SOME APPLICATIONS OF RANDOM FIELD THEORY

IN GEOTECHNICAL ENGINEERING

by

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ABSTRACT

An approach is presented for dealing with problems that can be formulated in terms of the variances and covariances of averages of random fields using the concept of the scale of fluctuation rather than that of the covariance function. This approach provides good approximations to the solution of such problems. In particular it is very suitable for use in geotechnical engineering because the usually limited amount of data does not permit the estimation of covariance functions and because it greatly simplifies the necessary computations. The application of the proposed approach to some problems of geotechnical engineering is outlined.

Several problems in geotechnical engineering can be formulated as problems involving random fields of which no theoretical solutions exist. The simulation of random fields is presented as a practical way to deal with such problems.

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CHAPTER 1

INTRODUCTION

Geotechnical engineering is often mentioned to be a "young" science. If the criterion of being "young" is the age in years, then perhaps this is no longer true: more than half a century has passed since the publication of Terzaghi's "Erdbaumechanik" in 1925. But if the criterion for a science to be characterized "young" is whether there is room for improvement of our state of knowledge, then geotechnical engineering definitely qualifies for this adjective.

From the point of view of the practicing engineer who recognizes that his methods do not give results as good as he would like them to be, the problem lies in at least one of the two: (a) in the insufficiency of the mathematical methods he is using; (b) in the inappropriateness of the parameters he inputs to his models. Of course the answer is that the problem lies in both of the above. But the question then arises as to where an attempt towards improvement will be more efficient.

In the early years of soil mechanics the main consideration clearly was how to get a better understanding of the physics of the problems at hand, so the most efficent way to attack them was to try to improve the mathematical modelling. But now it seems that the efficiency of modelling-oriented research is dropping. One could not easily substantiate the claim that concerning ourselves more with the parameters we input to the models would be more efficient, but at least the

idea becomes more and more appealing. Better parameter determiniation can of course mean improvement of the testing techniques or of the interpretation of existing ones. But still testing of samples amounts to testing a volume of soil whose order of magnitude is roughly $1/10^6$ of the volume of the soil stratum under consideration.

It should then appear as an attractive approach to the improvement of geotechnical predictions to try to take into account our uncertainty about soil properties. And of course we refer here not to the uncertainty due to poor testing techniques but to the uncertainty that is due to our inability (for practical and economic reasons) to obtain complete data describing a soil profile. Engineers have traditionally used empirical factors of safety to deal with this uncertainty. In this work the theory of random fields is seen as a tempting alternative. After all, probability theory is the only way to quantitatively formulate and solve the problem of design optimization which is the ultimate objective of engineering, i.e. the problem of maximizing the expected utility and/or minimizing the expected loss.

Chapter 2 presents a summary of the theory of random fields. A random field is a field (a function in, say, the three-dimensional space) whose values are random variables. This concept obviously covers several situations in geotechnical engineering where we are dealing with limited information about soil properties that vary in space.

Chapter 3 shows how it is possible to simplify the theory

of random fields in order to obtain simple approximations for practical use.

Chapter 4 deals with the important problem of simulation of random fields, which is presented as a practical method of answering questions for which no theoretical results are available.

Chapter 5 examines some applications of the material presented in Chapters 3 and 4 to some problems of geotechnical engineering.

CHAPTER 2

THE THEORY OF RANDOM FIELDS - BACKGROUND

The material in this chapter serves as a very brief introduction to the concepts that will be used later. Yaglom (1962), Matern (1960) and Veneziano (1978) cover what is presented here in much greater depth and breadth.

2.1 DEFINITION OF A RANDOM FIELD

Consider a real valued function $x(\omega,\underline{t})$ where $\underline{t}$ belongs to R^n, the n-dimensional space,* and ω belongs to the set Ω of the possible outcomes of a random experiment. Such a function is called a random field in R^n since its values are random numbers due to their dependence on ω. Usually, we are interested either in the study of the field (in the ordinary sense, i.e. non-random) resulting from fixing ω or in the study of the random variable resulting from fixing $\underline{t}$. We will denote the deterministic field resulting from fixing ω by $x(\underline{t})$ and the random variable resulting from fixing $\underline{t}$ by $X(\underline{t})$.**

As an example, consider the variation of the undrained shear strength in the volume of the soil under a foundation.

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* The symbol $\underline{t}$ is used to represent the point $(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are its Cartesian coordinates. The same symbol stands for the vector with Cartesian components $t_1, \ldots, t_n$.

** Random variables will be referred to by capital letters and deterministic variables by lower case letters.

We can consider the shear strength to be a random field $x(\omega,\underline{t})$ where $\underline{t} = (t_1, t_2, t_3)$ is a point in the three-dimensional space R^3 and t_i, $i = 1,2,3$ are its Cartesian coordinates. In this case $X(\underline{t})$ will be a random variable representing the shear strength at point $\underline{t}$, whereas $x(\underline{t})$ will represent the result of an experiment to determine the shear strength at point $\underline{t}$. (This example ignores the fact that soil tests are subject to experimental errors).

A random field is considered to be completely specified if for each $\underline{t}$ in R^n we are given (or can derive) the distribution function

$$F_{\underline{t}}(x) = P[X(\underline{t}) < x] \tag{2.1}$$

of the random variable $X(\underline{t})$, if for each $\underline{t}_1$, $\underline{t}_2$ in R^n we are given the joint distribution function

$$F_{\underline{t}_1,\underline{t}_2}(x_1,x_2) = P[X(\underline{t}_1) < x_1,\ X(\underline{t}_2) < x_2] \tag{2.2}$$

of $X(\underline{t}_1)$ and $X(\underline{t}_2)$, and so on, and in general if for any k points $\underline{t}_1, \ldots, \underline{t}_k$ in R^n we are given the joint distribution function

$$F_{\underline{t}_1, \ldots, \underline{t}_k}(x_1, \ldots, x_k) = P[X(\underline{t}_1)<x_1, \ldots, X(\underline{t}_k)<x_k] \tag{2.3}$$

of $X(\underline{t}_1), \ldots, X(\underline{t}_k)$.

2.2 HOMOGENEITY -ISOTROPY

A random field specified this way is called homogeneous if all the above distribution functions remain the same as the whole set of points $\underline{t}_1, \ldots, \underline{t}_k$ is translated (but not rotated) in R^n, i.e. if all the distribution functions depend only on the relative location of the points involved and not on absolute location. For random processes, i.e. one-dimensional random fields, the term stationary is used instead of the term homogeneous. If the distribution functions remain the same even when the group of points is rotated then the field is called isotropic.

2.3 THE MEAN AND THE COVARIANCE FUNCTION

In practice we do not know that much about the fields with which we are dealing. Instead all that we know (or assume that we know) are the mean and the covariance function of the random field. They are defined in the following way:

Mean:
$$m(\underline{t}) = E[X(\underline{t})] \tag{2.4}$$

Covariance:

$$B(\underline{t}_1,\underline{t}_2) = E[(X(\underline{t}_1)-m(\underline{t}_1))(X(\underline{t}_2)-m(\underline{t}_2))] = \\ = E[X(\underline{t}_1)X(\underline{t}_2)] - m(\underline{t}_1)m(\underline{t}_2) \tag{2.5}$$

Whereas getting a feeling of what the mean represents should be no problem, the concept of the covariance function needs some clarification. This is provided by the relation

$$B(\underline{t}_1,\underline{t}_2) = \rho(\underline{t}_1,\underline{t}_2)\ \sigma\ (\underline{t}_1)\ \sigma\ (\underline{t}_2) \tag{2.6}$$

where $\sigma^2(\underline{t}_1)$ and $\sigma^2(\underline{t}_2)$ are the variances (standard deviations squared) of the random variables $X(\underline{t}_1)$ and $X(\underline{t}_2)$ and $\rho(\underline{t}_1,\underline{t}_2)$

is the coefficient of correlation (called correlation function here) between these two variables. Always

$$-1 \leq \rho(\underline{t}_1, \underline{t}_2) \leq 1 \tag{2.7}$$

When dealing with a homogeneous field, we can write m instead of $m(\underline{t})$, i.e. the mean is constant, and $B(\underline{\tau})$ instead of $B(\underline{t}_1, \underline{t}_2)$, i.e. the covariance depends only on the vector

$$\underline{\tau} = \underline{t}_2 - \underline{t}_1 \tag{2.8}$$

and not on the location of the points $\underline{t}_1$ and $\underline{t}_2$. For isotropic fields we can simplify the notation further by writing $B(r)$ where

$$r = |\underline{\tau}| = |\underline{t}_2 - \underline{t}_1| \tag{2.9}$$

The covariance function is symmetric with respect to $\underline{t}_1$ and $\underline{t}_2$, i.e.

$$B(\underline{t}_1, \underline{t}_2) = B(\underline{t}_2, \underline{t}_1) \tag{2.10}$$

as it is obvious from equation (2.5).

Equation (2.7) implies that for a homogeneous random field

$$|B(\underline{t}_1, \underline{t}_2)| \leq \sigma^2 \tag{2.11}$$

where σ^2 is the (independent of $\underline{t}$) variance of the random variable $X(\underline{t})$. Also

$$B(\underline{t}_1, \underline{t}_1) = \sigma^2(\underline{t}_1) \tag{2.12}$$

in all cases and

$$B(\underline{t}_1, \underline{t}_1) = \sigma^2 \tag{2.13}$$

for homogeneous random fields.

In order to obtain some insight into the significance of

the covariance function consider the following example. Assume that a soil parameter can be modelled as a three-dimensional random field and that we have experimentally determined the value of this parameter at a point $\underline{t}_1$. Let this value be $x(\underline{t}_1)$. Assume further that the random field is homogeneous and has a covariance function

$$B(\underline{t}_1 - \underline{t}_2) = \sigma^2 \rho(\underline{t}_1 - \underline{t}_2) \ . \qquad (2.14)$$

Before making the measurement at point $\underline{t}_1$ we know that at another point $\underline{t}_2$ the value $X(\underline{t}_2)$ of the parameter has mean value m and variance σ^2. We want to determine the mean and the variance of $X(\underline{t}_2)$ after we have observed $X(\underline{t}_1)$. These are given by the equations

$$E[X(\underline{t}_2) | X(\underline{t}_1) = x(\underline{t}_1)] = m + \rho(\underline{t}_1 - \underline{t}_2)[x(\underline{t}_1) - m] \qquad (2.15)$$

$$Var[X(\underline{t}_2) | X(\underline{t}_1) = x(\underline{t}_1)] = \sigma^2[1 - \rho^2(\underline{t}_1 - \underline{t}_2)] \qquad (2.16)$$

Equations (2.15) and (2.16) are exact only for Gaussian fields (see Section 2.5) but they are good approximations in other cases. As can be seen from (2.15) and (2.16), the correlation function is a measure of how knowledge of the value of a realization of a random field at one point affects our knowledge of the value of the same realization at another point. If ρ is zero then knowledge of $x(\underline{t}_1)$ does not give us any information concerning $x(\underline{t}_2)$. If ρ is positive we expect $x(\underline{t}_2)$ to lie on the same side of the mean value of the field as $x(\underline{t}_1)$. If ρ is negative we expect $x(\underline{t}_2)$ and $x(\underline{t}_1)$ to lie on different sides of the mean value.

2.4 EXAMPLES OF COVARIANCE FUNCTIONS

The choice of functions that can be used as covariance functions of random fields is limited by necessary conditions such as (2.7). Sufficient conditions can be found in the references mentioned at the beginning of this chapter. In the present section some commonly used covariance functions are given. See Figure 2.1.

The following are admissible covariance functions of isotropic random fields of one, two or three dimensions:

a. Simple exponential

$$B(r) = \sigma^2 e^{-\frac{r}{a}}, \quad a > 0 \tag{2.17}$$

This covariance function is appropriate for spaces of any dimension.

b. Exponentially damped cosine function

$$B(r) = \sigma^2 e^{-\frac{r}{a}} \cos \beta r, \quad a > 0, \quad a < \frac{\sqrt{3}}{3\beta} \tag{2.18}$$

c. Double exponential

$$B(r) = \sigma^2 e^{-(\frac{r}{a})^2} \tag{2.19}$$

Two very simple models for use in random processes (one-dimensional random fields) are the box-shaped covariance function

$$B(r) = \begin{cases} \sigma^2 \text{ if } r < a, \ a > 0 \\ 0 \text{ otherwise} \end{cases}$$

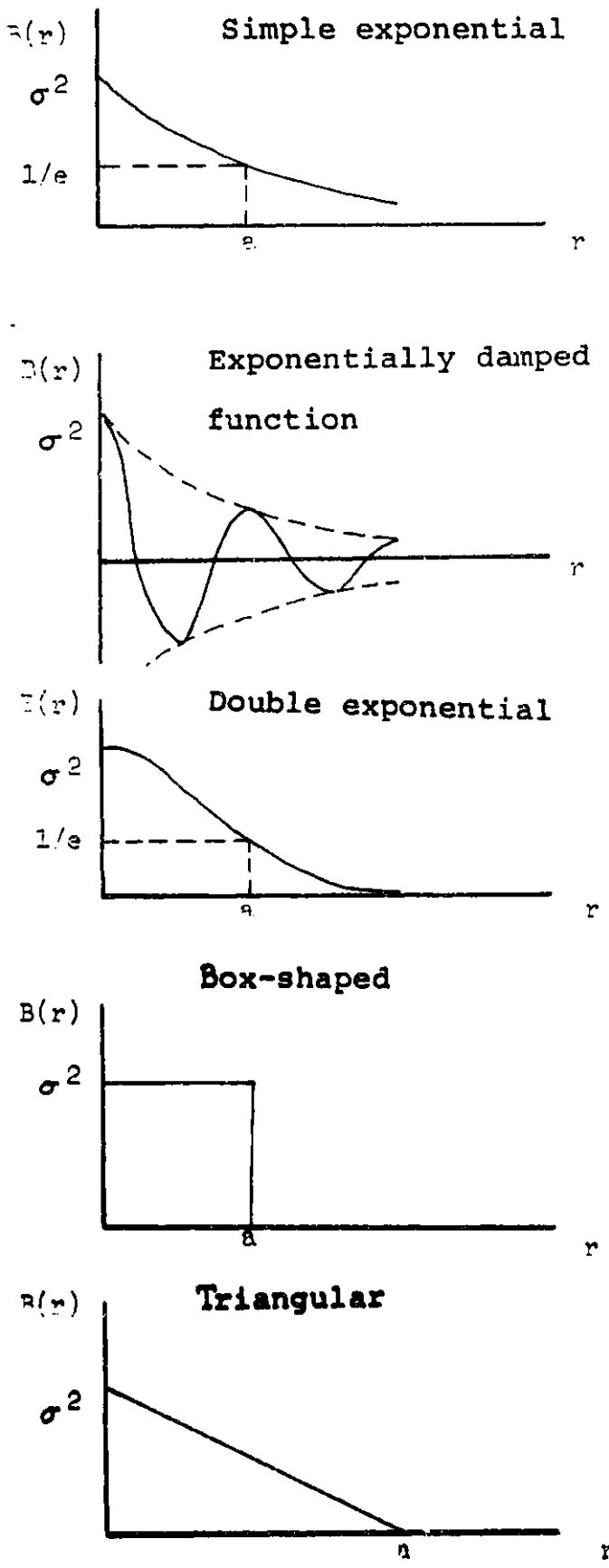

B(r)
Simple exponential
σ2
1/e
a
r
B(r)
Exponentially damped cosine
function
σ2
r
B(r)
Double exponential
σ2
1/e
a
r
Box-shaped
B(r)
σ2
a
r
B(r)
Triangular
σ2
a
r

FIGURE 2.1

and the triangular covariance function

$$B(r) = \begin{cases} \sigma^2(1-\frac{r}{a}) \text{ if } r \leq a,\ a > 0 \\ 0 \text{ otherwise} \end{cases} \tag{2.21}$$

However, these two models are not proper for random fields of dimension two or three.

2.5 GAUSSIAN RANDOM FIELDS

The mean and the covariance function are in general not sufficient to specify completely (in the sense previously discussed) a random field. However, if the field is Gaussian, i.e. if all the distribution functions specifying it are Gaussian distributions, then the mean and the covariance function are indeed enough for the complete specification of the field. The central limit theorem provides some justification for the assumption that the fields encountered in some applications are Gaussian.

2.6 ERGODICITY

We define the operator < > as follows:*

$$< f(\underline{t}) > = \lim_{D^n \to R^n} \frac{1}{D^n} \int_{D^n} f(\underline{t})\, d\underline{t} \tag{2.22}$$

where D^n is a subset of the space R^n and $f(\underline{t})$ is a function. This operator produces spatial averages, whereas, by comparison, the operator E produces ensemble averages. A homogeneous

---

* The symbol $\int_{D^n}$ is used instead of $\underbrace{\int \cdots \int}_{n \text{ times}}$ and the symbol $d\underline{t}$ is used instead of $dt_1 \cdots dt_n$. Also D_n stands for both the region and its measure, i.e. its length, area or volume, whichever is appropriate.

random field is called ergodic if these two operators produce the same results. Thus for an ergodic random field we will have, by analogy to (2.4) and (2.5):

$$m = \langle x(\underline{t}) \rangle \tag{2.23}$$

$$B(\underline{\tau}) = \langle (x(\underline{t})-m)(x(\underline{t}+\underline{\tau})-m) \rangle \tag{2.24}$$

for any realization $x(\underline{t})$ of the random field $X(\underline{t})$. In most cases random fields occuring in practice are assumed to be ergodic.

2.7 PROPERTIES IN THE WEAK VS. PROPERTIES IN THE STRONG SENSE

If a random field has any one of the previously mentioned properties (homogeneity, isotropy, ergodicity) in accordance with the respective definitions, we will say that the field has this property in the strong sense. If, however, this applies to the mean and the covariance function but not necessarily to other descriptors of the random field we will say that the field has this property in the weak sense. In what follows we will restrict our attention to fields described by their mean and covariance, so only weak homogeneity, isotropy and ergodicity will be of interest.

2.8 SPECTRAL REPRESENTATION OF RANDOM FIELDS

This section will be concerned with weakly homogeneous random fields. If $X(\underline{t})$ is such a field, we can write

$$X(\underline{t}) = \int_{R^n} \cos(\underline{\omega}\cdot\underline{t})\, Z(d\underline{\omega}) + m \tag{2.25}$$

where $\underline{\omega}\cdot\underline{t}$ denotes the dot product of the two vectors $\underline{\omega}$ and

$\underline{t}$. In this formula $Z(d\underline{\omega})$ is an uncorrelated random function with the following properties:

For all intervals $\Delta\underline{\omega}$

$$E[Z(\Delta\underline{\omega})] = 0 \tag{2.26}$$

For any disjoint intervals $\Delta_1\underline{\omega}$ and $\Delta_2\underline{\omega}$

$$Z(\Delta_1\underline{\omega} + \Delta_2\underline{\omega}) = Z(\Delta_1\underline{\omega}) + Z(\Delta_2\underline{\omega}) \tag{2.27}$$

and

$$E[Z(\Delta_1\underline{\omega})\ Z(\Delta_2\underline{\omega})] = 0 \tag{2.28}$$

On the other hand, if $B(\underline{\tau})$ falls off sufficiently rapidly at infinity, we can write

$$B(\underline{\tau}) = \int_{R^n} \cos\ (\underline{\omega}\cdot\underline{\tau})\ S(\underline{\omega})d\underline{\omega} \tag{2.29}$$

and

$$S(\underline{\omega}) = \frac{1}{(2\pi)^n} \int_{R^n} \cos(\underline{\omega}\cdot\underline{\tau})\ B(\underline{\tau})\ d\underline{\tau} \tag{2.30}$$

The function $S(\underline{\omega})$ is called the spectral density function of the random field. The two functions $Z(d\underline{\omega})$ and $S(\underline{\omega})$ are related to each other by the equation

$$S(\underline{\omega})d\underline{\omega} = E[Z^2(d\underline{\omega})] \tag{2.31}$$

This can be seen if one substitutes $X(0)$ and $X(\tau)$ from (2.25) into the equation

$$B(\underline{\tau}) = E[X(0)\ X(\tau)] - m^2 \tag{2.32}$$

Using (2.25) one arrives at

$$B(\underline{\tau}) = \int_{R^n} \cos(\underline{\omega}\cdot\underline{\tau})\ E[Z^2(d\underline{\omega})] \tag{2.33}$$

and equation (2.31) follows, since for each function $B(\underline{\tau})$ there is only one function $S(\underline{\omega})$ satisfying (2.29) and (2.30).

The function $S(\underline{\omega})$ is real and even. For isotropic fields $S(\underline{\omega})$ depends only on $\omega = |\underline{\omega}|$, hence we can write $S(\omega)$ instead of $S(\underline{\omega})$.

An interpretation of $S(\underline{\omega})$ is provided by (2.25) and (2.31). We can imagine $X(\underline{t})$ to be the result of superimposing to its mean value many functions of the form $\cos(\underline{\omega}\cdot\underline{t})$, each with a different $\underline{\omega}$ and each multiplied by a random amplitude $Z(d\underline{\omega})$ with zero mean and variance $S(\underline{\omega})\, d\underline{\omega}$. The chapter on simulation will use and clarify this observation.

2.9 LINEAR TRANSFORMATIONS OF RANDOM FIELDS

Consider a homogeneous random field $X(t)$ and let $Y(t)$ be another random field defined by

$$Y(\underline{t}) = \mathcal{L}[X(\underline{t})] \tag{2.34}$$

where $\mathcal{L}[\ ]$ is a linear operator independent of $\underline{t}$. We define the complex function $\ell(\underline{\omega})$ by the equation

$$\mathcal{L}[e^{i\underline{\omega}\underline{t}}] = \ell(\underline{\omega})e^{i\underline{\omega}\underline{t}} \tag{2.35}$$

Let

$$L(\underline{\omega}) = |\ell(\underline{\omega})|^2$$

Then the spectral density function $S_Y(\underline{\omega})$ of $Y(\underline{t})$ is given in terms of the spectral density function $S_X(\underline{\omega})$ of $X(\underline{t})$ by the equation

$$S_Y(\underline{\omega}) = L(\underline{\omega})\ S_X(\underline{\omega}) \tag{2.36}$$

Equation (2.36) is very useful since it enables us to find the spectral density function of random fields derived from a random field by integration, differentation, etc.

CHAPTER 3

SOME USEFUL APPROXIMATIONS

In the previous chapter it has been shown how it is possible to describe a random field in terms of its covariance function. Knowledge of the covariance function of a random field enables us to solve a great variety of problems. However, in many cases the form of the covariance function is not known, and this is certainly the case in geotechnical engineering where due to practical and economic limitations the available data are insufficient for the determination of covariance functions. The engineer is then faced with two options. The first is to assume a form for the covariance function, try to adjust it to the limited data and use the resulting covariance function together with the exact formulae of the theory of random fields to arrive at the results he wants. The second is to make reasonable assumptions about the results themselves, taking, of course, into account whatever data are available. The former approach has the disadvantages that it gives a false impression of accuracy and that it involves quite complicated computations. The latter avoids both of these. In what follows we will show how this is done.

3.1 THE VARIANCE OF AVERAGES OF RANDOM PROCESSES

Most of the material in this section appears in Vanmarcke (1977 , 1979).

Consider a zero-mean stationary random process $X(t)$ with covariance function $B(\tau)$. Let Y_T be the average of $X(t)$

over the length T, i.e.

$$Y_T = \frac{1}{T} \int_0^T X(t)dt \tag{3.1}$$

Clearly Y_T is a random variable. We are interested in determining the variance $Var[Y_T]$ of Y_T. It can be easily shown that

$$Var[Y_T] = \frac{1}{T^2} \int_0^T \int_0^T B(\tau_1-\tau_2)d\tau_1 d\tau_2 \tag{3.2}$$

or

$$Var\ [Y_T] = \frac{2\sigma^2}{T} \int_0^T (1-\frac{\tau}{T})\rho(\tau)d\tau \tag{3.3}$$

where

$$B(\tau) = \sigma^2 \rho(\tau) \tag{3.4}$$

Equation (3.3) can be written in the form

$$Var[Y_T] = \sigma^2 \Gamma^2(T) \tag{3.5}$$

If the covariance function is known, $\Gamma^2(T)$ can be determined as

$$\Gamma^2(T) = \frac{2}{T} \int_0^T (1-\frac{\tau}{T})\rho(\tau)d\tau \tag{3.6}$$

For example, if $\rho(\tau)$ is exponential, i.e.

$$\rho(\tau) = e^{-\frac{|\tau|}{a}} \tag{3.7}$$

then

$$\Gamma^2(T) = (\frac{a}{T})^2 \left[2\left(\frac{T}{a} - 1 + e^{-\frac{T}{a}}\right)\right] \tag{3.8}$$

If $\rho(\tau)$ if double exponential, i.e.

$$\rho(\tau) = e^{-(\frac{\tau}{a})^2} \tag{3.9}$$

then

$$\Gamma^2(T) = \left(\frac{a}{T}\right)^2 \left[\frac{T}{a}\sqrt{\pi}\ \mathrm{erf}\ \left(\frac{T}{a}\right) + e^{-\left(\frac{T}{a}\right)^2} - 1\right] \qquad (3.10)$$

An alternative way to arrive at the same results is the frequency domain approach. If $S(\omega)$ is the spectral density function of $X(t)$ then, using the results presented in Section 2.9, we get the formula

$$\mathrm{Var}[Y_T] = \int_{-\infty}^{+\infty} \left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2 S(\omega)\,d\omega \qquad (3.11)$$

hence

$$\Gamma^2(T) = \int_{-\infty}^{+\infty} \left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2 s(\omega)\,d\omega \qquad (3.12)$$

where

$$s(\omega) = \frac{S(\omega)}{\sigma^2} \qquad (3.13)$$

i.e.

$$s(\omega) = \frac{1}{2\pi}\int_{-\infty}^{-\infty} \cos(\omega\tau)\,\rho(\tau)\,d\tau \qquad (3.14)$$

and

$$\rho(\tau) = \int_{-\infty}^{-\infty} \cos(\omega\tau)\ s(\omega)\,d\omega \qquad (3.15)$$

Formulae (3.11) and (3.6) give the same results provided that $s(\omega)$ and $\rho(\tau)$ are derived from each other according to (3.14) and (3.15).

Because

$$\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2 \simeq \begin{cases} 1 \text{ if } T\to 0 \\ \\ \frac{2\pi\delta(\omega)}{T} \text{ if } T\to\infty \end{cases} \tag{3.16}$$

where δ is Dirac's δ-function, we have that

$$\Gamma^2(T) \simeq \begin{cases} \int_{-\infty}^{+\infty} s(\omega)\,d\omega = 1 \text{ if } T\to 0 \\ \\ \frac{2\pi s(0)}{T} \text{ if } T\to\infty \end{cases} \tag{3.17}$$

From (3.14) we get

$$s(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \rho(\tau)\,d\tau \tag{3.18}$$

Hence we can rewrite (3.17) as

$$\Gamma^2(T) \simeq \begin{cases} 1 \text{ if } T\to 0 \\ \\ \frac{\Theta}{T} \text{ if } T\to\infty \end{cases} \tag{3.19}$$

with

$$\Theta = 2\pi s(0) = \int_{-\infty}^{+\infty} \rho(\tau)\,d\tau \tag{3.20}$$

Equation (3.19) motivates the following approximation of $\Gamma^2(T)$:

$$\Gamma^2(T) \simeq \begin{cases} 1 \text{ if } T \leqq \Theta \\ \\ \frac{\Theta}{T} \text{ if } T \geqq \Theta \end{cases} \tag{3.21}$$

That this is indeed a good approxmiation can be seen from Figure 3.1 where equation (3.21) is compared with equations (3.8) and (3.10) and also with the equation

$$\Gamma^2(T) \simeq \begin{cases} 1 - \frac{T}{3\theta} \text{ if } T \leq \theta \\ \frac{\theta}{T}(1 - \frac{\theta}{3T}) \text{ if } T \geq \theta \end{cases} \qquad (3.22)$$

This last equation is exact if the correlation function of the process is triangular, i.e. if

$$\rho(\tau) = \begin{cases} 1 - \frac{|\tau|}{\theta} \text{ if } |\tau| \leq \theta \\ 0 \text{ otherwise} \end{cases} \qquad (3.23)$$

Also (3.21) is exact if the correlation function of the process is box-shaped, i.e. if

$$\rho(\tau) = \begin{cases} 1 \text{ if } |\tau| = \frac{\theta}{2} \\ 0 \text{ otherwise} \end{cases} \qquad (3.24)$$

Thus we see that either (3.21) or, better, (3.22) can be used as approximations for $\Gamma^2(T)$ with very good results, even if we do not know the exact form of $\rho(\tau)$. The only parameter which we need to know, apart from the variance σ^2 of the random process, is θ which is called scale of fluctuation. The physical interpretation of θ is very simple: two points at a distance to each other larger than θ are expected to have poorly correlated values of the random process, whereas two points at a distance to each other smaller than θ are expected to have highly correlated values of the random process.

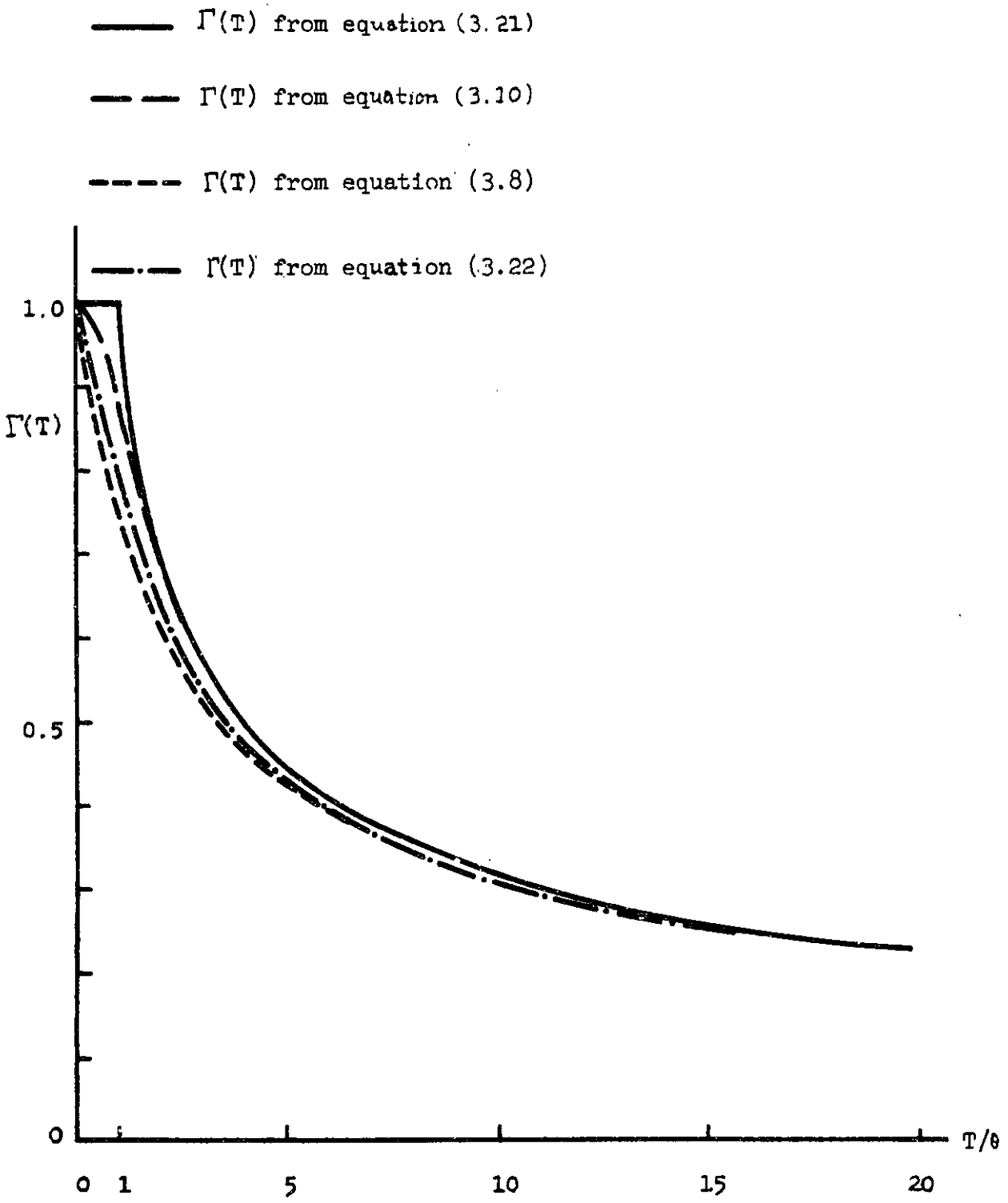

Γ(T) from equation (3.21)
Γ(T) from equation (3.10)
Γ(T) from equation (3.8)
Γ(T) from equation (3.22)
1.0
Γ(T)
0.5
0
0
1
5
10
15
20
T/θ

FIGURE 3.1

The approximations to $\Gamma^2(T)$ that have already been presented appear in Vanmarcke (1977) and were motivated from the exact expressions for $\Gamma^2(T)$ that correspond to certain types of correlation functions. An approximation to $\Gamma^2(T)$ which is not meant to be associated with a specific correlation function is:

$$\Gamma^2(T) = \exp\left\{ \frac{1}{2}\left[\ln \frac{\theta}{T} - \sqrt{\left(\ln \frac{\theta}{T}\right)^2 + \varepsilon^2} \right] \right\} \qquad (3.25)$$

This is a quite general approximation due to the presense of the parameter $\varepsilon \geq 0$ (see Fig. 3.1b). This parameter expresses how fast $\Gamma^2(T)$ decreases when T increases from 0 to ∞. This can be seen by evaluating (3.25) at $T = \theta$:

$$\Gamma^2(\theta) = \exp(-\frac{1}{2}\varepsilon) \qquad (3.26)$$

Therefore if $\varepsilon = 0$ then $\Gamma^2(\theta) = 1$. In this case (3.25) becomes

$$\Gamma^2(T) = \exp\left[\frac{1}{2}\left(\ln \frac{\theta}{T} - \left| \ln \frac{\theta}{T} \right| \right) \right] \qquad (3.27)$$

which is easily seen to be identical to (3.21). If $\varepsilon > 0$, then $\Gamma^2(\theta) < 1$. By appropriately selecting ε we can use (3.25) to approximate the exact $\Gamma^2(T)$ corresponding to any of the commonly used correlation functions. Of course the behavior of (3.25) when T is very small or very large is that described by (3.19).

3.2 THE VARIANCE OF AVERAGES OF RANDOM FIELDS

The results presented in the previous section can be

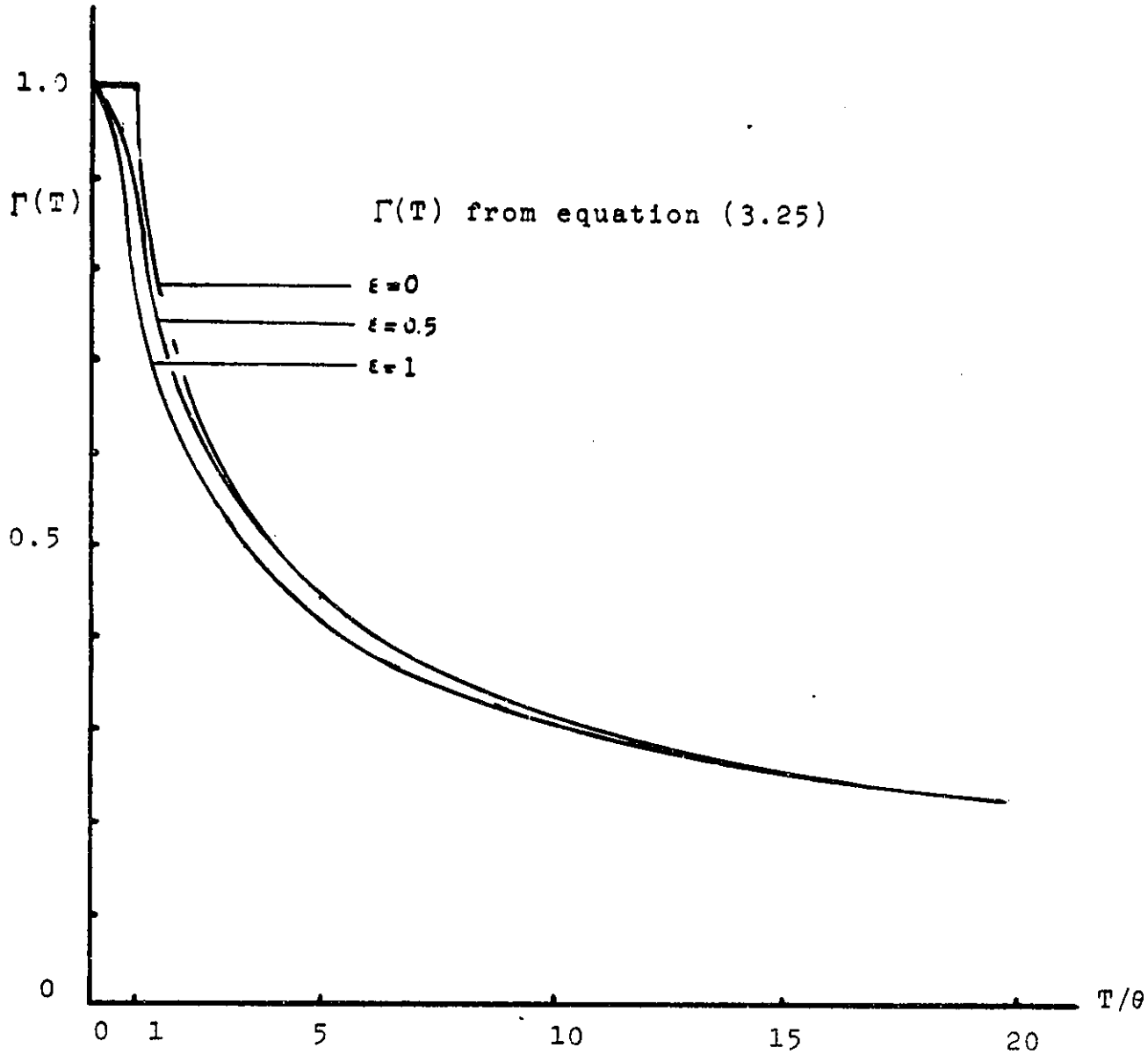


FIGURE 3.1b

generalized to two (and of course to more than two) dimensions. Consider a zero-mean homogeneous random field $X(t_1,t_2)$ with covariance function

$$B(\tau_1,\tau_2) = \sigma^2\rho(\tau_1,\tau_2) \tag{3.28}$$

and spectral density function $S(\omega_1,\omega_2)$ such that

$$S(\omega_1,\omega_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos(\omega_1\tau_1 + \omega_2\tau_2)\, B(\tau_1,\tau_2)\, d\tau_1 d\tau_2 \tag{3.29}$$

and

$$S(\tau_1,\tau_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos(\omega_1\tau_1 + \omega_2\tau_2)\, S(\omega_1,\omega_2)\, d\omega_1 d\omega_2 \tag{3.30}$$

Also let

$$s(\omega_1,\omega_2) = \frac{S(\omega_1,\omega_2)}{\sigma^2} \tag{3.31}$$

If Y_{T_1,T_2} is the average of $X(t_1,t_2)$ over a rectangle with sides T_1 and T_2, i.e., if

$$Y_{T_1,T_2} = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} X(t_1,t_2)dt_1dt_2 \tag{3.32}$$

then the variance $\mathrm{Var}[Y_{T_1,T_2}]$ can be found either from

$$\mathrm{Var}[Y_{T_1,T_2}] = \frac{4\sigma^2}{T_1T_2}\int_0^{T_1}\int_0^{T_2}(1-\frac{\tau_1}{T_1})(1-\frac{\tau_2}{T_2})\rho(\tau_1,\tau_2)d\tau_1d\tau_2 \tag{3.33}$$

or, using the results presented in section 2.9 from

$$\mathrm{Var}[Y_{T_1,T_2}] = \sigma^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{\sin\frac{\omega_1T_1}{2}}{\frac{\omega_1T_1}{2}}\right)^2\left(\frac{\sin\frac{\omega_2T_2}{2}}{\frac{\omega_2T_2}{2}}\right)^2 s(\omega_1,\omega_2)d\omega_1d\omega_2 \tag{3.34}$$

As in the one-dimensional case let

$$\theta_1 = \int_{-\infty}^{+\infty} \rho(\tau_1,0)d\tau_1 \tag{3.35}$$

and

$$\theta_2 = \int_{-\infty}^{+\infty} \rho(0,\tau_2)d\tau_2 \tag{3.36}$$

be the scales of fluctuation in the t_1 and the t_2 direction respectively. The two scales of fluctuation θ_1 and θ_2 are related to $s(\omega_1,\omega_2)$ by the equations

$$\theta_1 = 2\pi \int_{-\infty}^{+\infty} s(\omega_1,0)d\omega_1 \tag{3.37}$$

$$\theta_2 = 2\pi \int_{-\infty}^{+\infty} s(0,\omega_2)d\omega_2 \tag{3.38}$$

Let

$$\alpha = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{3.39}$$

Since

$$s(0,0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{3.40}$$

we have

$$\alpha = 4\pi^2\, s(0,0) \tag{3.41}$$

Observing that

$$\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2 \simeq \begin{cases} 1 \text{ if } T \to 0 \\ \frac{2\pi\delta(\omega)}{T} \text{ if } T \to \infty \end{cases} \tag{3.42}$$

We can arrive at interesting approximations of $\mathrm{Var}[Y_{T_1,T_2}]$, namely

$$\mathrm{Var}[Y_{T_1,T_2}] \simeq \begin{cases} \sigma^2 \text{ if } T_1 \to 0,\ T_2 \to 0 \\ \sigma^2 \dfrac{\theta_1}{T_1} \text{ if } T_1 \to \infty,\ T_2 \to 0 \\ \sigma^2 \dfrac{\theta_2}{T_2} \text{ if } T_1 \to 0,\ T_2 \to \infty \\ \sigma^2 \dfrac{\alpha}{T_1 T_2} \text{ if } T_1 \to \infty,\ T_2 \to \infty \end{cases} \tag{3.43}$$

Figure 3.2 compares the exact values of $\mathrm{Var}[Y_{T_1,T_2}]$ as obtained from (3.33) or (3.34) with the approximations introduced by (3.43) for the case where the correlation function is a simple exponential one.

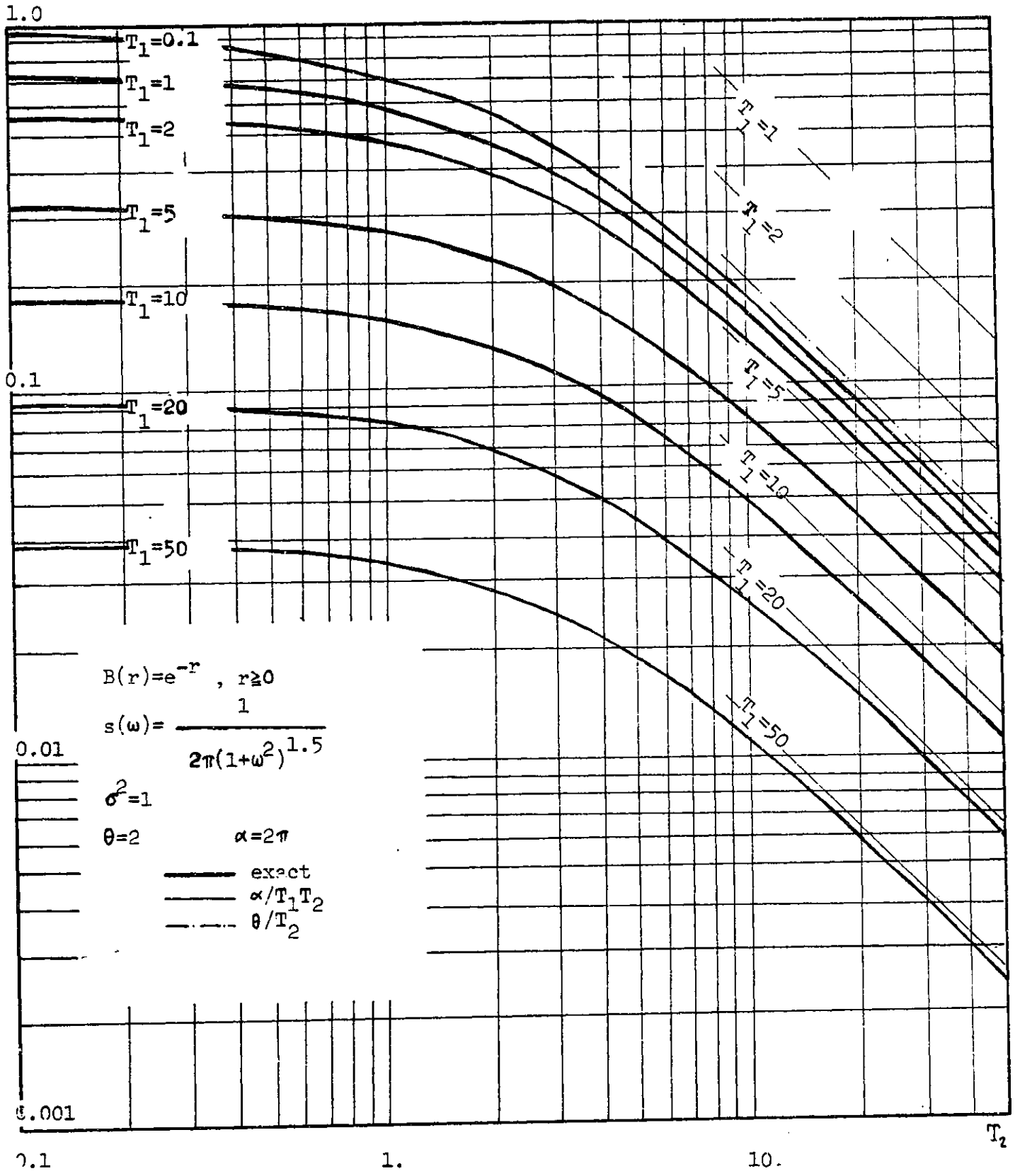

Var[$Y_{T_1T_2}$]
1.0
0.1
0.01
0.001
T_1=0.1
T_1=1
T_1=2
T_1=5
T_1=10
T_1=20
T_1=50
T_1=1
T_1=2
T_1=5
T_1=10
T_1=20
T_1=50
$B(r)=e^{-r}$, $r\geq 0$
$s(\omega)=\frac{1}{2\pi(1+\omega^2)^{1.5}}$
$\sigma^2=1$
$\theta=2$
$\alpha=2\pi$
exact
α/T_1T_2
θ/T_2
0.1
1.
10.
T_2

FIGURE 3.2

An approximate equation for finding $\Gamma^2(T_1,T_2)$ defined from

$$\mathrm{Var}[Y_{T_1,T_2}] = \sigma^2\ \Gamma^2(T_1,T_2) \tag{3.44}$$

is

$$\Gamma^2(T_1,T_2) = \Gamma^2(T_1)\ \Gamma^2(T_2) \tag{3.45}$$

where

$$\Gamma^2(T_1) = \exp\left\{ \frac{1}{2}\left[\ln \frac{\theta_1 k_1}{T_1} - \sqrt{\left(\ln \frac{\theta_1 k_1}{T_1}\right)^2 + \varepsilon_1^2}\ \right]\right\} \tag{3.46}$$

$$\Gamma^2(T_2) = \exp\left\{ \frac{1}{2}\left[\ln \frac{\theta_2 k_2}{T_2} - \sqrt{\left(\ln \frac{\theta_2 k_2}{T_2}\right)^2 + \varepsilon_2^2}\ \right]\right\} \tag{3.47}$$

and

$$k_1 = \frac{\sqrt{\alpha}}{\theta_1} - \left(\frac{\sqrt{\alpha}}{\theta_1} - 1\right) e^{-T_2/\theta_2} \tag{3.48}$$

$$k_2 = \frac{\sqrt{\alpha}}{\theta_2} - \left(\frac{\sqrt{\alpha}}{\theta_2} - 1\right) e^{-T_1/\theta_1} \tag{3.49}$$

It is easy to see that $\Gamma^2(T_1,T_2)$ as defined above exhibits the behavior described by (3.43). Also equation (3.25) can be considered as the special form of (3.45) when either $T_1 \to 0$ or $T_2 \to 0$. If the field is isotropic, then

$$\theta_1 = \theta_2 = \theta \quad , \quad \varepsilon_1 = \varepsilon_2 = \varepsilon \quad , \quad k_1 = k_2 = k \tag{3.50}$$

Notice that for an isotropic field k equals unity if

$$\alpha = \theta^2 \tag{3.51}$$

This is the case for the double exponential function. For the

simple exponential function

$$\alpha = 1.57\ \theta^2 \tag{3.52}$$

and (3.51) may be used as an approximation. In general, it may be assumed that α equals θ^2 whenever there is no reason to assume otherwise. The parameters k_1 and k_2 have been introduced in order to take the value of α into account in equation (3.45) and therefore no physical significance should be attached to them. Had k_1 and k_2 been ommitted, equation (3.45) would behave like $\theta_1\theta_2/T_1T_2$ instead of like α/T_1T_2 when $T_1 \to \infty$, $T_2 \to \infty$.

Equation (3.45) by no means represents the only possible approximation for $\mathrm{Var}[Y_{T_1,T_2}]$. Even though it is entirely artificial, it is justified on the basis of its limiting behavior and its flexibility.

The curves in Fig. 3.2 can be approximated by (3.45) with less than 10% error if ε is selected equal to unity.

3.3 THE COVARIANCE BETWEEN AVERAGES OF RANDOM FIELDS

Consider the problem of finding the covariance between the average Y_1 of a zero mean homogeneous two-dimensional random field $X(t_1,t_2)$ over a region D_1 and its average Y_2 over a region D_2. In other words, we want to find the covariance $\mathrm{Cov}[Y_1,Y_2]$ between

$$Y_1 = \frac{1}{D_1}\int_{D_1} X(t_1,t_2)\, dt_1\, dt_2 \tag{3.53}$$

and

$$Y_2 = \frac{1}{D_2}\int_{D_2} X(t_1,t_2)\, dt_1\, dt_2 \tag{3.54}$$

The solution to this problem is known to be:

$$\mathrm{Cov}[Y_1,Y_2] = \frac{1}{D_1 D_2}\int_{D_1}\int_{D_2} B(t_1 - t_1', t_2 - t_2')\, dt_1'\, dt_2'\, dt_1\, dt_2 \tag{3.55}$$

where D_1 and D_2 stand for the areas of the corresponding regions, (t_1,t_2) is a point in the region D_1, (t_1',t_2') is a point in the region D_2 and $B(t_1,t_2)$ is the covariance function of the random field (see Figure 3.3).

Evaluating the multiple integral in (3.55) is not a simple task. Besides, being much concerned with applying an exact formula of which the parameters (namely the covariance function) we do not know with great accuracy is contradictory. Hence, we will try to make assumptions about the output rather than the input of (3.55). To this end, consider the special case appear-

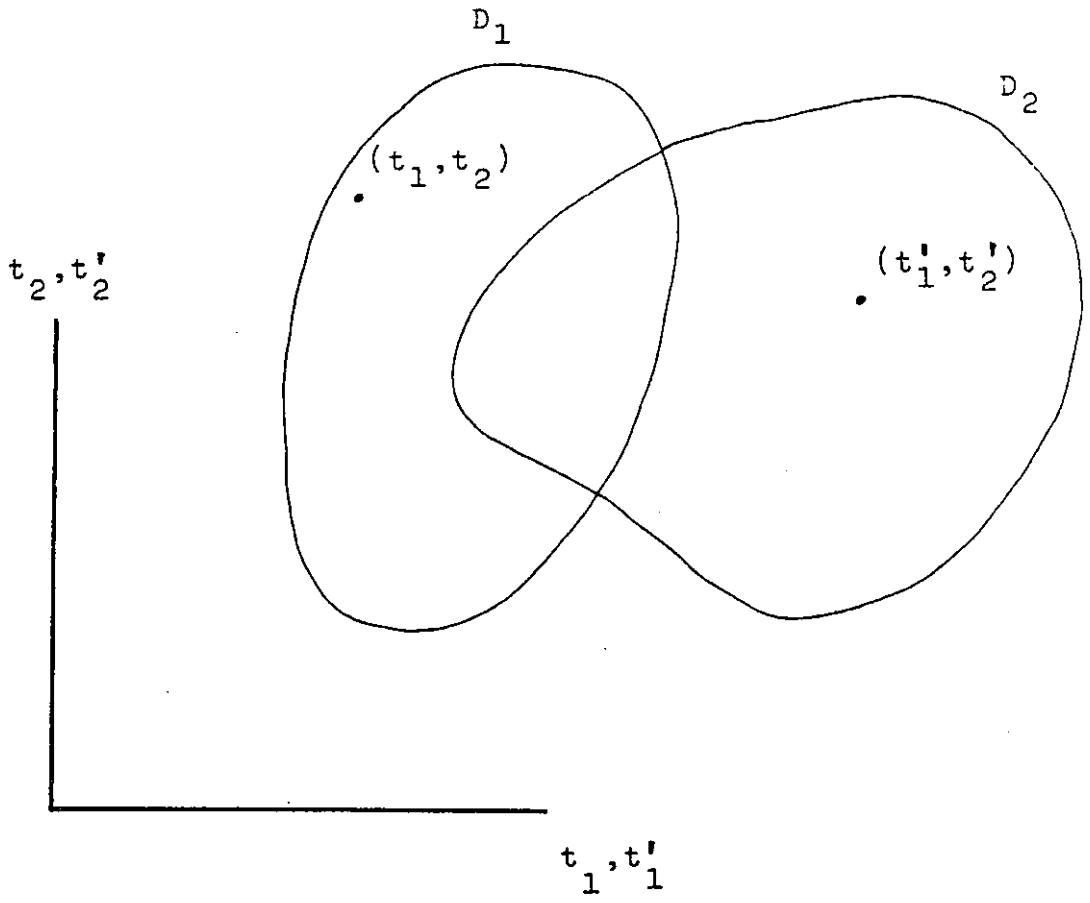


FIGURE 3.3

ing in Fig. 3.4, i.e., the case where the regions D_1 and D_2 are rectangles with parallel sides. It is possible to write:

$$(D_1Y_1)(D_2Y_2) + (D_3Y_3)(D_4Y_4) = \frac{1}{2}\sum_{i=1}^{16} k_i(\mathcal{D}_i\mathcal{Y}_i)^2 \qquad (3.56)$$

where the regions D_i, i = 1,2,3,4 are identified in Fig.3.4 and Y_i, i = 1,2,3,4 are the averages of the random field over these regions. The regions $\mathcal{D}_i$, i = 1, ... ,16 are all the regions having the forms appearing in Table 3.1 . In the same table, the coefficients k_i are given. By $\mathcal{Y}_i$, i = 1, ... ,16, we denote the averages of the random field over the regions $\mathcal{D}_i$, i = 1, ... , 16 respectively. The proof of (3.56) is straightforward if one expresses the products $\mathcal{D}_i\mathcal{Y}_i$, i = 1, ... ,16 in terms of the products D_iY_i, i = 1,2,3,4. Now consider again formula (3.55). This formula applied to the regions D_1 and D_2 of Fig. 3.4 becomes:

$$\mathrm{Cov}[Y_1,Y_2] = \frac{1}{D_1D_2}\int_{\gamma}^{\delta}\int_{a}^{b}\int_{\alpha}^{\beta}\int_{c}^{d} B(t_1 - t_1', t_2 - t_2')\, dt_1'\, dt_2'\, dt_1\, dt_2 \qquad (3.57)$$

The same formula applied to the regions D_3 and D_4 of Fig. 3.4 becomes:

$$\mathrm{Cov}[Y_3,Y_4] = \frac{1}{D_3D_4}\int_{\alpha}^{\beta}\int_{a}^{b}\int_{\gamma}^{\delta}\int_{c}^{d} B(t_1 - t_1', t_2 - t_2')\, dt_1'\, dt_2'\, dt_1\, dt_2 \qquad (3.58)$$

Obviously, it is always true that:

$$D_1D_2 = D_3D_4 \qquad (3.59)$$

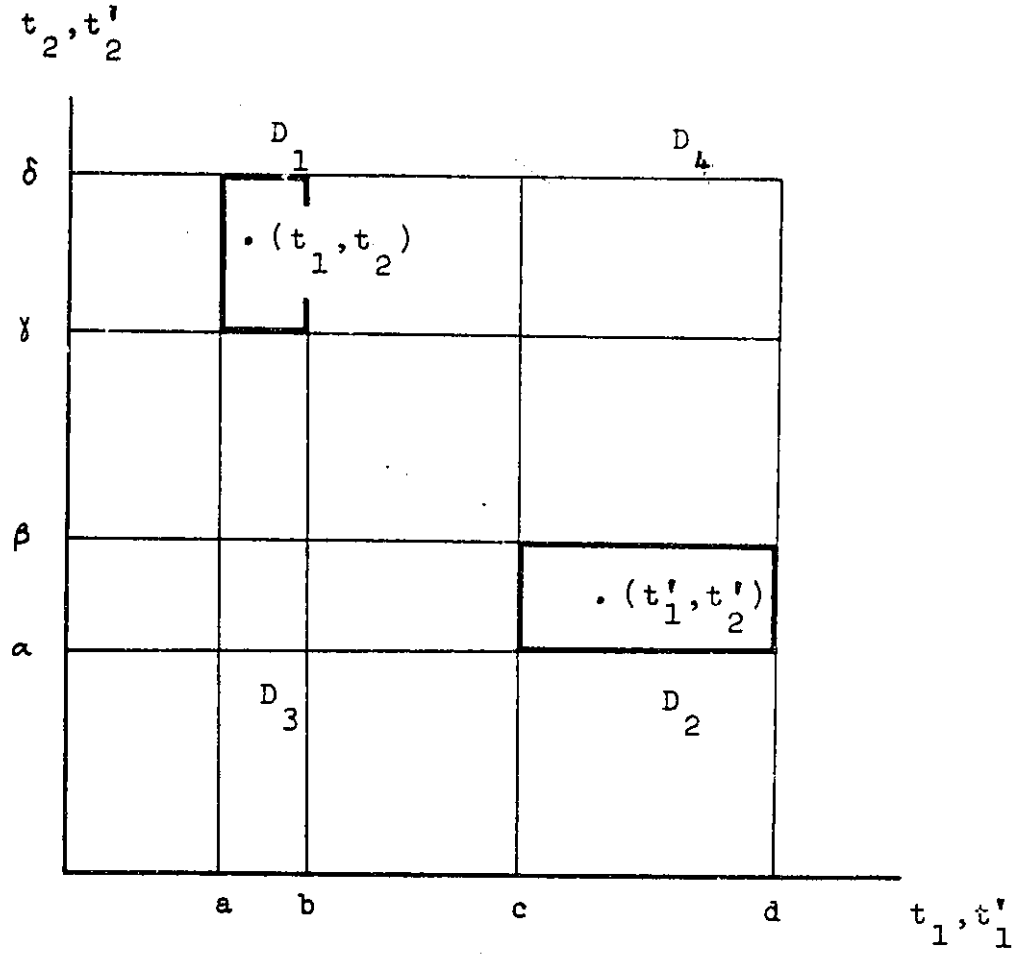


FIGURE 3.4

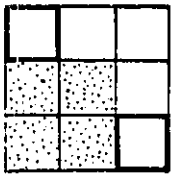
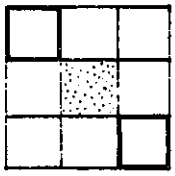
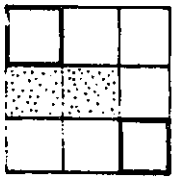
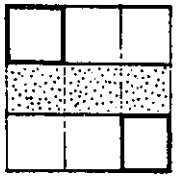
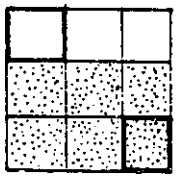
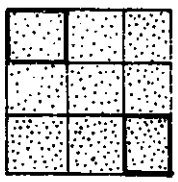
Form of $\mathcal{D}_i$	Number of regions with this form	Coefficient k_i
	4	1
	1	1
	4	-1
	2	1
	4	-1
	1	1

TABLE 3.1

On the other hand

$$B(t_1 - t_1', t_2 - t_2') = B(t_1 - t_1', t_2' - t_2) \tag{3.60}$$

Comparing (3.57) and (3.58) we see that

$$\mathrm{Cov}[Y_1, Y_2] = \mathrm{Cov}[Y_3, Y_4] \tag{3.61}$$

Using this and equation (3.56) we arrive at the following result

$$\mathrm{Cov}[Y_1, Y_2] = \frac{1}{4} \frac{\sum_{i=1}^{16} k_i \mathcal{D}_i^2 \mathrm{Var}[\gamma_i]}{D_1 D_2} \tag{3.62}$$

This is an exact formula if the exact values of $\mathrm{Var}[\gamma_i]$, i = 1, ... ,16 are used. However, we can write:

$$\mathrm{Var}[D] = \sigma^2 \Gamma^2(D) \tag{3.63}$$

as suggested in section 3.2. In (3.63) $\Gamma^2(D)$ is meant to show that Γ^2 depends on the region D and not only on its area. Note that regions and their areas are denoted by the same symbols. The coefficient of correlation between Y_1 and Y_2 can now be written as

$$\rho[Y_1, Y_2] = \frac{1}{4} \frac{\sum_{i=1}^{16} k_i \mathcal{D}_i^2 \Gamma^2(\gamma_i)}{D_1 D_2 \Gamma(D_1) \Gamma(D_2)} \tag{3.64}$$

Therefore, if we have an approximate expression for $\Gamma^2(D)$ we can find $\rho[Y_1, Y_2]$ from (3.64), a formula that involves only simple arithmetical operations as opposed to the integrations required when using (3.55). Of course, (3.64) can be used as an exact formula if the exact values of $\Gamma^2(D)$ are known, but its impor-

tance lies in providing us with a way to use approximate expressions for $\Gamma^2(D)$ to arrive at approximate values of $\rho[Y_1,Y_2]$.

It is possible to extend the above result to three dimensions. For this purpose, consider Fig. 3.5. We want to find an expression for $\mathrm{Cov}[Y_1,Y_2]$ where Y_1 and Y_2 are the averages of a three-dimensional field over two boxes with parallel sides, as in Fig. 3.5. Again we start from the observation that we can write

$$(D_1Y_1)(D_2Y_2) + (D_3Y_3)(D_4Y_4) + (D_5Y_5)(D_6Y_6) + (D_7Y_7)(D_8Y_8) =$$

$$\frac{1}{2}\sum_{i=1}^{64} k_i(\vartheta_i\varphi_i)^2 \qquad (3.65)$$

See Fig. 3.5 and Table 3.2 for identification of the regions involved. Using an argument similar to that used in the two-dimensional case, we can show that

$$\mathrm{Cov}[Y_1,Y_2] = \mathrm{Cov}[Y_3,Y_4] = \mathrm{Cov}[Y_5,Y_6] = \mathrm{Cov}[Y_7,Y_8] \qquad (3.66)$$

Also,

$$D_1D_2 = D_3D_4 = D_5D_6 = D_7D_8 \qquad (3.67)$$

Therefore

$$\mathrm{Cov}[Y_1,Y_2] = \frac{1}{8}\,\frac{\sum_{i=1}^{64} k_i \mathcal{D}_i^2\,\mathrm{Var}[\varphi_i]}{D_1D_2} \qquad (3.68)$$

The one-dimensional form of the above results appears in Vanmarcke (1977).

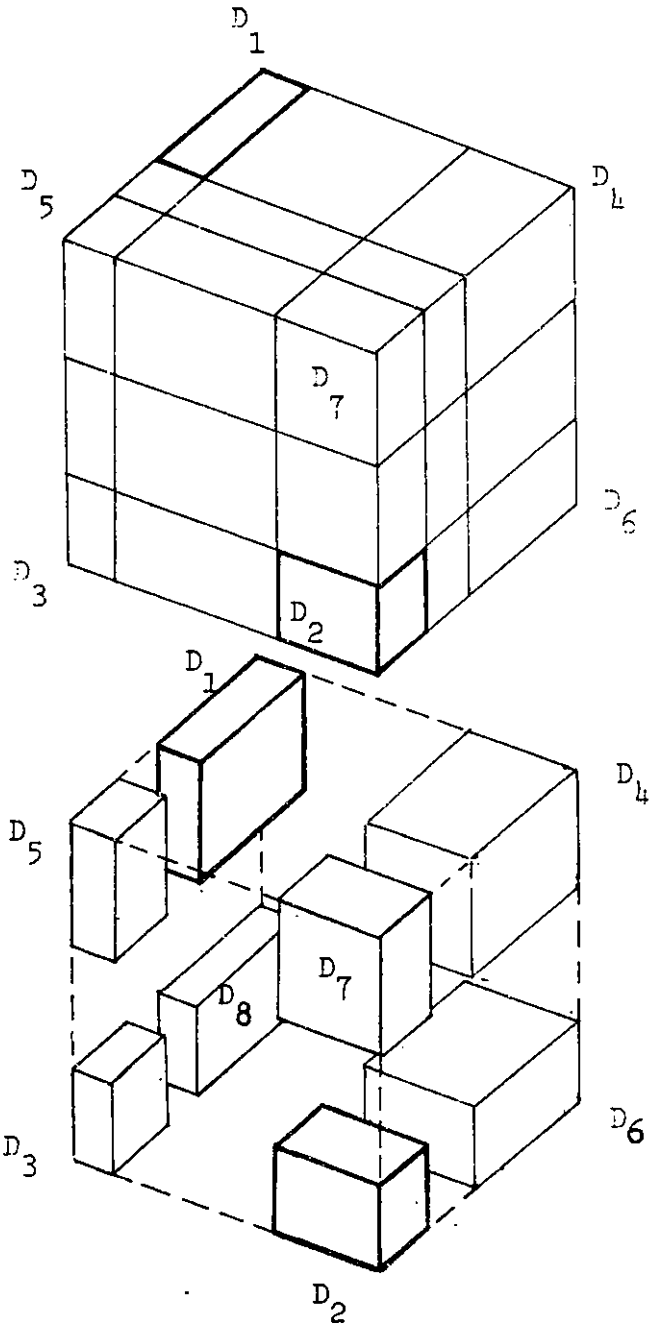


FIGURE 3.5

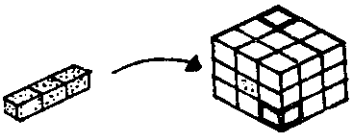
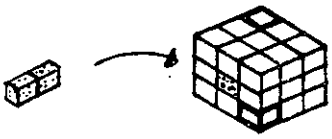
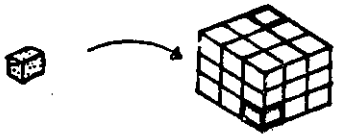
Form of $\mathcal{R}_i$	Number of regions with this form	Coefficient k_i
	1	1
	6	-1
	12	1
	3	1
	8	-1
	12	-1
	12	1
	3	1
	6	-1
	1	1

TABLE 3.2

Again, given an exact or approximate expression for $\Gamma^2(D)$ defined by

$$\mathrm{Var}[D] = \sigma^2 \, \Gamma^2(D) \tag{3.69}$$

we may write

$$\rho(Y_1,Y_2) = \frac{1}{8} \frac{\sum_{i=1}^{64} k_i \mathcal{D}_i^2 \, \Gamma^2(\mathcal{D}_i)}{D_1 D_2 \, \Gamma(D_1) \, \Gamma(D_2)} \tag{3.70}$$

Table 3.3 shows the general expressions for $\rho(Y_1,Y_2)$ when:

a. Y_1 and Y_2 are the averages of a random process over two intervals D_1 and D_2. See Figure 3.6a.

b. Y_1 and Y_2 are the averages of a two-dimensional random field over two rectangles with areas D_1 and D_2 and parallel sides. See Figure 3.6b.

c. Y_1 and Y_2 are the averages of a three-dimensional random field over two boxes with volumes D_1 and D_2 and parallel sides. See Figure 3.6c.

These expressions are of course valid for any position of the intervals, rectangles or boxes relative to each other and the same is true for all the expressions in this section. The formulae in Table 3.3 are just another way to write the already presented formulae, but have a form that makes them suitable to use with a programmable calculator, since given the coordinates of the points involved they provide the required correlation in a straightforward manner.

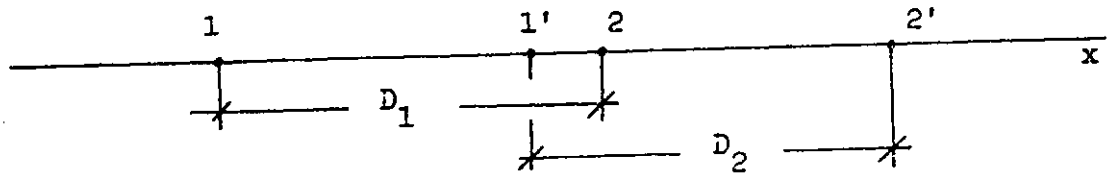
$$\rho(Y_1,Y_2) = -\frac{1}{2}\frac{\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^i(-1)^j(x_i-x'_j)^2\Gamma^2(|x_i-x'_j|)}{D_1D_2\Gamma(D_1)\Gamma(D_2)} \qquad (a)$$

$$\rho(Y_1,Y_2) = \frac{1}{4}\frac{\sum_{i=1}^{4}\sum_{j=1}^{4}(-1)^i(-1)^j[(x_i-x'_j)(y_i-y'_j)]^2\Gamma^2(|x_i-x'_j|,|y_i-y'_j|)}{D_1D_2\Gamma(D_1)\Gamma(D_2)} \qquad (b)$$

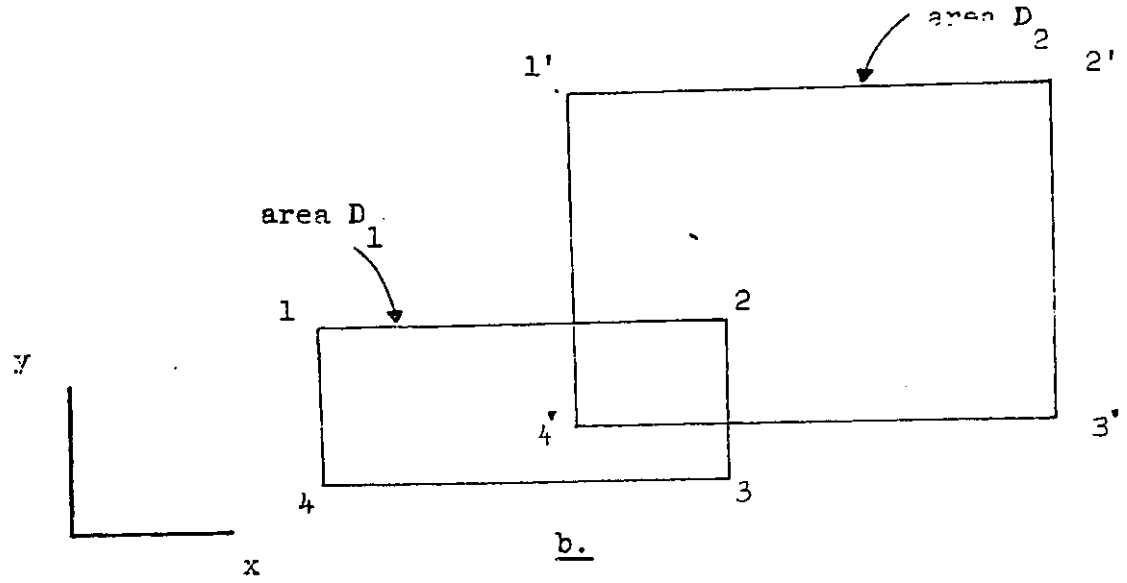
$$\rho(Y_1,Y_2) = \frac{1}{8}\frac{\sum_{i=1}^{8}\sum_{j=1}^{8}(-1)^i(-1)^j(-1)^{\delta_{ij}}[(x_i-x'_j)(y_i-y'_j)(z_i-z'_j)]^2\Gamma^2(|x_i-x'_j|,|y_i-y'_j|,|z_i-z'_j|)}{D_1D_2\Gamma(D_1)\Gamma(D_2)} \qquad (c)$$

δ_{ij} is Kronecker's delta

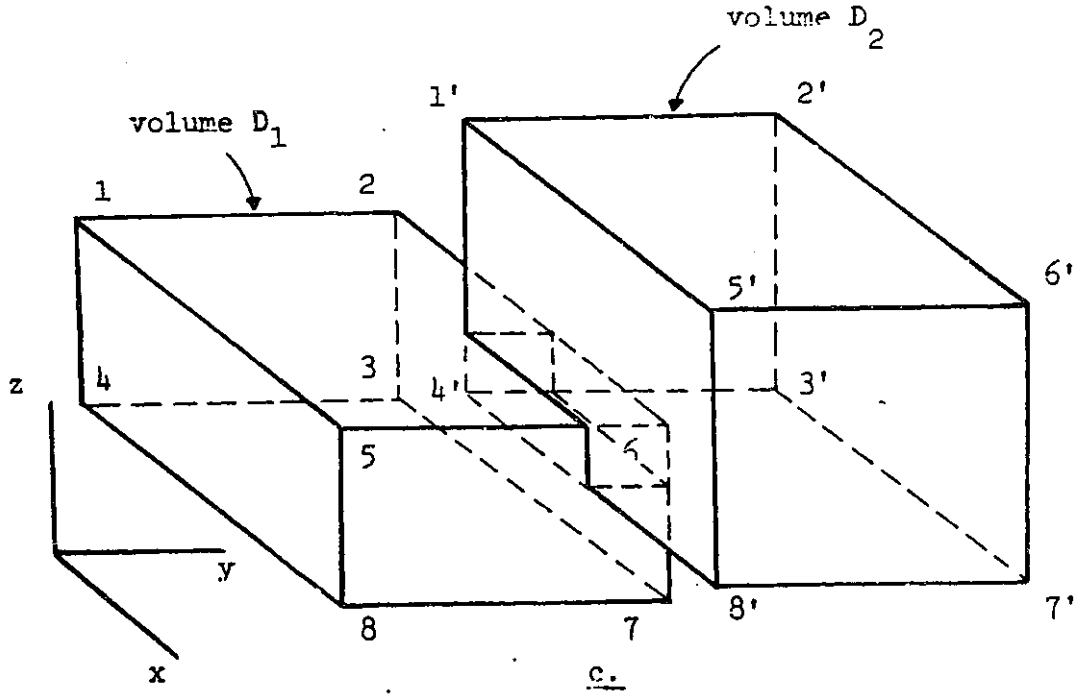
TABLE 3.3



a.



b.



c.

FIGURE 3.6

3.4 CONCLUSIONS

In this chapter, we have seen how it is possible to arrive at important results using a very simple description of a random field. Their applicability to several problems will be shown in Chapter 5. Only three parameters, σ^2, θ and α (and recall that we may assume that $\alpha = \theta^2$, which leaves only two parameters) can provide a satisfactory approximate description of a two-dimensional random field, thus substituting the covariance function. As it has been mentioned in the beginning of this chapter, this both simplifies the computations the engineer has to make and recognizes the fact that the limited data from soil exploration programs do not permit the engineer to determine covariance functions. On the other hand, it is hoped that for each soil type and for each soil property it will be possible to establish σ^2 and θ once and for all. For each site the engineer would then have only to determine the spatially varrying mean $m(\underline{t})$ of any soil property. Then he would use σ^2 and θ to describe the homogeneous and zero-mean field $X(\underline{t}) - m(\underline{t})$, where $X(\underline{t})$ is the soil property considered.

CHAPTER 4

SIMULATION OF RANDOM FIELDS

In this chapter the method of simulation of random fields as presented by Shinozuka and Jan (1972) is given. We will deal with zero-mean fields but clearly this does not restrict the general validity of the results.

Let $X(\underline{t})$ be a n-dimensional homogeneous random field with zero mean, covariance function $B(\underline{\tau})$ and spectral density function $S(\underline{\omega})$. According to (2.25) we may write:

$$X(\underline{t}) = \int_{R^n} \cos(\underline{\omega} \cdot \underline{t}) \; Z(d\underline{\omega}) \tag{4.1}$$

Recall also that

$$E[Z^2(d\underline{\omega})] = S(\underline{\omega}) \; d\underline{\omega} \tag{4.2}$$

We will show that $X(\underline{t})$ may be written as:

$$X(\underline{t}) = \sqrt{2} \int_{R^n} \cos[\underline{\omega} \cdot \underline{t} + \Phi(\underline{\omega})] \; \sqrt{S(\underline{\omega}) \; d\underline{\omega}} \tag{4.3}$$

where $\Phi(\underline{\omega})$ is a random angle uniformly distributed between 0 and 2π such that $\Phi(\underline{\omega}_1)$ and $\Phi(\underline{\omega}_2)$ are uncorrelated whenever $\underline{\omega}_1 \neq \underline{\omega}_2$. In order to do this we have to show that $X(\underline{t})$ as given by (4.3) has mean equal to zero and covariance function equal to $B(\underline{\tau})$. This can be done easily by using the definitions of the mean and the covariance function and observing that

$$E[\cos[\underline{\omega} \cdot \underline{t} + \Phi(\underline{\omega})]] = 0 \tag{4.4}$$

and

$$E[\cos[\underline{\omega}_1 \cdot \underline{t}_1 + \Phi(\underline{\omega}_1)] \cos[\underline{\omega}_2 \cdot \underline{t}_2 + \Phi(\underline{\omega}_2)]]$$

$$= \begin{cases} 0 \text{ if } \underline{\omega}_1 \neq \underline{\omega}_2 \\ \frac{1}{2} \cos \underline{\omega} \cdot (\underline{t}_1 - \underline{t}_2) \text{ if } \underline{\omega}_1 = \underline{\omega}_2 = \underline{\omega} \end{cases} \tag{4.5}$$

Thus we obtain

$$E[X(\underline{t})] = 0 \tag{4.6}$$

and

$$E[X(\underline{t}_2)\ X(\underline{t}_1)] =$$

$$= 2 \int_{R^n} \int_{R^n} E[\cos[\underline{\omega}_1 \cdot \underline{t}_1 + \Phi(\underline{\omega}_1)] \cos[\underline{\omega}_2 \cdot \underline{t}_2 + \Phi(\underline{\omega}_2)]] \sqrt{S(\underline{\omega}_1)\, d\underline{\omega}_1 S(\underline{\omega}_2)\, d\underline{\omega}_2}$$

$$= \int_{R^n} \cos[\underline{\omega} \cdot (\underline{t}_2 - \underline{t}_1)]\ S(\underline{\omega})\ d\underline{\omega} = B(\underline{t}_2 - \underline{t}_1) \tag{4.7}$$

Equations (4.4) to (4.7) are also true if we average spatially instead of averaging over the ensemble, i.e. if we write $\langle\ \rangle$ instead of $E[\ ]$, hence $X(\underline{t})$ as given by (4.3) is also ergodic.

Assume now that the spectral density function $S(\underline{\omega})$ has insignificant magnitude outside the region defined by

$$-\underline{\Omega} \leq \underline{\omega} \leq \underline{\Omega} \tag{4.8}$$

and let

$$\Delta\omega_i = \frac{2\Omega_i}{N_i} \quad , \ i = 1, \ \ldots \ , n \tag{4.9}$$

where N_i is the number of intervals $\Delta\omega_i$ in the ith direction of

the frequency domain. In this case we may use (4.3) to arrive at the following approximation $\hat{X}(\underline{t})$ of $X(\underline{t})$:

$$\hat{X}(\underline{t}) = \sqrt{2} \sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} \sqrt{S(\omega_{1k_1}, \ldots, \omega_{nk_n})\, \Delta\omega_1 \cdots \Delta\omega_n}$$

$$\cos(\omega_{1k_1} t_1 + \ldots + \omega_{nk_n} t_n + \ldots + \Phi_{k_1,\ldots,k_n}) \tag{4.10}$$

where $\Phi_{k_1, \ldots, k_n}$ is an independent random angle uniformly distributed between 0 and 2π and

$$\omega_{ik_i} = -\Omega_i + \left(k_i - \frac{1}{2}\right)\Delta\omega_i, \quad k_i = 1,\ldots N_i, \quad i = 1\ldots n \tag{4.11}$$

Equation (4.10) can be used directly for simulating $X(\underline{t})$ by generating a series of random angles $\Phi_{k_1,\ldots,k_n}$. The random fields generated using (4.10) are zero-mean, ergodic and have covariance function

$$B(\underline{\tau}) = \sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} \cos(\omega_{1k_1}\tau_1 + \ldots + \omega_{nk_n}\tau_n)$$

$$S(\omega_{1k_1}, \ldots, \omega_{nk_n})\, \Delta\omega_1 \cdots \Delta\omega_n \tag{4.12}$$

i.e., they have the same structure as the field $X(\underline{t})$ when $N_i \to \infty$, $\Omega_i \to \infty$, $i = 1, \ldots, n$. An additional feature of the random fields generated using (4.10) is that they are periodic with period $\pi/\Delta\omega_i$ in the ith direction. However, we can eliminate this periodicity by making the frequency domain grid slightly irregular, i.e. by adding a small random frequency

$\delta\omega_i$ to ω_{ik_i}, i = 1, ... ,n where $\delta\omega_i$ is uniformly distributed between $-\Delta\omega_i'/2$ and $\Delta\omega_i'/2$ and $\Delta\omega_i' << \Delta\omega_i$.

In the case of a two-dimensional random field, equation (4.3) becomes

$$X(t_1,t_2) = \sqrt{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cos[\omega_1 t_1 + \omega_2 t_2 + \Phi(\omega_1,\omega_2)] \sqrt{S(\omega_1,\omega_2)\, d\omega_1 d\omega_2} \tag{4.13}$$

and (4.10) becomes

$$\hat{X}(t_1,t_2) = \sqrt{2} \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \sqrt{S(\omega_{1k_1},\omega_{2k_2})\, \Delta\omega_1 \Delta\omega_2}\ \cos(\omega_{1k_1} t_1 + \omega_{2k_2} t_2 + \Phi_{k_1,k_2}) \tag{4.14}$$

Figure 4.1 shows the frequency domain grid used in 4.14. If the field is isotropic then the summations in (4.14) can be limited only to points with $\omega_{1k_1} > 0$ and $\omega_{2k_2} > 0$ if we substitute the $\sqrt{2}$ in front by $2\sqrt{2}$.

A computer program for simulating two dimensional isotropic random fields is listed in the Appendix.

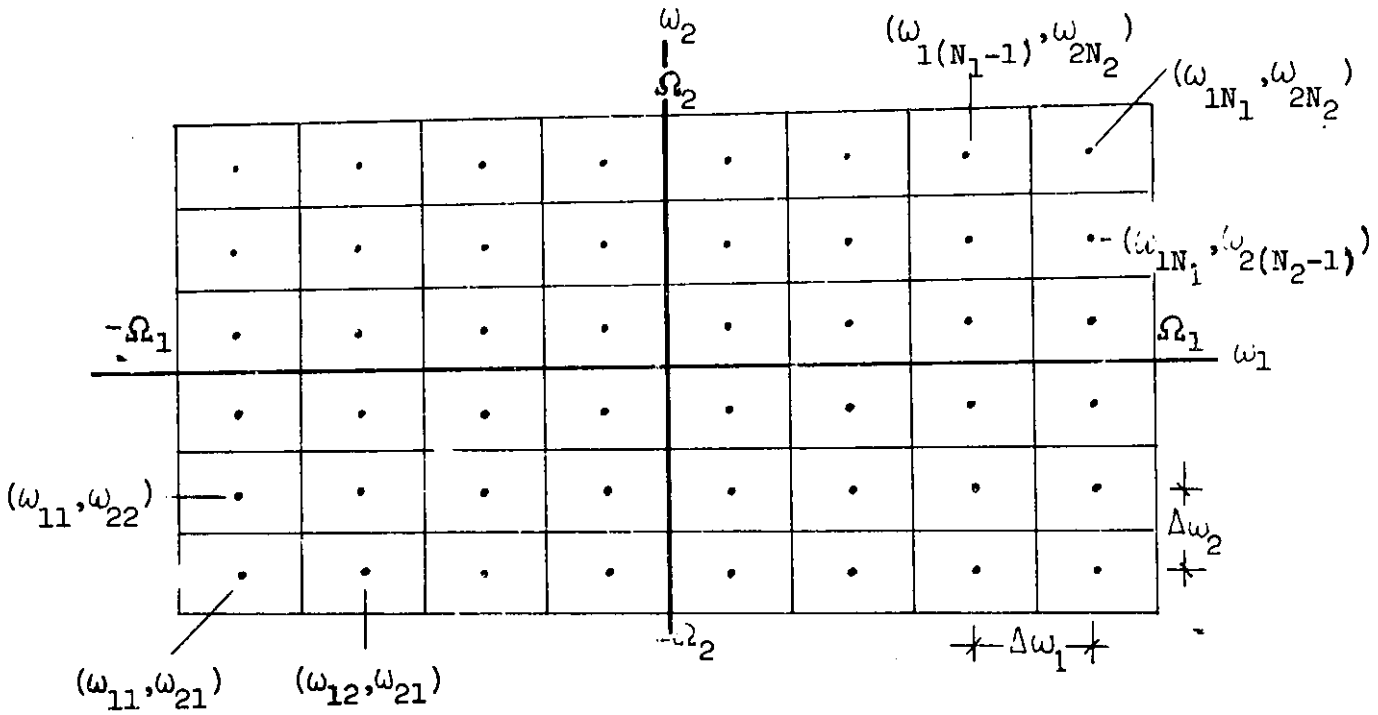

ω_2
Ω_2
$(\omega_{1(N_1-1)}, \omega_{2N_2})$
$(\omega_{1N_1}, \omega_{2N_2})$
$(\omega_{1N_1}, \omega_{2(N_2-1)})$
$-\Omega_1$
Ω_1
ω_1
$(\omega_{11}, \omega_{22})$
$\Delta\omega_2$
$-\Omega_2$
$\Delta\omega_1$
$(\omega_{11}, \omega_{21})$
$(\omega_{12}, \omega_{21})$

FIGURE 4.1

CHAPTER 5

SOME APPLICATIONS IN GEOTECHNICAL ENGINEERING

5.1 SETTLEMENTS OF FOOTINGS

Consider a building with n footings. We are interested in predicting the settlement of each footing and the differential settlement between any two footings. These will depend on the flexibility of the ground under each footing and on the loads carried by the building. Both the loads and the flexibility of the ground are usually not known exactly and should therefore be considered random variables. The settlement of each footing and the differential settlement between any two footings are then random variables too and our task is to determine their variances and mean values.

Following Diaz-Padilla (1974) let S_i be the (random) settlement of the ith footing, P_i the (random) load on the ith footing, t_{ij} the (deterministic) influence factor that is equal to the load at footing i due to a unit settlement at footing j (note that t_{ij} is equal to t_{ji}) and a_{ij} the (deterministic) factor that is equal to the fictitious load that should be applied on footing i in order to produce the settlement of footing i caused by a unit load acting on footing j (note that a_{ij} is equal to a_{ji}). F_i is the flexibility under the ith footing. Now let us form the following vectors and matrices*:

$$\underline{P} = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} \tag{5.1}$$

* We use _ and {} denote vectors, ~ and [] to denote matrices.

$$\underset{\sim}{F} = \begin{bmatrix} F_1 & 0 \cdots 0 \\ 0 & F_2 \cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & 0 \cdots F_n \end{bmatrix} \tag{5.2}$$

$$\underline{F} = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{Bmatrix} \tag{5.3}$$

$$\underline{S} = \begin{Bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_n \end{Bmatrix} \tag{5.4}$$

$$\underset{\sim}{t} = \begin{bmatrix} t_{11} & t_{12} \cdots t_{1n} \\ t_{21} & t_{22} \cdots t_{2n} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ t_{n1} & t_{n2} \cdots t_{nn} \end{bmatrix} \tag{5.5}$$

$$\underset{\sim}{a} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix} \tag{5.6}$$

Clearly

$$\underline{S} = \underset{\sim}{F}\ \underset{\sim}{a}\ (\underline{P} + \underset{\sim}{t}\ \underline{S}) \tag{5.7}$$

or

$$\underline{S} = (\underset{\sim}{I} - \underset{\sim}{\Omega}\ \underset{\sim}{t})^{-1}\ \underset{\sim}{\Omega}\ \underline{P} \tag{5.8}$$

where

$$\underset{\sim}{\Omega} = \underset{\sim}{F}\ \underset{\sim}{a} \tag{5.9}$$

Our objective is to determine the mean $\underline{m}_S$ (the ith component of the vector $\underline{m}_S$ is the mean value of the settlement S_1 of the ith footing) and the covariance matrix $\underset{\sim}{C}_S$ (the component C_{ij} of $\underset{\sim}{C}_S$ is equal to $Cov[S_i,S_j]$) of $\underline{S}$ given the mean $\underline{m}_P$ and covariance matrix $\underset{\sim}{C}_P$ of $\underline{P}$ and the mean $\underline{m}_F$ and covariance matrix $\underset{\sim}{C}_F$ of $\underline{F}$.

Assuming that there is no correlation between $\underline{F}$ and $\underline{P}$ we can arrive at the following approximate result (Diaz-Padilla, 1974)

$$\underline{m}_S = (\underset{\sim}{I} - \underline{m}_\Omega\ \underset{\sim}{t})^{-1}\ \underline{m}_\Omega\ \underline{m}_P \tag{5.10}$$

$$\underset{\sim}{C}_S = \underset{\sim}{\Lambda}\underset{\sim}{C}_F\underset{\sim}{\Lambda}^T + \underset{\sim}{\Theta}\underset{\sim}{C}_P\underset{\sim}{\Theta}^T \tag{5.11}$$

where

$$\underset{\sim}{\Theta} = (\underset{\sim}{I} - \underline{m}_\Omega\underset{\sim}{t})^{-1}\underline{m}_\Omega \tag{5.12}$$

and

$$\underline{\Lambda}_i = -(\underline{\underline{I}}-\underline{\underline{m}}_\Omega\underline{\underline{t}})^{-1}\left.\frac{\partial(\underline{\underline{I}}-\underline{\underline{\Omega}}\underline{\underline{t}})}{\partial F_i}\right|_{\text{at } \underline{F}=\underline{m}_F}(\underline{\underline{I}}-\underline{\underline{m}}_\Omega\underline{\underline{t}})^{-1}\,\underline{\underline{m}}_\Omega\underline{m}_P$$

$$+\,(\underline{\underline{I}}-\underline{\underline{m}}_\Omega\underline{\underline{t}})^{-1}\left.\frac{\partial\,\underline{\underline{\Omega}}}{\partial F_i}\right|_{\text{at } \underline{F}=\underline{m}_F}\underline{m}_P \qquad (5.13)$$

where $\underline{\Lambda}_i$ is the ith column of $\underline{\underline{\Lambda}}$.

Here we will focus our attention to the determination of $\underline{m}_F$ and C_F. Consider for concreteness the case of footings on sand. A commonly used deterministic formula to compute the settlement s in inches of a rectangular footing with the lesser side equal to B feet located at a depth d feet below the ground surface and loaded with q tons per square foot is

$$s = \frac{3q}{N}\left(\frac{2B}{1+B}\right)^2\left(1-\frac{d}{4B}\right) \qquad 5.14$$

where N is the average of the standard penetration test N-values down to a depth B feet below the footing. This formula is implied in the q vs. B design chart given by Terzaghi and Peck (1967). N is the average N-value down to a depth B at any point. Hence it is seen that the evaluation of the mean flexibility and of the variance of the flexibility under a footing involves evaluating the mean and the variance of the average of a random field over a rectangle. Also the evaluation of the covariance of the flexibilities under two footings involves evaluating the covariance between the averages of a random field over two rectangles. The applicability of the results of Section 3.3 to this problem is therefore obvious.

5.2 STABILITY OF SLOPES

The problem of slope stability has been investigated from a probabilistic viewpoint by several authors. See, for example, Morla-Catalan and Cornell (1976), Vanmarcke (1977b), Veneziano et al (1977). The purpose of this section is to show how the results of Chapter 3 can be utilized in this problem.

Refering to Fig. 5.1, the factor of safety is defined in a deterministic undrained analysis as

$$F = \frac{cLrb + R_e}{Wab} \tag{5.15}$$

If, following Vanmarcke (1977b), we ignore the randomness of the end sections' contribution R_e to the resisting moment, the only quantity about which we are uncertain is the average c of the shear strength over the cylindrical failure surface. Considering c as the realization of a random variable C we may write

$$E[F] = \frac{E[C]Lrb + R_e}{Wab} \tag{5.16}$$

and

$$Var[F] = Var\ [C] \left(\frac{Lr}{Wa}\right)^2 \tag{5.17}$$

But C is nothing but the average over the cylindrical failure surface of the random field representing the spatial variation of the shear strength, therefore the determination of its variance falls in the category of problems treated in Chapter 3. Provided that the radius r of the failure surface is a few times larger than the scale of fluctuation θ we may ignore

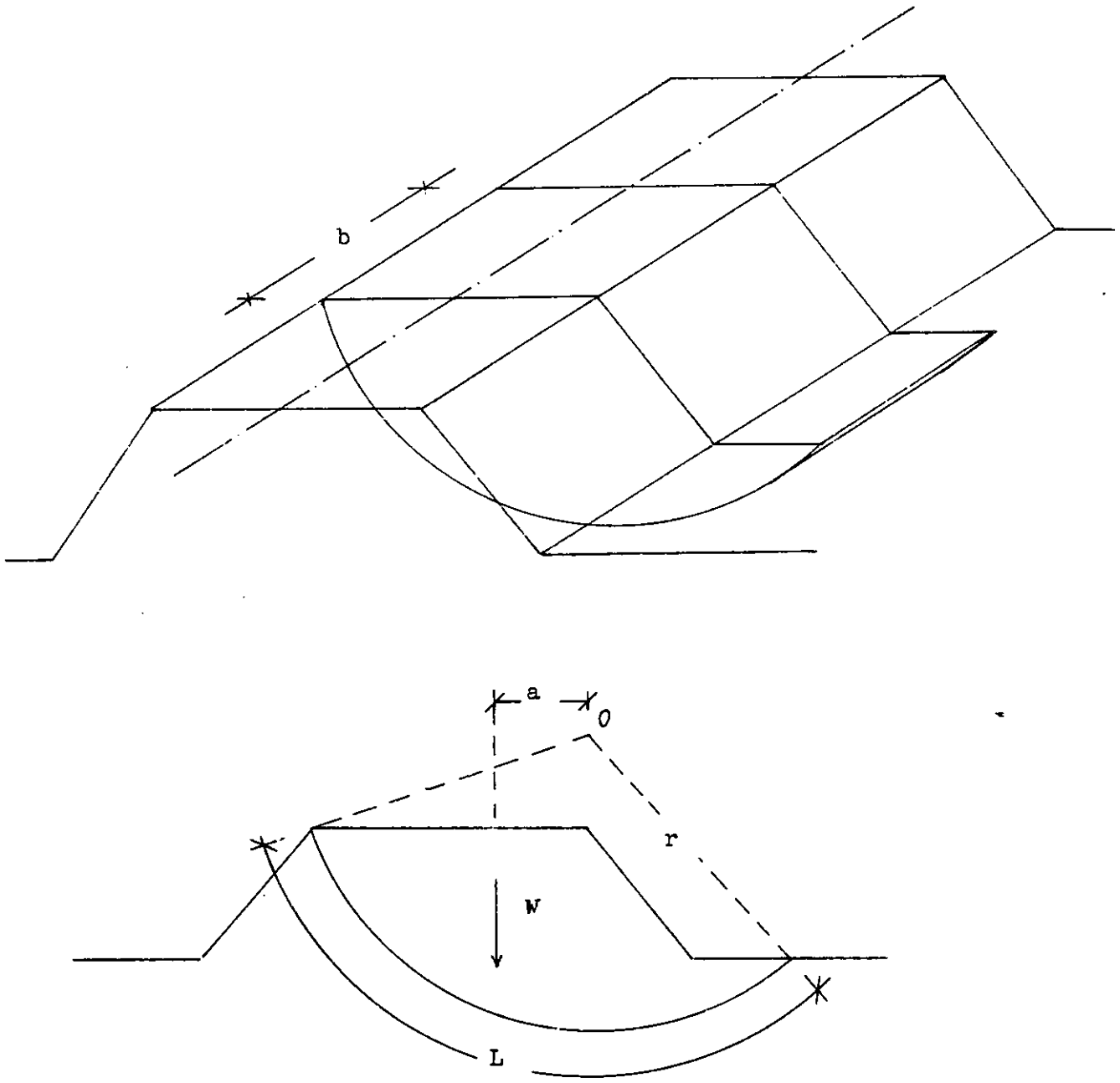


FIGURE 5.1

the fact that the failure surface is curved and compute the variance of $\bar{C}$ as the variance of the average of a two-dimensional random field over a rectangle b x L, hence we may apply the results of Section 3.3. Thus using (5.16) and (5.17) and assuming that the factor of safety has a Gaussian distribution we can find the probability distribution of the factor of safety corresponding to any mode of failure. To find an approximation for the probability of failure in any mode in any location along the embankment of Fig. 4.1 one may follow the procedure proposed by Vanmarcke (1977b), i.e. first assume that the location of 0 is such as to minimize the deterministic factor of safety in a two-dimensional anlaysis (i.e. assuming that $R_e = 0$ and $b \to \infty$), then determine the width b that will minimize the probability of failure at a given location along the embankment and lastly consider the factor of safety that corresponds to that width as a random process whose argument is the location along the embankment. The probability of failure anywhere within the length of the embankment will be equal to the probability that this random process becomes less than unity at least once within the length of the embankment.

5.3 EXPLORATION, SAMPLING AND TESTING

Most soil-testing procedures involve measuring of averages of soil properties. This is true for both laboratory tests (for example the oedometer test measures properties averaged over the volume of the sample) and field tests (for example the plate-loading test measures the average compressibility of the soil over a volume that depends on the size of the plate). Usually the engineer wants to use the test results to estimate either the variance of the random field that corresponds to the soil property under consideration given the variance of the average over the test volume, or the mean and variance of another average, over a volume different from the test volume, given that he has observed the average of the property over the test volume. Clearly both of these problems can be handled using the methodology presented in Section 3.3.

On the other hand an exploration program that will be of optimal efficiency in the sense of achieving high correlation between what is measured and what needs to be predicted and low correlation between different measurements (in order to avoid obtaining the same information twice) can be planned using again the methodology of Section 3.3.

5.4 APPLICATIONS OF SIMULATION

A quite general way to study the effect of the spatial variability of soil properties on the solution of a particular problem is to use a finite element discretization of the soil profile and assign to the elements values of the soil properties taken from random field simulations. By running the finite element program several times, each time with element properties taken from a different simulation, one can find the variance or even the probability distribution of any quantity of interest.

For example, one can apply this procedure to study the effect of the spatial variation of the soil deformability parameters or the effect of the spatial variation of permeability on the distribution of pore pressures and on the factor of safety against slope failure of an earth dam.

From a theoretical standpoint the simulation of random fields is an attractive (but very expensive) way to find approximate solutions to problems involving extremes of random fields. Almost all available solutions to such problems involve exceedances above very high levels. An investigation using simulation could find approximate results for problems involving exceedances above low levels and establish the limits of validity for the high level solutions.

CHAPTER 6

CONCLUSIONS

A simplified methodology for applying random field theory to problems of geotechnical engineering has been presented. It is based on using a constant, the scale of fluctuation, in the place of the covariance function in order to describe the correlation structure of a random field. Approximate expressions are presented for the variances and covariances of averages. These solutions are directly applicable when dealing with averages over rectangular regions, but obviously can be extended to regions of any shape by subdividing into rectangular subregions. Although approximate, these results are very appealing for use in problems of geotechnical engineering because they involve simpler computations than the exact results and because the generally limited amount of data on the spatial variation of soil properties makes the use of exact results superfluous. The approximate methodology that has been presented is an extention to two and three dimensional random fields of the methodology presented by Vanmarcke (1979) for random processes.

A number of applications is also presented, where the spatial variability of various soil properties can be dealt with the simplified approach.

The theory of simulation of random fields has been reviewed and a computer program for simulation of two-dimensional random fields is presented in the Appendix. Some applications of the simulation of random fields to geotechnical

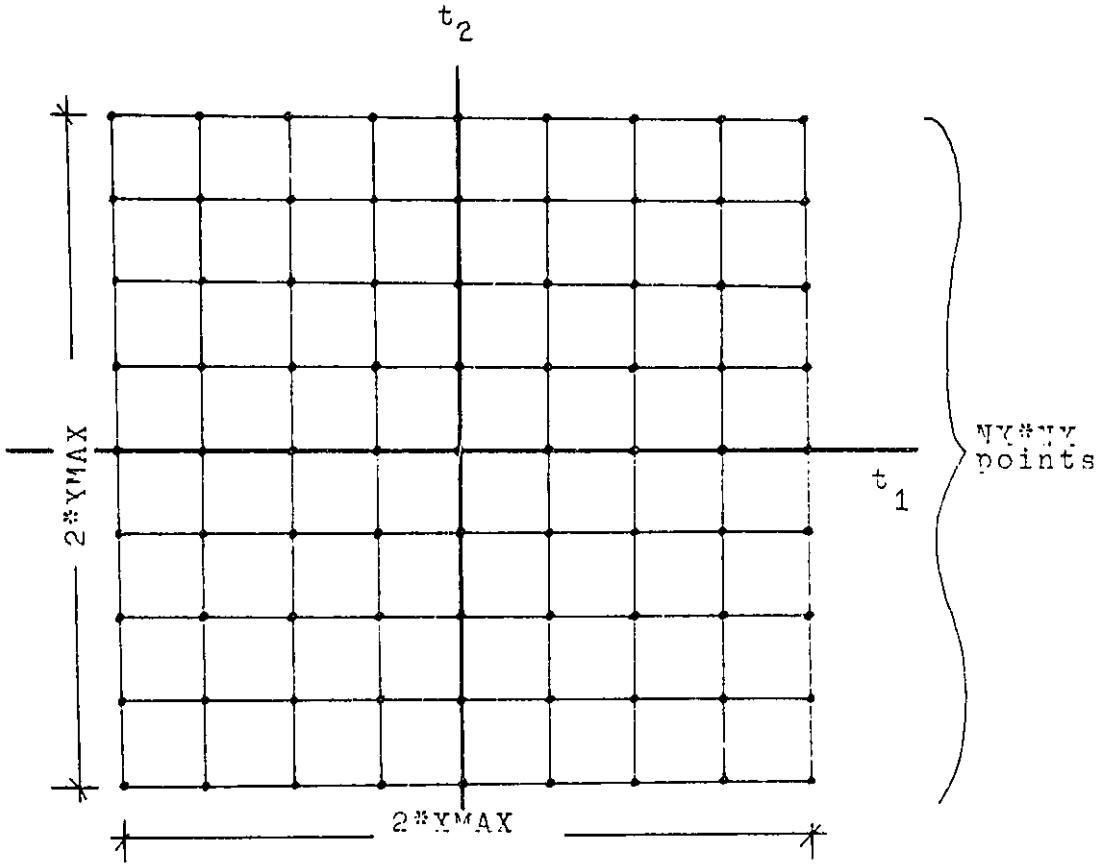
engineering have been suggested.

In general this thesis has attempted to present methodologies for use in practice-oriented rather than research-oriented geotechnical engineering work where the spatial variability of soil properties must be taken into account.

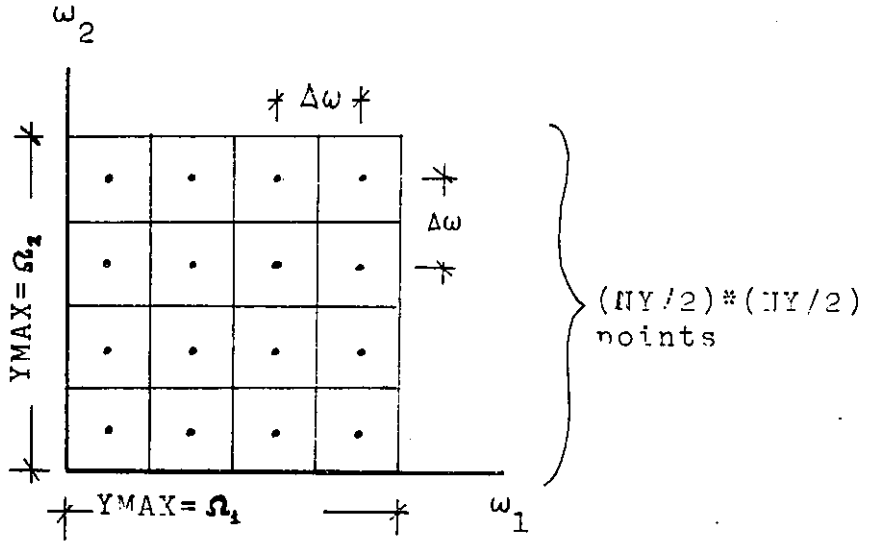
APPENDIX

An interactive computer program for generating two-dimensional isotropic fields is presented here. It has been limited to the isotropic case in order to increase computational efficiency. It is based on the modified form of equation (4.14) that has been described in chapter 4. Since the most time-consuming operation in applying (4.14) directly would be the evaluation of $N_1 \times N_2$ cosines for each point (t_1,t_2) the program has been made more efficient by having it evaluate relatively few trigonometric functions directly and using trigonometric identities for the indirect evaluation of the cosines required.

The program uses a time domain (i.e. the t_1,t_2 plane) and a frequency domain (i.e. the ω_1,ω_2 plane) grid as shown in Figure A.1. The input to the program consists of XMAX, NX (must be odd), YMAX and NY (must be even). The program simulates the values of the random field at the points of the time domain grid, computes their mean, variance and standard deviation and finds their maximum and minimum value. It can also find the number and percentage of points above any level FLEVEL the user requests. Finally it plots contour lines corresponding to levels C as requested by the user by interpolating among the simulated values at the points of the time domain. Two arbitrary integers, I1 and I2, must be input together with the initial data in order to initialize the random number generator. More than one simulation can be made in each run of the program and this saves computation time since some calculations are done



The time domain grid



The frequency domain grid

FIGURE A.1

only once and used in all the simulations in a run. Before using the program one should adjust line 138 so that the value $\sqrt{2}\ \sqrt{S(\omega_1,\omega_2)}$ is assigned to SQRTS(Y1,Y2) where Y1 and Y2 stand for ω_1 and ω_2 and $S(\omega_1,\omega_2)$ is the spectral density function of the field one wants to simulate. The matrices appearing in the DIMENSION statements must be at least of dimensions A(NY/2,NY/2), RY(NY/2), CC(NY/2,(NX-1)/2), SS(NY/2,(NX-1)/2),FIELD(NX,NX). The random fields that are generated by this program are non periodic. This is achieved by the method suggested in Section 4 using $\Delta\omega' = \Delta\omega/20$.

```
      DIMENSION A(25,25),RY(25),CC(25,25),SS(25,25),FIELD(51,51)
      PRINT 1001
      READ 1002,XMAX,NX,YMAX,NY
      PRINT 1003
      READ 1004,I1,I2
      DX=2.*XMAX/FLOAT(NX-1)
      DY=2.*YMAX/FLOAT(NY)
      JYMAX=NY/2
      JXMAX=(NX-1)/2
      DO 10 JY1=1,JYMAX
      Y1=(JY1-.5)*DY
      DO 10 JY2=1,JY1
      Y2=(JY2-.5)*DY
      A(JY1,JY2)=SQRTS(Y1,Y2)*DY
10    CONTINUE
      DO 20 JY=1,JYMAX
      Y=(FLOAT(JY)-.5)*DY
      CALL RANDU(I1,I2,RANDOM)
      RY(JY)=(RANDOM-.5)*DY/20.+Y
20    CONTINUE
      DO 30 JY=1,JYMAX
      CC(JY,1)=COS(RY(JY)*DX)
      SS(JY,1)=SIN(RY(JY)*DX)
      DO 30 JX=2,JXMAX
      CC(JY,JX)=CC(JY,JX-1)*CC(JY,1)-
     *                SS(JY,JX-1)*SS(JY,1)
      SS(JY,JX)=SS(JY,JX-1)*CC(JY,1)+
     *              CC(JY,JX-1)*SS(JY,1)
30    CONTINUE
      NSIM=0
999   CONTINUE
      DO 40 IX1=1,NX
      DO 40 IX2=1,NX
      FIELD(IX1,IX2)=0.
40    CONTINUE
```

```
      NSIM=NSIM+1
      PRINT 1005,NSIM
      DO 50 JY1=1,JYMAX
      DO 50,JY2=1,JYMAX
      MAXJY=MAX(JY2,JY1)
      MINJY=MIN(JY2,JY1)
      CALL RANDU(I1,I2,RANDOM)
      ANGLE=RANDOM*6.2831852
      CS=COS(ANGLE)
      SN=SIN(ANGLE)
      DO 50 IX1=1,NX
      JX1=IX1-JXMAX-1
      DO 50 IX2=1,NX
      JX2=IX2-JXMAX-1
      IF(JX1) 501,502,503
501   CS1=CC(JY1,-JX1)
      SN1=-SS(JY1,-JX1)
      GO TO 504
502   CS1=1.
      SN1=0.
      GO TO 504
503   CS1=CC(JY1,JX1)
      SN1=SS(JY1,JX1)
504   CONTINUE
      IF(JX2) 505,506,507
505   CS2=CC(JY2,-JX2)
      SN2=-SS(JY2,-JX2)
      GO TO 508
506   CS2=1.
      SN2=0.
      GO TO 508
507   CS2=CC(JY2,JX2)
      SN2=SS(JY2,JX2)
508   CONTINUE
```

```
      C=CS1*CS2-SN1*SN2
      S=SN1*CS2+CS1*SN2
      C=C*CS-S*SN
      FIELD(IX1,IX2)=FIELD(IX1,IX2)+A(MAXJY,MINJY)*C*2
50    CONTINUE
      FMEAN=0.
      FVAR=0.
      FMAX=0.
      FMIN=0.
      DO 60 IX1=1,NX
      DO 60 IX2=1,NX
      FMEAN=FMEAN+FIELD(IX1,IX2)
      FVAR=FVAR+FIELD(IX1,IX2)**2
      FMAX=MAX(FMAX,FIELD(IX1,IX2))
      FMIN=MIN(FMIN,FIELD(IX1,IX2))
60    CONTINUE
      FMEAN=FMEAN/FLOAT(NX**2)
      FVAR=FVAR/FLOAT(NX**2)-FMEAN**2
      FSTD=SQRT(FVAR)
      PRINT 1008,FMAX,FMIN
      PRINT 1009,FMEAN,FSTD,FVAR
70    CONTINUE
      PRINT 1010
      READ 1011,NANS
      IF(NANS.EQ.0) GO TO 90
      PRINT 1013
      READ 1014,FLEVEL
      NUMBER=0
      DO 80 IX1=1,NX
      DO 80 IX2=1,NX
      IF(FLEVEL.LT.FIELD(IX1,IX2)) NUMBER=NUMBER+1
80    CONTINUE
      RATIO=FLOAT(NUMBER)/FLOAT(NX**2)
      PRINT 1012,NUMBER,RATIO
      GO TO 70
90    CONTINUE
```

```
      PRINT 1015
      READ 1007,NANS
      IF(NANS.EQ.0) GO TO 110
      CALL PLOT(FIELD,NX,DX,XMAX)
110   CONTINUE
      PRINT 1006
      READ 1007,NANS
      IF(NANS.EQ.0) STOP
      GO TO 999
1001  FORMAT(/1X,'       XMAX    NX       YMAX    NY')
1002  FORMAT(F10.5,I5,F10.5,I5)
1003  FORMAT(/1X,'         I1         I2')
1004  FORMAT(2I10)
1005  FORMAT(/1X,'SIMULATION NO',I5)
1006  FORMAT(/1X,'DO YOU WANT ANOTHER SIMOULATION? 1=YES 0=NO')
1007  FORMAT(I1)
1008  FORMAT(1X,'MAX.VALUE=',F10.5,'MIN.VALUE=',F10.5)
1009  FORMAT(1X,'MEAN=',F10.5,'STD.DEVIATION=',F10.5,'VARIANCE=',F10.5)
1010  FORMAT(/1X,'DO YOU WANT TO INP UT A "FLEVEL" VALUE? 1=YES 0=NO')
1011  FORMAT(I1)
1012  FORMAT(/1X,'NUMBER=',I5,'RATIO=',F10.5)
1013  FORMAT(/1X,'    FLEVEL')
1014  FORMAT(F10.5)
1015  FORMAT(1X,'DO YOU WANT TO DRAW CO NTOUR LINES? 1=YES 0=NO')
      END
      FUNCTION SQRTS(Y1,Y2)
      SQRTS=SQRT(1./(6.2831852*(1.+Y1**2+Y2**2)**1.5))*1.4142135
      RETURN
      END
```

```
      SUBROUTINE PLOT(FIELD,NX,DX,XMAX)
      DIMENSION FIELD(51,51),A(2,2),X1(5),X2(5),F(5)
      JXMAX=(NX-1)/2
      NXM1=NX-1
      XRANGE=2.*XMAX
      XMIN=-XMAX
      CALL T4025
      CALL VARIAN
      CALL START_PLOT(3,'XXX')
      CALL DWINDO(XMIN,XRANGE,XMIN,XRANGE)
      CALL TWINDO(0,780,0,780)
      CALL MOVEA(XMIN,XMIN)
      CALL DRAWA(XMIN,XMAX)
      CALL DRAWA(XMAX,XMAX)
      CALL DRAWA(XMAX,XMIN)
      CALL DRAWA(XMIN,XMIN)
      DO 10 IX1=1,NX
      DO 10 IX2=1,NX
      JX1=IX1-JXMAX-1
      JX2=IX2-JXMAX-1
      XX1=JX1*DX
      XX2=JX2*DX
      CALL POINTA(XX1,XX2)
10    CONTINUE
999   CONTINUE
      PRINT 1001
1001  FORMAT(/1X,'DO YOU WANT TO INPUT A*C*VALUE? 1=YES 0=NO')
      READ 1002,NANS
1002  FORMAT(I1)
      IF(NANS.EQ.0) GO TO 100
      PRINT 1003
1003  FORMAT(1X,'          C')
      READ 1004,C
1004  FORMAT(F10.5)
```

```
      DO 70 IX1=1,NXM1
      DO 70 IX2=1,NXM1
      JX1=IX1-JXMAX-1
      JX2=IX2-JXMAX-1
      X1(1)=JX1*DX
      X1(2)=X1(1)+DX
      X1(3)=X1(2)
      X1(4)=X1(1)
      X1(5)=X1(1)
      X2(1)=JX2*DX
      X2(2)=X2(1)
      X2(3)=X2(1)+DX
      X2(4)=X2(3)
      X2(5)=X2(1)
      F(1)=FIELD(IX1,IX2)
      F(2)=FIELD(IX1+1,IX2)
      F(3)=FIELD(IX1+1,IX2+1)
      F(4)=FIELD(IX1,IX2+1)
      F(5)=F(1)
      N=0
      DO 50 I=1,4
      IF(((C.LT.F(I)).AND.(C.LT.F(I+1))).OR.((C.GT.F(I)).AND.
     *   (C.GT.F(I+1)))) GO TO 50
      N=N+1
      IF(F(I+1).EQ.F(I)) GO TO 50
      A(1,N)=X1(I)+(X1(I+1)-X1(I))*(C-F(I))/(F(I+1)-F(I))
      A(2,N)=X2(I)+(X2(I+1)-X2(I))*(C-F(I))/(F(I+1)-F(I))
      IF(N.EQ.2) GO TO 60
50    CONTINUE
      GO TO 70
60    CONTINUE
      CALL MOVEA(A(1,1),A(2,1))
      CALL DRAWA(A(1,2),A(2,2))
      CALL MOVEA(0.,0.)
```

```
70      CONTINUE
        GO TO 999
100     CONTINUE
        CALL END_PLOT
        RETURN
        END
```

A sample run of the program is presented in what follows. The field to be simulated has spectral density function

$$S(\omega_1,\omega_2) = \frac{1}{2\pi(1 + \omega_1^2 + \omega_2^2)^{1.5}} \qquad (A.1)$$

which is the spectral density function corresponding to

$$B(r) = e^{-r} \quad , \quad r > 0 \qquad (A.2)$$

Figure A.1 shows the time and frequency domain grids. Figure A.2 shows the corresponding interaction between the user and the computer. The user's responses are noted by arrows.

Figure A.3 shows a map of the simulated random field.

```
→ $ run simnew

        XMAX    NX        YMAX    NY
→ 2.            41        8.      40

          I1          I2
→       54673       28765

  SIMULATION NO     1
  MAX.VALUE=    1.22898MIN.VALUE=   -2.43144
  MEAN=   -0.23114STD.DEVIATION=    0.76295VARIANCE=    0.58209

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ 0.

  NUMBER=   754RATIO=    0.44854

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ .5

  NUMBER=   296RATIO=    0.17609

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ 1.

  NUMBER=    26RATIO=    0.01547

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ 1.5

  NUMBER=     0RATIO=    0.00000

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ -.5

  NUMBER= 1117RATIO=    0.66449

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
```

FIGURE A.2

```
→ 1

      FLEVEL
→ -1.

  NUMBER= 1398RATIO=   0.83165

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ -1.5

  NUMBER= 1549RATIO=   0.92148

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ -2.

  NUMBER= 1644RATIO=   0.97799

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 1

      FLEVEL
→ -2.5

  NUMBER= 1681RATIO=   1.00000

  DO YOU WANT TO INP UT A 'FLEVEL' VALUE? 1=YES 0=NO
→ 0
  DO YOU WANT TO DRAW CO NTOUR LINES? 1=YES 0=NO
→ 1

  DO YOU WANT TO INPUT A'C'VALUE? 1=YES 0=NO
→ 1
           C
→ 1.

  DO YOU WANT TO INPUT A'C'VALUE? 1=YES 0=NO
→ 1
           C

→ .5

  DO YOU WANT TO INPUT A'C'VALUE? 1=YES 0=NO
```

FIGURE A.2 continued

```
→ 1
                C
→ 0.

  DO YOU WANT TO INPUT A"C"VALUE? 1=YES 0=NO
→ 1
                C
→ -.5

  DO YOU WANT TO INPUT A"C"VALUE? 1=YES 0=NO
→ 1
                C
→ -1.

  DO YOU WANT TO INPUT A"C"VALUE? 1=YES 0=NO
→ 1
                C
→ -1.5

  DO YOU WANT TO INPUT A"C"VALUE? 1=YES 0=NO
→ 1
                C
→ -2.

  DO YOU WANT TO INPUT A"C"VALUE? 1=YES 0=NO
→ 0
  _SIA1:[P3331.KAFRITSAS]VARIAN.PLT;1

  DO YOU WANT ANOTHER SIMOULATION? 1=YES 0=NO
→ 0
  FORTRAN STOP
  $
```

FIGURE A.2 continued

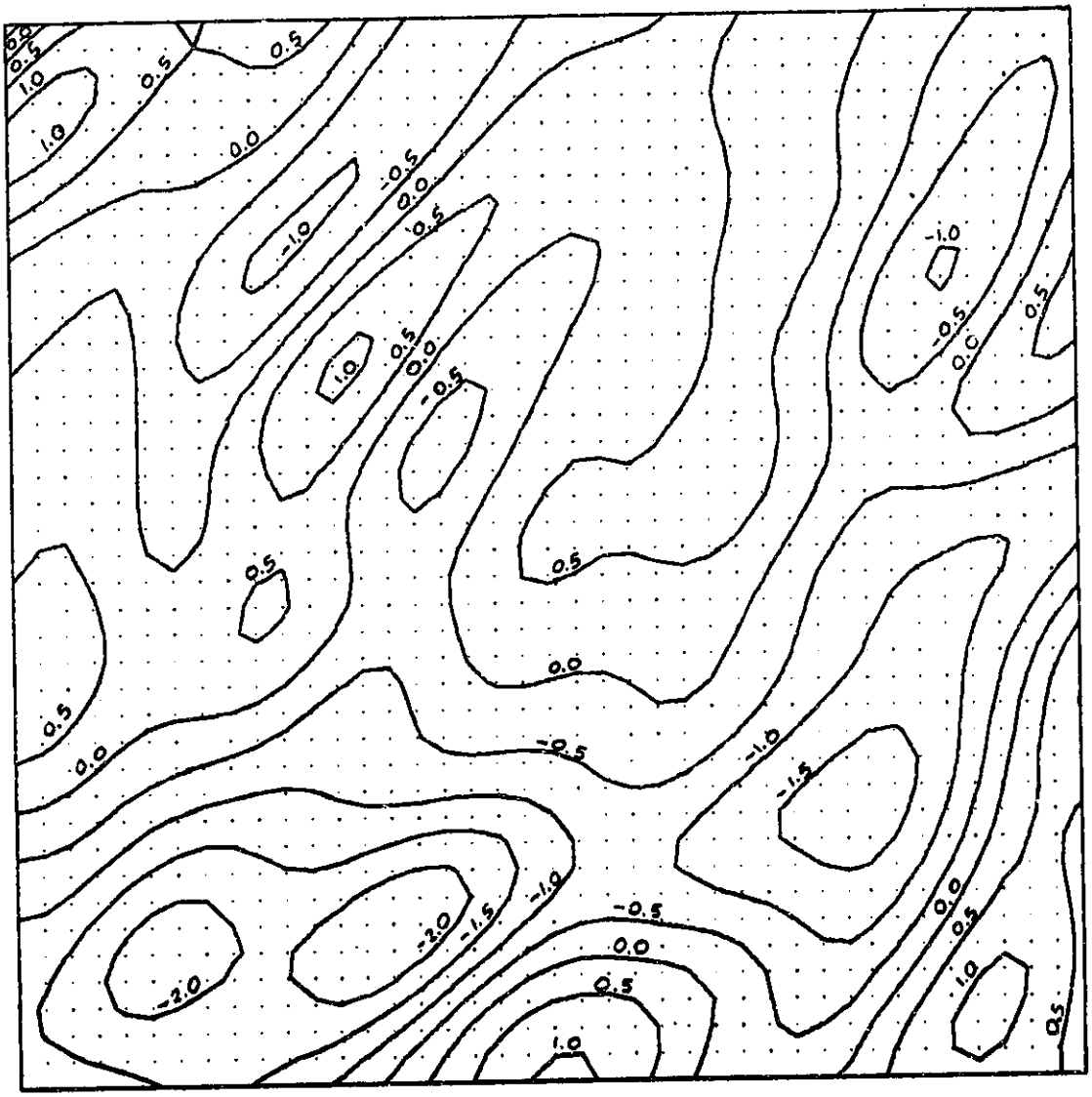

0.0
0.5
1.0
0.5
0.5
1.0
0.0
-0.5
0.0
0.5
-1.0
1.0
0.5
0.0
-0.5
-1.0
-0.5
0.0
0.5
0.5
0.5
0.0
-0.5
-1.0
-1.5
0.5
0.0
-0.5
-1.0
-2.0
-1.5
-2.0
-0.5
0.0
0.5
1.0
0.0
0.5
1.0
0.5

FIGURE A.3

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