\left(\frac{v}{\Omega r}\right) \quad ; \text{geometrical attack angle} \]  
(2.8)

\[ \alpha' = \alpha - \frac{u_i}{\sqrt{v^2 + \Omega^2 r^2}} \quad ; \text{effective attack angle} \]  
(2.5)

\[ U = \sqrt{v^2 + \Omega^2 r^2} - u_i \quad ; \text{effective speed} \]  
(2.10)

\[ m' = m/\sqrt{1 - (U/c)^2} \quad ; \text{Prandtl-Glauert rule} \]  
(2.4)

\[
C_L = \begin{cases} 
(C_L)_X, & \text{if } (C_L)_X < m'(\alpha' - \alpha_0) \\
m'(\alpha' - \alpha_0), & \text{if } m'(\alpha' - \alpha_0) \\
(C_L)_n, & \text{if } (C_L)_n > m'(\alpha' - \alpha_0) \end{cases} \quad ; \text{lift curve} \]
(2.9)

\[ \Gamma = \frac{1}{2} C_L U \chi \quad ; \text{jump in velocity potential or circulation} \]  
(2.6)

Notes

1. Propeller geometry and two dimensional airfoil section properties directly specify \( \chi, \beta, m, \alpha', (C_L)_X, \) and \((C_L)_n\) as functions of the distance from the axis, \( r \).

2. The operating conditions determine the constants \( \Omega, V, \) and \( c \).

3. The induced velocities \( u_i(r) \) and \( u_n(r) \) are functionals of the circulation \( \Gamma(r) \). Thus, equation (2.6) is, in general, an integral equation for \( \Gamma(r) \).
For the time being, therefore, the geometry of the shed vortex sheets may be assumed to be precisely as it was in the zero solidity limit. In this picture, the propeller wake consists of a collection of helical vortex lines of various radii, but each having a pitch equal to the geometrical pitch of the screw motion of the propeller itself. The flow field of an individual helical vortex line is thus the fundamental building block with which the complete wake flow field can be constructed. This flow field is simplified considerably by including a "return path" for the circulation of the helical line by means of a second vortex line which lies along the propeller axis. The combination of these two vortex lines represent a screw shaped sheet with a constant discontinuity across its surface; the flow field of a sheet of discontinuity seems to invariably be less complex than the flow field of an individual vortex filament.

The case of a helical vortex pair of infinite length will be examined first. The problem of the semi-infinite pair will be greatly simplified by this preliminary. Since the residual fluid motions associated with this vortex pair are quite small, the Mach numbers of the flow will be small, and an incompressible analysis is sufficient. Ampere's law, combined with the screw symmetry of the problem, can be used to learn a great deal about the flow field of an infinite helical vortex line with a return vortex along its axis. The screw symmetry implies that all flow parameters are constant along helical lines of the same pitch as the source vortex, i.e. along lines where

\[ r = \text{constant} \quad \text{and} \]
\[ z - \theta V/\Omega = \text{constant}. \]

The circulation around all loops of the general type ABCD illustrated in Fig. 2-10 is zero, since neither of the two vortex lines thread any such loop. The screw symmetry implies immediately that the line integral of the velocity along the
Figure 2-10: Infinite Helical Vortex Pair

\[ \frac{EF}{FP} = \frac{EF}{\sqrt{\frac{2\pi r}{2\zeta}}} = \frac{2\pi r}{\sqrt{(2\zeta)^2 + (v_{z1})^2}} \]

or

\[ \frac{EF}{FP} = \frac{2\pi r}{\sqrt{1 + \left(\frac{2\zeta}{V}\right)^2}} \]
outward radial leg BC must be equal and opposite to the line integral along the inward radial leg DA. Therefore, the line integral along the helical leg AB must be equal and opposite to the line integral along the other helical leg CD. Thus, the helical component of the velocity must be inversely proportional to r, the distance from the axis. However, it is clear that the circulation around very large loops enclosing the axis will decay considerably more rapidly than $1/r$ for this configuration of vortex lines (but not for a single helical vortex line!). Therefore, the helical component of the velocity is everywhere zero.

This beautiful geometrical simplicity is the reason that the equal and opposite return vortex will always be included along the axis. One further consequence of the screw symmetry will be mentioned now. By considering the symmetry of the vortex pair under a 180 degree rotation about any radial line which passes through the helical vortex line, it may be shown that the radial component of the velocity must be identically zero for all points on such a radial line. The velocity induced by a screw shaped vortex sheet (generated by a straight radial line) with any radial distribution of vorticity is therefore exactly perpendicular to its surface. This is a remarkable property of helical vortex sheets.

Unfortunately, there is no simple formula for the single non-zero component of the velocity field due to a helical vortex pair, evaluated at the screw shaped surface containing the pair. However, it is a simple matter to use Ampere’s law to compute the mean value of this perpendicular component at any given distance r from the axis. The loop EFGH shown in Fig. 2-10 is suitable for this purpose. Once again, the line integrals along the legs FG and HE cancel. Furthermore, the line integral along FG can be made arbitrarily small by taking this leg to be sufficiently distant from the axis. Provided that the points E and F are equivalent under the screw
symmetry, the line integral along this leg will simply be the product of the mean value of the velocity component along the EF direction with the length of EF. Meanwhile, the circulation around the entire loop, which is the same as the line integral of the velocity along EF, is simply equal to the circulation, \( d\Gamma \), of the vortex lines if the helical line threads the loop \((r < r')\), or to zero if the helical vortex does not thread the loop \((r > r')\).

The screw symmetry requires that the mean value of the radial component of the velocity at any given distance from the axis be zero. These findings about the flow field of an infinite helical vortex pair may be summarized as follows:

\[
\begin{align*}
\vec{v}_r &= 0, \text{ everywhere} \\
\langle \vec{v}_r \rangle &= 0, \text{ on radial lines passing through the helical vortex} \\
\langle \vec{v}_\perp \rangle &= \frac{d\Gamma}{H(2\pi r' - r) \sqrt{1 + (\Omega r/V)^2}} / 2\pi r 
\end{align*}
\]

Here \( \vec{v}_r \) is the helical component, \( \vec{v}_r \) the radial component, and \( \vec{v}_\perp \) the perpendicular component of the induced velocity field. \( H \) is the Heaviside unit step function. All of these formulas may be easily extended to any number of equal strength, equally spaced helical vortex lines, as would be shed from a number of identical, equally spaced propeller blades.

For incompressible flow, it is easy to extend this symmetry based analysis to the problem of the semi-infinite vortex pair shown in Fig. 2-11. The radial segment AB of this vortex line is of no interest, since it gives rise to no velocity along the line AB, which coincides with the propeller blade. Swept back propeller blades, which do not lie along such radial lines, are explicitly excluded from this treatment. A 180 degree rotation about line line AB, combined with a reversal of the direction of all circulations, results in the flow field of "the other half" of an infinite vortex pair. This fact allows the most important properties of the flow field in the semi-infinite
Figure 2-11: Semiinfinite Helical Vortex Pair
case to be inferred from the known properties of the flow field in the infinite case.

In particular, \( u(r,\theta) \) denotes the induced velocity in the disk plane of the semiinfinite vortex pair, then the symmetry about AB mentioned above implies

\[
\begin{align*}
(u_r(r,\theta))_R &= -(u_r(r,\theta))_L \\
(u_\theta(r,\theta))_R &= + (u_\theta(r,\theta))_L \\
(u_\perp(r,\theta))_R &= + (u_\perp(r,\theta))_L
\end{align*}
\]

Here the subscripts refer to the velocity induced by the right and left halves of the infinite helical vortex pair. The fact that the sum of these two velocity fields must be the velocity field of the infinite vortex pair relates the values of \( u \) in the disk plane to the values of \( v \) by

\[
\begin{align*}
u_\theta(r,\theta) + u_\theta(r,-\theta) &= v_\theta(r,\theta) = 0 \\
u_r(r,\theta) - u_r(r,-\theta) &= v_r(r,\theta) \\
u_\perp(r,\theta) + u_\perp(r,-\theta) &= v_\perp(r,\theta)
\end{align*}
\]

In particular, along the radial line AB, which corresponds to the propeller blade, and is the only place where the induced velocities are of any interest, these relationships become

\[
\begin{align*}
u_\theta &= 0 \\
u_\perp &= v_\perp/2
\end{align*}
\]  

(2-12)

Here, \( u \) and \( v \) are functions only of \( r \), since it is understood that \( \theta = 0 \). Once again, it is easy to extend (2-12) to the case of an arbitrary number of identical, equally spaced blades. The azimuthal averages of these two velocity components also satisfy the relationship (2-12). Explicitly,

\[
\begin{align*}
\langle u_\theta \rangle &= 0 \\
\langle u_\perp \rangle &= dr \ H(r'-r) \sqrt{1 + (\Omega r/V)^2}/4\pi r,
\end{align*}
\]

(2-13)

or, expressed in terms of the axial and azimuthal components \( u_\zeta \) and \( u_\phi \),
\[ u_z = -d\Gamma \frac{H(r'-r)}{4\pi V} \]
\[ u_\theta = +d\Gamma \frac{H(r'-r)}{4\pi r} \]  \hspace{1cm} (2-14)

Most physicists will find it convenient to think about these simple incompressible flow fields in terms of their magnetostatic analogues. For example, it is clear that the lines of magnetic flux diverge at the exit of a semiinfinite solenoid. This observation leads to the conclusion that the unknown radial component of the induced velocity in the disk plane, \( u_r \), must be negative. In aerodynamics, this radially inward flow in the disk plane is known as "slipstream contraction".

The actual flow field is compressible. Since the vortex lines which give rise to the induced velocity are continuously being created at a very rapid rate as the propeller advances, it is obvious that the assumption that the flow field is incompressible is not valid. For example, the finite speed of sound implies that the airfoil section at one radial location cannot possibly "feel" the induced velocity due to the most recently created sections of the vortex sheet at other radial locations. In principle, the equations for the compressible flow field of a vortex sheet which is continually being created (derived in the next chapter) could be used to compute the induced velocities. However, this much additional complexity would be computationally fatal for the purposes of this thesis.

Instead, the problem of the effect of compressibility on the induced velocity field encountered by the propeller blades will simply be ignored. Despite the fact that the tip speed of a typical light aircraft propeller is roughly Mach 0.7, this rather radical bit of theoretical surgery is not as bad as it first appears. The saving grace is the fact that the forward Mach number is generally quite small -- no more than Mach 0.2. Therefore, the entire section of the vortex sheet more than about 1/5 propeller diameter behind the disk plane will contribute to the flow field in the disk.
plane precisely as it would in incompressible flow. Since the magnetic field does not fall off too rapidly outside the mouth of a solenoid, it is clear that this more distant section of the shed vortex sheet actually accounts for the larger part of the induced velocity in the disk plane of the propeller.

A second mitigating factor relates to the approximation which will be employed in the following section. Since the exact incompressible flow at the blade, given by (2-12), is difficult to compute, the azimuthal average values in (2-13) will be used instead. The difference between the compressible and incompressible flow fields is that the propeller blades rotate through a finite angle before a section at one radius actually feels the velocity field due to vorticity shed at a distant radius. Therefore, it seems fair to assume that the actual induced velocity in the compressible case will be more nearly equal to the azimuthal average of the induced velocity than to the actual induced velocity at the location of the propeller blade in the incompressible case. Thus, the use of the azimuthal average velocity partially allows for the effect of the compressibility on the induced velocity.

The flow field will therefore be assumed to be incompressible for the remainder of this chapter, with the understanding that the approximation of azimuthal averaging yields results which apply even more accurately to the exact, compressible flow field than to the incompressible flow being modelled. However, this fortuitous circumstance applies only to the case that the forward Mach number, V/c, of the propeller is small. For propellers which advance at an appreciable fraction of the speed of sound, the following analysis is wholly inappropriate.
2.4.4 The Limit of Infinitesimal Sheet Spacing

In magnetostatics, the field of a tightly wound solenoid is easy to compute. The corresponding propeller flow field is caused by a propeller with very closely spaced vortex sheets relative to its diameter. This situation is usually called the case of "an infinite number of blades" in the literature, but it also applies to the case of a small number of blades operating at a low advance ratio. The latter situation is relevant to light aircraft propellers, which generally operate at rather low advance ratios (and have a small number of blades). The limit of an extremely high advance ratio, two bladed propeller could also be used as a starting point for propeller theory, since this is equivalent to a wing in rolling motion. However, as noted, the light aircraft propellers of interest here have relatively low advance ratios, so the low advance ratio theory will be presented here.

If the vortex sheets are closely spaced, then the flow field between the sheets must be very nearly uniform on the length scale of the sheet spacing. The value of the induced velocity at the surface of the sheets will therefore be very nearly equal to the mean value of the mean value of the velocity at the given radius. Each individual helical filament therefore contributes to the induced velocity field in the disk plane according to (2-13). If there are B identical blades, then the total induced velocity at a distance $r$ from the axis is

$$u_\perp(r) = B \int_\zeta \Gamma(r^2 - \zeta) \sqrt{1 + (\Omega \zeta/V)^2}/4\pi \zeta \, d\zeta$$

$$= (B \sqrt{1 + (\Omega \zeta/V)^2}/4\pi \zeta) \int_\zeta \, d\zeta$$

$$= B\Gamma(r) \sqrt{1 + (\Omega \zeta/V)^2}/4\pi r$$

since the circulation $\Gamma(r)$ drops to zero at $r = 0$ and $r = R$. Thus, in the limit that the sheets are closely spaced,
\[ u_\parallel = 0 \]
\[ u_\perp = Br(r) \sqrt{1 + (\Omega r/V)^2} / 4\pi r. \] (2-15)

The fact that the induced velocity, \( u_\parallel \), parallel to the local direction of motion is zero means that only the attack angles of the airfoil sections will be affected by the finite solidity correction. The fact that the induced velocity in the disk plane at radius \( r \) depends only on the circulation \( \Gamma(r) \) at the same radius is a tremendous simplification. There is no a priori reason why the induced velocity at one radius should not depend on the circulation at all other radii. Indeed, if the finite spacing between the sheets were accurately taken into account, there can be no question that this would actually occur. Equation (2-6) becomes an integral equation for \( \Gamma(r) \) if the induced velocity at one radius depends on the circulation at all other radii. However, in the present limit, this integral equation reduces to a simple algebraic equation.

2.4.5 A Correction for Low Advance Ratios

Although the preceding discussion technically applies to the limit of low advance ratios, the result (2-15) cannot be applied to such situations in reality. For a given propeller, the strength of the shed vortex sheet reaches a finite limit as the forward speed is reduced toward zero at fixed angular speed. Thus, the circulation around individual vortex filament reaches a finite limit as the forward velocity is decreased. The magnetostatic equivalent of reducing the forward velocity of a propeller is to wind the coil of the solenoid more and more tightly, while keeping the current through it fixed. Obviously, this results in a larger and larger field strength as the pitch of the winding is decreased without limit. This problem is reflected in the nonphysical behavior of (2-15) in the limit that \( V \to 0 \).
Physically, the self induced motion of the vortex sheets must become important in this limit. A fan or a hovering helicopter rotor produces a considerable flow through its disk plane, even though its forward speed is zero. This causes the shed vortex sheet to move out of the disk plane after it has been created, which prevents the pitch of the helical vortex lines from becoming smaller than some finite limit. Indeed, from (2-14), it is clear that the rate that the vortex sheets move out of the disk plane can never be smaller than about \( u_z = Br\Omega/4\pi V \). The effective advance ratio (pitch) of the vortex sheets can therefore never be less than about \( u_z/\Omega R = Br/4\pi VR \), even if the geometrical advance ratio \( V/\Omega R \) is smaller than this.

The preceding lower bound on the effective advance ratio of the shed vortex sheets is inversely proportional to \( V \), while the original estimate of the advance ratio of the sheets, \( V/\Omega R \), is directly proportional \( V \). Evidently, the changeover from the former regime to the latter will occur when the advance ratio estimated by either method is the same, that is when \( V/\Omega R = Br/4\pi VR \), or

\[
V/\Omega R = \sqrt{Br/4\pi\Omega R^2}.
\]

For advance ratios smaller than this, the self induced motion of the vortex sheets simply cannot be ignored. The parameter \( Br/2\pi\Omega R^2 \) which appears in this context will recur frequently in the following analysis. It is worthwhile to note that, if the speed of a typical airfoil section is largely the result of its rotational speed, then

\[
Br/2\pi\Omega R^2 = (C_L/2)(B\chi/2\pi r) = \sigma,
\]

where \( \sigma \) is the solidity of the propeller \( (C_L = 1) \). The self induced motion of the sheets therefore becomes important when the advance ratio is less than the square root of the solidity:

69
\[
\frac{V}{\Omega r} < \sqrt{\frac{B\Gamma}{2\pi\rho R^2}} \approx \sqrt{\sigma}
\] (2-16)

In the limit of zero solidity, the residual motion of the vortex sheets is unimportant all the way to zero advance ratio, as assumed above. However, due to the square root dependence in (2-16), the effects of finite solidity rapidly become important. For example, the solidity of a Cessna 172 propeller is 0.072, so that the self induced motion becomes important for advance ratios less than about 0.27. This is comparable to the advance ratios at which the propeller normally operates. It is therefore of the utmost practical importance to make some allowance for the residual motion of the vortex sheet.

To this end, the pitch of each individual vortex filament will be assumed to be given by \((V-v_\zeta)/(\Omega-v_\theta/r')\), rather than simply by \(V/\Omega\). (Here, \(v_\zeta<0\), while \(v_\theta>0\) for a thrusting propeller.) Again, \(v\) represents the induced velocity in the wake far from the disk plane. This assumption is motivated by the Helmholtz equation. The details of the self induced deformation of the vortex sheet will be ignored.

The contribution of such a vortex filament to the induced velocity field of an infinite, self deforming vortex sheet is given by (2-11), with the pitch \(V/\Omega\) replaced by its new value. Since the pitch may now vary with the radius \(r\), it is useful to use axial and azimuthal components to express the velocity field. Just as (2-14) was obtained from (2-13), (2-11) yields

\[
\begin{align*}
\langle dv_\zeta \rangle &= -d\Gamma \frac{H(r'-r)(\Omega-v_\theta/r')}{2\pi(V-v_\zeta)} \\
\langle dv_\theta \rangle &= +d\Gamma \frac{H(r'-r)}{2\pi r}.
\end{align*}
\]

The azimuthal average of the azimuthal component of the velocity may be immediately integrated, just as before, giving
\[ \langle v_\theta \rangle = B \Gamma(r)/2\pi r. \] (2-17)

The azimuthal average of the axial component is more difficult to integrate. Under the assumption that \( v_z = \langle v_z \rangle \) and \( v_\theta = \langle v_\theta \rangle \), the above differential yields

\[
\int (V - \langle v_z \rangle) dv_z = - \left( \frac{B \Omega}{2\pi} \right) \int_0^R \frac{d\Gamma}{r} H(r'_-r) \left( 1 - \frac{\langle v_\theta \rangle}{\Omega r'} \right)
\]

\[
(V - \frac{\langle v_z \rangle}{2}) \langle v_z \rangle = - \left( \frac{B \Omega \Gamma(r)}{2\pi} \right) \left( 1 + \frac{B}{\Gamma(r)} \int_0^R \frac{d\Gamma}{\Omega} \frac{H(r'-r) \Gamma(r')}{r'^2} \right)
\]

The troublesome integral on the right hand side is fortunately negligible. If it were not neglected, then equation (2-6) would become an integral equation for \( \Gamma(r) \), instead of a much to be preferred algebraic equation. To see that it is negligible, notice that

\[
\frac{B}{\Gamma(r)} \int_0^R \frac{d\Gamma}{r} \frac{\Gamma(r')}{\Omega r'^2} H(r'-r) \approx \frac{2B}{C_L \Omega \chi} \int_0^R \frac{(C_L \Omega \chi)^2}{\Omega} \frac{dr'}{r'}
\]

\[
\approx \frac{C_L}{2} \frac{B \chi}{2\pi r} \ln \frac{R}{r} \approx \frac{B \chi}{2\pi r} = \sigma << 1
\]

Therefore, the azimuthally averaged axial component of the velocity induced by the self deformed vortex sheet may be estimated as

\[
(V - \langle v_z \rangle/2) \langle v_z \rangle = -B \Omega \Gamma(r)/2\pi. \] (2-18)

One crucial step remains before (2-17) and (2-18) can be utilized. Namely, these estimates of the induced velocity far downstream of the disk plane must somehow be related to the induced velocities in the disk plane itself. Although the helical vortex lines certainly do not retain their self deformed pitch all the way to the trailing edge of the blade, there is no other simple course of action but to assume that they do have constant pitch. Consequently, the induced velocity in the disk plane will once again be assumed to have simply half the value it would have in the case of an infinite vortex sheet. Thus, the induced velocity \( u \) in the disk plane is
$$u_\theta = B\Gamma(r)/4\pi$$

$$V - u_z = -B\Omega(r)/4\pi$$

Now, in terms of components which are parallel and perpendicular to the geometrical direction of motion of the airfoil at radius \( r \),

$$u_\parallel = (u_\theta \Omega r + u_z V)/\sqrt{V^2 + (\Omega r)^2}$$

$$u_\perp = (u_\theta V - u_z \Omega r)/\sqrt{V^2 + (\Omega r)^2}.$$  \hspace{1cm} (2-20)

Next, from (2-19),

$$V - 2u_z = V\sqrt{1 + \left(\frac{B\Omega(r)}{\pi V^2}\right)},$$

so that

$$u_\parallel = \frac{V}{2} \frac{\Omega r}{\sqrt{V^2 + (\Omega r)^2}} \left\{ \left( \frac{\Omega R}{V} \right)^2 \left( \frac{B\Gamma}{2\pi \Omega R^2} \right)^2 \right\} \left( \sqrt{1 + 2\left( \frac{\Omega R}{V} \right)^2 \left( \frac{B\Gamma}{2\pi \Omega R^2} \right)^2} - 1 \right)$$  \hspace{1cm} (2-21)

$$u_\perp = \frac{V}{2} \frac{\Omega r}{\sqrt{V^2 + (\Omega r)^2}} \left\{ \left( \frac{R}{r} \right)^2 \left( \frac{B\Gamma}{2\pi \Omega R^2} \right)^2 \right\} \left( \sqrt{1 + 2\left( \frac{\Omega R}{V} \right)^2 \left( \frac{B\Gamma}{2\pi \Omega R^2} \right)^2} - 1 \right) \right\}.$$

It is easy to see that these formulas reduce to (2-15) if the parameter \((\Omega R/V)^2(B\Gamma/2\pi \Omega R^2) \ll 1\), as predicted in (2-16). The parallel component \( u_\parallel \) in (2-21) will be neglected. This is not necessary on the grounds of simplicity; I would perhaps be more consistent if I retained it. However, the major effect of the induced velocity is to change the attack angle of the airfoils slightly, not their relative speed through the fluid. The correction to the attack angle is \((a_0 - a_0 - u_\perp / U)\), while the correction to the speed is \((1-u_\parallel / U)\). Inasmuch as the attack angle \( a_0 - a_0 \) (in radians) is considerably less than 1 in all realistic situations, \( u_\perp \) has a considerably larger effect on the performance than does \( u_\parallel \). Furthermore, \( u_\parallel \) vanishes entirely in the high advance ratio limit, and is only of order \( \sigma \) in the low advance ratio limit, when \( u_\perp \) is of order \((\Omega R/V)\sigma\). Consequently,
\( u_\parallel \approx 0 \) in (2-21). \( \quad (2-22) \)

The relationship between the circulation \( \Gamma \) and the induced velocity \( u_\perp \) in (2-21) is remarkable. First, it is nonlinear; this is a consequence of the fact that the strength of the vortex sheet has finally been allowed to influence its shape. Second, despite the nonlinearity, the induced velocity at radius \( r \) is still a function only of the circulation around the blade at the same radius \( r \). This is fortunate indeed, since a nonlinear integral equation would be virtually intractable, while a nonlinear algebraic equation is simple.

2.4.6 A Correction for High Advance Ratios

As the advance ratio of the propeller increases, so does the spacing between its shed vortex sheets. When this occurs, the difference between the azimuthal mean of the induced velocity and the actual value of the induced velocity at the propeller blade can become fairly large. Again, a complete treatment of this effect would require an analysis in which the induced velocity at one radius was influenced by the circulation around the blade at all other radii. It would be a shame to sacrifice this considerable simplification at this late stage of the discussion. Thus, a gross engineering approximation will be introduced to simulate the effect of the finite sheet spacing. The resulting modifications to the induced velocity are large only for high advance ratios.

The technique which will be employed is a perversion of a suggestion made by Prandtl at the end of Betz' classic paper on propeller theory [Betz 1919]. Prandtl pointed out the similarity between the two dimensional flow past a row of evenly spaced, semiinfinite flat plates, and the three dimensional flow past the edges of a tightly wound screw shaped sheet. The similarity is enhanced by the fact that the
helical component of the velocity is identically zero in the latter flow field. The two dimensional flow is easily solved by a conformal mapping technique. The result is that the velocity component perpendicular to a plate, at the surface of the plate, differs from the average (along a line perpendicular to the plates) value of this component by a factor denoted by $F$. If $\Delta$ is the spacing between the sheets, and if $y$ is the distance from the edge of the sheet to the point in question, then

$$F = \frac{\langle u_\perp \rangle}{u_\perp} = (2/\pi) \cos^{-1}(\exp(\pi y/\Delta)).$$

This relationship will be taken to apply in the case of a screw shaped sheet, with $y$ the radial distance of the point from the edge of the sheet, and $\Delta$ the spacing between the edges of the vortex sheet as measured along the "\perp" direction. Thus,

$$F = (2/\pi) \cos^{-1}(\exp(-B\sqrt{1+(\Omega R/V)^{2}}(1 - r/R/2))).$$

(2-23)

Prandtl suggested that this correction factor be employed in the case that the vorticity distribution on the screw shaped sheets corresponds to the uniform motion of a rigid sheet. Following Prof. Larrabee, I shall apply the correction to arbitrary radial distributions of vorticity. Once again, the only justification of this assumption is computational simplicity in a problem whose exact solution is beyond the scope of this thesis.

Since the induced velocity of (2-21) was derived from an azimuthally averaged velocity in (2-17) and (2-18), it should be corrected by the factor $F$:

$$u_\parallel = 0$$

$$u_\perp = \frac{V}{2F} \frac{\Omega r}{\sqrt{V^{2} + (\Omega r)^{2}}} \left\{ \frac{R^{2}}{r} \left( \frac{BG}{2\pi \Omega R^{2}} \right) + \left( \sqrt{1 + 2\left( \frac{\Omega R}{V} \right)^{2}\left( \frac{BG}{2\pi \Omega R^{2}} \right) - 1} \right) \right\}$$

(2-24)

The factor $F$ is always less than 1, and $F$ deviates appreciably from 1 only for radii $r$ near the tip radius $R$, or for large values of the advance ratio $V/\Omega R$. 

74
2.4.7 Solution of the Propeller Performance Equations

Table 2-2 gives a summary of the relationship between the propeller geometry, the two dimensional characteristics of its airfoil sections, and the strength of the shed vortex sheet. Equation (2-24) provides the relationship between the strength of the sheet and the induced velocities which is necessary to close this system of equations. Since the induced velocity at radius r depends only on the circulation at the radius r under the assumptions which have been made, the complete system of equations are simple algebraic equations, rather than integral equations, as they would be in a more robust theory. Indeed, the theory developed here may be regarded as the most accurate description of propellers possible which does not require the solution of integral equations.

The problem which must be solved is easily visualized by the geometry shown in Fig. 2-12. Equations (2-5), (2-6), and (2-9) relate the induced angle $\alpha_\perp = \frac{u_\perp}{\sqrt{V^2 + (\Omega r)^2}}$ and the circulation around the airfoil section in question. This relationship merely reflects the lift curve of the section, that is, the amount by which the change in angle of attack due to the induced velocity, $\alpha_\perp$, changes its lift coefficient. Equation (2-24) also relates the circulation $\Gamma(r)$ to the induced angle $u_\perp = \frac{\sqrt{V^2 + (\Omega r)^2}}{\omega}$. This relationship is one branch of an inclined parabola, which represents the magnitude of the induced velocity which would be produced by a given circulation. There is only one value of the circulation which causes the airfoil section to operate at precisely the angle of attack for which it will have that circulation around it. Graphically, the self consistent solution to the problem is the point of intersection of the reflected lift curve and the parabola. (In certain bizarre circumstances involving backflow through the propeller disk, there may be no point of intersection; the theory cannot handle such situations.)
Figure 2-12: Solution to Propeller Performance Equations
The solution in the limit of zero solidity corresponds to the point \((0, \Gamma_0)\) in Fig. 2-12, since the induced angle is zero in that limit. It is useful to express the self consistent circulation \(\Gamma\) in terms of this naively computed circulation \(\Gamma_0\). Assuming that the intersection between the parabola and the lift curve lies on the stalled region of the lift curve, the equation for \(\Gamma\) is

\[
\frac{\Gamma}{\Gamma_0} = 1 - \frac{\alpha_f}{\alpha - \alpha_f} = \psi
\]

\[
= 1 - \frac{V}{2F(\alpha - \alpha_f)} \frac{\Omega r}{\sqrt{V^2 + (\Omega r)^2}} \left\{ \left[ \frac{\Omega R}{V} \right] \left[ \frac{\Omega R}{V} \right] + \sqrt{1 + 2 \left( \frac{\Omega R}{V} \right)^2 - 1} \right\}
\]

(2-25)

This quadratic equation in \(\Gamma/\Gamma_0\) can be solved after some tedious algebra. The final result is shown in Table 2-3, together with all formulas which are necessary to completely specify the jump in velocity potential \(\Gamma\) across the shed vortex sheet in terms of the propeller geometry and two dimensional section characteristics.

The profile drag coefficient is assumed to be the same function of the lift coefficient as it is in the two dimensional, incompressible flow past the given airfoil section. Thus, once the lift coefficient has been computed by means of the equations in Table 2-3, the profile drag coefficient is

\[
(C_D)_{\text{profile}} = C_D(C_L(r); r).
\]

(2-26)

The induced drag may be estimated as the "component of lift in the direction of motion" in the quasi two dimensional lifting line theory which has been used here. If the induced drag is associated with a drag coefficient, this coefficient will simply be given by the product of the lift coefficient and the sine of the (small) induced angle:

\[
(C_D)_{\text{induced}} = \alpha \perp C_L(r)
\]

(2-27)
Table 2.3. Solution of Propeller Performance Equations

Preliminary Definitions:

\[ \alpha = \beta - \tan^{-1}\left(\frac{V}{r}\right) \quad ; \quad (2.8) \]

\[ U = \sqrt{V^2 + \Omega^2 r^2} \quad ; \quad (2.10) \text{ plus } (2.23) \]

\[ m' = \frac{m}{\sqrt{1 - (U/c)^2}} \quad ; \quad (2.4) \]

\[ F \equiv \frac{2}{\pi} \cos^{-1}\left[ \exp\left( -\frac{B}{2} \sqrt{1 + \left(\frac{\Omega R}{V}\right)^2 \left(1 - \frac{r}{R}\right)} \right) \right] \quad ; \quad (2.22) \]

\[ \Gamma_0 = \frac{1}{2} m'(\alpha - \alpha_0) U \chi \quad ; \quad \text{circulation in zero solidity limit (unstalled airfoil)} \]

\[ \gamma_0 \equiv B \Gamma_0 / (2\pi \Omega R^2) \]

Solution to Equation (2.24)

\[ C_1 \equiv 2F(\alpha - \alpha_0) \left(\frac{\Omega R}{V}\right) \left(\frac{R}{r}\right) \left[ \left(\frac{V}{\Omega R}\right)^2 + \left(\frac{r}{R}\right)^2 \right] \]

\[ C_2 \equiv C_1 + \left(\frac{R}{r}\right)^2 \gamma_0 \]

\[ C_3 \equiv \left(\frac{\Omega R}{V}\right)^2 \frac{\gamma_0}{(C_2)^2} + \frac{1 + C_1}{C_2} \]

78
\[ \psi = C_3 \left[ 1 - \sqrt{1 - \frac{(2 + C_1)C_1}{(C_2C_3)^2}} \right] \]; correction factor for finite solidity effects

Grand Results:

\[ C_L = \begin{cases} 
(C_L)_x & ; \text{if } (C_L)_x < m'\psi(\alpha - \alpha_o) \\
 m'\psi(\alpha - \alpha_o) & ; \text{if airfoil stalled} \\
(C_L)_n & ; \text{if } (C_L)_n > m'\psi(\alpha - \alpha_o) 
\end{cases} \]

\[ \Gamma = \frac{1}{2} C_L U \chi \quad ; \quad (2.6) \]

\[ = \psi \Gamma_o \quad ; \quad \text{if air un stalled} \]

\[ \alpha_\perp = \frac{u_\perp}{\sqrt{v^2 + \Omega r^2}} \quad ; \quad \text{induced angle} \]

\[ = \frac{1}{2F} \frac{V\Omega r}{v^2 + \Omega^2 r^2} \left[ \left( \frac{R}{r} \right)^2 \left( \frac{B\Gamma}{2\pi\Omega R^2} \right) + \sqrt{1 + 2\left( \frac{\Omega R}{V} \right)^2 \left( \frac{B\Gamma}{2\pi\Omega R^2} \right) - 1} \right] \]

\[ = (1 - \psi)(\alpha - \alpha_o) \quad ; \quad \text{if airfoil is un stalled} \]
2.4.8 A Rough Allowance for a Nacelle

In practice, a propeller almost invariably operates in front or behind a nacelle which houses the engine to drive it. The flow past a propeller operating in front of such a nacelle is terribly complex, since the vortex sheet will be considerably distorted by this solid body. Any attempt to compute the unstable flow of a vortex sheet incident on a nacelle would be even more of a theoretical embarrassment than the preceding 20 pages of approximations for the same flow without this added complication.

The most important effect of the nacelle is to decrease the axial velocity of the flow through the propeller disk. In the absence of the propeller, this flow blockage is easy to estimate. A flow blockage factor $\delta$ can be defined as the ratio between the axial velocity $V(r)$ at radius $r$ in the disk plane and the forward speed $V$ of the propeller/nacelle combination:

$$V(r) = V \delta(r)$$  

(2-28)

A table of the values of $\delta(r)$ for the symmetric body employed in the experiment is provided in appendix E.

In terms of propeller operation, the primary effect of this flow blockage is to increase the angle of attack of each airfoil section. For the inboard sections of the blade, this increase in attack angle can be considerable. This should cause the inboard sections of any given propeller to be more heavily loaded in the presence of a nacelle than they would be without the nacelle. The simplest way to account for this effect is to simply replace all occurrences of the forward velocity $V$ in Table 2-3 by the partially blocked velocity $V(r)$ given by (2-28). Needless to say, this is an extremely rough approximation, but I know of no better way to make an allowance for the presence of a nacelle.
2.5 Summary

This chapter began with a thorough discussion of the nature of the flow field associated with a lifting wing. The flow is irrotational everywhere except in the very thin layer of fluid which actually passed through the boundary layer at the surface of the wing. The fluid in this thin layer plays a very important dual role in the flow field of the wing. First, it tends to be dragged along behind the wing as a result of the viscous and pressure drag forces which acted on this fluid when it was near the surface of the wing. This motion is associated with the profile drag of the wing. Second, a large scale vortex structure exists within this thin layer of fluid which can be considered a source for the large scale irrotational fluid motion external to the thin sheet. In this context, the fluid layer is referred to as the "shed vortex sheet". The large scale irrotational motion associated with this vortex sheet contains the fluid momentum which is the reaction to the lift force. The kinetic energy of this irrotational motion is responsible for the induced drag of the wing.

The acoustic theory in the following chapter predicts the irrotational compressible flow field associated with a vortex sheet in arbitrary motion, which is possibly being created or destroyed at its edges. However, the rotational flow in the vortex sheet itself lies outside the domain of linear acoustics; the convection of vorticity according to the Helmholtz equation is the quintessence of the nonlinearity in fluid mechanics. Therefore, the strength and motion of the vortex sheet must be supplied from outside the acoustic theory. Fortunately, aerodynamicists have considerable experience in estimating both the strength and position of the vortex sheet shed as an airfoil slices through a fluid.

In order to stress the irrotational nature of the flow field external to the shed vortex sheet, as well as to promote algebraic and conceptual simplicity, the vortex
sheet is here regarded as a surface of discontinuity in the velocity potential. The jump in velocity potential at any point on the sheet is simply the circulation between that point and the edge of the sheet. Conversely, vortex lines correspond to lines of constant jump in the velocity potential on the surface.

The Kutta condition states that the transverse component of the vorticity diffusing into the flow from the suction side of the airfoil must be equal and opposite to that diffusing into the flow from the pressure side. For an airfoil shape, this effectively requires that there be no flow around the sharp trailing edge of the section, which is the basis of the theory of two dimensional airfoil sections. This theory was briefly reviewed, including the Prandtl-Glauert correction for finite compressibility. For the two most important special cases of two dimensional airfoil theory, it was noted that the velocity potential external to the boundary layer and the momentum wake precisely solves the acoustic wave equation. These are the cases of incompressible flow past arbitrary airfoils and the case of compressible flow past thin airfoils. In the case of thin airfoils, even the pressure satisfies the wave equation. These facts emphasize the unity of the aerodynamic theory presented in this chapter and the acoustic theory presented in the following chapter.

If the aspect ratio of the wing is not too small, a quasi two dimensional analysis of its characteristics is possible. The idea is to use the two dimensional section properties of the airfoil sections, but to assume that each airfoil section acts as if it were moving with a velocity determined jointly by the geometrical forward motion of the wing and the velocity induced by the shed vortex sheet at the given section. This induced velocity primarily reduces the attack angles of the airfoil sections from their geometrical values. Since the amount of this reduction is proportional to the strength of the shed vortex sheet, which is itself proportional to the attack angles, the
overall effect is to reduce the lift curve slope of the wing. The larger the aspect ratio, the more nearly the two dimensional lift curve slope is approached, since the shed vortex sheet is weaker for high aspect ratios.

The transition from wings to propellers is merely a matter of geometry. The shed vortex sheets now lie along screw shaped surfaces. The concepts of attack angle and aspect ratio in wing theory are replaced by those of advance ratio and solidity in propeller theory. A detailed theory of propeller performance has been presented, with the aim of predicting the parameters necessary as inputs to the acoustic theory of the following chapter. These are the strength of the shed vortex sheet, the magnitude of the resulting induced drag, and the magnitude of the profile drag, all as functions of the distance from the propeller axis. The theory was based on the exact solution of the problem in the limit of zero solidity, and in the case in which the shed vortex sheets are wound very tightly.

Corrections were applied to this oversimplified theory to allow for the finite solidity. For low advance ratios, it is vital to allow for the self induced motion of the shed vortex sheet. As a rule of thumb, the self induced motion of the sheets becomes important when the propeller is operated at an advance ratio smaller than the square root of its solidity. This correction for self induced motion of the sheets at low advance ratios represents the only inclusion of a truly nonlinear effect in this thesis. A rough correction to allow for the actual, nonzero spacing between the vortex sheets was also included in the theory.

The propeller theory summarized in Table 2-3 has the virtue of extreme computational simplicity. With a little patience and a hand calculator, it is not difficult to work out the performance of an arbitrary propeller at one flow condition. With a computer, it is cheap to perform even a very extensive analysis of the
performance of a propeller over a wide variety of operating conditions. There are several major restrictions on the validity of the theory, however. First, the advance ratio must not be too high, and the solidity of the propeller must not be too great. Also, the Mach numbers of the blade sections cannot exceed about 0.7, or the Prandtl-Glauert rule for the high Mach number scaling of the section characteristics will fail. A less obvious restriction involves the compressibility of the flow. The theory should break down if the Mach number associated with the forward motion of the propeller becomes greater than a few tenths.
Chapter Three

General Aeroacoustic Theory

3.1 Introductory Remarks

The mathematics of sound production by moving objects will now be presented. Unfortunately, purely mathematical reasoning has very often obscured the physical processes which result in sound when a solid body moves through a fluid. After a thorough discussion of one of the more commonly used theoretical frameworks of aeroacoustics, a completely new viewpoint will be introduced. I profoundly regret contributing yet another set of equations to a field which is already mired in a mathematical swamp. I hope that the new picture presented here will help to clarify the nature of the sound caused by moving objects, rather than further obscuring the problem.

The starting point of any theory in fluid mechanics is the equations of mass and momentum conservation. Only a single component fluid will be considered here, which may be characterized at each point by its temperature, its density \( \rho \), and its bulk velocity \( \mathbf{u} \). In addition to the conservation of mass and of momentum, the first law of thermodynamics (conservation of energy), as well as a set of static and dynamic constitutive relations, are also necessary in order to determine the motion of the fluid. Heat conduction will be entirely neglected here. Viscosity will be considered only in the boundary layer adjacent to the surface of the solid boundary of the fluid, and in the remnant of this boundary layer in the wake of the moving object. The treatment of the small amount of fluid in the boundary layer is
restricted to the discussion presented in chapter 2. In the absence of heat conduction and viscosity, the First Law and the static constitutive relationship for the fluid reduce to
\[ p - p_0 = c^2 (\rho - \rho_0) \]  \hspace{1cm} (3-1)
where \( p \) is the isotropic pressure of the fluid and \( c \) is the isentropic sound speed. In all cases to be considered, the compressions or rarefactions do will be so small that \( c \) may be regarded as a constant.

The equations of motion (in Lagrangian form) are conveniently expressed in a hybrid matrix-vector notation as
\[ \left( \frac{\partial}{\partial t} \nabla \right) \begin{pmatrix} \rho & \rho \nabla \varepsilon \\ \varepsilon \nabla + p & p \nabla \varepsilon \end{pmatrix} = 0 \]  \hspace{1cm} (3-2)
The symbol \( \mathbf{ab} \) represents the outer product of vectors \( \mathbf{a} \) and \( \mathbf{b} \), also called a dyadic. The pressure tensor \( \mathbf{p} \) consists of the isotropic pressure \( \mathbf{1p} \), plus a term due to viscous stresses, which is negligible except in the boundary layer and its remnant. If sources of mass or of momentum were present within the fluid, the right hand side of (3-2) would not be zero. It will be necessary to assume that the magnitude of the fluid velocity is everywhere small compared to the sound speed:
\[ |u| \ll c \]  \hspace{1cm} (3-3)
This does not necessarily require that the speed of motion of the solid boundary surface is much less than sonic speed, as will be fully discussed in section 3.4 below.

The goal is to express the flow field in the bulk of the fluid in terms of its boundary values. As indicated in the previous chapter, the flow field is almost everywhere irrotational in the case of the motion of a streamlined object. Under all these assumptions, the velocity potential is found to satisfy the acoustic wave
equation everywhere the fluid is irrotational. The solution to the wave equation can be expressed in terms of its boundary values in much the same way as the solution to Laplace's equation can be so expressed. The formula for the solution in the case that the boundary surface is stationary (with respect to the fluid at infinity) is the Kirchhoff equation. This equation was generalized to the case of a moving boundary by Morgans. A much simplified derivation of Morgans equation will be presented below. Together with the realization that the vortex sheet shed by a lifting airfoil must be regarded as a part of the boundary of the irrotational flow field, this formula provides a coherent physical picture of the process of sound production by the arbitrary motion of a streamlined object.

Before introducing Morgans' equation, Lowson's work on the solutions of the inhomogeneous linearized equations of motion will be described. These solutions are the acoustic analogues of the Lienard Wiechert potentials of electromagnetic theory. In addition to the flow field of a moving point source of mass, which is analogous to the problem of a moving point charge, Lowson also studied moving point forces. While Lowson computed only the pressure field of these sources, the entire flow field associated with point sources of mass and point forces will be presented here. The study of the flow field of a moving point force is of particular interest, since it is found to include a shed vortex sheet, which embodies many of features discussed in chapter 2.

Also, the explicit formulas for the flow fields associated with moving couplets of mass sources will be worked out. If the separation of the couplet is spacial, it will be called a doublet, while if the separation of the couplet is temporal, it will be called a chronopole. I know of no previous discussion of the chronopole source type, either in acoustics or in electromagnetism.
The so-called "Lowson sources" have a completely transparent meaning, both mathematically and physically. However, the problem to be solved is the homogeneous wave equation in the presence of moving boundaries, not the inhomogeneous equation in an unbounded fluid. In the case of Laplace's equation, the solution to the homogeneous equation in a bounded region of space is expressible as a surface integral of the solutions of the inhomogeneous equation in unbounded space. By analogy, it should be possible to express the solution to the wave equation with boundary conditions on a moving surface as a sum of Lowson sources located at points of the boundary. Morgans' equation does just this; it provides the strengths and types of Lowson sources which must be associated with each point of the boundary surface in terms of the boundary conditions.

Two other theories for the types and strengths of the sources which should be located at the moving boundary surface will be presented. The first is the popular Flowcs Williams and Hawkings theory. The mathematics of this theory will be presented in some detail, since it appears to be correct. However, the theory deals with only the pressure field, and therefore provides little or no insight into the flow field which is being described. This lack of insight leads to frequent misapplications of the theory; in particular, the shed vortex sheet must be regarded as a portion of the boundary surface in addition to the solid boundary. The second theory is due to Powell. In contrast to Flowcs Williams and Hawkings theory, Powell theory contains tremendous physical insight into the mechanism of sound generation in the situations of interest. Unfortunately, the mathematics of this theory is flawed and will not be presented here.
3.2 Lowson Sources

3.2.1 Inhomogeneous Wave Equations

In regions of the fluid where heat conduction and viscosity can be neglected, the inhomogeneous equations of motion are

\[
\frac{\partial p}{\partial t} + \rho_0 c^2 \nabla \cdot \mathbf{u} = c^2 q \\
\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{f}
\]  

(3-4)

The equations have been linearized about the conditions at infinity, where the fluid is assumed to be at rest, and to have a uniform density \( \rho_0 \) and pressure \( p_0 \). The mass and momentum added to the fluid per unit volume per unit time are indicated by \( q \) and by \( f \), respectively. Since the fluid is almost everywhere irrotational in all situations of interest here, a velocity potential \( \Phi \) exists and satisfies

\[
\mathbf{u} = -\nabla \Phi
\]  

(3-5)

Any vortex sheet present must be regarded as a portion of the boundary surface in order for this equation to be valid.

Three important wave equations may be derived from (3-4), one for each of the field variables \( p, \mathbf{u}, \) and \( \Phi \). The equation for \( \Phi \) cannot easily be made to include the effects of a nonpotential force field \( f \); this point will be discussed in more detail below. The two scalar wave equations are:

\[
\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{1}{c^2} q
\]  

(3-6)
\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} p - \nabla^2 p = \frac{\partial}{\partial t} q - \nabla \cdot \vec{f}
\]  

(3-7)

The vector wave equation is:

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{u} - \nabla (\nabla \cdot \vec{u}) = -\frac{1}{\varepsilon} \nabla q + \frac{1}{\varepsilon c^2} \frac{\partial}{\partial t} \vec{f}
\]  

(3-8)

The pressure and the velocity potential are related via the equation \( p = \rho_0 \frac{\partial \Phi}{\partial t} \) in regions of the fluid where \( f = 0 \). For the case \( f = 0 \), equations (3-6) and (3-7) nicely illustrate a simple property of the wave equation which will be used repeatedly in the following analysis. Namely, if a function \( G(t,x) \) solves the wave equation with a source \( S(t,x) \), then the function \( DG(t,x) \) solves the wave equation with the source \( DS(t,x) \). Here, the operator \( D \) can be any linear operator which does not depend explicitly on the spacetime coordinates \( (t,x) \).

Lowson studied a particularly useful source distribution: a delta function in space which moves along a given arbitrary trajectory [Lowson 1965]. The trajectory will be parametrized in terms of the source time; the source is at the point \( x \) at time \( t \) if

\[ x = y(t) \]  

(3-9)

The mass and momentum distributions of interest are

\[ q(t,x) = Q(t) \delta(x-y(t)) \]  

\[ f(t,x) = F(t) \delta(x-y(t)) \]  

(3-10)

Here \( Q(t) \) represents the rate of addition of mass at the point \( y(t) \), while \( F(t) \) represents the force applied to the fluid at \( y(t) \).
3.2.2 Solutions to the Scalar Wave Equations

The solution of (3-6) with the moving point source \( Q \) of (3-10) is the well known Lienard Wiechert potential of electromagnetism [Jackson 1975a]:

\[
\Phi(\tau, \mathbf{r}) = \frac{1}{4\pi \varphi_0} \frac{Q(\tau)}{r \sqrt{1 - M_r^2}}
\]  

(3-11)

The source time \( \tau \) must be evaluated at the observer time \( t \) in this equation. The vector \( \mathbf{r} \) from source to observer, and the source Mach number \( M \) must also be evaluated at the so called retarded time \( \tau \). Thus,

\[
\tau = t - \frac{r(\tau)}{c}
\]  

(3-12)

\[
\mathbf{r} = r(\tau) = x(\tau) - y(\tau) ; \quad r = |r| ; \quad \hat{r} = \mathbf{r}/r
\]  

(3-13)

\[
M = M(\tau) = \frac{1}{c} \frac{\partial y}{\partial \tau} ; \quad M_r = \hat{r} \cdot \mathbf{M}
\]  

(3-14)

In the case that the radial component of the source Mach number \( M_r \) exceeds unity, there will invariably be more than one retarded time satisfying (3-12). In this case, (3-11) must be summed over all retarded times.

The other solutions of the scalar wave equation of interest here may all be obtained from the Lienard Wiechert result by differentiation. For example, the solution for the pressure field of a moving point source is given by

\[
p(\tau, \mathbf{r}) = \frac{1}{4\pi} \frac{\partial}{\partial \tau} \left( \frac{Q(\tau)}{r \sqrt{1 - M_r^2}} \right)
\]

It is desirable to actually carry out the indicated differentiations, for the purposes of the computations to be performed as outlined in chapter 4. A number of useful relationships for working out the necessary derivatives are given in Table 3-1, together with some additional notational conventions which will be adopted. These relationships seem straightforward, but their subtlety should not be underestimated.

Two new source types which were not studied by Lowson will be useful in the
Table 3.1. Useful Formulas and Notations

\((t, \vec{x}) = \) observer coordinates

\((\tau, \vec{y}(\tau)) = \) source coordinates

\[ \vec{r} = \vec{x} - \vec{y}(\tau) = \vec{r}(\tau) \; ; \; \hat{r} = \frac{\vec{r}}{|\vec{r}|} \]

\[ \vec{M} = \frac{1}{c} \frac{d}{d\tau} \vec{y} = \) source Mach number; \[ \hat{M} = \frac{d}{d\tau} \vec{M} \; ; \text{etc.} \]

\[ M_r = \hat{r} \cdot M \]

\[ \dot{M}_r = \hat{r} \cdot \frac{d}{d\tau} \vec{M} \quad \text{NOT} \quad \frac{d}{d\tau} M_r \]

\[ \ddot{M}_r = \hat{r} \cdot \frac{d^2}{d\tau^2} \vec{M} \quad \text{NOT} \quad \frac{d^2}{d\tau^2} M_r \]

\[ \frac{\partial}{\partial t} = \frac{1}{1 - M_r^2} \frac{d}{d\tau} \], \text{ assuming } \vec{x} \text{ is fixed}

\[ \nabla|_t = \nabla|_\tau - \frac{\hat{r}}{c} \frac{1}{1 - M_r} \frac{d}{d\tau} \]

Unless otherwise noted, \( \nabla \) refers to the derivative with respect to observer coordinates. When used without subscripts, \( \nabla \) means \( \nabla|_t \).

92
\[
\frac{\hat{\mathbf{r}}}{\hat{\tau}} = -c\mathbf{M} ; \quad \frac{\mathbf{r}}{\hat{\tau}} = -c\mathbf{M} \mathbf{r} ; \quad \frac{\hat{\mathbf{r}}}{\hat{\tau}} = \frac{c}{r} (\hat{\mathbf{M}} \mathbf{r} - \bar{\mathbf{M}})
\]

\[
\frac{d\mathbf{M}}{d\tau} = \dot{\mathbf{M}} + \frac{c}{r} (\mathbf{M}^2 - \bar{\mathbf{M}}^2) ; \quad \frac{d\mathbf{M}}{d\tau} = \dot{\mathbf{M}} + \frac{c}{r} (\dot{\mathbf{M}} \mathbf{M} - \dot{\bar{\mathbf{M}}} \bar{\mathbf{M}})
\]

\[
\frac{d}{d\tau} \left( \frac{1}{r^p (1 - M_r)^q} \right) = \frac{1}{r^p (1 - M_r)^{q+1}} \left\{ q\dot{\mathbf{M}} + \frac{c}{r} (p\mathbf{M} + (q-p)\mathbf{M}^2 - q\bar{\mathbf{M}}^2) \right\}
\]

\[\nabla |_{\tau} \mathbf{r} = \mathbf{e} ; \quad \nabla |_{\tau} \mathbf{r} = \hat{\mathbf{r}} ; \quad \nabla |_{\tau} \hat{\mathbf{r}} = \frac{1}{r} (\mathbf{e} - \hat{\mathbf{r}} \hat{r}) \]

\[\nabla |_{\tau} \mathbf{M} = 0 ; \quad \nabla |_{\tau} \mathbf{M} = -\frac{1}{r} (\hat{\mathbf{M}} \mathbf{r} - \bar{\mathbf{M}}) \]

\[\nabla |_{\tau} \dot{\mathbf{M}} = \frac{1}{r} \left( \dot{\mathbf{M}} - \hat{\mathbf{M}} \right) \]

\[\nabla |_{\tau} \left( \frac{1}{r^p (1 - M_r)^q} \right) = \frac{1}{r^{p+1} (1 - M_r)^{q+1}} \left\{ q\bar{\mathbf{M}} - \left[ p + (q-p)\mathbf{M} \right] \hat{\mathbf{r}} \right\} \]
following analysis. Both of these consist of couplets of mass sources, which are separated slightly along some direction in spacetime. The sources are taken to have equal and opposite strengths, which are inversely proportional to their separation. In the first case, the separation is spacial, and the couplet would be called a dipole in electromagnetic theory. In the second case, the separation between the sources in the couplet is temporal. This type of source has never been previously discussed to the best of my knowledge, so I have taken the liberty of christening it a "chronopole". Unfortunately, in the aeroacoustic literature, the term "dipole" usually refers to a point force, which is altogether different from a couplet of mass sources with a spacial separation. To avoid confusion, I shall refer to such a couplet as a "doublet" rather than as a "dipole" in the following discussion.

The formulas for the pressure and velocity potential fields resulting from moving point couplets can be worked out from the Lienard Wiechert result. The easiest way to do this is to work out each of the four possible components of a couplet by differentiating the formulas for a point mass source. Alternatively, a distribution of doublets $d(t,x)$ is equivalent to a mass source distribution $q(t,x) = -\nabla \cdot d(t,x)$. Similarly, a distribution of chronopoles $c(t,x)$ is equivalent to a mass source distribution $q(t,x) = -\partial c/\partial t$. Therefore, the velocity potential for a moving point doublet is identical to the formula for the pressure due to a moving point force; the pressure is simply the time derivative of this expression. Similarly, the velocity potential for a moving point chronopole has the same form as the pressure for a moving point mass source, except for a sign; the corresponding pressure field is the time derivative of this expression.

Four types of sources have been considered here: mass sources, body forces, doublet distributions, and chronopole distributions. The formulas for the pressure
and for the velocity potential for moving point mass sources, chronopoles, or doublets can be expressed as spacial and temporal derivatives of the Lienard Wiechert potential. The formula for the pressure field due to a moving point force may also be expressed in this fashion. The explicit formulas for each of these cases are given in Table 3-2. The velocity potential in the case of a moving point force is more difficult to compute; this is the subject of the next subsection.

3.2.3 Solutions to the Vector Wave Equation

An understanding of the nature of the flow field caused by a moving point force acting on the fluid can be obtained only from a study of the vector wave equation (3-8). This equation adds nothing to the understanding of the flow field in the case of mass sources, doublets, or chronopoles, since the scalar wave equation (3-6) for the velocity potential adequately describes each of these situations. Therefore, in this section, the only source type which will be considered is the body force \( \mathbf{f} \), specifically, the moving point force given by (3-10).

Taking the curl of both sides of equation (3-8) yields an equation for the vorticity \( \zeta = \nabla \times \mathbf{u} \):

\[
\rho \frac{\partial \zeta}{\partial t} = \nabla \times \mathbf{f}
\]  

(3-15)

This equation is the linearized version of the Helmholtz equation for the vorticity field. It excludes the convection of vorticity by the motion of the fluid, since that is a nonlinear phenomenon. If the body force \( \mathbf{f} \) is not irrotational, then vorticity is generated in the fluid according to (3-15). This explains the difficulty in writing an equation for the velocity potential when \( \mathbf{f} \) is not a potential force field.

Since the force field of interest, (3-10), is certainly not irrotational, a further study of (3-15) is enlightening. At any given instant of time, \( t \), the force distribution
Table 3.2. Formulas for Moving Point Sources

(a) **Mass Source** (monopole): \( q(t, \vec{x}) = Q(t) \, \delta(\vec{x} - \vec{y}(t)) \)

\[
\phi(t, \vec{x}) = \frac{1}{4\pi \rho_0} \frac{Q(\tau)}{r|1-M_r|}, \text{ where } \tau = t - \frac{1}{c} r(\tau), \\
\quad r = \vec{x} - \vec{y}(\tau)
\]

\[
p(t, \vec{x}) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left( \frac{Q(\tau)}{r|1-M_r|} \right) \\
= \frac{\text{sgn}(1-M_r)}{4\pi r(1-M_r)^2} \left\{ \dot{Q} + \frac{Q}{1-M_r} \left[ \ddot{M}_r + \frac{C}{r} \left( M_r - \bar{M}^2 \right) \right] \right\}
\]

(b) **Doublet**: \( q(t, \vec{x}) = -\nabla \cdot \left( \vec{D}(t) \, \delta(\vec{x} - \vec{y}(t)) \right) \)

\[
\phi(t, \vec{x}) = -\frac{1}{4\pi \rho_0} \nabla \cdot \left( \frac{\vec{D}(\tau)}{r|1-M_r|} \right) \\
= \frac{\text{sgn}(1-M_r)}{4\pi \rho_0 c r(1-M_r)^2} \left\{ \hat{r} \cdot \vec{D} + \frac{\hat{r} \cdot \vec{D}}{1-M_r} \left[ \ddot{M}_r + \frac{C}{r} \left( 1-\bar{M}^2 \right) \right] - \frac{C}{r} \bar{M} \cdot \vec{D} \right\}
\]

\[
p(t, \vec{x}) = -\frac{1}{4\pi} \frac{\partial}{\partial t} \nabla \cdot \left( \frac{\vec{D}(\tau)}{r|1-M_r|} \right) \\
= \frac{1}{4\pi c r|1-M_r|^3} \left\{ \hat{r} \cdot \vec{D} - 2 \frac{C}{r} \bar{M} \cdot \vec{D} + \frac{\hat{r} \cdot \vec{D}}{1-M_r} \right\} \\
\quad \cdot \left[ 3\ddot{M}_r + \frac{C}{r} \left( 1 + 2M_r - 3\bar{M}^2 \right) \right]
\]
Table 3-2: (Continued)

\[
- \frac{c}{r} \hat{\mathbf{M}} \cdot \hat{\mathbf{D}} - \frac{c}{r} \frac{\hat{\mathbf{M}} \cdot \hat{\mathbf{D}}}{1-M_r} \left[ 3 \hat{M}_r + \frac{c}{r} \left( 1 + 2M_r - 3\bar{M}^2 \right) \right] \\
+ \frac{\hat{r} \cdot \hat{D}}{(1-M_r)} \left[ \left( \hat{M}_r - 3 \frac{c}{r} \hat{M} \cdot \hat{M} \right) (1-M_r) + 3 \left( \hat{M}_r + \frac{c}{r} \left( 1-\bar{M}^2 \right) \right) \right] \\
\cdot \left( \hat{M}_r + \frac{c}{r} (M_r - M^2) \right) \right]
\]

(c) **Chronopole:** \( q(t, \bar{x}) = - \frac{\partial}{\partial t} \left[ C(t) \right] \delta(\bar{x} - \bar{y}(t)) \)

\[
\phi(t, \bar{x}) = - \frac{1}{4\pi \rho_o} \frac{\partial}{\partial t} \left[ \frac{C(t)}{r |1-M_r|} \right] \\
= - \frac{\text{sgn}(1-M_r)}{4\pi \rho_o r (1-M_r)^2} \left\{ \hat{C} + \frac{C}{1-M_r} \left[ \hat{M}_r + \frac{c}{r} (M_r - \bar{M}^2) \right] \right\}
\]

\[
p(t, \bar{x}) = - \frac{1}{4\pi} \frac{\partial^2}{\partial t^2} \left[ \frac{C(t)}{r |1-M_r|} \right] \\
= - \frac{-1}{4\pi r |1-M_r|^3} \left\{ \hat{\hat{C}} + \frac{\hat{C}}{1-M_r} \left[ 3 \hat{M}_r + \frac{c}{r} \left( M_r + 2M_r - 3\bar{M}^2 \right) \right] \right\} \\
+ \frac{C}{(1-M_r)^2} \left[ \left( \hat{M}_r - 3 \frac{c}{r} \hat{M} \cdot \hat{M} \right) (1-M_r) + 3 \left( \hat{M}_r - \frac{c}{r} \bar{M}^2 \right)^2 \right]
\]

97
Table 3.2: (Continued)

\[ + \frac{c}{r} \left( M_r + \frac{M_r^2}{M_T} \right) \left( \ddot{M}_r - \frac{c}{r} \ddot{M} \right) + 3 \frac{c^2}{r^2} M_r \]

(d) **Force:** \( \vec{f}(t, \vec{x}) = \vec{F}(t) \delta(\vec{x} - \vec{y}(t)) \)

\[
\phi(t, \vec{x}) = -\frac{1}{4\pi \rho_o} \int_t^{\infty} dt' \nabla \cdot \left( \frac{\vec{F}(\tau)}{r_1-M_r} \right)
\]

\[
= \frac{1}{4\pi \rho_o c} \frac{\hat{r} \cdot \vec{F}}{r_1-M_r} + \frac{1}{4\pi \rho_o} \int_{-\infty}^{T} dt' \frac{\hat{r}(t') \cdot \vec{F}(t')}{r(t')^2}
\]

\[
p(t, \vec{x}) = -\frac{1}{4\pi} \nabla \cdot \left( \frac{\vec{F}(\tau)}{r_1-M_r} \right)
\]

\[
= \frac{\text{sgn}(1-M_r)}{4\pi cr(1-M_r)^2} \left\{ \hat{r} \cdot \dddot{F} + \frac{\hat{r} \cdot \vec{F} \dddot{M}_r}{1-M_r} + \frac{c}{r} \left( 1-M_r^2 \right) - \frac{c}{r} \dddot{M} \cdot \vec{F} \right\}
\]
is a delta function centered at the point \( y(t) \). The curl of this distribution, 
\[-\mathbf{F}(t) \times \nabla \delta(x-y(t)),\] may be visualized by means of approximate delta functions. As illustrated in Fig. 3-1, \( \nabla \times \mathbf{f} \) is a toroidal vector field, where the size of the torus is determined by the extent of the approximate delta function employed in the visualization. While the details of this toroidal field depend on the particular approximate delta function, the magnitude of the dipole moment of this field does not:

\[
\text{moment} = \frac{1}{2} \oint d^3 x \times (\nabla \times \mathbf{f}) \\
= -\frac{1}{2} \int d^3 x \times (\mathbf{F} \times \nabla \delta(x)) \\
= -\frac{1}{2} \int d^3 x \left\{ -(\mathbf{F} \times \nabla) \times x \delta(x) \right\} + \delta(x)(\mathbf{F} \times \nabla) \times x_0 \\
= + \frac{1}{2} \int d^3 x \mathbf{F} \times \hat{x} \delta(x) - \frac{1}{2} \int d^3 x \delta(x) \{-2 \mathbf{F}\} \\
= + \mathbf{F} 
\]

According to (3-15), the vorticity field is the time integral of the toroidal vector field described in the preceding paragraph. Consequently, if the force is an impulse, \( \mathbf{F}(t) = I \delta(t) \), an infinitesimal vortex ring will be created at \( t = 0 \). While the strength of the vorticity in this ring depends on the approximate delta function used to visualize it, the preceding analysis shows that the dipole moment of the vortex distribution is simply \( I/\rho_0 \). After it has been created, the vortex ring remains stationary and persists indefinitely according to (3-15). This is not surprising, considering that the nonlinear terms in the Helmholtz equation which give rise to the convection of vorticity have been omitted here.

A general time dependent force \( \mathbf{F}(t) \) may be regarded as a succession of impulses. Each such impulse creates an infinitesimal ring vortex at the point \( y(t) \), whose dipole moment is given by the strength of that particular impulse. None of the vortex rings so created move after their creation; all persist indefinitely. The moving point force therefore deposits a trail of vorticity at the points of its
The curl of a localized force distribution $F_\delta(x)$ is equal to $-\nabla \times \delta(x)$.

- **Gradient of Delta Function (Vector Field)**
  - Two dimensions
  - Three dimensions

- **Approximate Delta Function in Two Dimensions (Scalar Field)**

**Figure 3-1: Infinitesimal Vortex Rings**
trajectory, but the flow field remains irrotational everywhere except along this continuously lengthening curve (see Fig. 3-2). If the force $F$ is always parallel to its direction of motion, the deposited vorticity resembles an infinitely thin solenoid. Such a distribution of vorticity is associated with a jetlike flow inside this vortex tube (solenoid), which resembles the jetlike flow associated with the profile drag force. Similarly, if the force $F$ is always perpendicular to its direction of motion, the deposited vorticity resembles the vortex sheet shed behind an lifting wing of infinitesimal span (an infinitesimal "horseshoe vortex").

Since the vorticity is nonzero everywhere except at the points of the trajectory $y(t)$ for times $t$ in the past, it should be possible to express the velocity field in terms of a velocity potential everywhere except at these points. Indeed, when the Green function for the vector wave operator in (3-8) is calculated, it is found to have precisely this character. The calculation is performed in appendix A. The result is that the velocity field for a distributed momentum source $f(t,x)$ is given by

$$ u(t,x) = -\nabla \phi(t,x) + \left(1/\rho_0\right) \int_{-\infty}^{t} f(\tau,x) \, d\tau $$

where the velocity potential $\phi$ is

$$ \phi(t,x) = \int d\tau \, \mathbf{v}(\tau,y) \cdot \mathbf{G}(t-\tau,x-y) $$

The vector Green function $\mathbf{G}(t,x)$ is given in the appendix.

For the case of a moving point force (3-10), the velocity potential completely describes the flow field everywhere except at the points where the point force has deposited vorticity. The expression for the velocity potential is easily worked out from the force distribution (3-10) and the vector Green function given in the appendix:

$$ \Phi(t,x) = \frac{1}{4\pi \epsilon_0 c} \int_{-\infty}^{t} \frac{d\tau}{r} \cdot \mathbf{F}(\tau) \cdot \left[ \frac{\delta(t-\tau - \frac{r}{c})}{r} + \frac{\delta(t-\tau - \frac{r}{c})}{r^2} \right] $$

$$ = \frac{1}{4\pi \epsilon_0 c} \frac{\delta \cdot \mathbf{F}(t)}{r \left| \mathbf{M} \right|} + \frac{1}{4\pi \epsilon_0 c} \int_{-\infty}^{t} \frac{d\tau}{r^2} \frac{\mathbf{r}(\tau) \cdot \mathbf{F}(\tau')}{r(\tau)^2} $$

101
Flow field is irrotational except for points on the source trajectory.

Case 1: Force perpendicular to its direction of motion.

Case 2: Force parallel to its direction of motion.

Figure 3-2: Flow Field of a Moving Point Force
The remaining integral in this equation represents the effects of the vorticity deposited in the wake of the moving force. Later, it will be demonstrated that in real situations, in which this residual vorticity moves, this integral should be replaced by a similar integral over the actual location of the shed vorticity, rather than over the trajectory of the point force.

The explicit expressions for both the velocity potential and the pressure fields (i.e. for the complete flow field) of moving point monopoles (mass sources), forces, doublets, and chronopoles are given in Table 3-2. These expressions are used as Green functions in the following analysis.

3.3 Ffowcs Williams' and Hawking's Theory

3.3.1 The Approach

Before the solution of the wave equation, expressed in terms of the conditions on a moving boundary, is treated as a straightforward generalization of the usual discussion of Laplace's equation in three dimensions, the more cumbersome approach due to Ffowcs Williams and Hawking [Ffowcs Williams 1969] will be described. This approach is presented here only because it is very popular in the aeroacoustic literature [Goldstein 1976, Farassat 1975]. Technically, the Ffowcs Williams and Hawking equation is correct, but physically it has little content. Even though some of the most important properties of the flow fields to which this
equation is applied cannot be visualized within the framework of this theory, the Ffowcs Williams and Hawkings equation bears a strong resemblance to the more correct theory which will be presented in section 3.5. Indeed, the calculations of propeller noise which will be performed in chapter 4 are equivalent to most theories which are based on the Ffowcs Williams and Hawkings equation, except for the treatment of induced drag forces.

The major dissimilarity between the problem solved by Lowson and the actual flow field being modelled is the fact that the real fluid has a boundary surface. Ffowcs Williams' and Hawkings' idea was to mathematically remove the boundary surface from the model of the actual flow field. This is accomplished by imagining a moving surface S which encloses the solid boundaries at all times. The actual interior of the surface S is discarded, to be replaced by part of an arbitrary, different flow field whose boundary surfaces are all external to the surface S. The splicing together of two distinct flow fields in this manner is illustrated in Fig. 3-3. It is imagined that the arbitrary fluid interior to S is chemically the same material as the actual fluid exterior to S. Notice that the new, spliced flow field has no boundary surface, but that sources of mass and of momentum must be placed at the surface S to account for the difference in the fluxes of these two quantities through the surface S in the two flow fields.

Thus, the flow field with boundary conditions, but no sources, has been converted to a problem in an unbounded fluid with sources of mass and of momentum at the moving surface S. The problem has thereby been converted into the problem solved by Lowson. The cost is that an arbitrary, new flow field had to be introduced to fill the interior of the surface S; since the solution to the unbounded flow problem will yield both the actual exterior flow and the arbitrary
Figure 3-3: Splicing Together Two Flow Fields
interior flow, the complexity of the problem seems to have doubled. However, since
the interior flow field is completely arbitrary, it is a simple matter to choose it to be
some trivial flow field (e.g. homogeneous and at rest), as Flowcs Williams and
Hawkinings originally did. This should not be done at an early stage of the
calculation, though, since the freedom to choose any interior flow field for the
interior of S can be used to great advantage. For now, the actual, exterior flow field
will be denoted by a subscript "1", its homogeneous conditions at infinity by a
subscript "0", and the arbitrary interior flow field by a subscript "2".

The magnitudes of the sources which must be placed at the surface S are easily
computed. The fluxes of mass and of momentum through a surface with unit
normal n which moves locally with velocity v are

\[
\text{mass source/unit area} = [\rho_1(u_1\cdot v) - \rho_2(u_2\cdot v)]\cdot n
\]

\[
\text{momentum source/unit area} = [\rho_1 u_1(u_1\cdot v) - \rho_2 u_2(u_2\cdot v)]\cdot n
\]

respectively. The difference between these fluxes evaluated just inside and just
outside S represents the mass and momentum which must be added to the fluid per
unit time per unit area of the surface at S. In order to obtain the mass and
momentum sources per unit volume, some infinitesimal thickness must be assigned
to the surface S. Since the actual value of this thickness will not affect the final
results, it is convenient to introduce a delta function \( \delta(n) \), where n is the coordinate
normal to the surface (n = 0 on S). This delta function may be read as "per unit
thickness of the surface S". The source distributions on S are then

\[
q = [\rho_1(u_1\cdot v) - \rho_2(u_2\cdot v)]\cdot n\delta(n)
\]

\[
f = [\rho_1 u_1(u_1\cdot v) - \rho_2 u_2(u_2\cdot v)]\cdot n\delta(n)
\]

(3-16)

If the first law of thermodynamics were important, a source of heat would also be
required at the surface S. However, this does not affect the following analysis, since
only the equations of motion will be considered.
3.3.2 The Equation

Flowes Williams and Hawkings follow the lead of Lighthill [Lighthill 1952] and rewrite the exact equations of motion in a form suggesting the linearized equations of acoustics. All terms which do not conform to the linearized equations of motion are considered to be sources of momentum (or mass) in an idealized, perfectly linear fluid (the so called acoustic analogy). When the sources of mass and of momentum which arose from the splicing operation at the surface $S$ are also included as source terms in the equations of motion, the equations become

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = q$$
$$\frac{\partial (\rho \mathbf{u})}{\partial t} + c^2 \nabla p = -\nabla \mathbf{T} + \mathbf{f}$$

where

$$\mathbf{T} = p + \rho \mathbf{uu} - 4c^2 \rho$$

is the so called Lighthill tensor.

Next, a wave equation for the density is obtained in a manner analogous to the derivation of (3-7). The result is

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) c^2 (\mathbf{e} - \mathbf{e}_0) = \nabla \nabla \times (\mathbf{T} - \mathbf{T}_0) + \frac{\partial}{\partial t} q - \nabla \cdot \mathbf{f}$$

The constants $\rho_0$ and $T_0$ have been introduced here in order to ensure that $\rho - \rho_0$ and $T - T_0$ vanish at infinity. This device greatly simplifies integration by parts in the following analysis.

Equation (3-19) can be rewritten as an integral equation by means of the free space Green function $G$ for the scalar wave operator.
\[ G(t,x) = \delta(t - r/c)/4\pi r \]

(3-20)

as

\[ c^2 (\varphi - \varphi_0) = \int_{\text{spacetime}} d\tau d^3y \left( \mp \mp \right) \nabla \nabla G + \int d\tau d^3y a \frac{\partial G}{\partial \xi} - \int d\tau d^3y F \cdot \nabla G \]

Owing to the delta function in \( G \), one of the four integrals in each of the last two terms may be easily performed. Also, \( \int d^3y \delta(n) = dA \), the element of area on the surface \( S \). Thus,

\[ c^2 (\varphi - \varphi_0) = \int_{\text{spacetime}} d\tau d^3y \left( \mp \mp \right) \nabla \nabla G + \int d\xi d\eta \frac{\partial G}{\partial \xi} \left( \frac{Q(\tau, \xi, \eta)}{4\pi r |1 - M_r|} \right) - \int d\xi d\eta \nabla \cdot \left( \frac{\mathbf{F}(\tau, \xi, \eta)}{4\pi r |1 - M_r|} \right) \]

(3-21)

Several new notations have been introduced with this equation. First, the surface \( S \) has been parametrized in terms of the curvilinear coordinates \((\xi, \eta)\) and the source time \( \tau \). That is, \( x \) lies on the surface \( S \) at the time \( \tau \) if and only if

\[ x = y(\tau, \xi, \eta) \]

(3-22)

where \((\xi, \eta)\) is any point in the curvilinear coordinate space describing the surface at time \( \tau \). In (3-21), the retarded time \( \tau \), the range vector \( r \), and the source Mach number \( M \) are given by the analogues of (3-12), (3-13), and (3-14):

\[ \tau = t - r(\tau, \xi, \eta)/c \]

(3-23)

\[ r = r(\tau, \xi, \eta) = x - y(\tau, \xi, \eta) \]

(3-24)

\[ v = v(\tau, \xi, \eta) = (\partial y/\partial \tau)\xi,\eta; M = v/c \]

(3-25)

Furthermore, the element of surface area \( dA \) on the surface \( S \) is given by
\[ \hat{n}dA = (\partial y / \partial \xi)_{\eta, \tau} \times (\partial y / \partial \eta)_{\xi, \tau} d\xi d\eta = \hat{n}A(\tau, \xi, \eta) d\xi d\eta \]  

(3-26)

And the sources of mass and momentum located at the patch of surface corresponding to \(d\xi d\eta\) have been written as

\[ q(\tau, \xi, \eta) = [\rho_1(u_1 - v) - \rho_2(u_2 - v)] \cdot \hat{n}A(\tau, \xi, \eta) \]
\[ F(\tau, \xi, \eta) = [\rho_1 u_1(u_1 - v) - \rho_2 u_2(u_2 - v)] \cdot \hat{n}A(\tau, \xi, \eta) \]  

(3-27)

(It helps to imagine that the coordinates \(\xi\) and \(\eta\) are dimensionless when interpreting this notation; \(A(\tau, \xi, \eta)\) is then the surface area of the patch.) The form of the equation given in (3-21) relies on the assumption that the boundary of the region in \((\xi, \eta)\) space which maps into \(S\) via (3-22) is independent of time. The \(d\xi d\eta\) integrals are shown to have the form given in (3-21) in appendix B.

A glance at Table 3-2 identifies the two \(d\xi d\eta\) integrals in (3-21): The first is simply a sum of moving point mass sources, one for each point of the surface \(S\), with strengths given by (3-27). The second is a similar sum of moving point forces located on the surface \(S\). In view of the mass and momentum sources which were placed at \(S\) in the process of splicing the interior and exterior flow fields, this interpretation of the terms in (3-21) is hardly surprising. The \(d\tau d^3y\) integral in (3-21) may be regarded as a sum of moving point sources located in the bulk of the fluid. The corresponding source type was not discussed in section 3.2 above; it is referred to as a "quadrupole" source in the aeroacoustics literature. (This is a misnomer in the same sense that the term "dipole" is a misnomer for "force".)

Flows Williams and Hawkings assume that the surface \(S\) coincides with an impervious solid boundary surface in the exterior flow problem. Furthermore, they choose the arbitrary interior flow to be homogeneous and at rest, with the same density and pressure as the fluid in the exterior field at infinity. (That is, conditions "2" are assumed to be identical to conditions "0".) Under these assumptions, the source terms (3-27) become
\[ Q(\tau, \xi, \eta) = \rho_0 \nu \cdot \hat{n} A(\tau, \xi, \eta) \]
\[ F(\tau, \xi, \eta) = (p_1 - 1 p_0) \cdot \hat{n} A(\tau, \xi, \eta) \]

In this specialized form, the integrands of (3-21) reference only the parameters of the physical, exterior flow field. Equation (3-21) with the source terms (3-28) is called the Flowcs Williams and Hawkings equation.

3.3.3 Logical Completion of the Theory

The Flowcs Williams and Hawkings equation (3-21) is simply a restatement of the equations of motion as an integral equation. However, this restatement is incomplete, since the value of the density alone is insufficient to determine the integrands on the right hand side of the equation. Also, it is clear that a single, scalar equation cannot have the same information content as the four equations of mass and momentum conservation. In order to logically complete the integral equations (3-21), the following vector wave equation may be derived from (3-17) in a manner analogous to the derivation of (3-8):

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla \nabla \right) \cdot \mathbf{e} \mathbf{u} = - \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \cdot (\nabla - \nabla_0) + \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{F} - \nabla q
\]

This equation may be reexpressed in integral form by means of the Green function \( G_L \) for the vector wave equation worked out in Appendix A. In short hand form, the result is

\[
\mathbf{e} \mathbf{u} = - \frac{1}{c^2} \int d\tau d\gamma (\nabla - \nabla_0) \cdot \nabla \frac{\partial}{\partial t} \mathbf{G}_L + \frac{1}{c^2} \int d\gamma d\eta \frac{\partial}{\partial t} \left( \int d\tau \mathbf{G}_L \cdot \mathbf{F} \right) \]

(3-29)

\[
- \int d\gamma d\eta \nabla \cdot \left( \int d\tau \mathbf{G}_L Q \right)
\]

Together, equations (3-21) and (3-29) could be used in lieu of the equations of mass and momentum conservation; the information content is identical. The only conceivable advantage of these integral equations is that they could be used
iteratively. That is, a starting guess for the flow field could be inserted into the integrals on the right hand sides of the equations, and used to generate a new guess for the mass and momentum densities. The Born approximation is an example of the first step of such an iterative procedure. In the present case, there is no guarantee that the iterative process would converge, of course. Also, yet another integral equation to replace the differential form of the conservation of energy would be necessary if the philosophy outlined here were to be carried to its ultimate extreme. This equation could take the form of an integral equation for the temperature, which also satisfies a wave equation in the limit of linearized acoustics.

3.3.4 Discussion and Criticism

The Ffowcs Williams and Hawkings equation frequently serves as the basis for theories of propeller and helicopter rotor noise. As will be fully discussed in the following section, the approximations of linear acoustics apply almost everywhere in these situations. In particular, \( T - T_0 = 0 \) (or is very small) throughout the fluid for all practical purposes. Therefore, the Ffowcs Williams and Hawkings equation becomes

\[
\rho \chi \left( c - c_0 \right) = \rho - \rho_c = \int d\xi \ d\eta \ \frac{\partial}{\partial t} \left( \frac{\rho \chi A(\tau, \xi, \eta)}{4\pi r |1 - M|} \right) - \int d\xi \ d\eta \ \nabla \cdot \left( \frac{(\rho - \rho_c) \chi A(\tau, \xi, \eta)}{4\pi r |1 - M|} \right)
\]

(3-30)

This equation appears to have achieved the original goal of expressing the pressure field everywhere in terms of its value on the boundary surface.

In fact, this expression is very nearly equivalent to the expression which will eventually be used in this thesis to compute propeller noise. However, the flow field suggested by the equation (3-30) is a very poor model for the actual flow field.
associated with a solid, impermeable object in motion. The major conceptual problem with the FLOWCS Williams and Hawkings theory is the failure to deal directly with the vortex sheet shed by a lifting body. Of course, the moving point forces which are summed in the second \( d\xi d\eta \) integral automatically shed a vortex sheet as they move, as discussed in section 3.2.3. However, the real vortex sheet, unlike the sheet associated with a linearized moving point force, moves after it has been created. This residual motion accounts for the induced drag force associated with the lift force, so it is clear that induced drag forces cannot be properly represented by (3-30).

Moreover, the derivation of equation (3-30) from (3-21) is suspect. The equation (3-21) has generated an enormous amount of confusion in the aeroacoustics literature, which is most evident in the misnomers which are commonly used to refer to the three integrals. The worst of these misnomers is the common reference to the integral involving the Lighthill tensor as "the volume integral". Indeed, the similar integral \( \int d\tau d^3y (T-T_0)G \) is a volume integral, by virtue of the delta function in \( G \). However, the integral

\[
\int d\tau d^3y (T-T_0)\nabla \nabla G = \int d\tau d^3y G \nabla \nabla (T-T_0)
\]

invariably contains a finite contribution from the surface of discontinuity \( S \), in addition to contributions from the bulk of the fluid. Only in the case that \( T-T_0 \) is continuous across \( S \) is there no contribution to this integral from the surface \( S \).

Indeed, there is a contribution to this integral from any surface of discontinuity in \( T-T_0 \), including a shed vortex sheet in the real exterior flow field, as well as the solid boundary surface \( S \). The problem is that the term \( \rho uu \) in \( T-T_0 \) is not precisely zero, even in the limit that \( p-p_0 = c^2(\rho-\rho_0) \) exactly. The discontinuity of this term in the Lighthill tensor across a shed vortex sheet gives rise to a finite contribution to
the "volume integral" at the surface of the vortex sheet. Additionally, the discontinuity of $\rho u u$ across S itself gives rise to contributions from the surface S which are additional to the two surface integrals in (3-30). If these contributions are ignored, then the $\rho u^2/2$ contribution to $p_2 - p_1$ in (3-30) must also be ignored for the sake of consistency.

Therefore, while (3-21) is mathematically a correct expression, it can be physically misleading. The equation is merely a mathematical tautology, with no more physical content than the equations of motion (indeed, with less according to the discussion of section 3.3.3). In fact, the constant $c^2$ in (3-21) is totally arbitrary; it need not be the square of the speed of sound. Despite its extreme conceptual weaknesses, the Flowcs Williams and Hawking's equation has become solidly entrenched in the literature. For this reason, a mathematical identity of considerable utility in dealing with (3-21) or (3-30) will be mentioned here.

Choosing various interior flow fields is a powerful way to generate alternative, equivalent representations of the equation (3-21). This freedom will be doubly exploited here by choosing both the internal and the external flow fields to be homogeneous and at rest. In this case, all velocities are zero, and all flow parameters are constants. Since the surface S is a complicated, moving surface, (3-21) is not a trivial statement, even in this case. If the pressure and density of the interior flow are related to those of the exterior flow by $p_2 - c^2 \rho_2 = p_1 - c^2 \rho_1$, then the integrand involving the Lighthill tensor $T - T_0$ is identically zero, and (3-21) becomes

$$c^2 (e - e_0) = \begin{cases} 0 & \text{outside } S \\ c^2 (e - e_0) & \text{inside } S \end{cases} = (e_2 - e_0) \int_e^d \frac{\partial}{\partial \tau} \left( \frac{\nabla \cdot A(\xi, \xi, \eta)}{\nabla \cdot r \mid 1 - \nabla \cdot r} \right) + (p_2 - p_1) \int_e^d \frac{\partial}{\partial \eta} \nabla \cdot \left( \frac{\nabla A(\xi, \xi, \eta)}{\nabla \cdot r \mid 1 - \nabla \cdot r} \right)
$$

Taking into account the assumed relationship between the interior and exterior
pressures and densities, this yields the identity
\[
\frac{1}{c^2} \int_S d\xi \, d\eta \, \frac{\partial}{\partial \tau} \left( \frac{\vec{\nabla} \cdot \vec{A}(\xi, \eta)}{4\pi r |1-M_r|} \right) + \int_S d\xi \, d\eta \, \nabla \cdot \left( \frac{\vec{\nabla} \cdot \vec{A}(\xi, \eta)}{4\pi r |1-M_r|} \right) = \begin{cases} 
0, & \text{for observer locations outside } S \\
1, & \text{for observer locations inside } S 
\end{cases}
\tag{3-31}
\]

By means of this identity, (3-30) could be rewritten in the equivalent form
\[
c^2 (\varphi - \varphi_c) = -\int_S d\xi \, d\eta \, \nabla \cdot \left( \frac{(\rho_1 - \rho_2 + c^2 \rho_2) \vec{\nabla} \cdot \vec{A}(\xi, \eta)}{4\pi r |1-M_r|} \right) 
\tag{3-32}
\]

among several possibilities. An alternative proof of (3-31) is given in appendix C beginning from a more fundamental starting point. By taking other combinations of spliced trivial flow fields, a number of similar identities can easily be generated. The identity (3-31) has an interesting geometrical significance. As shown in appendix C, it is a four dimensional analogue of the three dimensional concept of solid angle:
\[
\int_S dA \, \frac{\vec{\nabla} \cdot \vec{A}}{r^2} = \begin{cases} 
0, & \text{if origin is outside } S \\
4\pi, & \text{if origin is inside } S 
\end{cases}
\]

Users of the Fflowcs Williams and Hawking's equation will find (3-31) to be a useful test to ensure that computer algorithms designed to compute the two types of surface integrals are working properly.

3.4 The Flow Field of a Moving Object, Powell's Theory

The theory of vortex generated sound proposed by Powell [Powell 1964] is diametrically opposite to the Fflowcs Williams and Hawking's theory: The only thing wrong with the former is the mathematics, while the only thing right with the latter is the mathematics. Powell begins with a picture of a flow field and tries to
develop a theory around that picture. Only the picture will be presented here. The mathematical flaws in the theory will not be detailed. The theory to be developed in section 3.5 goes hand in hand with Powell's beautiful physical picture of the process of sound production.

This picture is based on the idea that the distribution of vorticity in a flow field determines the velocity distribution. Strictly speaking, this idea applies only to the case of an incompressible flow; in a compressible flow, the velocity distribution at any given time depends on the entire past history of the vorticity distribution. The major simplification achieved by thinking in terms of the vorticity is that vorticity is usually confined to a relatively small volume of the fluid. The flow fields of wings and of propellers described in chapter 2 are relevant examples of this fact. Because of its limited spacial extent, it is reasonable to think of the vorticity as a "source" field for the entire flow field.

Powell's premise is that sound is generated whenever vortex loops are created, or when they are accelerated. Vorticity in uniform motion is associated with a steady flow field, not with a sound field. Free distributions of vorticity -- for example, the vortex sheet shed behind a lifting airfoil -- are invariably accelerated by virtue of their own instability (at least at high Reynolds numbers). This is the mechanism of sound generation by turbulence, which will not be discussed here. However, the creation of vorticity by a lift force, or by the profile drag mechanism also produces sound, which is very relevant to the present discussion. Also, a moving object is invariably associated with a sheet of vorticity at its surface due to the abrupt change in the tangential velocity in the boundary layer. Changes in the motion of an object will therefore cause changes in the surface vorticity which generate sound even in the absence of any lift or profile drag forces.
develop a theory around that picture. Only the picture will be presented here. The mathematical flaws in the theory will not be detailed. The theory to be developed in section 3.5 goes hand in hand with Powell's beautiful physical picture of the process of sound production.

This picture is based on the idea that the distribution of vorticity in a flow field determines the velocity distribution. Strictly speaking, this idea applies only to the case of an incompressible flow; in a compressible flow, the velocity distribution at any given time depends on the entire past history of the vorticity distribution. The major simplification achieved by thinking in terms of the vorticity is that vorticity is usually confined to a relatively small volume of the fluid. The flow fields of wings and of propellers described in chapter 2 are relevant examples of this fact. Because of its limited spacial extent, it is reasonable to think of the vorticity as a "source" field for the entire flow field.

Powell's premise is that sound is generated whenever vortex loops are created, or when they are accelerated. Vorticity in uniform motion is associated with a steady flow field, not with a sound field. Free distributions of vorticity -- for example, the vortex sheet shed behind a lifting airfoil -- are invariably accelerated by virtue of their own instability (at least at high Reynolds numbers). This is the mechanism of sound generation by turbulence, which will not be discussed here. However, the creation of vorticity by a lift force, or by the profile drag mechanism also produces sound, which is very relevant to the present discussion. Also, a moving object is invariably associated with a sheet of vorticity at its surface due to the abrupt change in the tangential velocity in the boundary layer. Changes in the motion of an object will therefore cause changes in the surface vorticity which generate sound even in the absence of any lift or profile drag forces.
For the problems of interest here, the vorticity is always distributed in thin sheets. These vortex sheets are equivalent to a discontinuity in the velocity potential across the sheet. This intimate relationship to the "source" of the flow field gives the velocity potential a tremendous advantage over, say, the pressure field as a description of the flow. Additionally, the velocity potential field is linear both in linear acoustics and in incompressible flow theory, while the pressure field is linear only in the former case. Specifically, if the compression of fluid elements is not large (i.e. if the fluid velocity is much less than sonic) then

$$p - p_0 = \rho_0 \Phi / \partial t - \rho_0 u^2 / 2$$

(3-33)

This nonlinear equation for the pressure applies to any irrotational flow field, whether acoustic or incompressible.

Assuming again that the fluid velocity is much less than sonic speed, conservation of mass may be written in the form

$$\frac{1}{c^2} \frac{\partial}{\partial t} (\rho - \rho_2) + \frac{1}{c^2} \nabla \cdot (\rho \nabla (\rho - \rho_2) + (\rho - \rho_2) \nabla \cdot \mathbf{v} + \rho_2 \nabla \cdot \mathbf{v} = 0$$

(3-34)

Far from the object, the fluid velocity is very small, and all fluid parameters are very nearly equal to their values at infinity. Therefore, only the first term on the right hand side of (3-33), and the first and fourth terms on the left hand side of (3-34) need be retained. Thus, in the far field, the equations of linear acoustics apply, and the velocity potential satisfies

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 0$$

(3-35)

In fact, this linear wave equation for the velocity potential applies under considerably more general circumstances. The flow fields of interest are those
generated by a solid object in motion through the fluid. In this chapter, the reference frame in which the bulk of the fluid is at rest and the object is in motion will always be adopted. Suppose that the object has a characteristic speed $U$, a characteristic dimension parallel to its motion $\chi$, and a characteristic dimension transverse to its direction of motion $\varepsilon$. For an object consisting of airfoil sections, $\chi$ would be a typical chordlength, while $\varepsilon$ would be a typical thickness.

Close to the object, the magnitude of the fluid velocity $u$ may be estimated as $U\varepsilon/\chi$. The term $\partial \Phi/\partial t$ is of order $(U\varepsilon)U/\chi$, so that the pressure $p-p_0$ may be estimated to be of order $\rho_0 U^2 \varepsilon/\chi$, according to (3-33). This estimate assumes that $\varepsilon$ is no larger than $\chi$, and that $M^2(\varepsilon/\chi)$ is much less than unity, where the Mach number $M$ is defined as $U/c$. The ratios of the four terms in (3-34) are therefore of order

$$M^2 : (\varepsilon/\chi)M^2 : (\varepsilon/\chi)M^2 : 1$$

The two middle terms can be neglected, provided only that the parameter $(\varepsilon/\chi)M^2$ is much less than unity. Thus, the acoustic wave equation (3-35) for the velocity potential applies everywhere throughout the fluid, provided that one of two conditions is met: Either the motion of the object through the fluid is much slower than sonic speed ($M^2 \ll 1$), or the object is very thin $(\varepsilon/\chi \ll 1)$, or both.

The wave equation (3-35) is therefore valid throughout the fluid when the flow in the immediate vicinity of the object is nearly incompressible. This is quite obvious, since in the limit that the sound speed becomes infinite, (3-35) reduces to Laplace's equation, which describes an incompressible flow field. Indeed, the method of matched asymptotic expansions which is often employed [Obermeier 1979] to determine the sound field at a great distance from the nearly incompressible flow field in the immediate vicinity of a slowly moving object seems
totally unnecessary: The wave equation for the velocity potential applies uniformly in such situations, not just in the far field. (It is the pressure field which does not satisfy the wave equation in the near field of a slowly moving object.) Secondly, the wave equation for the velocity potential is uniformly valid throughout the fluid in the case of the motion of a thin airfoil, even when the speed of the airfoil approaches sonic speed, since $\epsilon/\chi \ll 1$.

The boundary of the region of irrotational flow may include a free vortex sheet, as well as the solid boundary surface. In this case, there is a discontinuity in the velocity potential across the sheet, so that the wave equation (3-35) does not apply at the vortex sheet. However, provided that the residual motion of the free vortex sheet is much less than sonic velocity, the order of magnitude estimates given above show that the wave equation (3-35) does describe the velocity potential everywhere in space outside the vortex sheet.

In summary, the wave equation for the velocity potential applies throughout the irrotational region of the fluid flow in the situations of greatest interest here. These physical arguments provide a background for the mathematical theory developed in the following section which was lacking in the case of the Flowcs Williams and Hawkings theory. Now that the conditions under which the wave equation (3-35) applies are known, it is a relatively straightforward task to express the solution to this wave equation in terms of the boundary conditions. The boundary, of course, must include not only the solid boundary surface, but also any free vortex sheets.
3.5 Morgans' Extension of Kirchhoff's Formula

3.5.1 Tooling Up

In the case that the boundary surfaces are stationary, the classical formula of Kirchhoff [Jackson 1975b] is the desired solution to the wave equation (3-35) in terms of conditions at the boundary. This formula was first extended to the general case in which the boundary surfaces are in arbitrary motion by Morgans [Morgans 1930]. The allegation in Flowcs Williams and Hawkings [Flowcs Williams 1969] that Morgans' solution to this problem is incorrect is itself quite false; the extension to Kirchhoff's formula given by Morgans is flawless. The mathematical tools used by Morgans in his derivation, as well as the tools used in all of the literature on aeroacoustics, are entirely inadequate for the task at hand.

In order to fully comprehend the nature of the wave equation when the boundary surfaces are in motion, it is necessary to introduce a few rudimentary concepts of differential geometry [Misner 1973]. I have been unable to devise a notation which is at once as powerful as the notation of differential forms, and is immediately clear to someone who has no experience with differential geometry. I apologize for the piebald notation I shall use; I hope that it will prove comprehensible to those without training in differential forms. The basic problem with all the mathematical treatments of the wave equation in aeroacoustics literature is that they do not recognize the nature of an integral over spacetime. A thorough understanding of the nature of multiple integrals can come only after the study of differential geometry.

I do not propose to write a tract on that subject. However, only a very minor, yet exceedingly powerful result from the theory of differential forms is needed here.
This result is easily understood as the generalization of the divergence theorem, Stokes' theorem, Green's theorem, the fundamental theorem of calculus, and several other commonly used theorems. There is really only one truth behind all of these theorems: The integral of the derivative of a function can be expressed in terms of the values of the function itself at the boundaries of the region of integration. Working in spacetime with only spacial gradient operators and partial time derivatives as tools is precisely equivalent to working in three dimensions using only partial derivatives along the coordinate directions. This lack of the proper mathematical tool is absolutely crippling; it is well known that Isaac Newton himself was stumped by as simple a result as Gauss' divergence theorem because of the lack of vector concepts.

In spacetime, the necessary concepts are those of four vectors and forms. It must be emphasized that only affine four vectors are necessary in integration theory; no symmetry is necessary under Galilean transformations. Indeed, the divergence theorem applies to any three dimensional affine space; symmetry under rotations is absolutely unnecessary. The coordinate four vector, and the four gradient operator (which is actually a form, not a vector), will be denoted by

\[ \mathbf{X} = (t, \mathbf{r}) \quad \text{and} \quad \nabla = \left( \frac{\partial}{\partial t}, \nabla \right) \]

respectively. Since only affine (i.e. volumetric) symmetry is important here, the fact that the dimensions of the timelike and spacelike parts of these objects differ is of no importance. An arbitrary, four dimensional volume will be denoted by \( V \), while its three dimensional boundary surface will be called \( S \).

In three dimensions, a general form of the divergence theorem for a three dimensional volume \( V \) and its two dimensional boundary \( S \) reads
\[ \int d^3 y \nabla(\text{anything}) = \int d\Sigma(\text{anything}) \]

It is a common and heinous misconception that the element of surface area \( dA \) is a vector perpendicular to the surface \( S \). Actually, \( dA \) is a form whose level surfaces are parallel to the surface \( S \), just as \( \nabla(\text{function}) \) is a form whose level surfaces are parallel to surfaces of constant (function). Indeed, if the surface \( S \) is expressed parametrically in terms of the curvilinear coordinates \((\xi, \eta)\) (via (3-22) with \( \tau \) fixed), then the directed element of surface area should be written

\[
d\vec{A} = \det \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} \, d\xi \, d\eta
\]

where \( y = (x \ y \ z) \).

This analysis, unlike the usual vector analysis, extends easily to the case of four dimensions, even though there is no notion of perpendicularity to a three surface \( S \) in spacetime. The required four dimensional analogy to the generalized three dimensional divergence theorem is

\[
\int d\tau d^3 y \vec{\nabla}(\text{anything}) = \int d\Sigma(\text{anything}) \tag{3-36}
\]

The element of three dimensional "surface area" \( d\Sigma \) is given by
Here, the boundary S has been parametrized by \( X = X(\xi, \eta, \zeta) \), and \( X = (t \ x \ y \ z) \). The proof of (3-36) is not difficult, but it is profound. I would spoil its beauty and generality by attempting to prove it in a few lines using the present bastardized notation. A true understanding of the divergence theorem in three dimensions renders (3-36) and (3-37) obvious.

In order to put (3-36) into a form which is computationally convenient, the three dimensional boundary surface S will be parametrized by \((\tau, \xi, \eta)\), as in (3-22). For this particular parametrization, (3-37) becomes

\[
\alpha \Sigma = \det \begin{vmatrix}
\hat{\tau} & \hat{x} & \hat{y} & \hat{z} \\
\frac{\partial \hat{x}}{\partial \xi} & \frac{\partial \hat{y}}{\partial \xi} & \frac{\partial \hat{z}}{\partial \xi} \\
\frac{\partial \hat{x}}{\partial \eta} & \frac{\partial \hat{y}}{\partial \eta} & \frac{\partial \hat{z}}{\partial \eta} \\
\frac{\partial \hat{x}}{\partial \zeta} & \frac{\partial \hat{y}}{\partial \zeta} & \frac{\partial \hat{z}}{\partial \zeta}
\end{vmatrix} d\tau \, d\xi \, d\eta
\]

\[
= \left\{ \left( \frac{\partial \hat{x}}{\partial \xi} \times \frac{\partial \hat{x}}{\partial \eta} \right) \cdot (\nabla \cdot \vec{\tau} - \hat{x} \hat{x} - \hat{y} \hat{y} - \hat{z} \hat{z}) \right\} d\tau \, d\xi \, d\eta
\]

The sign of the element of surface \( d\Sigma \) depends on a convention for right and left handedness. In order that this convention agree with the usual convention of an outward normal \( \hat{n} \) in the three dimensional case, the element of surface volume will be taken as
\[ d\Sigma = (\mathbf{\mathbf{v}} \cdot \mathbf{n}) A(\tau, \xi, \eta) d\tau d\xi d\eta \]  
\[ (3-38) \]

With this particular parametrization and sign convention, the four dimensional generalization of the divergence theorem may be rewritten in the form
\[ \int \nabla \cdot \mathbf{f} d^3 y = \int \frac{\partial}{\partial \tau} \mathbf{f} \cdot d\Sigma \]
\[ (3-39) \]

3.5.2 Derivation of Morgans' Equation

With the appropriate mathematical tools, generalizing the Kirchhoff formula to the case of a moving boundary surface is quite simple. The development exactly follows the analysis of Laplace's equation given in Jackson [Jackson 1975b]. With the notation of appendix A, the analogue of Green's identity is
\[ \phi \Box G - G \Box \phi = \nabla \cdot M \cdot (\phi \Box G - G \Box \phi) \]  
\[ (3-40) \]

Here, \( M \) is the four by four constant matrix
\[ M = \begin{pmatrix} \frac{1}{c^2} & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \]  
\[ (3-41) \]

Equation (3-40) follows from the fact that
\[ \Box = \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} - \nabla^2 = \nabla \cdot M \cdot \nabla \]

which also explains the meaning of the "\( \cdot \)" contraction operator. The equation (3-40) is true for any two functions \( \phi \) and \( G \), but \( \phi \) will be chosen to be the velocity potential and \( G \) will be taken to be the Green function (3-20).

Under the conditions detailed in section 3.4, the velocity potential may be assumed to satisfy the acoustic wave equation throughout the region of irrotational flow. Therefore, if (3-40) is integrated over all of spacetime \( V \) exterior to the three dimensional boundary \( S \), the result is

123
\[
\int d\tau d^3y \Phi \Box G = \int d\tau d^3y \nabla \cdot M \cdot (\Phi \nabla G \cdot \nabla \phi)
\]
by the definition of the Green function \(G\). Or, according to the generalized divergence theorem (3-36),
\[
\Phi(t,x) = \int d\Sigma \cdot M \cdot (\Phi \nabla G \cdot \nabla \phi)
\]  
(3-42)
This is the desired generalization of Kirchhoff's formula to the case of boundary surfaces in arbitrary motion. The surface \(S\) need not include a branch at infinity if the velocity potential \(\phi\) is selected to be zero at infinity.

Introducing the specific parametrization of the boundary which lead to (3-38),
\[
\int d\Sigma \cdot M \cdot \nabla = d\tau d\xi d\eta A(\tau, \xi, \eta) \left[ \frac{\hat{h} \cdot \nabla}{c^2} \frac{\partial \Phi}{\partial \tau} + \hat{h} \cdot \nabla \Phi \right]
\]  
(3-43)
The sign of this expression has again been reversed, since the unit vector \(\hat{h}\) of (3-38) would point away from the fluid. The unit normal vector in (3-43) is the outward normal at the boundary surface, which points into the fluid. The final expression equivalent to (3-42) in more conventional notation reads
\[
\Phi(\tau, \xi) = -\int d\tau d\xi d\eta A(\tau, \xi, \eta) \left( \frac{1}{c^2} \frac{\partial \Phi}{\partial \tau} + \hat{h} \cdot \nabla \Phi \right)G
\]  
+ \int d\tau d\xi d\eta A(\tau, \xi, \eta) \left( \frac{\hat{h} \cdot \nabla \Phi}{c^2} \right) \left( -\frac{\partial G}{\partial \tau} \right)
+ \int d\tau d\xi d\eta A(\tau, \xi, \eta) \left( \hat{h} \Phi \right) \cdot (-\nabla G)
\]  
(3-44)

3.5.3 Interpretation of the Result

The Green function \(G\) describes the velocity potential due to an impulsive mass source, while the functions \(-\partial G/\partial t\) and \(-\nabla G\) describe the velocity potential due to an impulsive chronopole source and doublet source, respectively. Morgans'
equation (3-44) therefore reduces to the statement that a boundary surface may be represented as

\[ -(\frac{\hat{n} \cdot \nabla}{c^2} \nabla \Phi + \hat{n} \cdot \nabla \Phi)A(\tau, \xi, \eta) \]

(3-45)

\[ + (\frac{\hat{n} \cdot \nabla}{c^2} \Phi)A(\tau, \xi, \eta) \]

plus

\[ (\hat{n} \cdot \nabla \Phi)A(\tau, \xi, \eta) \]

a mass source of strength

a chronopole of strength

a doublet of strength

The analogy of this result to the case of the electrostatic potential is evident: The charge on a surface is given by the jump in \(-\hat{n} \cdot \nabla \Phi\), while the dipole strength \(\hat{n} \cdot \Phi\) on a surface causes a jump in velocity potential of magnitude \(\Phi\) across the surface. The analogy is most accurate when it is recognized that the right hand side of (3-44) yields \(\Phi(t, \mathbf{x}) = 0\) for field points \((t, \mathbf{x})\) which are interior to the boundary surface \(S\) (where there is no fluid). It should not be surprising that the fourth component of the dipole is necessary to get the correct jump in the velocity potential in the four dimensional case.

Since the Green functions in (3-44) are nonzero only at the intersection of the three dimensional surface \(S\) and the past "light cone" whose apex is at the observer location \((t, \mathbf{x})\), the integrals may be reduced to double integrals. The procedure is slightly more complex for the case of the chronopole and doublet integrals than for the mass source integral, since the former contain derivatives of the delta function. These more complicated generalized functions lead to Leibnitz terms from the edges of the boundary surface, if the boundary in \((\xi, \eta)\) parameter space changes as a function of time. The edge of the boundary surface will be parametrized by the single variable \(\mu\). The velocity \(\mathbf{w}\) of this edge, and its arc length \(ds\) are given by the formulas analogous to (3-25) and (3-26) for the two dimensional surface:

125
\( x = z(\tau, \mu) \) on the edge \( (3-46) \)
\( w = (\partial z / \partial \tau)_\mu \) \( (3-47) \)
\[ ds = (\partial z / \partial \mu)_\tau d\mu = s(\tau, \mu) d\mu \] \( (3-48) \)

In appendix B, the integrals in (3-44) are written as one and two dimensional integrals over the surface (3-22) and the edge (3-46). The resulting expression is identical to the equation given in Morgans' paper, except that the restriction that the boundary of the surface in \((\xi, \eta)\) parameter space be independent of time has been lifted:

\[
\Phi(\tau, \vec{x}) = \int_S d\xi \, d\eta \left[ -\frac{\hat{\mathbf{n}} \cdot \vec{V} \, \frac{\partial \Phi}{\partial \tau} + \hat{\mathbf{n}} \cdot \nabla \Phi}{4\pi r |1 - M_r|} \right] \\
- \int_S d\xi \, d\eta \, \frac{\partial}{\partial t} \left[ \frac{\hat{\mathbf{n}} \cdot \vec{V} \, A}{4\pi r |1 - M_r|} \right] - \int_S d\xi \, d\eta \, \nabla \cdot \left[ \frac{\hat{\mathbf{n}} \cdot \vec{V} \, A}{4\pi r |1 - M_r|} \right] \\
+ \int_\xi d\mu \left[ \frac{(\hat{\mathbf{r}} - \vec{V}) \cdot (\vec{V} - \hat{\mathbf{r}}) \times \Phi}{4\pi c r |1 - \hat{\mathbf{r}} \cdot \vec{V} | (1 - \hat{\mathbf{r}} \cdot \vec{V})} \right]
\] \( (3-49) \)

If the surface velocity \( \vec{v} \) is the same as the edge velocity \( w \), then the surface is not being created at its edge, and the line integral in (3-49) vanishes. This line integral therefore represents the contribution of an edge at which new boundary surface is being created. The three surface integrals will be recognized as sums of moving Lowson sources from Table 3-2. In particular, they represent moving mass sources, chronopoles and doublets, one of each type for each point of the surface, with strengths given by (3-45). A moving boundary surface may be regarded as a set of moving points in an infinitude of different ways. Since all parametrizations of a particular boundary will share the same normal velocity \( \hat{\mathbf{n}} \cdot \vec{v} \), the expression (3-44) and the source strengths (3-45) are manifestly invariant under changes in the parametrization of the boundary surface which is chosen.
However, expression (3-49) is very definitely not independent of the parametrization, since $M_\rho$, $\hat{r}\cdot w$, and $w-v$ depend on the indeterminate component of the velocity parallel to the sheet or to the edge. When this equation is to be used as the basis of a calculation, it is therefore of vital importance to choose a parametrization of the boundary surface which maximizes the understandability of the calculation. The boundary surface will invariably consist of an impermeable solid surface, plus a free vortex sheet. The solid surface will usually be parametrized in such a way that the points $(\xi,\eta)$ on this section of the boundary correspond to actual points on the solid surface. Therefore, for this section of the boundary, $v=w$ and the line integral vanishes. On the other hand, in the case of the shed vortex sheet, it is convenient to regard the points of the sheet to have the small residual velocity given by the Helmholtz vorticity equation. This requires that the edge of the vortex sheet move at a different velocity $w$ than the velocity of the sheet $v$ immediately after its creation. Therefore, the line integral in (3-49) will be entirely the result of the creation of vorticity in the wake of the object.

While the surface integrals can be identified as sums of moving point sources, the line integral does not correspond to any type of a moving point source. For an infinitely thin free vortex sheet with jump in velocity potential $\Gamma$, equation (3-49) reduces to
\[ \Phi(t, \mathbf{x}) = \int_S d\xi d\mathbf{h} \left\{ \frac{\mathbf{n} \cdot \mathbf{\nabla} A}{c^2} \frac{\partial}{\partial t} \left( \frac{\mathbf{n} \cdot \mathbf{\nabla} A}{4\pi r} \right) \right\} 
- \int_S d\xi d\mathbf{h} \frac{\mathbf{n} \cdot \mathbf{\nabla} A}{4\pi r} \frac{\partial}{\partial t} \left( \frac{\mathbf{n} \cdot \mathbf{\nabla} A}{4\pi r} \right) 
- \int_S d\xi d\mathbf{h} \mathbf{\nabla} \cdot \left( \frac{\mathbf{n} \cdot \mathbf{\nabla} A}{4\pi r} \right) 
+ \int_E d\mu \left\{ \frac{\mathbf{\hat{n}} \cdot \mathbf{\nabla} \times \mathbf{w}}{4\pi c r} \right\} \right\} 
- \int_S d\xi d\mathbf{h} \mathbf{\nabla} \cdot \left( \frac{\mathbf{n} \cdot \mathbf{\nabla} A}{4\pi r} \right) \] 

The fact that the jump in the normal velocity across such a sheet is zero has been used here, in addition to the fact that the unit normal vector \( \mathbf{n} \) points in opposite directions on the two sides of the sheet. In (3-50), \( \Gamma \) is the velocity potential on the side of the sheet toward which \( \mathbf{n} \) points minus the velocity potential on the opposite side.

The expression (3-50) is easily interpreted in the case that the residual velocity of the vortex sheet after its creation, \( v \), is zero. Then (3-50) becomes

\[ \Phi(t, \mathbf{x}) = \int_E d\mu \left\{ \frac{\mathbf{\hat{r}} \cdot \mathbf{w} \times \mathbf{\nabla} \cdot \mathbf{n}}{4\pi c r} \right\} \] 

This will be recognized as a sum of the solutions for a set of moving point forces located at the points of the moving edge. The integral over the trajectory of the point force in the expression for the velocity potential given in Table 3-2(d) has become an integral over the surface swept out by the moving edge. In each case, this integral represents the vorticity deposited in the fluid by the action of a force; the vorticity never dissipates in the absence of viscosity. The force acting over a length \( ds \) of the edge is seen to be \( \rho_0 \Gamma w \times ds \), exactly as would be expected for a bound vortex line of strength \( \Gamma \) moving with a velocity \( w \).

If the sheet is allowed to have a residual motion \( v \) after its creation, then the situation is no longer so simple. The shed vortex sheet is no longer a motionless doublet layer; it becomes a moving sheet of mass sources and chronopoles as well as
doublets. However, despite its motion and the fact that the types of sources required at its surface are somewhat different, the vortex sheet remains a simple surface of discontinuity of the velocity potential.

Comparisons of this theory to experiments usually involve pressure measurements made by microphones. Thus, it is important to compute the pressure field as well as the velocity potential field associated with the flow. For the equivalent sources distributed on the boundary surfaces, the Lowson formulas for the pressure are adequate for the far field. (Very close to the object, it may be necessary to use (3-33) to compute the pressure, but such situations will not be discussed here.) However, for the non-Lowson sources located at an edge where boundary surface is being created (e.g. a lifting line), the formulas for the pressure must be worked out anew.

The pressure field may be computed by taking the observer time derivative of expression (3-50). New Leibnitz terms arise from the surface integrals in (3-50), as detailed in appendix D. If the points at the two ends of this edge were moving at a different speed than the edge itself (i.e. if the edge were being continually created at its ends), then the pressure field would have finite contributions from these points, just as the velocity field had finite contributions not only from the surface of the sheet, but also from the edges where it was created. I see no practical advantage to choosing a parametrization of the edge in which it is created at its endpoints, so this situation will not be discussed here. The time derivative may therefore be assumed to commute with the line integral in (3-50). The time derivatives of the surface integrals, of course, yield new line integrals.

The surface integrals in the expression for the pressure field corresponding to the velocity field (3-50) are straightforward sums of the appropriate Lowson formulas,
as given in Table 3-2. The line integrals are given in Table 3-3, which may be regarded as the extension of Table 3-2 to include the "source type" which must be placed at an edge where a vortex sheet is being created. In practical applications, these non-Lowson sources may be regarded as the sources which correspond to a lifting line, although it must be realized that a lifting line is inseparable from the vortex sheet which is shed behind it.

In the limit that the shed vortex sheet remains motionless after its creation, \( v = s' = (\partial \Gamma / \partial \tau)_{\xi, \eta} = 0 \), and the expression for the pressure field given in Table 3-3 reduces to the Lowson formula for a moving point force (Table 3-2(d)). If the residual velocity \( v \) of the sheet is not zero, however, the formula for the pressure due to a lifting line does not reduce to Lowson's formula. Since \(-\rho_0 \mathbf{v} \times \mathbf{s} \Gamma\) represents the induced drag force, a part of the formula given in Table 3-3 is recognizable as the Lowson force formula. The additional terms and factors introduced here must be interpreted as the difference between the induced drag and the profile drag. Since the mechanisms of the two phenomena are so different, it is physically very appealing to treat them separately.

Unfortunately, the expression for the pressure field given in Table 3-3 depends not only on the lift and induced drag forces, \( \rho_0 \mathbf{w} \times \mathbf{s} \Gamma \) and \(-\rho_0 \mathbf{v} \times \mathbf{s} \Gamma\), but also on several other properties of the sheet and its parametrization. Nevertheless, this expression certainly represents the effect of creating a vortex sheet with a residual motion, just like the shed vortex sheet associated with the lift and induced drag forces. Since the expression reduces to the Lowson formula for a lift force in the limit of a stationary vortex sheet, the difference between the formula in Table 3-3 and the formula for a Lowson lift force may be regarded as the effect of the residual motion of the sheet. This problem will be dealt with further in chapter 4.

130
Table 3.3. The Lifting Line Source

Whether or not a vortex sheet is being created at its edge is strictly a matter of how it is parametrized. There are two types of source time derivatives at the creation edge, which will be denoted as follows:

\[ f'(\tau, \mu, \nu_0) \equiv \frac{\partial f}{\partial \tau} \bigg|_{\mu, \nu} \]

\[ \hat{f}(\tau, \mu, \nu_0(\tau)) \equiv \frac{\partial f}{\partial \tau} \bigg|_{\mu} = f' + \frac{\partial f}{\partial \nu} \bigg|_{\mu, \tau} \frac{\partial \nu_0}{\partial \tau} \]

The two types of velocity are

\[ \vec{w} = \vec{y} \equiv c\vec{N} = \text{velocity of edge (lifting line)} \]

\[ \vec{v} = \vec{y}' \equiv c\vec{M} = \text{velocity of sheet (at edge)} \]

The Joukowski force on the lifting line is defined by

\[ \vec{F} = \rho_o (\vec{w} - \vec{v}) \times \vec{s} \Gamma \]

The velocity potential at the lifting line is (3.48):

\[ \phi(t, \vec{x}) = \frac{(\hat{r} - \vec{M}) \cdot \vec{F}}{4\pi \rho_o c r |1 - N_r| (1 - M_r)} \]
Table 3-3: (Continued)

The corresponding pressure field, according to Appendix D (D.19) is

\[
p(t, \vec{x}) = \frac{\partial}{\partial t} \frac{\hat{r} \cdot \vec{F}}{4 \pi cr |1-N_r| (1-M_r)} + \frac{\hat{\tau} \cdot \vec{F}}{4 \pi r^2 |1-N_r| (1-M_r)} \\
- \frac{\vec{M} \cdot (\vec{w} - \vec{v}) \times \vec{s}}{4 \pi cr |1-N_r| (1-M_r)} \rho_0 \frac{\partial \Gamma}{\partial \tau} \bigg|_{\xi, \eta} \\
+ \frac{\rho_0}{4 \pi c |1-N_r| (1-M_r)} \left\{ (\vec{w} - \vec{v}) \times \vec{s} \cdot \frac{\partial}{\partial \tau} \left[ \frac{\hat{r} \cdot \vec{M}}{r (1-M_r)} \right] \bigg|_{\xi, \eta} \\
+ \frac{\hat{r} \cdot \vec{M}}{1-M_r} \cdot \left[ (\vec{w} - \vec{v}) \times \vec{s} + (\vec{v} - \vec{v}') \times \vec{s} \right] \Gamma \right\}
\]

Expanding this equation yields the detailed expression for the pressure field in terms of the velocity and acceleration of the sheet and of the edge:

\[
p(t, \vec{x}) = \frac{\text{sgn}(1-N_r)}{4 \pi cr (1-N_r)^2 (1-M_r)} \left\{ \hat{r} \cdot \vec{F} + \frac{\hat{\tau} \cdot \vec{F}}{1-N_r} \left[ \frac{\hat{r} \cdot \vec{N} + \frac{c}{r} (1-N^2_r)}{r (1-M_r)} \right] - \frac{c}{r} \vec{N} \cdot \vec{F} \\
- \vec{M} \cdot \hat{r} + \frac{\hat{r} \cdot \vec{M}}{1-N_r} \left[ \frac{\hat{r} \cdot \vec{N} + \frac{c}{r} (N_r - N^2_r)}{1-M_r} \right] + \left[ \frac{r - \vec{M} \cdot \vec{F}}{1-M_r} \left( M_r N_r - \vec{M} \cdot \vec{N} \right) \right] \right\}
\]

132
Table 3-3: (Continued)

\[ + \frac{\rho_0}{4\pi cr |1-N_r| (1-M_r)} \left[ \frac{\hat{r} \cdot \vec{M}}{1-M_r} - \overline{\vec{M}} \right] \cdot (\vec{w} - \vec{v}) \times \hat{s} \frac{\partial \gamma}{\partial \xi} \xi, \eta \]

\[ + \frac{1}{4\pi cr |1-N_r| (1-M_r)^2} \left\{ (\hat{r} - \vec{M}) \cdot \left[ (\vec{w} - \vec{v}) \times \hat{s}' + (\vec{v} - \vec{v}') \times \hat{s} \right] \right\} \rho_0 \gamma \]

\[ + \frac{(\hat{r} - \vec{M}) \cdot \vec{F}}{1 - M_r} \left[ \hat{r} \cdot \vec{M}' + \frac{c}{r} (M_r - \vec{M}^2) \right] - \vec{M}' \cdot \vec{F} + \frac{c}{r} \left( \hat{r} \cdot \vec{F} \right) \]

133
A vortex sheet which is continually being created by the motion of a lifting line (the edge of the sheet) is a more correct replacement for the concept of a set of moving Lowson point forces at the points of the lifting line. In the vortex sheet picture, the induced drag force is a direct consequence of the residual motion of the shed vortex sheet. This matches the physical picture for the mechanism of induced drag forces given in chapter 2 much more closely than the jetlike flow associated with any Lowson type drag force. In addition to the slowly moving sources which must be placed at the surface of the vortex sheet, the non-Lowson sources shown in Table 3-3 must be placed at the lifting line itself. Since the motion of the lifting line is generally much more rapid than the residual motion of the sheet, these non-Lowson sources are more important to the sound production than are the sources distributed on the surface.

Table 3-4 gives a summary of the exact solution of the wave equation in the case in which the boundary surfaces are in motion. The velocity potential at all points of spacetime have been expressed in terms of the conditions at the moving boundary surfaces. The boundary surface consists of both a solid surface, and a free vortex sheet. The free vortex sheet contributes to the flow field not only from moving point sources distributed over its surface, but also from the "lifting line" source type of Table 3-3, which must be associated with the edge at which the sheet is created.

3.5.4 Consistent Approximations

Although the results presented in Table 3-4 provide an exact solution to the wave equation (3-35), this equation itself holds only approximately, according to the arguments given in section 3.4. Some of the terms given in the solution to the wave equation are smaller than others by the same order of magnitude as the nonlinear
Table 3-4: Equivalent Sources at a Fluid Boundary

Suppose \( \Box \Phi = 0 \) everywhere outside a closed solid boundary surface \( S \), except at a thin free vortex sheet \( \Sigma \), where the jump in velocity potential is \( \Gamma \). Then the flow field may be modelled as:

1. A moving point mass source at each point of \( S \) of strength
   \[
   - \frac{\hat{n} \cdot \nabla}{c^2} \frac{\partial \Phi}{\partial \hat{n}} - n \cdot \nabla \Phi \text{ per unit area, and at each point of } \Sigma \text{ of strength }
   - \frac{\hat{n} \cdot \nabla}{c^2} \frac{\partial \Gamma}{\partial \hat{n}} \text{ per unit area.}
   \]

2. A moving point chronopole at each point of \( S \) of strength \( \rho_0 \frac{\hat{n} \cdot \nabla}{c^2} \Phi \) per unit area, and at each point of \( \Sigma \) of strength \( \rho_0 \frac{\hat{n} \cdot \nabla}{c^2} \Gamma \) per unit area.

3. A moving point doublet at each point of \( S \) of strength \( \hat{n} \Phi \) per unit area, and at each point of \( \Sigma \) of strength \( \hat{n} \Gamma \) per unit area.

4. A lifting line source at each point of the edge where the free vortex sheet is being created. The lifting line has circulation \( \Gamma \) and coincides with the creation edge.

Formulas for the velocity potential and pressure fields associated with the first three source types may be found in Table 3-2. The corresponding formulas for the fourth source type are given in Table 3-3.
terms which were already neglected in the derivation of the wave equation. There is no real point in retaining such terms, so the order of magnitude analysis of section 3.4 will be applied to equation (3-44) in order to identify them. In the notation of section 3.4, relative magnitudes of the various terms in (3-44) are:

\[
\begin{array}{ccc}
\begin{array}{c}
\hat{\mathbf{r}} \cdot \frac{\partial \Phi}{\partial t} \\
\epsilon/\chi M^2
\end{array}
&
\begin{array}{c}
\hat{n} \cdot \nabla \Phi \\
1
\end{array}
&
\begin{array}{c}
\frac{\hat{A} \times \hat{\mathbf{r}}}{c^2} \\
(\epsilon/\chi) M^2
\end{array}
\end{array}
\]

Therefore, the first part of the mass source term and the chronopole term are smaller than either the second part of the mass source term or the doublet term by precisely the same factor as the nonlinear terms which were neglected in the derivation of the wave equation. These terms need not be retained in problems involving the sound generated by the motion of a relatively rigid object through the fluid. If the surface of the object were vibrating, so that there were a time scale other than \( \chi / U \), then these terms might be important.

The lifting line source of Table 3-3 can be of considerably greater importance than either the mass source or the doublet source mentioned above. On the same scale as (3-51), the terms of the lifting line source have the approximate magnitudes:

\[
\begin{array}{ccc}
\begin{array}{c}
w \times s \Gamma \\
C_L (\epsilon/\chi) M
\end{array}
&
\begin{array}{c}
\epsilon \times s \Gamma \\
C_L (AR) (\epsilon/\chi) M
\end{array}
\end{array}
\]

Here, \( C_L \) is a typical lift coefficient (which determines the magnitude of the jump in velocity potential \( \Gamma \) as a fraction of \( U \chi \)), and \( (AR) \) is the aspect ratio.

The comparison between (3-51) and (3-52) is quite interesting. Depending on the relative magnitudes of the thickness ratio, aspect ratio, and Mach number, the lift force can be either significantly more important or less important than the
doublet or mass source terms. For the regime which will be important in propeller noise, the Mach number is near unity, and the thickness ratio is small. However, the thickness ratio tends to be of the same order as the inverse of the aspect ratio. Hence, the lift force will dominate, and the induced drag, mass source and doublet terms will tend to be of the same order.

3.6 Summary

This chapter began with a discussion of sources of mass or of momentum in the context of the linearized equations of motion. Such sources are called Lowson sources if they are concentrated at a moving point. Particular care was taken to discuss both the pressure and the velocity fields for such sources. In the case of the moving point force, the velocity field had a considerably different character than for the other types of sources. Namely, the moving point force was found to deposit a trail of vorticity in its wake. This shed vorticity resembles the vortex sheet shed behind a lifting wing in the case that the force is perpendicular to its direction of motion, or the jetlike flow associated with profile drag in the case that the force is parallel to its direction of motion. Since the equations of motion are linearized, the shed vorticity remains motionless after being deposited.

The meaning of the moving Lowson sources is easy to understand, but it is not clear how these sources can be distributed in order to represent the flow field of a moving object. The Flowcs Williams and Hawkings theory purports to explain how equivalent sources should be distributed on a moving surface. While this theory is mathematically correct, it provides little insight into the nature of the flow field associated with a moving object. Furthermore, the vorticity which is shed into the wake behind a lifting airfoil is not explicitly recognized in this theory. The moving
point forces at the surface of the object in Ffowcs Williams' and Hawking's theory automatically create vorticity in the wake of the object, of course. However, the shed vortex sheet remains stationary in this theory. This leads to an incorrect treatment of induced drag in the Ffowcs Williams and Hawking's theory, since it is the residual motion of the shed vortex sheet which is responsible for induced drag.

A systematic discussion of the relative orders of magnitude of the various terms in the equations of motion reveals that the acoustic wave equation for the velocity potential holds more generally than the wave equation for the pressure. By working with the velocity potential, which is linear both in acoustics and in incompressible flow theory, it is immediately recognized that the boundary surface of the irrotational part of the flow field, rather than simply the boundary of the fluid, is important. That is, not only should the solid surface of a wing or of a propeller be regarded as a boundary surface, but also the vortex sheet shed behind such a moving object should be included as a part of the boundary. This insight goes hand in hand with Powell's theory of vortex generated sound. Creation or acceleration of the vorticity in a shed vortex sheet, as well as the motion of the solid boundaries, gives rise to sound.

These ideas were formulated mathematically in the framework of Morgans' extension of the Kirchhoff-formula to the case of a moving boundary surface. The generalization of the Kirchhoff formula provides an expression for the velocity potential at all points of spacetime in terms of the history of the conditions on the boundary surface. In addition to the monopole and dipole (doublet) sources at the surface familiar from electrostatics, a new source type was necessary. It was christened a "chronopole", and represents a dipole pointing in a timelike direction instead of a spacelike direction.
When the boundary surface is created at an edge, a new type of source is found to be necessary at the creation edge. This is of particular relevance to the case of a shed vortex sheet, since it is quite reasonable to imagine that such a sheet is being continually created at the trailing edge of a lifting airfoil. The equivalent sources necessary at a such a creation edge, or lifting line, may loosely be considered as a new source type. Unlike the equivalent sources necessary at a two dimensional boundary surface, it is not possible to identify the lifting line source at a creation edge with any Lowson source. However, in the case that the vortex sheet is stationary, and created at an edge, the combination of the lifting line source and the sources at the stationary surface of the sheet correspond exactly to the formula for a moving point force at the lifting line.

When the shed vortex sheet has a residual motion, the resulting modifications to this picture are suggestive of the changes which would be expected due to an induced drag force. These induced drag terms arise naturally as a consequence of the residual motion of the vortex sheet, just as the induced drag should. A more detailed analysis of the terms arising from the residual motion of the shed vortex sheets is given in the following chapter. The theory based on Morgans' extension of the Kirchhoff result, itself extended to the case in which the boundary surface can be created, thereby provides a much more correct physical picture of the flow field than does the Ffowcs Williams and Hawkings theory. In the following chapter, this theory will be specialized to the case of a light aircraft propeller.
4.1 Historical Introduction

The first successful attempt to explain the basic nature of the sound produced by a propeller was made by Gutin in 1936 [Gutin 1948]. Gutin discovered that the sound field of a point force moving in a circular trajectory could roughly explain both the amplitude and directionality of propeller sound. His paper is of some historical interest to the physicist: Since Gutin worked with frequency rather than time, his equations are identical to the frequency spectrum of the synchrotron radiation which would be produced by a rotating dipole. It is interesting that synchrotron radiation became a problem of great interest in the 1950s, just as the jet engine began to make the study of propeller noise obsolete.

Within a few years of Gutin's original work, it was recognized that even a completely unloaded propeller blade gives rise to sound by simply displacing the fluid as it moves. Among others, Deming worked out the contribution of this mechanism to the sound field [Deming 1937]. The part of the sound field which could be modelled as a rotating force came to be known as "loading noise", while the displacement effect was called "thickness noise". Both of these mechanisms give rise to a sound which has a period equal to the blade passage time. Any type of propeller sound source which has this periodic character is called "rotational noise".

Experimentally, it was found that propellers give rise to broadband noise as well as to rotational noise [Stowell 1935]. This aperiodic component is sometimes called
"vortex noise", although the precise mechanism which gives rise to this component of the observed sound field is still disputed. Vortex noise cannot be either predicted or measured nearly as reliably as rotational noise. For light aircraft propellers, the vortex noise is usually of considerably smaller amplitude than the rotational noise. However, the fact that vortex noise involves considerably higher frequencies than rotational noise makes it relatively easy to observe. No attempt will be made here to develop a theory to predict the vortex noise [Amiet 1976, Hanson 1974].

In order to explain the details of the rotational noise produced by a particular propeller, it is natural to try to add up a number of rotating point forces instead of using only one as Gutin did. Additionally, a number of rotating mass sources and sinks can be used to simulate the opening up and snapping shut of the fluid as a propeller blade cuts through it. The magnitudes of the necessary forces and mass sources can be determined by considering the mass and momentum flux distributions through a thin disk surrounding the spinning propeller. This approach is taken by Morse and Ingard, for example [Morse 1968]. The resulting theory is a natural meld of classical actuator disk theory and of Gutin’s idea. The direct physical reasoning used to determine the source strength and distribution guarantees that the sound field predicted by such a theory will closely agree with the actual rotational noise produced by a propeller.

All further theoretical developments are basically variations on this theme. The goal of this continuing theoretical banter is to provide a more detailed understanding of precisely what type of acoustic sources should be placed where in order to best model the actual flow field. The F Flowcs Williams and Hawkings theory does not really fulfill this goal; it is basically an extension of the old idea that the situation is best modelled by merely distributing sources of mass and
momentum in the fluid which are related to the aerodynamic forces on the blades and to their geometry.

In contrast, the general theory developed in the previous chapter provides a completely fresh perspective. The ideas developed there will be specialized to the case of the sound generated by a propeller in this chapter. It will be found, of course, that the formulas derived from the looser physical arguments are recovered to a large extent. However, some potential "sources" of sound which are not recognized in the rougher treatments are recognized here. Unfortunately, the strength and distribution of these additional sources is not related to the aerodynamic forces or to the blade geometry directly.

First, in one special case, the general aeroacoustic theory of chapter 3 is found to be identical to the physical model of the propeller as a source of momentum and mass, distributed according to the aerodynamic loads and the blade geometry. This is the case of a very lightly loaded propeller, in which the residual motion of the shed vortex sheet can be ignored. It is no accident that this is precisely the limit in which the performance theory of chapter 2 was simplest. When the shed vortex sheet is stationary after its creation, the flow can be considered to be generated by sources of mass and of momentum which are distributed according to the aerodynamic loads and blade geometry, just as in the physically motivated theories. In order to obtain this result, it is also necessary to assume that the airfoil sections which comprise the blade are very thin.

Next, the possibility that the shed vortex sheet has some small residual velocity is considered. If this residual motion is not considered, then there can be no induced drag force. The addition of a small sheet motion can therefore be regarded as the allowance for induced drag in the acoustic theory. Just as in the performance
theory, the major effect of this residual motion is to change the relative motion of
the lifting line through the fluid. Unlike the performance theory, the spanwise
component of the induced velocity may have a significant effect on the sound
production by the creation of the shed vortex sheet. Since this component is
independent of the aerodynamic forces on the blade, it represents a source of sound
which could not be computed from a knowledge of the blade loading alone.
Unfortunately, this fact requires that the new term be dropped until a more
sophisticated aerodynamic theory is devised which can predict the necessary
parameters.

Finally, an implementation of this theory will be outlined. The computer model
combines the aerodynamic theory of chapter 2 with the aeroacoustic theory of
chapter 3. Computers did not exist when aircraft propeller noise was of greatest
interest. For this reason, older theories preferentially use a frequency domain
analysis, rather than the time domain analysis which is used here. For a computer,
the two approaches are complementary; time domain analysis requires the
evaluation of a simple formula at a great many times, while frequency domain
analysis requires the evaluation of a (computationally) complicated formula for a
relatively small number of harmonics. The time domain has been selected here for
its greater similarity to the aerodynamic theory, as well as for its greater simplicity in
a near field calculation.

4.2 Sound from a Lightly Loaded Propeller
4.2.1 Overview

The most detailed two dimensional airfoil section data which is generally available is the pressure coefficient as a function of chordwise position along the section. The information required to compute the sound field by means of the theory summarized in Table 3-4, however, is the velocity potential at the surface of the blade. Additionally, the strength and motion of the shed vortex sheet is a necessary input to the acoustic calculations. It is possible to bridge this gap in information between the outputs of the aerodynamic theory and the inputs to the acoustic theory only in the limit of a lightly loaded propeller with very thin airfoils.

A velocity potential coefficient could be defined and tabulated for airfoil sections much as the pressure coefficients are tabulated. For thin airfoils, this is unnecessary, since the velocity potential coefficient as a function of chordwise position is simply the integral of the pressure coefficient. However, for thick airfoils, the velocity potential coefficient provides different information than the pressure coefficient. Moreover, this information is theoretically as interesting as the information provided by the pressure coefficient. The need to assume that the airfoil sections are thin in what follows is only partially due to this lack of tabulated "velocity potential coefficients" for airfoil sections, however. When the Mach numbers are high, the assumption of thin sections is necessary in order for linearized acoustics to be applicable in the first place.

In the case that the blades are lightly loaded, the residual motion of the shed vortex sheets may be neglected. This simplifies the calculation of the sound produced by the creation of the sheet considerably. In fact, in this limit, the only sources which contribute to the pressure field are the mass sources, couplets, and lifting lines, which all move along with the surface of the propeller. (The stationary
doublets on the shed vortex sheet contribute only to the velocity field, not to the pressure field, in the linearized approximation.) According to the discussion of section 3.5.4, the chronopole and a part of the monopole source associated with the moving surface of the blade may be neglected. From Table 3-4, the sound field may thus be expressed as

\begin{equation}
\hat{A} \cdot \nabla \Phi \text{ at each point of the surface of the blades,}
\end{equation}

A mass source of strength $-\hat{n} \cdot \nabla \Phi A$ at each point of the surface of the blades,

A doublet source of strength $\hat{n} \Phi A$ at each point of the surface of the blades,

A stationary vortex sheet which is created by each propeller blade as it advances,

A distribution of Lowson point forces on the surface of the blades corresponding to the profile drag force.

The profile drag sources give rise to vorticity which creates a jetlike flow in the wake of each propeller blade. If the magnitude of the forces is taken to be the profile drag force, then this jetlike flow is an accurate model of the momentum wake shed behind an actual propeller blade. This model is obviously not perfect. First, the actual growth of the thickness of the momentum wake by turbulent and viscous diffusion is not taken into account. Second, it is necessary to assume that the airfoils are thin, since the vortex tubes associated with Lowson drag forces lie exactly along the trajectories of the points (see Fig. 4-1). The actual boundary layer which these forces are modelling obviously follows the contours of the surface of the blade.

The doublet sources at the surface of the propeller represent the vortex sheet associated with the rapid change in the tangential velocity at that surface. The shed vortex sheet is an integral part of this vortex system. The separation of these two
moving point forces located on airfoil surface

vortex tubes created by moving point forces lie precisely along the trajectories of the points

Flow Field Used to Model Profile Drag

boundary layer

actual distribution of vorticity in the momentum wake which results from profile drag

Actual Flow Field Associated with Profile Drag

Figure 4-1: Profile Drag Model
terms in (4-1) is therefore purely an artifact of the analysis. It is more convenient to divide the complete vortex structure into the following two parts: The shed vortex sheet is a part of a vortex system which involves the so-called bound vorticity on the blades. This bound vorticity may be considered algebraically as the part of the vorticity at the surface of the blade which is symmetric between the pressure and suction sides of the blade. The remaining antisymmetric part of the vorticity at on the blades consists of vortex loops which run along the span on the suction side, then return along the span on the pressure side; that is, the loops all close on the surface of the blade. On the other hand, the vortex lines associated with the symmetric part of the vorticity distribution are closed by the shed vortex sheet. Therefore, the shed vortex sheet and the symmetric part of the vorticity distribution on the blade surface should be considered as a part of the same vortex system, distinct from the antisymmetric part of the vorticity distribution.

This separation of the vortex structure is indicated in Fig. 4-2. Mathematically, the velocity potential \( \Phi \) at the surface can be written as the sum of a symmetric part \( \Phi_S \) and an antisymmetric part \( \Phi_A \), where

\[
\begin{align*}
\Phi_S &= \frac{1}{2}(\Phi_{\text{suction}} + \Phi_{\text{pressure}}), \text{ on suction side} \\
\Phi_A &= \frac{1}{2}(\Phi_{\text{suction}} - \Phi_{\text{pressure}}), \text{ on both sides}
\end{align*}
\]

The symmetric part of the vorticity distribution includes the shed vortex sheet, and may be considered as a distribution of doublets of strength \( \hat{\Phi}_S \) on the surface of the blades, which merge smoothly with the doublets representing the shed vortex sheet. Similarly, the antisymmetric part is associated with doublets \( \hat{\Phi}_A \) at the blade surface only. The symmetric (antisymmetric) part of the vorticity distribution roughly corresponds to the antisymmetric (symmetric) part of the pressure or velocity distributions.

147
Vortex lines at the surface of a moving airfoil and a stationary shed vortex sheet associated with the lift force

Antisymmetric part of the vorticity distribution: vortex loops close on the blade surface

Symmetric part of the vorticity distribution; vortex lines close on the shed vortex sheet

Figure 4-2: Decomposition of Vortex Structure
4.2.2 Modelling the Lift Force

Instead of regarding the vortex sheet as being created at the trailing edge of the airfoil sections, the division of the vortex structure of the propeller flow field into its symmetric and antisymmetric parts suggests a somewhat different view. Namely, the symmetric part of the vorticity distribution (including the shed vortex sheet) can be viewed as a vortex sheet which is created gradually beginning at the leading edge of the blade and reaching its full strength at the trailing edge. Here, the free vortex sheet is assumed to be stationary. Provided that the blades are very thin, the transverse motion of the part of the vorticity distribution on their surface may be ignored. As indicated in Fig. 4-3, the separation between the pressure and suction sides of the blades may be ignored in the case of the symmetric part of the vorticity distribution in this limit.

This gradually created vortex sheet may be regarded as a linear superposition of a number of stationary vortex sheets created at lifting lines which lie in the region between the leading and trailing edges of the blade, where the strength of the sheet is changing. This is indicated in Fig. 4-3. Each lifting line increases the jump in potential across the sheet slightly. According to Table 3-3, such a distribution of lifting line sources is equivalent to a distribution of Lowson point forces, since the vortex sheets being created have no residual motion. These forces are perpendicular to the sheet being created, and have a magnitude given by the Joukowski rule. Therefore, the forces can be identified with the lift force on the blade. Furthermore, the distribution of the point forces is determined by the rate of increase of the strength of the vortex sheet in the chordwise direction, just as is the chordwise distribution of lift.

The distribution of the lift forces on the surface of the blade will be denoted by
Symmetric part of the vorticity distribution...

may be considered to be a single vortex sheet if the airfoil section is very thin.

Such a vortex sheet may be considered to be created by a distribution of lifting lines between the leading and trailing edges of the original airfoil section.

Figure 4-3: Lift Force Model
\( L(\tau, \xi, \eta) d\xi d\eta \). The coordinates \((\xi, \eta)\) parameterize the surface of the blade in a manner which is independent of the source time \(\tau\). According to Table 2-2(d), or Table 3-3 in the case that the residual sheet velocity is zero, the pressure field resulting from this distribution of lift forces is

\[
\rho(t, \xi) = \sum_{propeller \ blades} \int_{\text{blade span and chord}} d\xi d\eta \left\{ \frac{\sinh((1-M_c^2))}{4\pi \rho c \tau (1-M_c^2)} \left[ \hat{r} \cdot \hat{C} \right] \\
+ \frac{\hat{r} \cdot \hat{C}}{(1-M_c^2)} \left( \frac{\dot{M}_c}{\tau} - \frac{c}{\tau} (1-M_c^2) - \hat{M} \cdot \hat{C} \right) \right\}
\]

where, \( L \) represents the force applied to the fluid, which is opposite in direction to the lift force. Since the airfoil has been assumed to be quite thin, this expression for the pressure is valid throughout the fluid. Of course, in reality, it cannot apply to observer locations which are actually inside the momentum wake shed behind the blade.

### 4.2.3 Modelling the Profile Drag Force

As discussed in section 4.2.1, the profile drag force will be modelled as a distribution of Lowson point forces, denoted by \( D_p(\tau, \xi, \eta) \). Once again, this is expected to be accurate only in the limit that the blades are very thin. Otherwise, it would be necessary to regard the boundary of the region of irrotational flow to be the surface of the boundary layer, rather than the solid surface of the blade. This complicates matters considerably, since the boundary conditions at the surface of the boundary layer are not nearly so simple as the no penetration condition at the solid surface. In the present approximation, the profile drag makes the following contribution to the pressure field:
\[ p(t, \mathbf{x}) = \sum_{\text{propeller blades}} \int_{\text{blade}} \rho \, \delta v \left\{ \frac{\gamma_n (1 - M_p)}{4 \pi c r (1 - M_r)^2} \left[ \hat{r} \cdot \hat{D}_p \right. \\
+ \frac{\hat{x} \cdot \hat{D}_p}{(1 - M_r)} (\vec{M}_{r+} + \frac{1}{2} (1 - M_r^i) \frac{\mathbf{v}}{c} \cdot \hat{M} \cdot \hat{D}_p) \right\} \right\} \]

Here, \( D_p \) is parallel (not antiparallel) to the local section Mach number \( M \).

Since there is no induced drag when the shed vortex sheet has no residual motion, the lift and profile drag are the only forces acting on the propeller blades. Therefore, the sum of (4-2) and (4-3) represents the loading noise of a lightly loaded propeller.

### 4.2.4 Modelling the Thickness Noise

The remaining acoustic sources in (4-1) are the antisymmetric part of the vorticity distribution and the mass source, both on the surface of the blades. Both of these are associated with the thickness noise. The mass source term will be dealt with in this section. In the following section the antisymmetric part of the vorticity distribution will be shown to be smaller than the mass source term by the order of the thickness to chord ratio of the blades, which has already been assumed small. Therefore, the result of this section actually represents a complete expression for the thickness noise of a propeller blade with thin airfoils.

Since the propeller blades are impermeable, the velocity potential satisfies the no penetration boundary condition at their surface. That is, \( \hat{n} \cdot v = -\hat{n} \cdot \nabla \phi \) at the surface of the blades. The magnitude of the mass source at the surface of the blades is therefore \( \hat{n} \cdot v \), so that from (3-44) the contribution of this term to the velocity potential in the bulk of the fluid is

152
\[ \Phi(t,x) = \int \frac{d\tau d\xi d\eta}{\mathcal{A}} \hat{n} \cdot \nu G \]

where \( S \) represents the moving surface of the propeller blades. This integral involves only geometrical quantities; no fluid properties such as the velocity potential appear. The form of the integral is extremely inconvenient for numerical work, since the sign of \( \hat{n} \cdot \nu \) changes between the leading and trailing edges of the blade surface. In fact, for a rigid object, the mean value of \( \hat{n} \cdot \nu \) at one instant of (source) time is invariably zero. Therefore, if the integral is to be evaluated as a sum, that sum will contain terms which very nearly cancel one another, which is very unfavorable from a computational standpoint.

This problem was first recognized and solved by Dr. Succi [Succi 1979]. The basic idea is to rewrite the troublesome integral so that its integrand does not change sign. This is easily done using the powerful four dimensional analogue to the divergence theorem developed in chapter 3. In particular, from (3-37) and (3-39), any closed three dimensional surface \( S \) in spacetime, which encloses a four volume \( V \) satisfies

\[
\int d\tau d^3y \left( \frac{\partial}{\partial \tau} \nabla_y \right) G = \int d\tau d\xi d\eta A(-\hat{n} \cdot \nu \hat{n}) G
\]

or

\[
\int d\tau d^3y \left( \frac{\partial}{\partial \tau} \nabla_x \right) G = \int d\tau d\xi d\eta A(\hat{n} \cdot \nu -\hat{n}) G
\]

The timelike part of this expression has the desired form:

\[
\int d\tau d\xi d\eta A \hat{n} \cdot \nu G = \int d\tau d^3y (\partial G/\partial t)
\]  

(4-4)

The right hand side of this expression is recognized as a sum of chronopoles of strength \(-d^3y\) distributed throughout the volume interior to the boundary surface. (This relationship is also applicable to the mass source term in the Ffowcs Williams and Hawkings equation, which has a form identical to the left hand side of (4-4).) The chronopole source strength, unlike the equivalent mass source strength, never changes sign. When the integral on the right hand side of (4-4) is evaluated as a
sum, the numerical problems associated with a small difference between large quantities are thereby avoided.

When the time integrals in (4-4) are performed, the relationship becomes

\[ \int_S d\xi d\eta \Delta \hat{n} \cdot \mathbf{v} \frac{1}{4\pi r|1-M_\lambda|} = \int_V d\xi d\eta dl \frac{\partial}{\partial \xi} \left( \frac{V(\xi, \eta, \xi, \xi)}{4\pi |1-M_\lambda|} \right) \] (4-5)

Here \( V \) is the three volume enclosed by \( S \), and \((\xi, \eta, \zeta)\) parametrizes the volume at a source time \( \tau \). The volume of the region corresponding to \( d\xi d\eta d\zeta \) at time \( \tau \) is denoted by \( V(\tau, \xi, \eta, \zeta) d\xi d\eta d\zeta \). The region of \((\xi, \eta, \zeta)\) space which maps into \( V \) has been assumed independent of \( \tau \) in order to interchange the integration and observer time derivative. Aside from this restriction on the parametrization, (4-5) applies to any closed surface, rigid or pliable. For thin airfoil sections, the integral in the thickness direction may be performed at once. Thus, \( V(\tau, \xi, \eta) d\xi d\eta \) will denote the volume of the patch of the blade which has an extent \( d\xi d\eta \) on the surface of the blade. (As in (4-2) and (4-3), the blade surface is here regarded as a single sided surface.)

According to (4-5) and Table 3-2(c), the flow field corresponding to the mass source distribution may be regarded as a chronopole of strength \(-\rho_0 V(\tau, \xi, \eta) d\xi d\eta\) distributed along the chord and span of the blade. The resulting contribution to the pressure field, which represents the thickness noise, is
\[ p(z, \bar{r}) = \sum \int_{\text{propeller blades}} F_{\text{propeller blades}} \left\{ \frac{1}{4\pi c r (1-M_t)^2} \left[ \hat{v} + \frac{\hat{f}}{(1-M_t)^2} (3\hat{M}_r) \right] + \frac{c}{r} (M_r^2 + 2M_r - 3M_r^2) + \frac{V}{(1-M_t)^2} (1-M_t)(M_r - 3 \frac{c}{r} \hat{M}_r \cdot \hat{M}) + 3(M_r - \frac{c}{r} M_r^2) + \frac{c}{r} (M_r - \frac{c}{r} M_r^2)(1 + 4M_r + M_r^2) + 3(\frac{c}{r} M_r^2) \right\} \] (4-6)

At first sight, it might appear that the representation of the thickness noise as a sum of chronopole sources is merely the result of some mathematical sleight of hand. Actually, this representation is quite natural. For example, in incompressible flow theory, the flow past an impermeable sphere is equivalent to the flow field of a moving doublet located at the center of the sphere. In spacetime, a doublet in uniform motion along its axis and of unchanging strength is equivalent to a chronopole moving along the same trajectory, as shown in Fig. 4-4.

A moving chronopole differs from a moving doublet only in the case that the strength of the doublet is changing, or that its motion is accelerated. A stationary sphere whose radius oscillates can be represented as a time varying mass source at the center of the sphere. Since the strength of this mass source is proportional to the rate of change of the sphere's volume, it is equivalent to a stationary chronopole whose strength is proportional to the volume of the sphere. Thus, both a stationary pulsating sphere and a sphere in uniform motion are naturally represented by a chronopole source at the center of the sphere, with strength proportional to the volume of the sphere. Chronopole sources are therefore a reasonable way to think about the effects of the displacement of fluid by a solid volume; they are not just a mathematical convenience.
The flow field of a moving doublet simulates the flow past a sphere in an incompressible fluid...

and a chronopole in uniform motion is equivalent to a doublet moving along the same trajectory.

Figure 4-4: Thickness Noise Model
4.2.5 Antisymmetric Part of the Vorticity Distribution

The one remaining source term is the antisymmetric part of the vorticity distribution on the surface of the blades. This term represents a set of vortex lines roughly parallel to the span of the blade which all close on the surface of the blade itself. Mathematically, the source is represented by the doublet distribution of strength $\hat{n} \Phi_A$. Since $\Phi_A$ is of the same sign on both the pressure and suction sides of the blade, the vectors $\hat{n} \Phi_A$ are very nearly equal and opposite at corresponding points on opposite sides of the blade. When the airfoil is thin, the cancellation between the pressure and suction sides renders the contribution of the antisymmetric part of the vorticity distribution much smaller than the contribution of the mass source term.

Explicitly, the contributions to the velocity potential which are to be compared are, from (3-44),

$$-\int d\tau d\xi d\eta \hat{n} \cdot \nabla \Phi G$$

According to the analysis of the previous section, and that of chapter 3, these may be rewritten

$$\int_{\text{blade}} d\eta \left( \frac{3}{3\epsilon} \left( \frac{V}{4\pi r \|n - M\|} \right) \right)$$

and

$$-\int_{\text{blade}} d\eta \nabla \cdot \left( \frac{(\hat{n}_s + \hat{n}_p) \cdot \nabla \hat{n}}{4\pi r \|n - M\|} \right)$$

where $\hat{n}_p$ and $\hat{n}_s$ are the unit normals on the pressure and suction sides of the airfoils, and the integrals are carried out over only one side of the (thin) airfoil surface. From the geometry of Fig. 4-5, the magnitude of $(\hat{n}_s + \hat{n}_p)$ is of order $\epsilon / \chi$ in the notation of section 3.4. With this notation, the relative orders of magnitude of the two terms are

157
Figure 4-5: Thin Airfoil Geometry
\[ \text{mass source } \hat{n} \cdot \nabla \phi : \text{doublet source } \hat{n} \Phi_A \]
\[ (U/\chi)\varepsilon : (1/\chi)(\varepsilon/\chi)(U\varepsilon) \]

since \( \Phi_A \) is of order \( U\varepsilon \), and \( V \) is of order \( \text{Area} \) times \( \varepsilon \).

Therefore, the contribution of the antisymmetric part of the vorticity distribution on the surface of the blade is smaller than that of the mass source term by a factor of order \( \varepsilon/\chi \). For thin airfoil sections, this contribution to the thickness noise is therefore negligible. (A more detailed analysis shows that for observer locations within a distance \( \varepsilon \) of the blunt leading edge of such an airfoil, the term is not negligible, but this fact will be of no importance here.)

It should be emphasized that the theory of chapter 3 is valid for thick airfoil sections, provided that they move at much less than Mach one. In such a situation, the antisymmetric part of the vorticity distribution cannot be neglected. A case in point is the incompressible flow past a sphere in uniform motion mentioned at the end of the previous section. In that case, the contribution of the antisymmetric part of the vorticity distribution to the flow field is exactly half as large as the contribution of the mass source distribution. This clearly demonstrates that the antisymmetric part of the vorticity distribution is associated with the thickness noise, since there is no force on the sphere. The common practice of referring to the mass source integral in the Ffowcs Williams and Hawkings equation as the "thickness noise" term is therefore misleading. This term is associated with the thickness noise only in the case that the object to be modelled is quite thin in the dimension transverse to its motion.
4.3 A Correction for Finite Blade Loading

4.3.1 Overview

The major approximation made in section 4.2 was to ignore the residual motion of the shed vortex sheet. This assumption implies that the induced drag on the blades is negligible. In reality, the induced drag force is usually of the same order of magnitude as the profile drag force, which is of the order of several percent of the lift force. It is therefore desirable to introduce at least a first order correction for the residual motion of the shed vortex sheet. This will be the goal of the present section.

It is hopeless to attempt to evaluate the sound which results from the residual motion of the doublet sources at the surface of the shed vortex sheet. It can only be hoped that the unstable motion of this sheet is sufficiently slow that the resulting sound is negligible. The motion of these doublet sources depends in part on the parametrization which is chosen for the sheet. Specifically, the motion of the doublets in the plane of the sheet is completely arbitrary. They will make the smallest contribution to the pressure field if the sheet is parametrized in such a way that the magnitude of the velocity of the doublets is as slow as possible. This is accomplished when the motion of each point of the shed vortex sheet is assumed to be exactly perpendicular to the sheet.

According to the analysis of chapter 2, the induced velocity for a straight bladed propeller is normal to the surface of the shed vortex sheet in the developed wake. Therefore, it makes sense to parametrize the vortex sheet in such a way that the motion of each point of its surface is given by the actual induced velocity at that point. This choice of parametrization should minimize the contribution of the
doublet distribution on the sheet itself to the pressure field. The residual velocity of the sheet, v, will therefore be assumed to be identical to the induced velocity at the sheet.

As shown in Fig. 4-6, the sources in (4-1) may once again be decomposed into a mass source at the surface, an antisymmetric distribution of vorticity on the surface, a profile drag force on the surface (not illustrated), and a symmetric distribution of vorticity on the surface, which logically includes the shed vortex sheet. The analysis of sections 4.2.3, 4.2.4, and 4.2.5 applies unchanged to this situation. However, the analysis of the symmetric part of the vorticity distribution given in section 4.2.2 must be altered to take into account the residual motion of the shed vortex sheet.

4.3.2 Modelling the Induced Drag Force

Considerable care must be taken when the assumption that the airfoil sections are thin is used to consider the pressure and suction sides of the blade to coincide. Just as in section 4.2.2, the symmetric part of the vorticity distribution on the blade (the bound vorticity) will be considered to be an extension of the shed vortex sheet. In order to minimize the contribution of the dipoles which are created between the leading and trailing edges of the blade, it is once again necessary to minimize their residual velocity after creation. This can be accomplished by taking the combined bound and shed vortex sheet to be smooth at the trailing edge of the blade, as shown in Fig. 4-7. If there were a discontinuity in the slope of the sheet at the trailing edge, then the acceleration of the dipoles created upstream of the trailing edge when they reached the trailing edge would make an additional contribution to the pressure field. The upshot of this reasoning is that the motion of the part of the combined vortex sheet which corresponds to the bound vorticity should also be taken to have a
Figure 4-6: Decomposition of Sources for Finite Loading
Figure 4-7: Symmetric Component of Vorticity Distribution
residual velocity equal to the induced velocity at the surface of the blade.

Once again, the vortex sheet may be considered to be created gradually beginning at the leading edge of the blade and increasing to its full strength at the trailing edge. The pressure field can then be estimated as the linear superposition of lifting line sources distributed on the surface of the blade. Unlike the analysis of section 4.2.2, however, the lifting line sources do not reduce to simple Lowson forces in the case that the vortex sheet is created with a finite residual velocity. From Table 3-3, to first order in the induced velocity \( v \) (the residual velocity of the sheet), the contribution of a lifting line source to the pressure field is

\[
p(t, z) \approx \frac{2\pi n (1 - N_d)}{4\pi cr (1 - N_d)} \left\{ \begin{array}{l}
\hat{\rho} \cdot \hat{C} + \frac{\hat{\rho} \cdot \hat{C}}{(1 - N_d)} \left( \hat{N}_r + \frac{\hat{N}}{r} (1 - N_d) \right) - \frac{\hat{N} \cdot \hat{C}}{r} \\
+ \frac{\hat{\rho} \cdot \hat{D}_z}{(1 - N_d)} \left( \hat{N}_r + \frac{\hat{N}}{r} (1 - N_d) \right) - \frac{\hat{N} \cdot \hat{D}_z}{r} \\
+ M_r \left[ \hat{\rho} \cdot \hat{C} + \frac{\hat{\rho} \cdot \hat{C}}{(1 - N_d)} \left( \hat{N}_r + \frac{\hat{N}}{r} (1 - N_d) \right) - \frac{\hat{N} \cdot \hat{C}}{r} \right] \\
- M_r \hat{I} - M_r \hat{I} - \frac{M_r \hat{C}}{(1 - N_d)} \left( \hat{N}_r + \frac{\hat{N}}{r} (N_r - N_d) \right) \\
+ \frac{1}{4\pi cr (1 - N_d)} \left\{ \hat{\rho} \cdot \hat{\omega} \times \hat{\omega} \cdot \frac{\partial \hat{\rho}}{\partial z} \right\}_{\hat{y}, \hat{z}} + \hat{\rho} \cdot \hat{w} \times \hat{w} + \hat{\rho} \cdot \hat{v} \times \hat{v} \\
+ \frac{\hat{\rho}}{r} \left( 2 \hat{\rho} \cdot \hat{r} M_r - \hat{M} \cdot \hat{C} \right) \right\}
\]

where \( L = \rho_0 w \times s \hat{r} \) is the lift force and \( D_I = \rho_0 v \times s \hat{r} \) is the induced drag force.

The first line of this expression is simply the contribution of the lift force, and gives precisely the same pressure field as (4-2). This term need no longer be considered. The rest of the expression represents the correction to the analysis of section 4.2.2 to first order in the residual motion of the shed vortex sheet.

The second line is precisely the contribution which would be expected from a Lowson force \( D_I \). However, many of the remaining terms are also proportional to
the magnitude of the induced drag force, as will be shown in the following section. The induced drag is directly related to the component of the induced velocity which is normal to the shed vortex sheet. However, the component of the induced velocity in the spanwise direction is invariably of the same order of magnitude as the perpendicular component. Since this spanwise induced velocity does not affect the performance of the propeller, it was ignored in the analysis of chapter 2. The sixth line of (4-7) and all terms containing $M_r$ involve the spanwise component of the induced velocity.

An aerodynamic theory which predicts only the airloads on the propeller inherently makes no prediction of the spanwise component of the induced velocity. Such an aerodynamic theory therefore provides too little boundary value information to compute the pressure field by means of the aeroacoustic theory of chapter 3. For this reason, the spanwise component of $v$ (or $M$) in (4-7) will be ignored here. This component does not necessarily make a negligible contribution to the pressure field, but its contribution cannot be estimated without a more sophisticated theory of propeller aerodynamics than that presented in chapter 2. The reason for discarding the spanwise component of the induced velocity is basically the same as the reason for ignoring the residual motion of the doublets in the shed vortex sheet. In neither case is there an aerodynamic theory available to predict the necessary parameters.

Under the assumption that the residual velocity of the shed vortex sheet is normal to the sheet, (4-7) simplifies considerably. In order to make the notation consistent with that of section 4.2, the Mach number $M = v/c$ will be renamed $m$, where $m$ is now assumed to be parallel (not antiparallel!) to the lift force $L$. This allows $N$ to be renamed $M$ as in section 4.2. Summing up the contributions from all
of the lifting lines, the effect of the component of the induced velocity perpendicular to the shed vortex sheet is found to be

\[
p(r, \vec{x}) = \sum_{\text{propeller}} \int d\vec{y} \, d\vec{h} \frac{\text{sgn}(1-M_j)}{4\pi cr(1-M_j)^2} \left\{ \hat{r} \cdot \hat{D} + \frac{\hat{r} \cdot \hat{D}_z}{(1-M_j)} (\hat{M}_r + \frac{\hat{r} \cdot \hat{E}}{r} (1-M_j) - \frac{\hat{r}}{r} \hat{M} \cdot \hat{E}) \right. \\
- \frac{\hat{r} \cdot \hat{D}_z + m_r [\hat{r} \cdot \hat{D} - \frac{\hat{r} \cdot \hat{E}}{r} (\hat{M}_r + \frac{\hat{r} \cdot \hat{E}}{r} (1-M_j) - \frac{\hat{r}}{r} \hat{M} \cdot \hat{E})]}{(1-M_j)} (\hat{M}_r + \frac{\hat{r}}{r} (M_r-M_j)) \\
+ \hat{r} \cdot \hat{E} (\hat{M}_r + \frac{\hat{r}}{r} (m_r M_r - \hat{M} \cdot \hat{E})) + \frac{\hat{r}}{r} (1-M_j)(2m_r \hat{r} \cdot \hat{E} - \hat{M} \cdot \hat{E}) \right\}
\]

All of the terms in this expression are proportional to the magnitude of the induced drag \(D_I = |D_I|\). To see this, note that if \(L = |L|\), \(M = |M|\), and \(m = |m|\), then \(mL = MD_I\) (4-9)

To reiterate, the spanwise component of the induced velocity along the blade has been ignored in this analysis. This is tantamount to ignoring the radial contraction of the shed vortex sheet. This was necessary, since an aerodynamic theory capable of determining the radial component of the induced velocity has not been developed here. The present analysis should serve as a warning that a knowledge of the forces on the blade and its displacement (volume) alone may be insufficient to compute the sound produced by the propeller. The simple analyses based on the momentum transferred to the fluid by the blade do not make this prediction. Such theories are evidently valid only in the limit that the residual motion of the shed vortex sheet is negligible. That is, in the limit that the induced drag force can be ignored.
4.4 Sources in Helical Motion

4.4.1 Geometry of Source and Observer

The pressure at any event \((t,x)\) in spacetime may be estimated by adding the effects of the lift force \((4-2)\), the induced drag force \((4-8)\), the profile drag force \((4-3)\), and the blade thickness \((4-6)\). Each of these contributions is itself a sum (integral) of contributions from each point of the surface of each propeller blade. The blades are assumed to be thin, so that only one side need be considered in these sums. The quantities \(L, D_l, D_p, \) and \(V\) which are required in the formulas are all predicted by the theory of propeller aerodynamics presented in chapter 2.

In this section, the vector formulas of the previous sections will be reduced to specific formulas for the case of a propeller spinning at a constant speed and advancing along its axis of rotation. The formulas for the lift, profile drag, and thickness may easily be extended to more complex geometries. For example, it is not difficult to repeat the following analysis for the geometry appropriate to a helicopter rotor. However, the aerodynamic theory of chapter 2 cannot easily be extended to this case. Also, it is no longer clear how to interpret the induced drag formula \((4-8)\). (That is, what is being ignored?)

The geometry of the motion of a single point on the surface of the propeller is shown in Fig. 4-8. Each point moves in a helical trajectory with angular speed \(\Omega\) and forward speed \(U\). The source coordinates at time \(t\) are given by
Figure 4-8: Helical Trajectory Geometry
\[ x = r_s \cos(\Omega t + \phi_s) \tag{4-10} \]
\[ y = r_s \sin(\Omega t + \phi_s) \]
\[ z = Ut + z_s \]

This is simply the vector function \( y(t) \) in equation (3-9) for this specific case. The variables subscripted "s" stand for the source coordinates at \( t = 0 \). General values of \( z_s \) and of \( \phi_s \) are allowed to facilitate the later summation of sources at different locations on the surface of the blades.

For convenience, the observer will be assumed to move in the \( z \)-direction with speed \( U \) as the propeller advances. The pressure field is then perceived to be periodic with period \( 2\pi/\Omega \). This simplifies the interpretation of the results of the calculations, since one full period completely specifies the sound heard by the observer (listener!). The observer coordinates will be subscripted "o"; they are given as a function of time by

\[ x = 0 \tag{4-11} \]
\[ y = r_o \sin \theta_o \]
\[ z = r_o \cos \theta_o + Ut \]

The time \( t \) when a sound signal emitted from the source at time \( \tau \) reaches the observer is the sum of \( \tau \) and the propagation time:

\[ t = \tau + |y_o(t) - y_s(\tau)|/c \]

where \( y_o(t) \) and \( y_o(\tau) \) are given by (4-11) and (4-10) respectively. This equation is quadratic in \( t - \tau \), and is easily solved to yield the relationship between observer time and source time for events connected by a sound ray:

\[ t = \tau + r/c \tag{4-12} \]

where
\[ r = (1/\gamma) \left\{ M_z (r_0 \cos \theta_0 - z_s) + \sqrt{(r_0 \cos \theta_0 - z_s)^2 + \gamma (r_s^2 + (r_0 \sin \theta_0)^2 - 2 r_0 r_s \sin \theta_0 \sin (\Omega \tau + \phi_s))} \right\} \]  

(4-13)

The following notations have been adopted:

\[ M_z = U/c \quad \text{and} \quad \gamma = 1 - M_z^2 \]

The assumption that \( M_z \), the forward Mach number, is less than unity has been made in the derivation of (4-13). The distance \( r \) defined in (4-13) is the actual distance travelled by a sound ray between emission from the source at time \( \tau \) and reception by the observer at time \( t \).

The source Mach number \( M \), its radial component \( M_r \), and its derivatives \( M_r \) and \( M_r \) may easily be worked out from the geometry shown in Fig. 4-8 (see also Table 3-1). The results of these calculations are given in Table 4-1, along with some other useful parameters. Among the useful parameters are the unit vector in the "lift direction", \( \hat{\lambda} \), the unit vector in the radial direction, \( \hat{R} \), and the unit vector in the direction of motion, \( \hat{M} = M/M \). These three unit vectors are mutually orthogonal and provide the most convenient basis set for expressing the vector quantities required in the formulas of sections 4.2 and 4.3 above.

4.4.2 The Four Source Terms

The generalized coordinates \((\xi, \eta)\) which will be used to describe points on the surface of the blades are the spanwise and chordwise position. As illustrated in Fig. 4-9, the source coordinates \((r_s, \varphi_s, z_s)\) at \( \tau = 0 \) of a source located at \((\xi, \eta)\) are
Table 4.1. Formulas for Helical Source Trajectories

\[ M_z \triangleq \frac{V}{c} = \text{forward Mach number} \]

\[ M_\phi \triangleq \frac{r_s \Omega}{c} = \text{rotational Mach number} \]

\[ \phi \triangleq \Omega t + \phi_s = \text{source phase} \]

\[ \gamma \triangleq 1 - M_z^2 \]

\[ r = \frac{1}{\gamma} \left\{ M_z \left( r_o \cos \theta_o - z_o \right) + \sqrt{\left( r_o \cos \theta_o - z_o \right)^2 + \gamma \left( r_s^2 + r_o^2 \sin^2 \theta_o \right) - 2 r_o r_s \sin \theta_o \sin \phi} \right\} \]

\[ \bar{r} = (-r_s \cos \phi, r_o \sin \theta_o - r_s \sin \phi, r_o \cos \theta_o + M_z r - z_o) \]

\[ t = \tau + r/c = \text{observer time} \]

\[ \bar{M} = \frac{1}{c} \frac{d}{dt} \bar{\gamma} = (-M_\phi \sin \phi, M_\phi \cos \phi, M_z) = \text{source Mach number} \]

\[ \dot{\bar{M}} = -\Omega \bar{M}_\phi (\cos \phi, \sin \phi, 0); \quad \ddot{\bar{M}} = \Omega^2 \bar{M}_\phi (\sin \phi, -\cos \phi, 0) \]

\[ \bar{M}^2 = M_\phi^2 + M_z^2; \quad M = \sqrt{\bar{M}^2}; \quad \bar{M} \cdot \bar{M} = 0; \quad \bar{M} \triangleq \bar{M}/M \]

171
\[ M_r = \hat{r} \cdot \hat{M} = M_z \left( M_z - \frac{z_s}{r} \right) + \frac{r_o}{r} \left( M_\phi \sin \theta_0 \cos \phi + M_z \cos \theta_0 \right) \]

\[ \dot{M}_r = \hat{r} \cdot \dot{\hat{M}} = M_\phi \left( -\frac{e}{r} M_\phi - \Omega \frac{r_o}{r} \sin \theta_0 \sin \phi \right) \]

\[ \ddot{M}_r = \hat{r} \cdot \ddot{\hat{M}} = -\Omega^2 M_\phi \frac{r_o}{r} \sin \theta_0 \cos \phi \]

\[ \Lambda = \hat{\Lambda} = \frac{1}{\bar{M}} \left( -M_z \sin \phi, M_z \cos \phi, -M_\phi \right); \quad \bar{\Lambda} \cdot \bar{M} = 0 \]

\[ \dot{\bar{\Lambda}} = \frac{M_z}{M \bar{M}_\phi} \dot{\bar{M}}; \quad \ddot{\bar{\Lambda}} = \frac{M_z}{M \bar{M}_\phi} \ddot{\bar{M}}; \quad \bar{\Lambda} \cdot \ddot{\bar{\Lambda}} = 0 \]

\[ \Lambda_r = \frac{1}{\bar{M}} \left\{ -M_\phi \left( M_z - \frac{z_s}{r} \right) + \frac{r_o}{r} \left( M_z \sin \theta_0 \cos \phi - M_\phi \cos \theta_0 \right) \right\} \]

\[ \Lambda_r = \hat{r} \cdot \Lambda = \frac{M_z}{M \bar{M}_\phi} M_r \]

\[ \bar{R} = \bar{R} \left( \cos \phi, \sin \phi, 0 \right) = \frac{1}{\Omega \bar{M}_\phi} \dot{\bar{M}}; \quad \bar{R} \cdot \bar{\Lambda} = \bar{R} \cdot \bar{M} = 0 \]

\[ R_r = \hat{r} \cdot \bar{R} = \frac{1}{\Omega \bar{M}_\phi} \dot{M}_r; \quad \dot{R}_r \Delta \hat{r} \cdot \dot{\bar{R}} = \frac{1}{\Omega \bar{M}_\phi} \ddot{M}_r \]
\[ r_s = \xi \]
\[ \varphi_s = (-\eta + \kappa \chi)(\cos \beta) / \xi \]
\[ z_s = (-\eta + \kappa \chi)(\sin \beta) \]

Here, \( \chi = \chi(r) \) is the airfoil chord at radius \( r \), while \( \beta = \beta(r) \) is the blade angle (i.e. the angle between the disk plane and the chordline) at radius \( r \). Finally, \( \kappa = \kappa(r) \) is the "stacking parameter" at radius \( r \). That is, \( \kappa \) is the chordwise location of the point of the airfoil section at radius \( r \) which lies on the radial line passing through the tip of the blade.

With this simple choice for \((\xi, \eta)\), the area on the blade surface is just \( d\xi d\eta \). Thus, \( L(\tau, \xi, \eta), D_l(\tau, \xi, \eta), D_p(\tau, \xi, \eta), \) and \( V(\tau, \xi, \eta) \) represent simply the lift, induced drag, profile drag, and volume per unit area of the surface. Since the lifting lines (the bound vorticity) are essentially radial, the edge parameter \( \mu \) is the same as \( \xi \), and \( s = (\partial x / \partial \mu) \tau = \mathbf{R} \), the unit vector in the radial direction. The sources are

\[ L = \hat{\lambda} l(\tau, \xi, \eta) \]
\[ D_l = \hat{M} D_l(\tau, \xi, \eta) = \hat{M}(m/M)L(\tau, \xi, \eta) \]
\[ D_p = \hat{M} D_p(\tau, \xi, \eta) \]
\[ V = \varepsilon(\tau, \xi, \eta) \]

where \( \hat{\lambda} \) and \( \hat{M} \) are the unit vectors of Table 4-1, and \( L, D_l, \) and \( D_p \) are the scalar values of the lift and drag per unit area of the blade. Lastly, \( \varepsilon \) is the thickness of the blade, which is the volume per unit area.

The Mach number \( m \) associated with the residual motion of the shed vortex sheet will be assumed to be perpendicular to the sheet, as discussed above. Thus,

\[ m = \hat{\lambda} m \]

Once again, this ignores the radial contraction of the shed vortex sheet. The unstable roll up of the sheet will also be ignored in the following calculations.

The equations for the sound pressure due to the lift, drag, and thickness simplify
Figure 4-9: Blade Coordinate System
considerably in the specific geometry considered here. Combining equations (4-2), (4-3), (4-6), and (4-8) with the formulas of Table 4-1 and equations (4-15) and (4-16), results in the expressions given in Table 4-2. The possibility for a time varying lift and profile drag is included for reference. However, it will always be assumed in this chapter that the scalar values of $L$, $D_l$, $D_p$, and $\epsilon$ are independent of time. Thus, in Table 4-2, $L$ and $D_p$ will be assumed to be zero for now.

If the propeller were subjected to a nonuniform inflow, or if its axis were not aligned with its direction of forward motion, the lift and drag forces would become time dependent, even in a reference frame corotating with the blades. This case is of great importance in the theory of helicopter rotors, or for marine propellers. Note that if the frequency of the unsteadiness is much larger than the angular frequency of the propeller, then the analysis which lead to the neglect of the antisymmetric part of the vorticity distribution on the surface of the blade may no longer be valid. Thus, in addition to even greater problems in dealing with the induced drag contribution, the antisymmetric part of the vorticity distribution may be confronted in an attempt to generalize the results of this section to the case of unsteady loads.

The profile drag force is equivalent to a distribution of mass sources of magnitude $D_p/Mc$ on the surface of the blade, as can be seen from Table 3-2(a). This result is quite independent of the helical geometry employed here. A point force of constant magnitude which moves along an arbitrary trajectory, and which always remains parallel to its direction of motion, is equivalent to a mass source of fixed strength moving along the same trajectory. Such a point force creates a vortex tube in its wake, as per the analysis of section 3.2.3. The fact that such a vortex tube is equivalent to a point mass source at its end is analogous to the notion that a "string" of dipoles is equivalent to a point charge in electromagnetic theory. In the
Table 4.2. Formulas for Propeller Sound

Total sound pressure at an observer location \((t, \bar{x})\):

\[
p(t, \bar{x}) = p_L(t, \bar{x}) + p_p(t, \bar{x}) + p_T(t, \bar{x}) + p_I(t, \bar{x}), \quad \text{where}
\]

(a) Lift noise (Lowson point forces)

\[
p_L(t, \bar{x}) = \sum_{\text{propeller blades}} \int_{\text{blade chord & span}} d\xi d\eta \frac{\text{sgn}(1-M_r)}{4\pi cr(1-M_r)^2}
\]

\[
\cdot \left\{ \Lambda_r \frac{d^2L}{dt^2} + L(\tau, \xi, \eta) \left[ \Lambda_r + \frac{\Lambda_r}{1-M_r} \left[ \frac{\tau}{M_r} + \frac{c}{r} (1-M^2) \right] \right] \right\}
\]

(b) Profile drag noise (Lowson point forces) (or mass sources)

\[
p_p(t, \bar{x}) = \sum_{\text{propeller blades}} \int_{\text{blade chord & span}} d\xi d\eta \frac{\text{sgn}(1-M_r)}{4\pi cr(1-M_r)^2}
\]

\[
\cdot \left\{ \frac{D_p}{M} M_r + \frac{D_p(\tau, \xi, \eta)}{M(1-M_r)} \left[ \frac{\tau}{M_r} + \frac{c}{r} (M_r - M^2) \right] \right\}
\]
(c) Thickness noise (moving part chronopoles)

\[ p_T(t, \bar{x}) = \sum_{\text{propeller blades}} \int_{\text{blade chord \& span}} d\xi d\eta \frac{\rho_0 v(\tau, \xi, \eta)}{4\pi r |1-M_r|^5} \]

\[ \cdot \left\{ (1-M_r)M_r^\prime + \left[ M_r - \frac{C}{r} M^2 \right] \left[ 3 \left( M_r - \frac{C}{r} M^2 \right) + \frac{C}{r} \left( 1+4M_r+M_r^2 \right) \right] \right. \]

\[ + 3 \left( \frac{C}{r} M_r \right)^2 \left\} \right. \]

(d) Induced drag noise (part of lifting line source)

\[ p_I(t, \bar{x}) = \sum_{\text{propeller blades}} \int_{\text{blade chord \& span}} d\xi d\eta \frac{\text{sgn}(1-M_r)}{4\pi cr (1-M_r^2)} \frac{D_r(\tau, \xi, \eta)}{M} \]

\[ \cdot \left\{ \frac{1-M_r^2}{1-M_r} \left[ \frac{r}{M_r} + \frac{C}{r} \left( M_r - \frac{C}{r} \right) \right] + M^2 \left[ \frac{\Lambda_r}{r} \left[ 2\Lambda_r + \frac{\Lambda_r}{1-M_r} \right] \right] \right. \]

\[ \cdot \left[ \left( M_r + \frac{C}{r} \left[ 3(1-M_r) + M_r^2 - M^2 \right] \right) - \frac{C}{r} \left( 1-M_r \right) \right] \left\} \right. \]

177
fluid flow problem, the mass source represents the fact that the boundary layer has a finite thickness at the trailing edge of the airfoil. The irrotational flow around the outside of the boundary layer does not quite come together at the trailing edge.

Also, in the limit that $M^2$ is negligible, the pressure field associated with the induced drag forces takes on the same form as the expression for the pressure field due to the profile drag forces. In this low speed limit, the prescription for computing propeller sound given in Table 4-2 becomes identical to the result of a simple, physically motivated theory. In either case, the surface of the propeller is modelled simply as a source of momentum equal to the aerodynamic forces on the blades, plus a term to account for the thickness noise. However, at high speeds, the present treatment differs from such theories. This is due to the explicit recognition of the difference between the mechanisms of profile drag and of induced drag in the present theory.

Additional terms due to the radial contraction of the shed vortex sheet, and the roll up of the shed vortex sheet are predicted by the aeroacoustic theory of chapter 3. These terms must be overlooked until improvements in the aerodynamic theory of chapter 2 are made.

4.5 A Computer Model of Propeller Rotational Noise

4.5.1 Implementation of the Theory

The formulas of Tables 4-1 and 4-2 express the sound as a function of observer time in terms of the lift, induced drag, profile drag, and thickness distributions on the propeller blades. Each source point follows a helical trajectory, and the observer
is assumed to have the same forward velocity as the sources. The spanwise and chordwise distribution of thickness is known from the given propeller geometry. The spanwise distributions of lift, induced drag, and profile drag may be computed from the propeller geometry and the two dimensional characteristics of its airfoil sections. The explicit formulas are given in Table 2-3, and by equations (2-26) and (2-27). The chordwise distribution of the loads could be taken from more detailed two dimensional airfoil section data. Instead, for simplicity, the rough estimate of the chordwise load distribution proposed in the following section will be used in all calculations presented here.

The integrals of Table 4-2 will be performed numerically as finite sums. The four integrands have been carefully chosen to facilitate this approximation; the sign of each integrand tends to be the same for source points in any given region of the surface of one blade. The pressures produced by the various sources must be summed at the same observer time, not at the same source time. The goal of the calculation is to find the pressure as a function of time for the given observer. The observer time corresponding to each source time for a given point source is an elementary function (eqs. (4-12) and (4-13)). However, this function cannot be inverted analytically to yield the source time as a function of observer time.

Therefore, it is far simpler to compute the pressure as a function of observer time parametrically in terms of source time. This calculation will be repeated for each source point on the surface of the blade for source times extending over one complete revolution of the propeller. The contributions from each source point to the pressure at a given observer time can then be added at constant observer time, without the necessity of performing a single difficult calculation of the retarded time. This procedure is far simpler and less expensive than attempting to compute
the retarded times for each source point which correspond to a particular observer time. Although the case of helical Mach numbers greater than unity is not important here, the parametric calculation which will be used is far, far simpler in that case than an equivalent retarded time calculation.

It is expedient to use only a single grid of source points on the surface of the blades in order to estimate all four integrals in Table 4-2. This minimizes the number of observer times which must be computed for a given source time. Because of the care with which the representation of the sources has been chosen, only about twenty or thirty grid points per blade are required in order to estimate the values of the four source integrals with an accuracy of one or two percent, uniformly over all observer times. This remark applies to parameters appropriate for light aircraft propellers. More source points would be necessary in order to accurately model the source integrals for propellers with higher solidity, or with higher tip Mach numbers. (These facts were determined by a series of numerical experiments with the computer program outlined below.)

The loading noise tends to be the dominant rotational noise production mechanism for light aircraft propellers. This is the reason that Gutin’s original theory was successful. The fastest convergence of the sound pressure as the number of grid points is increased is obtained when the area of the blade surface corresponding to one grid point carries roughly the same load as the areas corresponding to every other grid point. The load is proportional to the dynamic pressure, or to the square of the section velocity. The section velocity is approximately proportional to its distance \( r \) from the axis, for the low advance ratios typical of light aircraft propellers. Therefore, a simple and effective way to choose the grid points is to divide the blade into patches which have equal values of
(radius)²(area).

Three model propellers were constructed for the wind tunnel experiments detailed in chapter 5. For the theoretical calculations of the rotational noise produced by these blades, the 20 source point grids shown in Fig. 4-10 were used. The coordinates of these 20 points are given in appendix. Each of the nine radial stations has a value of (radius)²(area) which is proportional to the number of chordwise divisions at that station. The number of chordwise stations increases toward the tip, for reasons which will be explained in more detail below. At each of the twenty grid points, the lift, induced drag, and profile drag forces acting on the corresponding patches of the surface, and the volume of the patch of blade, determine the strengths of the four sources which are placed at that grid point.

In order to sum the sound pressures from different source points at the same observer time, the procedure shown in Fig. 4-11 is fast and accurate. Observer time may be divided into a large number of intervals of equal duration. The observer time at the end of each such interval will be called a "sample time". It is convenient to measure observer time from the time when a sound signal emanating from $r_s = z_s = 0$ at time $\tau = 0$ reaches the observer. From (4-12) and (4-13), this zero time is

$$t^* = \{M^*r_0 \cos \theta_0 + \sqrt{(r_0 \cos \theta_0)^2 + \gamma(r_0 \sin \theta_0)^2}\}/\gamma$$

(4-17)

The first sample time will be assumed to fall at this instant, with sample times before $t^*$ counted negative and those after $t^*$ counted positive. The goal will be to compute the pressure observed at each sample time, much as a digital oscilloscope would sample the output of a microphone.

Numerically, the list of pressures at all of the sample times, which will be termed the "pressure vector", represents the pressure as a function of observer time. The
Figure 4-10: Source Point Grids
Figure 4.11: Flow Chart for Sound Calculation Program
pressure vector will be taken to begin with the sample time corresponding to $t^*$, and to end with the last sample time before one period of the rotational sound has elapsed. If the observer time falls outside this one period window, it can always be reduced to an observer time within the window by virtue of the periodicity of the pressure.

If there are a number of identical equally spaced blades, then the period of the rotational sound will equal the blade passing period, rather than the shaft rotation period. For two blades, the source time can be allowed to vary over one full shaft rotation, while the observer time varies over two full blade passing periods. The second blade passing period can then be interpreted as the effect of the second blade. If the contribution to the pressure from the second half of the blade passage is simply added to the appropriate component of the pressure vector, with the observer time reduced to one blade passage, then there is no need to have a separate network of grid points for the second blade.

The explanation of the flow chart of Fig. 4-11 is as follows: For each source point,

1. The current source time is initially set to zero.

2. The observer time corresponding to the current source time for the current grid point is computed. This is the current observer time.

3. On the first pass through the loop, steps 4 and 5 are taken to initialize the loop. On subsequent passes, the loop is aborted early if the current observer time is less than the current sample time. An early abort passes control to step 14, otherwise the loop continues with step 6.

4. The initial sample time is the smallest sample time larger than the current (initial) observer time. This is the sample time which will first be passed as the source time increases. The final sample time is one full
shaft rotation period after the initial sample time, less one sample. One
full rotation of the propeller is sampled at equally spaced intervals in
this way.

5. The current sample time is set to the initial sample time.

6. The values of the four integrands in Table 4-2 are computed at the
current source time, for this grid point. Their sum is the contribution of
this grid point to the pressure at the current observer time and location.

7. If the current sample time is greater than the current observer time (as it
always is on the first pass through the loop), then the pressure at the
current sample time cannot yet be computed, and control passes to step
13.

8. Otherwise, the current sample time lies between the previous observer
time and the current observer time. In this case, the pressure is assumed
to vary linearly between the previous pressure contribution and the
current pressure contribution. Thus, the contribution to the pressure
field at the current sample time due to this grid point is estimated by
linear interpolation.

9. Each sample time corresponds to a particular element of the pressure
vector. The element corresponding to the current sample time is
computed by reducing the sample time to a standard period using the
periodicity of the pressure.

10. The linearly interpolated contribution to the pressure at the current
sample time is added to the appropriate element of the pressure vector.
If there are N identical equally spaced blades, each element of the
pressure vector receives exactly N such increments from each grid point.

11. If the current sample time is the final sample time, the program gets the
next grid point and begins again from step 1.

12. Otherwise, the current sample time is incremented, and control passes
back to step 7.
13. The inner loop will eventually terminate when the sample time increments past the current observer time. When this happens, the previous observer time is set to the current observer time, and the previous contribution to the pressure is set to the current contribution to the pressure. This sets up the next linear interpolation.

14. The source time is incremented. In the calculations presented below, the source and sample time increments were both taken to be 1/256th of a blade passing period, which gave acceptable resolution for the plots of pressure versus time. However, there is no reason why the sample and source times could not be incremented by different amounts. Moreover, it may be more efficient to increment the source time by a different amount depending on the rate of change of the pressure. This possibility was not explored here.

This calculation algorithm may be used for any geometry; it is in no way specific to the helical source trajectories considered here.

4.5.2 Chordwise Load Distributions

The effect of distributing the loads or volume in various ways along the chord will now be explored. Suppose that a point load of magnitude $L$ located at the radius $r_s$ of interest gives rise to a pressure $L f(\Omega t)$ at a given observer location. The function $f$ has a period of $2\pi$ for the moving observers considered here. If the load is now distributed in the azimuthal direction, so that $L(\varphi)\,d\varphi$ represents the magnitude of the load at azimuth $\varphi_s = \varphi$, then the pressure is

$$p(t) = \int f(\Omega t - \varphi) L(\varphi)\,d\varphi$$

If $L(\varphi)$ is nonzero only over a narrow range of azimuths (near $\varphi = 0$, say), then
\[ p(t) \approx Lf(\Omega t) + (Q/r_s)f'(\Omega t) + \mathcal{O}((\chi/r_s)^2) \]

where

\[ L = \int L(\varphi)d\varphi \quad \text{and} \quad Q = \int (r_s \varphi)L(\varphi)d\varphi \]

The quantity \( L \) is simply the total load and \( Q \) is the moment of the load distribution about \( \varphi = 0 \). The origin \( \varphi = 0 \) may always be chosen in such a way that \( Q = 0 \). Therefore, to first order, a load which is distributed along the azimuthal direction gives rise to a pressure signature (i.e. pressure as a function of time) with a shape identical to the signature of a point load. The effective azimuthal location of the azimuthally distributed load is at the centroid of the load distribution (defined by \( Q = 0 \)).

The shape of the pressure signature due to an load which is distributed along the chord of an airfoil differs from that of a point load located at the centroid of the distribution only to second order in the ratio of the chordlength \( \chi \) to the source radius \( r_s \). Since the performance theory includes corrections to the case of zero solidity only to first order in the chord to radius ratio (the solidity), it is questionable whether a grid with more than a single point in the chordwise direction can be consistent with earlier approximations. Nevertheless, the second order shape changes due to distributing the load along the chordwise direction are visible, though not large, for the case of a light aircraft propeller, and will be included in the calculations.

For a flat plate at an angle of attack, the distribution of lift along the chord is given by

\[ L_0(\eta) = \frac{(4/\pi)^{3/2}}{\sqrt{1 - (\chi-\eta)/\eta}} \quad (4-18) \]

This expression is normalized to integrate to unity. The centroid of this distribution occurs one quarter of the chordlength from the leading edge; this point is called the
quarter chord point. The actual lift distribution on a thin airfoil is the sum of this "angle of attack" distribution and a second distribution depending on the detailed shape of the camberline of the section. For most commonly used airfoil sections, the distribution of lift due to camber ramps to zero at the trailing edge of the section. All told, unless the airfoil is operating very near zero lift, the distribution of lift along the chord will be qualitatively very similar to (4-18).

Therefore, (4-18) will be adopted as the chordwise load distribution in all calculations made here. As noted above, the most important effect of this assumption is that the center of all load distributions will occur at the quarter chord point. Numerical experiments reveal only a slight change in the shape of the pressure signature between the chordwise distribution (4-18) and a flat chordwise distribution (which represents the opposite extreme).

If the chordwise distribution is represented as a number of discrete points, it is convenient to choose the points in pairs which have no net moment about the quarter chord point. Each pair will be chosen here to have the same total load. For a given number of points along the chord, these criteria determine the chordwise locations \( \eta_i/\lambda \), and the fraction of the load \( f_i \) to be placed at each point. For one through five chordwise divisions, the discrete distributions used to represent the angle of attack distribution (4-18) are given in Table 4-3. All calculations presented here used the chordwise locations and fractions shown in this table to divide the load at a given radial station among the points along the chord at this station. The volume was distributed along the chord in a manner appropriate to the distribution of the thickness along the chord for the given airfoil section. These discrete thickness distributions are tabulated in appendix E for the three blades of interest.
Table 4.3. Adopted Chordwise Load Distribution

<table>
<thead>
<tr>
<th>Number of Chordwise Divisions</th>
<th>Chordwise Station, $\eta_i/\chi$</th>
<th>Fraction of Load $f_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>0.08</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>0.76</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>0.16</td>
</tr>
<tr>
<td>4</td>
<td>0.03</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>0.18</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>0.46</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>0.91</td>
<td>0.12</td>
</tr>
<tr>
<td>5</td>
<td>0.02</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>0.13</td>
<td>0.27</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>0.61</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>0.94</td>
<td>0.10</td>
</tr>
</tbody>
</table>
4.5.3 Relative Importance of Various Source Terms

A few of the more informative numerical experiments performed with the computer model outlined above will now be briefly described. Parameters typical of light aircraft propellers will be assumed in the discussion. All of the plots displayed in this section were made from the 20 grid point model of the model Cessna propeller blade operating at 10,000 RPM at a forward speed of 30 m/s. This operating condition is similar to the conditions most commonly used in the wind tunnel experiments of chapter 5.

The relative importance of the four types of sources is shown in Figs. 4-12, 4-13, and 4-14. The observer is located at various azimuths, one propeller diameter from the hub in these graphs. In the near field, the positive mean pressure behind the disk plane and the negative mean pressure in front of the disk plane are primarily the result of the lift forces on the blades. These mean values are a consequence of the thrust produced by the propeller; the expression used to compute the pressure is valid arbitrarily close to the blades (although extremely close to the blades, more grid points would be required in the numerical work).

The relative importance of the four source types is not drastically different from what is shown here when the observer is located in the far field. At low rotational speeds, the thickness noise becomes less important than it is here. If the propeller were operating at a lower advance ratio, the lift force is less dominant than indicated in these pictures.

Figs. 4-15, 4-16, and 4-17 illustrate the change in the character of the pressure signatures between the near and the far field. The near field traces are at the same observer locations as above, while the far field traces are for an observer ten diameters from the hub. The vertical scales for the far field signatures have been
Figure 4.12: Relative Importance of Source Types: -40°
Figure 4.13: Relative Importance of Source Types: 0°

- Cessna/Symmetric

- L: lift forces
- I: induced drag forces
- P: profile drag forces
- T: thickness noise

Microphone location:
- Radius: 48.3 cm
- Azimuth: 0°

Operating conditions:
- Angular speed: 10 kRPM
- Forward speed: 30 m/s
- Sound speed: 348 m/s
Figure 4-14: Relative Importance of Source Types: +40°
operating conditions:
- angular speed: 10 kRPM
- forward speed: 30 m/s
- sound speed: 348 m/s

microphone location:
- radius: 48.3 cm (N), 483 cm (F)
- azimuth: -40°

(vertical scale of far field trace magnified by 10)
Figure 4.16: Transition from Near to Far Field, 0°

Cessna/Symmetric

(vertical scale of far field trace magnified by 10)

Sound Pressure (Pa)

operating conditions:
- angular speed: 10 kRPM
- forward speed: 30 m/s
- sound speed: 348 m/s

microphone location:
- radius: 48.3 cm (N), 483 cm (F)
- azimuth: 0°
enhanced by a factor of ten to allow for the geometrical 1/r effect. The mean value of the pressure disappears at all azimuths in the far field; it is a strictly near field phenomenon.

The shift in phase between the near and far field (especially in the disk plane, where the shape of the signature does not change markedly) can be understood as a geometrical effect. The lag in phase in the near field signature relative to the far field signature is caused by the fact that the propeller blades must rotate slightly beyond the vertical before they approach the near field observer most rapidly (assuming that the plane determined by the observer and the propeller axis is horizontal).

4.6 Summary

The aerodynamic theory and the aeroacoustic theory have been combined in this chapter. The aeroacoustic theory provides a relationship between the flow conditions at the boundary of a region of irrotational flow and the flow parameters anywhere in the bulk of the fluid. The aerodynamic theory provides the necessary boundary values as inputs to the aeroacoustic theory. Unlike theories in which the propeller is viewed as simply a source of momentum and of mass (to simulate its displacement of fluid), the blade geometry and the airloads on the blades do not provide a sufficient amount of information to completely determine the sound field (or the flow field near the blades) in the aeroacoustic theory developed in chapter 3. Therefore, the theory presented here is, in a sense, not complete. A more detailed performance theory than that presented in chapter 2 would be required in order to completely determine all of the necessary inputs to the aeroacoustic theory.
However, there is one limiting case in which the aerodynamic theory of chapter 2 does provide all of the information necessary in order to determine the entire flow field. Namely, if the blade is very lightly loaded, then the shed vortex sheet is essentially stationary after its creation. Thus, the doublet layer associated with this sheet has a known location and, if the load distribution is known, a known strength. If, additionally, the airfoil sections of the blade are exceedingly thin, then the blade geometry and the airloads predicted by the aerodynamic theory of chapter 2 are sufficient to determine all of the boundary conditions necessary as inputs to the aeroacoustic theory. With these two approximations, the propeller blade can be modelled as sums of three types of sources. These are lift forces, profile drag forces, and chronopole sources.

Each of these three source types is a moving point source. The lift force creates a stationary shed vortex sheet, which is identical to the actual, stationary shed vortex sheet discussed in chapter 2 for this limit. The profile drag force creates a thin momentum wake which coincides with the shed vortex sheet, much as an actual momentum wake. And the chronopole source represents the transverse displacement of the fluid as the blade cuts through it in a very natural and appealing way. Unfortunately, if the vortex sheet is regarded as stationary, it is impossible to account for the phenomenon of induced drag, since the origin of that force is rooted in the residual motion of the fluid in the wake.

In order to model the effect of the induced drag, the vortex sheet has been allowed to have a small residual motion. The parametrization of the sheet was chosen to minimize the contribution of the doublets located at the surface of the sheet to the pressure field. The contribution of these doublet sources on the free vortex sheet is then ignored. Next, the first order effects of the residual motion of
the vortex sheet are included in the expression for the pressure field associated with a lifting line source.

Unfortunately, when this is done, the pressure field is found to depend not only on the component of the induced velocity which is perpendicular to the shed vortex sheet, but also on the spanwise component. The induced drag is determined by the perpendicular component of the induced velocity alone. The spanwise component of the induced velocity gives rise to the radial contraction of the shed vortex sheet (the so called wake contraction). Thus the first order correction for the residual motion of the shed vortex sheet produces an effect due to the induced drag, as desired, but it also produces an effect due to the contraction of the wake. Since the performance theory of chapter 2 does not predict the wake contraction, this term is dropped here, much as the residual motion of the doublet sources associated with the vortex sheets has been dropped for lack of information.

The four source mechanisms retained here are the ones which are predicted by the simple performance theory of chapter 2. These are the lift force, the induced drag force, the profile drag force, and the thickness effect. The first and third are modelled as sums of Lowson point forces, the fourth as a sum of moving chronopoles, and the second as a first order correction to the lifting line source for the residual motion of the vortex sheet being created. The new form for the induced drag source presented here reduces to the form of the profile drag source in the limit of low Mach numbers.

Finally, a very efficient algorithm for computing the pressure as a function of time has been outlined. The sources have been carefully chosen to minimize the fineness of the mesh which must be used to evaluate the source integrals. The primary innovations of this technique are the fact that no retarded times need be
computed, and that the thickness noise is expressed as a sum of moving chronopoles. The latter expression provides another illustration of the utility of the powerful mathematical techniques for dealing with four dimensional integrals developed in chapter 3.
Chapter Five

An Experiment with Propeller Noise

5.1 Overview

The theoretical problem of applying the general theory of continuum fluid dynamics to the computation of the sound generated by a solid object in motion through the fluid has been discussed at length. For the special case of a streamlined object, there appears to be some hope that a reasonably accurate estimate of the sound field can be made. A rather extensive experiment with propellers for light aircraft has been performed, which will now be described in the framework of the theory developed above.

The experimental problem of measuring the sound field produced by such a propeller is no less formidable than the theoretical problems which have been dealt with previously. There are several important reasons for performing such experiments. The first is to ward off the simple neurotic fear that some major mistake or oversight has been made in all of the theorizing. Not everyone enjoys an Einsteinian confidence in the correctness of abstract reasoning. More importantly, an experiment provides a different and valuable perspective of a problem. Indeed, only by performing experiments can one be lead to the correct questions for any theory to try to answer. In turn, the theory can lead to new questions to be explored experimentally.

One legitimate problem with the theory which has been developed so far is its determinism. No allowance is made for the phenomenon of turbulence, aside from
a prayer that it not be "important". The experiment described below shows that for light aircraft propellers in most situations, the most distinctive features of the sound field are adequately predicted without turbulence. However, some features of propeller noise are evidently not predicted by so simple a theory. These probably result from flow instabilities such as a Karman vortex street. Finally, in one instance the experiments disclosed a rather violent flow instability which dominated the sound produced by the propeller. In this case, of course, the theory is not adequate at all.

Before embarking on a detailed description of the experiment, a few introductory remarks are in order. The propellers which were tested were 1:4 scale models of propellers designed for a Cessna 172 airplane. Since the Reynolds numbers of this flow were fairly large, the flow field is a function primarily of the geometry and of the Mach numbers. The flow field associated with a scale model will therefore be similar to the full scale situation if only the Mach numbers are the same. That is, time must be scaled by the same factor as distance, 1:4 in the present case. For this reason, the model propellers were spun with four times the angular velocity of the full scale ones. Since velocities were to be the same for the two situations, the model propeller should advance at the same speed that the full scale airplane moves forward, which is over 60 m/s in a "cruise" condition. Unfortunately, the wind tunnel which was used was not capable of wind speeds in excess of about 30 m/s, which corresponds roughly to the speed of the Cessna 172 at takeoff.

The 1:4 scaling factor was not at all arbitrary. Due to the rather small chord lengths of an aircraft propeller, the Reynolds numbers of its airfoil sections are rather small. In the model propellers, the Reynolds numbers were smaller by the
scale factor. This begins to be a problem, since the airfoil section characteristics do not attain their high Reynolds number limits until rather large values of that parameter have been reached. For this reason, the scale factor could not have been chosen to be too much smaller than 1:4, without entering the flow regime in which airfoil section characteristics become Reynolds number dependent.

With the 1:4 scale which was used, the Reynolds numbers of the airfoil sections near the tip of the propeller were about half a million (based on chord and the local flow speed), at the highest speeds which were tested (corresponding to tip Mach numbers of about .7). The section Reynolds numbers decrease roughly in proportion to the distance of the section from the propeller axis, and in proportion to the angular speed. These Reynolds numbers are already dangerously low from the standpoint of being able to easily estimate the performance of the airfoil sections for most of the blade.

Conversely, the upper limit on the model scale was set by the size of the available anechoic wind tunnel and electric motor to turn the propeller. These two constraints were about equally prohibitive of propellers with a scale larger than 1:4. The power required to turn a scale model propeller is proportional to the square of the scale factor, so in this case the model propellers required a motor about one sixteenth as powerful as the reciprocating Lycoming engine which powers a Cessna 172. This is at the limits of the capability of the inexpensive surplus jet engine starter/generator which was used to turn the model propellers in this experiment. It should also be pointed out that the motor to some extent creates a lower bound on the scale factor, as well as this upper bound. This is due to the fact that the 1:4 scale model must spin at four times the angular speed of the actual propeller; machinery which operates at such high angular speeds is already difficult to find for the 1:4
The wind tunnel had to be large enough to hold the 1:4 scale model. Additionally, it had to be relatively "dead" acoustically to avoid confusing sound reflections. The 5'×7.5' wind tunnel in the Aeronautics Department's Experimental Projects Laboratory was the only available tunnel which fulfilled these two requirements. As mentioned above, the highest wind speed attainable in this tunnel is only about half of the cruising speed of a typical light airplane. For a number of reasons, the inability to simulate the cruise condition was not a significant problem. In fact, higher wind speeds create higher incidental flow noise which masks the sound produced by the propeller. The magnitude of this flow noise, even at only half of the "cruise" speed, precluded good measurements of the propeller noise unless the microphone was located quite close to the propeller. For this reason, all of the data which will be reviewed below was taken with the microphone located about one propeller diameter from the hub.

Furthermore, from the standpoint of this thesis, which is oriented toward science rather than toward engineering, it is completely irrelevant that the model propeller cannot exactly model the real propeller. Indeed, it is not necessary to regard the propellers which were tested as "models" at all. From this point of view, it is only necessary to point out that the approximate flow regime of the propellers which were tested is identical to the flow regime of a typical light aircraft propeller; a theory capable of describing the results of these experiments is almost certain to work when applied to light aircraft propeller noise.

In retrospect, the fact that all of the propellers which were tested were constructed as scale models of flightworthy propellers was an unnecessary and mildly damaging restriction. At the time the experiment was designed, it was not
expected that the measured sound would show such startlingly close agreement with the predictions of so simple a theory. It now appears that a much more detailed comparison of the theory to a similar experiment would have been possible, had less time and money been spent on conscientiously trying to build and test "models" of flight hardware. Only a small fraction of the data which was taken will be discussed here, and some more specific problems with too exactly simulating flyable propellers will be pointed out.

Before the experimental results are presented, a detailed description of the apparatus which was necessary to carry out the acoustic part of the experimental program will be given. This description is divided into two parts. First, the heavy mechanical hardware and high power electrical equipment will be discussed. Second, the more delicate sensing devices and electronics will be described. After this, the data processing facilities and techniques which were employed will be mentioned.

The discussion of the experimental results will begin with the repeatability of the sound measurements. This is intended to provide an index for what "good" agreement between theory an experiment might mean. Some comparisons between measured and predicted sound pressures will then be made at various observer locations and propeller operating conditions. Next, the residual "noise" which is not predicted by the theory will be briefly examined. The one case mentioned above in which this "noise" dominated the propeller sound will then be examined in considerably greater detail, although the experiments necessary to completely explain the nature of the phenomenon could not be performed with the existing apparatus.
5.2 Apparatus

5.2.1 Wind Tunnel

The anechoic wind tunnel located in the basement and first floor of building 33 at M.I.T. was the site of this experiment. It is a closed circuit wind tunnel (Fig. 5-1), which is driven by a six bladed fan (Fig. 5-1a). The fan is about 2.1 m in diameter, and is located in the basement together with a large generator to power it. Immediately downstream of the fan are five irregularly spaced straightening vanes, which serve both to support the motor which drives the fan, and to remove much of the swirl velocity which the fan imparts to the flow. Moving downstream of the fan, the tunnel widens gradually from a circular duct which fits tightly around the fan to a 4.6 m by 2.7 m rectangular duct, which turns upward at a right angle about 16 m from the fan. At this bend of the tunnel, and at each of the other three bends, a row of curved sheet metal turning vanes (b) is mounted diagonally across the bend. Also, a layer of sound absorbing material (c) is attached to the duct walls at each turn.

From the first bend, the tunnel proceeds directly upward from the basement to the first floor, where it is again turned through a right angle. Immediately after this second turning, the flow passes through two wire mesh screens (d) which help to smooth out any turbulence created by the fan or turning vanes. Shortly after it passes these screens, the flow is contracted by a 2.4 m long nozzle to a 2.3 m by 1.5 m rectangular cross section. The test section of the tunnel begins at the end of this nozzle and is about 3 m in length. After the test section, the flow is slowly reexpanded over a distance of 7 m before it passes through a chickenwire screen (this simply prevents large objects from being ingested into the fan) and turns

206
Figure 5.1: Anechoic Wind Tunnel

207
downward toward the basement. Immediately after making the fourth and final turn, the flow passes through the fan and begins the cycle again.

At the top tunnel speed of about 30 m/s (in the test section), approximately three seconds are required for each particle of air to make the round trip. Hence, the air circulates around the tunnel on the order of a hundred times in the few minutes it takes to perform one series of measurements. Since there is no provision for cooling this recirculating air mass, essentially all of the tens of kilowatts which are put into the large fan drive motor go into increasing the temperature of the air in the circuit. Fortunately, the volume of air in the tunnel is fairly large (amounting to roughly 350 kg), so this temperature increase proceeds fairly slowly. Typically, the temperature rose one or two degrees Celsius for each minute of tunnel operation at the highest airspeeds, amounting to a total change of as much as five degrees during one series of measurements. This property of the tunnel is rather annoying, since it places the temperature of the airstream beyond experimental control.

The wind tunnel is constructed entirely of wood. Its major feature is the fact that the 3 m long test section has been cut away entirely. Thus, the test section is really an open jet 2.3 m wide by 1.5 m high. A 6.8 m by 3.5 m by 3 m anechoic chamber has been constructed around this open jet. This chamber and the other acoustic treatments which have been applied to the tunnel have been described in detail in the report by Widnall and Bauer [Bauer 1972]. Briefly, all six internal surfaces of the chamber consist of perforated metal sheets, backed by a fine mesh and about 10 cm of glass fiber, which make a relatively non-reflective surface acoustically. Covering these surfaces on three walls and on the ceiling porous cloth bags containing small blocks of glass fiber batting have been lashed to a depth of about 30 cm (Fig. 5-1e). Since the floor and walls are nonstructural, a heavy steel mesh of
very low solidity has been laid on the floor in order to make it possible to walk about inside. When an acoustic measurement is to be made, however, large blocks of porous foam are spread out to cover the floor to a depth of about 15 cm. Additionally, the entire chamber is vibrationally isolated from the rest of the building; it is not rigidly attached to the floor, walls, or ceiling of the laboratory.

As anechoic chambers go, however, this one is rather spoiled by the large, rectangular holes in opposite walls where the wind tunnel jet enters and exits. The tunnel itself has hard plywood walls which are good sound reflectors. For this reason, the acoustics of the chamber vary quite considerably at different points within it. Subjectively, when one stands away from the jet openings, the room is quite "dead" acoustically, as well as quite well isolated acoustically from the outside world. However, standing in line with the jet (with the wind off), the reverberation time is very noticeably larger, though it is still somewhat more "dead" than an average room.

When the wind is on, the anechoic chamber becomes a truly bizzare place. The jet issues from the nozzle in the south wall of the room and disappears into the hole in the north wall, creating a gale force wind through the center of the room. It is relatively calm outside the jet, although obviously there is a certain amount of turbulence which escapes into the rest of the room. The shear layer at the boundary of the jet grows as it passes through the test section, and impacts on a semicircular porous lip Fig. 5-1(f) which surrounds the exit hole. This lip prevents the jet from exciting the tunnel resonances like a gigantic organ pipe (and yes, the need for this lip was discovered empirically, I am assured).

One can hear a deep rumble from the fan in the basement, as well as feel/hear an annoying very loud "tone" just below the threshold of hearing due to the turbulent
breakup of the shear layer around the jet. This turbulence also creates the invariable rushing sound of a strong wind, but this is not especially noticeable when one is standing outside the jet. Standing inside the jet (which isn't easy), the rushing noise caused by the turbulent flow past head and ears is quite loud, and also quite effective at masking any other background sounds.

In addition to the holes in the north and south walls of the anechoic chamber, there is also a hole in its floor directly beneath the jet. This exposes a large steel beam Fig. 5-1(g) which is used to mount the model to be tested. Recall that none of the surfaces inside the chamber is structural; the was perhaps the most vexing feature of the tunnel. For these experiments, the propeller thrust stand (h) described below, as well as the lower bearing for the microphone frame, were solidly bolted to this steel beam. When an acoustic measurement was to be made, this hole in the floor was filled with bags of the fiberglass batting and partially covered with foam like the rest of the floor. The only other provision for mounting equipment in the anechoic chamber was an overhead rack (i) consisting of three pieces of steel channel bolted into a U-shape about 1.5 m wide and 2.3 m long. The open end of this U was bolted to the lip of the nozzle in the south wall of the chamber; the base of the U was suspended in midair by a system of guy wires. The upper bearing of the microphone frame was attached to this overhead U-shaped rack.

The vital statistics of the wind tunnel and its anechoic chamber are summarized in the following table. The "cutoff frequency" is the frequency above which sounds in the chamber are strongly attenuated.
Table 5-1: Wind Tunnel Characteristics

Jet cross section: 2.3 m × 1.5 m
Maximum airspeed: 30 m/s
Turbulence inside jet: <1% of airspeed
Cutoff frequency: 200 Hz

5.2.2 Propeller Thrust Stand

The major piece of hardware which was constructed specifically for this experiment is the fixture which holds the model propeller and its drive motor in place inside the wind tunnel. This elaborate apparatus was designed primarily by Prof. E. Eugene Larrabee, and is described in great detail in several of the NASA contract reports. Structurally, it consists of four separate assemblies, as depicted in an exploded isometric view in Fig. 5-2. These are the propeller assembly Fig. 5-2(a), the motor and shaft assembly (b), the support post assembly (c), and the body and fairing assembly (d). The position of the thrust stand in the wind tunnel is shown in Fig. 5-1(h).

The propeller assembly begins with a smooth elliptical nose cone, called the the spinner in propeller parlance, which screws into the tip of the propeller shaft. This nose cone covers the large nut which holds the propeller hub seated firmly against a lip machined into the shaft. The combination of the nut in front and the lip behind the hub absolutely precludes any fore-aft motion of the hub relative to the shaft. The hub is prevented from rotating relative to the shaft by means of two Woodruff keys set in the shaft 180 degrees apart, which fit into matching keyways in the hub.

The propeller hub itself splits in half along a plane perpendicular to the shaft when the four screws which hold it together are removed. This allows the two
Figure 5-2: Propeller Thrust Stand
propeller blades to be removed from their sockets in the hub. The propeller blades
are prevented from flying out of the hub radially by means of a flange machined
into their shank. The hub must be completely disassembled in order to remove the
blades. However, the blades are prevented from twisting in their sockets only by
means of a tight friction fit when the two halves of the hub are screwed together.
This enables the attack angle of the blades to be altered by merely loosening the
screws which hold the hub together.

The operation of setting the blades to the desired pitch could therefore be carried
out while the hub was in place on the propeller shaft. This operation was performed
by reflecting a laser beam off of a mirror temporarily attached to each blade. The
blade angles could be set to within 0.25° by this technique.

Three pairs of aluminum propeller blades were fabricated and tested. The first
was an exact 1:4 scale model of the McCauley 1C160 propeller which is commonly
used to pull a Cessna 172 airplane. This will henceforth be called the Cessna blade.
The other two blades were designed by Dr. George Succi as alternative "quieter"
propeller blades for a Cessna 172. The first of these was quickly dubbed "the
Bruiser" when it was realized that it produced much more thrust (and required
much more power) than the Cessna blade under most off-design conditions. The
second was more nearly matched to the requirements of the Cessna 172 airplane,
and was named "the Windsong" by its designer in order to prevent it, too, from
acquiring a derogatory nickname.

Despite the failure of the Bruiser to be flightworthy, it is a much more interesting
blade than its flightworthy successor. The Windsong is so similar to the Cessna that
it demonstrated almost nothing scientifically. The Bruiser, on the other hand,
illustrates that the theory which has been developed applies to propellers with
significantly higher disk loadings than any actual light aircraft propeller.

The most complex of the four assemblies which make up the thrust stand is the motor and shaft assembly [Larrabee 1979b]. It may be further broken down into three major sections. Proceeding from upstream to downstream (fore to aft), these are the shaft extension and its housing, the slip ring assembly and its housing, and the motor itself.

The purpose of the shaft extension is to increase the separation between the propeller and the motor. This minimizes the aerodynamic interference of the motor on the propeller (in the minimum body configuration), as well as minimizing the effects of the sound produced by the motor. As described above, the propeller assembly is rigidly attached to the forward end of the shaft extension, which is a hollow steel shaft about 30 cm long and 3 cm in diameter. The torsional rigidity of this shaft is extremely large, and there is absolutely no freedom of motion of the propeller hub relative to the shaft.

There is a bearing at each end of the shaft housing; the propeller hub butts against a roller bearing at the forward end, and the aft end of the shaft is supported by a deep groove ball bearing which carries the propeller thrust. The inner race of the thrust bearing was held in place on the aft end of the propeller shaft by means of a bearing nut. The interrupter disk for the optical trigger described in the section on instrumentation was screwed to this bearing nut, which was prevented from rotating relative to the extension shaft (rigidly connected to the propeller itself) by means of a special lockwasher keyed onto the shaft. This was a rather unfortunate choice, since the orientation of the interrupter disk was changed whenever the shaft was removed for servicing. The aft end of the extension shaft has a square hole machined in it, which accepts the square ground forward end of the smaller section.
of shaft which passes through the slip ring assembly.

The housing for the slip ring assembly has a somewhat greater diameter than the shaft extension housing, which is bolted to its forward end. The aft end of the slip ring housing bolts directly onto the motor. The six studs which actually make this connection are machined into the bases of the six torque flexures. The slip rings themselves are irrelevant to this discussion, except for the two ball bearings in their assembly which support the small section of shaft between the motor and the shaft extension. As described above, the forward end of this shaft mates into the shaft extension; its aft end mates with the spline on the end of the motor shaft. The few degrees of play between the motor shaft and the propeller is entirely due to the imperfection of these two connections.

In particular, the interrupter disk for the optical trigger is forward of both these connections; there is no play at all between the interrupter disk and the propeller. The LED-phototransistor pair which detects the passage of the slot in the interrupter disk is screwed to the inside of the slip ring housing, near its forward end. The operation of the phototransducer is described in appendix F.

The motor itself is a rebuilt jet engine starter and generator. It is about 18 cm in diameter, 30 cm long, and weighs about 30 kg. As a motor it can produce as much as 10 kw of shaft power at up to 12,000 RPM for short periods of time. Electrically, it is a DC six pole compensated shunt motor. Thus, it requires a high current power supply for its armature circuit, plus a separate smaller power supply for the stationary field windings. The motor is provided with a duct at its aft end, through which cooling air may be drawn.

A vacuum cleaner hose connected this duct to an industrial vacuum cleaner (Fig.
5-1j) located outside the tunnel, which drew air into the motor through vents in its forward end. When this system was devised, it was not realized that the armature of the motor is designed to provide a centrifugal pumping action in the direction opposite to the suction provided by the vacuum cleaner. This fact was discovered during the post-mortem on the second of the three motors which were destroyed during the course of the experiment... Motor overheating limited the duration of each sequence of measurements to a maximum of about five minutes. After such a "run", at least five or ten minutes were required for the motor to become cool enough to safely do the next "run".

The armature power supply is a heavy duty Westinghouse welding supply. It is capable of producing 400 A at 72 V of rectified three phase power. The output is unfiltered, but rectified three phase has only a 13% ripple about its mean value, which is acceptable. The only control is a large crank which screws the iron core in and out of the transformer. With the crank set in any given position, the power supply will produce a particular current almost independent of the impedance of its load, until its maximum voltage (determined by the turns ratio of its transformer) of 72 V is reached. The supply therefore acts as a constant current source, until 72 V or more is demanded, at which point it becomes a 72 V constant voltage source. This strange characteristic makes a welding supply far from optimal as an armature power supply. This power supply was placed just outside the wind tunnel (Fig. 5-1k).

The shunt field power supply was placed on top of the armature power supply. It is a homemade constant voltage source which can produce 12 A at 0-24 V. The output voltage, and hence the shunt field strength in the motor, could be controlled by means of a knob on the front of the supply. This knob, together with the crank
on the armature supply, were used jointly to control the speed of the motor. By fairly intricate manipulations of these two controls, the motor speed could be varied from about 4000 RPM to over 10,000 RPM.

The third structural assembly is the support post. This consists of a sleeve which surrounds the motor, a heavy steel tube which holds the model in the center of the jet, and a pair of sector plates which are bolted to the large steel beam Fig. 5-1(g) beneath the test section. In the measurements which will be discussed below, the steel tube was vertical, and the propeller axis was coincident with the centerline of the jet.

The top of the steel tube was not connected directly to the motor sleeve. Instead, four flexures (the thrust flexures) extended from a horizontal flat plate welded to the top of the tube, to the bottom of the motor sleeve. The heads of the six torque flexures are screwed to a plate attached to the forward end of the motor sleeve. In principle, the torque flexures allow the entire motor and shaft extension assembly to twist slightly around the propeller axis. Similarly, the thrust flexures allow the entire motor and shaft extension assembly plus the motor sleeve to translate slightly along the propeller axis. In practice, the magnitude of these rotations and displacements is invisibly small, but the strains these virtual motions produce in the flexures themselves are measureable by means of strain gauges which were glued to several of the flexures.

The body and fairing assembly is completely nonstructural. It serves only to smooth the flow past the various structural components described above. There is really no single aerodynamic "assembly"; in contrast to the three assemblies described above, these fairings are very lightly constructed. They can easily be assembled and disassembled into three different configurations. In all cases, the
fairing and bodies are connected only to the support post assembly. In fact, except for the minimum body configuration, no connection is ever made to the motor sleeve; the fairing and the bodies do not contribute to the torque or the thrust acting on the flexures. The three bodies are shown in Fig. 5-3.

The symmetric, airfoil shaped fairing enshrouding the support tube was always in place while measurements were being made. It was constructed of perforated metal sheet and was stuffed with foam. Two strips of duct tape (each about 6 cm width) were placed along the leading edge of this airfoil to make it nonporous there. This fairing enclosed all of the wiring as well as the vacuum cleaner hose for the motor cooling air.

In the minimum body configuration, a small nose bowl was fitted over the slip ring housing. This nose bowl was attached to the forward end of the motor sleeve. A 1 cm annular gap was left between the shaft extension housing and this nose bowl in order to admit the cooling air for the motor. Additionally, a light metal sleeve terminated with a fiberglass tailcone was attached to the aft end of the motor sleeve in order to cover the motor itself. Finally, the forward half of the fairing described in the previous paragraph was extended upward a few centimeters by taping a curved piece of sheet aluminum to its upper edge.

In the symmetric body configuration, a much larger radially symmetric body was clamped to the upper end of the support tube. It extended from about 2.5 cm behind the propeller disk plane to 1.5 m behind the disk plane, and was a maximum of 30 cm in diameter. Again, a 1 cm annular gap was left between the nose of this body and the shaft extension housing to admit the motor cooling air. The symmetric body configuration was used most frequently, since all three of the propellers were designed to operate in front of it.
Figure 5-3: Bodies Tested with Propellers
For a final touch of realism, the asymmetric body configuration could be employed. This was identical to the symmetric body except that the radially symmetric nose was replaced by an idealized scale model of a light aircraft engine cowling and windshield.

5.2.3 Microphone Holder

The last major piece of equipment necessary for the experiment is the device which positions the microphone. Due to the nonstructural nature of all the interior surfaces of the anechoic chamber, positioning the microphone at a known location relative to the propeller was more difficult than might be imagined. Once again, Prof. Larrabee devised a scheme which is simple mechanically, and which performs adequately.

The design is based on a rectangular frame, roughly 1.9 m square, which is constructed of 1 inch steel pipe clamped together at the four corners (see Fig. 5-4). The plane of this rectangle is always vertical, and can be rotated about the vertical axis passing through the center of the propeller hub. A short piece of 1 inch steel shaft is clamped at the center of each of the two horizontal members of the frame. These two shafts are collinear and determine the axis of rotation. The upper shaft passes through a journal bearing which is clamped to a light perforated steel beam, which is in turn clamped to the overhead rack above the test section. The lower shaft passes into a gearbox, and may be driven by a small servo motor. The gearbox and servo motor are bolted to a heavy steel tongue, which is in turn bolted onto the steel beam in the floor of the test section.

The entire rectangular frame can be oriented at any angle between about -42 and +42 degrees relative to the propeller disk plane. The controls for the servo motor
Figure 5-4: Microphone Holder
are located outside the tunnel in the instrumentation cage (Fig. 5-1n). Additionally, a 10 turn potentiometer was geared onto the servo motor. By measuring the division ratio of this pot, it is possible to remotely determine the orientation of the rectangular frame. The voltmeter used for this purpose was located near the servo motor controls, to provide the person controlling the microphone location with the necessary feedback.

The microphone boom itself is clamped to the port side (i.e. the left hand side, facing upstream) of the rectangular frame. This boom consists of a 56 cm length of the 1 inch steel piping. A U-shaped sheet metal "fork" has been welded to one end of this pipe. An open balsa wood wind vane is free to rotate about the vertical axis determined by the tips of the two "tines" of the sheet metal fork. When the wind is blowing, the wind vane forces the bullet nose cone of the microphone to be aligned with the airstream, even though the orientation of the rectangular frame might be changing. Also, the missle shaped microphone fits into this wind vane in such a way that the microphone's pressure ports lie at the center of the vane's axis of rotation. Therefore, the microphone pressure ports are always the same distance from the propeller hub, no matter what the frame orientation.

The center of mass of the wind vane together with the microphone also lies on the axis of the vane. Thus, vibrations of the boom have no tendency to cause the wind vane to oscillate. Actually, the boom and the rectangular frame showed no observable vibrations, even at the highest wind speeds. This was in part due to a thick piece of plastic covered wire which was wrapped in a loose spiral around the pipes constituting the rectangular frame and the microphone boom. Such spiral wrappings prevent the formation of Karman vortices which are coherent over long sections of the pipe. Not only do such coherent shed vortices produce significant
buffeting forces on smooth cylinders, but also they produce an audible tone. Indeed, before the spiral wrappings were applied, a distinct howling sound emanated from these pipes when the wind blew past them.

The performance of the complete microphone holder was adequate in the sense that it held the microphone in place to within a millimeter or two, once it had been positioned. Unfortunately, as will be discussed in section C below, it was impossible to measure the actual microphone location with this accuracy.

5.3 Instrumentation

5.3.1 Wind Tunnel

The wind speed is the only characteristic of the airstream which is under direct control. A control panel is mounted outside the wind tunnel adjacent to the location of the armature power supply (Fig. 5-1m). The controls for the large fan in the basement are on this panel. The wind tunnel fan and the model propeller interact considerably, since the model is capable of driving the entire wind tunnel at about 7 m/s even with the fan off. Additionally, there is time lag of several seconds between control inputs and their effects on the airspeed. Controlling the airspeed therefore requires the constant attention of one experienced person whenever data is to be taken.

The wind speed, or more accurately, the dynamic pressure of the airstream, is sensed by means of a pitot static tube. This tube is about 9 cm long and 1 cm in diameter. It is held in place by a slender vertical strut whose base is screwed to the floor of the nozzle about 35 cm from the beginning of the jet (just after the airstream
has reached its maximum contraction). The strut is roughly 50 cm tall and 50 cm from the starboard wall of the nozzle (facing upstream, as always). Two thin flexible 10 m lengths of gas tubing communicate the static and dynamic pressures at the pitot tube to the two ends of an alcohol manometer, which is located on the wind tunnel control panel within reach of the person who controls the wind speed. Due to the length of these tubes, an additional delay of a couple of seconds is introduced between a change in airspeed, and the indication of that change on the manometer.

In order to determine the airspeed from this dynamic pressure, it is also necessary to measure the temperature and the static pressure of the airstream. The static pressure was taken to be equal to the barometric pressure, which must be very nearly true by virtue of the fact that the test section is an open jet. The barometric pressure was measured using either a small aircraft barometric altimeter which was available, or the barometer which is supplied with the Bruei and Kjar pistonphone calibrator. The temperature of the airstream was measured just before and just after a series of measurements was performed. An ordinary laboratory thermometer was mounted on the port wall of the nozzle, roughly 15 cm upstream of the beginning of the jet, and vertically centered. The temperature at the time a particular sound measurement was made was crudely estimated by assuming equal temperature increments between measurements within a series.

The three important parameters which characterize the airstream are summarized in the following table, together with their estimated uncertainties. The uncertainty of the barometric pressure is due primarily to the infrequency with which it was measured (once or twice each day). The uncertainty in the temperature is from not knowing how evenly spaced were the temperatures in a series of measurements. The constant part of the error in the dynamic pressure represents
the probable accuracy with which the manometer scale can be read. The variable error in the dynamic pressure becomes appreciable only at high tunnel speeds. It is due to the fluctuations in the density of the alcohol used in the manometer with its temperature, which was not measured. The density of this alcohol, plus its red dye, is 0.812 g/cm³ minus 0.0006 g/cm³ for each degree Fahrenheit above 60. A density of 0.805 g/cm³ was assumed in the reduction of all the data which will be presented.

**Table 5-2: Airstream Properties**

<table>
<thead>
<tr>
<th>Property</th>
<th>Estimated Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic Pressure</td>
<td>.01&quot; of alcohol plus 1% of value</td>
</tr>
<tr>
<td>Temperature</td>
<td>1 degree Celsius</td>
</tr>
<tr>
<td>Barometric Pressure</td>
<td>1 mm of mercury</td>
</tr>
</tbody>
</table>

### 5.3.2 Microphone

Before describing the microphone and its associated electronics, the uncertainties in the knowledge of its location relative to the propeller will be discussed. Since the sensitive region of the microphone is confined to a region of space about 1.5 mm in radius, it makes sense to speak of the "microphone location" with this precision. However, it was impossible to keep track of the actual microphone location this accurately.

Since the microphone holder rotates about a vertical axis passing through the center of the propeller hub, it is convenient to describe the microphone location in the cylindrical coordinate system with that axis. The azimuthal angle of this coordinate system will be measured from the propeller disk plane, with positive angles forward of the disk plane, and negative angles aft of the disk plane. The altitude will be measured from the horizontal plane containing the propeller axis
(with positive altitude upward); and the cylindrical radius is measured from the axis of rotation of the microphone holder. Since the microphone boom was attached to the port side of the rectangular frame, this is a left handed coordinate system.

The microphone altitude was always as small as it could be made. All of the measurements reported below were made with the microphone radius equal to about 48.5 cm, which is very nearly one propeller diameter. Due to the lack of solid reference points inside the anechoic chamber, it was difficult to accurately measure the microphone altitude, radius, and to accurately align the axis of rotation of the microphone holder.

The microphone holder axis was aligned with the vertical, and caused to pass through the center of the propeller hub by means of a plumb bob. This procedure allowed the axis to be made vertical to within 1/3 of a degree, and to pass through the center of the propeller hub to within 0.5 cm. The change in altitude and in radius, respectively, which these alignment errors would cause in a 45 degree rotation of the microphone would therefore be no more than 0.2 cm and 0.4 cm, respectively.

The microphone altitude and radius themselves were measured when the microphone was approximately in the disk plane. A primitive procedure involving sightings along the horizontal and measurements with meter sticks was evolved for this purpose. The resulting measurements were probably accurate to within about 0.5 cm for both of these parameters, in the propeller disk plane.

The microphone azimuth was the only one of its coordinates which could be varied within a series of measurements. When the propeller azimuth was to be held fixed, the microphone was visually positioned in the disk plane, and the microphone
holder was lashed firmly in this position. By using the crack in the hub and the tip of the propeller as a gunsight, this positioning could be done to within about 0.5 cm, or 0.6 degree. The accuracy of the knowledge of the microphone azimuth during an angular survey, in which the azimuth was changed remotely, is somewhat less than this disk plane value.

This is due to the uncertainty with which the rate of change of the division ratio of the 10 turn potentiometer with microphone azimuth was known. The division ratio was measured by measuring the voltage of the wiper in the potentiometer when 10.00 V was placed across its ends. The calibration constant (i.e. the change in microphone azimuth corresponding to a unit change of the division ratio) was measured on five widely separate occasions over the course of the experiment, by five different groups of people using five slightly different techniques. All of these techniques involved measuring the division ratio when the microphone was positioned at a number of known azimuths. The calibration constants so measured ranged from 170 to 184 degrees, with a mean value of 179 degrees. Since different constants were used at different times, some systematic errors were obviously made. These errors will be estimated to be about 9 degrees out of 180 degrees, or about 0.5 degrees for each 10 degree change in propeller azimuth.

The uncertainties in the microphone location are summarized in the following table. The uncertainty in the angle of the propeller blades at the instant the phototransducer triggered the oscilloscope is also shown (see appendix F).

The microphone itself is a 1/8 inch diameter Brueel and Kjar model 4138 condenser microphone. It is fitted with a model UA 0355 bullet nose cone in place of its normal protective grid, and is mounted on a model 2618 preamplifier with the model UA 0160 adaptor. A 10 m B&K type AO 0028 extension cable connects the
Table 5-3: Microphone Location

<table>
<thead>
<tr>
<th>Parameter</th>
<th>-Uncertainty-</th>
<th>-Disk Plane- Per Each 10° Azimuth-</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Altitude</td>
<td>0.5 cm</td>
<td>0.05 cm</td>
</tr>
<tr>
<td>2. Radius</td>
<td>0.5 cm</td>
<td>0.1 cm</td>
</tr>
<tr>
<td>3. Azimuth</td>
<td>0.6 degree</td>
<td>0.5 degree</td>
</tr>
<tr>
<td>4. Trigger Angle</td>
<td>1.5 degree</td>
<td></td>
</tr>
</tbody>
</table>

preamplifier, which was taped to the microphone boom, with a B&K model 2801 power supply, which was located on top of the diffuser section of the wind tunnel just outside the anechoic chamber. The "direct" output of this power supply is connected via 10 m of coaxial cable to the input of a B&K model 2604 amplifier, located in the instrumentation cage (Fig. 5-1n). The "recorder" output of the 2604 amplifier is a signal of a few volts magnitude (typically), which is a suitable input to the digital oscilloscope. This circuit is shown in the Fig. 5-5.

Notice that a positive pressure on the microphone diaphragm gives rise to a negative voltage. For this reason, the amplified microphone signal was usually connected to the inverting input of the oscilloscope. The resistor R1 shown in Fig. 5-5 is exceedingly large, and does not affect the response of the microphone at frequencies above about 10 Hz. The preamplifier is provided with a switch which can be used to select -20, 0, or +20 dB of gain. In these experiments, the preamp gain was always set at +20 dB. The 2604 amplifier was usually set to provide an additional 20 dB gain before the scope amplifier.

The microphone response at several different angles of incidence is provided by B&K, and is shown in Fig. 5-6. The amplitude response takes into account both the presence of the bullet shaped nose cone, and the difference between the sound field
Figure 5-5: Microphone Circuit
Figure 5-6: Microphone Frequency Response
in the presence of the microphone and what it would have been without the microphone. Unfortunately, the phase response which B&K provides does not take into account either of these two effects. It is shown here merely as an indication of the magnitude of the phase shifts introduced by the microphone. The magnitude of the background noise caused by flow at several velocities past the bullet nose cone is also provided by B&K, and is shown in Fig. 5-7.

The preamplifier, power supply, and amplifier do not appreciably affect the microphone response, except at frequencies either too high or too low to be of any interest. However, the 30 Hz high pass filter which was connected across the "external filter" jacks of the 2604 amplifier does cause a significant change in the phase of the lower harmonics of the propeller sound. This filter was included in the circuit in order to mitigate the effects of the extremely loud tone at 10 to 20 Hz caused by the turbulent breakup of the shear layer at the boundary of the jet. At the highest tunnel speeds, the amplitude of the pressure fluctuations associated with this phenomenon was comparable to the propeller noise itself. A single pole filter was employed, since sharper filters created unacceptably large phase shifts in the propeller sound. A Krohn-Hite model 335 active filter was used for this purpose.

The microphone together with all of the electronics just described was calibrated on several occasions. Two different calibrators were used: a B&K model 4220 pistonphone, and a General Radio model 1562A calibrator. A special adaptor had to be built to fit the B&K microphone into this calibrator (the two companies apparently disagree about the meaning of "1/8 inch"), so the calibration performed with this calibrator is not as reliable as the other. The results of these calibrations are shown in Table 5-4. There is a slight difference between the two power supplies which were used, presumably due to a few volts difference in their polarization
Fig. 9.22. Wind-induced noise of the eighth-inch microphone fitted with Nose Cone UA 0355. 0° incidence

Figure 5-7: Microphone Flow Noise Spectrum
voltages. The uncertainties quoted correspond to the 0.2 dB accuracy claimed by B&K for their pistonphone. During all of these calibrations, the electronics was set precisely as is was for a propeller sound measurement. All of the calibrations quoted were performed at 250 Hz.

When the data was stored in its final form on magnetic tape, slightly different values for the microphone calibration constant were used, which cannot be inferred from the data presented in Table 5-4. Specifically, the constant was taken to be 17.384 Pa/V if power supply #314430 was in use, 18.118 Pa/V if power supply #728674 was in use, and 17.774 Pa/V if the number of the power supply in use was not recorded. Since these values lie within the uncertainties associated with the calibrations, no attempt was made to use more rational values for these constants here. It seems reasonable to simply consider the error in the microphone calibration constant to be 0.5 Pa/V, which means that all of the sound pressures reported here have an uncertainty of about 2.5%. Notice that to within the accuracy of the measurements, the microphone calibration constant did not change over the entire year during which the experiments were performed.

5.3.3 Digital Oscilloscope

A Nicolet Explorer III digital oscilloscope was used to digitize and permanently store the output of the 2604 amplifier. It was located in the instrument cage (Fig. 5-1n), along with the 30 Hz filter and the 2604 amplifier. This instrument has three sections: the amplifier and A-D (analog to digital) converter, the display hardware, and a "minifloppy" magnetic disk drive. The unit also has a standard RS232C serial line interface and can thereby mimic a computer terminal in order to send data to a computer exactly as if someone were typing in the data at a terminal. Through an
### Table 5.4. Microphone Calibrations

B&K 4138 Cartridge Serial #711897

<table>
<thead>
<tr>
<th>Date</th>
<th>Calibrator</th>
<th>Calibrating Pressure (250 Hz)</th>
<th>Microphone Sensitivity (Pa/V)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>dB re 20 μPa</td>
<td>Pa peak-peak</td>
</tr>
<tr>
<td>5/1/79</td>
<td>Gen Rad</td>
<td>114</td>
<td>28.35</td>
</tr>
<tr>
<td>7/3/79</td>
<td>B&amp;K pistonphone</td>
<td>123.81</td>
<td>87.72</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/6/79</td>
<td>B&amp;K pistonphone</td>
<td>123.92</td>
<td>88.83</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/18/80</td>
<td>B&amp;K pistonphone</td>
<td>123.9</td>
<td>88.62</td>
</tr>
</tbody>
</table>

(a) 2801 power supply serial #728674
(b) 2801 power supply serial #314430
extremely involved process, the data from the digital scope could be made available at both the Joint Computer Facility and the Information Processing Center where it could be processed.

The amplifier section of the oscilloscope samples the incoming signal at a rate determined by the equivalent of the sweetime control of a ordinary oscilloscope. The controls can be set so that 512, 1024, 2048, or 4096 samples are taken at evenly spaced intervals, before the scope stops and waits for its next trigger signal. At each of these sample times, a 12 bit A-D converter determines the instantaneous value of the input voltage and stores this value in a memory. The result of a single sweep is therefore a list of 12 bit integers which correspond to the voltages at equally spaced intervals of time after the trigger. All of the data to be reported below originally consisted of a list of either 2048 or 4096 12 bit integers in this fashion.

The display section of the oscilloscope consists of more memory, plus the hardware necessary to display the contents of this memory on its CRT screen. The contents of the display memory are updated from the contents of the amplifier section memory continually in normal operation. This gives rise to a display which is essentially identical to the display produced by an ordinary oscilloscope. However, it is possible to stop updating the display memory, thereby "freezing" a single sweep on the display. During an acoustic measurement, the microphone signal was frozen in this way when it was determined that the airspeed, propeller RPM, and microphone azimuth had all reached their target values.

The minifloppy drive section of the scope was then activated. This action permanently stores the contents of the display memory on small magnetic disks. The storage operation requires only about two seconds, after which the next airspeed, RPM, and azimuth are set, and the next measurement in the series is
made. At a later date, the information stored on the minifloppies can be recalled into the display memory to be reviewed, or to be transferred to a computer by the serial interface.

While it is a nice instrument, the Explorer III is not ideal for the acoustic measurements which have been made. Its one great deficiency is its inability to average data. As will be seen, it is necessary to extract the periodic part of the measured sound pressure in order to compare it to the predictions of the theory. This process would have been greatly facilitated if the oscilloscope had been capable of averaging several hundred consecutive sweeps, since aperiodic noise would be averaged out by this process. (This would entail a negligible increase in the complexity of the circuitry inside the oscilloscope.) In a more accurate study of propeller sound (which appears to be feasible on the basis of the results presented below), such an averaging capability would be an absolute necessity.

5.4 Data Processing

5.4.1 Conventions

All of the data processing, as well as all theoretical prediction, which will be described in this thesis was performed at the Joint Computer Facility at M.I.T. on a VAX 11/780 computer. All programs were written for the FORTRAN IV-Plus compiler which is supplied with the VAX/VMS operating system by the Digital Equipment Corporation.

It has already been explained that the raw data consist of many lists of either 2048 or 4096 integers. Each such list represents a few milliseconds of the
microphone signal, taken with the propeller operating at a particular condition, and with the microphone located at a particular point. A unique sequence number has been assigned to each such measurement of the sound pressure. For convenience, the sequence numbers will frequently be used to specify the data collected on a particular run, made on a particular date, under particular conditions. The sequence numbers are between 001 and 999, and are preceded by the letter "A" if the original data list contained 4096 points, and by the letter "H" if the original list contained 2048 points. ("A" is for "all" and "H" is for "half" of the oscilloscope display memory capacity.) Within each alphabetic prefix, the sequence numbers were usually assigned in the order in which the data was taken.

When the raw data reached the VAX, it was immediately reformatted to make it intelligible to FORTRAN programs. This format begins with two FORTRAN records which serve as a heading, and which contain all the pertinent facts about the propeller operating conditions and microphone location. This heading is followed by the reformatted list of integers which represent the sound pressure measured as a function of time. The ANSI standard magnetic tapes which contain the experimental results in this format fulfill the goal of producing a library of detailed propeller sound data.

Each of the sound measurements made is thus available for further processing as a data file, with a sequence number as its file name. Since the outputs of the theoretical calculations were also data files, it was most convenient to keep them in the same format as the raw data files. Indeed, data which represented the sound pressure as a function of time was always stored in the same format as the raw data. This standard format greatly simplified the program which plotted all of the graphs presented below. Any number of waveforms from any source (i.e. raw data,
processed data, or theory) could be plotted together for comparison.

Two conventions in the plots about to be presented must be mentioned. First, since the microphone is unable to detect the time average pressure, the mean value of the sound pressure has been subtracted from all curves, experimental and theoretical alike, before plotting them. In particular, in all the plots in which a single blade passing period is shown, the mean value of every trace is guaranteed to be precisely zero. Second, in the plots of a single blade passage, care has been taken to ensure that the region of most rapid variation of the pressure occurs near the center of the plot. In order to meet this aesthetic requirement, it was necessary to ascribe a very definite and significant meaning to the zero in the time scale in these plots.

Twice during each revolution of the two bladed propellers tested here, the blades are precisely normal to the plane determined by the microphone and the propeller axis. Imagine that a very brief, loud explosion were to take place at the center of the propeller hub at the instant the blades passed this vertical orientation. The resultant pressure spike would propagate outward along with the propeller sound, reaching the location of the microphone at a well defined instant of time. This instant of time is plotted as time equals zero in all the plots of a single blade passage shown below. The theoretical expression for the time which elapses while the sound from this hypothetical explosion propagates from the hub to a given microphone location is given by equation (4-17). This expression has been used to determine the point on each of the experimentally measured waveforms which should be plotted at time equals zero. Notice, therefore, that errors in the phase of the measured waveforms can be caused not only by errors in the trigger angle, but also by errors in the airstream temperature (i.e. in the speed of sound), or in the microphone location.
5.4.2 Processing Program

The bulk of this thesis is directed toward understanding the production of sound in situations in which turbulence and flow instabilities can be neglected. The theory which has been developed applies only to the periodic component of propeller noise, sometimes called the rotational noise. In order to compare the experimental results to such a theory, it is first necessary to extract from the data the best possible estimate of the periodic part of the measured sound. Conceptually, this amounts to simply averaging the several blade passages recorded in each sound pressure measurement. Each data file contains the sound from between two and thirty blade passages. The averaging of these blade passages to form the best estimate for the rotational noise was performed by a computer program which will now be described in detail.

The data processing program begins with a formatted raw data file. The first problem is to use the data in this file to estimate the blade passing period. The subroutine which is supposed to perform this task has never been fully debugged, so in the cases in which an accurate estimate of the blade passing period (that is, the propeller RPM) is crucial, this estimate has been provided "by hand" by reviewing the data directly on the digital oscilloscope.

Because the angular position of the propeller is known only at the instant the oscilloscope is triggered, the required accuracy of its angular velocity is a function of the number of periods in the entire stored waveform. If there are only a few blade passages in the waveform, the necessary accuracy in the blade passing period is not too great. Conversely, if there are a large number of blade passages in the data file, then even a small error in the knowledge of how much time constitutes one blade passage can lead to a significant error in the accuracy with which the first and last
blade passages in the waveform are aligned when they are averaged.

For example, consider a data file which contains 25 blade passages. This represents 4500 degrees of propeller rotation. Consequently, a 0.1% error in the assumed RPM (i.e. 10 RPM at 10,000 RPM) would lead to a misalignment of 4.5 degrees between the first and last blade passages when the average sound pressure over one period was calculated. The erroneous average would therefore show a phase shift of 2.25 degrees over the true average waveform. This error would be nearly indistinguishable from an error in the trigger angle of the same magnitude. Since the estimated uncertainty in the trigger angle is less than 1.5 degrees of propeller rotation, this is already a significant error. The difference between the actual RPM and the recorded target RPM was often as great as 40 RPM, hence the need to use the measured sound pressure itself to determine the instantaneous value of the RPM becomes significant if there are more than 2 or 3 blade passages in the measurement. It should be noted that no indication has ever been found that the RPM changed significantly during the few milliseconds recorded in any one data file.

After the blade passing period has been determined, the remainder of the processing program works very reliably. First, the number of complete blade passages contained in the waveform is determined. The last few points of the raw data file, which do not constitute a complete blade passing period, are temporarily disregarded. The remaining points are "stretched" back into an array containing the same number of points as the original raw data (either 2048 or 4096). This proportional stretching operation is performed under the assumption that the sound pressure varies linearly with time at times in between the sample points. A Fast Fourier Transform (FFT) is applied to the stretched waveform. (An FFT is merely
an exceedingly efficient algorithm; the "transform" is precisely a lattice Fourier transform.)

Since the stretched waveform contains an integer number, N, of blade passages, only every Nth harmonic of its Fourier series would be non zero if it were perfectly periodic. That is, the desired periodic part of the signal is contained in only these Nth harmonics in the transform. The first 64 such harmonics (of the blade passage frequency) are collected into an array of complex numbers, which represent the Fourier components of the periodic part of the measured signal. Harmonics higher the the 64th are far down into the background noise, and are discarded in order to avoid meaningless high frequency noise from spoiling the plots of measured sound pressure.

An 8 bit inverse FFT is applied to these 64 harmonics, which produces a 256 point waveform representing exactly one period of the periodic part of the measured sound pressure. This periodic part is the average of the sound pressure over the several blade passages recorded, with all frequencies above the 64th harmonic of the blade passage frequency removed. Next, the periodic part of the signal is subtracted from each point of the raw waveform, resulting in the purely aperiodic part of the signal. Again, it is assumed that the sound pressure varies linearly with time between the 256 sample points of the averaged period during this subtraction process. The aperiodic part of the measured sound pressure is the first of three outputs of the processing program.

The operations discussed so far are shown in Figs. 5-8 and 5-9. for the typical case of the measurement with sequence number H072. The first figure shows the raw data exactly as it appeared on the oscilloscope screen. The second figure shows the periodic and aperiodic parts into which it was decomposed by the section of the
Cessna/Symmetric
11072 Raw Data
disk plane

operating conditions:
- angular speed: 6971 RPM
- forward speed: 20.4 m/s
- sound speed: 349 m/s

Figure 5-8: Example of Raw Data
Figure 5.4: Example of Data Processing

Cessna/Symmetric
H072 Periodic and Aperiodic parts of signal
as separated by the processing program
processing program discussed so far. (Only the first 20 ms of the computed aperiodic part are shown.)

The periodic component of the measured waveform was not used as the best estimate of the periodic part of the sound pressure. The 30 Hz high pass filter which was used to minimize the low frequency background noise also causes a noticeable phase shift in the first few harmonics of the blade passage frequency, particularly at the lower propeller RPMs. Since the processing program neatly extracts the frequency components of the periodic part of the signal, it was quite simple to allow for the effects of this filter. The first 64 harmonics of the blade passing frequency are multiplied by the inverse of the transfer function for a single pole, 30 Hz high pass filter. The resulting 64 "defiltered" harmonics are the second output of the processing program.

Finally, an 8 bit inverse FFT is applied to these "defiltered" frequency components to obtain a 256 point waveform which represents the appearance of the periodic component of the measured sound pressure before it was filtered. This is the third and final output of the data processing program. The waveform represents the best estimate which can be made on the basis of the measurement of the periodic part of the sound pressure as a function of time. The "experimentally measured" rotational noise in all of the plots in the following two sections of this chapter was obtained from the data in this manner.

5.5 A Review of the Data
5.5.1 Repeatability

One of the primary yardsticks against which any experiment must be measured is its reproducibility. In the present case, the experimental reproducibility reflects the precision with which the various parameters such as the microphone location, airspeed, etc., were measured. Since this experiment was primarily a survey of light aircraft propeller noise over a wide variety of operating conditions, very little of the data is repetitive. The one exception to this rule is the Cessna propeller spinning at 10,000 RPM in front of the symmetric body with a forward speed of about 29.5 m/s. The sound produced in this one case was measured numerous times throughout the experiment, and the repeatability of these measurements will be the subject of this section.

The data taken at this standard operating condition with the microphone located in the disk plane has already been used to determine the delay in the optical trigger pulse, as discussed in appendix F. Since this procedure involves translating the observed sound pressure waves in time until the steep negative going zero crossings coincide, it is meaningless to try to infer anything about the repeatability of the phase of the measured sound by comparing these particular waveforms. However, the reproducibility of the shape and amplitude of the sound wave measured in this situation can be studied.

Four such repeated measurements, H104, A081, A230, and A287, are shown in Fig. 5-10. A time span of about nine months elapsed between the H104 measurement and the A287 measurement, with A081 and A230 near the midpoint of this interval. Some differences between these four curves are to be expected, particularly because the propeller RPM and airstream temperature were not the same at the instants the waveforms were collected. A third difference is more
Figure 5.10: Experimental repeatability.

- 2.5% absolute uncertainty in microphone sensitivity
- 1.5° uncertainty in trigger angle

Cessna/Symmetric
10 kRPM, 2.5" alcohol disk plane, experiment

<table>
<thead>
<tr>
<th>Data</th>
<th>Date</th>
<th>(°C)</th>
<th>RPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>H104</td>
<td>4/79</td>
<td>36</td>
<td>10,000</td>
</tr>
<tr>
<td>A081</td>
<td>7/79</td>
<td>35</td>
<td>9998</td>
</tr>
<tr>
<td>A230</td>
<td>8/79</td>
<td>33</td>
<td>9980</td>
</tr>
<tr>
<td>A287</td>
<td>1/80</td>
<td>33</td>
<td>10,132</td>
</tr>
</tbody>
</table>
subtle; the number of blade passages which were averaged in order to obtain the plotted waveform was different in each case. By far the most different of the four curves is the one corresponding to the A287 measurement. The RPM and temperature for this run also combine to give the propeller considerably higher Mach numbers than for the other three cases. Can the larger amplitude of this signal be entirely ascribed to this difference in Mach number?

The only way to answer this question is to use the theory which has been developed to investigate the expected sensitivity of the sound to small changes in the temperature or in the propeller RPM. It can be argued that even if the theory is not precisely correct, it should certainly be able to predict the orders of magnitude and signs of the changes produced by small changes about a target condition. Fig. 5-11 shows the changes predicted by the theory for a 5 degree Celsius increase in temperature, and for a 100 RPM increase in propeller speed. A similar theoretical study showed that any reasonable errors in the measured values of the barometric pressure or in the tunnel airspeed cause virtually imperceptible differences in the sound produced by the propeller. Errors in the measured values of the microphone location could have a more noticeable effect.

It seems clear that the differences between A287 and the other three waveforms shown can indeed be adequately explained on the basis of the measured differences between the temperature and RPM when this data was collected. The fact that the 1.8% change in Mach number which these differences represent can cause the observed 20% difference in the amplitude of the sound produced underscores the importance of the Mach number to propeller noise. Also, the fact that the airstream temperature could not be controlled at all and that the propeller RPM could be controlled only approximately obviously places an important constraint on the
Cessna/Symmetric
10 kRPM, 2.5\" alcohol
disk plane, theory

Figure 5.11: Theoretical Repeatability, 0°

<table>
<thead>
<tr>
<th>(°C)</th>
<th>RPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>30</td>
</tr>
<tr>
<td>2)</td>
<td>30</td>
</tr>
<tr>
<td>3)</td>
<td>35</td>
</tr>
</tbody>
</table>
experimenter. Propeller noise experiments would be much easier to perform and to interpret if these two variables were under control.

The fact that all acoustic measurements were made in series of six to ten measurements can be used to advantage in a study of their reproducibility. For example, Fig. 5-12 shows a comparison of H103, A080, A229, and A286. These measurements were made, in each case, a minute or two before the measurements shown in the previous graph. The microphone azimuth was -10° instead of 0°, but all other controllable parameters were the same as before. Notice that the negative zero crossing, which is no longer constrained by the trigger calibration procedure, no longer occurs at the same instant for all four curves. The observed spread in the zero crossing time is now consistent with the estimated uncertainty of 1.5 degrees in the trigger angle. The estimated uncertainties of the microphone coordinates are also consistent with these variations. Since the RPMs in this group of measurements agree with one another much more closely than A287 agreed with the other measurements in that plot, these four curves are much more nearly coincident than were the previous four.

The comparisons which have been made here are fairly representative of the extent to which a particular measurement of the sound pressure can be repeated. Most of the differences between one measurement and another seem to be attributable to the differences in the temperature and in the precise value of the RPM, which were both beyond experimental control. The effect of inadequate averaging over many blade passing periods can also impose an important limitation on the accuracy of the measured sound pressure. This effect is very noticeable in Fig. 5-13 of the next section, in which the residual background noise after averaging is clearly visible. The accuracy with which the other airstream and microphone
Figure 5-12: Experimental Repeatability: -10°
parameters were measured is adequate in the sense that no additional limitation is thereby placed on the extent to which the sound measurements can be repeated.

5.5.2 Trends in the Data

Before the detailed quantitative comparison of the propeller sound data to theoretical predictions, some qualitative trends in the data will be exhibited. The intent here is to gain an understanding of the qualitative features of propeller noise from a purely experimental standpoint. The major ideas developed from a theoretical point of view are nicely illustrated in this manner. The changes in the sound produced by changing the forward speed of the propeller while holding its RPM fixed, and by changing both RPM and airspeed at fixed advance ratio are particularly instructive.

Fig. 5-13 shows the sound produced by the Cessna propeller spinning at 7,000 RPM in front of the symmetric body, with three different airspeeds. The microphone is located in the disk plane in all three cases. Since the rotational velocity any given blade section is generally much greater than its forward velocity, the Mach numbers are substantially the same in the three cases. However, the attack angle of any given blade section decreases as the forward speed is increased. Consequently, the lift force on each blade section decreases as the forward speed is increased. The proportionality of the sound produced to these lift forces is nicely exhibited by the monotonic decrease of the measured sound pressure amplitude as the forward speed of the propeller increases. The fact that there are no gross changes in the shape of the sound waves accompanying the changes in blade loading indicates that the loading noise is indeed dominant over thickness noise in this situation.
Figure 5-13: Effect of Advance Ratio
Fig. 5-14 shows the same propeller with the same microphone location at several different RPMs. The smallest amplitude curve shown is the same waveform plotted as the middle curve in the previous figure. The Mach number has been increased from 7,000 RPM to 10,000 RPM in 1,000 RPM increments, while the advance ratio is held fixed. Only the Mach numbers of the blade sections are altered in this process; attack angles remain substantially unchanged. The observed increase in amplitude of the sound pressure is partially due to the loading effect discussed in the previous paragraph.

Specifically, the dynamic pressure available to any given blade section in the 7,000 RPM case is only half as great as that available to the same section in the 10,000 RPM case. A factor of two increase in the sound pressure is therefore expected from 7,000 RPM to 10,000 RPM on the basis of the proportionality of the sound to the blade loading alone. As predicted by the theory, however, the sound measured at 10,000 RPM is considerably more than twice as loud as the sound measured at 7,000 RPM. Moreover, the shape of the pressure wave measured at 10,000 RPM is much spikier than the shape of the wave at 7,000 RPM. The theory of loading noise is simply a quantification of these two ideas; the sound pressure is proportional to the load at a fixed Mach number, and that increasing the Mach number for a fixed load increases the sound level and impulsiveness of the sound.

One other qualitative feature of the data will be mentioned here. The presence of the large, rigid, symmetric body behind the propeller has been taken into account in only the most rudimentary way in the theory. No allowance for the reflection or diffraction of the propeller noise by this body has been made at all. Yet most of the acoustic measurements have been performed in the presence of the body, because all of the propellers tested were designed to operate in front of it.
Cessna/Symmetric constant advance ratio, experiment

Figure 5.14: Effect of Mach Number

<table>
<thead>
<tr>
<th>Data</th>
<th>Angular speed (RPM)</th>
<th>Forward speed (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>H073</td>
<td>10,024</td>
</tr>
<tr>
<td>2)</td>
<td>H070</td>
<td>8998</td>
</tr>
<tr>
<td>3)</td>
<td>H071</td>
<td>8000</td>
</tr>
<tr>
<td>4)</td>
<td>H072</td>
<td>6971</td>
</tr>
</tbody>
</table>
Figures 5-15, 5-16, and 5-17 show the sound measurements made at microphone azimuths of -30, 0, and +30 degrees for the Cessna propeller both with the symmetric body and with the minimum body. The differences, fortunately, are very slight, as would be expected on the basis of the rudimentary theory. The one significant difference is in the graph of the sound recorded 30 degrees aft of the disk plane (Fig. 5-15). The difference takes the form of a slight shift in the phase of one curve relative to the other. It is likely that this difference represents a real effect of sound diffraction (or reflection) caused by the presence of the symmetric body. A difference of this type is not predicted by the rudimentary theory, which predicts only a very slight change in the amplitude of the sound due to the presence of the body. The observed effect amounts to a phase shift equivalent to roughly 4 degrees of propeller rotation.

A body of this size apparently has very little effect on the sound field of the propeller. This may be due to the fact that propeller sound is concentrated in the disk plane, away from the axis that the body surrounds. The body seems to affect the phase of the radiated sound more than its amplitude. The comparison of the sound measurements to theoretical predictions would be considerably more credible without the symmetric body. Since the presence of this body changes the sound field by so little, there is no excuse for the body to be present in any further studies of propeller noise (assuming that the theory being tested does not really allow for the presence of such a body). Unfortunately, in the comparisons of the theory to the sound measurements presented below, the symmetric body will always be present.
Figure 5.15: Effect of a Nacelle: -30°

Cessna Propeller
10 kRPM, 29.5 m/s
Microphone azimuth -30°

S: H100 Symmetric Body
M: H052 Minimum Body
Figure 5-16: Effect of a Nacelle: 0°
Figure 5-17: Effect of a Nacelle: +30°
5.6 Comparison to Theoretical Predictions

5.6.1 Introduction

An exhaustive comparison of all the data collected to the predictions of the theory is impractical and will not be attempted. It will soon become apparent that the theory is in excellent agreement with the measured sound, and attention will be shifted to the small residual differences between the two. All of the measurements which will be discussed in this section were made with the symmetric body in place, and with the microphone a distance of one propeller diameter from the hub.

The theoretical predictions were all made using a network of 20 point sources, which is described in detail in section 4.5 and in appendix E. Four distinct types of sources were placed at each of these 20 locations: a lift force, an induced drag source, a profile drag force, and a thickness source. The radial distributions of the various loads were computed by means of the propeller performance theory outlined above. This theory is based on the estimated two dimensional characteristics of the airfoil sections which comprise the blade, together with a correction to allow for the effect of the finite (nonzero) solidity of the blade.

The chordwise distributions of all the loads were simply guessed (see section 4.5.2). All three types of loads were assumed to have the chordwise distribution of the lift force on an inclined flat plate in ideal two dimensional flow (equation 4-18). This so-called "angle of attack distribution" has its centroid at the quarter chord point and is sharply peaked at the leading edge of the section. The difference between this distribution and a flat chordwise distribution is very slight acoustically. Experience suggests that the actual loading distribution is somewhere between these two extremes. Since the angle of attack distribution is probably closer to the truth
than the flat distribution, it has been adopted here.

5.6.2 Cessna Blade

The first sequence of three plots (Figs. 5-18, 5-19, and 5-20) shows the Cessna propeller spinning at 7,000 RPM at three different forward speeds, as heard in the propeller disk plane. These are the same three measurements discussed in section 5.5.2 above. The good agreement of the measured and computed sound pressure curves indicates that the loading predicted by the performance theory corresponds to the actual radial loading distribution. Moreover, the fact that the magnitude of the changes of the sound as the forward speed changes are correctly predicted indicates the the actual loads change with attack angle much as the theory predicts. Both the magnitude of the lift force and its rate of change with attack angle are thus in agreement with the theory.

The next sequence of figures (Figs. 5-19, 5-21, 5-22, and 5-23) (which begins with the middle figure of the previous sequence) shows the effect of increasing the Mach number at a constant advance ratio. The agreement seems to be best at 7,000 RPM, both for the shape and for the amplitude of the pulses. It is noteworthy that the discrepancies between measured and computed sound pressures lie mostly in the negative pressure half of the pulses. At the higher propeller RPMs, the shape of this negative half of the pressure pulse departs considerably from the predicted shape. This difference in shape takes the form of a steeper positive going edge (pressure recovery) than the theory predicts for this part of the pulse.

It may well be that this is the beginning of the nonlinear steepening which affects intense sounds. The possible shock steepening does occur during the section of the waveform where the positive rate of change of the pressure is greatest. In the 9,000
Figure 5-18: Cessna Blade: 7kRPM, 29 m/s, 0°
Figure 5.19: Cessna Blade: 7K RPM, 20 m/s, 0°

Cessna/Symmetric H072

E: Experiment
T: Theory

Operating conditions:
- Angular speed: 6,971 RPM
- Forward speed: 20.4 m/s
- Sound speed: 349 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: 0°
Figure 5-20: Cessna Blade: 7kRPM, 15 m/s, 0°
Figure 5.21: Cessna Blade: 8RPM, 23 m/s, 0°

operating conditions:
- angular speed: 8,000 RPM
- forward speed: 23.3 m/s
- sound speed: 349 m/s

microphone location:
- radius: 48.4 cm
- azimuth: 0°
Figure 5-22: Cessna Blade, 9KRPM, 26 m/s, 0°

Cessna/Symmetric H070

E: Experiment
T: Theory

Operating conditions:
- Angular speed: 8,998 RPM
- Forward speed: 26.2 m/s
- Sound speed: 348 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: 0°
Figure 5.23: Cessna Blade: 10K RPM, 29 m/s, 0°

Cessna/Symmetric A081

E: Experiment
T: Theory

operating conditions:
- angular speed: 9,998 RPM
- forward speed: 29.6 m/s
- sound speed: 352 m/s

microphone location:
- radius: 48.4 cm
- azimuth: 0°
and 10,000 RPM cases, it may be argued that the positive going part of the positive half of the pulse is also steeper than the theory predicts. If these effects are indeed due to nonlinear sound propagation, then the negative going edges of the pressure pulses should be less steep than predicted. This is very definitely not observed; the slope of the sharp negative going part of the waveforms is the closest point of agreement between the predicted and measured sound. A very definite example of shock steepening may be found in the article by Boxwell [Boxwell 1979], where high speed rotor noise measurements are discussed. Both a steepening of the positive going edges and a relaxing of the negative going edges are evident in the sound pressure measurements presented there.

A second possible reason for the observed discrepancy is that there may have been reflections or diffraction of the sound by the solid objects inside the test section. The effects of such reflections are certainly not pronounced, but for such small deviations of the sound pressure from its expected value as those under consideration, the possibility cannot be disregarded. The up-down and right-left asymmetries in the test section, combined with the handedness of the propeller rotation, would produce an extreme port-starboard asymmetry in any postulated diffracted or reflected sound field. The simplest way to test the importance of reflections is therefore to compare the sound measured at symmetric locations on the port and starboard sides of the model.

Such a comparison is shown in Fig. 5-24. The differences between the sound on the port and starboard sides are not very large, but are of the same order as the differences between the measured and predicted waveforms. Thus, some small residual differences between observed and computed sound may well be the result of reflection or diffraction of the sound from objects in the tunnel.
Figure 5-24: Port-Starboard Comparison
On the other hand, it may be that the actual loads do not agree perfectly with the computed loads. Or the approximations of ignoring the residual motion of the vortex wake, or of neglecting the finite thickness of the blade in computing the sound (see sections 4.2 and 4.3) may not be quite good enough to obtain better agreement than that shown here. A case in which the loads were definitely not predicted correctly will be presented in the discussion of the Bruiser blade.

The nearly perfect agreement between the predicted and measured times for the sharp negative going zero crossing is gratifying. It indicates that the centroid of the chordwise loading distribution is not too far from the quarter chord point, and also that the computed magnitude of the thickness noise is nearly correct. The experimentally measured waveforms do systematically lag the theoretically predicted curves slightly, however. The amount of this lag varies from almost zero in H072 to 3.0 degrees of propeller rotation in H071. This indicates that the actual center of lift is indeed somewhat behind the quarter chord point, as it should be. For comparison, the blade chord at the 80% radius corresponds to 8 degrees of propeller rotation. A lag of 1.5 degrees in the measured sound pressure would therefore result if the centroid of the actual chordwise load distribution were just forward of midchord, instead of at the quarter chord.

The final series of plots (Figs. 5-23 and 5-25 through 5-28) of the sound from the Cessna blade illustrates the directionality of the sound field. Again, the theory predicts the observed changes in the shape of the waveform quite well over the entire range of microphone azimuths where the sound could be measured. The agreement between the predicted and measured pulse shapes is extraordinary at large angles either forward or aft of the disk plane. However, the disagreement between theory and experiment in the recovery from the negative excursion of the
Figure 5-25: Cessna Blade: 10k RPM, 29 m/s, -40°
Figure 5.26: Cessna Blade, 10k RPM, 29 m/s, -20°

Cessna/Symmetric A079

- E: Experiment
- T: Theory

Operating conditions:
- Angular speed: 10,019 RPM
- Forward speed: 29.5 m/s
- Sound speed: 351 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: 20°
Figure 5.27: Cessna Blade: 10k RPM, 29 m/s, +20°

Cessna/Symmetric
A083

E: Experiment
T: Theory

Operating conditions:
- Angular speed: 9,985 RPM
- Forward speed: 29.6 m/s
- Sound speed: 353 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: +20°
Figure 5-28: Cessna Blade, 10k RPM, 29 m/s, +40°

Operating conditions:
- Angular speed: 9,992 RPM
- Forward speed: 29.7 m/s
- Sound speed: 353 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: +40°
pressure is evident both in the disk plane (Fig. 5-23) and 20° forward of the disk plane (Fig. 5-27). It is interesting that this disagreement should occur at precisely those times and for precisely those microphone locations when the propeller blades most rapidly approach the microphone. The differences could be the result of failure of the approximations made in the acoustic theory, or of inaccuracies in the predicted loads, or of reflection or diffraction of the sound in the test section.

It should be noted here that the directional characteristics shown in these plots are extremely distorted by the close proximity of the microphone to the propeller. In the acoustic far field, the variation of the pulse shape as the microphone azimuth departs from zero more closely resembles the variation of the pulse shape as the propeller RPM is decreased with the microphone in the disk plane (see Fig. 5-14). The fact that the "acoustic theory" predicts the pressure field correctly this close to the propeller underlines the unity of acoustics and aerodynamics which has been stressed throughout this thesis.

5.6.3 Windsong Blade

Figures 5-29, 5-30, 5-31, and 5-32 show comparisons of the sound produced by the Windsong blade to theoretical predictions. As mentioned above, the similarity between the Windsong and the Cessna blades is so great that the commentary of the previous section applies almost word for word to these plots. The very detail of the similarity between the two blades is striking.

One important difference between the two blades, on the basis of these plots, is the significant tendency for the sound from the Windsong blade to lead the theoretical predictions. This discrepancy amounts to 3 or 4 degrees of propeller rotation. It is caused by the delay in the optical trigger, which could not be
Figure 5-29: Windsong Blade: 7kRPM, 20 m/s, 0°

Windsong/Symmetric H510

E: Experiment
T: Theory

Operating conditions:
- Angular speed: 7,003 RPM
- Forward speed: 20.8 m/s
- Sound speed: 352 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: 0°
Figure 5-30: Windsong Blade: 10kRPM, 29 m/s, -30°
Figure 5.31: Windsong Blade: 10kRPM, 29 m/s, 0°

Windsong/Symmetric H525

E: Experiment
T: Theory

Operating conditions:
- Angular speed: 10,022 RPM
- Forward speed: 29.7 m/s
- Sound speed: 352 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: 0°
Figure 5-32: Windsong Blade: 10kRPM, 29 m/s, +30 deg
determined for these data, since no "calibration" runs (i.e. sound measurements made with the Cessna propeller at 10,000 RPM and 2.5" alcohol dynamic pressure) exist near the time the Windsong measurements were made. Quite apparently, the Windsong data represents another epoch in the history of the delay time of the optical trigger (see appendix F). The delay time for this epoch will remain unknown until at least one sound measurement of the Windsong is repeated with a known trigger angle.

5.6.4 Bruiser Blade

The first three plots (Figs. 5-33, 5-34, and 5-35) compare the Bruiser blade at 10,000 RPM and 2.5" alcohol dynamic pressure to the theoretical predictions at various microphone azimuths. The Bruiser data were collected on the same day that the dynamic calibration of the trigger angle was performed. Therefore, the trigger angle is once again accurately known for all of the measurements presented in this section. Again, the agreement of the measured and predicted waveforms follows the same pattern as for the Cessna and Windsong blades. However, the amplitude of the sound produced by the Bruiser is considerably greater than for the other two blades; it lives up to its name.

Notice the possible shock steepening effect seen in the recovery of the pressure from its negative excursion. The steepening effect is more pronounced for the Bruiser than it was for either the Windsong or the Cessna. A nonlinear effect would have just this characteristic; it should be more pronounced for larger amplitude disturbances.

A very interesting effect was observed when the Bruiser blade was spun very slowly. The raw data shown in Fig. 5-36 was taken at three different angular speeds,
Figure 5-33: Bruiser Blade: 10kRPM, 29 m/s, -40°
Bruiser/Symmetric
A306

E: Experiment
T: Theory

operating conditions:
angular speed: 9,966 RPM
forward speed: 29.6 m/s
sound speed: 350 m/s

microphone location:
radius: 48.4 cm
azimuth: 0°
Bruiser/Symmetric A310

Figure 5.35: Bruiser Blade: 10k RPM, 29 m/s, +40°

Operating conditions:
- Angular speed: 10,020 RPM
- Forward speed: 29.6 m/s
- Sound speed: 351 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: +40°

F: Experiment
T: Theory
Figure 5-36: Blade to Blade Asymmetry
but at one advance ratio. The scope amplifier gain was higher for the lower RPM cases, so the amplitudes are not directly comparable. Although the two Bruiser blades produce very nearly identical pressure pulses at 7,000 RPM and at 4,000 RPM, the two blades are markedly different at 6,000 RPM. This blade-to-blade variation had a very smooth onset as the propeller speed is decreased below 7,000 RPM, or increased above 4,000 RPM (indeed, a small difference between the blades is discernable in these two cases as well). This effect is undoubtedly associated with the very low Reynolds numbers at which the airfoil sections are operating. Airfoil section properties often change abruptly when the Reynolds number passes below a critical value. Presumably, as the propeller RPM is decreased, the flow transition occurs first for the inboard sections and slowly works its way outboard, giving rise to the observed smooth change. The two blades, which may differ slightly in shape, go through this flow transition at slightly different propeller speeds, so that the loads on the two blades are different between the occurrence of the transition on one blade and on the other.

The question is, does the blade which has made the transition produce more or less sound than the blade which has not made the transition? In other words, which of the two blades in the 6,000 RPM measurement retains its high Reynolds number characteristics, and which has made the transition to its low Reynolds number characteristics? This question can be answered by comparing the predicted sound, which is computed on the basis of the high Reynolds number section characteristics, with the measured sound. The sequence of three plots in Figs. 5-37, 5-38, and 5-39 show this comparison for the three waveforms shown in the previous figure. In Fig. 5-37 - the high Reynolds number case - the theory agrees extremely well with the measurement.
Figure 5.37: Bruiser Blade: 7kRPM, 20 m/s, 0°

Bruiser/Symmetric
A316

E: Experiment
T: Theory

Sound Pressure (Pa)

operating conditions:
- angular speed: 6,998 RPM
- forward speed: 20.8 m/s
- sound speed: 351 m/s

microphone location:
- radius: 48.4 cm
- azimuth: 0°
Figure 5-38: Bruiser Blade: 6k RPM, 18 m/s, 0°

- Bruiser/Symmetric A317
- 1: Experiment (Blade #1)
- 2: Experiment (Blade #2)
- T: Theory

Operating conditions:
- Angular speed: 6,011 RPM
- Forward speed: 17.8 m/s
- Sound speed: 351 m/s

Microphone location:
- Radius: 48.4 cm
- Azimuth: 0°
Figure 5-39: Bruiser Blade: 4kRPM, 15 m/s, 0°
The second figure (5-38) was prepared by forcing the processing program to extract the component of the signal with the shaft rotation frequency rather than the blade passing frequency. The two blade different blade passages were then plotted separately; the discontinuities at 1.5 ms are the unavoidable consequence of this procedure. The theory agrees quite well with the quieter of the two blades, indicating that the flow transition increases the sound at this microphone location. The argument is clinched by the third figure (5-39), in which both blades have undergone the transition. The theory again underpredicts the sound produced by the blades at low Reynolds numbers.

The most probable explanation of this phenomenon is that the boundary layer becomes laminar in the low Reynolds number flow. This is not desirable in airfoils, since it causes the separation point of the flow to move forward on the surface of the airfoil section. This, in turn, increases the drag coefficient of the section. It is easily believable that this flow transition takes place at slightly different propeller speeds for corresponding sections on the two blades. The considerably higher drag coefficients cause a larger profile drag force than that used in the computations, which causes the sound to be underpredicted in the low Reynolds number case. Usually, the lift coefficient of an airfoil section decreases at the same time than the drag force is increased. Since the drag force is most important to the sound exactly in the disk plane, any decrease in lift accompanying the increase in drag may have gone unobserved. In future experiments, it would be interesting to see how the amplitude of the sound out of the disk plane is changed by this low Reynolds number transition. Such measurements would settle the question of the changes in both the lift and the drag forces which accompany this flow transition.
5.7 Vortex Noise

The combined performance and acoustic theories are capable of predicting the periodic, low frequency component of propeller sound. This part of the sound field has been aptly called "rotational noise". It has long been recognized that in addition to rotational noise, propellers also produce high frequency, aperiodic noise. This component of propeller noise is often referred to as "vortex noise", since it may originate from the Karman vortex street shed by the blades as they cut through the air. Alternatively, the source of the vortex noise may be pressure fluctuations in the turbulent boundary layer at the surface of the blades [Tam 1974].

The experimental problems associated with performing meaningful measurements of propeller vortex noise have not been solved. Not only are the interesting quantities much more difficult to measure than the sound pressure, but also the phenomenon is far more sensitive to the surface roughness of the blades and other hard-to-control factors than is the rotational noise. For these reasons, rather than for any lack of importance as a sound production mechanism, a detailed analysis of the vortex noise observed during this experiment will not be presented here.

However, the aperiodic part of the measured sound, which is removed from the periodic part by the processing program, is definitely not simply background noise. In order to demonstrate this fact, the propeller blades were removed from one of the hubs, and the sound was measured with the same motor speed and wind tunnel speed as for some of the runs made with the propeller in place. This residual noise is partially due to the noise of the flow past the bullet nosed microphone, partially to the propeller drive motor, and partially due to the background noise inside the tunnel. In the disk plane at a distance of one microphone diameter, the results of
this background measurement are compared to the aperiodic part of the measured Windsong noise with identical motor and wind speeds (Fig. 5-40). There can be no doubt that the larger part of the aperiodic component of the propeller sound is indeed generated by the propeller itself; in no sense is this simply background noise.

Subjectively, the vortex noise is associated with a tearing or crackling sound. This crackling is barely audible when the model propeller is spinning slower than 7,000 RPM, but it becomes more and more pronounced as the tip speed is increased. The crackling is a very intermittent, variable sound. It seems to occur in variable length bursts, especially at the higher tip speeds, although these bursts are almost run together. These modulations in the subjective noise level have a time scale of the order of a large fraction of a second, and may be associated with the ingestion of turbulence into the propeller. Whenever a particularly loud "rip" is heard, the high frequency part of the microphone signal displayed on the oscilloscope becomes markedly larger. The visual effect on the scope is an increase in the "furriness" of the signal; trying to capture the most ragged "rip" on the screen became something of a sport.

The sequence of plots in Figs. 5-41 and 5-42 shows the aperiodic part of the sound for the Windsong blade as heard in the disk plane at several Mach numbers. The vortex noise shows a sharp increase with increasing Mach number, just as the rotational noise does. If these figures indeed represent the noise caused by the shedding of Karman vortices, then the primary noise generating mechanism would be the accompanying fluctuations in the lift force on the blade elements. The noise created by such a rotating, fluctuating lift force could be computed using the equations derived in chapter 4. Unfortunately, there are at least three important parameters describing such a vortex shedding process which are unknown.
Figure 5-40: Vortex Noise
Figure 5-41: Mach Number Dependence of Vortex Noise
Figure 5-42: Mach Number Dependence of Vortex Noise (cont.)
The first of these parameters is the frequency with which the Karman vortices are shed. The process of vortex shedding is a flow instability, and it is therefore not surprising that it is not an absolutely periodic phenomenon. However, the rate at which vortices are shed from alternate sides of a moving cylinder is roughly determined by the thickness of its turbulent wake, together with the speed at which it is moving. The product of the vortex shedding frequency with the time required to move across a transverse dimension of the object at the speed the object moves through the fluid is called the Strouhal number of the flow. Empirically, for circular cylinders and other blunt objects, the Strouhal number is found to be roughly 1/5.

At 10,000 RPM the airfoil section at the 80% radius of a model propeller cuts through the air at 180 m/s. For the Windsong blade, its thickness is roughly 3 mm, which indicates a vortex shedding frequency of about 12 kHz if the Strouhal number based on thickness is assumed to be 1/5.

Of course, the blade is alternately approaching and receding from the microphone at a large fraction of the speed of sound. The frequency of the sound caused by vortex shedding is thus Doppler shifted up and down as the blade rotates. Moreover, the sound produced by a fluctuating force tends to be largest along its direction of motion, so that even if the frequency and amplitude of the fluctuating lift forces were constant in the frame of the propeller blade, neither frequency nor amplitude would be constant in the frame of the microphone. Actually, not only is the frequency and amplitude of the vortex shedding from one section expected to be a random variable, but also the expectation values of frequency and amplitude are certainly different for the airfoil sections located at different points along the span.

The second unknown parameter which is important to the vortex shedding process is the amplitude of the shed vortices. Unfortunately, there are no order of
magnitude estimates which can be made for the amplitude, unlike the frequency. The frequency is determined by the growth rates of small perturbations to the steady, unstable flow, and can therefore be understood in terms of the geometry of the situation. The amplitude, on the other hand, is determined by some nonlinear process, and is more difficult to estimate in terms of known quantities. Airfoil sections are designed in such a manner that the amplitude of the shed Karman vortex street is very small. The sharp trailing edge constrains the circulation around the blade to have very nearly one particular value, which automatically inhibits the shedding of a vortex street.

The third important unknown in the vortex shedding process is the spanwise distance over which the vortex shedding is coherent. The sound which is produced increases roughly as the square root of the number of coherence areas along the span of the blade. The variation of the thickness of the blade, together with the changes in the speed of the airfoils along the span ensures that the correlation length for the vortex shedding process cannot be too large. The number of coherence areas along the span is probably not fewer than 10 or 20 for this reason.

The frequency of the vortex shedding is likely to be insensitive to subtle variations in the experimental setup. However, both the amplitude of the shed vortices and their spanwise coherence certainly depend on such factors. Specifically, the roughness of the surface of the blades has a large impact on the coherency of the vortex shedding process. The residual turbulence in the airstream of the wind tunnel would probably also affect these parameters significantly. Indeed, the fluctuations of the crackling sound specifically implicate the residual turbulence in the airstream. Not only are such parameters difficult to control in experimental situations, but they are also wildly variable in real life situations. The results of such
experiments, while interesting in their own right, are therefore not easy to interpret realistically.

The expected modulation of the vortex noise as the blade moves toward and away from the microphone can easily be seen in Figs. 5-41 and 5-42. The approximate times when a propeller blade was most rapidly approaching the microphone are indicated by the arrows. The maxima in the vortex noise envelope clearly coincide with this time, as they should. Also, notice that the amplitude modulation, as well as the overall magnitude, is greater for the higher propeller speeds. Since there is a continuous distribution of frequencies in this aperiodic component of the noise, it is difficult to see the fluctuations in the frequency which undoubtedly accompany these amplitude fluctuations.

The sound measurements which have been made are geared to the determination of the rotational noise. The data is not really adequate for the study of the aperiodic vortex noise. Since this phenomenon is at best a statistical process, a power density spectrum would be far preferable to the simple sound pressure measurements which have been made. However, the most distinctive feature of the vortex noise is the modulation of its amplitude as the blade approaches and recedes from the listener. A simple power density spectrum would discard this vital characteristic of the sound. It is possible that some technique for measuring an effective power spectrum of the vortex noise at each instant of a blade passing period could be found. Such a technique would press the limits of modern digital data acquisition equipment, and would represent a real tour de force in propeller sound measurement. Attempting to extract such detailed information from the existing data, on the other hand, would be an exercise in futility.

However, the data shown in the plots above can be used to estimate the
magnitude of the fluctuating lift forces. The sound pressure due to a rapidly fluctuating, rotating lift force is given in Table 4-3. If the fluctuation frequency is very much greater than the rotation frequency, then only the term involving the rate of change of the lift force, \( L \), need be considered. The resulting expression for the sound pressure is

\[
\rho = \frac{\frac{V}{c}(r_o \sin \theta_o) \cos \Omega \tau - \frac{r_o \Omega}{c}(r_o \cos \theta_o + V(\tau - \tau))}{4\pi (1 - \frac{1}{M_p})^2 r_o^2 \sqrt{(r_o \Omega)^2 + V^2}} L
\]

Since the forward Mach number is relatively small in this case, it is tempting to drop the terms in \( v/c \) (including \( v(\tau - \tau) = (v/c)r \)). The sound pressure would then be proportional to \( r_o \cos \theta_o \), which has caused many writers to comment that the vortex noise is maximum along the propeller axis, and minimum in the disk plane. Because of the strong dependence of the Doppler factors, \( |1 - M_p| \), in the denominator on observer azimuth, this statement is misleading, especially if the tip Mach number is large. Furthermore, the modulation of the vortex noise at the rotational frequency is obviously most extreme near the disk plane, not on the axis.

For the purposes of estimating the magnitude of the force needed to explain the observed vortex noise, equation (5-1) may be estimated as

\[
\rho = \frac{(V/r_o \Omega)(\sin \theta_o \cos \Omega \tau) - (\cos \theta_o + V/c)}{(1 - \frac{1}{M_p})^2 r_o^2} \frac{\nu}{2c} L
\]

Here, \( L \) is now the amplitude of the fluctuating part of the lift force, which is assumed to have frequency \( \nu \). For the 80% radius at 10,000 RPM, a forward speed of 29.5 m/s, and a sound speed of 350 m/s, the Doppler factor in the denominator fluctuates between about 0.5 and 1.5. At a microphone radius \( r_o = 48.5 \) cm in the disk plane, and a source frequency of 12 kHz, the height of the envelope varies between about 2.5 and 20 Pa per newton of fluctuating lift force. Evidently, at most
one newton of fluctuating lift is sufficient to explain the observed magnitude of the vortex noise.

For comparison, the computed lift coefficient for the airfoil at the 80% station of the Windsong blade under these conditions is 0.93. The implied lift force per unit span is 15 nt/cm, while the total thrust produced by the blade is roughly 200 nt. The observed vortex noise therefore indicates that the lift force on any given airfoil section fluctuates by no more than a fraction of a percent of its mean value under this operating condition. A strict lower limit could be placed on the amplitude of the fluctuating force by assuming coherent vortex shedding along the entire span. Since the sound decreases without limit as the correlation length of the shedding process decreases, no strict upper limit can be placed on the magnitude of the lift fluctuations for any given airfoil section. In any case, rapidly fluctuating forces are obviously far more effective at generating sound than steady forces.

5.8 A Violent Flow Instability

5.8.1 The Phenomenon

In one case, the aperiodic high frequency component of the measured sound was comparable in magnitude to the rotational noise itself. A closer examination of this phenomenon revealed a strikingly coherent oscillation, very unlike the stochastic vortex shedding discussed in the previous section. This effect was explored in some detail, since the simple sound pressure measurements which could be made provide a much more complete description of this phenomenon than of the vortex noise. The goal here is to clarify some of the ideas introduced in the previous section concerning acoustic sources which have an intrinsic time dependence, as well as a
rapid rotational motion.

When the first measurements were made of the sound produced by the model Cessna propeller, a extremely large amplitude, high frequency sound was noticed at the highest tip speeds. This high frequency component of the signal appeared in a brief burst coinciding with the steep negative slope of the rotational noise in the disk plane. Upon closer examination, these peculiar high frequency bursts were found to be almost perfect, pure sine waves. This was a great contrast with the vortex noise, which has a rather broad frequency spectrum. Since the main emphasis of the experiment was the measurement of the rotational noise, the high frequency bursts were not studied until much later in the course of the experiment.

Several months later, much to everyones dismay, the phenomenon had disappeared. Finally, Dr. Succi had the idea that the thin layer of grease and dust which had accumulated on the surface of the propeller might be inhibiting the oscillations. Cleaning the blades with steel wool and a degreaser had a spectacular effect on the sound produced by the blades. The following two figures show the oscilloscope trace as it appeared before (Fig. 5-43) and after (Fig. 5-44) the blades were cleaned. The Cessna propeller was spinning at 10,000 RPM in front of the symmetric body, with a wind speed of 29.5 m/s. The sensitivity of the phenomenon to the surface roughness of the blades is impressive.

Unlike any other phenomenon which has been discussed, these high frequency bursts are binary; a blade either produces such a tone burst or it does not. The change from "off" to "on" as the propeller RPM increases is a dramatic thing to watch on the oscilloscope. There is no discernable growth rate; the steep negative zero crossings instantly become "fuzzy" as some threshold propeller speed is crossed. There may have been a very slight hysteresis in the effect in the sense that

299
Cessna/Symmetric
A070: 10 kRPM, 29 m/s
microphone azimuth +10°

Blades "Dirty"
Figure 5-44: Cessna Blade: Clean Surface
the "fuzziness" persisted to a lower RPM than originally necessary for it to "turn on". Such an effect, if it existed, was so small that it could not be measured; it is merely a subjective impression. The binary behavior could only be the result of the sudden onset of a flow instability.

Not only is the frequency inside the burst almost a pure tone, but also the shape of the envelope itself is nearly the same from blade passage to blade passage. Again, this is in very marked contrast to the case of the vortex noise. Whatever flow instability gives rise to these tone bursts, its amplitude and frequency must be nearly independent of time. This is the only reason that the simple sound pressure measurements we were equipped to perform are adequate, meaningful measurements to make for these high frequency tone bursts.

In order to test the presumed time independence of the source of these tone bursts, the microphone was temporarily moved to the starboard side of the model. As expected the bursts still coincided with the steep negative slope in the rotational noise. However, the second region of high frequency activity (arrow in Fig. 5-44) was missing from the starboard side measurements. It is clear that the high frequency part of the sound was reflecting from the floor of the test section and returning to the microphone on the port side after a delay of 6 or 7 ms due to the extra travel time. The reflection was absent on the starboard side, since there are no solid objects on the ceiling of the test section. The large steel tongue which supports the lower bearing of the microphone holder, and the steel base of the thrust stand were quickly identified as the source of the reflections. When these surfaces were covered by a few centimeters of foam, the reflections on the port side disappeared, as shown in Fig. 5-45.

An expanded view of one burst is shown in Fig. 5-46. The microphone was
Figure 5-45: Elimination of Reflections

Region shown in next Fig.

Cessna/Symmetric
A231: 10kRPM, 29 m/s
microphone azimuth +10°
Figure 5-46: Expanded View of One Tone Burst
located 10 degrees forward of the disk plane. The fact that the sound is almost a pure tone has already been mentioned. The frequency of this tone is roughly 40 kHz, which corresponds to a wavelength of about 9 mm. This explains why the reflections off of the small exposed areas of steel plate, which have no noticeable effect on the rotational noise, are important for the high frequency bursts.

One final remark must be made at this point. The free field correction to the microphone response is 6 dB at 50 kHz (see Fig. 5-6). The microphone is therefore enhancing the amplitude of these high frequency tone bursts by nearly a factor of two. Moreover, since the Doppler shift as the blade rotates continuously changes the frequency of the sound incident on the microphone, the shape of the envelope of the burst will be somewhat distorted by the rapidly varying microphone response. Also, the transient response of the microphone might significantly affect the onset and decay of the burst.

5.8.2 The Location of the Instability

A simple experiment was performed in order to determine what part of the propeller blades is responsible for the high frequency tone bursts. A small square of scotch tape, 1 cm on a side, was placed on the suction side (i.e. forward side) of one blade, about 1 mm from the leading edge. The idea is to "trip" the boundary layer, forcing it to become turbulent sooner than it would without the tape. The effect is identical to the effect of the roughness of the surface of the blades when they were dirty. As mentioned above, the violent flow instability does not occur if the blades are dirty; the presence of the scotch tape should have a similar effect.

Incidentally, scotch tape was a poor choice of material for the purpose of tripping the boundary layer. The combination of the large aerodynamic suction force near
the leading edge of the airfoil and the prodigious centrifugal force (roughly 25,000 gravities) tends to peel the plastic sheet away from the blade surface. This results in an exceedingly loud buzzing noise, until the tape is ripped away entirely. As the sensitivity of the instability to the cleanliness of the blades demonstrates, the plastic backing of the tape is completely unnecessary. The boundary layer could have been tripped just as well by applying a roughened smear of rubber cement or other adhesive to the blade surface.

The square of tape was first centered at 95% of the tip radius, so that it extended from the 93% radius to the 97% radius. The instability never "turned on" for the blade which was taped, although the untaped blade both "turned on" and "turned off" normally. The boundary layer on the suction side of the blade near the tip is evidently crucial to the onset of the flow instability. The large modulation of the envelope of the tone bursts is, of course, also indicative that the acoustic source is near the tip of the propeller.

The square of tape was next placed at the 75% radius, extending from the 73% station to the 77% station. This time, the instability appeared on both blades, exactly as it had in the absence of any tape. The flow instability therefore does not involve sections of the blade this far inboard.

Finally, the tape was placed at the 85% radius (from the 83% radius to the 87% radius). When the propeller RPM was increased, the instability first appeared on both blades, just as it had for the 75% radius. However, after a few seconds of operation, the instability disappeared from the taped blade, and could not be recovered at any RPM. Repeated attempts to attach the tape more securely to the blade resulted in similar occurrences. When the propeller was stopped, the tape did not appear to have been peeled back, although it is possible that its leading edge did
pull back slightly under the aerodynamic suction, then reseat itself elastically when the blade stopped spinning. In any case, it is clear that the 87% radius is extremely close to the radius at which the flow instability begins.

These experiments establish directly that the flow instability involves the airfoil sections in roughly the final 10% of the blade, from the 90% radius to the tip. The boundary layer on the suction side of the blade is definitely involved in the instability. Unfortunately, the scotch tape was never applied to the pressure side of the airfoils. It is extremely likely that such a disturbance of the boundary layer on the pressure side of the blade would not have affected the flow instability at all, but it would have been nice to have a direct experimental confirmation of this fact.

5.8.3 The Boundaries of the Instability

The simplest property of the violent flow instability which causes the high frequency tone bursts is its binary nature. After the sudden onset of the instability, the amplitude and frequency of the tone bursts changed only slowly with the propeller speed. Specifically, the amplitude of the bursts decreases slightly as the propeller RPM increases above the onset value. The "turn off" of the bursts at very high RPM, though sudden, is considerably less spectacular than "turn on" for this reason. Nevertheless, for the purposes of this section, the instability will be regarded as a strictly binary phenomenon.

The "turn on" and subsequent "turn off" of the instability as the propeller RPM is increased will now be regarded as a part of the boundary of the flow instability in the "operating state space" of the propeller. This state space, strictly speaking, has three dimensions, since it may be described by the advance ratio, Mach number, and Reynolds number of the flow. (A fourth dimensionless parameter would be
necessary if the propeller axis were not aligned with the airstream.) Experimentally, the three degrees of freedom may be considered to be the propeller RPM, the airspeed, and the temperature.

Because a three dimensional state space would be unmanageably large for the purpose of mapping the boundaries of the flow instability, a two dimensional state space will be employed here. This is an experimental convenience as well, since the airstream temperature is not under direct experimental control, making it difficult to move at will through a three dimensional state space. As in the rudimentary performance theory developed in chapter 2, the parameter which will be discarded is the Reynolds number.

In light of the known sensitivity of the flow instability to the boundary layer, the assumption that the instability does not depend on the Reynolds number of the flow may seem ridiculous. However, the difference between the boundary layer of a smooth blade and that of a blade whose surface has been roughened is far greater than the difference between the boundary layer of a smooth blade and that of the smooth blade at a slightly different (but always very large) Reynolds number. Therefore, as long as the surface of the blade is smooth, it is expected that the flow instability will be independent of Reynolds number, at least in the high Reynolds number limit.

It is convenient to consider the two dimensional state space in terms which are more meaningful than simply advance ratio and Mach number. Specifically, since there is no question that the flow instability is confined to the outermost 10% of the blade, the two degrees of freedom can be interpreted as the attack angle and the Mach number of the airfoil sections near the tip of the propeller. The airfoil section at the 95% radius will be adopted as a reference section. Experimentally, the two
most convenient parameters are the forward Mach number of the propeller and the rotational Mach number at the 95% radius. Note that the forward Mach number depends only on the dynamic pressure of the airstream, which was measured and controlled directly, and on the barometric pressure, which was roughly constant on any given day. The rotational Mach number depends only on the propeller RPM, which was under direct control, and on the airstream temperature.

This experimentally convenient two dimensional state space, referenced to the 95% airfoil section, is shown in Fig. 5-47. The attack angle of this airfoil section can be estimated by means of the formulas given in the figure, and several lines corresponding to particular values of the section attack angle are shown. Actually, for Mach numbers, M, near unity, flows past thin airfoils are similar when the parameters \( \alpha/\sqrt{1-M^2} \), rather than simply the attack angle \( \alpha \), match. For this reason, several lines of constant \( \alpha/\sqrt{1-M^2} \) are also shown as dashed lines on the plot.

For reference, it should also be mentioned that the Reynolds number for this airfoil section is directly proportional to its total Mach number. However, the constant of proportionality is itself directly proportional to the density of the airstream, a degree of freedom which does not appear on the two dimensional plot. Assuming a typical airstream density of 1.16 kg/m\(^3\), the Reynolds number of the 95% station airfoil (which has a chord of 1.9 cm) is 280,000 when its Mach number is 0.68.

At several different airstream dynamic pressures (i.e. forward Mach numbers), the propeller RPM (i.e. rotational Mach number) was slowly increased, to move along a horizontal line in the two dimensional state space. The circles on the plot show the points at which a waveform was "frozen" on the oscilloscope screen. The
BOUNDARIES OF INSTABILITY

- both blades off
- blade 2 on, blade 1 off
- both blades on

- boundary for blade 1
- boundary for blade 2

$\beta = 15.0^\circ$
$\alpha_0 = -4.0^\circ$
$\alpha = \beta - \tan^{-1} \frac{V}{\Omega r}$

**Figure 5.47: Boundaries of Instability**

$\text{tip radius } = R$
$= 24.13 \text{ cm}$

Forward Mach Number ($M_f$)

Rotational Mach Number ($\Omega r/c$)

$(r = 0.95R)$
markings on the interior of these circles indicate whether the instability was "on" or "off" at that point. Note that one propeller blade "turned on" significantly earlier than the other. The oscilloscope trace in Fig. 5-48 shows this remarkable effect.

There are two anomalies in the plot of the instability in the two dimensional state space. First, the individual point marked by a small question mark seems to be out of place; only one blade, not both, should have exhibited the instability at this point. Since the determination of the propeller RPM is not infallible, especially when the signals are as "fuzzy" with the tone bursts as these, this point will simply be ignored. The more important anomaly is the presence of the second, disconnected region of instability in the upper left hand corner of the state space, indicated by the large question mark. The amplitude of the tone bursts observed in this section of the state space was considerably smaller than that in the larger region of instability in the upper right hand part of the plot. It is very likely that this region represents a completely distinct phenomenon from the violeat flow instability which must give rise to the tone bursts observed in the larger region of state space. For example, the vortex shedding process may become very coherent in this region of state space for some unknown reason. In any case, I do not believe that the tone bursts in this smaller region of state space have the same origin as those in the larger region. Consequently, I shall ignore this small region of instability in the remainder of the discussion.

The plot suggests very strongly that the "turn on" of the flow instability occurs when a particular critical Mach number is exceeded. (The total Mach number of the airfoil is very nearly equal to its rotational Mach number, so that the vertical segment of the boundary of the instability represents a particular value of the total Mach number.) The difference in the Mach number at which the two Cessna blades
Figure 5-48: Instability on One Blade Only
"turn on" can be regarded as indirect evidence of this idea. High subsonic Mach number flow similarity suggests that it is the quantity $\epsilon/\sqrt{1-M^2}$, rather than simply the airfoil thickness $\epsilon$ itself which determines the nature of the flow past an airfoil section. Thus, a difference in thickness of only about 0.8% (0.02 mm), would be sufficient to cause the observed difference in the Mach number at "turn on". A difference in the thickness of the two Cessna blades of this order of magnitude is detectable. The thicker of the two blades is indeed the one which "turns on" at the smaller Mach number.

The plot also suggests strongly that the "turn off" of the instability occurs when a critical attack angle is exceeded. Note, once again, that the transonic scaling of the attack angle by the factor of $\sqrt{1-M^2}$ is quite noticeable. The boundary layer on the suction side of an airfoil section can be very greatly affected by the angle of attack, as in the extreme case of stall. Thus, it seems likely that the disappearance of the flow instability at high attack angles is another manifestation of its sensitivity to the structure of the boundary layer on the suction side of the blade.

5.8.4 The Frequency of the Instability

After its binary nature, the most intriguing thing about this flow instability is the purity of the tone inside the burst envelope. The frequency of the tone is surely Doppler shifted; it is natural to ask what the frequency of the flow instability must be in order to produce the observed tone bursts. The answer to this question is strongly affected by the radius of rotation of the acoustic source corresponding to the instability. If the acoustic source were located at one particular radius, then the observed phase of the high frequency sound in the tone burst could be regarded as a direct measure of the source time at this source point. In this case, the observed
phase as a function of time would be a direct measurement of the relationship between source time and observer time. Since this relation (equation 4-12) is crucial to the validity of the free space Green functions which have been advocated throughout this thesis, the opportunity to check it directly by these observations is welcome.

Unfortunately, the acoustic source is undoubtedly distributed over roughly the final 10% of the tip of the blade, along with the flow instability. However, as a working hypothesis it will be assumed that the acoustic source is a point located at the 95% radius of the blade. More information about the spacial structure and physical origin of the flow instability would be necessary in order to significantly improve upon this hypothesis. The information provided by the tone bursts themselves is not sufficient to completely characterize the phenomenon.

The first task is to determine the phase of the high frequency tone bursts as a function of observer time for several different microphone azimuths. In order to accomplish this task, the rotational noise is first discarded by the processing program. Expanded views of the bursts, in which the rotational noise has been removed, are shown in Figs. 5-49 through 5-55. Only the bursts for the thinner blade (i.e. the one which "turns on" at the higher Mach number) are shown. Note that the constant given in each figure must be added to the time scale in the plot in order to obtain the time measured from the standard zero time introduced in section 5.5.

For each microphone azimuth from -10 degrees to +40 degrees, the observer times corresponding to each positive peak in the tone burst marked by an "x" were read from these graphs. The resulting lists of observer times correspond to one cycle increments of the source phase. These relationships between source phase and
Cessna/Symmetric
A228: 10 kRPM, 29 m/s
microphone azimuth: -20°
microphone radius: 48.4 cm

Periodic, low frequency part
of signal removed
Cessna/Symmetric A229: 10 kRPM, 29 m/s
microphone azimuth: -10°
microphone radius: 48.4 cm

Periodic, low frequency part of signal removed

Add -1.238 ms to time scale

Figure 5-50: Measured Tone Burst: -10°
Figure 5-51: Measured Tone Burst: 0°
Cessna/Symmetric
A231: 10 kRPM, 29 m/s
microphone azimuth: +10°
microphone radius: 48.4 cm

Periodic, low frequency part of signal removed

Add -1.196 ms to time scale
Figure 5-53: Measured Tone Burst: +20°
Cessna/Symmetric
A234: 10 kRPM, 29 m/s
microphone azimuth: +40°
microphone radius: 48.4 cm

Periodic, low frequency part of signal removed

Add -1.151 ms to time scale
Figure 5-56: Source Phase vs. Observer Time.
observer time are shown in Fig. 5-56. The Doppler shift is evident in this plot, principally by the difference in slope of the curves for different azimuths. The frequency indicated by the rate of change of the phase with the source time is maximum at the 10 degree microphone azimuth, and minimum at the 40 degree azimuth. The maximum frequency is 38 kHz (at +10°), and the minimum is 25 kHz (at +40°). Additionally, the curvature of the relationships between the source phase and the observer time is probably the result of the change in Doppler shift as the propeller blade rotates.

The geometry of the situation is shown in Fig. 5-57. The trajectories traced out by the microphone in coordinates in which the blade is fixed are indicated. The point labeled $t^*$ on each of these trajectories is the position of the microphone at the instant that a sound emitted from the propeller hub when the blade is at the position shown would reach it. This location therefore corresponds to the standard zero time adopted above. The segment of each of the circular microphone trajectories shown in the figure corresponds roughly with the observer time scale shown as the vertical axis in Fig. 5-56.

The plane $P$ cuts through the 95% radius of the blade, and is perpendicular to the radial line passing through the center of the hub. The line of intersection of this plane with the disk plane is labelled $D$. The airfoil section at the 95% radius is moving through the fluid along the line labelled $V$, and its chordline lies along the line labelled $C$. Since the velocity line, $V$, passes between the 0 and 10 degree azimuth microphone trajectories, it is clear why the maximum Doppler shift is observed at the 10 degree azimuth. Note that, for geometrical clarity, this picture does not allow for the slight convection of the sound by the axial flow.

Next, the source time at which each positive pressure peak in these tone bursts
Figure 5-57: Microphone Trajectories in Blade Fixed Coordinates
was emitted from a hypothetical acoustic source at the 95% radius can be computed from equation (4-12). This program yields a relationship between source phase and source time for each microphone azimuth, which is shown in Fig. 5-58. The slope of these relationships now represents the intrinsic frequency of the acoustic source which produced the observed tone bursts. Notice that the slope of the relationships for each azimuth are much more nearly identical in this plot than in the previous plot. The lines drawn in the plot all have an identical slope corresponding to 12.6 kHz. This frequency is the mean value of the slopes of the best fit lines for each azimuth; the standard deviation of the slopes is 0.5 kHz.

There is a systematic tendency for the rate of change of the source phase with the computed source time to be larger at the larger microphone azimuths. In fact, the standard deviation quoted above is caused by a steady progression of the slope from 11.8 kHz for the -10 degree azimuth to 13.1 kHz for the +40 degree azimuth. This systematic shift could be an indication that the acoustic source is not actually located at the 95% radius. However, in order for the slopes of the best fit lines to be roughly equal, the acoustic source would have to be located close to the 80% radius. Not only is this small a radius ruled out by the results of section 5.8.2, but also the linearity of each individual relationship is not nearly as good if the acoustic source is assumed to be this far inboard. It is more likely that the actual frequency of the instability slowly changed with time by this small amount, since the measurements at successive azimuths were made in order of increasing azimuth, with roughly 1 minute of real time between successive measurements.

5.8.5 The Nature of the Acoustic Source

In order to be able to interpret the details of the high frequency tone bursts,
All lines have a 12.6 kHz slope.

Figure 5-58: Source Phase vs. Source Time
some theoretical guidance is necessary. Several types of acoustic sources - mass sources, mass doublets, external forces, etc. - were discussed in chapter 3. Unfortunately, it is not known which, if any, of these source types is appropriate in the present case. The observed high frequency bursts are merely a consequence of a flow instability; it is essentially impossible to use these acoustic measurements to gain an understanding of the physics underlying the instability itself.

The following are the pertinent facts concerning this flow instability: The instability involves only the airfoil sections between the 90% radius and the tip of the propeller. The boundary layer on the suction side of these airfoils plays a crucial role in the instability. The onset of the instability occurs when the Mach number of the sections near the tip of the propeller exceeds roughly 0.68. The instability disappears if the attack angle exceeds a critical value. Finally, the frequency of the instability in the reference frame of the blade is roughly 12.5 kHz.

Prof. Ingard has proposed a mechanism which can probably account for each of these observations. The instability known as transonic buffeting is a flow instability involving oscillating shock waves on transonic airfoil sections [Wood 1960]. This instability hampered the original development of transonic and supersonic aircraft. If this is indeed the cause of the high frequency tone bursts which have been observed, then the phenomenon of transonic buffeting may contribute significantly to the sound produced by aircraft propellers operating with transonic tips. The solution to this problem in the case of wings was to sweep them; swept back propellers may be the way to reduce the high frequency tone bursts. Further experiments designed to test this hypothesis have been proposed, but not yet performed.

It is not at all clear that the theory developed in this thesis is appropriate for
computing the sound radiated by an oscillating shock wave. Obviously, a shock wave lies outside the domain of linear acoustics. It is just as obvious, however, that the theory of chapter 3 would apply everywhere except in a small region in the immediate vicinity of the hypothetical shock. Therefore, it would come as no surprise if one of the source types which has been developed just happened to describe the sound radiated by an oscillating shock wave. An exploratory study has been undertaken to determine if an oscillating source of a type developed above would, fortuitously, account for the shapes of the observed tone bursts.

Since transonic buffeting is associated with fluctuating forces, point forces in both the lift and the drag directions have been considered. Also, a simple point source of mass was considered, since many diverse phenomena radiate sound as if they were mass sources. Obviously, higher multipoles could be considered, in particular, quadrupoles might be of interest. However, the number of independent types of acoustic multipoles increases rapidly with the number of poles. For example, there are three orientations of lateral quadrupoles, and three more longitudinal ones, not to mention all of their linear combinations. Searching through all these source types to find a combination which happens to match the observed tone bursts would be a waste of time.

Hence, only point sources of mass, lift force, and drag force will be considered here. These point sources will be assumed to have a sinusoidal time dependence with a frequency of 12.6 kHz. Only the terms in the derivatives of the source strength need be considered, since this frequency is so much greater than the rotational frequency. The envelope produced by a fluctuating point force is given by equation (5-1); the corresponding equations for a drag force, D, and a mass source, Q, are:
\[
\rho = \frac{r_e \Omega}{c} (r_e \sin \theta_2) \cos \Omega \tau - \frac{V}{c} (r_e \cos \theta_2 + V(t - \tau)) \cdot \hat{D} \quad (5-2)
\]

\[
\rho = \frac{1}{4\pi (1 - \beta^2)} \cdot \hat{Q} \quad (5-3)
\]

The burst envelopes which would be generated by point sources of these types are compared to the observed tone burst envelope at the +10 degree microphone azimuth in Fig. 5-59. The amplitude of the observed burst has been approximately corrected for the microphone response at 40 kHz. The source strengths used to generate the computed curves are given in the figure. Note that all three of these source types would give rise to a tone burst of width considerably greater than the observed width. This could indicate that the acoustic source is a higher multipole.

However, it is more likely that the approximation that the source is pointlike is simply not viable. After all, the wavelength corresponding to 12.6 kHz is only about 2.5 cm, and it is almost certain that the source (the flow instability) is distributed over a region of space of this size. Clearly, then, there is a strong possibility that the sources at different points, whatever their types, will produce an interference pattern. And such an interference pattern would have different directional characteristics than those of the point sources giving rise to it.

Fig. 5-60 shows the computed drag force envelope together with the observed envelopes at microphone azimuths of -10 degrees and +40 degrees. The strength of the source used in the computation is the same as in the previous plot. From the picture of the geometry (Fig. 5-57, the -10 degree and +40 degree azimuths are seen to be roughly equally disposed above and below the chordline of the airfoil section.
Figure 5-59: Measured and Expected Burst Envelopes: +10°
Figure 5-60: Measured and Expected Burst Envelopes: -10°, +40°
at the 95% radius.

Evidently, the amplitude of the envelope of the observed bursts falls considerably more rapidly than might be expected as the microphone is moved beneath the chordline of the airfoil. This reenforces the notion that the acoustic source lies on the suction side of the blade, as the oscillating shock does in the transonic buffeting effect. Since the wavelength of the sound is actually somewhat larger than the chord of the blade near the tip, the blade cannot prevent sound from being heard when the microphone is aft of it, but it can reduce the sound level significantly.

Considerably greater understanding of the nature and spacial location of the flow instability is necessary before the amplitude of the high frequency tone bursts can be explained. Simple, time dependent point sources of lift force, drag force, or mass, do not even come close to describing the observed directional characteristics of these tone bursts. The fact that the acoustic source is distributed over a region of space comparable in size to the wavelength of the sound being generated undoubtedly must be taken into account if the directionality of the tone bursts is to be understood. Also, the asymmetry of the bursts observed on the suction and pressure side of the blade indicates that diffraction of the sound around the solid surface of the blade must also be important.

5.9 Conclusions

This chapter began with a detailed description of an experiment with propeller noise. The anechoic wind tunnel, propeller thrust stand, and the microphone holder used in the experiment were fully described. The instrumentation included a 1/8
inch condenser microphone, its associated amplifiers, and a digital oscilloscope which could store the sound pressure measurements. The data representing the sound pressure as a function of time was processed on a VAX 11/780 computer in order to separate its periodic and aperiodic components.

The major thrust of the experiment was to study the periodic component of the measured sound pressure. The theory developed earlier in this thesis only predicts the periodic part of the sound, called the rotational noise. The greatest experimental difficulty which was encountered in the measurement of the rotational noise involved the optical trigger which initiated the oscilloscope trace. It is extremely important to know the precise angle of the propeller blades at the instant this trigger pulse is generated. The phase of the sound wave, which contains at least as much information as the amplitude has been discarded in many earlier investigations of propeller noise. In this experiment the phase was known to within 1.5 degrees of propeller rotation for most of the measurements.

The sound pressure measurements were very repeatable, even when a period of several months had elapsed between the repetitions. In fact, the residual differences in repeated measurements can usually be attributed to small measured differences in the operating conditions of the propeller, without the need to invoke random measurement errors. In particular, the temperature of the airstream and the precise angular speed of the propeller were not under direct experimental control. A more sophisticated motor speed control, capable of holding the propeller RPM at a desired value to within 5 or 10 RPM would have been a particularly useful addition to the experimental apparatus.

When the measured sound pressures are compared with theoretical predictions, the agreement is striking. Both the changes in the sound with changing propeller
speed and with changing microphone location are accurately described by the theory. The fact that the theory applies even in the acoustic near field where all the measurements were made underscores the unity of the acoustic and aerodynamic representations of the flow field associated with a propeller. Indeed, the measurements were performed so close to the propeller that the the average value of the static pressure at the microphone was of the same order as the amplitude of the fluctuations in pressure about this mean. Also, the close agreement between the predictions of the theory and the measured sound held for each of the three different propeller blades which was tested.

Two points of disagreement between experiment and theory are noteworthy. First, for all three blades tested, the positive going part of the observed pressure pulse at the highest propeller speeds was somewhat steeper than predicted by theory. This may be a precursor to the nonlinear shock steepening effect which has been observed in higher speed rotors by Boxwell. The theory developed here is, of course, incapable of dealing with such nonlinear effects in an adequate way. It would be interesting to extend the measurements made here to higher tip speeds to discover just how high the tip speed can be before this theory seriously breaks down.

Second, a flow transition was observed at the lowest propeller RPMs at which the Bruiser propeller was tested. This transition is almost certainly the result of the very low Reynolds numbers at which the airfoil sections were operating. Unlike the shock steepening effect, it is expected that the theory would correctly predict the sound in this case, if the correct low Reynolds number airfoil section characteristics were known. An interesting possibility would be to use the acoustic measurements themselves to determine these section properties. Indeed, a future study might use the measured sound pressure at several microphone locations to compute the
magnitudes and distributions of the loads on the blades, which could then be directly compared with the predictions of the performance theory. The solution of this inverse problem would require extremely careful acoustic measurements, but the quality of the data presented here indicates that this may be possible. If so, then acoustic measurements could provide a valuable diagnostic tool in the study of propeller performance theories.

While the rotational noise is dominant in most of the sound measurements which have been made, there is a significant aperiodic, high frequency component of propeller noise. This component becomes more and more significant at higher tip speeds. It is associated with a distinctive crackling sound, which can be more annoying than the rotational noise, even if its amplitude is lower. One possible cause of this component of the sound is the shedding of a Karman vortex street by the blades as they cut through the air. For this reason, the term "vortex noise" is used to describe the phenomenon. Measuring the vortex noise in a meaningful way is a virtually intractable problem. Due to its statistical nature, sound pressure measurements of the type performed here are do not provide particularly useful information. However, since the strong modulation of the vortex noise as the blade revolves, both in amplitude and in frequency, is one of its most distinctive features, a power density spectrum is not a very complete description of the phenomenon either.

A more pronounced flow instability than this vortex shedding process was observed with the Cessna propeller blade. A nearly pure, high frequency tone burst was observed each time a blade approached the microphone most rapidly as it rotated. The flow instability which gives rise to these high frequency tone bursts was found to localized to the outer most 10% of the span of the blades. Unlike the
statistical vortex noise, the simple sound pressure measurements which could be performed with the experimental setup described here can provide meaningful information about these tone bursts.

A very revealing study of the boundaries of this violent flow instability in the two dimensional "operating state space" of the propeller was carried out. This study shows that the instability begins when the Mach number of the airfoil sections near the tip of the propeller exceeds a certain critical value (roughly 0.68). The critical Mach number was slightly different for the two blades, which may differ in thickness very slightly. The instability does not persist if the angle of attack of the airfoil sections near the tip of the propeller becomes too large. The angle of attack at which the instability "turns off" decreases with increasing Mach number, in precisely the manner expected from transonic flow similarity considerations.

Different Doppler shifts of the frequency inside the tone bursts were observed at different microphone azimuths. The assumption that the acoustic source was concentrated at a point located at 95% of the tip radius is consistent with these observed Doppler shifts. This fact may be regarded as a direct check of the relationship between source time and observer time which is so crucial to the validity of the free space Green functions employed throughout this thesis. The frequency of the instability in the reference frame of the blade was inferred to be 12.6 kHz. On the other hand, the shape of the observed tone bursts cannot be explained by a simple point source of mass, lift force, or drag force. The greater directionality of the observed bursts is probably a result of both interference of the sound from different spacial locations within its source, and diffraction of the sound around the solid surface of the blade.

The inability of the theory to explain (let alone to predict) the details of the
sound radiated by this violent flow instability stands in sharp contrast to its predictive power in the case of the rotational noise. The important difference between the two situations is the lack of a theoretical model for the types, locations and strengths of the acoustic sources in the case of the flow instability. In the case of the rotational noise, this information was provided by the propeller performance theory. Without a similar, external knowledge of the acoustic sources associated with the flow instability, no detailed theoretical predictions can be made. Further experiments have been proposed to try to shed more light on the structure of this flow instability, but they have not yet been carried out. If the instability is indeed associated with the transonic buffeting phenomenon, it may be a very common source of noise in propellers operating at high tip speeds.
Chapter Six

Concluding Remarks

A theory has been developed for the production of sound by moving objects. It is basically a linear acoustic theory for the velocity potential, which reduces quite naturally to incompressible flow theory in the limit of infinite sound speed. The limit of incompressible flow is quite important, since the type of motion considered here is frequently modelled as incompressible flow, when the sound field is not of interest. Specifically, the general theory has been applied to the case of light aircraft propellers. This is quite a different application than the better known situation in which the displacement of the boundary surface is negligible. In the latter case, the classical formula due to Kirchhoff relates the conditions anywhere within the fluid to the time history of the conditions at the stationary boundary surface.

The present theory is based on an extension of Kirchhoff's formula to the case in which the boundary surfaces are in motion. This extension is originally due to Morgans, but Morgans' formula has itself been extended here to allow for the boundary surface to be created at an edge in addition to simply being in motion. (The distinction between being created and simply moving lies only in the parametrization of the surface.) The velocity potential satisfies the acoustic wave equation, and therefore Morgans' integral equation, even when the velocity of the boundary surface is comparable to the speed of sound.

For the case of a propeller, the specific condition which must be met in order for the theory to be applicable is that $M^2e/X << 1$ for all its airfoil sections. Here, $M$ is
the section Mach number, $\epsilon$ is the section thickness, and $\chi$ is the section chord. This condition is easily satisfied by a typical light aircraft propeller, where the parameter in question never exceeds a few percent. The meaning of the condition is that the fluid motion caused by the near sonic speed of the propeller blades is itself much slower than the speed of sound.

In order to utilize Morgans' equation to predict the conditions anywhere in the fluid, the time history of the conditions at the boundary of the flow field must be determined. In addition to the solid boundary surface, whose position is assumed to be given, the boundary of the irrotational part of the flow field also includes any vortex sheet shed into the fluid from the boundary layer of the object. The motion of these free vortex sheets is an exceedingly complex, nonlinear phenomenon which is intimately related to turbulence. Fortunately, the residual motion of the sheets is rather slow, and is not very important as a source of sound for light aircraft propellers. However, the creation of the shed vortex sheets, as described by the equations for the creation of boundary surface, is the primary source of propeller noise.

The creation of vorticity is associated with body forces (of a nonpotential nature) acting on the fluid. The strength of the shed vortex sheet created by a propeller blade, as well as other boundary conditions necessary as inputs to Morgans' equation, have been obtained by means of a simple aerodynamic theory of propellers. This theory is based on the idea of lifting line theory, which represents a first order correction to the limit of zero solidity. The only contribution to propeller aerodynamics made in this thesis is the derivation of a closed form expression for the distributions of lift and drag on a low solidity propeller blade in terms of the blade geometry and operating conditions. This closed form expression is applicable
only in the case that the advance ratio of the propeller is relatively small, which is true for typical light aircraft propellers.

The aerodynamic theory is based on exact solution for the problem of an infinitesimal solidity propeller. In this limit, the airfoil sections of the blade behave as if they were a part of an infinite aspect ratio wing. The effective attack angle of the sections is simply their geometrical attack angle. In order to correct the two dimensional airfoil section characteristics for the effects of high Mach numbers, the Prandtl-Glauert rule is applied to the low Mach number characteristics. In the limit that the advance ratio of the propeller is small, it is possible to derive an analytic correction to this zero solidity limit, which is valid to first order in the solidity.

This correction involves reducing the effective angles of attack of the airfoil sections in order to allow for the effect of the velocity induced by the shed vortex sheets. (Such a correction is the heart of lifting line theory.) The factor by which the attack angles are reduced is the quantity which is derived analytically; this factor is analogous to the ratio between the lift curve slope of a finite aspect ratio wing and that of an infinite aspect ratio wing. Since the analytic form of this correction factor is valid only in the case of a vanishingly small advance ratio, a rough correction is made to allow for the effects of a finite advance ratio.

A result of this analysis which I believe to be original is the criterion for when the distortion of the shed vortex sheets by their self induced motion becomes significant. This criterion states that the self induced deformation of the vortex sheets shed behind a propeller must be taken into account if the geometrical advance ratio of the propeller is less than the square root of the solidity. A nonlinear correction term for this effect is included in the analytical expression for the finite solidity correction factor. Finally, a very rough allowance is made for the
effects of a nacelle behind a propeller.

Unlike theories of propeller and rotor sound which treat the blades as sources of momentum in a linear acoustic medium, the airloads on the blades do not provide a sufficient amount of information about the boundary conditions to specify the flow field by means of the extended Morgans' equation. The problem lies in the fact that the boundary surface consists not only of the surface of the propeller blades, but also of the surface of the shed vortex sheets. If the sheets remain stationary after their creation, then the airloads and the blade geometry are sufficient to predict the pressure field by means of Morgans' equation, and the present analysis becomes identical to those other theories of propeller sound.

On the other hand, when a first order allowance is made for the residual motion of the shed vortex sheets, similar to the allowance for the finite propeller solidity, the aerodynamic theory presented here no longer provides sufficient boundary value information for the aeroacoustic theory. The induced drag force predicted by the aerodynamic theory is entirely due to the residual motion of the fluid in the wake. The part of the correction for the residual motion of the vortex sheets which is proportional to the induced drag force has been derived here. A correction of roughly equal magnitude due to the radial contraction of the shed vortex sheets is expected, but cannot be computed here, since the forces on the propeller blades (which are predicted by the aerodynamic theory) do not directly determine the magnitude of this contraction. Similarly, the contribution of the complicated, unstable roll up of the vortex sheets to the pressure field has been overlooked because the simple aerodynamic theory does not predict the details of the process.

The results of this analysis have been used to develop a computer model of propeller rotational noise. This model incorporates the four types of sources
identified by the aeroacoustic theory:

- lift forces

- profile drag forces

- thickness noise

- induced drag sources

A fifth source due to the contraction of the wake, and a sixth source due to the roll up of the vortex sheet could not be computed. The induced drag sources, which are expected to affect the pressure field by an amount similar to the neglected fifth source term, had a minimal impact on the computed pressure field.

The contribution of the profile drag forces was also found to be considerably smaller than that of the lift forces for most situations. While the induced drag forces are roughly proportional to the lift forces, the profile drag forces increase in importance for operating conditions in which the lift happens to be small.

The modelling of the lift and the profile drag forces is quite standard. The blade is regarded as the sum of a number of point forces distributed on its surface (as predicted by Morgans' formula). The pressure field is then the sum of the fields due to helically moving point sources of momentum, a problem originally solved by Lowson. The vortex sheets automatically shed by the lift forces, and the vortex tubes shed by the profile drag forces, accurately model the shed vortex sheet and the momentum wake of the actual propeller flow field.

The induced drag force, however, results from the residual motion of the fluid in the wake. In the present analysis, the effect of the induced drag is modelled as a part of the first order correction for the residual motion of the shed vortex sheet.
The explicit recognition of the difference between the mechanisms of induced drag and of profile drag is one of the contributions of the present analysis to the understanding of propeller noise.

A second major innovation in this thesis is the explicit recognition of the chronopole source. This concept was valuable in interpreting the nature of the sources which appear in Morgans' equation. In spacetime, the chronopole represents the timelike component of a couplet of mass sources. Later, the thickness noise was found to be best modelled as a sum of chronopoles. When the thickness noise is modelled in terms of chronopoles, a nasty numerical integral is replaced by one much more amenable to inexpensive evaluation.

The experiment with propeller noise described here was less innovative than the theoretical development. A large amount of acoustic data was taken for three different propeller blades over a wide range of operating conditions. The microphone was located one propeller diameter from the hub, at a variety of azimuths. In all cases, the computed pressure signature accurately matches the microphone measurement.

The comparisons to theory were made on the basis of the average of between six and twenty seven consecutive blade passing periods. This averaging extracted the periodic part of the signal, which should correspond to the computed rotational noise. One distinctive feature of this data is the care which was taken to record the precise angular position of the propeller blades at a known instant of time in the sound pressure data. For most of the data, this angle was determined to within about 1.5 degrees of propeller rotation. This level of accuracy was important in the comparisons to theory, which agreed with the data with about this precision.
Although the aperiodic part of the sound produced by a light aircraft propeller generally has a much smaller amplitude than the periodic part, it can be quite important subjectively. Probably the greatest need for further development in the study of propeller sound is in this area.

Theoretically, it seems likely that the aeroacoustic theory developed here will be applicable to the vortex noise problem. However, a new theory is necessary which can predict the fluctuating boundary conditions associated with this aperiodic component of the sound. This is a difficult problem, since such fluctuations can come only from boundary layer turbulence, or from self sustaining unstable flow oscillations. Experimentally, a meaningful measurement of vortex noise presents a formidable problem. The broadband noise of interest has stochastic aspects as well as periodic aspects, which makes it extremely difficult to measure.

The one new experimental result reported here is the discovery of coherent, high frequency bursts of sound associated with a narrow range of operating conditions of one of the propellers tested. The flow instability which gives rise to these bursts may be related to the phenomenon of transonic buffeting. The further study of these coherent oscillations may shed some light on the more difficult problem of the vortex noise. The problems associated with the measurement of a coherent phenomenon are less severe than those of measuring a stochastic phenomenon.

One lesson to be learned from the experimental program described here is that, at least for light aircraft propellers, the presence of the nacelle behind the propeller has only a very minimal effect on the sound field. To account for the nacelle in the theory is essentially impossible, so its presence can only detract from the credibility of the theory.
Possible directions for future propeller noise research include:

- a theoretical prediction of the contribution of the contraction of the shed vortex sheet to the sound field

- a theoretical estimate of the magnitude of the sound produced by the unstable roll up of the shed vortex sheet

- the development of sound pressure measurements as a diagnostic tool to determine propeller load distributions

- investigation of the breakdown of the linear acoustic theory at higher tip Mach numbers than those of light aircraft propellers

The last program has been underway for some time in several laboratories [Hanson 1976]. A worthwhile theoretical development would be the refinement of the four dimensional integration theory which was used to such advantage in the derivation of Morgans’ equation, and in the manipulation of spacetime integrals. A less clumsy notation than that used here, which is less abstract than the notation of differential geometry, would simplify the expression of the ideas still further.
References


Appendix A

Green Functions for Vector Wave Equations

The Green function for the scalar wave operator is very well known. Namely, if

\[ \square \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \]

\[ \square G = \delta(t) \delta(\vec{x}) \] is solved by

\[ G = \frac{1}{4\pi r} \delta \left( t - \frac{r}{c} \right). \quad (A.1) \]

Here, only solutions representing outgoing waves will be considered. There are three distinct generalizations of the scalar wave operator which can operate on vectors. With their conventional descriptions, followed by the equivalent operator in the dyadic notation which will be adopted here, these are:

\[ \square_L \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \text{grad} \cdot \text{div} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla \nabla \]

\[ \square_T \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \text{curl} \cdot \text{curl} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla \times \nabla \times \quad (A.2) \]

\[ \square \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - (\text{grad} \cdot \text{div} - \text{curl} \cdot \text{curl}) = \bar{\Pi} \square \]

The operator \( \square \) is a scalar or a dyadic, depending on
whether it operates on a scalar or a vector, respectively. The operator \( \Box_L \) may be called the longitudinal wave operator, while \( \Box_T \) may be called the transverse wave operator. The former describes the velocity field of a sound wave; only the longitudinal polarization is propagated. The latter describes the electric or magnetic field of a light wave; only the two transverse polarizations are propagated. The scalar operator \( \Box \), operating on a vector field, will propagate all three polarizations.

The Green functions for the longitudinal and transverse wave operators are much less well known than (A.1). These Green functions are themselves dyadics, as described in Morse and Feshbach, "Methods of Theoretical Physics," pp. 1769-1783. The following analysis is far simpler than presented in that reference. The final results also differ somewhat; whether this is due to a subtle difference in the problem being solved, or to a mistake, I do not know. The Green functions sought here are defined by

\[
\Box_L \cdot \mathbb{G}_L = \mathbb{1} \delta(t) \delta(\mathbf{x})
\]

\[
\Box_T \cdot \mathbb{G}_T = \mathbb{1} \delta(t) \delta(\mathbf{x})
\]  

(A.3)

\[
\Box \cdot \mathbb{G} = \mathbb{1} \delta(t) \delta(\mathbf{x})
\]
The solution to the third equation is immediate:

$$\tilde{G} = \tilde{\Gamma} G = \frac{\tilde{\pi}}{4\pi r} \delta \left( t - \frac{r}{c} \right)$$ \hspace{1cm} (A.4)

The Green functions $\tilde{G}_L$ and $\tilde{G}_T$ may be determined from $\tilde{G}$ with very little computational effort (in contrast to the treatment in Morse and Feshbach). The calculation rests on the amusing identity

$$\square_L \cdot \square_T = \square_T \cdot \square_L = \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$ \hspace{1cm} (A.5)

which follows immediately from the fact that \text{div curl} = \text{curl grad} = 0. From (A.5), $\tilde{G}_L$ and $\tilde{G}_T$ may be expressed in terms of $\tilde{G}$:

$$\tilde{G}_L = c^2 \int_{t_0}^{t} dt \int_{t_0}^{t} dt \, \square_T \cdot \tilde{G}$$ \hspace{1cm} (A.6)

$$\tilde{G}_T = c^2 \int_{t_0}^{t} dt \int_{t_0}^{t} dt \, \square_L \cdot \tilde{G} ,$$

as may be verified by direct substitution into (A.3).

Although it is straightforward to find the explicit expressions for $\tilde{G}_L$ and $\tilde{G}_T$ directly from (A.4) and (A.6), a large amount of tedious algebra is avoided by a slightly clever approach. First, a small amount of algebra is
unavoidable. The required identities are

\[ \nabla \frac{\hat{r}}{r} = \frac{\hat{1}}{r^2} - 2 \frac{\hat{\hat{r}}}{r^2} \]

\[ \nabla \frac{1}{r} = - \frac{\hat{r}}{r^2} \quad (A.7) \]

\[ \nabla \nabla \frac{1}{r} = - \frac{\hat{1}}{r^3} - \frac{\hat{\hat{r}}}{r^3} - \frac{4\pi}{3} \delta(\vec{x}) \]

(The delta function in the latter formula is obtained by virtue
of the fact that \[ \int_{\text{sphere}} \nabla \frac{1}{r} \, dV = - \int \frac{\hat{\hat{r}}}{r} \, dA = - \int \cos^2 \theta \, d\Omega \]
= \[ -\frac{\pi}{3} \frac{4\pi}{3} \, . \])

The expressions for \( G_L \) and \( G_T \) may now easily be worked
out by noticing that

\[ \Box_T = \Box + \nabla \nabla , \quad \text{and} \]
\[ \Box_L = \Box - \nabla \times \nabla \times \]

First, \( \Box_T \):

From (A.6),

\[ \Box_T = c^2 \int^t dt \int^t \text{dt} (\Box - \nabla \times \nabla \times) \Box \]

\[ = c^2 \int^t dt \int^t \text{dt} \hat{1} \delta(t) \delta(\vec{x}) - c^2 \int^t dt \int^t \text{dt} \nabla \times \nabla \times \Box \]

\[ = \hat{1} c^2 t H(t) \delta(\vec{x}) - c^2 \int^t dt \int^t \text{dt} \nabla \times \nabla \times \Box \]
Where $H(t)$ is the unit step function. A choice presents itself at this stage. Often, a transverse vector field is expressed as the curl of a vector potential field. The second term in the equation above may be written in this form with no labor at this stage in the calculation:

$$c^2 \int_0^t dt \int_0^t dt \nabla \times \nabla \times \vec{G} = \nabla \times \left( -c^2 \int_0^t dt \int_0^t dt \nabla \times \vec{G} \right)$$

Now, from (A.4) and (A.7),

$$\nabla \times \vec{G} = \left\{ \nabla \left[ \frac{\delta \left( t - \frac{r}{c} \right)}{4\pi r} \right] \right\} \cdot \vec{x} = -\frac{\hat{r}_x}{4\pi cr} \left( \frac{\delta'}{t - \frac{r}{c}} + \frac{c}{r} \delta \left( t - \frac{r}{c} \right) \right),$$

so that

$$\vec{G}_T = \vec{G} = \int c^2 t H(t) \delta(x) + \nabla \times \left\{ \frac{c \hat{r}_x}{4\pi r} \left[ H \left( t - \frac{r}{c} \right) + \frac{c}{r} \left( t - \frac{r}{c} \right) H \left( t - \frac{r}{c} \right) \right] \right\}$$

Alternatively, from (A.6)

$$\vec{G}_T = c^2 \int_0^t dt \int_0^t dt \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla \nabla \right) \vec{G} = \vec{G} = c^2 \int_0^t dt \int_0^t dt \nabla \nabla \vec{G}$$

Now,

$$\nabla \nabla = \nabla \left[ \frac{\delta \left( t - \frac{r}{c} \right)}{4\pi r} \right] = -\nabla \left[ \frac{\hat{r}_x}{4\pi cr} \left( \frac{\delta'}{t - \frac{r}{c}} + \frac{c}{r} \delta \left( t - \frac{r}{c} \right) \right) \right]$$

355
\[ = \frac{rr}{4\pi c^2 r} \delta'' \left( t - \frac{r}{c} \right) - \frac{1-3rr}{4\pi c^2 r} \frac{\delta'}{r} \left( t - \frac{r}{c} \right) + \frac{c}{r} \delta \left( t - \frac{r}{c} \right) \]

\[ - \frac{1}{3} \delta \left( t - \frac{r}{c} \right) \delta(\bar{x}), \]

from (A.7). Noting that \( \delta \left( t - \frac{r}{c} \right) \delta(\bar{x}) = \delta(t) \delta(\bar{x}) \), this gives

\[\bar{G}_T = \frac{1}{3} \frac{\epsilon}{c^2} c^2 t H(t) \delta(\bar{x}) + \frac{1-3rr}{4\pi r} \delta \left( t - \frac{r}{c} \right) + \frac{1-3rr}{4\pi r} \frac{c}{r} \left( t - \frac{r}{c} \right) H \left( t - \frac{r}{c} \right) \]

\[ (A.9) \]

Next, the longitudinal Green function \( \bar{G}_L \). From (A.6),

\[\bar{G}_L = c^2 \int_0^t dt \int_0^t dt (\nabla + \nabla) G = \frac{1}{3} c^2 t H(t) \delta(\bar{x}) + c^2 \int_0^t dt \int_0^t \nabla \nabla G \]

\( \nabla \nabla G \) was just worked out; the result gives

\[\bar{G}_L = \frac{2}{3} \frac{\epsilon}{c^2} c^2 t H(t) \delta(\bar{x}) + \frac{\hat{r}r}{4\pi r} \delta \left( t - \frac{r}{c} \right) - \frac{1-3\hat{r}r}{4\pi r} \frac{c}{r} \left( t - \frac{r}{c} \right) H \left( t - \frac{r}{c} \right) \]

\[ (A.10) \]

Just as transverse waves are often expressed at the curl of a vector potential, longitudinal waves are often expressed as the gradient of a scalar potential. The Green function \( \bar{G}_L \) may be easily rewritten to accommodate such an expression by noting that

356
\[ c^2 \int_0^t dt \int_0^t dt \, \nabla G = \nabla c^2 \int_0^t dt \int_0^t dt \, \nabla G, \]

and that
\[
\nabla G = \nabla \left( \frac{\delta(t - \frac{r}{c})}{4\pi r^2} \right) = - \frac{\hat{r}}{4\pi cr} \left[ \delta'(t - \frac{r}{c}) + \frac{c}{r} \delta(t - \frac{r}{c}) \right]
\]

whence
\[
\tilde{G}_L = \tilde{c}^2 t \, H(t) \, \delta(\bar{x}) - \nabla \left( \frac{\hat{r}}{4\pi} \frac{1}{r} \frac{c}{r} H(t - \frac{r}{c}) + \frac{c}{r} H(t - \frac{r}{c}) \right)
\]  \hspace{1cm} (A.11)

As a sample application of these results, consider the equation governing the acoustic velocity \(\tilde{u}\) in an unbounded fluid under the action of a body force \(\bar{f}\):

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tilde{u} - \nabla (\nabla \cdot \tilde{u}) = \Box_L \cdot \tilde{u} = \frac{1}{\rho_0 c^2} \frac{\partial}{\partial t} \bar{f}
\]

The solution to this equation is

\[
\tilde{u}(t, \bar{x}) = \int_{\text{all spacetime}} d\tau \, d^3y \, \tilde{G}_L(t - \tau, \bar{x} - \bar{y}) \cdot \frac{1}{\rho_0 c^2} \frac{\partial}{\partial \tau} \bar{f}(\tau, \bar{y})
\]

\[
= \int d\tau \, d^3y \, \frac{1}{\rho_0 c^2} \bar{f}(\tau, \bar{y}) \cdot \frac{\partial}{\partial \tau} \tilde{G}_L(t - \tau, \bar{x} - \bar{y})
\]

Now, from equation (A.11),

357
\[ \frac{\partial \tilde{G}_l}{\partial t} = \tilde{G}(t) H(t) \delta(x) - \nabla \left\{ \frac{c \hat{r}}{4\pi r} \left( \delta \left( t - \frac{r}{c} \right) + \frac{c}{r} H \left( t - \frac{r}{c} \right) \right) \right\}. \]

Therefore,

\[ \bar{u}(t, x) = \frac{1}{\rho_0} \int_{-\infty}^{t} d\tau \ f(\tau, x) - \nabla \phi(t, x), \quad \text{where} \]

\[ \phi(t, x) = \frac{1}{4\pi \rho_0 c} \int_{\text{all spacetime}} d\tau \ d^3y \ \bar{f}(\tau, y) \cdot \frac{\hat{r}}{r} \left( \delta \left( t - \frac{r}{c} \right) + \frac{c}{r} H \left( t - \frac{r}{c} \right) \right) \]

(A.12)

The physical meaning of the result (A.12) is discussed in Chapter 3.
Appendix B

Moving Source Sheets

There are three generic forms for the integrals in the expression (3.44) for the velocity potential:

\[ \int_S \text{d}\tau \text{d}\xi \text{d}\eta \, \mathcal{F}(\tau, \xi, \eta) \cdot \mathbf{n} \mathbf{A}(\tau, \xi, \eta) \, G(t - \tau, \mathbf{x} - \mathbf{y}(\tau, \xi, \eta)) \quad (B.1) \]

\[ \int_S \text{d}\tau \text{d}\xi \text{d}\eta \, \mathcal{F}(\tau, \xi, \eta) \cdot \mathbf{n} \mathbf{A}(\tau, \xi, \eta) \frac{\partial G}{\partial t} \, , \text{ and} \quad (B.2) \]

\[ \int_S \text{d}\tau \text{d}\xi \text{d}\eta \, \mathcal{F}(\tau, \xi, \eta) \cdot \mathbf{n} \mathbf{A}(\tau, \xi, \eta) \cdot \nabla G \]

The third type may immediately be expressed in terms of the other two by virtue of the identity

\[ -\nabla G = \frac{\hat{r}}{c} \frac{G}{t} + \frac{\hat{r}}{r} G ; \quad \text{for, the free space} \quad (B.3) \]

Green function \( G = \frac{1}{4\pi t} \delta \left( t - \frac{r}{c} \right) \). Furthermore, the second type of integral is merely the time derivative of the first type:

\[ \int_S \text{d}\tau \text{d}\xi \text{d}\eta \, \mathbf{F} \cdot \mathbf{n} \mathbf{A} \frac{\partial G}{\partial t} = \frac{\partial}{\partial t} \int_S \text{d}\tau \text{d}\xi \text{d}\eta \, \mathbf{F} \cdot \mathbf{n} \mathbf{A} \, G \quad (B.4) \]
The moving surface $S$ is parametrized by

$$\bar{x} = \bar{y}(\tau, \xi, \eta) . \tag{B.5}$$

This surface has an "edge", which, at time $\tau$, is simply the image of the boundary of $S$ in $(\xi, \eta)$ space. The edge will be parametrized by

$$\bar{x} = \bar{z}(\tau, \mu) . \tag{B.6}$$

A second convenient means of conveniently describing this edge is by a parameter $\nu = \nu(\xi, \eta)$ which is constant along the edge:

$$\nu = \nu(\xi, \eta) = \nu_0(\tau) , \quad \text{on the edge at time } \tau . \tag{B.7}$$

As illustrated in Fig. B.1, $(\mu, \nu)$ is simply an alternative curvilinear coordinate system to $(\xi, \eta)$, which describes a slice through the boundary $S$ at time $\tau$. The element of area $\hat{n}A d\xi d\eta$ has its counterpart in $(\mu, \nu)$ coordinates:

$$\hat{n}A(\tau, \xi, \eta) \ d\xi \ d\eta = \hat{n}A'(\tau, \mu, \nu) \ d\mu \ d\nu , \quad \text{where}$$

$$\hat{n}A'(\tau, \mu, \nu) = \left. \frac{\partial \bar{y}}{\partial \mu} \right|_{\nu, \tau} \times \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \tau} \quad \text{360} \tag{B.8}$$
Figure B.1: Edge and Surface Geometry

Parameter Space

Real Space at time $\tau$

lines of constant $\mu$

slice through $S$ at time $\tau$
The first type of integral, (B.1), may easily be reduced from a triple to a double integral:

\[
\int_{\mathcal{S}} d\tau \, d\xi \, d\eta \, \vec{f} \cdot \hat{n} \, A \, G = \int_{\mathcal{S}} d\tau \, d\mu \, d\nu \, \vec{f} \cdot \hat{n} A' \frac{\delta(\tau - t - \frac{r(\tau, \mu, \nu)}{c})}{4\pi r} \\
= \int_{-\infty}^{\infty} d\tau \, d\mu \, d\nu \, H(\nu - \nu_0(\tau)) \vec{f} \cdot \hat{n} A' \frac{\delta(\tau - t - \frac{r}{c})}{4\pi r}
\]

where it is assumed, in addition to (B.7), that \(\nu > \nu_0(\tau)\) everywhere on \(\mathcal{S}\). (\(H\) is the unit step function.)

\[
= \int_{-\infty}^{\infty} d\mu \, d\nu \left\{ \frac{\vec{f} \cdot \hat{n} A'}{4\pi r \left| 1 + \frac{1}{c} \frac{\partial r}{\partial \tau} \right|_{\mu, \nu}} \right\}_{\tau = t - \frac{r}{c}}
\]

(B.9)

\[
= \int_{0}^{\infty} d\xi \, d\eta \left\{ \frac{\vec{f} \cdot \hat{n} A'}{4\pi r \left| 1 + \frac{1}{c} \frac{\partial r}{\partial \tau} \right|_{\xi, \eta}} \right\}_{\tau = t - \frac{r}{c}}
\]

by (B.8), and since \(\frac{\partial r}{\partial \tau}\)_{\mu, \nu} = \(\frac{\partial r}{\partial \tau}\)_{\xi, \eta} = -\hat{r} \cdot \vec{v} = -cM_x.\) Thus,
\[ \int_S d\tau \ d\xi \ d\eta \ \mathbf{F} \cdot \mathbf{n} \mathbf{A} \mathbf{G} = \int_S d\xi \ d\eta \ \frac{\mathbf{F}(\tau, \xi, \eta) \cdot \mathbf{n} \mathbf{A}(\tau, \xi, \eta)}{4\pi r|1 - M_r|}, \quad (B.10) \]

where \( S \) is the intersection of the "light cone" with apex at \((t, \bar{x})\), and the three dimensional surface \( S \) (also called the "acoustic planform"). The integrand in (B.10) is to be evaluated at the retarded time, \( \tau \).

The second type of integral, (B.2), may now be evaluated according to (B.4). Starting with (B.9),

\[ \int_S d\tau \ d\xi \ d\nu \ \mathbf{F} \cdot \mathbf{n} \mathbf{A} \frac{\partial G}{\partial \tau} = \frac{3}{\partial t} \int_{-\infty}^{\infty} \ d\mu \ d\nu \ H(\nu - \nu_0(\tau)) \ \frac{\mathbf{F} \cdot \mathbf{n} \mathbf{A}'}{4\pi r|1 - M_r|} \]

\[ = \int_{-\infty}^{\infty} \ d\mu \ d\nu \ H(\nu - \nu_0(\tau)) \ \frac{\partial}{\partial t} \left. \mathbf{F} \cdot \mathbf{n} \mathbf{A}' \right|_{\mu, \nu} \ + \int_{-\infty}^{\infty} d\mu d\nu \left. \frac{\partial H}{\partial t} \right|_{\mu, \nu} \ \frac{\mathbf{F} \cdot \mathbf{n} \mathbf{A}'}{4\pi r|1 - M_r|} \]

\[ = \int_S d\xi \ d\eta \ \frac{\partial}{\partial \tau} \left. \mathbf{F} \cdot \mathbf{n} \mathbf{A} \right|_{\xi, \eta, x} \ + \int_{-\infty}^{\infty} d\mu d\nu \left. \frac{\partial H}{\partial t} \right|_{\mu, \nu} \ \frac{\mathbf{F} \cdot \mathbf{n} \mathbf{A}'}{4\pi r|1 - M_r|} \]

The second term in this equation is a Leibnitz term which describes the possible motion of the boundary in \((\xi, \eta)\) space, i.e. - the creation of boundary surface in real coordinate space \(\bar{x}\).

In particular,
\[ \frac{\partial}{\partial t} H(\nu - \nu_0(\tau)) = -\delta(\nu - \nu_0(\tau)) \frac{\partial}{\partial t} \nu_0(\tau) = -\delta(\nu - \nu_0) \frac{\partial v_0}{\partial \tau} \frac{\bar{w}}{1 - \hat{r} \cdot \frac{\bar{w}}{c}}, \]

where \( \bar{w} \) is the velocity of the edge.

The velocity \( \bar{v} \) of the edge of the boundary is not necessarily equal to the velocity \( \bar{v} \) of the boundary itself, evaluated at the edge:

\[ \bar{v} = \frac{\partial \bar{v}}{\partial \tau} \bigg|_{\xi, \eta} = \frac{\partial \bar{v}}{\partial \tau} \bigg|_{\mu, \nu} \]

(B.11)

\[ \bar{w} = \frac{\partial \bar{w}}{\partial \tau} \bigg|_{\mu}. \]

Indeed, the "creation" of new boundary surface is distinguished from the motion of old boundary surface by whether or not the velocities \( \bar{v} \) and \( \bar{w} \) (or, strictly speaking, their components normal to the edge) are equal.

The Leibnitz term is therefore

\[
\int_{-\infty}^{\infty} d\mu \ d\nu \left( \frac{\partial}{\partial t} \bigg|_{\mu, \nu} H \right) \frac{\bar{f} \cdot \hat{n} A'}{4\pi r |1 - M_r|} = - \int d\mu \ \frac{\partial v_0}{\partial \tau} \frac{\bar{w}}{1 - r \cdot \frac{\bar{w}}{c}} \frac{\bar{f} \cdot \hat{n} A'}{4\pi r |1 - M_r|}
\]

which has the expected character of a line integral over the edge of the boundary surface. Furthermore, \( \partial v_0 / \partial \tau \) is nonzero if and only if this edge is moving in \( (\xi, \eta) \) space, i.e. if the boundary surface is being created.
Let the edge of the boundary surface have an element of arc length $d\bar{s}$ defined by:

$$d\bar{s} = \frac{\partial \bar{z}}{\partial \mu} \, d\mu = \bar{s}(\tau, \mu) \, d\mu$$  \hspace{1cm} (B.12)

The parameter $\mu$ may as well be chosen in such a way that the direction of $\bar{s}$ on the edge corresponds to a circulation around the surface which is positive relative to the sense of the normal vector $\hat{n}$ on the surface. The geometry of the situation shown in Fig. B.2 shows that the element of area which is newly created in a short time $d\tau$ by the motion of a small segment $d\mu$ of the edge is

$$((\bar{w} - \bar{v})d\tau) \times (\bar{s} \, d\mu) = -d\mu \, dv_o \, \hat{n}A'$$  \hspace{1cm} (B.13)

Equation (B.13) allows the integrand of the Leibnitz term to be recognized as

$$d\mu \, \frac{\partial v_o}{\partial \tau} \, \hat{n}A' = - (\bar{w} - \bar{v}) \times \bar{s} \, d\mu .$$

When this expression is to be evaluated at the retarded time, care must be taken to assure the correct sign if either $\bar{w}$ or $\bar{v}$ exceeds sonic velocity. More thought shows that, in the context of the integrand,
Figure B-2: The Creation of Boundary Surface
\[ \frac{\partial \nu_0}{\partial t} \hat{\mathbf{n}} \mathbf{A'} = -\text{sgn}(1 - M_r) \text{sgn}\left(1 - \hat{r} \cdot \frac{\mathbf{w}}{c}\right) (\mathbf{w} - \mathbf{v}) \times \hat{s} \, d\mu . \] (B.14)

Therefore,

\[ \int_{-\infty}^{\infty} \frac{d\mu}{\partial \nu} \frac{\partial H}{\partial t} \frac{\bar{f} \cdot \hat{n} \mathbf{A'}}{4\pi r^2 |1 - M_r|} = + \int d\mu \frac{\bar{f} \cdot (\mathbf{w} - \mathbf{v}) \times \hat{s}}{4\pi r^2 |1 - \hat{r} \cdot \frac{\mathbf{w}}{c}| (1 - M_r)} \]

The complete result for the second type of integral, (B.2), is thus

\[ \int_{S} d\tau d\xi d\eta \frac{\partial G}{\partial t} = \int_{S} d\xi d\eta \frac{\partial}{\partial t} \left( \frac{\bar{f} \cdot \hat{n} \mathbf{A}}{4\pi r^2 |1 - M_r|} \right)_{\xi, \eta} + \int d\mu \frac{\bar{f} \cdot (\mathbf{w} - \mathbf{v}) \times \hat{s}}{4\pi r^2 |1 - \hat{r} \cdot \frac{\mathbf{w}}{c}| (1 - M_r)} \]

(B.15)

From this result together with (B.10) and (B.3), the third type of integral is given by

\[ \int_{S} d\tau d\xi d\eta f \hat{n} \mathbf{A} \cdot \nabla G = \int_{S} d\xi d\eta \nabla \cdot \left( \frac{f \hat{n} \mathbf{A}}{4\pi r^2 |1 - M_r|} \right) - \int d\mu \frac{f \hat{r} \cdot (\mathbf{w} - \mathbf{v}) \times \hat{s}}{4\pi cr^2 |1 - \hat{r} \cdot \frac{\mathbf{w}}{c}| (1 - \hat{r} \cdot \frac{\mathbf{v}}{c})} \]

(B.16)
Appendix C

A Four Dimensional Analogue to Solid Angle

The following is a fundamental proof of the identity (3.31) using the ideas developed in section 3.5.1.

\[ \Box G = \nabla \cdot M \cdot \nabla G = \delta(t) \delta(x), \quad \text{hence} \]

\[ \int_V d\tau \, d^3y \, \Box y G = \int_V d\tau \, d^3y \, \nabla \cdot M \cdot \nabla G = \begin{cases} 1, & \text{if } (t,x) \in V \\ 0, & \text{if } (t,x) \notin V \end{cases} \]

From (3.36),

\[ \int_S d\Sigma \cdot M \cdot \nabla G = \]

From (3.38),

\[ -\int_S d\tau \, d\xi \, d\eta \left\{ A \frac{\hat{n} \cdot \nabla G}{c^2} \frac{\partial G}{\partial \tau} + \hat{n} \cdot \nabla G \right\} = \]

\[ + \int_S d\tau \, d\xi \, d\eta \left\{ A \frac{\hat{n} \cdot \nabla G}{c^2} \frac{\partial G}{\partial \tau} + \hat{n} \cdot \nabla G \right\} = \begin{cases} 1, & \text{if } (t,x) \text{ inside } S \\ 0, & \text{if } (t,x) \text{ outside } S \end{cases} \]

(C.1)

Here, \( V \) is assumed to be the (finite) volume interior to \( S \), so that \( \hat{n} \) is the unit outward normal to a spacelike slice through \( S \). The final equation (C.1) is therefore identical to (3.31).

(Note: If \( V \) were the region exterior to \( S \), then an additional surface integral at infinity would have to be...
included. The final result will again be (C.1); the integral is zero in the unbounded region. If $\mathcal{S}$ is not a closed surface, then the situation is more complicated and (C.1) may not apply.)
Appendix D

The Pressure Field for Creation of Boundary Surface

Morgan's equation (3.44) has a simply expressed pressure field (via (3.45) and Table 3.2) only if the boundary surface is not being created at its edge. As shown in Appendix B, the case of surface creation at an edge introduces greater mathematical complexity into the problem. When a derivative of a surface integral is performed in this case, Leibnitz terms are required at the edge of the surface. These terms gave rise to the line integral of (3.49). This line integral may be considered as the sum of a new acoustic "source type" - a lifting line source - at each point of the creation edge. Since the creation and in plane motion of a boundary surface is completely a matter of viewpoint rather than of mathematical necessity, this "lifting line source" is highly dependent on how one chooses to think about the boundary - i.e. on how one parametrizes the boundary surface.

Indeed, the pressure field, which is the time derivative of the velocity potential field (in the far field), includes line integrals arising both from the line integral in the expression (3.49), and from new Leibnitz terms from the surface integrals in that expression. Thus, the contribution of a point on the "lifting line" to the pressure field is not simply the time derivative of its contribution to the velocity potential field. This is a dramatic contrast to
the flow fields of the Lowson sources given in Table 3.2 (except for the point force, Table 3.2(d)). The reason for this difference is that the very existence of sources at the lifting line depends on the specific parametrization of the boundary surface.

The pressure field, in the far field, is given by

\[ p(t, \vec{x}) = \rho \cdot \frac{\partial \Phi}{\partial t}, \quad \text{where } \Phi(t, \vec{x}) \text{ is given by (3.49).} \]

\[ (D.1) \]

The first surface integral, and the line integral may be differentiated easily, by the methods of Appendix B:

\[ \frac{\partial}{\partial t} \int_S d\xi d\eta \frac{\hat{A}}{4\pi r |1-M_r|} \left\{ -\frac{\vec{v}}{c^2} \frac{\partial \Phi}{\partial t} - \nabla \Phi \right\} = \int_S d\xi d\eta \frac{\partial}{\partial t} \left\{ \frac{\hat{A}}{4\pi r |1-M_r|} \right\} \]

\[ + \int_E d\mu \frac{(\vec{w} - \vec{v}) \times \vec{s}}{4\pi cr |1 - \hat{r} \cdot \frac{\vec{w}}{c} | (1-M_r)} \]

\[ (D.2) \]

\[ \frac{\partial}{\partial t} \int_E d\mu \frac{\left[ \hat{r} - \frac{\vec{v}}{c} \right] \times (\vec{w} - \vec{v}) \times \hat{r} \Phi}{4\pi cr |1 - \hat{r} \cdot \frac{\vec{w}}{c} | (1-M_r)} = \int_E d\mu \frac{\partial}{\partial t} \left\{ \frac{\left[ \hat{r} - \frac{\vec{v}}{c} \right] \times (\vec{w} - \vec{v}) \times \hat{r} \Phi}{4\pi cr |1 - \hat{r} \cdot \frac{\vec{w}}{c} | (1-M_r)} \right\} \]

\[ (D.3) \]
Expression (D.3) assumes that the creation edge $E$ is not itself being created at its endpoints. (Again, this is a matter of the parametrization which is adopted, not any physical property of the edge.)

The other two terms are more involved. The most subtle problem is that there are two types of time derivatives at the creation edge: time rates of change with the edge coordinate, $\mu$, held constant, and time rates of change with the surface coordinates $(\xi, \eta)$ held constant. The edge velocity $\vec{w}$ and the surface velocity $\vec{v}$ evaluated at the edge, respectively, are examples of these two different types of time derivatives. The line integrals in (D.2) and (D.3) may be explicitly written, as, respectively,

$$\int_E \frac{d\mu}{4\pi r|1 - N_T(1 - M_r)|} \left\{ \frac{\vec{v}}{c^2} \frac{\partial \phi}{\partial \tau} |_{\xi, \eta} - \nabla \phi \right\}, \quad \text{and} \quad (D.4)$$

$$\int_E \frac{d\mu}{|1 - N_r|} \frac{\partial}{\partial \tau} \left[ \left( \frac{\hat{r} - \vec{v}}{c} \right) \cdot (\vec{w} - \vec{v}) \times \vec{s} \right] \phi_{\mu} \left( \frac{1 - N_T}{4\pi cr(1 - N_T)(1 - M_r)} \right) \right]$$

For convenience, the following notation as been adopted

$$\vec{N} = \frac{\vec{w}}{c}, \quad N_r = \hat{r} \cdot \frac{\vec{w}}{c}. \quad (D.6)$$
From Table 3.1,

\[
\nabla \cdot \left( \frac{\hat{n}\phi A}{4\pi r|1-M_r|} \right) = \nabla \cdot \left( \frac{\hat{n}\phi A}{4\pi r|1-M_r|} \right) - \frac{\hat{r}}{c} \cdot \frac{\partial}{\partial t} \left( \frac{\hat{n}\phi A}{4\pi r|1-M_r|} \right)
\]

\[
= \frac{\hat{n} \cdot A}{4\pi r|1-M_r|} \left( \frac{\hat{r}}{r} - \frac{\partial}{\partial t}(\hat{r} \cdot c) \right) - \frac{\hat{r}}{c} \cdot \frac{\partial}{\partial t} \left( \frac{\hat{n}\phi A}{4\pi r|1-M_r|} \right)
\]

\[
\nabla \cdot \left( \frac{\hat{n}\phi A}{4\pi r|1-M_r|} \right) = - \frac{\hat{r} \cdot \hat{n} A \phi}{4\pi r^2|1-M_r|} - \frac{\partial}{\partial t} \left( \frac{\hat{r} \cdot \hat{n} A \phi}{4\pi cr|1-M_r|} \right)
\]

Therefore, the doublet integral in (3.47) becomes

\[
- \int_S d\xi d\eta \ \nabla \cdot \left( \frac{\hat{n} A \phi}{4\pi r|1-M_r|} \right) + \int_S d\xi d\eta \ A\hat{n} \cdot \frac{\hat{r}\phi}{4\pi r^2|1-M_r|} + \int_S d\xi d\eta \ \frac{\partial}{\partial t} \left( \frac{\hat{r} \cdot \hat{n} A \phi}{4\pi cr|1-M_r|} \right)
\]

The first term of this expression is differentiable by the methods of Appendix B:

\[
\frac{\partial}{\partial t} \int_S d\xi d\eta \ A\hat{n} \cdot \frac{\hat{r}\phi}{4\pi r^2|1-M_r|} = \int_S d\xi d\eta \ \frac{\partial}{\partial t} \left( \frac{A\hat{n} \cdot \hat{r}\phi}{4\pi r^2|1-M_r|} \right)
\]

\[
+ \int_E du \ \frac{(\overrightarrow{w} - \overrightarrow{v}) \times \overrightarrow{S} \cdot \hat{r} \phi}{4\pi r^2|1-N_r||1-M_r|}
\]

The second term is of the same form as the chronopole integral in (3.49); their sum is
\[ \int_S d\xi \, dn \frac{\partial}{\partial t} \left( \frac{\hat{r} - \frac{\vec{V}}{c}}{4\pi cr|1 - M_r|} \cdot \hat{n}A\phi \right) \]

It remains only to determine the time derivative of this integral. The generic form is, with the notation of Appendix B,

\[ \int_S d\xi \, dn \frac{\partial}{\partial t} \left( \hat{n}A' \cdot \vec{f}(\tau, \xi, \eta) \right)_{\tau = t - \frac{r}{c}} = \int \, du \, dv \, H(v - v_0) \frac{\partial}{\partial t} \left( \hat{n}A' \cdot \vec{f} \right) \]

\[ = \int \, du \, dv \, H(v - v_0) \frac{\partial}{\partial \tau} \left( \hat{n}A' \cdot \vec{f} \right) \frac{\xi, \eta}{(1 - M_r)} \]

The time derivative of this type of integral is

\[ \frac{\partial}{\partial t} \int_S d\xi \, dn \, (\hat{n}A' \cdot \vec{f}) = \int_S d\xi \, dn \frac{\partial^2}{\partial t^2} (\hat{n}A' \cdot \vec{f}) - \int \, du \, dv \, v_0 \frac{\partial}{\partial t} \frac{\partial (\hat{n}A' \cdot \vec{f})}{(1 - M_r) \xi, \eta} \]

\[ (D.8) \]

Now, as in Appendix B, \( \frac{\partial v_0(\tau)}{\partial t} = \frac{1}{(1 - N_r)} \frac{dv_0}{d\tau} \), since \( \tau \) represents the source time at the edge, not on the surface, in \( v_0(\tau) \). The surface integral is

\[ -\int \, du \frac{1}{(1 - N_r)(1 - M_r)} \frac{dv_0}{d\tau} \frac{\partial}{\partial \tau} (\hat{n}A' \cdot \vec{f})_{\xi, \eta} = -\int \, du \frac{1}{(1 - N_r)(1 - M_r)} \left( \frac{\partial v_0}{\partial \tau} \hat{n}A' \right) \cdot \frac{\partial \vec{f}}{\partial \tau}_{\xi, \eta} \]

\[ -\int \, du \frac{1}{(1 - N_r)(1 - M_r)} \frac{\partial v_0}{\partial \tau} \frac{\partial}{\partial \tau} (\hat{n}A')_{\xi, \eta} \]

\[ (D.9) \]

374
The first term on the right hand side is simplified by equation (B.14) of Appendix B:

\[ + \int d\mu \frac{\overline{\omega} - \overline{v}}{|1 - N_r| |1 - M_r|} \cdot \frac{\partial F}{\partial \tau} |_{\xi, \eta} \]  

(D.10)

The second term is of a new form.

The area element is given (to within a sign) by the formula (3.26) of the text:

\[ \hat{A}' = \left. \frac{\partial \overline{y}}{\partial \mu} \right|_{\nu, \tau} \times \left. \frac{\partial \overline{y}}{\partial \nu} \right|_{\mu, \tau} \equiv \tilde{s}(\tau, \mu, \nu) \times \left. \frac{\partial \overline{y}}{\partial \nu} \right|_{\mu, \tau} \]

\[ \left. \frac{\partial}{\partial \tau} (\hat{A}') \right|_{\xi, \eta} = \left. \frac{\partial}{\partial \tau} (\hat{A}') \right|_{\nu, \tau} = \left. \frac{\partial \overline{s}}{\partial \tau} \right|_{\mu, \nu} \times \left. \frac{\partial \overline{y}}{\partial \nu} \right|_{\mu, \tau} + \tilde{s} \times \left. \frac{\partial}{\partial \nu} \left( \frac{\partial \overline{y}}{\partial \tau} \right|_{\mu, \nu} \right) \]

\[ = \left. \frac{\partial \overline{s}}{\partial \tau} \right|_{\mu, \nu} \times \left. \frac{\partial \overline{y}}{\partial \nu} \right|_{\mu, \tau} + \tilde{s} \times \left. \frac{\partial \overline{y}}{\partial \nu} \right|_{\mu, \tau} \]

Next, \( \overline{z}(\tau, \mu) \) of (3.46) and \( \overline{y}(\tau, \mu, \nu) \) of (3.24) are related via

\[ \overline{z}(\tau, \mu) = \overline{y}(\tau, \mu, \nu_0(\tau)) \]  

(D.11)

Therefore,

\[ \left. \frac{\partial \overline{z}}{\partial \mu} \right|_{\mu} = \left. \frac{\partial \overline{y}}{\partial \tau} \right|_{\mu, \nu} + \left. \frac{\partial \overline{y}}{\partial \nu} \right|_{\mu, \tau} \frac{d\nu_0}{d\tau}, \text{ or} \]
\[ \bar{w} = \bar{v} + \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \tau} \frac{dv_0}{dt}, \quad \text{so that} \]

\[ \bar{w} - \bar{v} = \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \tau} \frac{dv_0}{dt} \tag{D.12} \]

Furthermore, from (D.11),

\[ \left. \frac{\partial \bar{z}}{\partial \mu} \right|_\tau = \bar{s}(\tau, \mu) = \left. \frac{\partial \bar{y}}{\partial \mu} \right|_{\tau, \nu} = \bar{s}(\tau, \mu, \nu_0(\tau)) \tag{D.13} \]

Taking the cross product of (D.12) and (D.13) provides an alternative proof of equation (B.14) of Appendix B, to within a sign. Note that

\[ \frac{dv_0}{dt} \frac{\partial}{\partial \tau} (\hat{A}')_{\xi, \eta} = \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \nu} \times \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \tau} \frac{dv_0}{dt} + \bar{s} \times \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \tau} \frac{dv_0}{dt} \]

\[ = \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \nu} \times (\bar{w} - \bar{v}) + \bar{s} \times \left. \frac{\partial \bar{y}}{\partial \nu} \right|_{\mu, \tau} \frac{dv_0}{dt} \]

Now \( \bar{v} = \bar{v}(\tau, \mu, \nu) = \bar{v}(\tau, \mu, \nu_0(\tau)) \) at the edge, so that

\[ \left. \frac{\partial \bar{v}}{\partial \tau} \right|_\mu = \left. \frac{\partial \bar{v}}{\partial \tau} \right|_{\mu, \nu} + \left. \frac{\partial \bar{v}}{\partial \nu} \right|_{\mu, \tau} \frac{dv_0}{dt}. \]

Thus,
\[
\frac{d \nu_0}{d \tau} \frac{\partial}{\partial \tau} (\hat{\mathbf{u}}')_\xi, \eta = \frac{\partial \bar{s}}{\partial \tau}_{\mu, \nu} \times (\bar{w} - \bar{v}) + \bar{s} \times \left( \frac{\partial \bar{v}}{\partial \tau}_{\mu} - \frac{\partial \bar{v}}{\partial \tau}_{\mu, \nu} \right) \\
= \frac{\partial \bar{s}}{\partial \tau}_{\xi, \eta} \times (\bar{w} - \bar{v}) + \bar{s} \times \left( \frac{\partial \bar{v}}{\partial \tau}_{\mu} - \frac{\partial \bar{v}}{\partial \tau}_{\xi, \eta} \right)
\]

(D.14)

With the sign of (D.14) adjusted to agree with the sign of (B.14) from Appendix B, the second integral on the right hand side of (D.9) becomes

\[
+ \int_{E} d\mu \frac{\bar{f}}{|1-N_r||1-M_r|} \left\{ (\bar{w} - \bar{v}) \times \frac{\partial \bar{s}}{\partial \tau}_{\xi, \eta} + \left( \frac{\partial \bar{v}}{\partial \tau}_{\mu} - \frac{\partial \bar{v}}{\partial \tau}_{\xi, \eta} \right) \right\} \times \bar{s}
\]

(D.15)

For brevity, the following notation will be used:

\[
\frac{\partial \bar{s}}{\partial \tau}_{\xi, \eta} \equiv \bar{s}' = \text{contraction rate of vortex sheet (immediately after its creation)}
\]

\[
\frac{\partial \bar{v}}{\partial \tau}_{\mu} \equiv \frac{\dot{v}}{v} = \text{rate of change of "induced velocity" at lifting line}
\]

\[
\frac{\partial \bar{v}}{\partial \tau}_{\xi, \eta} \equiv \frac{\ddot{v}'}{v'} = \text{acceleration of vortex sheet (immediately after its creation)}
\]

The time derivative of (D.7) becomes, upon combining (D.15), (D.10), (D.9), and (D.8):

377
\[ \frac{\partial}{\partial t} \int_{S} d\xi dn \left( \frac{\tau - \nu}{c} \cdot \hat{n}\Lambda \Phi \right) + \int_{S} d\xi dn \frac{\partial^{2}}{\partial t^{2}} \left( \frac{\tau - \nu}{c} \cdot \hat{n}\Lambda \Phi \right) + \int_{E} d\mu \left( \frac{(\hat{w} - \nu) \times \hat{s}}{1 - N_{r}(1 - M_{r})} \cdot \frac{\partial}{\partial t} \left( \frac{\tau - \nu}{c} \cdot \hat{n}\Lambda \Phi \right) + \frac{\left( \tau - \nu \right) \cdot \left( \left( \hat{w} - \nu \right) \times \hat{s} \right) \left( \hat{w} - \nu \right) \cdot \left( \left( \hat{w} - \nu \right) \times \hat{s} \right) }{4\pi cr(1 - M_{r})} \right) \]

\[ = \int_{S} d\xi dn \rho_{0} \frac{\partial}{\partial t} \left( \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \cdot \left[ \frac{\nu}{c^{2}} \cdot \hat{n}\Phi \right]_{\xi, \eta} \right) \]

\[ - \int_{S} d\xi dn \rho_{0} \frac{\partial^{2}}{\partial t^{2}} \left( \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \right) - \int_{S} d\xi dn \rho_{0} \frac{\partial}{\partial t} \left( \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \right) \]

\[ + \rho_{0} \int_{E} d\mu \frac{\partial}{\partial t} \left( \frac{(\hat{w} - \nu) \times \hat{s}}{4\pi cr(1 - N_{r}(1 - M_{r})} \cdot \left[ \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \right] \right) + \rho_{0} \int_{E} d\mu \frac{\hat{r} \cdot \left( (\hat{w} - \nu) \times \hat{s} \right) \left( \hat{w} - \nu \right) \cdot \left( (\hat{w} - \nu) \times \hat{s} \right) }{4\pi cr(1 - N_{r}|1 - M_{r})^{2}} \]

The pressure field corresponding to (3.49), and excluding the \( \frac{1}{2} \rho \bar{u}^{2} \) nonlinear term, is thus, from (D.2), (D.3), (D.6) and (D.12),

\[ p(t, \mathbf{x}) = \int_{S} d\xi dn \rho_{0} \frac{\partial}{\partial t} \left( \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \cdot \left[ \frac{\nu}{c^{2}} \cdot \hat{n}\Phi \right]_{\xi, \eta} \right) \]

\[ - \int_{S} d\xi dn \rho_{0} \frac{\partial^{2}}{\partial t^{2}} \left( \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \right) - \int_{S} d\xi dn \rho_{0} \frac{\partial}{\partial t} \left( \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \right) \]

\[ + \rho_{0} \int_{E} d\mu \frac{\partial}{\partial t} \left( \frac{(\hat{w} - \nu) \times \hat{s}}{4\pi cr(1 - N_{r}(1 - M_{r})} \cdot \left[ \frac{\hat{n}\Lambda \Phi}{4\pi r|1 - M_{r}|} \right] \right) + \rho_{0} \int_{E} d\mu \frac{\hat{r} \cdot \left( (\hat{w} - \nu) \times \hat{s} \right) \left( \hat{w} - \nu \right) \cdot \left( (\hat{w} - \nu) \times \hat{s} \right) }{4\pi cr(1 - N_{r}|1 - M_{r})^{2}} \]
The surface integrals are the pressure fields of the Lowson sources of the corresponding terms in the expression for the velocity potential. For a free vortex sheet, the line integrals constitute a sum of "lifting line" sources at the points of the edge where the sheet is being created. If the circulation round this lifting line is \( \Gamma \), the line integrals are

\[
p(t, x) = \rho_0 \int_E \frac{\partial}{\partial t} \left( \frac{\mathbf{r} - \mathbf{v}}{c} \cdot (\mathbf{w} - \mathbf{v}) \times \mathbf{s} \frac{\Gamma}{4\pi cr |1 - N_r| (1 - M_r)} \right) + \rho_0 \int_E \frac{\mathbf{v} \cdot (\mathbf{w} - \mathbf{v}) \times \mathbf{s}}{4\pi c^2 r |1 - N_r| (1 - M_r)} \frac{\partial \Gamma}{\partial \tau} |\xi, \eta|
\]

\[
+ \rho_0 \int_E \left( \frac{\mathbf{w} - \mathbf{v}}{|1 - N_r| (1 - M_r)} \cdot \frac{\partial}{\partial \tau} \left( \frac{\mathbf{r} - \mathbf{v}}{c} \Gamma \right) \right) + \frac{\mathbf{r} - \mathbf{v}}{c} \cdot [((\mathbf{w} - \mathbf{v}) \times \mathbf{s} \mathbf{\dot{s}}') + (\mathbf{v} - \mathbf{v}') \times \mathbf{s}] \frac{\Gamma}{4\pi cr |1 - N_r| (1 - M_r)^2} \bigg|_{\xi, \eta}
\]

(D.19)

When the residual velocity of the sheet, \( \mathbf{v} \), is zero, only the first two terms survive. For arbitrary \( \mathbf{v} \), these terms strongly resemble Table 3.2(d) for a point force of magnitude \( \rho_0 (\mathbf{w} - \mathbf{v}) \times \mathbf{s} \Gamma \). Only the \( 1/(1 - M_r) \) factor, and the \( -\mathbf{v}/c \) in the first term differ.
Appendix E

Propeller Blade Geometries

Cessna Propeller Geometry

(1:4 scale model of McCauley 1C160)

Radius (R) = .2413 m, 2 Blades

Hub at r/R = .132

<table>
<thead>
<tr>
<th>Radius r/R</th>
<th>Chord c/R</th>
<th>Blade Angle β (degrees)</th>
<th>Zero Lift Angle α₀ (degrees)</th>
<th>Lift Curve Slope, m (per degree)</th>
<th>Airfoil Section (Area)/R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>.132</td>
<td>.1470</td>
<td>31.50</td>
<td>2.14</td>
<td>0.224</td>
<td>0.66E-02</td>
</tr>
<tr>
<td>.237</td>
<td>.1505</td>
<td>30.90</td>
<td>-7.30</td>
<td>0.074</td>
<td>0.34E-02</td>
</tr>
<tr>
<td>.316</td>
<td>.1513</td>
<td>29.50</td>
<td>-8.80</td>
<td>0.083</td>
<td>0.26E-02</td>
</tr>
<tr>
<td>.395</td>
<td>.1521</td>
<td>27.10</td>
<td>-7.07</td>
<td>0.086</td>
<td>0.22E-02</td>
</tr>
<tr>
<td>.474</td>
<td>.1508</td>
<td>24.60</td>
<td>-5.87</td>
<td>0.088</td>
<td>0.18E-02</td>
</tr>
<tr>
<td>.632</td>
<td>.1389</td>
<td>20.30</td>
<td>-4.82</td>
<td>0.089</td>
<td>0.13E-02</td>
</tr>
<tr>
<td>.789</td>
<td>.1179</td>
<td>17.30</td>
<td>-4.58</td>
<td>0.087</td>
<td>0.84E-03</td>
</tr>
<tr>
<td>.868</td>
<td>.1021</td>
<td>16.10</td>
<td>-4.21</td>
<td>0.088</td>
<td>0.61E-03</td>
</tr>
<tr>
<td>.947</td>
<td>.0776</td>
<td>15.10</td>
<td>-3.96</td>
<td>0.091</td>
<td>0.35E-03</td>
</tr>
<tr>
<td>.999</td>
<td>.0150</td>
<td>14.30</td>
<td>-3.92</td>
<td>0.092</td>
<td>0.81E-05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Radius r/R</th>
<th>(C_L)_x</th>
<th>(C_L)_n</th>
<th>Stacking Parameter, ξ</th>
<th>Symmetric Body Flow Blockage, S</th>
</tr>
</thead>
<tbody>
<tr>
<td>.132</td>
<td>1.20</td>
<td>-0.63</td>
<td>.4586</td>
<td>0.535</td>
</tr>
<tr>
<td>.237</td>
<td>1.20</td>
<td>-0.73</td>
<td>.4413</td>
<td>0.615</td>
</tr>
<tr>
<td>.316</td>
<td>1.26</td>
<td>-0.80</td>
<td>.4406</td>
<td>0.705</td>
</tr>
<tr>
<td>.395</td>
<td>1.24</td>
<td>-0.80</td>
<td>.4406</td>
<td>0.795</td>
</tr>
<tr>
<td>.474</td>
<td>1.22</td>
<td>-0.80</td>
<td>.4406</td>
<td>0.850</td>
</tr>
<tr>
<td>.632</td>
<td>1.20</td>
<td>-0.80</td>
<td>.4400</td>
<td>0.925</td>
</tr>
<tr>
<td>.789</td>
<td>1.20</td>
<td>-0.80</td>
<td>.4389</td>
<td>0.950</td>
</tr>
<tr>
<td>.868</td>
<td>1.20</td>
<td>-0.80</td>
<td>.4380</td>
<td>0.960</td>
</tr>
<tr>
<td>.947</td>
<td>1.20</td>
<td>-0.80</td>
<td>.4237</td>
<td>0.965</td>
</tr>
<tr>
<td>.999</td>
<td>1.20</td>
<td>-0.80</td>
<td>.3990</td>
<td>0.970</td>
</tr>
</tbody>
</table>
Cessna Propeller Geometry (continued)

(1:4 scale model of McCauley 1C160)

Section Drag Coefficients (100xC_D)

<table>
<thead>
<tr>
<th>Radius r/R</th>
<th>-0.80</th>
<th>-0.40</th>
<th>0.00</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>.132</td>
<td>10.00</td>
<td>3.00</td>
<td>2.90</td>
<td>2.90</td>
<td>3.20</td>
<td>3.70</td>
<td>4.10</td>
<td>10.00</td>
</tr>
<tr>
<td>.237</td>
<td>10.00</td>
<td>3.00</td>
<td>2.80</td>
<td>2.60</td>
<td>2.88</td>
<td>3.10</td>
<td>3.61</td>
<td>10.00</td>
</tr>
<tr>
<td>.316</td>
<td>10.00</td>
<td>3.00</td>
<td>2.70</td>
<td>2.30</td>
<td>2.20</td>
<td>2.70</td>
<td>3.14</td>
<td>5.00</td>
</tr>
<tr>
<td>.395</td>
<td>10.00</td>
<td>3.00</td>
<td>2.00</td>
<td>1.85</td>
<td>1.81</td>
<td>2.13</td>
<td>2.57</td>
<td>7.50</td>
</tr>
<tr>
<td>.474</td>
<td>10.00</td>
<td>3.00</td>
<td>1.90</td>
<td>1.65</td>
<td>1.62</td>
<td>1.96</td>
<td>2.33</td>
<td>9.00</td>
</tr>
<tr>
<td>.632</td>
<td>10.00</td>
<td>3.00</td>
<td>1.70</td>
<td>1.46</td>
<td>1.44</td>
<td>1.80</td>
<td>2.23</td>
<td>10.00</td>
</tr>
<tr>
<td>.789</td>
<td>10.00</td>
<td>3.00</td>
<td>1.55</td>
<td>1.39</td>
<td>1.41</td>
<td>1.76</td>
<td>2.24</td>
<td>10.00</td>
</tr>
<tr>
<td>.868</td>
<td>10.00</td>
<td>3.00</td>
<td>1.55</td>
<td>1.45</td>
<td>1.43</td>
<td>1.75</td>
<td>2.25</td>
<td>10.00</td>
</tr>
<tr>
<td>.947</td>
<td>10.00</td>
<td>3.00</td>
<td>1.58</td>
<td>1.44</td>
<td>1.51</td>
<td>1.78</td>
<td>2.27</td>
<td>10.00</td>
</tr>
<tr>
<td>.999</td>
<td>10.00</td>
<td>3.00</td>
<td>1.60</td>
<td>1.48</td>
<td>1.62</td>
<td>1.83</td>
<td>2.30</td>
<td>10.00</td>
</tr>
</tbody>
</table>
Windsong Propeller Geometry

(Designed by G. Succi)

Radius (R) = .2381 m, 2 Blades

Hub at r/R = .133

<table>
<thead>
<tr>
<th>Radius r/R</th>
<th>Chord X/R</th>
<th>Blade Angle β (degrees)</th>
<th>Zero Lift Angle α₀ (degrees)</th>
<th>Lift Curve Slope, m (per degree)</th>
<th>Airfoil Section (Area)/R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>.133</td>
<td>.1640</td>
<td>48.80</td>
<td>-4.00</td>
<td>0.087</td>
<td>0.75E-02</td>
</tr>
<tr>
<td>.200</td>
<td>.1839</td>
<td>41.51</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.64E-02</td>
</tr>
<tr>
<td>.300</td>
<td>.1966</td>
<td>35.38</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.60E-02</td>
</tr>
<tr>
<td>.400</td>
<td>.1981</td>
<td>31.37</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.53E-02</td>
</tr>
<tr>
<td>.500</td>
<td>.1943</td>
<td>27.76</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.43E-02</td>
</tr>
<tr>
<td>.600</td>
<td>.1842</td>
<td>24.86</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.33E-02</td>
</tr>
<tr>
<td>.700</td>
<td>.1674</td>
<td>20.87</td>
<td>-3.31</td>
<td>0.107</td>
<td>0.23E-02</td>
</tr>
<tr>
<td>.800</td>
<td>.1430</td>
<td>17.74</td>
<td>-1.95</td>
<td>0.100</td>
<td>0.15E-02</td>
</tr>
<tr>
<td>.900</td>
<td>.1067</td>
<td>15.29</td>
<td>-1.40</td>
<td>0.092</td>
<td>0.56E-03</td>
</tr>
<tr>
<td>.950</td>
<td>.0740</td>
<td>14.04</td>
<td>-1.09</td>
<td>0.087</td>
<td>0.18E-03</td>
</tr>
<tr>
<td>.999</td>
<td>.0350</td>
<td>12.80</td>
<td>-0.75</td>
<td>0.083</td>
<td>0.25E-04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Radius r/R</th>
<th>(Cₐ)ₙ</th>
<th>(Cₐ)ₙ</th>
<th>Stacking Parameter, k</th>
<th>Symmetric Body Flow Blockage, S</th>
</tr>
</thead>
<tbody>
<tr>
<td>.133</td>
<td>1.00</td>
<td>-1.00</td>
<td>.4500</td>
<td>0.535</td>
</tr>
<tr>
<td>.200</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.570</td>
</tr>
<tr>
<td>.300</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.670</td>
</tr>
<tr>
<td>.400</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.790</td>
</tr>
<tr>
<td>.500</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.855</td>
</tr>
<tr>
<td>.600</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.910</td>
</tr>
<tr>
<td>.700</td>
<td>1.50</td>
<td>-1.10</td>
<td>.4500</td>
<td>0.940</td>
</tr>
<tr>
<td>.800</td>
<td>0.95</td>
<td>-1.00</td>
<td>.4500</td>
<td>0.950</td>
</tr>
<tr>
<td>.900</td>
<td>1.10</td>
<td>-0.90</td>
<td>.4500</td>
<td>0.960</td>
</tr>
<tr>
<td>.950</td>
<td>1.10</td>
<td>-0.90</td>
<td>.4500</td>
<td>0.965</td>
</tr>
<tr>
<td>.999</td>
<td>1.10</td>
<td>-0.90</td>
<td>.4500</td>
<td>0.970</td>
</tr>
</tbody>
</table>
Bruiser Propeller Geometry

(Designed by G. Succi)

Radius \((R) = 0.2381\) m, 2 Blades

Hub at \(r/R = 0.133\)

<table>
<thead>
<tr>
<th>Radius (r/R)</th>
<th>Chord (\chi/R)</th>
<th>Blade Angle (\beta) (degrees)</th>
<th>Zero Lift Angle (\chi_0) (degrees)</th>
<th>Lift Curve Slope, m (per degree)</th>
<th>Airfoil Section (Area)/(R^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.133</td>
<td>.0800</td>
<td>51.10</td>
<td>-4.00</td>
<td>0.000</td>
<td>0.50E-02</td>
</tr>
<tr>
<td>.200</td>
<td>.2291</td>
<td>43.53</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.79E-02</td>
</tr>
<tr>
<td>.300</td>
<td>.2453</td>
<td>37.25</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.77E-02</td>
</tr>
<tr>
<td>.400</td>
<td>.2475</td>
<td>33.14</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.69E-02</td>
</tr>
<tr>
<td>.500</td>
<td>.2432</td>
<td>29.38</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.58E-02</td>
</tr>
<tr>
<td>.600</td>
<td>.2309</td>
<td>26.37</td>
<td>-3.98</td>
<td>0.104</td>
<td>0.44E-02</td>
</tr>
<tr>
<td>.700</td>
<td>.2099</td>
<td>22.18</td>
<td>-3.31</td>
<td>0.107</td>
<td>0.30E-02</td>
</tr>
<tr>
<td>.800</td>
<td>.1795</td>
<td>18.88</td>
<td>-1.95</td>
<td>0.100</td>
<td>0.20E-02</td>
</tr>
<tr>
<td>.900</td>
<td>.1339</td>
<td>16.26</td>
<td>-1.40</td>
<td>0.092</td>
<td>0.71E-03</td>
</tr>
<tr>
<td>.950</td>
<td>.0979</td>
<td>14.91</td>
<td>-1.09</td>
<td>0.087</td>
<td>0.24E-03</td>
</tr>
<tr>
<td>.999</td>
<td>.0280</td>
<td>13.56</td>
<td>-0.75</td>
<td>0.083</td>
<td>0.19E-04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Radius (r/R)</th>
<th>((C_L)_f)</th>
<th>((C_L)_h)</th>
<th>Stacking Parameter, (K)</th>
<th>Symmetric Body Flow Blockage, (\delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.133</td>
<td>1.00</td>
<td>-1.00</td>
<td>.5000</td>
<td>0.535</td>
</tr>
<tr>
<td>.200</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.570</td>
</tr>
<tr>
<td>.300</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.670</td>
</tr>
<tr>
<td>.400</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.790</td>
</tr>
<tr>
<td>.500</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.855</td>
</tr>
<tr>
<td>.600</td>
<td>1.60</td>
<td>-1.20</td>
<td>.4500</td>
<td>0.910</td>
</tr>
<tr>
<td>.700</td>
<td>1.50</td>
<td>-1.10</td>
<td>.4500</td>
<td>0.940</td>
</tr>
<tr>
<td>.800</td>
<td>0.95</td>
<td>-1.00</td>
<td>.4500</td>
<td>0.950</td>
</tr>
<tr>
<td>.900</td>
<td>1.10</td>
<td>-0.90</td>
<td>.4500</td>
<td>0.960</td>
</tr>
<tr>
<td>.950</td>
<td>1.10</td>
<td>-0.90</td>
<td>.4500</td>
<td>0.965</td>
</tr>
<tr>
<td>.999</td>
<td>1.10</td>
<td>-0.90</td>
<td>.4500</td>
<td>0.970</td>
</tr>
</tbody>
</table>
Windsong and Bruiser Propeller Geometry (continued)

(Designed by G. Succi)

Section Drag Coefficients (100×C₀)

<table>
<thead>
<tr>
<th>Radius r/R</th>
<th>-1.20</th>
<th>-0.90</th>
<th>-0.60</th>
<th>-0.30</th>
<th>0.00</th>
<th>0.50</th>
<th>1.10</th>
<th>1.40</th>
<th>1.60</th>
</tr>
</thead>
<tbody>
<tr>
<td>.133</td>
<td>80.00</td>
<td>80.00</td>
<td>80.00</td>
<td>80.00</td>
<td>80.00</td>
<td>80.00</td>
<td>80.00</td>
<td>80.00</td>
<td>80.00</td>
</tr>
<tr>
<td>.200</td>
<td>10.00</td>
<td>1.33</td>
<td>1.05</td>
<td>0.85</td>
<td>0.70</td>
<td>0.50</td>
<td>0.76</td>
<td>2.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.300</td>
<td>10.00</td>
<td>1.33</td>
<td>1.05</td>
<td>0.85</td>
<td>0.70</td>
<td>0.50</td>
<td>0.76</td>
<td>2.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.400</td>
<td>10.00</td>
<td>1.33</td>
<td>1.05</td>
<td>0.85</td>
<td>0.70</td>
<td>0.50</td>
<td>0.76</td>
<td>2.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.500</td>
<td>10.00</td>
<td>1.33</td>
<td>1.05</td>
<td>0.85</td>
<td>0.70</td>
<td>0.50</td>
<td>0.76</td>
<td>2.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.600</td>
<td>10.00</td>
<td>1.33</td>
<td>1.05</td>
<td>0.85</td>
<td>0.70</td>
<td>0.50</td>
<td>0.76</td>
<td>2.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.700</td>
<td>10.00</td>
<td>1.28</td>
<td>0.98</td>
<td>0.78</td>
<td>0.66</td>
<td>0.55</td>
<td>1.03</td>
<td>2.00</td>
<td>18.00</td>
</tr>
<tr>
<td>.800</td>
<td>10.00</td>
<td>7.85</td>
<td>5.51</td>
<td>3.17</td>
<td>1.06</td>
<td>0.78</td>
<td>9.00</td>
<td>10.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.900</td>
<td>10.00</td>
<td>3.73</td>
<td>2.67</td>
<td>1.62</td>
<td>0.62</td>
<td>1.26</td>
<td>9.59</td>
<td>10.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.950</td>
<td>10.00</td>
<td>3.73</td>
<td>2.67</td>
<td>1.62</td>
<td>0.62</td>
<td>1.26</td>
<td>9.59</td>
<td>10.00</td>
<td>10.00</td>
</tr>
<tr>
<td>.999</td>
<td>10.00</td>
<td>3.73</td>
<td>2.67</td>
<td>1.62</td>
<td>0.62</td>
<td>1.26</td>
<td>9.59</td>
<td>10.00</td>
<td>10.00</td>
</tr>
</tbody>
</table>
## Cessna Propeller Grid

### Source Locations for 20 Point Grid Used in Calculations

<table>
<thead>
<tr>
<th>Radius $\xi/R$</th>
<th>Number of Chordwise Stations</th>
<th>Chordwise Coordinates $\eta_l/\chi$</th>
<th>Fraction of Thickness $f_l$ *</th>
</tr>
</thead>
<tbody>
<tr>
<td>.280</td>
<td>1</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>.398</td>
<td>1</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>.490</td>
<td>2</td>
<td>0.08 0.76</td>
<td>0.45 0.55</td>
</tr>
<tr>
<td>.575</td>
<td>2</td>
<td>0.08 0.76</td>
<td>0.45 0.55</td>
</tr>
<tr>
<td>.645</td>
<td>2</td>
<td>0.08 0.76</td>
<td>0.45 0.55</td>
</tr>
<tr>
<td>.724</td>
<td>3</td>
<td>0.05 0.25 0.85</td>
<td>0.13 0.57 0.30</td>
</tr>
<tr>
<td>.801</td>
<td>3</td>
<td>0.05 0.25 0.85</td>
<td>0.13 0.57 0.30</td>
</tr>
<tr>
<td>.875</td>
<td>3</td>
<td>0.05 0.25 0.85</td>
<td>0.13 0.57 0.30</td>
</tr>
<tr>
<td>.950</td>
<td>3</td>
<td>0.05 0.25 0.85</td>
<td>0.13 0.57 0.30</td>
</tr>
</tbody>
</table>

*See Table 4-3 for the equivalent fraction for the chordwise load distribution*
Windsong and Bruiser Propeller Grids

Source Locations for 20 Point Grid Used in Calculations

<table>
<thead>
<tr>
<th>Radius $r/R$</th>
<th>Number of Chordwise Stations</th>
<th>Chordwise Coordinates $\eta_i/\chi$</th>
<th>Fraction of Thickness $f_i$ *</th>
</tr>
</thead>
<tbody>
<tr>
<td>.215</td>
<td>1</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>.335</td>
<td>1</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>.452</td>
<td>2</td>
<td>0.08</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.76</td>
<td>0.47</td>
</tr>
<tr>
<td>.550</td>
<td>2</td>
<td>0.08</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.76</td>
<td>0.47</td>
</tr>
<tr>
<td>.620</td>
<td>2</td>
<td>0.08</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.76</td>
<td>0.47</td>
</tr>
<tr>
<td>.698</td>
<td>3</td>
<td>0.05</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.85</td>
<td>0.28</td>
</tr>
<tr>
<td>.775</td>
<td>3</td>
<td>0.05</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.85</td>
<td>0.38</td>
</tr>
<tr>
<td>.852</td>
<td>3</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.85</td>
<td>0.41</td>
</tr>
<tr>
<td>.935</td>
<td>3</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.85</td>
<td>0.41</td>
</tr>
</tbody>
</table>

*See Table 4-3 for the equivalent fraction for the chordwise load distribution*
Appendix F

Optical Trigger Calibration

The most important instrument associated with the propeller thrust stand is the "optical trigger", consisting of the phototransducer and its interrupter disk. This device has a dual purpose: First, the frequency of these pulses provides the person operating the armature and shunt field power supplies with the feedback necessary to control the motor speed. Second, the leading edge of each pulse provides a trigger signal for the oscilloscope which records the output of the microphone. Hence, it was important to carefully measure the angle of the propeller blades at the instant this trigger pulse occurred.

Physically, the phototransducer consists of an infrared light emitting diode (LED) which illuminates a phototransistor. The LED and the phototransistor are moulded into a U-shaped piece of plastic, such that they face one another across the gap at the top of the U. The U is mounted in a plane containing the propeller axis, with the axis "above" the U. The interrupter disk is a circular aluminum plate which is perpendicular to the shaft and spins with it. It fits between the two arms of the U, thereby preventing the LED from illuminating the phototransistor. A single slot is machined in the interrupter disk, so that the phototransistor is briefly illuminated once during each revolution of the propeller. The disk is painted matte black in order to minimize reflections. The phototransistor response is shaped and buffered by a tiny circuit which was attached to the base of the U-shaped plastic package.

I made a serious error in the design of this shaping circuit. Fortunately, it proved
possible to recover from this mistake, but a rather lengthy digression is necessary to explain the corrections which have been applied to the data. Simply stated, the electronics of the shaping circuit introduces a fixed delay between the instant the slot in the interrupter disk exposes the phototransistor, and the time at which the trigger pulse is generated. Therefore, the propeller angle at which the trigger pulse for the oscilloscope occurs depends on the angular speed of the propeller.

Unfortunately, this delay time changed over the course of the experiment. In fact, the problem was discovered during the detailed study of the repeatability of the experiment which is explained in section 5.5. There are two types of changes in the delay time associated with the trigger pulses:

First, there were catastrophic changes which occurred whenever the phototransducer unit suffered some exceptional abuse. Twice during the experiment, the arm of the U containing the phototransistor moved into contact with the interrupter disk (probably as a result of vibrations). This rubbing contact apparently increased the opacity of the plastic surrounding the transistor, thereby decreasing its sensitivity and increasing the delay time of the circuit. These episodes of rubbing were detected rapidly and corrected, but it was not realized at the time that any delay existed at all. Also, long periods of disuse apparently could increase the opacity of the LED or phototransistor packages with similar effects. Such intervals will also be regarded as "exceptional abuse".

Second, the circuit was slightly temperature sensitive. Since the temperature of the phototransducer bears no relation to the airstream temperature (which was the only temperature ever recorded), it is impossible to correct for this effect. The temperature effect appears to be important only for the first run made on any given day, which seems to be time enough for the phototransducer to approach some sort
of equilibrium temperature.

The shaping circuit is shown in Fig. F-1, along with its characteristic response to the passage of the slot in the interrupter disk. The leading edge of the trigger pulse, corresponding to the transition of the phototransistor from darkness into light, was normally used to trigger the oscilloscope. The current which the phototransistor passes is proportional to the light incident on it. This current is converted to a voltage by passing it through the resistor R\textsubscript{1}. The voltage developed across R\textsubscript{1} is used as the input to a 555 Precision Timer, which serves both as a line driver and as a pulse shaper (a Schmitt trigger). When the input to the 555 exceeds 8 V (2/3 of its supply), its output switches from its normally high state to ground. The output of the 555 does not return to its high state until its input drops below 4 V (1/3 of its supply), which provides the Schmitt trigger action.

The input capacitance and conductance of the 555 are completely negligible in this circuit. The 555 does introduce a noticeable delay time itself, however this delay is on the order of 5 to 10 \(\mu\text{s}\), and is thus not particularly important. For comparison, at the highest angular speed (10,000 RPM) one degree of propeller rotation requires about 16 \(\mu\text{s}\). The problematical delay is caused by the large effective capacitance of the phototransistor. In tests of the circuit, the exponential rise and fall times of the voltage across R\textsubscript{1} proved to be roughly proportional to the value of R\textsubscript{1}, and roughly independent of the power supply voltage or the LED brightness, just as if a simple capacitor were being charged through the resistance R\textsubscript{1}. These time constants were of the order of 100 \(\mu\text{s}\), which represents a considerable angular displacement of the propeller blades at 10,000 RPM.

However, notice that the delay between the exposure of the phototransistor and the generation of the trigger pulse does depend on the LED brightness and/or the
Figure F-1: Optical Trigger Circuit
phototransistor sensitivity. Even though the time constant associated with the circuit does not change, the time required for the voltage across R1 to reach 8 V and trigger the 555 depends on the value of the final current which will flow through the phototransistor. It was inferred from the data that the trigger level was reached in less than one time constant in the first experiments, while requiring perhaps two time constants in the last experiments.

The procedure by which the delay times associated with the shaping circuit were inferred will now be outlined. In all data taken prior to January 1980, it had been assumed that the trigger angle (i.e. the angle of the propeller blades at the instant the trigger pulse occurs) was independent of the propeller RPM. The trigger angle was therefore measured simply by rotating the blades by hand very slowly until the trigger pulse was generated. The angle of the line connecting the propeller tips at the instant this pulse was generated relative to the horizontal could be measured by means of a bubble level with an accuracy of about ±0.6 degrees. The trigger angle at zero RPM is therefore known for all of the data, within this uncertainty. (This angle changed whenever the thrust stand was disassembled.) The point at which the trailing edge of the trigger pulse occurs was measured on several occasions by this same procedure. It was found that the width of the trigger pulse at zero RPM corresponds to 15.7 ±0.6 degrees of propeller rotation. (The width of the slot in the interrupter disk never changed.)

In January 1980, when the delay problem was discovered, acoustic measurements were made with the Cessna propeller blade operating in front of the symmetric body. The trigger angle as a function of RPM was measured as accurately as possible by means of a stroboscope flash which was fired by the optical trigger pulse. An upper limit of 3 μs was placed on the delay between the trigger pulse and the
appearance of the strobe flash by measuring the response of a photocell to the flash. The duration of the strobe flash was only a fraction of a microsecond, judging by the complete absence of any "blurring" of marks on the propeller shaft even at 10,000 RPM.

The propeller blades were removed from one of the hubs for the dynamic trigger angle calibrations. Scratch marks were made on this hub and on the shaft extension housing immediately adjacent to the hub, in such a way the the marks were aligned at the instant the trigger pulse was generated at zero RPM. Additionally, scratch marks were scribed on the shaft housing at 5 degree intervals. When the shaft was spinning and illuminated by the strobe flashes, the apparent position of the scratch mark on the hub was compared to the shaft marks on the stationary housing. In this way, the change in the trigger angle from its static value at any given RPM could be directly measured. Simultaneously with this visual measurement, two consecutive trigger pulses were "captured" on the digital oscilloscope, which enabled the RPM at the time of the measurement to be determined precisely. The width of each individual trigger pulse was also measured in order to determine the angle through which the propeller rotated between the leading and trailing edges of the trigger pulse.

The results of a series of such measurements are shown in Fig. F-2. The fact that the change in the trigger angle with RPM is linear indicates that the phenomenon is indeed caused by a simple, constant delay time between the illumination of the phototransistor and the generation of the trigger pulse. Note that the delay time associated with the leading edge of the pulse is greater than that associated with the trailing edge. The reason for this becomes clear upon examination of the phototransistor circuit response shown in Fig. F-1. In all further dynamic
Figure F-2: Trigger Angle Calibration
calibrations of the trigger angle, the linearity of the relationship between the change in trigger angle and the RPM was taken for granted, and the angle was measured at only two or three different RPMs.

Next, three identical series of acoustic measurements were performed at one standard operating condition. Before and after each series of measurements, the Cessna propeller was replaced by the hub with the calibration scratch, and a dynamic calibration of the trigger angle was performed. The four values of the trigger delay so measured were in close agreement with one another, except for the first of the four. When the phototransducer was cold, the delay time was about 10% shorter than after the first series of measurements had warmed up the motor and everything connected to it. This change in the delay time with temperature was visible as a phase shift in the sound pressure measured as a function of time.

When one of these acoustic measurements is compared with an identical measurement made at an earlier epoch in the history of the experiment, the amplitude and shape of the pressure signatures agree quite well (see section 5.5) However, a phase shift is evident, due to the change in the trigger delay time between the two epochs. By measuring this phase shift, the unknown delay time during the earlier epoch can be inferred, since the delay time during the calibration epoch was measured. All such calibrations were made using the sound pressure recorded in the disk plane at a distance of one propeller diameter (48.5 cm) from the hub, with the Cessna propeller, the symmetric body, at an angular speed of 10,000 RPM, and with an airspeed of 2.5 inches of alcohol. The phase was determined by measuring the times (relative to the scope trigger) at which the sharp negative zero crossings occurred. The phase was corrected for differences in the ambient sound speed, and in the static trigger angle.
This single operating condition and microphone location were the only ones which were repeated regularly throughout the course of the experiment. It is very fortunate that there was such a point; in retrospect it would have been wise to have chosen a specific data point to be repeated numerous times throughout the experiment, to serve in just the capacity that this data point has served.

Five experimental epochs exist in which at least one measurement was made at the same operating condition as the calibrations. The trigger delay has been back calibrated in four of these five epochs by the phase comparison described above. The sequence numbers of the acoustic measurements used in the calibrations are given in Table F-1, together with the inferred trigger delay times. A sixth epoch, for which there exists no acoustic measurement at the calibration operating condition, is also shown (see section 5.5.2). The estimated uncertainty in the trigger angle for the four epochs which have been calibrated is 1.5 degrees of propeller rotation.
### Table F-1: Epochs in the History of the Optical Trigger

<table>
<thead>
<tr>
<th>Epoch</th>
<th>Dates</th>
<th>Data Sequence # of Calibration Files</th>
<th>Inferred Delay Time (µs)</th>
<th>Static Trigger Angle (degrees)</th>
<th>Dynamic Correction (degrees per 1000 RPM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3/13/79-3/24/79</td>
<td>H073, H104</td>
<td>78.0</td>
<td>20.5 ± 0.6</td>
<td>0.648</td>
</tr>
<tr>
<td>3</td>
<td>7/31/79*</td>
<td>A071, A081*</td>
<td>81.5</td>
<td>23.5</td>
<td>0.489</td>
</tr>
<tr>
<td>4</td>
<td>8/1/79 &amp; 8/2/79</td>
<td>A142</td>
<td>-not available-</td>
<td>7.8</td>
<td>-not available-</td>
</tr>
<tr>
<td>5</td>
<td>8/3/79</td>
<td>A230</td>
<td>144.2</td>
<td>9.2</td>
<td>0.865</td>
</tr>
<tr>
<td>6</td>
<td>1/10/80-1/22/80</td>
<td>A281, A296</td>
<td>Delay times accurately measured</td>
<td>Measured Dynamic Trigger Angle. 21.2 degrees</td>
<td></td>
</tr>
</tbody>
</table>

*Triggered on trailing edge of trigger pulse.

Estimated uncertainty in trigger angle is ±1.5 degree (for epochs 1, 3, 5, and 6 due to all causes)