NUMERICAL STUDY OF STOKES' WAVE DIFFRACTION

AT GRAZING INCIDENCE

by

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ABSTRACT

The diffraction of finite amplitude water waves by rigid obstacles is studied for the special case when the angle of incidence is small.

In Part I, the incidence of uniform Stokes' waves is considered. The governing equation for the amplitude function is derived using a parabolic approximation and shown to satisfy a cubic nonlinear Schrödinger equation. Numerical computations are performed for a variety of geometries. Of special interest is the demonstration of Mach stem phenomenon for a thin wedge which is in qualitative agreement with existing experiments. An analytic model based on shock theory is proposed which successfully predicts the essential features.

In Part II, the more general incidence of a transient one-dimensional wave packet is studied. The relevant two (space) dimensional Schrödinger governing equation is derived and a numerical solution scheme based on the alternating-direction-implicit (ADI) method is proposed and analyzed. This is applied to the numerical study of diffraction of envelope solitons by an oblique channel and a converging channel. In both cases, there are clear evidences of two-dimensional recurrence. A further example of a broader envelope entering an oblique channel is found to give rise to a recurrent final state of two partially separated pulses.

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I. INTRODUCTION

The diffraction of finite amplitude waves in water is a matter of increasing practical importance in oceanographic engineering. Large structures for offshore exploration must be able to withstand the forces due to powerful storm waves. Breakwater entrances must also be designed for large waves. Current design practices are mostly based on model experiments in the laboratory and on theoretical estimates of the linearized theory. Theoretical developments of diffraction of finite amplitude waves are so far scarce. For example in ship hydrodynamics, a second order theory is available only for a two-dimensional floating body (Lee, 1968). For three-dimensional motions the second order theory for the simplest geometry of a vertical circular column is still controversial (Raman et al., 1977; Chakrabarti, 1978; Issackson, 1977); see, however, Molin (1979).

For long waves in shallow water, considerable experiments have been performed for solitary waves incident obliquely on a straight wall by Perroud (1957) and Chen (1961) and reported by Wiegel (1964 a,b). It is observed that for angles of incidences less than 45°, the familiar picture that the incident and the reflected wave meet at the wall symmetrically about the normal to the wall, is replaced by a three-wave-crest system. In particular, there is now a third wave crest (called the stem) which intersects the wall normally; the incident wave, the reflected wave and the stem meet at a point some distance away from the wall. For incidence angles less that 20° or so, the reflected crest disappears, leaving only the incident crest and the stem (See Figure
(1.1)). Because of its geometrical resemblance to the reflection of shock waves in gas dynamics (see Lighthill, 1949; Whitham, 1974) the phenomenon in shallow water waves has also been called Mach stem effect by Wiegel.

Similar experiments of oblique incidence of periodic waves in finite water depth have been performed by Nielsen (1962) and more recently by Berger and Kohlhase (1976). The kinematics of the wave crests is much the same as in the case of solitary waves. From the experiments the following features are noted. The wave amplitude along the barrier, i.e., the stem height, increases downwave for a finite distance and then levels off gradually. At any station, this amplitude increases with angle of incidence. The width of the stem region, which generally increases with distance along the wall, increases with decreasing incidence angle and with decreasing water depth. The stem width appears to be greater for longer incident waves. It is important to stress that there is substantial scatter in the experimental data, so that only general trends and qualitative conclusions can be made. In particular, the dependence on incident wave height is often ambiguous, although selected experiments indicate decreasing relative stem heights and increasing stem widths for higher waves. Aside from the usual concern for dissipation and errors in measurements, the scattering of data has been attributed to possible reflections due to the finite length and width of the wave flume and diffraction from the far edge of the barrier, although the instability of Stokes' waves may be another source of complication.

In shallow water a theory for the diffraction of solitary waves is
Figure 1.1: Illustration of Mach reflection of solitary waves diffracted by an oblique wall.
now available by Miles (1977 a,b) who applied Whitham's method of geometrical shock dynamics. For shallow water and linearized periodic waves at grazing incidence on a slender body, Mei and Tuck (1979) have recently found it expedient to employ the parabolic approximation first devised for propagation of radio waves over the earth by Leontovitch and Fock (1944, 1945, 1960) and now extended to many problems in acoustics (see Tappert, 1977 and references therein.)

In Part One (Chapters 2 - 4) we extend the parabolic approximation to study the diffraction of steady nonlinear Stokes' waves. The formulation is presented in Chapter 2 where we show that the nonlinear diffraction for grazing incidence is governed by a cubic Schrödinger equation, a fact that can be anticipated in light of Mei and Tuck (1979). In Chapter 3, we present numerical solutions for the case of a vertical wedge with a small apex angle. These results are qualitatively comparable to existing experimental observations. A simplified analytical model of a stationary shock is applied to corroborate and help understanding of the computed features. Further numerical results for a grazing incidence of Stokes' waves on a thin parabola as well as an island of finite length are given in Chapter 4. As a final example, we show the results for the edge diffraction by a thin semi-infinite breakwater when the incidence is nearly normal.

While the existence of uniform Stokes' waves has been rigorously demonstrated (Levi-Civita, 1925; Struik, 1926; Kraskovskii, 1960, 1961), it is well known now that they are unstable subject to two-dimensional side-band disturbances in deep water (Zakharov, 1967; Benjamin and Feir, 1967; Benjamin, 1967; and Feir, 1967) and unstable to three-dimensional
perturbations in any water depth (Benney and Roskes, 1969), so that there is a tendency to formation of groups (Lake, Yuen, Rungaldier and Ferguson, 1977.) The representation of ocean waves as uniform Stokes' waves is hence subject to limitations. A more realistic approach is to model the waves as packets (groups of periodic waves modulated by an envelope of limited extent), and to study the diffraction of such packets by barriers, as well as their subsequent evolution and interactions. The potential applicability of such results in coastal engineering problems is particularly encouraging in light of recent developments in the study of evolution of wind wave spectra (Mollo-Christensen and Ramamonjiarisoa, 1978 and Lake and Yuen, 1978) which suggests that there exists a coherent behavior in wind waves which may be relevant to the interaction of envelope solitons of Stokes' waves. A study of the diffraction of nonlinear sea waves by an ocean structure hence requires first a good understanding of the diffraction of a single wave packet, analogous to the consideration of Fourier frequency components in the wave spectrum for linear theory. This will be the task of Part Two.

Before deriving the equations governing the slow two-dimensional modulation of gravity waves in finite constant depth (Chapter 6), we first present some background for the two-dimensional nonlinear Schrödinger equation, and review many of its important theoretical properties in Chapter 5. Numerical methods for the solution of the one- and two-space dimension Schrödinger equations are discussed in Chapter 7, and their stability and convergence properties are analyzed. Partly to confirm the very abstract theory in existence and to enhance concrete
understanding, we concentrate in Chapter 8 on the one-dimensional Schrödinger equation and examine the nonlinear evolution problems for (a) a plane pulse envelope, and (b) a (normal) progressing wave front. Chapters 9 and 10 are the central parts of Part Two and are devoted to the numerical study of two specific examples of grazing diffraction of envelope solitons and soliton-like pulses in deep water. In the first example, two one-dimensional pulses of different envelope profiles are sent into a uniform channel which is at an oblique angle to the direction of the incident waves. The second example is that of a plane soliton entering a slowly converging channel which leads into a narrower uniform channel.
PART ONE

DIFFRACTION OF STEADY STOKES' WAVES
AT GRAZING INCIDENCE

II. FORMULATION

We consider the diffraction of a plane Stokes' wave (primary wave number \( k_0 \), amplitude \( A_0 \)) from \( x \sim -\infty \), by a thin long strut of width \( B \), length \( L \) (along the \( x \)-axis), and in water depth \( h \). (See Figure 2.1.)

It is assumed from the outset that the body is slender

\[
B/L << 1 \tag{2.1}
\]

and long compared to the incoming wave

\[
k_0L >> 1 \tag{2.2}
\]

II.1 Derivation of the Approximate Governing Equation

Assuming inviscid, irrotational flow, and ignoring surface tension, the exact equations for the velocity potential \( \phi \) and the free-surface elevation \( \zeta \) are

\[
\nabla^2 \phi = 0 \quad -h < z < \zeta(x,y,t) \tag{2.3.a}
\]

\[
g\phi_z + \phi_{tt} + |\nabla \phi|^2 + \frac{1}{2} (\nabla \phi \cdot \nabla) |\nabla \phi|^2 = 0 \quad z = \zeta(x,y,t) \tag{2.3.b}
\]

\[
\phi_z = 0 \quad z = -h \tag{2.3.c}
\]

and

\[
\phi_t + g\zeta + \frac{1}{2} |\nabla \phi|^2 = 0 \quad z = \zeta(x,y,t) \tag{2.3.d}
\]

The gravitational acceleration is designated by \( g \).
Figure 2.1: Definition sketch.
To fix ideas, let us recall the salient features of the linearized diffraction of short waves by a long body. Given vertical side walls for the entire sea depth, and linearizing Eqs. (2.3), the velocity potential may be factored as

\[ \phi(x,y,z,t) = \phi(x,y,t) \cosh k_0(z+h) \quad -h < z < 0 \quad (2.4) \]

Eq. (2.3.a) leads to the Helmholtz equation

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k_0^2 \phi = 0 \quad -h < z < 0 \quad (2.5) \]

Since the longitudinal axis of the body is in the direction of wave propagation, and \( k_0 L \gg 1 \) and \( B/L \ll 1 \); we expect the backward scattering to be small so that the waves remain essentially propagating in the forward direction with the amplitude and phase modulated slowly in \( x \) (with the scale of \( L \)) and in \( y \).

It is, therefore, reasonable to assume

\[ \phi = \text{Re}(\psi e^{-ikx}) \quad (2.6) \]

where

\[ \psi_x \ll k_0 \psi \quad (2.7) \]

Substituting (2.6) into (2.5) we obtain

\[ 2ik \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0 \quad (2.8) \]

Clearly

\[ 2ik \frac{\partial \psi}{\partial x} / \frac{\partial^2 \psi}{\partial x^2} = 0(k_0 L) \gg 1 \quad (2.9) \]
and the third term in (2.8) may be omitted with a relative error of
\[ O(k_0 L)^{-1} \]
\[ 2i k_0 \psi_x + \psi_{yy} = O(k_0 L)^{-1} \psi_x \quad k_0 \]

which implies that the length scale of transverse modulation \( L_y \) is given by
\[ \frac{2i k_0}{L} \sim O(\frac{1}{L^2}) \quad \text{or} \quad \frac{L_y}{L} = O(k_0 L)^{-1/2} \]  

This is called the parabolic approximation which was first derived for
the diffraction across the shadow boundary when waves are normally
incident on a semi-incident screen, Leontovich (1944), Fock (1960).

Now, for Stokes' wave, nonlinearity affects the wave at leading order
through the phase over a distance of \( O(k_0 \varepsilon^{-2})^{-1} \), where \( \varepsilon \) is of the order
of the wave slope. In order that nonlinear modulation and spatial modu-
lation imposed by the body be equally important, we choose
\[ (k_0 \varepsilon^{-2})^{-1} = O(L) \quad \text{or} \quad k_0 L = O(\varepsilon^{-2}) \]  

and from (2.11)
\[ k_0 L \sim O(\varepsilon^{-1}) \]  

Returning to the governing equations (2.3), we introduce the small order-
ing parameter \( \varepsilon \) and slow coordinates \( X = \varepsilon^2 x \) and \( Y = \varepsilon y \) as suggested by
the preceding argument, and use multiple-scale expansions as follows:
\[ \phi = \sum_{n=1}^{\infty} \varepsilon^n \phi_n (x,X,Y,z,t) \]  

and
\[ \zeta = \sum_{n=1}^{\infty} \varepsilon^n \zeta_n(x, X, Y, z, t) \]  \hspace{1cm} (2.15)

with

\[ \frac{\partial}{\partial x} + \frac{\partial}{\partial x} + \varepsilon^2 \frac{\partial}{\partial x} ; \quad \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial y} \]  \hspace{1cm} (2.16)

Substituting Eqs. (2.14) - (2.16) into Eq. (2.3), expanding in Taylor series about \( z = 0 \), we obtain a sequence of perturbation problems after many algebraic steps which are similar to those needed in deducing the governing equation for the evolution of Stokes' waves in \( x \) and \( t \):

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \phi_n = F_n \quad -h < z < 0 \]  \hspace{1cm} (2.17.a)

where

\[ F_1 = 0 \]  \hspace{1cm} (2.17.b)

\[ F_2 = 0 \]  \hspace{1cm} (2.17.c)

\[ F_3 = -(\phi_{1YY} + 2\phi_{1XX}) \]  \hspace{1cm} (2.17.d)

\[ \Gamma \phi_n = G_n \quad \text{on } z = 0 \]  \hspace{1cm} (2.18.a)

where

\[ \Gamma = (g \frac{\partial}{\partial z} + \frac{\partial^2}{\partial t^2}) \]  \hspace{1cm} (2.18.b)

\[ G_1 = 0 \]  \hspace{1cm} (2.18.c)

\[ G_2 = -\left[ \zeta_1 \Gamma_z \phi_1 + (\phi_{1x}^2 + \phi_{1z}^2) t \right] \]  \hspace{1cm} (2.18.d)

\[ G_3 = -\left[ \zeta_1 \Gamma_z \phi_1 + \zeta_1 \Gamma_z \phi_2 + \frac{1}{2} \zeta_1 \Gamma_{zz} \phi_1 + 2(\phi_{1x} \phi_{2x} + \phi_{1z} \phi_{2z}) t \right. \]

\[ + \left. \zeta_1 (\phi_{1x}^2 + \phi_{1z}^2) t + \frac{1}{2} (\phi_{1x} \frac{\partial}{\partial x} + \phi_{1z} \frac{\partial}{\partial z})(\phi_{1x}^2 + \phi_{1z}^2) \right] \]  \hspace{1cm} (2.18.e)
\[ \phi_n z = 0 \quad z = -h \quad (2.19) \]

and

\[-g\zeta_n = H_n \quad (2.20.a)\]

where

\[ H_1 = \phi_1 t \quad (2.20.b) \]

\[ H_2 = \phi_2 t + \frac{1}{2} (\phi_{1x}^2 + \phi_{1z}^2) + \zeta_1 \phi_{1zt} \quad (2.20.c) \]

\[ H_3 = \phi_3 t + \phi_{1x} \phi_{2x} + \phi_{1z} \phi_{2z} + \zeta_1 \phi_{2zt} + \zeta_2 \phi_{1zt} \]

\[ + \frac{1}{2} \zeta_1^2 \phi_{1zzt} + \frac{1}{2} \zeta_1 (\phi_{1x}^2 + \phi_{1z}^2) z \quad (2.20.d) \]

Finally, we introduce the Fourier expansions

\[ (\phi_n, F_n, \zeta_n, G_n, H_n) = \sum_{m=-n}^{n} e^{im\psi}(\phi_{mn}, F_{mn}, \zeta_{mn}, G_{mn}, H_{mn}) \quad (2.21) \]

where \((\cdot \cdot)_{-m,n}^m, (\cdot \cdot)_{m,n}^m\) are complex conjugates

\[ \psi = k_o x - \omega t \quad (2.22) \]

\[ \omega^2 = k_o g \tanh k_o h \quad (2.23) \]

Note that \(\phi_{mn}, F_{mn}\) are now complex functions of \((X, Y, z)\), and \(\zeta_{mn}, G_{mn}, H_{mn}\) that of \((X, Y)\) only. If we substitute Eq. (2.21) into Eqs. (2.17) - (2.19) using (2.20) we obtain at each \(m\) and \(n\) a boundary value problem in \(z\):

\[ \left( \frac{\partial^2}{\partial z^2} - m^2 k_o^2 \right) \phi_{mn} = F_{mn} \quad -h < z < 0 \quad (2.24.a) \]

\[ + \]
\[(g \frac{\partial}{\partial z} - m^2 \omega^2) \phi_{mn} = G_{mn} \quad z = 0 \quad (2.24.b)\]

\[\frac{\partial}{\partial z} \phi_{mn} = 0 \quad z = -h \quad (2.24.c)\]

At any \(n\), non-trivial homogeneous solutions to the above boundary value problem will only exist for \(m = 0\) and \(m = \pm 1\), in which case \(F_{mn}\) and \(G_{mn}\) must satisfy solvability conditions to avoid secularity. In particular we require

\[\frac{1}{g} G_{on} = \int_{-h}^{0} F_{on} \, dz \quad \text{for } m = 0 \quad (2.25)\]

and

\[\frac{1}{g} G_{mn} = \int_{-h}^{0} F_{mn} \frac{\cosh k_{o}(z+h)}{\cosh k_{o}h} \, dz \quad \text{for } m = \pm 1 \quad (2.26)\]

For \(n = 1\), \(F_1 = G_1 = 0\) and for \(m = 0\), Eqs. (2.24) require

\[\frac{\partial}{\partial z} \phi_{01} = 0 \quad (2.27.a)\]

so that

\[\phi_{01} = \phi_{01}(X,Y) \quad (2.27.b)\]

for \(m = 1\), the homogeneous solution is

\[\phi_{11} = \frac{g \cosh Q}{2\omega \cosh \omega} \, iA \quad (2.28.a)\]

where

\[A = A(X,Y) \quad \text{and} \quad (2.28.b)\]

†The details of \(F_{mn}, G_{mn}\) are given in Appendix A.
\[ Q = k_0 (z+h); \quad q = k_0 h \]  

\[ \phi_1 = - \frac{g \cosh Q}{2 \omega \cosh q} (iA e^{i\psi} + \ast) + \phi_{01} \]  

(2.29.a)

and

\[ \zeta_1 = \frac{1}{2} (Ae^{i\psi} + \ast) \]  

(2.29.b)

from Eqs. (2.26), where * denote the complex conjugate of the preceding term. For \( n = 2 \), \( P_2 = 0 \) and evaluating Eq. (2.18.d) using Eqs. (2.29), we have

\[ G_2 = \frac{3\omega^2}{4 \sinh^2 q} (iA^2 e^{2i\psi} + \ast) \]  

(2.30)

Solving Eqs. (2.24) now give

\[ \frac{3}{\partial z} \phi_{02} = 0 \quad \text{for } m = 0 \]

or

\[ \phi_{02} = \phi_{02}(X,Y) \]  

(2.31)

We discard homogeneous solutions of the type \( ^+ \)

\[ \phi_{12} = - \frac{g \cosh Q}{2 \omega \cosh q} iB; \quad B = B(X,Y) \quad \text{for } m = 1 \]  

(2.32)

as well as \( \phi_{02} \) and only keep the particular solution:

\[ \phi_{22} = - \frac{3\omega}{16} \frac{\cosh 2Q}{\sinh q} iA^2 \quad \text{for } m = 2 \]  

(2.33)

\( ^+ \) By carrying out the perturbation analysis to the 4th order, it can be shown that both \( \phi_{01} \) and \( B \) can at best be known functions of \( X \) only, given by the incident wave, and in fact, \( A \) determines the entire wave field to \( O(\varepsilon^2) \).
whence

\[ \phi_2 = -\frac{3\omega \cosh 2Q}{16 \sinh q} \left( iA^2 e^{2i\psi} + * \right) \]  

and from Eqs. (2.20)

\[ \zeta_2 = -\frac{k_o}{2 \sinh 2q} |A|^2 + \frac{k_o \cosh q (2 \cosh^2 q + 1)}{8 \sinh^3 q} \left( A^2 e^{2i\psi} + * \right) \]  

where the first and last terms correspond respectively to the second order mean set down and second harmonic.

Proceeding to the third order, we have

\[ F_{03} = -\frac{3}{2y^2} \phi_{01} \]  

(2.35.a)

\[ G_{03} = 0 \]  

(2.35.b)

and requirement (2.25) with (2.27) gives

\[ \frac{3}{2y^2} \phi_{01} = 0 \]  

(2.36)

For \( m = 1 \),

\[ F_{13} = -\frac{\omega \cosh Q}{\sinh q} \left( \frac{\partial A}{\partial x} - \frac{i}{2k_o} \frac{\partial^2 A}{\partial x^2} \right) \]  

(2.37.a)

and

\[ G_{13} = \frac{\omega^3 k_o \cosh q (\cosh 4q + 8 - 2 \tanh^2 q)}{16 \sinh^5 q} \left( i|A|^2 + \right) \]  

(2.37.b)

so that solvability condition (2.26) gives, after simplification
\[ 2 \frac{\partial A}{\partial X} - \frac{i}{k_0} \frac{\partial^2 A}{\partial Y^2} + iK' |A|^2 A = 0 \] 

(2.38)

where

\[ K' = k_0^3 \frac{C_0}{C_{go}} \frac{\cosh 4q + 8 - 2 \tanh^2 q}{8 \sinh^4 q} \] 

(2.39.a)

\[ C_o = \frac{\omega}{k_0} \]

(2.39.b)

and

\[ C_{go} = \frac{\partial \omega}{\partial k_0} = \frac{\omega}{k_0 \sinh 2q} \left( \frac{\sinh 2q}{2} + q \right) \]

(2.39.c)

We have finally obtained the governing equation (2.38) for the first order complex amplitude A. This equation is formally identical to the well known cubic-nonlinear Schrodinger equation encountered in the study of nonlinear evolution of dispersive waves (Zakharov and Shabat, 1972) where X would take the role of time and Y the distance in the coordinate system traveling at group velocity. We remark that since \( K' \) as defined by (2.39a) is always positive, this corresponds to nonlinear dispersive waves without self-focusing or transverse side band stability (Karpman, 1975). We now turn to the boundary and initial conditions for (2.38).
II.2 Boundary and Initial Conditions

Let the body be symmetric about x-axis and the walls be vertical throughout the sea depth and given by

\[ y = \pm y_B(x) \]
\[ y_B = 0 \quad \text{for} \quad x \leq 0 \]
\[ y_B > 0 \quad \text{for} \quad x > 0 \]  \hspace{1cm} (2.40)

In the stretched coordinates, the body boundary is at

\[ Y = \pm \varepsilon k_B y_B(X) \]  \hspace{1cm} (2.41.a)

where we have let

\[ y_B = (k_B)Y_B \]  \hspace{1cm} (2.41.b)

with

\[ k_B Y_B = O(1) \]  \hspace{1cm} (2.41.c)

Considering the symmetric half problem for y positive, the boundary condition on the strut is

\[ \frac{D}{Dt} (y_B - y) = \phi_x y_B - \phi_y = 0 \quad \text{or} \]
\[ \phi_y = \phi_x y_B \quad \text{on} \quad y = y_B \]  \hspace{1cm} (2.42.a)

(2.42.b)

Using Eqs. (2.41), (2.42.b) becomes in normalized variables

\[ \varepsilon \phi_y = \left[ \frac{3}{2} + \varepsilon \frac{2}{2} \frac{3}{2X} \right] \varepsilon^2 k_B y_B'(X) \quad \text{on} \quad Y = \varepsilon k_B Y_B(X) \]  \hspace{1cm} (2.43)

and substituting in the expression for \( \phi \) from the previous section, we obtain
\[
\epsilon \left[ -\frac{g \cosh Q}{2\omega \cosh q} \left( iA_x e^{i\psi} + * \right) + \phi \right] + O(\epsilon) \] 

\[
= \left\{ -\frac{g \cosh Q}{2\omega \cosh q} [i(ik_o)Ae^{i\psi} + *] + O(\epsilon) \right\} \epsilon^2 (k_o B) Y_B^i(X) 
\]

on \( Y = \epsilon(k_o B) Y_B^i(X) \) \( \quad (2.44) \)

We now equate the first harmonic terms in (2.44) so that to leading order,

\[
\frac{\partial A}{\partial Y} = ik_o A \epsilon k_o B Y_B^i(X) \quad \text{on} \quad Y = \epsilon k_o B Y_B(X) \quad (2.45)
\]

We now have two possibilities for the body width compared to wavelength:

(i) thin strut: \( k_o B = 1 \), and (2.45) becomes

\[
\frac{\partial A}{\partial Y} = \epsilon ik_o A Y_B^i(X) \quad \text{on} \quad Y = \epsilon Y_B \quad (2.46)
\]

so that the body only affects the waves at \( O(\epsilon) \).

(ii) moderately thin strut: \( k_o B = \epsilon^{-1} \). Note that in this case, \( B/L \sim O(\epsilon) \) and the body is still thin compared to length. The boundary condition (2.45) now affects \( A \) at the leading order and becomes

\[
\frac{\partial A}{\partial Y} = ik_o A Y_B^i(X) \quad \text{on} \quad Y = Y_B(X) \quad (2.47)
\]

which, must be applied exactly on the body.

We shall henceforth concentrate on the moderately thin strut and consider (i) as only a special case of (ii).

Far away from the body, at large \( Y \), we expect no transverse variations, and the boundary condition is
\[
\frac{\partial A}{\partial X} + 0 \quad Y \rightarrow \infty \tag{2.48,a}
\]

or using Eq. (2.38),

\[
A = A_0 e^{-i k A_0^2 X/2} \quad Y \rightarrow \infty \tag{2.48,b}
\]

which is simply the undisturbed uniform Stokes' wave. The appropriate initial condition just ahead of the body is

\[
A = A_0 \quad X = 0 \tag{2.49}
\]

We further introduce the non-dimensional variables

\[
\bar{A} = A/A_0 \quad , \quad \bar{X} = k_0 X = \varepsilon^2 k_0 x \quad , \quad \bar{Y} = k_0 Y = \varepsilon k_0 y \quad , \quad \bar{Y}_B = k_0 Y_B
\]

The normalized initial-boundary value problem may be stated as follows:

\[
2 \frac{\partial \bar{A}}{\partial \bar{X}} - i \frac{\partial^2 \bar{A}}{\partial \bar{Y}^2} + i K \bar{A}^2 = 0 \quad \bar{Y} > \bar{Y}_B(\bar{X}), \quad \bar{X} > 0 \tag{2.51.a}
\]

\[
\bar{A} = 1 \quad \bar{X} = 0 \tag{2.51.b}
\]

\[
\frac{\partial \bar{A}}{\partial \bar{Y}} = i \bar{Y}_B(\bar{X}) \bar{A} \quad \bar{Y} = \bar{Y}_B(\bar{X}) \tag{2.51.c}
\]

\[
\frac{\partial \bar{A}}{\partial \bar{Y}} + 0 \quad \text{or} \quad \bar{A} = e^{-i K \bar{X}/2} \quad \bar{Y} \rightarrow \infty \tag{2.51.d (i), (ii)}
\]

The parameter K is now given by

\[
K = \left( \frac{k A_0}{\varepsilon} \right)^2 \Theta(k_0 h) \tag{2.52}
\]

with
\[ \Theta(k_o h) = \frac{c}{c_o} \frac{\cosh 4k_o h + 8 - 2 \tanh^2 k_o h}{8 \sinh^4 k_o h} \]  

(2.53)

Figure (2.2) is a plot of \( \Theta \) vs. \( k_o h \). Note that \( \Theta \) approaches 2 for \( k_o h \to \infty \) but grows as \( (k_o h)^{-4} \) for \( k_o h \to 0 \) in shallow water.

Eqs. (2.51) must now be solved for the first order wave envelope \( \tilde{A}(\tilde{x}, \tilde{y}) \).
Figure 2.2: Plot of $\theta$ vs. $k_0 h$. 
II.3 Method of Numerical Solution

Since the boundary condition (2.51.c) is applied on the body surface, analytic solution to the nonlinear equation is difficult and one must in general integrate Eqs. (2.51) numerically.

For computation, we restrict the $\bar{Y}$ domain to a finite distance $\bar{Y}_\infty$, which is chosen, for a given range of $\bar{X} \leq \bar{X}_R$, to be sufficiently large so that Eqs. (2.51) are satisfied smoothly and further increase of $\bar{Y}_\infty$ produces no significant changes in the solution.

In principle, we may replace Eqs. (2.51.d) by either

\[
\frac{3\bar{\bar{A}}}{\partial \bar{Y}} = 0 \quad \text{or} \quad (2.54.a)
\]

\[
\bar{\bar{A}} = e^{-ik\bar{X}/2} \quad \text{on } \bar{Y} = \bar{Y}_\infty \quad (2.54.b)
\]

Eq. (2.54.b), which yields excellent results, is used throughout.

For Eq. (2.51.a), we employ an implicit scheme of Crank-Nicholson type for integration in $\bar{X}$, and centered second-order differencing in $\bar{Y}$:

\[
2\bar{\bar{A}}_{j+1}^{n+1} = 2\bar{\bar{A}}_j^n + \frac{\Delta \bar{X}}{2} \left[ (i \frac{\bar{\bar{A}}_{j+1}^n - 2\bar{\bar{A}}_j^{n+1} + \bar{\bar{A}}_{j-1}^{n+1}}{\Delta \bar{Y}^2} - iK|\bar{\bar{A}}_{j+1}^n|^2 \bar{\bar{A}}_{j-1}^{n+1}) 
\right.
\]

\[
\left. + (i \frac{\bar{\bar{A}}_{j+1}^n - 2\bar{\bar{A}}_j^n + \bar{\bar{A}}_{j-1}^n}{\Delta \bar{Y}^2} - iK|\bar{\bar{A}}_j^n|^2 \bar{\bar{A}}_{j-1}^n) \right] + O(\Delta \bar{X}^3, \Delta \bar{Y}^2) \quad (2.55)
\]

where

\[
\bar{\bar{A}}_{j+1}^{n+1} = \bar{\bar{A}}_j^n + \frac{\Delta \bar{X}}{2} \left[ (i \frac{\bar{\bar{A}}_{j+1}^n - 2\bar{\bar{A}}_j^n + \bar{\bar{A}}_{j-1}^n}{\Delta \bar{Y}^2} - iK|\bar{\bar{A}}_j^n|^2 \bar{\bar{A}}_{j-1}^n) \right] + O(\Delta \bar{X}^2, \Delta \bar{Y}^2) \quad (2.56)
\]
\[ A_j^n \equiv A(n\Delta x, j\Delta y) \quad (2.57) \]

\[ n = 1, 2, \ldots, N \quad (2.58) \]

\[ j = j_B^n + 1, j_B^n + 2, \ldots, J - 1 \quad \bar{y}_B(n\Delta x) = j_B^n \Delta \bar{y} \]

\[ \bar{y}_\infty = J \Delta \bar{y} \quad (2.59) \]

For \( j = j_B \), Eq. (2.55) is modified by boundary condition (2.51.c):

\[ \frac{A_j^{m+1} - A_j^{m-1}}{2\Delta \bar{y}} = i\bar{y}_B'(m\Delta x) A_j^m + 0(\Delta \bar{y})^2 \quad m = n, n+1 \quad (2.60) \]

and, of course, for \( j = J \), Eq. (2.54.b) gives

\[ \frac{A_j^m}{A_j^m} = e^{-iK m\Delta x/2} \quad m = n, n+1 \quad (2.61) \]

Eq. (2.55), which has a global truncation error of \((\Delta x^2, \Delta y^2)\), is well known to be unconditionally stable for the linear problem, and is found to be stable for reasonable choices of \( \Delta x \) and \( \Delta y \) in the fully nonlinear case. From Eqs. (2.51) the conservation of energy may be derived

\[ \frac{\partial}{\partial x} \int_{\bar{y}_B(x)}^{\bar{y}_\infty} |\bar{A}|^2 d\bar{y} = 0 \quad (2.62) \]

which reduces to

\[ \int_{\bar{y}_B(x)}^{\bar{y}_\infty} |\bar{A}|^2 d\bar{y} = \bar{y}_\infty - \bar{y}_B(0) = \bar{y}_\infty \quad (2.63) \]

for a finite region. Eq. (2.63) is used as a measure of the total error due to discretization, round-off, and truncation of \( \bar{y} \); and is satisfied to within a few percent for all computed cases in the following sections.
Sample computer code (in Multics FORTRAN) for the special case of a wedge is included in Appendix B.1.
III. DIFFRACTION BY A THIN WEDGE

III.1 Numerical Results

For a train of Stokes' waves incident along the axis of a vertical wedge of small half-angle \( \alpha \), (i.e., \( y_B = x \tan \alpha \)), we define the width-to-length ratio \( \varepsilon \equiv \tan \alpha \), so that

\[ \bar{y}_B(x) = \bar{x} \]  \hspace{1cm} (3.1)

and Eqs. (2.51.a), (2.51.c) become

\[ 2 \frac{\partial \bar{A}}{\partial \bar{x}} - i \frac{\partial \bar{A}}{\partial \bar{y}} + iK|\bar{A}|^2 \bar{A} = 0 \hspace{1cm} \bar{y} > \bar{x} > 0 \]  \hspace{1cm} (3.2.a)

\[ \frac{\partial \bar{A}}{\partial \bar{y}} = i\bar{A} \hspace{1cm} \text{on} \hspace{0.5cm} \bar{y} = \bar{x} \]  \hspace{1cm} (3.2.b)

The governing equations now contain a single parameter \( K \) given by (2.52), which depends only on the incident wave characteristics and the wedge angle. In particular, for deep water (\( k_0h \rightarrow \infty \))

\[ K \sim 2\left( \frac{\alpha_0\omega_0}{\varepsilon} \right)^2 \]  \hspace{1cm} (3.3)

which is simply twice the square of the wave steepness relative to wedge thickness.

Eqs. (3.2), (2.51.b) and (2.54.b) are solved numerically for a range of \( K \) with \( \bar{X}_R = 8 \) and \( \bar{Y}_\infty = 100 \).

Figure (3.1) shows a plot of \( |\bar{A}|^2 \) along the wedge (on \( \bar{Y} = \bar{X} \)) for the different \( K \) values. For \( K = 0 \) (the linear case), an analytic solution can be directly obtained, with
Figure 3.1: Magnitude squared of the envelope at the wall for a wedge:
$|\tilde{A}(\bar{x}, \bar{y}=\bar{x}; K)|^2$, for different values of $K$. 
\[ |A(\bar{x}, \bar{y} = \bar{x})|^2 = \left| 1 + \text{erf} \left( -i\bar{x}/2 \right) \right|^2 \]  \hspace{1cm} (3.4)

which oscillates about 4 and approaches it as \( \frac{1}{\bar{x}} \) for \( \bar{x} \gg 1 \). This is shown in Figure (3.2). Eq. (3.4) is indistinguishable from the computed linear curve in Figure (3.1). Note in Figure (3.1) that in contrast to the linear case, the amplitude along the wedge for \( K > 0 \) grows for a short distance, then flattens out to a constant value. This value decreases with increasing \( K \) and we define for further reference

\[ E_\pm(K) = |\bar{A}(\bar{x} = 8, \bar{y} = \bar{x}; K)|^2 \]  \hspace{1cm} (3.5)

Also, since \( K \) is inversely proportional to the square of the wedge slope, the qualitative features of Figure (3.1) for the wave height at the wall are in agreement with earlier experimental observations.

For a global picture of the wave envelope, we display its magnitude squared, \( |\bar{A}(\bar{x}, \bar{y})|^2 \) (which is also proportional to the second-order mean set-down), in three-dimensional and contour plots for \( K = 0, 2, 6 \).

In the three-dimensional plots, Figures (3.3.a,b,c) the wedge is given a height of 5 (note that \( |\bar{A}|^2 = 1 \) at \( \bar{x} = 0 \)) to provide a vertical measure. Figures (3.4.a,b,c) show the corresponding contour plots where a contour increment of 0.2 is used throughout. (Note that the contours become somewhat unreliable in the small region of tiny rippling just ahead of the main oscillations.)

The linear wave envelope exhibits the oscillating behavior typical of Fresnel diffraction with \( |\bar{A}|^2 \) attaining a maximum value at the barrier, which decreases immediately with distance from the wall. After a short initial distance, the width of this first hump at the body remain almost
Figure 3.2: Analytic solution for $|\tilde{A}(\bar{x}, \bar{y}=\bar{x})|^2$ for a wedge (Eq. (3.4)).
Figure 3.3: Three-dimensional plots of $|\bar{A}(\bar{X},\bar{Y})|^2$ for a wedge in the region $0 < \bar{X} < 6$, $\bar{X} < \bar{Y} < 15$, for (a) $K = 0$, (b) $K = 2$, and (c) $K = 6$. All vertical scales are the same. The wedge is given a height of 5.
Figure 3.4: Contour plots of $|\bar{A}(x, y)|^2$ for a wedge for (a) $K = 0$, (b) $K = 2$ and (c) $K = 6$. Contours are at 0.2 increments.
constant (as evidenced by contours nearly parallel to the wedge) even though its height is varying with distance.

For the nonlinear cases \((K > 0)\), the overall amplitudes are much smaller than the linear case (decreasing as \(K\) increases), and there is now a clearly defined region along the wall within which \(|\tilde{A}|^2\) has a nearly constant value (approximately \(E_0(K)\)). The width of this flat region (which is roughly that bounded by the first contour from the wedge) appears to increase linearly with \(\tilde{x}\), at an angle which increases with \(K\). Outside of the region, the envelope amplitude profile is not qualitatively different from that observed for the linear case. A more quantitative picture of these features can be obtained by studying successive cross-sections of \(|\tilde{A}(\tilde{x} = \tilde{x}_1, \tilde{y})|^2\), and defining (rather arbitrarily) a width \(\tilde{y} = M(\tilde{x};K)\) corresponding to the edge of the flat region. Figures (3.5.a,b,c) show some examples for \(K = 0, 2, 6\) at \(\tilde{x}_1 = 4\) and \(8\). (The increasing width with \(K\) and \(\tilde{x}\) is evident.) We summarize the results for \(M(\tilde{x},K)\) in Figure (3.6) where the points are from the computed results, and the straight lines their least square linear fit. We see that the linear correlations are excellent, suggesting an empirical relationship of the form

\[
M(\tilde{x};K) = \beta(K)\tilde{x}
\]  

(3.6)

The values of \(\beta\) for different \(K\) are tabulated in Table 3.1. We shall return to the dependence \(\beta(K)\) in §3.2.

For more physical information, we compute the first order instantaneous free surface elevation

\[
\xi_1(k_0 x, k_0 y, t=0) = \text{Re}(\tilde{A}(\varepsilon k_0 x, \varepsilon k_0 y) e^{-i k x})
\]

(3.7)
Figure 3.5: Cross-sections of $|\tilde{A}(\bar{x},\bar{y})|^2$ for a wedge at (i) $\bar{x}_1 = 4$ and (ii) $\bar{x}_1 = 9$, $(\bar{x} \leq \bar{y} \leq 50)$ for (a) $K = 0$, (b) $K = 2$, and (c) $K = 6$. (Note that the curves start at the wall of the wedge given by $\bar{y} = \bar{x}_1$.)
Figure 3.6: Width of the Mach stem region $M(\bar{x}; K)$. $+$: the edge of the flat region; $-$: linear best fit by least squares.
<table>
<thead>
<tr>
<th>K</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.00</td>
</tr>
<tr>
<td>0.5</td>
<td>1.03</td>
</tr>
<tr>
<td>1</td>
<td>1.23</td>
</tr>
<tr>
<td>2</td>
<td>1.50</td>
</tr>
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</tr>
<tr>
<td>6</td>
<td>2.28</td>
</tr>
<tr>
<td>10</td>
<td>2.82</td>
</tr>
</tbody>
</table>
using the fact that \( \bar{x} = \varepsilon^2 k_o x \) and \( \bar{y} = \varepsilon k_o y \). Note that for the same \( k_o \), the incoming Stokes' wave has a longer wavelength for increasing nonlinearity given by the asymptotic behavior (2.31.d.(ii)). Again, we display three-dimensional and contour plots of \( \zeta_1 \), but this time in uniformly scaled \( k_o x \) and \( k_o y \) coordinates, so that all physical angles are undistorted (see Figures (3.7) and (3.8)). The parameter chosen is for \( K = 0, 1, 2 \); and for wedge slopes corresponding to \( \varepsilon^2 = 0.1 \) and 0.15 (apex half-angles of \( \alpha = 17.55^\circ \) and \( 21.17^\circ \) respectively). We caution that these values of \( K \) and \( \varepsilon^2 \) are selected here to accentuate the important features, and may correspond to somewhat steep waves. In particular, for the steepest case presented of \( K = 2, \varepsilon^2 = 0.15 \), and assuming infinite depth, we have a wave slope of \( k_o A_o \approx 0.387 \), which is near breaking. It is useful to point out, however, that even when the parameters used correspond to much milder or to unrealistically steep waves, the qualitative features are similar to those presented only less or more pronounced.

From these figures, we note that for \( K = 0 \), the wave crests are rather straight except near the wedge, where the first hump lags behind slightly and is tilted forward so that it becomes almost normal at the wall. The lag increases with each successive wave. This first crest at the wall, however, remains quite narrow and its width is almost constant in the direction of the wave. The picture for the nonlinear cases becomes dramatically different. Here the waves are in general smaller (decreasing for larger \( K \)), and the equal phase lines are noticeably bent forward to somewhat beyond 90° along the wedge, forming stems with near horizontal crest lines. The width of these stems grows linearly with
Figure 3.7 (i): Three-dimensional plots of $\zeta_1(k_x, k_y, t=0)$ for a wedge for (a) $K = 0$, (b) $K = 1$, (c) $K = 2$ in uniformly scaled $k_0 x$, $k_0 y$ coordinates. $\varepsilon^* = 0.1$ ($a = 17.55^\circ$). The region shown is for $0 \leq \bar{x} \leq 6, \bar{x} \leq \bar{y} \leq 18$ corresponding to $0 \leq k_0 x \leq (60,40)$, $k_0 x \leq k_0 y \leq (56.92,46.48)$ for $\varepsilon^2 = (0.1,0.15)$. All vertical scales are the same and body is given a height of 0.
Figure 3.7 (ii): Three-dimensional plots of $\zeta_1(k_x, k_0y, t=0)$ for a wedge for (a) $K = 0$, (b) $K = 1$, (c) $K = \frac{1}{2}$ in uniformly scaled $k_0x$, $k_0y$ coordinates. $\epsilon^2 = 0.15$ ($\alpha = 21.17^\circ$). The region shown is for $0 \leq \bar{x} \leq 6$, $\bar{x} \leq \bar{y} \leq 18$ corresponding to $0 \leq k_0x \leq (60,40)$, $k_0x \leq k_0y \leq (56.92,46.48)$ for $\epsilon^2 = (0.1,0.15)$. All vertical scales are the same and the body is given a height of 0.
Figure 3.8 (i): Contour plots of $\zeta_1(k_0x, k_0y, t=0)$ for a wedge for (i) $\varepsilon^2 = 0.1 (\alpha = 17.55^\circ)$, (ii) $\varepsilon^2 = 0.15 (\alpha = 21.17^\circ)$; and for (a) $K = 0$, (b) $K = 1$, (c) $K = 2$ in uniformly scaled $k_0x, k_0y$ coordinates. The contour increment is 0.2. The direction of $\rightarrow\leftarrow$ is defined by Eq. (3.35).
Figure 3.8 (ii): Contour plots of $\xi_2(k_0x, k_0y, t=0)$ for a wedge for (i) $\varepsilon^2 = 0.1 \ (\alpha = 17.55^\circ)$, (ii) $\varepsilon^2 = 0.15 \ (\alpha = 21.17^\circ)$; and for (a) $K = 0$, (b) $K = 1$, (c) $K = 2$ in uniformly scaled $k_0x$, $k_0y$ coordinates. The contour increment is 0.2. The direction of $--.--$ is defined by Eq. (3.35).
distance so that they all remain inside another wedge whose apex angle increases with $K$ (and $\varepsilon^2$). The exact form of dependence of this angle as well as the angle of bending on $K$ and $\varepsilon^2$ is not explicit, and will be discussed in the next section. To compare these features to earlier results for the envelope magnitude $|\tilde{A}|^2$, the lines $\tilde{Y} = M(\tilde{X}; K)$ with the boundary of the stems, which we shall call Mach stems from now on, is noted. We have hence shown that the numerical empirical formula (3.6) for $M$, in fact, governs the width and rate of growth of the Mach stem. Indeed, if we interpret the change of wavelength, incidence angle, water depth and incident wave height in terms of $K$, the qualitative conclusions of Nielsen (1962) and Berger and Kohlhase (1976) for the stem width are again confirmed by Eq. (3.6) and Figure 3.8.

The above computed results show that the very simple Eqs. (3.2) is a good model for the diffraction of periodic waves by a thin wedge. In particular, the nonlinearity of the incoming waves is shown to have an important role in the observed phenomenon associated with "Mach reflection" for small angle incidences.

We remark in closing that while only a small number of values of $K$ are chosen and selected details displayed, more extensive computations have been made which show that the solutions to the Schrödinger equation (3.2) always vary smoothly with $K$. 

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III.2 Approximating the Mach Stem Region by a Shock

The computed features associated with the Mach stems are so systematic that some analytical examinations are called for. The sharp bending of wave crests and the near horizontal plateau of $|\vec{A}|^2$ inside a triangular stem region suggest that the region may be modelled as one side of a straight shock joining two discontinuous constant states. The undulatory variations outside clearly cannot be characterized as a constant region. Nevertheless let us ignore these undulations and examine whether a simple shock model will be able to predict the region of Mach stems. First, we rewrite (3.2.a) by letting

$$\frac{\vec{A}}{a} = a e^{i\psi} \quad (3.8)$$

where $a$, $\psi$ are real. Substituting (3.8) into (3.2.a) and equating real and imaginary terms respectively, we obtain

$$\frac{2a_x}{X} + \frac{2a_y\psi}{Y} + a\frac{\psi}{YY} = 0 \quad (3.9)$$

and

$$2a\psi_x - a\frac{\psi}{YY} + a\psi_x^2 + ka^3 = 0 \quad (3.10)$$

which can be manipulated to give the conservation equations

$$\frac{E}{X} + (EW) \frac{Y}{Y} = 0 \quad (3.11)$$

and

$$W_x + \frac{\partial}{\partial Y} \left[ - \frac{a}{2a} + \frac{W^2}{2} + \frac{KE}{2} \right] = 0 \quad (3.12)$$
where
\[ E \equiv a^2 \]  \hspace{1cm} (3.13)
\[ W \equiv \psi_Y \]  \hspace{1cm} (3.14)

Eqs. (3.11) and (3.12) resemble the conservation equations of Chu and Mei (1970) for time-evolving Stokes waves where it is known that omission of the term \( a_{\frac{YY}{YY}} \) leads to shocks (Whitham, 1967).

We now assume that there is a line of discontinuity along \( \bar{Y} = \beta \bar{X} \), and that \( E, W \) have constant values on either side of this line. It is well known that for a conservation equation
\[ \frac{\partial P}{\partial \bar{X}} + \frac{\partial Q}{\partial \bar{Y}} = 0 \]  \hspace{1cm} (3.15)
the shock condition is
\[ \beta [P] = [Q] \]  \hspace{1cm} (3.16)

where
\[ [P] = P_+ - P_- \]  \hspace{1cm} (3.17.a)
\[ [Q] = Q_+ - Q_- \]  \hspace{1cm} (3.17.b)

and
\[ P_\pm = P(\bar{Y} \gtrless \beta \bar{X}), \text{ etc.} \]  \hspace{1cm} (3.17.c)

Thus from (3.11) and (3.12) (ignoring the \( a_{\frac{YY}{YY}} \) term) we have
\[ \beta (E_+ - E_-) = E_+ W_+ - E_- W_- \]  \hspace{1cm} (3.18)
\[ \beta(W_+ - W_-) = \frac{1}{2} (W_+^2 - W_-^2) + \frac{K}{2} (E_+ - E_-) \]  \hspace{1cm} (3.19)

We now use the known asymptotic values

\[ E_+ = E(\bar{Y} \sim +\infty) = 1 \]  \hspace{1cm} (3.20)

\[ W_+ = \psi_\bar{Y}(\bar{Y} \sim +\infty) = 0 \]  \hspace{1cm} (3.21)

Furthermore on the wall the boundary condition

\[ \frac{\partial A}{\partial \bar{Y}} = iA \quad \text{on} \quad \bar{Y} = \bar{X} \]  \hspace{1cm} (3.22)

implies that

\[ W = \psi_\bar{Y} = 1 \quad \text{on} \quad \bar{Y} = \bar{X} \]  \hspace{1cm} (3.23)

Consistent with the shock assumption we take

\[ W_- = 1, \quad \text{for} \quad \bar{X} < \bar{Y} < \beta \bar{X} \]  \hspace{1cm} (3.24)

Entering (3.20), (3.21) and (3.24) in (3.18) and (3.19) we obtain

\[ \beta(1 - E_-) = -E_- \]  \hspace{1cm} (3.25)

\[ -\beta = -\frac{1}{2} + \frac{K}{2} (1 - E_-) \]  \hspace{1cm} (3.26)

It follows after eliminating \( \beta \) that

\[ E_- = \frac{1}{2K} \left(2K + 1 + \sqrt{8K + 1}\right) \]  \hspace{1cm} (3.27)

where the positive sign has been chosen for the radical so that \( E_- \) is always positive. The shock slope is then
\[
\beta = \frac{E}{E - 1} = (3 + \sqrt{8K + 1})/4 \quad (3.28)
\]

using (3.27). Knowing \(K\) from the characteristics of the incident waves and the wedge angle, the squared amplitude inside the Mach stem region as well as the slope of the stem boundary are thus given theoretically by (3.27) and (3.28). These are compared with the results from the numerical experiments in Figure 3.9. The agreement is surprisingly good, despite the gross simplification of the diffraction region outside the Mach stems.

Finally, let us derive theoretically the angle of bending of the free surface crests across the shock. Integrating Eq. (3.24) with respect to \(Y\), we get

\[
\psi_+ = \bar{Y} + f(\bar{X}) \quad \quad \quad \beta \bar{X} > \bar{Y} > \bar{X} \quad (3.29)
\]

On \(\bar{Y} = \beta \bar{X}\), we require \(\psi_- = \psi_+\) so that

\[
\psi_- \bigg|_{\bar{Y} = \beta \bar{X}} = \beta \bar{X} + f(\bar{X}) = \psi_+ \bigg|_{\bar{Y} = \beta \bar{X}} = \psi \bigg|_{\bar{Y} = \infty} = -K\bar{X}/2 \quad (3.30)
\]

using the boundary condition. Hence,

\[
f(\bar{X}) = -K\bar{X}/2 - \beta \bar{X} \quad (3.31)
\]

Now the free surface displacement is

\[
\zeta_1(k_o x, k_o y) = \text{Re}(A e^{i k_o y}) = a \cos (\psi + k_o x) \quad (3.32)
\]

so that
Figure 3.9: Comparison of numerical values for the slope of the Mach stem region $\beta$ (*) and the amplitude squared along the wall $E_-$ ($\Delta$) with that predicted by shock theory (---).
\[ \zeta_\perp = a_\perp \cos [(k_0 x)(1 - \varepsilon^2 x^2/2 - \varepsilon^2 \beta)] + \varepsilon (k_0 y) \]  \quad (3.33)

If we denote by \( \theta \) the angle through which the crests bend across the shock (see Fig. (3.10)), we have

\[
\tan \theta = \frac{\varepsilon}{1 - \varepsilon^2 \left( \frac{K}{2} + \beta \right)} = \frac{\varepsilon}{1 - \varepsilon^2 \left( \frac{K}{2} + \frac{3+\sqrt{8K+1}}{4} \right)} > \varepsilon \quad (3.34)
\]

on using Eq. (3.28) for \( \beta \).

To the leading order \( \tan \theta = \varepsilon \equiv \tan \alpha \), so that the Mach stems are approximately orthogonal to the wall. To assess the effect of \( K \), we define the angle away from the normal by \( \delta = \theta - \alpha \), then

\[
\tan \delta = \tan (\theta - \alpha) = \frac{\varepsilon^3 \left( \frac{K}{2} + \frac{3+\sqrt{8K+1}}{4} \right)}{1 - \varepsilon^2 \left( \frac{K}{2} + \frac{\sqrt{8K+1} - 1}{4} \right)} \quad (3.35)
\]

which is an \( O(\varepsilon)^3 \) quantity. \( \delta \) is plotted as a function of \( \alpha \) and \( K \) in Figure (3.10). To compare with computed results, the directions of the stem given by Eq. (3.35) are added as broken lines (---) in the contours of \( \zeta_\perp \) in Figures (3.8). The agreement is again satisfactory.
Figure 3.10: Plot of $\delta(\alpha; K)$, the forward inclination of the Mach stems relative to the normal, as predicted by shock theory.
III.3 Comparison with Experiments

Here we compare our numerical results with experiments for diffraction of periodic waves by an oblique wall by Nielsen (1962) and Berger and Kohlhase (1976). Nielsen's experiments were performed in a ripple tank over a set of values for the incidence angle ($\alpha = 5^\circ, 10^\circ, 15^\circ, 20^\circ$), water depth ($h = .16 \text{ ft, .24 ft}$), wavelength ($\lambda = 2\pi/k_o$: $\lambda/h = 4.2, 5.9, 7.5, 9.3$) and incidence height ($A^*_I$: $A^*_I/h \sim 0.05 - 0.22$). Note that the water depth is only about two inches and the highest incoming amplitude is less than 0.5 inch. The Stokes' parameter, $\sigma_s = \frac{k_o A^*_I}{(k_o h)^3}$, varies from 0.02 to 0.3. Wave gauge measurements for amplitude were only made at five fixed locations near the wall and the wave patterns were determined from refraction of light rays by the free surface. The more recent experiments of Berger and Kohlhase (1976) used a thermosensitive probe movable at constant speeds (5 m/min or 20 m/min) so that the wave height along the wall was measured at a large number of stations. They used a single water depth $h = 25 \text{ cm (.82 ft)}$ for $\alpha = 10^\circ, 15^\circ, 20^\circ, 25^\circ$, $\lambda/h = 4, 6, 8$ and $A^*_I/h = .09, .14, .17$ ($\sigma_s$ is between .04 and .09).

All the experimental data contain a large degree of scatter. In both sets of experiments, there were evidences of longitudinal and transverse oscillations set up in the tank, and in the case of Berger and Kohlhase, the amplitudes at the wall measured by the moving gauge shows distinct oscillations over a length scale of $\lambda$. The effect of friction may also be important, especially in the experiments of Nielson where there was a clear trend of decreasing wave amplitude with distance along the wall, and for the shallower depth case.

By isolating the effects of each parameter, certain qualitative
trends can be discerned. Here, we choose to interpret the different sets of experimental conditions in terms of the single nonlinear parameter K.

Furthermore, because of the rather arbitrary and somewhat imprecise definitions for the width of the Mach stem in the experiments, we shall only concentrate on the amplitude variations on the wedge. All the experimental data by Nielsen and Berger and Kohlhase are included in Figures 3.11 where the square amplitude of the waves on the wall $|\tilde{A}(\tilde{x},\tilde{y}=\tilde{x})|^2$ are grouped with respect to K and plotted as a function of $\tilde{x}$. The theoretical curves of Figure 3.1 are included. The degree of scatter in the experimental data is evidently quite large, so that meaningful comparisons are possible only in a few isolated instances.
Figure 3.11 (i): Comparison between theory and experiment for the magnitude squared of the enveloped at the wall of a wedge: \( |\tilde{A}(x, y=X)|^2 \). +: experimental data from B (Berger & Kohlhase, 1976), and N (Nielsen, 1962). —: numerical curves of Figure 3.1 for different values of K.
Figure 3.11 (ii): Comparison between theory and experiment for the magnitude squared of the envelope at the wall of a wedge: $|A(X,Y=x)|^2$. $+:$ experimental data from B (Berger & Kohlhase, 1976) and N (Nielsen, 1962). $-$: numerical curves of Figure 3.1 for different values of $K$. 

(e) B: $K = 0.742$

(f) B: $K = 1.159$

(g) B: $K = 1.525$

(h) B: $K = 1.912$
Figure 3.11 (iii): Comparison between theory and experiment for the magnitude squared of the envelope at the wall of a wedge: $|A(X, Y = X)|^2$. +: experimental data from B (Berger & Kohlhase, 1976) and N (Nielson, 1962). ---: numerical curves of Figure 3.1 for different values of K.
Figure 3.11 (iv): Comparison between theory and experiment for the magnitude squared of the envelope at the wall of a wedge: 
\(|A(X, Y=X)|^2\). +: experimental data from B (Berger & Kohlhase, 1976), and N (Nielsen, 1962). ---: numerical curves of Figure 3.1 for different values of K.
Figure 3.11 (v) Comparison between theory and experiment for the magnitude squared of the envelope at the wall of a wedge: $|A(\bar{X}, Y=\bar{X})|^2$. +: experimental data from B (Berger & Kohlhase, 1976), and N (Nielsen, 1962). -: numerical curves of Figure 3.1 for different values of K.
IV. OTHER GEOMETRIES

IV.1 Parabolic Half Body

To see if the features similar to Mach reflection for a wedge are present for open bodies of a different geometry, we consider a half body with a parabolic coastline:

\[ k_o y_B = \sigma \sqrt{k_o x} \]  \hspace{1cm} (4.1)

where \( \sigma = O(1) \) is a relative measure of the island width. At any given \( x \) the width-to-length ratio is \( \frac{y_B}{x} = \sigma \frac{1}{\sqrt{k_o x}} \), so that as long as \( x = O(1) \), \( k_o x = O(\varepsilon) \), \( \varepsilon^2 \frac{y_B}{x} = O(\varepsilon) \), and the slenderness requirement is satisfied.

Since the body boundary and the governing equation are both parabolic, it is possible to absorb the coefficient \( K \). Thus we identify \( \varepsilon \) with the nonlinear wave characteristics and the thickness parameter:

\[ \varepsilon \equiv \sigma k_o A_o \left( \frac{1}{2} \Theta(k_o h) \right)^{1/2} > 0 \]  \hspace{1cm} (4.2)

and define the normalized variables

\[ \tilde{A} = A/A_o ; \quad \tilde{X} = \varepsilon^2 k_o x ; \quad \tilde{Y} = \varepsilon k_o y/\sigma \]  \hspace{1cm} (4.3)

The body (4.1) is given by

\[ \tilde{Y} = \tilde{Y}_B = \sqrt{\tilde{X}} \]  \hspace{1cm} (4.4)

and the governing equations for the \{ linear \} problems are
\[
2 \frac{\partial \bar{A}}{\partial \bar{X}} - \frac{1}{\sigma^2} \frac{\partial^2 \bar{A}}{\partial \bar{Y}^2} + \{ \frac{Q}{2t|\bar{A}|^2 \bar{A}} \} = 0 \\
\bar{Y} > \sqrt{\bar{X}} > 0 \quad (4.5.a)
\]

\[
\bar{A} = 1 \\
on \bar{X} = 0 \quad (4.5.b)
\]

\[
\frac{\partial \bar{A}}{\partial \bar{Y}} = \frac{\sigma^2 \bar{A}}{2 \sqrt{\bar{X}}} \\
on \bar{Y} = \sqrt{\bar{X}} \quad (4.5.c)
\]

\[
\bar{A} = \{ e^{-\frac{\bar{X}}{2}} \} \\
on \bar{Y} = \bar{Y}_\infty \quad (4.5.d)
\]

They are solved numerically for \( \bar{X}_R = 4, \bar{Y}_\infty = 50 \). The linear problem can be solved by the method of similarity (Mei & Tuck, 1979):

\[
\bar{A} = 1 + \int_\eta^\infty e^{it^2/2} / \left( \frac{2i}{\sigma} e^{i\sigma^2/2} - \int_\sigma^\infty e^{i\sigma^2/2} dt \right) \quad (4.6)
\]

where
\[
\eta = \sigma \sqrt{\bar{X}} = k_0 y / \sqrt{k_0 x} \quad (4.7)
\]

and is independent of the relative scaling parameter \( \epsilon \). Note that \( \bar{A} \) is a constant (which depends on \( \sigma \) but is always greater than 1) on the body where \( \eta = \sigma, \bar{X} > 0 \). Thus along the wall, \( \bar{A} \) has a finite jump from the initial value \( \bar{A} = 1 \) at \( \bar{X} = 0 \).

Figures (4.1.a) and (4.1.b) present the squared magnitude of the envelope along the strut \( |\bar{A}(\bar{X}, \bar{Y} = \sqrt{\bar{X}})|^2 \) for \( \sigma^2 = 0.5 \) and 1 respectively. Note that the numerical curve for the linear case starts from 1 but does not immediately reach the analytic value (horizontal dashed line). This discrepancy for small \( \bar{X} \) is associated with the discontinuity along the tip of the body and is reduced by refining the numerical discretization.
Figure 4.1: Magnitude squared of the envelope along the wall of a parabola: $|\tilde{A}(\tilde{x}, \tilde{y}/\sqrt{\tilde{x}})|^2$ for (a) $\sigma^2 = 0.5$, (b) $\sigma^2 = 1$.

- e: analytic linear solution;
- k: computed linear solution - coarse discretization;
- L: computed linear solution - fine discretization;
- n: nonlinear solution - fine discretization;
- N: nonlinear solution - coarse discretization.
In the nonlinear situation, the amplitude again reaches some initial value but then quickly attenuates with distance. From the figures, it appears that for large distance, the nonlinear amplitude will approach a constant asymptote greater than 1. Comparing Figures (4.1.a) and (4.1.b), we note that the curves are qualitatively very similar, and the effect of a larger $\sigma$ (fatter parabola) is mainly an increase of amplitude through the boundary condition (4.5,c). The qualitative similarity is also present in the overall envelope profiles, and we shall therefore focus our discussion on further results for $\sigma = 1$ only.

Figures (4.2,a,b) and (4.3,a,b) show respectively three-dimensional and contour pictures of $|\bar{A}(\bar{x},\bar{y})|^{2}$ for $\sigma = 1$, and for zero and finite amplitude waves. For contrast, the body of the island is set at 2.5 in the three dimensional plots. In the contour figures, the increment $\Delta|\bar{A}|^{2}$ is 0.1.

Let us examine the variation of $|\bar{A}|^{2}$ for a fixed $\bar{x}$. In the linear case, the wave envelope (which can also be computed from Eq. (4.7)) has an amplitude which is maximum at the wall, and decreases and oscillates as typical diffraction fringes in the transverse direction. The contours here are parabolas given by (4.8). For very small $\sqrt{\bar{x}}$ and $\bar{y} = 0(1)$ (i.e., for large $\eta$), $\bar{A}^{2}$ approaches 1 as $\eta^{-1}$ in a rapidly oscillatory manner so that the numerical solution is unreliable.

For nonlinear waves the maximum $|\bar{A}|^{2}$ is not at the wall, but is attained along a line (shown as a dotted curve in Figures (4.3,b)) some distance away. The amplitude along this maximum line is not constant and decreases in distance. Inside this line, the envelope amplitude is quite flat, while some distance outside the line, the solutions resembles
Figure 4.2: Three-dimensional plots of $|\tilde{A}|^2$ for a parabola ($c = 1$) in the region $0 \leq \tilde{X} \leq 4$, $\sqrt{\tilde{X}} \leq \tilde{Y} \leq 10$, for (a) linear solution, (b) nonlinear solution. All vertical scales are the same and body is given a height of 2.5.
Figure 4.3: Contour plots of $|\bar{A}|^2$ for a parabola ($\sigma = 1$) for (a) linear solution, (b) nonlinear solution. Contours are at 0.1 increment. ---- (in (b)): location of maximum $|\bar{A}|^2$. 
the linear result, but with the contours forming steeper parabolas. In fact, these contours in the outer region resemble those of a linear diffraction by a larger parabola. Thus nonlinearity displaces the diffraction zone outwards. To further illustrate the above features, the linear and nonlinear cross-section profiles of $|\tilde{A}|^2$ are plotted at $\tilde{X} = 2$ and 4 (See Figures (4.4.a,b). Note that the mild edge at the body is absent for $K = 0$ and has a width which increases with $\tilde{X}$ with nonlinearity.

For the instantaneous free surface elevation, we set $c^2 = 0.1$ and $\sigma = 1$ and give the results in Figures (4.5,a,b) and (4.6.a,b). For the linear case, the wave crests are approximately normal to the body and the diffraction pattern follows the parabolic nature of its envelope with humps along the crests which decrease in width and amplitude in the transverse direction away from the body, falling within successive parabolic rays. Unlike wedge diffraction, the contours here show very little shift of crest lines and the waves remain quite straight.

In the nonlinear case, the qualitative features remain, but the diffraction pattern now follow much steeper parabolas. In particular, the first humps at the wall fall within a much wider region forming ridges of almost constant height. Within this parabolic region, the crest lines are slightly bent forward, so that they meet the wall at an angle close to (and somewhat on the waveward side of) the perpendicular. This bending, however, is very small and the transition quite mild as compared to that for a straight wedge. The less pronounced effect of nonlinearity may be attributed to the decreasing slope of the parabola with increasing distance.
Figure 4.4: Cross-sections of $|\tilde{A}(\tilde{x}_1, \tilde{y})|^2$ for a parabola at (i) $\tilde{x}_1 = 2$, (ii) $\tilde{x}_1 = 4$ for (a) linear, (b) nonlinear waves. (Note that the curves start at the wall of the body given by $\tilde{y} = \sqrt{\tilde{x}_1}$.)
(a) linear

(b) nonlinear

Figure 4.5: Three-dimensional plots of $r(x, y, t=0)$ for a parabola ($\sigma = 1$) with $\epsilon^2 = 0.1$ for (a) linear, (b) nonlinear waves, in uniformly scaled $k_x, k_y$ coordinates. The region shown is for $0 \leq \tilde{X} \leq 4, \sqrt{X} \leq \tilde{Y} \leq 12$ corresponding to $0 \leq k_0X \leq 40, \sqrt{k_0X} \leq k_0Y \leq 37.95$. All vertical scales are the same and the body is given a height of 0.
Figure 4.6: Contour plots of $\zeta_1(k_0x, k_0y, t=0)$ for a parabola ($\sigma = 1$) with $\varepsilon^2 = 0.1$ for (a) linear, (b) nonlinear waves, in uniformly scaled $k_0x, k_0y$ coordinates. The contour increment is 0.25.
IV.2 Slender Island of Finite Length

We turn to the case of head seas incident on a long and slender island. Of particular interest is the shadow zone and the associated wave pattern behind the island. Assuming a symmetric island of length $L$ and half-width $B$, the shoreline can be written as

$$y_B = B \xi \left( \frac{2}{L} \right)^{1/2} \tag{4.9}$$

where

$$0 < \xi < 1 \quad \text{for} \quad 0 < x < L \tag{4.10}$$

and $\xi = 0$ otherwise. We define

$$\varepsilon = \frac{B}{L} << 1 \quad \tag{4.11.a}$$

and insist that

$$k_0 B = O(\varepsilon)^{-1} \quad k_0 L = O(\varepsilon^{-2}) \quad \tag{4.11.b}$$

Furthermore, we introduce a parameter $\sigma$ which is the square root of the ratio of two small parameters $\varepsilon$ and $(k_0 B)^{-1}$

$$\sigma = \frac{k_0 B}{\sqrt{k_0 L}} = \sqrt{k_0 B \varepsilon} \sim O(1) \quad \tag{4.12}$$

The convenient normalizations are

$$\bar{A} = A/A_0, \quad \bar{x} = \frac{x}{L} \quad \text{and} \quad \bar{y} = \frac{y}{B} \quad \tag{4.13}$$

so that
\[ 2 \frac{\partial \overline{A}}{\partial \overline{X}} - \frac{1}{\sigma^2} \frac{\partial^2 \overline{A}}{\partial \overline{Y}^2} + 1 \kappa | \Delta |^{2-} = 0 \quad \overline{Y} > \xi(\overline{X}), \quad \overline{X} > 0 \] \hspace{1cm} (4.14)

\[ \frac{\partial \overline{A}}{\partial \overline{Y}} = i \sigma^2 \xi^*(\overline{X}) \overline{A} \quad \text{on} \quad \overline{Y} = \xi(\overline{X}) \] \hspace{1cm} (4.15)

and the conditions at \( \overline{X} = 0 \) and \( \overline{Y} = \overline{Y}_\infty \) are as in Eqs. (4.5). The expression for \( \kappa \) is the same as Eq. (2.52), but with the present definition for \( \varepsilon \).

For simplicity we consider a half-sine island given by

\[ \xi(\overline{X}) = \begin{cases} \sin (\pi \overline{X}) & 0 \leq \overline{X} \leq 1 \\ 0 & \overline{X} \geq 1 \end{cases} \] \hspace{1cm} (4.16)

As with the case for the parabolic island, the qualitative picture of the solutions for different thickness parameters \( \sigma \) are very similar, and only the case for \( \sigma = 1 \) will be presented here. The computation domain used is \( \overline{X}_R = 4, \overline{Y}_\infty = 50 \). Unlike the semi-infinite bodies considered so far, where the widths are always increasing with \( \overline{X} \), special numerical treatment is necessary here when integrating through the second half of the island \( \left( \frac{1}{2} \leq \overline{X} \leq 1 \right) \). In this region, the body boundary is receding and new grids must be added at successive steps (in \( \overline{X} \)). To fill in the new values, (see Eqs. (2.55),(2.56)), the results from the previous \( \overline{X} \) step are extrapolated numerically. There is an associated increase in truncation error which appears as small fluctuations on the solution around the trailing end of the body. These fluctuations are reduced by refining grid sizes, and are acceptably small (although noticeable) in the following results.

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The square of the envelope amplitude along the shore and on the centerline behind the island, \( |\tilde{A}(\bar{x}, \bar{y}) = \xi(\bar{x})| \) \(^2\) is shown in Figure (4.7). The magnitude increases gradually from 1 at the tip and reaches a maximum at about a quarter of the distance along the body. This then decreases to a minimum value below 1 at the end of the island, leading into a lee of diminished amplitude which slowly returns to the ambient level 1 after approximately one island length. The presence of non-linearity reduces the wave height along the island and the location of maximum value is shifted slightly towards the front. The curves, however, have very similar shapes, and reach the same minimum value at the end of the body for all \( K \). The difference in the lee is quite small.

Figures (4.8.a(i),(ii) and 4.8.b(i),(ii)) show
\[ |\tilde{A}(\bar{x}, \bar{y})| \] \(^2\) for \( K = 0 \) and 6 from two different directions, while Figures (4.9.a and b) are the contour plots. In the three-dimensional picture the maximum height for \( K = 0 \) may be seen at the middle of the leading face of the island and is slightly higher than the top of the body which is assigned a value of 3.5. The run-up for the nonlinear case is clearly much smaller. Referring to Figures (4.9.a and b), we note that in the nonlinear curve there are two relatively flat regions (marked A,B in Figure (4.9.b)) behind the island bounded by nearly straight and divergent contours. This is absent in the linear case where the variation is more gradual in the shadow zone and the contour lines are still parabolic.

Finally we calculate the free surface contours for \( \varepsilon^2 = 0.05 \) (which corresponds to an island of aspect ration \( B/L = \varepsilon = 0.22 \)), see Figures (4.10). As with the previous cases, the wave crests near the island are almost perpendicular to the shore. In spite of that,
Figure 4.7: Magnitude squared of the envelope $|\tilde{A}|^2$ for an island ($\sigma = 1$) on $\tilde{y} = \xi(\tilde{x})$ for $0 < \tilde{x} \leq 1$ and $\tilde{y} = 0$ for $\tilde{x} > 1$. (Note: Small fluctuations near the trailing end at $\tilde{x} = 1$ are numerical noise.)
Figure 4.8 (a): Three-dimensional plots of $|\vec{A}|^2$ for an island ($\sigma = 1$) for (a) $K = 0$, (b) $K = 6$ as viewed from (i) $\vec{X} \sim -\infty$ (ii) $\vec{Y} \sim \infty$. The domain shown is $0 \leq \vec{X} \leq 4$ and $\vec{Y} \leq 10$. All vertical scales are the same and the island is given a height of 3.5.
Figure 4.8 (b): Three-dimensional plots of $|\vec{A}|^2$ for an island ($\sigma = 1$) for (a) $K = 0$, (b) $K = 6$ as viewed from (i) $\vec{X} \sim -\infty$, (ii) $\vec{Y} \sim \infty$. The domain shown is $0 \leq \vec{X} \leq 4$ and $\vec{Y} \leq 10$. All vertical scales are the same and island is given a height of $3.5$. 
Figure 4.9: Contour plots of $|\tilde{A}|^2$ for an island ($\sigma = 1$) for (a) $K = 0$, (b) $K = 6$. Contour increments are 0.2.
Figure 4.10: Contour plots of $\zeta_1(k_0x, k_0y, t=0)$ for an island ($\sigma = 1$) with $\varepsilon^2 = 0.05$ for (a) $K = 0$, (b) $K = 6$, in uniformly scaled $k_0x$, $k_0y$ coordinates at contour increments of 0.25.
the linear crest lines remain quite straight along the distance of the body. Behind the body, the contours shift backwards slightly in the shadow zone, which diminishes as the zone widens, and becomes almost unnoticeable after about one body length. In the nonlinear case, the waves are bent somewhat forward along the leading half of the island and curves backwards as the body width decreases. In the shadow zone, there is a distinct backward shift of the contour lines. These shifted crests widen linearly with distance forming a wedge but with no apparent decrease in the phase difference even after 1.5 body lengths.
IV.3 Semi-Infinite Breakwater

The present parabolic approximation is ideally suited to study the classical problem of diffraction of a plane wave normally incident upon a half-infinite screen. Historically, the linear parabolic approximation was originated in the linear diffraction by an edge at oblique incidence, for describing the shadow boundary. Extension to nonlinear diffraction by a slit was anticipated by Karpman (1975, p. 128) who gave no quantitative results however.

In this case, the only small parameter and horizontal length scale are associated with the steepness and length of the incident waves. As with the parabolic island, we define \( \varepsilon \) by the nonlinear wave steepness and the water depth parameter \( \Theta \):

\[
\varepsilon = \kappa_0 \sqrt{\frac{1}{2} \Theta(k_0 h)} \quad (4.17)
\]

and the normalized coordinates

\[
\bar{x} = \varepsilon^2 k_o x \quad , \quad \bar{y} = \varepsilon k_o y \quad (4.18)
\]

Assuming a semi-infinite wall at \( \bar{x} = 0, \bar{y} < 0 \), we have then

\[
2 \frac{\partial^2 \bar{A}}{\partial \bar{x}^2} - \left( \frac{\partial^2 \bar{A}}{\partial \bar{y}^2} + \frac{1}{2} \frac{\bar{A}^2}{\bar{A}} \right) = 0 \quad \bar{x} > 0, \quad |\bar{y}| < \bar{y}_\infty \quad (4.19.a)
\]

\[
\bar{A} = 1 \quad \bar{x} = 0, \quad \bar{y} > 0 \quad (4.19.b)
\]

\[
\bar{A} = 0 \quad \bar{x} = 0, \quad \bar{y} < 0
\]

\[
\bar{A} = \left\{ \frac{1}{e^{-i\bar{x}/2}} \right\} \quad \bar{y} = \bar{y}_\infty, \quad \bar{x} > 0 \quad (4.19.c)
\]
for \{ linear \ \text{nonlinear}\} \ \text{case.}

The linear equations of (4.19) have a simple solution in terms of the similarity variables \( \eta = \frac{x}{\sqrt{X}} \):

\[
\bar{A} = \frac{1}{2} \left( 1 + \frac{\int_0^\infty e^{-it^2/2} \, dt}{\int_0^\infty e^{-it^2/2} \, dt} \right)
\]

(4.20)

From experience, a direct application of the initial condition (4.19.b) with a finite step at the origin is found to introduce relatively large numerical errors. Hence for computational purposes, we replace (4.19.b) by

\[
\bar{A} = \frac{1}{2} \left[ 1 + \tanh \frac{\bar{Y}}{\Delta} \right] \quad \text{on } \bar{X} = 0
\]

(4.21)

where \( \Delta \) is roughly equal to the grid size \( \Delta \bar{Y} \).

Figure (4.11) plots \( |\bar{A}|^2 \) along the \( \bar{X} \)-axis directly behind the tip. The linear analytic formula (4.21) for \( \bar{Y} = 0 \) (\( \eta = 0 \)) is \( \bar{A} = \frac{1}{2} \), and the curve \( |\bar{A}(\bar{X}, \bar{Y} = 0)|^2 \) for \( K = 0 \) is indistinguishable from 0.25. For the nonlinear case, the amplitude \( |\bar{A}| \) increases from \( \frac{1}{2} \) and reaches an asymptotic value of about 0.65 (\( |\bar{A}|^2 = 0.42 \)). From the profiles of \( |\bar{A}(\bar{X}, \bar{Y})|^2 \) (Figs. (4.12.a,b)(4.13.a,b)), we see that the major influence of nonlinearity is in a small approximate parabolic region along the centerline (\( \bar{Y} = 0 \)). The linear profile (which can also be obtained from (4.20)) shows a steep drop from the first (and highest) peak on the illuminated side (\( \bar{Y} > 0 \)) into the shadow zone. In the nonlinear case, there is a much milder transition zone which penetrates much more deeply into the illuminated and shadow regions; the height of the first peak is
Figure 4.11: Magnitude squared of the envelope $|\tilde{A}(\bar{x}, \bar{y}=0)|^2$ behind the tip of a semi-infinite screen. N: nonlinear, L: linear results.
Figure 4.12: Three dimensional plots of $|\tilde{A}|^2$ for a semi-infinite screen for (a) linear, (b) nonlinear incident waves for $0 \leq \tilde{X} \leq 4$, $-7 \leq \tilde{Y} \leq 14$. The same vertical scales are used for (a) and (b).
Figure 4.13: Contour plots of $|\bar{A}|^2$ for a semi-infinite screen for (a) linear, (b) nonlinear incident waves. The contours are at 0.1 increments.
reduced and the amplitudes behind the wall are higher.

This is clearly shown in the contour plots of $|\tilde{A}|^2$. By plotting the cross-sections of the $|\tilde{A}|^2$ surface at some distance $\tilde{X}$, the above features are further illustrated. See Figures (4.14a,b). Note the depressed peak and slower slope near $\tilde{X} = 0$ for the nonlinear amplitude, and the similarity of the two profiles away from the transition.

Figures (4.15), (4.16) show the free surface elevation using $\varepsilon^2 = 0.1$. For the linear case, the waves show the familiar ring-like diffraction pattern. The crests are quite parallel (and straight) until some distance before the wall. It is possible to identify a small parabolic region along the centerline (discussed earlier) within which we have most of the amplitude decrease. These crests finally curve around the tip as circular rings into the shadow. For finite amplitude waves, the picture is very similar but the transition is much more gradual.
Figure 4.14: \( \bar{Y} \) cross-sections of \( |\bar{A}|^2 \) at \( \bar{X} = 4 \) for a semi-infinite screen for (a) linear, (b) nonlinear incident waves.
Figure 4.15: Three-dimensional plots of the free surface $\zeta_1$ at $t = 0$ behind a semi-infinite screen with $\varepsilon^2 = 0.1$ for (a) linear, (b) nonlinear waves. The domain shown corresponds to $0 \leq \bar{X} \leq 4$, $-10 \leq \bar{Y} \leq 10$ and the viewpoint is located at $\bar{X} \sim \bar{Y} \leq 0$. 

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Figure 4.16: Contour plots of the free surface $\zeta_1$ at $t = 0$ behind a semi-infinite screen with $\epsilon^2 = 0.1$ for the (a) linear, (b) nonlinear problems. The contour increments are 0.25 and the scales for $k_0x, k_0y$ are the same.
PART TWO

DIFFRACTION OF TRANSIENT STOKES' WAVES
AT GRAZING INCIDENCE

V. SLOWLY MODULATED NONLINEAR DISPERSIVE GRAVITY WAVES

V.1 Background

Permanent waves which propagate in shallow water without change of form have been known for a long time (called solitary waves or cnoidal waves). According to the linear theory for arbitrary depth, any localized disturbance may be regarded as the superposition of a spectrum of simple harmonic waves, each of which will disperse at its own phase velocity so that the extent of the disturbance becomes continuously broader. This is (linear) frequency dispersion. On the other hand, the nonlinear theory of Airy for shallow water (Airy, 1845) predicts the opposite tendency that any initial hump would steepen forward and eventually break. It was Korteweg and de Vries (1895) (see also Boussinesq, 1871; Rayleigh, 1876; Ursell, 1953) who successfully brought out the important interplay between nonlinearity and (frequency) dispersion. It is now understood that finite amplitude waves of permanent form are the consequence of perfect dynamic equilibrium between nonlinearity and frequency dispersion,

Uniform Stokes' waves (Stokes, 1847) in deep water are also a case of periodic permanent waves. However, theoretical investigations by Benjamin and Feir (1967), Benjamin (1967) and Zakharov (1967), as well as experiments by Feir (1967), have demonstrated the inherent instability of such equilibrium for Stokes' waves subject to side-band disturbances. These developments have led to surging interests in the study of nonlinear
evolution of transient Stokes' wave trains.

The first work in this direction appears to be due to Zakharov (1968) who applied a method similar to van der Pohl's method (multiple scales) and derived the single two-dimensional cubic-nonlinear Schrödinger equation governing the slowly varying complex two-dimensional modulation amplitude of periodic gravity waves in deep water:

\[
\text{i} \left( \frac{\partial A}{\partial t} + \frac{\omega}{2k_o} \frac{\partial A}{\partial x} \right) - \frac{\omega}{8k_o^2} \frac{\partial^2 A}{\partial x^2} + \frac{\omega}{4k_o^2} \frac{\partial^2 A}{\partial y^2} - \frac{1}{2} \omega k_o^2 |A|^2 A = 0 \tag{5.1}
\]

where the free surface elevation \( \eta(x,y,t) \) is given in terms of the complex envelope function \( A(x,y,t) \) via

\[
\eta(x,y,t) = \text{Re}\{A(x,y,t)e^{\text{i}k_x x - \text{i}\omega t} \} \tag{5.2}
\]

and \( \omega, k_o \) are the carrier frequency and corresponding wave number; and \( y, t \) the transverse distance and time respectively. Eq. (5.1) can be rewritten as the one-dimensional nonlinear Schrödinger equation:

\[
\text{i} \frac{\partial A}{\partial t} - \frac{\omega}{8k_o^2} (1 - 3 \sin^2 \alpha) \frac{\partial^2 A}{\partial z^2} - \frac{1}{2} \omega k_o^2 |A|^2 A = 0 \tag{5.3}
\]

after the substitution

\[
z = (x - \frac{\omega}{2k_o} t) \cos \alpha + y \sin \alpha \tag{5.4}
\]

which for \( 1 - 3 \sin^2 \alpha > 0 \) can be rescaled directly to give the more familiar form (in dimensionless variables):

\[
\text{i} \frac{\partial A}{\partial \tau} - \frac{1}{8} \frac{\partial^2 A}{\partial \xi^2} - \frac{1}{2} |A|^2 A = 0 \tag{5.5}
\]

Equations of this form are also obtained in the study of various nonlinear dispersive systems (see for example, Karpman and Krushkal, 1969;
For initial conditions that diminish sufficiently rapidly as $|\xi| \to \infty$, i.e., for pulse envelopes, the nonlinear Eq. (5.5) was solved exactly by Zakharov and Shabat (1972) using an inverse-scattering method first discovered by Gardner, Green, Kruskal and Miura (1967) for the Korteweg-de Vries (KdV) equation. The most important result of Zakharov and Shabat is to show the fundamental role played by particular solutions of Eq. (5.5) called solitons, which are progressive envelope pulses of permanent profiles whose heights and widths are inversely proportional to each other, but are unrelated to the speed of propagation $U$ relative to group velocity:

$$A = \text{sech}\left[\sqrt{2} a(\xi - Ut)\right] e^{-i\frac{a^2 t}{4-4iUx+2iU^2 t}}$$ (5.6.a)

for the dimensionless Eq. (5.5) or for Eq. (5.1):

$$A = \text{sech}\left\{\sqrt{2} k^2 a[x - \left(\frac{\omega}{2k_o} + U\right)t]\right\} \cdot$$

$$e^{-ik^2 a^2 \omega t/4 - 4ik^2 U[x - \left(\frac{\omega}{2k_o} + \frac{U}{2}\right)t]/\omega_o}$$ (5.6.b)

in physical variables in a coordinate system fixed in space.

Zakharov and Shabat (1972) found that such solitons are stable to one-dimensional perturbations (in $x$). Furthermore, they are stable formations and survive collisions with each other. On the other hand, any arbitrary initial pulse envelope will disintegrate into a definite number of solitons plus an oscillatory tail which (like linear dispersion) spreads out linearly with time with an amplitude decay of $\frac{1}{\sqrt{t}}$.

For finite but constant water depth, Hasimoto and Ono (1972)
allowed slow modulation only in the propagating direction and deduced a one-dimensional depth-dependent version of Eq. (5.1). Their results are, however, incomplete for finite depth in that they ignored the mean drift current which may be of importance here. The generalization of two-dimensional modulated wave packets to finite depth was first made by Benney and Roskes (1969) (see also Davey and Stewartson, 1974) who extended the results of Benjamin and Feir (1967) and established the two-dimensional modulational instability of uniform Stokes' waves. They obtained two coupled differential equations involving the complex envelope function and the drift current. These equations are uncoupled and reduced to Eq. (5.1) in the deep water limit, and to the results of Hasimoto and Ono (1972) in the one-dimensional case of no transverse variations and ignoring mean current. In a somewhat different approach, Chu and Mei (1970, 1971) applied a WKB-perturbation scheme and derived one-dimensional modulation equations similar to those based on Whitham's theory (Whitham, 1965a,b; 1967a,b), but containing extra dispersive terms, which were shown to be of the same order as the nonlinear correction. Their inclusion altered the nature of the Whitham equations and removed the singularity encountered in earlier applications of Whitham equations (Lighthill, 1967; Howe, 1967, 1968). The apparent difference in forms of Whitham's equations as modified by Chu and Mei and the Schrödinger equation was resolved by Davey (1972). Yuen and Lake (1975) showed the equivalence of Eq. (5.5) and higher order theory of Whitham's method, so that the earlier discrepancies were merely the result of expansions to different orders.

The vigorous theoretical developments of this subject have also
been complemented by important experimental and numerical investigations. The first experiments were by Feir (1967) who studied the evolution of an envelope pulse. Chu and Mei (1971) solved their Whitham–like equations numerically for the initial condition of a pulse and compared their results qualitatively to those of Feir. One of the Chu–Mei equations, however, contains the envelope amplitude in the denominator, and the numerical calculation can only be carried out for a finite time up to the first occurrence of any node in the envelope. Yuen and Lake (1975) performed elaborate wave pulse experiments, and compared these results quantitatively with those from numerical calculations of the Schrödinger equation (1.1), as well as its theoretical predictions based on the exact solution of Zakharov and Shabat (1972). The good agreement showed that the nonlinear Schrödinger equation indeed provides a satisfactory model for the long-time evolution of wave packets, and that computations using the Schrödinger equation seem to be more expedient than those based on the Whitham–Chu–Mei equations (see also the comment by Roskes (1975)).

For the post-stability development of a continuous Stokes' wave train, inverse-scattering method for Eq. (5.5) cannot be applied and no analytic solution is known. From experimental evidence, Benjamin (1967) conjectured that the exponential growth of unstable side-band modulations would eventually lead to complete disintegration and incoherence and the spread of energy over the frequency spectrum (see also Hasselmann, 1963, 1967). This was shown numerically not to be so by Chu and Mei (1971) and more completely by Lake, Yuen, Rungaldier and Ferguson (1977) (see also Lake and Yuen, 1977) who did detailed experiments which corroborated with their numerical solutions of the nonlinear Schrödinger equation. Lake et al.
(1977) established that the long-time evolutions of a nonlinear wave train exhibits the Fermi-Pasta-Ulam (FPU, see Fermi, Pasta and Ulam, 1940) recurrence phenomenon so that the initial wave train is reconstructed after one recurrence period of modulation and demodulation, and that an irreversible equipartition of energy to a random final state does not occur. The nonlinear feature of recurrence is of course familiar in the numerical calculations of the KdV equation (Zabusky and Kruskal, 1965).

For the evolution of two-space-dimensionally modulated wave packets, the FPU recurrence phenomenon was again demonstrated numerically by Yuen and Ferguson (1978), who computed Eq. (5.1) for a variety of two-dimensional initial conditions and (artificial) periodic boundary conditions.

In order to help understand our own result, it is useful to discuss some known theoretical properties of the Schrödinger equations (Eqs. (5.1,3,5)).
Some Known Theoretical Properties of the Schrödinger Equation

Conservation Laws

Zakharov and Shabat (1972) demonstrated that the one-dimensional form of Eq. (5.1), Eq. (5.5), possesses an infinite number of conservation laws, an enumerable set of which are so-called polynomial laws conserving certain spatial integrals of polynomials of $A$ and its derivatives. The first several conserved integrals are:

\[ C_1 \sim \int_{-\infty}^{\infty} |A|^2 \, dx \quad (5.7,a) \]

\[ C_2 \sim \int_{-\infty}^{\infty} (A^{*}A_{\xi} - A^{*}_{\xi}A) \, dx \quad (5.7,b) \]

\[ C_3 \sim \int_{-\infty}^{\infty} C|A_{\xi}|^2 - \frac{1}{4} |A|^4 \, dx \quad (5.7,c) \]

\[ C_4 \sim \int_{-\infty}^{\infty} (A^{*}_{\xi\xi\xi} + \frac{3}{4} A A^{*}_{\xi} |A|^2) \, dx \quad (5.7,d) \]

\[ C_5 \sim \int_{-\infty}^{\infty} \left| A_{\xi \xi} \right|^2 + \frac{1}{8} |A|^6 - \frac{1}{4} (|A_{\xi}|^2)^2 - \frac{3}{2} |A_{\xi}|^2 |A|^2 \, dx \quad (5.7,e) \]

As in Gardner et al. (1967, 1968, 1970) for the KdV equation, Zakharov and Shabat identified the first three conserved quantities $C_1$, $C_2$, $C_3$ with the number of particles (mass), the momentum and the energy respectively. This turns out to be erroneous for the energy for both the KdV and the Schrödinger equation, as pointed out by Ablowitz and Segur (1979). To see this, let us compute the mass, (horizontal) momentum and energy of the wave motion (in physical variables):

\[ \text{mass: } M = \rho \int_{-\infty}^{\infty} dx \, \zeta \quad (5.8.a) \]
horizontal momentum: \[ m_x = \rho \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz (\phi_x - \frac{\omega}{2k_0}) \] (relative to group velocity) \hfill (5.8.b)

total energy: \[ E = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx (g\xi^2 + \int_{-\hbar}^{\xi} dz |\nabla\phi|^2) \] \hfill (5.8.c)

The first term in Eq. (5.8.c) being the potential energy and the second term the kinetic energy. We now substitute in the explicit expressions of \( \phi \) and \( \xi \) in terms of \( A \) (see Appendix A) and express \( M, m_x, E \) in a perturbation series in powers of \( \varepsilon \). Note that any harmonic terms contribute only at higher order as can be shown by repeated integration by parts:

\[ \int_{-\infty}^{\infty} dx \text{ } F(x_1,y_1,t_1)e^{in\psi} = -\frac{1}{Ink}\int_{-\infty}^{\infty} dx \frac{\partial F}{\partial x} e^{in\psi} \]

\[ = -\frac{\omega}{Ink}\int_{-\infty}^{\infty} dx \frac{\partial F}{\partial x_1} e^{in\psi} \] \hfill (5.9)

The result, after some algebra and ignoring constant factors, gives

\[ M \sim \varepsilon a_1 C_1 + \varepsilon^2 a_2 C_2 + O(\varepsilon^3) \] \hfill (5.10.a)

\[ m_x \sim \varepsilon^2 a_3 C_2 + O(\varepsilon^3) \] \hfill (5.10.b)

\[ E \sim \varepsilon a_4 C_1 + \varepsilon^2 a_5 C_2 + O(\varepsilon^3) \] \hfill (5.10.c)

where \( a_i \) are constants. Eqs. (5.10) are identical to those given by Ablowitz and Segur (1979), except for a typographical error, and provide the correct physical interpretation for the conserved integrals \( C_1, C_2, C_3 \). Thus the interpretation of Zakharov and Shabat is correct only to the leading order for mass and momentum.
V.2.2 Plane Permanent Solutions: Solitons, Oblique Solitons and Their Stability

The two-dimensional Eq. (5.1) admits exact solutions of plane permanent form modulating the wave group at any oblique angle $\alpha$ with respect to the direction of propagation of the packet. Referring to Eq. (5.3), we see that there is a critical angle

$$\alpha = \alpha_c \equiv \sin^{-1} \left( \frac{1}{\sqrt{3}} \right) = 35.26^\circ$$  \hspace{2cm} (5.11)

through which the coefficient of the dispersive term $A_{zz}$ changes sign.

For $\alpha < \alpha_c$, Eq. (5.3) possesses the periodic elliptic $cn$ and $dn$ solutions and their limiting solitary sech forms; while for $\alpha > \alpha_c$, only periodic elliptic $sn$ and negative solitary tanh solutions are possible. These solutions are derived in detail in Hui and Hamilton (1979) using a pair of equations derived from Eq. (5.1). The only envelope that decays as $|z| \to \infty$ is the sech profile soliton (Eqs. (5.6)) which exists only when $\alpha < \alpha_c$.

In terms of the angle $\alpha$ and the oblique coordinate $z$, the expression for the soliton is:

$$A = a \text{ sech} \left[ k_o^2 a \sqrt{\frac{2}{1-3\sin^2 \alpha}} (z - Ut) \right] \cdot \exp \left\{ -i(k_o a)^2 \omega t/4 - 4ik_o^2 U(z-U/2)/\omega(1-3\sin^2 \alpha) \right\}$$  \hspace{2cm} (5.12)

Hence the amplitude and length of the soliton are inversely proportional but bears no direct relationship to its (second order) speed $U$ relative to group velocity, unlike the solitary wave solitons of the KdV equation in shallow water. Using inverse-scattering transform methods, multi-soliton solutions to Eq. (5.3) (with different relative positions, phases,
amplitudes and speeds) can also be constructed (Zakharov and Shabat, 1972; Ablowitz et al., 1974). Furthermore, Zakharov and Shabat (1972) showed that solitons are stable formation in the sense that they can survive collisions with only a subsequent shift in position and phase, but with no change of profile or speed, a phenomenon similar to that observed for collisions of KdV solitons. The stability of a plane soliton to general small perturbations, however, does not immediately follow. For plane (one-space-dimensional) perturbations along the same direction as a soliton, Zakharov and Shabat (1972) proved that the soliton is stable or neutrally stable (i.e., stable but tends asymptotically to a soliton with parameters slightly perturbed) depending on whether or not the perturbation corresponds to a continuous spectrum in the related scattering problem, and that the perturbation would decrease like $\frac{1}{\sqrt{t}}$ as $t \to \infty$.

Vakhitov and Kolokolov (1973) generalized these results and showed stability for one-dimensional plane perturbation that can be at any oblique angle to the soliton.

For two-dimensional applications, the stability (and hence the practical relevance) of one-dimensional solitons to two-dimensional perturbations is of obvious importance. To clarify ideas, let us return to the two-dimensional equation (5.1) and define dimensionless variables

$$
\tau = \omega t, \quad \xi = 2k_0 (x - \frac{\omega}{2k_0} t), \quad \eta = 2k_0 y, \quad \tilde{A} = \frac{k_0 A}{\sqrt{2}}
$$

(5.13)

so the Eq. (5.1) can be written in the standard form:

$$\tilde{A}_\tau - \frac{1}{2} \tilde{A}_{\xi \xi} + \tilde{A}_{\eta \eta} - |\tilde{A}|^2 \tilde{A} = 0 \quad (5.14)$$

Consider a plane soliton with $\alpha = 0$, $U = 0$ (clearly with no loss of
generality since any other case with $|\alpha|^2 < \alpha_c^2$ and $U \neq 0$ can be reduced to this case by rescaling the independent variables). The soliton in the new variables is

\[ \tilde{A}(\xi, \tau) = \text{sech}(a_0 \xi) e^{-i \alpha^2 \tau / 2} \]  \hspace{1cm} (5.15)

Zakharov and Rubenchick (1974) considered the stability of (5.15) subject to Eq. (5.14) and two-dimensional perturbations of the form

\[ \hat{A}(\xi, \eta, \tau) = \psi(\xi) e^{i \Omega \tau - i p \eta} \]  \hspace{1cm} (5.16)

with $|\psi| \ll 1$ and vanishes as $|\xi| \to \infty$ as required. They demonstrated that there is a cutoff transverse wave number $p_c$, below which there exists an odd (in $\xi$) unstable mode of $\psi$ so that $\Omega^2 < 0$. That is, any plane soliton solution of the form (5.15) is unstable to sufficiently long transverse perturbations. Saffman and Yuen (1978) performed numerical calculations on the eigenvalue problem given by Zakharov and Rubenchik (1974), and confirmed their conclusions. In particular, it was found that the cutoff wave number for the unstable odd eigenmode is

\[ \Omega^2 < 0 \quad \text{for} \quad 0 < p^2 < p_c^2 = 0.59 \ a^2 \]  \hspace{1cm} (5.17)

and that there exists a most unstable mode corresponding to

\[ p^2 = p_{\text{max}}^2 = 0.325 \ a^2 \]  \hspace{1cm} (5.18)

where

\[ |\text{Im} \ \Omega|_{\text{max}} \approx 0.33 \ a^2 \]  \hspace{1cm} (5.19)

It is important to point out that in the one-dimensional limit $p^2 = 0$, the
unstable mode has an eigenvalue of $\Omega^2 = 0$. This is in accordance with
the one-dimensional analysis ($p^2 = 0$) of Vakhitov and Kolokolov (1973)
who showed that $\Omega^2$ is always real and non-negative. Similar results
showing instability of plane solitons to two-dimensional small perturba-
tions have now been extended to the case of finite depth by Ablowitz
and Segur (1979). It may be pointed out that such two-dimensional
stability of plane solitons has not been observed in experiments probably
because the wave tanks are not sufficiently wide to admit the long
transverse perturbations required by Eq. (5.17).

The transverse length scale implied by Eq. (5.17) for the in-
stability of solitons must hence be relevant to the study of two-dimension-
al wave packets. Consequently, for any two-dimensional problem, such as
diffraction of a plane pulse by a body, we speculate that transverse
instability is important only if the lateral extent of fluid is large
compared to the pulse length.

To simplify matters, we shall only study channels whose widths $b$
are somewhat smaller than the longitudinal length scale of the incident
group in the sense of Eq. (5.17).

V.2.3 Similarity Solutions for Decaying Oscillations: Radiation

The one-dimensional Eq. (5.5) (or Eq. (5.3)) also have similarity
solutions which decay as $1/\sqrt{\tau}$ and contain two arbitrary constants $A_o$ and
$\theta_o$ in the amplitude and phase respectively (Benney and Newell, 1967):

$$A(\xi,\tau) = \frac{A_o}{\sqrt{\tau}} \exp\{ -i(\xi^2/\tau + A_o^2 \ln\tau + \theta_o)/2 \} \quad (5.20)$$

This together with the localized soliton solutions play a significant
role in the nonlinear evolution of a one-dimensional pulse as will be discussed in the next section.

For the two-dimensional case, Eq. (5.14), an analogous unsteady solution exists but with a decay rate of $1/\tau$:

$$A(\xi, \eta, \tau) = \frac{A_0}{\tau} \exp\left(-\frac{1}{\tau} \left[\frac{\xi^2}{8} - \frac{\eta^2}{4} - A_0^2 + \tau o\right]\right)$$  \hspace{1cm} (5.21)

As in the one-dimensional case, solutions of the form (5.21) should be important in the understanding of the evolution of a two-dimensional envelope.

V.2.4 Nonlinear Evolution of a One-Dimensional Envelope Pulse

The governing equation is Eq. (5.5) with a localized initial condition. This problem has been solved exactly by Zakharov and Shabat (1972). Since then, there have been numerous theoretical extensions and clarifications, notably by Ablowitz et al. (1974), Manakov (1974), Segur and Ablowitz (1976) and Segur (1976), so that the analytic understanding is essentially complete. Furthermore, there are wave pulse experiments by Feir (1967) and Yuen and Lake (1975) and several numerical confirmations of theory and experiments by Chu and Mei (1971), Yuen and Lake (1975) and Roskes (1975). To provide a background for the numerical studies in this thesis, the essential features of the evolution of a plane wave packet is summarized as follows:

(a) An arbitrarily shaped one-dimensional initial envelope pulse will eventually disintegrate into a definite number of permanent progressive sech solitons of the form (5.6) in the background of some dispersive, decaying oscillations.
(b) These background oscillations disperse *linearly* so that the amplitude decays as $1/\sqrt{\tau}$. This is often referred to as radiation, and can in general be considered as a slowly varying modulation of solutions of the form (5.20) where the amplitude and phase $A_o, \theta_o$ are now functions of $(\xi/\tau)$. (Segur and Ablowitz, 1976).

(c) When the initial profile $A(\xi,0)$ is real and contains no complex phase, the number of solitons in the asymptotic state can be roughly estimated by the formula (Yuen and Lake, 1975):

$$N_s = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} A(\xi,0) d\xi$$

(5.22)†

In particular, for small enough $N_s$, no permanent waves can arise and the evolution leads to a pure oscillatory decay. (Incidentally, this is always the case for physical systems where the nonlinear and dispersive terms in the Schrödinger equation have opposite signs.) (Manakov, 1974).

(d) For the case when the final state contains one soliton and decaying oscillations, the asymptotic solution near the center behaves like the soliton, while away from that, the profile is dominated by the decaying solution as a slowly varying form of Eq. (5.20) (Segur, 1976). When several solitons are present, the asymptotic state exhibits recurrence of a localized multi-solution group (Roskes, 1975). Since individual solitons in the group propagate at the original group velocity regardless of amplitude or length, they remain together and do not disperse, in contrast to multi-soliton solutions in shallow water waves (Gardner et al.,

†This is not to be confused with $\int |A| d\xi$ for any complex initial data, a quantity which depending on the phase need have no relationship with the eventual number of solitons.
1967; Madsen and Mei, 1969). The frequencies of a multi-soliton recurrence are completely characterized by all the possible frequency differences of the component solitons \( N_s(N_s - 1)/2 \) frequencies for \( N_s \) solitons confined together. For example, a group containing 2 solitons is periodic with one recurrence frequency, while a 3-soliton group is characterized by 3 recurrence frequencies (Zakharov and Shabat, 1972).

(e) The time scale for the initial pulse to reach the asymptotic state is directly proportional to the length of the pulse and inversely proportional to its amplitude.
VI. GOVERNING EQUATIONS

VI.1 Deriving the Two-Dimensional Schrödinger Equation

Adapting the same coordinate orientations and notations, the basic governing equations here are as stated in Part One, Eq. (2.3).

We restrict ourselves to the case of a packet of nearly one-dimensional waves progressing in the x direction with the principal wave number \( \mathbf{k}_o = (k_x, 0) \). Furthermore, we assume the group to be long compared with the principal wavelength, i.e., a slowly modulated group, so that the variation of wavenumber within the group \( \delta k \) is small compared to \( k_o \). Hence, for \( \mathbf{k} = (k_o + \delta k, \ell) \)

\[
|\ell|/k_o \ll 1 \quad \text{for nearly one-dimensional waves} \quad (6.1)
\]

and

\[
\delta k/k_o \ll 1 \quad \text{for slow modulation} \quad (6.2)
\]

We apply a multiple-scales method to Eq. (2.3). The details are very close to that of Davey and Stewardson (1974) (see also Benney and Roskes, 1969; Chu and Mei, 1970; Hasimoto and Ono, 1972; Djordjevic and Redekopp, 1977) and are given in Appendix A. The results are:

\[
\phi = e^{i\phi_0} - \frac{g \cosh Q}{2\omega \cosh q} (iA e^{i\Psi} + *) + e^{2i\phi_0} - \frac{g \cosh Q}{2\omega \cosh q} \frac{3A}{\partial x_1} e^{i\Psi} + * \\
+ \frac{\omega}{2k_o^2 \sinh q} (Q \sinh Q - q \tanh q \cosh Q) \frac{3A}{\partial x_1} e^{i\Psi} + * \\
- \frac{3\omega \cosh 2Q}{4} (iA^2 e^{2i\Psi} + *) + \ldots \quad (6.3)
\]
\[ \zeta = \varepsilon \left( A e^{i\psi} + \star \right) + \varepsilon^2 \left( -\frac{1}{g} \frac{\partial \phi_{01}}{\partial t_1} - \frac{k_o}{2 \sinh 2q} |A|^2 \right) \]
\[ + \frac{1}{2} \left( B + \frac{1}{\omega} \frac{\partial A}{\partial t_1} \right) e^{i\psi} + \star + \frac{k_o \cosh q (2 \cosh^2 q + 1)}{8 \sinh^3 q} A e^{2i\psi} + \star \]
\[ + \ldots \]

(6.4)

Here \( x_1, y_1, t_1 \) are the slow variables

\[ x_1 = \varepsilon x, \quad y_1 = \varepsilon y, \quad t_1 = \varepsilon t \]

(6.5)

and \( \phi_{01}, \phi_{02}, A, B \) are functions of the slow variables \( x_1, y_1, t_1 \) only. The small ordering parameter \( \varepsilon \) is taken to be the wave steepness

\[ \varepsilon \sim O(k_o \zeta) \]

(6.6)

so that

\[ k_A, k_B, \frac{\omega}{g} \phi_{01}, \frac{\omega}{g} \phi_{02} \sim O(1) \]

(6.7)

The relevant governing equations are

\[ A_{t_1} + C_{g_0} \frac{\partial A}{\partial x_1} = 0 \]

(6.8)

which requires the wave packet to travel at its linear group velocity, and the two coupled equations for complex group envelope \( A \) and the drift current potential \( \phi_{01} \):

\[ \left[ \frac{\partial^2}{\partial t_1^2} - g h \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) \right] \phi_{01} = \frac{\omega^2}{2} \left( -\frac{\omega}{2k_o \tanh q} + \frac{C_{g_0}}{2 \sinh^2 q} \right) |A|^2 \]

(6.9.a)
\[
\frac{1}{\varepsilon} \left[ (A + \varepsilon B)_{t_1} + C_0 (A + \varepsilon B)_{x_1} \right] + \frac{C_0}{2k_0} A_{y_1} y_1 \\
+ \left[ -\frac{w k_o}{k_o \sinh 2q} (\tanh^2 q - q \tanh q) - \frac{C_0^2}{2w} \right] A_{x_1} x_1 \\
+ \left( -\frac{k_0^2}{2w \cosh^2 q} \phi_{01} t_1 - k_0 \phi_{01} x_1 \right) A - \frac{w k_o^2 (\cosh 4q + 8 - 2 \tanh^2 q)}{16 \sinh^4 q} |A|^2 A \\
= 0
\]

(6.9.b)

Eq. (6.9.a) is a forced wave equation for \( \phi_{01} \) which is related to the mean sea-level and the wave-induced drift current. The second-order first harmonic envelope \( B \) in Eq. (6.9.b) can be readily absorbed by redefining \( A \). In the absence of ambient mean current, the appropriate initial conditions are

\[
A(x_1, y_1, t_1 = 0) = f(x_1, y_1) \\
\phi_{01}(t_1 = 0) = 0
\]

(6.10.a,b)

and the correct boundary conditions for the case of localized \( f \) are

\[
A \to 0 \\
\text{as } x_1^2 + y_1^2 \to \infty \\
|\nabla \phi_{01}| \to 0
\]

(6.11.a,b)

The shallow-water limit of Eqs. (6.9), \( k_o h \to 0 \) (but maintaining \( \varepsilon \ll (k_o h)^2 \)), yields equations of the Korteweg-de Vries form, with plane wave solutions of weak cnoidal type (see Hasimoto and Ono, 1972). The non-uniformity of approaching the double limits \( k_o h \to 0, \varepsilon \to 0 \), however, presents special difficulties as extra terms are introduced in the
asymptotic expansions (depending on the magnitude of \( \epsilon \)) as \( k_0 h \to 0 \).

This problem is addressed satisfactorily by Freeman and Davey (1975).

In this thesis, we restrict ourselves to the case of deep water.

In this limit, \( k_0 h \to \infty \), Eq. (6.9) become

\[
\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) \phi_{01} = 0 \tag{6.12}
\]

and

\[
\frac{1}{\epsilon} \left[ (A+\epsilon B) \tau_1 + \frac{\omega}{2k_0} (A+\epsilon B) x_1 \right] + \frac{\omega}{4k_0^2} A y_1 y_1 - \frac{\omega}{8k_0} A x_1 x_1 
- k_0 \frac{\partial \phi_{01}}{\partial x_1} A - \frac{\omega k^2}{2} |A|^2 A = 0 \tag{6.13}
\]

In the absence of any forcing, Eq. (6.12) with Eqs. (6.10.b, 6.11.b) yields

\[
\phi_{01} \equiv 0 \tag{6.14}
\]

and the coupling term \( \frac{\partial \phi_{01}}{\partial x_1} A \) in Eq. (6.13) vanishes. We shall further non-dimensionalize Eq. (6.13) with

\[
\tilde{A} = (A+\epsilon B)/(\zeta_0/\epsilon) \quad \text{where } \zeta_0 \text{ is the maximum free-surface amplitude}
\]

\[
(\tilde{x}_1, \tilde{y}_1) = k_0 (x_1, y_1) = \epsilon k_0 (x, y) \tag{6.15}
\]

\[
\tilde{\tau}_1 = \omega t_1 = \epsilon \omega t
\]

and obtain to \( 0(\epsilon) \)

\[
\frac{1}{\epsilon} \left( \frac{\tilde{A}}{\tau_1} + \frac{1}{2} \frac{\tilde{A}}{x_1} \right) + \frac{\omega}{4} \tilde{A} y_1 y_1 - \frac{1}{8} \tilde{A} x_1 x_1 - k|\tilde{A}|^2 \tilde{A} = 0 \tag{6.16}
\]

where
\[ \kappa = \frac{1}{2} \left( \frac{k \zeta}{\varepsilon} \right)^2 > 0 \]  

(6.17)

Alternatively, we may adopt a coordinate system moving with the packet at the group velocity:

\[ \xi = \bar{x}_1 - \frac{1}{2} \bar{t}_1 = \varepsilon k_0 (x - \frac{w}{2k_0} t) \]
\[ \eta = \bar{y}_1 = \varepsilon k_0 y \]  

(6.18)

and

\[ \tau = \varepsilon \bar{t}_1 = \varepsilon^2 \omega t \]

Eq. (6.16) becomes more compact:

\[ i\bar{A}_\tau + \frac{1}{4} \bar{A}_{\eta\eta} - \frac{1}{8} \bar{A}_{\xi\xi} - k|\bar{A}|^2 \bar{A} = 0 \]  

(6.19)

Eqs. (6.16, 6.19) are the two-dimensional Schrödinger equation first derived by Zakharov (1968). Note that the two second-order space derivatives are of opposite signs. This is due to the fact that the dispersion relationship for deep gravity waves have curvatures of opposite signs in the longitudinal and transverse directions.
VI.2  Boundary Conditions

We restrict ourselves to grazing incidence on a rigid permeable wall. Following Part One, Chapter 2, let the wall be given by

\[ y = y_B(x_1) \]  \hspace{1cm} (6.20)

so that \( \frac{\partial}{\partial x} y_B = 0(\varepsilon) \). We now substitute the expressions for \( \phi \) (Eq. (6.3)) into Eq. (2.42) and obtain

\[ \varepsilon [\varepsilon \phi_{01} y_1 - \frac{g \cosh q}{2\omega \cosh q} (ie_A e^{i\psi} + \ast) + O(\varepsilon)^2] \]

\[ = \varepsilon [\varepsilon \phi_{01} x_1 - \frac{g \cosh q}{2\omega \cosh q} [ie_A e^{i\psi} + \ast + i(ik_o)e^{i\psi} + \ast] + O(\varepsilon)] \]

\[ \cdot \varepsilon y_B(x_1) \hspace{1cm} \text{on} \hspace{0.5cm} y_1 = \varepsilon y_B(x_1) \]  \hspace{1cm} (6.21)

Separating terms of different harmonics, we obtain to leading order

\[ \phi_{01} y_1 = 0 \hspace{1cm} \text{on} \hspace{0.5cm} y_1 = \varepsilon y_B(x_1) \]  \hspace{1cm} (6.22.a, b)

\[ A_{y_1} = ik_o y_B(x_1)A \]

Since Eq. (6.22.a) is homogeneous, the earlier result leading to Eq. (6.14) remains valid. In dimensionless form, Eq. (6.22.b) becomes (to leading order)

\[ \tilde{\mathcal{A}}_{y_1} = i \tilde{y}_B(x_1) \tilde{A} \hspace{1cm} \text{on} \hspace{0.5cm} \tilde{y}_1 = \varepsilon \tilde{y}_B(x_1) \]  \hspace{1cm} (6.23)

where \( \tilde{y}_B = k_o y_B \).
VI.3 Two-Dimensional Initial-Boundary-Value Problems for Two Special Geometries

VI.3.1 An Oblique Channel

Consider a wave group incident upon a uniform channel of width b tilted at an oblique angle $\alpha$ to the direction of propagation (x). The banks of the channel are given by:

$$y_B = x \tan \alpha \,, \quad b + x \tan \alpha$$

(6.24)

We require $\alpha$ to be small and define

$$\varepsilon \equiv \tan \alpha$$

(6.25)

so that $y_B'(x_1) = 1$.

It is convenient to define orthogonal coordinates $x_1', y_1'$ parallel and normal respectively to the channel walls (see Figure 6.1):

$$x_1' = x_1 + \varepsilon y_1$$

$$y_1' = y_1 - \varepsilon x_1$$

(6.26.a,b)

The initial boundary value problem (IBVP) in dimensionless variable (but dropping (') from now on) in the channel in terms of the new coordinates is then

$$\frac{1}{\varepsilon} (A_{t_1} + \frac{1}{2} A x_1') - \frac{1}{2} A y_1' + \frac{1}{4} A y_1'y_1' - \frac{1}{8} A x_1'x_1' - K|A|^2 A = 0$$

$$0 < y_1' < b \,, \quad x_1' > 0 \,, \quad t_1 > 0$$

(6.27.a)

$$A = 0 \quad \text{on } t_1 = 0$$

(6.27.b)

$$A = P(x_1' = 0, y_1', t_1) \quad \text{on } x_1' = 0$$

(6.27.c)
Figure 6.1: Definition sketch for an oblique channel.

Figure 6.2: Definition sketch for a converging channel.
\[ A = 0 \quad \text{as } x'_1 \to \infty \quad (6.27.d) \]

and

\[ A_{y'_1} = iA \quad \text{on } y'_1 = 0, b \quad (6.27.e) \]

where \( P \) is the incident pulse envelope with \( t_1 \) defined so that Eq. (6.27.b) is satisfied. To be consistent with the perturbation scheme, all higher order terms are ignored and the main result of introducing the rotated coordinates (6.26) is the term proportional to \( A_{y'_1} \) in Eq. (6.27.a).

VI.3.2 A Converging Channel

We consider a channel that converges slowly from a uniform width \( 2b \) into a narrower uniform channel of width \( 2B \). Assuming symmetry about the center line, we consider only the lower half of the problem (see Figure 6.2). The converging wall forms a small oblique angle \( \alpha \) with the \( x \)-axis again given by Eq. (6.25). The initial-boundary-value problem in the converging channel is governed by

\[
\frac{1}{\varepsilon} \left( A_{t_1} + \frac{1}{2} A_{x'_1} \right) + \frac{1}{4} A_{y'_1 y'_1} - \frac{1}{8} A_{x'_1 x'_1} - iK |A|^2 A = 0
\]

\[
x'_1 > 0, \quad t'_1 > 0, \quad b > y'_1 > \begin{cases} 0 & \text{if } x'_1 = \text{const}, \quad 0 < x'_1 < L = (b-B)/\varepsilon \\
B & \text{if } x'_1 = \text{const}, \quad x'_1 > L 
\end{cases}
\]

(6.28.a)

\[ A = 0 \quad \text{at } t'_1 = 0 \quad (6.28.b) \]

\[ A = \{ P(x'_1=0,y'_1,t'_1) \} \quad \text{on } x'_1 = 0 \]

\[ A = \{ 0 \} \quad \text{as } x'_1 \to \infty \quad (6.28.c,d) \]
\[ A_{y_1} = \begin{cases} \text{iA} & \text{on } y_1 = \varepsilon x_1, \ 0 < x_1 < L \\ 0 & \text{on } y_1 = B, \ x_1 > L \end{cases} \quad (6.28.e) \]

\[ A_{y_1} = 0 \quad \text{on } y_1 = b \quad (6.28.f) \]

The IBVP Eqs. (6.28) is now defined in an irregular domain, and special solution technique is required.
VII. NUMERICAL METHOD FOR SOLUTION

VII.1 One-Dimensional Problem

While the one-dimensional Schrödinger equation (5.5) has been solved analytically using inverse scattering transform method for any initial condition that decays as \( |\xi| \to \infty \) (Zakharov and Shabat, 1972), the actual computation for arbitrary initial states via the analytic formulas can be quite tedious and it is desirable to integral Eq. (5.5) directly using numerical differencing. Furthermore, when the initial data is not compact, such as in the case of a continuous wave train or a propagating front, the inverse scattering method is not applicable and an analytic solution to Eq. (5.5) is as yet unavailable. One must then resort to numerical solutions.

A variety of numerical methods can be applied to solve the nonlinear equation (5.5) (for example, a modified split-step spectral method, Lake et al., 1977). Here we choose a modified Crank-Nicolson implicit scheme with second-order centered spatial finite differencing:

\[
A_j^{n+1} = A_j^n - \frac{\Delta t}{2} \left( \frac{1}{8} A_{j+1}^{n+1} - 2A_j^{n+1} + A_{j-1}^{n+1} \right) A_n^{n+1} + \frac{1}{2} |A_n^{n+1}|^2 A_n^{n+1} \\
+ \frac{1}{8} A_{j+1}^n - 2A_j^n + A_{j-1}^n \right) \frac{\Delta \xi^2}{\Delta t^2} A_n^n \\
+ 0(\Delta t^3, \Delta \xi^2) \tag{7.1}
\]

where

\[
A_j^n = A(j \Delta \xi, n \Delta t) \tag{7.2}
\]

To maintain a local truncation error of \( 0(\Delta t^3) \), the implicit nonlinear
term in Eq. (7.1) is estimated using an Euler scheme:

\[ A_{n+1} = A_n - \Delta t \left( \frac{A_{n+1}}{8} - \frac{2A_n}{\Delta \xi^2} + \frac{A_{n-1}}{2} \right) + \frac{i}{2} |A_n|^2 A_n \]

\[ + 0(\Delta t^2, \Delta \xi^2) \quad (7.3) \]

It can be readily verified by Taylor series that Eq. (7.1) is convergent with global truncation error of \( O(\Delta t^2, \Delta \xi^2) \). Because of the complication of the cubic nonlinear term, we are unable to show the stability of (7.1), however, it is well known that the Crank-Nicolson scheme is stable for the linear Schrodinger equation. For convenience, the linear Neumann stability analysis is outlined here.

Consider periodic boundary conditions for \( 0 < \xi < \pi \), and a Fourier mode \( p \) given by

\[ A_j^n = G^n e^{ip(j\Delta \xi)} \quad j = 0,1,...,J ; \quad J = \frac{\pi}{\Delta \xi} \quad (7.4) \]

where \( G \) is a complex constant representing the growth factor. Substituting Eq. (7.4) into the linear part of (7.1) (ignoring nonlinear terms), we have

\[ G^{n+1} e^{ip(j\Delta \xi)} = G^n e^{ip(j\Delta \xi)} - \frac{\Delta t}{2} \left[ \frac{i}{8} (G^{n+1} + G^n) \right] \\
+ \frac{e^{ip(j+1)\Delta \xi} - 2e^{ip(j\Delta \xi)} + e^{ip(j-1)\Delta \xi}}{\Delta \xi^2} \quad (7.5) \]

which can be readily factored to give

\[ G^{n+1}(1 - \frac{i\Delta t}{16} \sin^2 \frac{1}{2} p\Delta \xi) = G^n(1 + \frac{i\Delta t}{16} \sin^2 \frac{1}{2} p\Delta \xi) \quad (7.6) \]

so that
\[
\left| \frac{A_n}{A_0} \right| = |G|^n = \left| 1 + \frac{i\Delta t}{16} \sin^2 \frac{1}{2} \frac{t}{p\Delta \xi} \right|^n = 1
\]

and the Fourier mode is bounded in modulus from the initial condition. Hence the scheme (7.1) is unconditionally stable.

Numerical experience with the full Eq. (7.1) gave satisfactory results for all reasonable choices of \( \Delta \xi \) and \( \Delta t \), and the quantity \( C_1 \) (Eq. (5.7.a)) is conserved to within acceptable limits for all the computations performed.

The computer programs are written in Multics FORTRAN, and the code for the study of a plane propagating wave front is included in Appendix B.2 as an example.

VII.2 Two-Dimensional Nonlinear Schrödinger Equation

To our knowledge, the two-space-dimensional Schrödinger equation (5.1) has not been solved analytically for general initial conditions. While the one-dimensional problem has been successfully integrated using inverse scattering transform, Ablowitz and Segur (1979) conjectured, using an argument based on the nature of singularities of Eq. (5.1), that an exact solution for the full equation (5.1) via IST is infeasible. One must in general resort to numerical solution for two-dimensional problems even with the simplest boundary conditions such as Eq. (6.11.a).

As pointed out earlier, Eq. (5.1) (or equivalently Eqs. (6.16), (6.19)) which are well-posed with spatial boundary conditions, has the somewhat uncommon feature of having opposite signs for the spatial derivative terms, and special care must be taken in any direct numerical
solution. To illustrate this point, let us apply a general semi-implicit integration scheme to Eq. (6.19):

\[
\begin{aligned}
    iA^{n+1} &= iA^{n} - \Delta t \left[ \lambda \left( \frac{1}{4} A^{n+1}_{\eta\eta} - \frac{1}{8} A^{n+1}_{\xi\xi} - K|A^{n+1}|^{2} A^{n+1} \right) \\
    &+ (1-\lambda) \left( \frac{1}{4} A^{n}_{\eta\eta} - \frac{1}{8} A^{n}_{\xi\xi} - K|A^{n}|^{2} A^{n} \right) \right]
\end{aligned}
\]  

(7.8)

where \( \lambda \) is some relaxation parameter.

At any given time step, then, we have (except for the purely explicit case of \( \lambda = 0 \)) the wave equation

\[
A^{n+1}_{\eta\eta} = \frac{1}{2} A^{n+1}_{\xi\xi} + \text{nonlinear and terms of lower order derivatives} = \text{known explicit terms}
\]  

(7.9)

which has to be solved with boundary conditions in \( \xi \) and \( \eta \). This presents possible difficulties since it is well known that the linear hyperbolic equation is ill-posed as a boundary value problem. To eliminate this difficulty we employ the split-step alternating direction implicit (ADI) method which treats each spatial direction implicitly in alternating steps but never together. This also has the important advantage that the inversion of large matrices for all the (two-dimensional) spatial unknowns of \( A \) simultaneously is avoided.

VII.2.1 The Alternative Direction Implicit (ADI) Scheme

The ADI scheme, also known as the Peaceman-Rachford scheme, was originally proposed for the solution of parabolic and elliptic differential equations by Peaceman and Rachford (1955). It has since been applied extensively to numerous linear and nonlinear problems in fluid mechanics.
(e.g., Reynolds' gas bearing problem, de Vries (1975), three-dimensional hypersonic flows, Nardo and Cresci (1971), and shallow water flow problems, Gustafsson (1971)). Here, we apply this method to the IBVP's Eqs. (6.27) and Eqs. (6.28). For Eq. (6.27.a) we write (using \((x,y,t)\) for \((x_1,y_1,t_1)\) for brevity):

\[
\frac{1}{\epsilon} \left( A^{n+\frac{1}{2}} - A^n \right) = \frac{\Delta t}{2} \left( - \frac{1}{2\epsilon} A^n_x + \frac{1}{2} A^{n+\frac{1}{2}}_y - \frac{1}{4} A^{n+\frac{1}{2}}_{yy} + \frac{1}{8} A^{n+\frac{1}{2}}_{xx} 
+ \kappa |A^n|^{2} A^n \right) \quad n = 0, 1, 2, \ldots \quad (7.10.a)
\]

\[
\frac{1}{\epsilon} \left( A^{n+1} - A^{n+\frac{1}{2}} \right) = \frac{\Delta t}{2} \left( - \frac{1}{2\epsilon} A^{n+\frac{1}{2}}_x + \frac{1}{2} A^{n+\frac{1}{2}}_y - \frac{1}{4} A^{n+\frac{1}{2}}_{yy} + \frac{1}{8} A^{n+\frac{1}{2}}_{xx} 
+ \kappa |A^{n+\frac{1}{2}}|^{2} A^{n+\frac{1}{2}} \right) \quad n = 0, 1, 2, \ldots \quad (7.10.b)
\]

and

\[
A^{n+1} = A^{n+\frac{1}{2}} + \frac{\Delta t}{2} \left( - \frac{1}{2\epsilon} A^{n+\frac{1}{2}}_x + \frac{1}{2} A^{n+\frac{1}{2}}_y + \frac{1}{4} A^{n+\frac{1}{2}}_{yy} + \frac{1}{8} A^{n+\frac{1}{2}}_{xx} 
+ \kappa |A^{n+\frac{1}{2}}|^{2} A^{n+\frac{1}{2}} \right) \quad n = 0, 1, 2, \ldots \quad (7.10.c)
\]

Thus, each time step is subdivided into two half-steps. In the first half-step, the \(x\)-derivative terms and the nonlinear term are explicit while the \(y\)-derivative terms are totally implicit. \(A^{n+\frac{1}{2}}\) is hence computed from \(A^n\) by line-by-line inversions of equations in \(y\) for each \(x\). The situation is completely opposite for the second half-step where the \(y\)-terms are now explicit, the \(x\)-terms implicit and the nonlinear term semi-implicit via the extrapolation step (7.10.c). For every \(y\)-level we again have a one-dimensional boundary value problem in \(x\) to be solved for \(A^{n+1}\) using the values of \(A^{n+\frac{1}{2}}\). This procedure (7.10.a) - (7.10.c) is
repeated for successive integral steps. For computation, the spatial derivatives in Eqs. (7.10) are replaced by second-order center differences:

\[
A_{x,j,k} = \frac{A_{j+1,k} - A_{j-1,k}}{2\Delta x} + O(\Delta x)^2 \tag{7.11.a}
\]

\[
A_{y,j,k} = \frac{A_{j+1,k} - A_{j-1,k}}{2\Delta y} + O(\Delta y)^2 \tag{7.11.b}
\]

\[
A_{xx,j,k} = \frac{A_{j+1,k} - 2A_{j,k} + A_{j-1,k}}{\Delta x^2} + O(\Delta x)^2 \tag{7.11.c}
\]

\[
A_{yy,j,k} = \frac{A_{j+1,k} - 2A_{j,k} + A_{j-1,k}}{\Delta y^2} + O(\Delta y)^2 \tag{7.11.d}
\]

and

\[
A_{j,k}^n = A(j\Delta x, k\Delta y, n\Delta t) \quad j = 0,1,2,\ldots;
\]

\[
k = 0,1,2,\ldots,k_B; \quad k_B = \frac{b}{\Delta y}
\]

\[
n = 0,1,2,\ldots \tag{7.11.e}
\]

The boundary conditions of Eqs. (6.27) are imposed as follows. First we truncate the x domain at \(x_\infty\), where \(x_\infty\) is chosen so that Eq. (6.27.d) is satisfied smoothly and further increase of \(x_\infty\) causes negligible change in the solution. On the boundaries, some of Eqs. (7.11) cannot be evaluated and the unknown grid values are eliminated by using the boundary conditions Eq. (6.27.c-f) consistently. For example, on the vertical boundaries we have

\[
A_{0,k}^n = \text{prescribed} \quad k = 0,1,\ldots,k_b \tag{7.12.a}
\]

\[
A_{j,\infty,k}^n = 0 \quad \text{where} \quad j = \frac{x_\infty}{\Delta x} \tag{7.12.b}
\]
while on the channel walls, we have from Eq. (6.27,e,f):

\[
\frac{A_{j,k-1}^n}{2\Delta y} - \frac{A_{j,k-1}^n}{2\Delta y} = \pm A_{j,k}^n + O(\Delta y)^2 \quad \text{where } k = 0 \text{ or } k_b \quad j = 0, 1, \ldots, j_{\infty} \tag{7.13}
\]

which gives

\[
A_{yy,j,0} = \frac{2}{\Delta y^2} [A_{j,1} - (1+i\Delta y)A_{j,0}] + O(\Delta y) \tag{7.14}
\]

e tc., so that on the boundaries \( k = 0 \) and \( k_b \), there is only first-order accuracy.

The procedure for the converging channel case Eqs. (6.28) is similar except for the absence of the terms proportional to \( A_y \) in Eqs. (7.10). The non-uniform boundary of Eqs. (6.28), however, requires special treatment in implementing the ADI scheme (7.10) with Eqs. (7.11). One way is to approximate the lower boundary by using variable vertical grid sizes and replace Eqs. (7.11) by variable grid formulas on this boundary and proceed in the manner of Eqs. (7.13) and (7.14). These variable grid difference formulas to the required order are somewhat involved, and we choose instead to perform vertical extrapolation for the unknown nodes making use of the boundary condition (6.28.e) but maintaining constant \( \Delta y \). For example:

\[
A_{j,-1} = A_{j,1} - 2i\Delta y S A_{j,0} + O(\Delta y)^3 \tag{7.15.a}
\]

and

\[
A_{j,0} = \frac{4A_{j,1} - A_{j,2}}{3 + 2i\Delta y S} + O(\Delta y)^3 \tag{7.15.b}
\]

where
\[ S = (1,0) \text{ for } x (<,>) L. \]

Substituting Eq. (7.15.a) into Eq. (7.11.d) on the bottom y boundary again gives \( O(\Delta y) \) accuracy consistent with Eq. (7.14). Eq. (7.15.b) is used to extrapolate for the prescribed boundary condition (7.12.a) on the sloping wall. For consistency, it is necessary to use the nodal values from the same time step in evaluating Eq. (7.15.b). This is achieved for the \( x \)-implicit step (7.10.b) by solving first the higher \( y \) levels, i.e., in the order \( k = k_b, k_b - 1, \ldots \).

Before proceeding to study the convergence and stability of the ADI scheme (7.10), we present the mass conservation law for Eqs. (6.27), (6.28) respectively. For Eq. (6.27) we have

\[
C_1 = \int b \int_0^x dx \int_0^y |A|^2 = \frac{1}{2} \int t dt \int b \int_0^y dy |P(x=0,y,t)|^2
\]

and for Eq. (6.28):

\[
C_1 = \left( \int L dx \int_0^b dy + \int \int_0^x dx \int_0^b dy \right) |A|^2
\]

\[
= \frac{1}{2} \int t dt \int b dy |P(x=0,y,t)|^2 + \frac{i\epsilon}{8} \int t dt \int L dx (A^*_x - A^*Ax) \bigg|_{y=\epsilon x}
\]

Eqs. (7.16) and (7.17) are used to estimate the numerical errors, which are kept at a tolerable level for all the cases studied.

VII.2.2 **Convergence and Stability of the ADI Scheme**

It is desirable to show that our present ADI method Eqs. (7.10) together with (7.11) and boundary conditions converges and is stable. We shall, however, deduce the truncation error and proved the unconditional
stability of the scheme only for the linearized equation (i.e., \( K \equiv 0 \)) under simplified boundary conditions.

Consider the linear Neumann stability analysis inside the square \( x, y \in [0, \pi] \). We apply periodic boundary conditions in both directions and study the growth of a particular mode \((p,q)\):

\[
A_{j,k}^n = G^n e^{ip\Delta x} e^{iq\Delta y} \\
A_{j,k}^{n+\frac{1}{2}} = \hat{G} A_{j,k}^n
\]

(7.18.a,b)

where

\[
j = 0, 1, \ldots, j_1 \quad \Delta x = \frac{\pi}{j_1} \\
k = 0, 1, \ldots, k_1 \quad \Delta y = \frac{\pi}{k_1}
\]

Substituting Eqs. (7.18) into (7.10.a) (with \( K \equiv 0 \)) and using (7.11) we have:

\[
\frac{1}{\epsilon} (\hat{G} - 1) G^n e^{ip\Delta x} e^{iq\Delta y} = \frac{\Delta t}{2} \left\{ -\frac{1}{2\epsilon} G^n e^{iq\Delta y} \cdot \frac{1}{2\Delta x} \left[ e^{ip(j+1)\Delta x} - e^{ip(j-1)\Delta x} \right] + \frac{1}{2} G^n \hat{G} e^{ip\Delta x} \cdot \frac{1}{2\Delta y} \left[ e^{iq(k+1)\Delta y} - e^{iq(k-1)\Delta y} \right] \\
- \frac{1}{4} G^n \hat{G} e^{ip\Delta x} \frac{1}{\Delta y^2} \left[ e^{iq(k+1)\Delta y} - 2e^{iq\Delta y} + e^{iq(k-1)\Delta y} \right] + \frac{1}{8} G^n e^{iq\Delta y} \frac{1}{\Delta x^2} \left[ e^{ip(j+1)\Delta x} + 2e^{ip\Delta x} + e^{ip(j-1)\Delta x} \right] \right\}
\]

(7.19)

Factoring out the common terms and using trigonometrical identities for the exponentials, Eq. (7.19) reduces to

\[
\hat{G} - 1 = i(\alpha \hat{G} + \beta)
\]

(7.20)

where \( \alpha, \beta \) are real and given by
\[ \alpha = \frac{c \Delta t}{2} \left( \frac{\sin q \Delta y}{2 \Delta y} - \frac{\sin^2 \frac{1}{2} q \Delta y}{\Delta y^2} \right) \]

\[ \beta = \frac{c \Delta t}{4} \left( \frac{\sin^2 \frac{1}{2} p \Delta x}{\Delta x^2} - \frac{\sin p \Delta x}{\Delta y} \right) \quad (7.21.a,b) \]

whence

\[ \hat{G} = \frac{1 + i \beta}{1 - i \alpha} \quad (7.22) \]

Similarly, if we perform the algebra for Eq. (3.10.b) for \( A^{n+\frac{1}{2}} \) and \( A^{n+1} \), we obtain

\[ G - \hat{G} = i(\alpha \hat{G} + \beta \hat{G}) \quad (7.23) \]

so

\[ G = \frac{1 + i \alpha \hat{G}}{1 - i \beta} \quad (7.24) \]

using Eq. (7.22). Therefore

\[ |G| = 1 \quad (7.25) \]

\[ |A|^n = |G|^n = 1 \quad (7.26) \]

and the ADI scheme is unconditionally stable. For the case of Eq. (6.28.a) or when the equations are expressed in a coordinate system moving with group velocity, the analysis is similar with only a redefinition of Eqs. (7.21) and the stability conclusion follows.

To simplify the algebra for the truncation error analysis for Eqs. (7.10), we define the linear operators

\[ \Lambda_x = -\frac{1}{4} \frac{\partial}{\partial x} - \frac{i \kappa}{16} \frac{\partial^2}{\partial x^2} \quad (7.27.a) \]
\[ \Lambda_y = \frac{\varepsilon}{4} \frac{\partial^2}{\partial y^2} + \frac{i\varepsilon}{8} \frac{\partial^2}{\partial y^2} \]  

(Eqs. (7.10.a,b) (with \( K = 0 \)) then read

\[ A^{n+\frac{1}{2}} - A^n = \Delta t \left( \Lambda_x A^n + \Lambda_y A^{n+\frac{1}{2}} \right) \]

\[ A^{n+1} - A^{n+\frac{1}{2}} = \Delta t \left( \Lambda_x A^{n+1} + \Lambda_y A^{n+\frac{1}{2}} \right) \]  

(7.28.a,b)

or

\[ (1 - \Delta t \Lambda_y)A^{n+\frac{1}{2}} = (1 + \Delta t \Lambda_x)A^n \]

\[ (1 - \Delta t \Lambda_x)A^{n+1} = (1 + \Delta t \Lambda_y)A^{n+\frac{1}{2}} \]  

(7.29.a,b)

Eliminating \( A^{n+\frac{1}{2}} \) from Eqs. (7.29), we can write formally

\[ A^{n+1} = (1 - \Delta t \Lambda_x)^{-1}(1 + \Delta t \Lambda_y)(1 - \Delta t \Lambda_y)^{-1}(1 + \Delta t \Lambda_x)A^n \]  

(7.30)

For small \( \Delta t \), the inverse operators can be formally expanded

\[ (1 - \Delta t \Lambda) = 1 + \Delta t \Lambda + \Delta t^2 \Lambda^2 + \ldots \]  

(7.31)

Substituting Eq. (7.31) into (7.30) and arranging in powers of \( \Delta t \), we finally obtain

\[ A^{n+1} = \{1 + 2 \Delta t(\Lambda_x + \Lambda_y) + 2\Delta t^2(\Lambda_x^2 + \Lambda_y A_x + \Lambda_y \Lambda_x + \Lambda_y^2) + 0(\Delta t)^3\}A^n \]  

(7.32)

This has to be compared to the non-discretized Eq. (6.27.a) (linearized), which can be written using the operators (7.27) as

\[ A_t = 2(\Lambda_x + \Lambda_y)A \]  

(7.33)
Expanding Eq. (7.33) in Taylor series, we write

\[
A(t + \Delta t) = A(t) + \Delta t A_t (t) + \frac{\Delta t^2}{2} A_{tt} (t) + O(\Delta t^3)
\]

Comparing Eqs. (7.32) and (7.34), we obtain a truncation error of \(O(\Delta t)^3\) for each time step so that the global error is \(O(\Delta t)^2\) for \(t = O(1)\). It is interesting to point out that by examining each half-step of the ADI procedure (7.10.a, b) separately, we find that the truncation error is \(O(\Delta t)^2\) for Eqs. (7.10.a), (7.10.b) respectively, which however combines to give an error of \(O(\Delta t)^3\) for an integral step. We remark here that if the nonlinear term is kept in Eqs. (7.10), the foregoing procedure becomes very tedious but essentially valid and the truncation error is still \(O(\Delta t)^3\).

To give an example of how the algorithm is coded, sample Multics FORTRAN source for the case of a converging channel is listed in Appendix B.3.
VIII. TWO ONE-DIMENSIONAL EVOLUTION PROBLEMS

The governing equation is the one-space-dimensional Schrödinger equation (5.5),

\[ A_T + \frac{1}{8} A_{\xi\xi} + \frac{1}{2} |A|^2 A = 0 \]  

(8.1)

where \( \tau = \varepsilon^2 \tau_o = \varepsilon^2 \omega t^* \), \( \xi = \varepsilon \xi_o = \varepsilon k_o (x^* - C_g t^*) \), \( A = A^*/A_o \) and the maximum wave steepness defines \( k_o A_o \equiv \varepsilon_c \), and ( )* represent dimensional physical variables. The initial condition is

\[ A(\xi, \tau=0) = f(\xi) \]  

(8.2.a)

where

\[ \max_{\xi} f(\xi) = 1 \]  

(8.2.b)

because of normalization. Eq. (8.1) is solved numerically for the evolution \( A(\xi, \tau) \) using the difference scheme of Section 7.1 for two different types of initial data \( f(\xi) \).

VIII.1 Nonlinear Evolution of Pulse Envelopes

Here we consider spatially-confined initial profiles of symmetric sech form:

\[ f(\xi) = \text{sech} (\sqrt{2}\xi/\sigma) \]  

(8.3)

On account of symmetry, we shall solve Eq. (4.1) only in the domain \( 0 \leq \xi < \infty \), with the boundary conditions
\[ A_\xi = 0 \quad \text{and} \quad \xi = 0 \quad (8.4.a) \]

Note that for Eq. (8.3), \( \sigma = 1 \) is the permanent soliton, while a broader profile, say \( \sigma = 2 \), would correspond physically to a pulse whose length is twice that of a soliton of the same height or alternatively one which is twice as high as a soliton of the same length. Furthermore, the integral for estimating theoretically the number of solitons in the asymptotic state is now simply

\[ N_s = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} f(\xi) d\xi = \sigma \quad (8.5) \]

The exact solution for \( \sigma = 1 \) is

\[ A(\xi, \tau) = \text{sech} (\sqrt{2}\xi) \ e^{-i\tau/4} \quad (8.6) \]

This is used as a further check of the numerical program and to limit the discretization error.

VIII.1.1 Example of Pure Radiation

As a first example, we consider a steeper-than-soliton initial sech profile corresponding to \( \sigma = \frac{1}{2} \) \((N_s \sim \frac{1}{2})\). Theoretically, the asymptotic state contains no permanent soliton and the entire initial data decays smoothly in time as so-called radiation. This evolution picture is obtained numerically and shown in Figure 8.1. To study the rate of amplitude decay, the maximum amplitude along the center-line of
Figure 8.1: Evolution of a steeper-than-soliton initial sech pulse
($\sigma = \frac{1}{2}$): $A(\xi, \tau=0) = \text{sech}(2\sqrt{2}\xi)$. The region shown is for
$0 < \xi < 10$ and $0 < \tau < 20$. 

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the envelope is plotted and compared to the asymptotic rate \( \frac{1}{\sqrt{\tau}} \) as predicted theoretically (Figure 8.2). For \( \tau > \sim 10 \), the deviation from the asymptotic trend is quite small. Based on this decay rate, and the conservation of energy of the pulse:

\[
C_1 \sim \int_{-\infty}^{\infty} |A|^2 d\xi
\]

one can readily deduce that the longitudinal length scale of the envelope would increase linearly with time, i.e., a linear dispersion in \( \xi \). To show this, we plot the locus of points \( \xi_1(\tau) \) where the evolution profile amplitude is a constant fraction \( \gamma_1 \) of the center-line amplitude:

\[
|A(\xi_1(\tau),\tau)| = \gamma_1 |A(\xi=0,\tau)|
\]

This is shown in Figure 8.3. The lines are clearly quite straight except near \( \tau = 0 \), and where the amplitude is very small and numerical noise is dominant. Lastly, we display the amplitude and phase of \( A \) at several times (Figures 8.4.a,b). We define the phase \( p(\xi,\tau) \) by

\[
A(\xi,\tau) = |A(\xi,\tau)| \exp[ip(\xi,\tau)]
\]

The initial profile has no phase \( (p(\xi,0)=0) \), while \( p \) generally decreases continuously with \( \xi \) for subsequent times.

**VIII.1.2 Example of a Single Frequency Recurrence Due to Interaction of Two Solitons**

As a confirmation of the analytical theory we first study the evolution of an initial sech profile which is twice as long as a soliton of the same height, \( \sigma = 2 \left( N_s \sim 2 \right) \). This same case was first studied
Figure 8.2: Decay rate for the centerline amplitude of a steeper-than-soliton initial sech pulse ($\sigma = \frac{1}{2}$): $A(\xi, \tau=0) = \text{sech} \left(2\sqrt{2}\xi\right)$. 

--- is the $1/\sqrt{\tau}$ asymptote.
Figure 8.3: Locus of points $\xi_1(\tau)$ satisfying $|A(\xi_1(\tau),\tau)| = \gamma_1 |A(\xi=0,\tau)|$, $\gamma_1 = 0.9, 0.8, \ldots, 0.1$ for the evolution of a steeper-than-soliton initial sech profile ($\sigma = \frac{1}{2}$): $A(\xi,\tau=0) = \text{sech}(2\sqrt{2}\xi)$. 

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Figure 8.4: Details of $|A(\xi, \tau)|$ for a $\sigma = \frac{1}{2}$ initial sech profile:
$A(\xi, 0) = \text{sech } (2\sqrt{2} \xi)$, at $\tau = 0, 5, 10$.

(a) magnitude: $|A(\xi, \tau)|$ ; (b) phase: $p(\xi, \tau)$. 
numerically by Chu and Mei (1971), who calculated the initial development but were unable to show the long time behavior. Roskes (1976) finally computed the complete evolution history and showed numerically that this initial data evolve into a "periodic, nondiverging multi-soliton, bound state", a result which can be anticipated from the analytic solution of Zakharov and Shabat (1972).

The evolution picture is shown in Figure 8.5, and a Fermi-Pasta-Ulam recurrence is quite evident. A more quantitative picture is obtained by plotting the center-line amplitude of the pulse as a function of time (Figure 8.6). We see that there is a single recurrence period of \( \tau_0 \approx 12.6 \), after which the profile is repeated with no noticeable change in magnitude. This indicates that the initial pulse completely evolves into exactly two interacting solitons (as evidenced by a single recurrence period), with no residual oscillatory decay; a direct consequence of the choice of the parameter \( \sigma = N_s = 2 \). The spatial profiles of \( A \) at three recurring states \( a, b, c \) (see Figure 8.6) are shown in Figures 8.7.a,b for the magnitude and phase respectively. Note that all the "a" states, which correspond to minimum center-line amplitudes, are complete reconstructions of the initial sech profile with only a small but constant phase shift. The envelope attains a maximum amplitude of about 2 at states \( c \) on \( \xi = 0 \).

A noteworthy feature here is the presence of a node (or zero) of the amplitude near \( \xi \approx 0.9 \). By examining the phase, we see that there is a sudden jump by \( \approx \pi \) at that point, connecting two sides of nearly constant phase. This means that \( A \) has an almost constant phase angle at \( c \) and passes through the origin (in the complex plane) near \( \xi \approx 0.9 \), so that \( A \) has, in fact, no discontinuity of its derivative across the node.
Figure 8.5: Evolution of an initial sech profile with $\sigma = 2$: $A(\xi, 0) = \text{sech}(\xi/\sqrt{2})$. The region shown is for $0 < \tau < 30$, $0 < \xi < 6$. 
Figure 8.6: Centerline amplitude $|A(\xi=0, \tau)|$ for the evolution of an initial sech profile with $\sigma = 2$: $A(\xi, 0) = \text{sech} \left( \frac{\xi}{\sqrt{2}} \right)$. 

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Figure 8.7: Details of $|A(\xi, \tau)|$ for an initial sech profile with $\sigma = 2$: 
$A(\xi, 0) = \text{sech} \left( \frac{\xi}{\sqrt{2}} \right)$ at the recurring states; 
$a_1: \tau = n \times \tau_o$; 
$b_0: \tau = (n + 1/4)\tau_o$; 
$b_1: \tau = (n + 3/4)\tau_o$; 
$c_n: \tau = (n + 1/2)\tau_o$ 
where $\tau_o = 12.6$ for (a) magnitude: $|A(\xi, \tau)|$, (b) phase: 
$p(\xi, \tau)$. 

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We remark in closing that the above numerical result showing long
time recurrence from an initial pulse envelope has as yet not been
checked by experiments (Yuen and Lake, 1976). Aside from difficulties
in producing the required initial profile accurately, Yuen and Lake
(1976) estimated that typical experimental dissipation time scales are
significant compared to the recurrence time scale $\tau_0$, so that the systematic recurrent features are easily obliterated.

VIII.1.3 Multiple-Frequency Recurrence

The example here is an initial envelope of a very flat sech form
with $\sigma = 3$ ($N_s = 3$). As summarized in Section 5.2.4, the analytical
theory predicts an asymptotic state of three solitons bound together in
a recurring state characterized by three frequencies. The numerical
solution is shown in Figures 8.8.a,b for one complete periodic interval
(where the initial profile is just repeated). The features are more
complex than those for $N_s = 2$, and in particular, there are times now
where the maximum amplitude of the pulse is not along the center-line but
on the two sides of it. The complete recurrence history is best illus-
trated by a plot of the center-line amplitude (Figure 8.9). Here we can
identify three distinct states a,b,c corresponding to extrema of
$|A(0,\tau)|$, each with recurring periodicity of $\tau_0 \approx 28.5, 13.5$ and 14.9
alternatively, and 28.1 respectively. Again in this case, the same
amplitudes in successive recurrences indicate an absence of decaying
radiation. The details of the profiles at different instants are shown
in Figures 8.10.a,b. Note in Figure 8.10.a that a,b,c correspond

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Figure 8.8: One period of the evolution of an initial sech profile $(\sigma = 3)$: $A(\xi, 0) = \text{sech}(\sqrt{2}\xi/3)$. (a), (b) show the result from two slightly different view angles. The region shown is for $0 < \xi < 6, 0 < \tau < 30$. 
Figure 8.9: Centerline amplitude $|A(\xi=0,\tau)|$ for the evolution of an initial sech profile ($\sigma = 3$): $A(\xi,0) = \text{sech}(\sqrt{2}\xi/3)$. 
Figure 8.10 (a): Details of $|A(\xi, \tau)|$ for an initial sech profile with
\[ \sigma = 3: \quad A(\xi,0) = \text{sech} \left( \sqrt{2\xi}/3 \right). \]

(a) magnitude: $|A(\xi, \tau)|$ at recurring states $a, b, c$

(b) phase: $p(\xi, \tau)$ at $a, b, c$
Figure 8.10 (b): Details of $|A(\xi, \tau)|$ for an initial sech profile with $\sigma = 3$: $A(\xi, 0) = \text{sech}(\sqrt{2}\xi/3)$.

(a) magnitude: $|A(\xi, \tau)|$ at recurring states a, b, c
(b) phase: $p(\xi, \tau)$ at a, b, c states.
respectively to 1, 3 and 4 humps in the total envelope, and in particular
for c, the maximum peak is not at $\xi = 0$. Figures 8.10.b show the phase
variation. At the "a" states, the near constant curves again confirm
the complete reconstruction of initial (sech form) data. As in the pre-
vious section, the nodes in the amplitudes of the profile for b and c
states correspond to phase jumps of $\sim \pi$ so that A varies smoothly in $\xi$
throughout. Based on the results of this and the last section, one can
now generalize and deduce the essential qualitative features for even more
complex recurrence situations corresponding to initial profiles with
$N_s \sim 4, 5, \ldots$.

VIII.1.4 Recurrence with Decaying Oscillations

Here we present a case where the initial data breaks down into
a soliton plus a background radiation. Here, we choose $\sigma = 1.5$ ($N_s = 1.5$).
The qualitative evolution picture (Figure 8.11) is again one of recurrence
quite similar to that for $N_s = 2$. On examining the time dependence of
$|A(0, \tau)|$, however, Figure 8.12 shows a decay of amplitude variation with
each successive (partial) recurrence. By approximate extrapolation, we
deduce that $|A(0, \tau)|$ will approach an asymptotic value of $\sim 1.28$, but the
decay rate is much longer than the typical recurrence periodicity. For
further evidence of the presence of a decaying radiation, we look at the
amplitude profile at successive recurrent times $b, b'$ (Figure 8.13.a) and
c, c' (Figure 8.13.b). In both figures, we note that the tails of the pro-
files corresponding to a greater time, (the ( )' cases), have larger
amplitudes regardless of their relative values at $\xi = 0$. Since bound
Figure 8.11: Evolution of an initial sech profile, $\sigma = 1.5$: $A(\xi,0) = \text{sech}(2\sqrt{2}\xi/3)$. The region shown is for $0 < \tau < 30$ and $0 < \xi < 6$. 
Figure 8.12: Centerline amplitude $|A(\xi=0,\tau)|$ for the evolution of an initial sech profile ($\sigma = 1.5$): $A(\xi,0) = \text{sech}(2\sqrt{2}\xi/3)$. 
Figure 8.13: Details of $A(\xi, \tau)$ for an initial sech profile ($\sigma = 1.5$): $A(\xi, 0) = \text{sech}(2\sqrt{2}\xi/3)$ for amplitude $|A(\xi, \tau)|$ and phase $\text{p}(\xi, \tau)$ at b, b' and c, c' states and at $\tau = 0, 10$. 
solitons have tails which remain constant with time (see Figures 8.7.a, 8.10.a), while a decaying oscillation actually increases its amplitude at a fixed but large $\xi$ (corresponding to a given range of $\tau$) (see Figure 8.4.a), we have indeed the situation of a soliton bound in a background of decaying radiation. Finally, we plot the phase of $A$ for several times in Figures 8.13.c,d.

VIII.1.5 Creation of a Single Soliton From Two Initial Pulses

Lastly, we give an example of fusion of two initial sech profiles into a single soliton plus some oscillatory decay. Let the initial condition be:

$$f(\xi) = \text{sech} \left[ 2\sqrt{2} (\xi - \xi_o) \right] + \text{sech} \left[ 2\sqrt{2} (\xi + \xi_o) \right]$$  \hspace{1cm} (8.10)

Here we set $\xi_o = 2.5$ so that the two pulses are initially separated ($|A(0,0)| = 0.003$). Each of the two similar sech profiles is the same as that of Section 8.1.1 ($\sigma = \frac{1}{2}$, $N_s \sim \frac{1}{2}$) and is narrower than a soliton of the same height, so that individually, we expect them to evolve into a pure radiation decay similar to that observed in Section 8.1.1. Together, however, they give $N_s = 1$ and the theoretical prediction is for the creation of one permanent soliton. To confirm this, we compute the evolution history (considering only the symmetric half $\xi > 0$) and the picture for $|A|$ is shown in Figure 8.14. Note that the sech profile at $\xi_o$ initially decays into oscillations resulting in a build-up along the center-line ($\xi = 0$) eventually forming a single permanent hump as the oscillation in the background decay and recede. The variation of $|A|$
Figure 8.14: Fusion of two initially separate sech humps. The region shown is for $0 < \tau < 40$ and $0 < \xi < 4$. 
along $\xi = 0$ is shown in Figure 8.15. The amplitude in the middle increases from 0 to a peak of $\sim 0.51$ at $\tau = 12$ and then drops gradually like $\frac{1}{\sqrt{\tau}}$ reaching an asymptotic value for a soliton height there of $\sim 0.40$. In Figures 8.16 for the profiles of $|A|$ at different times, the evolution from 2 separate off-center pulses to a single hump in the center in the background of oscillatory radiation, to the establishment of a single permanent soliton envelope is clearly shown.

In closing, we remark that the single choice of sech profiles in the foregoing examples is chiefly for convenience, and a variety of other profiles such as Gaussian, sine, etc., have also been studied with essentially similarly features and the dominant role still played by the integral estimate $N_s$ of Eq. (5.22).
Figure 8.15: Centerline amplitude $|A(\xi=0, \tau)|$ for the fusion of two initially separate sech humps.
Figure 8.16 (1): Details of $|A(\xi, \tau)|$ for the fusion of two sech humps.
Figure 8.16 (ii): Details of $|A(\xi, \tau)|$ for the fusion of two sech humps.

---: soliton profile.
VIII.2 *Nonlinear Evolution of a Propagating Wave Front*

The physical problem is that of a travelling wave front which may be generated by the abrupt starting (or stopping) of a wavemaker. This problem was studied theoretically by Wu (1957) and Miles (1962). Experiments by Longuet-Higgins (1974) showed a larger maximum wave amplitude behind the front than that given by the linear theory and pointed out the importance of nonlinearity. We assume here that the initial envelope of the progressive wave train is of the form:

\[
f(\xi) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{\xi}{\sigma} \right) \right]
\]

(8.12)

The appropriate boundary conditions for \( A \) are

\[
A \rightarrow 0 \quad \xi \rightarrow \infty \quad (8.13.a)
\]

\[
A \rightarrow e^{-i\tau/2} \quad \xi \rightarrow -\infty \quad (8.13.b)
\]

Such a problem is not of the ISI type and no exact solution is known. As in Section 8.1, the only parameter is \( \sigma \), which now is the length scale over which the initial step jump (8.12) occurs. Since the slow variables are normalized by the maximum wave steepness \( \varepsilon \equiv k_o A_o \) (\( \xi = \varepsilon \xi_o = \varepsilon k_o (x^* - C g_o t^*) \); \( \tau = \varepsilon^2 \tau_o = \varepsilon^2 \omega t^* \)), two problems \( A^{(i)} \) with different \( \sigma^{(i)} \) \((i = 1, 2)\) can also be thought of as having an initial profile with a transition over the same physical length \( \xi_o \) but of different wave steepnesses \( \varepsilon^{(i)} \) given by

\[
\frac{\varepsilon^{(1)}}{\sigma^{(1)}} = \frac{\varepsilon^{(2)}}{\sigma^{(2)}} \quad (8.14.a)
\]

or step heights related by
\[
\frac{A_0^{(1)}}{\sigma_0^{(1)}} = \frac{A_0^{(2)}}{\sigma_0^{(2)}}
\] (8.14.b)

That is, a larger \(\sigma\) could physically correspond to a higher step with the same transition length. On the other hand, since

\[
\xi_0 = \xi/\varepsilon \quad \text{and} \quad \tau_0 = \tau/\varepsilon^2
\]

the same \(\xi\) and \(\tau\) correspond to larger \(\xi_0\) and \(\tau_0\) for smaller \(\varepsilon\). In particular, the solution for the near linear limit of small amplitude, \(\varepsilon \ll 1\), can be deduced from a nonlinear solution for \(\sigma \ll 1\) in the neighborhood of small normalized time and longitudinal distance (from the step) \(\tau = \varepsilon^2 \tau_0 \ll 1\) and \(\xi = \varepsilon \xi_0 \ll 1\).

For the linear problem (i.e., dropping the cubic term in Eq. (8.1)), the exact similarity solution corresponding to an initial profile of a unit Heaviside step is:

\[
A(\eta) = \frac{1}{2} - \frac{C(\sqrt{\frac{3}{\pi}} \eta) - iS(\sqrt{\frac{3}{\pi}} \eta)}{1 - i}
\] (8.15)

in terms of the variable

\[
\eta = \xi/\sqrt{\tau}
\] (8.16)

where \(C, S\) are the Fresnel integrals.

The amplitude \(|A(\eta)|\) based on Eq. (8.15) is plotted in Figure 8.17. Note the familiar Fresnel integral type oscillations and an overshoot of about 18% just behind the front. Clearly nonlinearity would be most important behind the front where the amplitude is large. The full nonlinear equation (8.1) is solved numerically subject to (8.12), (8.13) for \(\sigma = 0, 0.5\) and 1. The results are shown in time sequence plots of the
Figure 8.17: Plot of $|A(n)|$ as given by Eq. (8.15).
amplitude in Figures 8.18 where the linear result according to Eq. (8.15) is included for comparison. The nonlinear results are dramatically different from the linear case. The following observations are made:

(i) Initially, the nonlinear disturbances spread from the jump at a rate somewhat slower than $\sqrt{\tau}$ according to the linear theory. This is particularly true for the milder nonlinear transitions, i.e., larger $\sigma$. (Aside from a small leading front, the results at small $\tau$ for $\sigma = 0$ are qualitatively similar to the linear similarity solution.)

(ii) As time increases, the nonlinear envelope forms distinct lobes behind the front which oscillate between approximately 0 and 2, and are hence about 5 times larger than the linear oscillations. The number of such lobes increases with time as the nonlinear effect move away from the step transition. The lobes, however, remain of almost equal widths and heights (and troughs) regardless of their positions or time. This is in direct contrast with the decay of amplitude, and the lengthening ($\sim \sqrt{\tau}$) of the linear disturbances. Consequently, for large time, the nonlinear lobes are narrower and closer together than the linear oscillations close behind the front, while the opposite is true further away. In particular, the first peak behind a nonlinear wave front moves slightly ahead of the linear counterpart as $\tau$ increases; a phenomenon also observed in experiments (Longuet-Higgins, 1974).

(iii) For large time, the amplitude immediately forward of the front is much smaller for all the nonlinear cases.

(iv) The effect of a larger $\sigma$ (milder transition) in the initial condition seems to be mainly a time lag in the early development, so that for large time, the essential features for example for the case $\sigma = 1$
Figure 8.18 (a): Time sequence for the magnitude of the evolution $|A(\xi, \tau)|$ for a travelling wave front.
Figure 8.18 (b): Time sequence for the magnitude of the evolution $|A(\xi, \tau)|$ for a travelling wave front.
Figure 8.18 (c): Time sequence for the magnitude of the evolution $|A(\xi,\tau)|$ for a travelling wave front.
Figure 8.18 (d) Time sequence for the magnitude of the evolution \(|A(\xi, \tau)|\) for a travelling wave front.
are similar to that for \( \sigma = 0 \) at a fixed time earlier. This is quite reasonable, as one expects the variations in the initial data to become less important as time increases.

The nonlinear results are thus very different from the linear solution (8.15). In particular, the presence of lobes of such large amplitudes, which also have unique and near constant length, amplitude and time (for generation) scales, is remarkable. For convenience of comparison, we present selected details of linear and nonlinear \((\sigma = 0)\) profiles for the unit step initial condition in Figures 8.19.

It is unfortunate that the experimental data of Longuet-Higgins are not sufficiently detailed to provide quantitative comparisons.
Figure 8.19: Details of the amplitude $|A(\xi, \tau)|$ for a travelling wave front for $\tau = 5, 10, 15$ for (a) linear result, (b) nonlinear result for $\sigma = 0$. 

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IX. DIFRACTION IN AN OBLIQUE CHANNEL

The geometry is that described in Section 6.3.1 and the initial boundary value problem is Eqs. (6.27). In the absence of ambiguity, we shall use \((x, y, t)\) in place of \((x', y', t')\) for the sake of brevity. The width \(b\) of the channel, which should be of \(O(1)\) but not too wide as to cause transverse instability, cf. Section 5.2.2, is fixed to be \(b = 1\).

The other geometric parameter, \(\varepsilon\) is chosen to be \(\varepsilon = 0.2\) which corresponds to a bending angle of \(\alpha = \tan^{-1} \varepsilon \approx 11.31^\circ\). For the incoming envelope pulse, we consider a fixed peak amplitude corresponding to \(K = 1.125\), or \(k_0 \zeta_0 = 0.3\) (for \(\varepsilon = 0.2\)), and study different envelope shapes.

For computations, Eqs. (6.27) is solved using discretizations of \(\Delta x = \Delta y = 0.05\) and \(\Delta t = 0.1\). As we integrate in time, it is necessary to repeatedly increase the extent of the longitudinal domain of computation so that the boundary conditions Eq. (6.27.c,d) can be imposed smoothly and the solution near the center of the group \(x_g\) is not affected by the boundary at \(x_\infty\). For a typical integration time of \(0 \leq t \leq 0(60)\) (\(\sim 600\) time steps), the \(x\) domain would be between \(x_g \pm 0(25)\) (a total of \(0(2 \times 10^4)\) unknowns for \(b = 1\)). The number of operations per unit progression in \(t\) here is clearly very large, and requires typical CPU time of \(\sim 1000\) seconds on the Honeywell 6180 computer under the Multics operating system.

IX.1 Soliton Incidence

Here, the incident wave envelope is a one-dimensional soliton pulse given by
\[ P(x,y,t) = \text{sech} [2\sqrt{K} (x - \frac{1}{2} t - x_0)]e^{-i\xi Kt/2} \quad (9.1) \]

To satisfy zero initial condition in the oblique channel, Eq. (6.37.b), the soliton peak is given a starting position at \( x_0 = -4 \). The IBVP (6.37) is solved numerically for up to \( t = 40 \). It is found that the envelope remains essentially confined and propagates as a single pulse at group velocity but exhibits special periodic two-dimensional variations.

The amplitude of the envelope \( |A| \) near the center of the group \( x_g \equiv x_0 + \frac{1}{2} t \) is shown for a sequence of time in Figures 9.1. Note that the first frame is \( t = 8 \) and \( x_g = 0 \) so that the peak of the soliton has just entered the channel. Several systematic features can be observed here: the incident pulse does not break up or decay but remains as a single hump which sloshes in the transverse direction. This sloshing has a clear periodicity with a recurrence period of approximately 10. This periodic feature is best illustrated by examining the maximum amplitude at \( x \sim x_g \) for different \( y \) as a function of \( t \), as shown in Figure 9.2 for \( y = 0, .25, .5, .75 \) and 1. Note that the peak amplitude is 1 for any \( y \) before entry into the channel (\( t < 8 \)). For ease of reference, we identify different states: \( a_i \), corresponding to the times when the amplitude at both banks are equal; \( b_i \), when the peak amplitude achieves a maximum (the maximum is at \( y = 0 \) for \( i = 1, 3, 5 \) and at \( y = 1 \) for \( i = 2, 4, 6 \)); and \( b_i' \), when the peak is minimum (at \( y = 0 \) for \( i = 2, 4, 6 \) and at \( y = 1 \) for \( i = 1, 3, 5 \)).

From Figure 9.2, the following can be noted:

(1) There is clear evidence of recurrence except for some small oscillations initially and an irregularity near \( y = 1 \) at \( b_4 \). The two-dimensional near recurrence has an average period of \( t_o \sim 10.2 \).
Figure 9.1 (i): Envelope amplitude $|A|$ following a plane soliton entering a bent channel at unit time increments. The regions shown are for $0 < y < b$ and $0 < x < 5$ for $t < 13$ and $x_g - 2.5 < x < x_g + 2.5$ for $t \geq 13$. 

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Figure 9.1 (ii): Envelope amplitude $|A|$ following a plane soliton entering a bent channel at unit time increments. The regions shown are for $0 \leq y \leq b$ and $0 \leq x \leq 5$ for $t \leq 13$ and $x_g - 2.5 \leq x \leq x_g + 2.5$ for $t \geq 13$. 

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Figure 9.2: Maximum amplitude along $y = 0, .25, .5, .75$ and $1$ as a function of time for a plane soliton entering a bent channel.
(2) For each \( y \), the profile is periodic with period \( \tau_0 \) with the interesting exception near the center of the channel \( y = \frac{1}{2} \) where the peak amplitude oscillates about 1 at about twice the frequency, indicating possible presence of a second periodicity. The variations are greatest along the two banks and has an average maximum and minimum of \( \approx 1.31 \) and \( \approx 0.65 \) respectively.

(3) As with the case of one-dimensional evolution, the near constant amplitudes at successive recurrences indicate that the two-dimensional pulse is completely bound in the sense that there is no decaying loss due to radiation.

The fact that the recurrence period is approximately 10.2 is somewhat surprising because this is also the periodicity of a cross wave in the same channel for a linear uniform harmonic incident wave train — a fact completely determined by geometry. Referring to Figure 9.3, simple geometric constructions for the rays based on the law of reflection (zero normal derivatives) at the walls give a spatial period of \( \frac{b}{\tan \alpha} \) between which the rays would have travelled a distance of \( \frac{b}{\sin \alpha} \). Since the group velocity is \( \frac{1}{2} \), the periodicity is simply \( \frac{2b}{\sin \alpha} \), which for \( b = 1 \), \( \tan \alpha = 0.2 \) gives 10.198!

To understand this better, we return to Eqs. (6.37) and introduce coordinates moving with the group:

\[
\tau = \varepsilon \tau_1, \quad \xi = x_1 - \frac{1}{2} \tau_1 \quad (9.2.a,b)
\]

Eq. (6.37.a) then becomes

\[
iA_{\tau} - \frac{1}{2} A_y + \frac{1}{4} A_{yy} - \frac{1}{8} A_{\xi \xi} - K|A|^2 A = 0 \quad (9.3)
\]
Figure 9.3: Geometric construction for linear harmonic cross-waves in a bent channel.
It may be readily verified that

\[ A = \text{sech} \left( 2a \sqrt{K} \xi \right) e^{-iK a^2 \tau /2} e^{i t /4 + iy} \]  

(9.4)

is an exact solution to Eq. (9.3) subject to the boundary conditions of the IBVP (6.37), so that Eq. (9.4) has been used extensively to check the accuracy of the two-dimensional diffraction program.

Returning to Eq. (9.3) we further assume small amplitude and long pulses so that nonlinearity and the term \( A_{\xi \xi} \) may be neglected. The problem reduces to that of a one-dimensional IBVP in \( y \):

\[ iA_\tau - \frac{i}{2} A_y + \frac{1}{4} A_{yy} = 0 \quad \quad 0 < y < b \]  

(9.5)

for a profile \( A(y, \tau) \) bound between two points with the boundary conditions (6.37.e,f):

\[ A_y = iA \quad \quad \text{on } y = 0, b \]  

(9.6)

We impose the initial condition:

\[ A(y, 0) = 1 \]  

(9.7)

The general solution to Eq. (9.5), (9.6) and (9.7) can be expressed in the series

\[ A = \sum_{n=0}^{\infty} a_n e^{i\Omega a^2 \tau} e^{iy} \cos \left( n \pi y / b \right) \]  

(9.8.a)

where

\[ a_n = \frac{i \varepsilon_n \left[ (-1)^n e^{-ib} -1 \right]}{b \left[ 1 - \left( \frac{n \pi}{b} \right)^2 \right]} \]  

(9.8.b)

and \( \varepsilon_n \) is the Jacobi symbol: \( \varepsilon_0 = 1, \varepsilon_n = 2 \) for \( n = 1, 2, \ldots \). The
result of Eqs. (9.8) for \( b = 1 \) is shown in Figure 9.4 where the magnitude \( |A| \) is plotted against \( t = \tau / \varepsilon \) (\( \varepsilon = 0.2 \)) for \( y = 0, .25, .5, .75 \) and 1. The profiles are quite jagged but shows a clear periodicity of \( t_0 \sim 12.5 \) with maximum and minimum of about 1.38 and 0.43 respectively. The qualitative resemblances between Figures 9.2 and 9.4 in terms of the periodicity, amplitude and the higher frequency near the center are noteworthy. It seems reasonable to conclude, therefore, that the lateral sloshing observed for the nonlinear problem is not a consequence of nonlinearity. The preservation of a single recurrent pulse with no decay or breakdown is, of course, a nonlinear phenomenon, in striking contrast to the linear theory which would predict the flattening and eventual disappearance of the group due to dispersion.

Finally, we study the details of the two-dimensional envelope by showing further results at 4 particular instants in a sample recurrence period: \( a_4, b_5, a_5 \) and \( b_6' \) (corresponding to \( t = 28.5, 30.7, 33.6 \) and 35.8 respectively). In Figure 9.5, we plot the profile of the envelope amplitude \( |A| \) in the \( y \) direction along the ridge of maximum amplitude at \( x \sim x_8 \). During the time when the envelope is most tilted transversely (at \( b_5', b_6' \)), the profiles can be closely approximated by half of a cosine curve of the form:

\[
f(y) = f_1 + f_2 \cos (\pi y/b) \tag{9.9}
\]

where the constants \( f_1, f_2 \) are adjusted to match the maximum and minimum values at the walls (\( y = 0, b \)). These are plotted in Figure 9.5 for \( b_5 \) and \( b_6' \), (\( f_1 \) equal to or slightly less than 1 and \( f_2 \sim 0.32 \)), and the correspondence is very good. Half-way between such states of maximum
Figure 9.4: Plot of $|A(y,t)|$ as given by Eqs. (9.8).
Figure 9.5: Transverse profiles of $|A|$ along the peak $x \sim x_0$ for the cases $a_4$, $a_5$, $b_5$ and $b_6'$ for a plane soliton entering a bent channel. 

---: approximation by cosine curve.
transverse slope, at \( a_1 \), the peak of the pulse becomes almost horizontal and is now nearly a full cosine curve somewhat above 1 given by

\[
g(y) = g_1 + g_2 \cos (2\pi y/b)
\]  \( (9.10) \)

as shown in the figure. By examining the profiles within other recurring periods, we see that the top of the pulse gradually tilts back and forth between the curves \( b_5 \) and \( b_6' \), and becoming similar to the \( a_1 \) profiles as the envelope levels out.

The longitudinal profiles for \(|A|\) are plotted in Figures 9.6. For clarity, only the curves for \( y = 0, 0.5 \) and 1 are shown. The profiles for intermediate of values lie between the above curves and generally have similar shapes as is evident from Figure 9.1. To provide some comparison, we include in the plots two sech profiles which correspond to (plane) solitons with respective amplitudes equal to the maximum peak amplitude and unit amplitude (same as the incident pulse Eq. (9.1)). (The latter curves are omitted for \( a_1 \) cases since the peak amplitude is already near unity.) Three main features can be noted:

(1) Near the center, the profiles for different \( y \) values are very similar, and in the case of states \( a_1 \), almost identical. Furthermore, for cases \( b_5 \) and \( b_6' \), the higher peaks are also somewhat wider than the lower ones in contrast to soliton profiles whose widths are inversely proportional to their heights.

(2) Away from the center, small oscillatory tails are present, which become quite complex and appear to decay rather slowly with longitudinal distance.

(3) All the profiles are narrower than corresponding soliton
Figure 9.6: Longitudinal profiles of $|A|$ for $y = 0, 0.5$ and $1$ at different states. $\cdots$ : soliton sech profiles of amplitude $1$ and amplitude equal to the maximum peak amplitude.
profiles of the same height. This is particularly true for the $a_1$ cases, so that although the pulses are nearly one-dimensional, they do not correspond to any plane soliton.
IX.2  

**Incidence of a Sech Envelope Twice as Long as a Soliton of the Same Height**

The incident pulse is one-dimensional and given by

\[ P(x, y, t) = \text{sech} \left( \sqrt{K} \left( x - \frac{1}{2} t - x_0 \right) \right) e^{-i\xi t/2} \]  \hspace{1cm} (9.11)

Because of the nondimensional scaling, this is also equivalent physically to a sech hump twice as high as that of a soliton of the same length.

We start the incident group at an upstream position \( x_0 = -7 \) so that there is essentially no disturbance in the oblique channel at \( t = 0 \). The complete evolution of this wide pulse under the governing equation and boundary conditions of Eqs. (6.37) is computed up to \( t = 66 \). The two-dimensional profiles for the amplitude \( |A| \) are shown in Figures 9.7 at time intervals of 2 (starting from \( t = 14 \) when the peak of the envelope has just entered the channel). Note that for \( t \geq 24 \), we have followed the group centered at \( x_g = x_0 + t/2 \) moving at group velocity \( C_g = \frac{1}{2} \). The envelope initially remains as a single hump near \( x \sim x_g \) which goes through two transverse (tilting) oscillations similar to that observed in the previous section. The important difference here is that the peak becomes significantly narrower than the incoming pulse and the maximum amplitude increases to over twice the incident height for the second oscillation (see Figures 9.7 for \( t = 26, 28 \); also compare with Figures 9.1). At \( t \sim 30 \), the single peak begins to break down, and we enter a transition period marked by the emergence of many small humps. By carefully studying Figures 9.7 (and also results for intermediate times), we observe the interesting feature that each of these disturbances exhibits some sloshing in \( y \) regardless of their amplitudes,
Figure 9.7 (i): Envelope amplitude $|A|$ following a plane wider-than-soliton pulse entering a bent channel at time intervals of 2. The regions shown are for $0 < y < b$ and $0 < x < 10$ for $t < 24$ and $x_g - 5 < x < x_g + 5$ for $t \geq 24$. 
Figure 9.7 (ii): Envelope amplitude $|A|$ following a plane wider-than-soliton pulse entering a bent channel at time intervals of 2. The regions shown are for $0 \leq y \leq b$ and $0 \leq x \leq 10$ for $t \leq 24$ and $x_g - 5 \leq x \leq x_g + 5$ for $t \geq 24$. 

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Figure 9.7 (iii): Envelope amplitude $|A|$ following a plane wider-than-soliton pulse entering a bent channel at time intervals of 2. The regions shown are for $0 \leq y \leq b$ and $0 \leq x \leq 10$ for $t \leq 24$ and $x_g - 5 \leq x \leq x_g + 5$ for $t \geq 24$. 

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further confirming the suggestion that such "sloshing" motion in a channel is also inherent in the linear problem. As time increases further, the situation greatly simplifies, and a clear picture emerges: the single peak at the center (see \( t = 24 \) for example) has evolved into two major separate humps both of which slosh transversely (but with somewhat different periodicities). A typical picture is that at \( t = 50 \) (here the forward peak has just reached a maximum at \( y = 1 \) and the leeward peak is nearly level in the \( y \) direction). The forward peak is slightly ahead of the center of the group \( x_g \), and is steeper (narrower) and higher than the second peak. (At \( t \approx 50 \), the maximum amplitude is \( \approx 2 \) at \( y = 1 \) and \( x \approx x_g + 0.65 \).) The leeward hump is some distance behind \( x_g \) and has a broader but lower peak which goes through complicated multi-peak stages. (At \( t \approx 50 \), this peak is at \( x \approx x_g - 2.5 \), and the average height is \( \approx 0.7 \).) As time increases, these 2 peaks tend to move further away from the center and reaches average positions at about \( x_g + 1 \) and \( x_g - 3 \) for large times. There is partial evidence of near complete two-dimensional recurrence (see for example \( t = 54 \) and 65).

From Section 8.1.2 for the one-dimensional evolution of the same initial profile (9.11), we recall that the envelope pulse remains symmetric and always returns to a single hump profile at some time in the recurrence (this is also true regardless of the number of solitons confined). This is a marked contrast to the present case of a two-dimensional envelope diffracted by parallel oblique walls, where two separate asymmetric humps evolve away from the center after a long time, each undergoing periodic transverse sloshing with different frequencies. We have not, however, established evidence of fission into two solitons or independent
sloshing pulses.
X. DIFFRACTION OF A PLANE SOLITON BY A CONVERGING CHANNEL

The channel geometry is that described in Section 6.3.2. Here we fix the half-width of the outer channel to $b = 2$ and that of the uniform downwave channel to be $B = 1$. In the converging channel, the oblique walls have a slope of $\varepsilon = 0.2$ (\( \alpha \approx 11.31^\circ \)) so that this section has a length of $5$. As with the last chapter, the maximum amplitude of the incident group is rather large $K = 1.125$ ($k_{0,0} = 0.3$) in order to accentuate nonlinearity. The envelope is that of a one-dimensional soliton given by Eq. (9.1) ($x_0 = -4$). The initial-boundary-value problem Eqs. (6.28) for this case is integrated for $0 < t < 52$.

The grid sizes and time step are the same as for Chapter 9. Because of the presence of a wider converging section, and the need for extrapolation on the irregular boundary, the computation effort is, however, somewhat more involved for the present case.

The evolution of the soliton propagating through the converging channel is shown in Figures 10.1 for the amplitude $|A|$ at $t = 8, 10, 12, 16, 18, 20$. Because of symmetry, only the envelope in the lower half of the channel is shown. At $t = 8$, $x_g = x_0 + t/2 = 0$, the peak begins to enter the converging section; at $t = 18$, $x_g = 5$, the center of the group is just entering the narrower uniform channel. The envelope remains as a single pulse but becomes sharper.

There is also a transverse sloshing where the peak near the wall rises above then drops below the peak at the center of the channel. In Figure 10.2 we plot the $y$-variations of the amplitude $|A|$ at the peak $x = x_g$. Note that the curves for $t > 8$ start at the wall given by

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Figure 10.1 (i): Envelope amplitude $|A|$ of a plane soliton entering a converging channel at time intervals of 2. The domain shown is for $0 \leq x \leq 10$, and the half-channel $y \leq 1$ and $y \geq 0.2x$ (for $x \leq 5$) and $y \geq 1$ (for $x \geq 5$).
Figure 10.1 (ii): Envelope amplitude $|A|$ of a plane soliton entering a converging channel at time intervals of 2. The domain shown is for $0 \leq x \leq 10$, and the half-channel $y \leq 1$ and $y \geq 0.2x$ (for $x \leq 5$) and $y \geq 1$ (for $x \geq 5$).
Figure 10.2: Transverse profiles of $|A|$ along the envelope peak ($x = x_g$) as a soliton propagates into a converging channel.
\( y = \varepsilon \cdot x_g(t) \). The amplitude along the oblique wall builds up gradually with distance of propagation. This upheaval gradually spreads transversely towards the center of the channel \( y = 2 \). The amplitude at the center-line, then, initially oscillates, reaches a minimum of about 0.8 at \( t \sim 14 \) after which it increases steadily and continues to do so after entry into the downwave uniform channel \( t > 18; \text{see } t = 20 \text{ in Figure 10.1} \). The time variation of the center-line peak amplitude is shown in Figure 10.3. The wetted profile along the converging wall is of engineering interest and is plotted for a series of time in Figure 10.4. The peak amplitude at \( x = x_g \) increases gradually as it moves along the wall reaching a maximum of about 1.9 at \( t = 17 \) \( (x_g = 4.5) \). This increase in height is followed by a narrowing of the profile.

After entering the uniform channel, recurrence is a distinct feature of the evolution. Referring to Figure 10.3, we observe an approximate periodicity with a period of \( \approx 9 \). Furthermore, the nearly constant amplitudes for successive periods suggest recurrence with very little decaying radiation in \( x \). To see that there indeed is recurrence at instants \( (a,d,h,k \text{ and } b,f,j, \text{ etc } ) \) separated by the recurrence period, we plot three-dimensional pictures of \( |A(x,y,t)| \) at \( (a,b,\ldots,k,l) \) in Figures 10.5, but now for the full width of the channel (using symmetry). Although the computation domain is much longer, only the region \( x_g - 2 \leq x \leq x_g + 2 \) which contains the major features, is shown. The transverse variation of the center of the incoming pulse "a" begins with one peak at the center, then the center drops while the sides rise until there is a trough at the center. In between these two limits there is a three peak terrain. This process is then reversed until \( t = 27.2 "d" \text{ when the} \)
Figure 10.3: Maximum amplitude along the center of the channel:

$$\max_x \left| A(x, y=1, t) \right|.$$
Figure 10.4: Wetted profile along the wall of a converging channel

\( y = \varepsilon x \) \((0 \leq x \leq 5)\): \( |A(x, y=\varepsilon x, t)| \).
Figure 10.5 (i): Two-dimensional variations of $|A|$ in the uniform channel at selected times. The domains shown are for the full channel width $1 \leq y \leq 3$ and $x_g - 2 \leq x \leq x_g + 2$. 
Figure 10.5 (ii): Two dimensional variations of $|A|$ in the uniform channel at selected times. The domains shown are for the full channel width $1 < y < 3$ and $x - 2 < x < x + 2$. 
Figure 10.5 (iii): Two-dimensional variations of $|A|$ in the uniform channel at selected times. The domains shown are for the full channel width $1 \leq y \leq 3$ and $x \leq 2 \leq x \leq x_{g}$. 

$\text{i: } t = 39.4$

$\text{j: } t = 40.9$

$\text{k: } t = 47.4$

$\text{l: } t = 49.1$
initial state recurs. From here on, the same cycle is repeated. The near complete recurrences at a,d,h; b,f,j; c,g and e,i,l respectively are quite evident.

To further establish recurrence, we display the y cross-section of |A| along the envelope peak (near *) as a function of time. This is shown in Figures 10.6, again for the full channel. Figure 10.6.a shows the result between t = 19.5 "a" and t = 27.5 "d". This transverse variation of the envelope peak is nearly repeated in Figure 10.6.b (27.5 ≤ t ≤ 35.5) giving a different view confirming the earlier observations of recurrence.

In summary, we have shown that the incoming two-dimensional envelope as produced by sending a plane soliton through a mildly converging channel, exhibits two-dimensional FPU-type recurrence as it propagates down a uniform channel. Our work provides further evidence to Yuen and Ferguson (1968) (who treated periodic boundary conditions only) that recurrence is an intrinsic feature of two-dimensional slowly modulated Stokes' waves whatever the initial and/or boundary conditions. Thus, in ocean surface waves, there is a strong tendency to maintain coherence rather than randomization.
Figure 10.6: Time variation of the transverse profiles of $|A|$ along the envelope peak $(x \sim x_0) \max_x |A(x,y,t)|$ for (a) $19.5 \leq t \leq 27.5$, (b) $27.5 \leq t \leq 35.5$ and for the full channel $1 \leq y \leq 3$.
XI. SUMMARY AND CONCLUSIONS

In this thesis extensive numerical experiments have been performed on the basis of parabolic approximation which describes the slow modulation of non-linear Stokes' waves. Specifically, in Part One, we showed that the steady state diffraction of Stokes' waves by a thin body is governed by the cubic Schrödinger equation (2.38) with a mixed diffraction boundary condition of the form (2.47). The efficacy of this rather simple model for diffraction at grazing angles is tested for several geometries. For the case of a thin wedge, the interesting phenomenon of Mach reflection (Weigel, 1964b) is shown to be an important nonlinear effect. The numerical results are in good qualitative agreement with existing experiments, although quantitative corroboration cannot be made due to large scatter in the experimental measurements. A better understanding of the factors causing these scatter and/or new well controlled systematic experiments based on the single (nonlinear) parameter K to confirm the present results is desirable. The systematic features of the Mach stem in the numerical solution suggest a shock model, and the resultant comparison is remarkably satisfactory in light of the gross simplifications involved. For oblique incidence upon the other geometries of a parabola, a thin island of finite length and normal incidence on a semi-infinite screen, the computed features are somewhat less dramatic. The significant effects of nonlinearity, however, are again underscored.

The diffraction of transient envelopes of Stokes' waves (for grazing incidences) is considered in Part Two. This is of particular
importance in view of the side-band instability of steady Stokes' waves (Benjamin and Feir, 1967; Benney and Roskes, 1969), and the recent interest in modelling observed wind waves structures and evolution in terms of wave packets (Mollo-Christensen and Ramamonjiarisoa, 1978; Lake and Yuen, 1978). As groundwork for actual computation of diffraction of pulse envelopes, we presented the governing Schrödinger equation for slow two-dimensional modulation of Stokes' waves, reviewed some of its important theoretical properties and studied several numerical one-dimensional nonlinear evolution problems for illustration. Of particular interest are the examples of multi-soliton recurrent bound asymptotic states of a plane pulse, and the generation of large lobes in the nonlinear evolution of a plane wave front.

For the two-dimensional nonlinear Schrodinger equation, we proposed a new numerical solution scheme based on a modified alternating-direction-implicit (ADI) method. The algorithm has a second-order accuracy and is unconditionally stable (for the linear case). Furthermore, it has the advantage that it does not require a simultaneous solution of all the unknowns in the two-dimensional domain.

Because of the instability of plane envelope solitons to long transverse disturbances, we restrict ourselves to examples of diffraction by channels whose widths are large compared with wavelength but not sufficiently large compared with the longitudinal dimensional of the envelope to allow lateral instability of solitons. The first example is that of a one-dimensional wave packet entering a uniform channel which is at an oblique angle to the direction of propagation. When the incoming group is a soliton, the resulting envelope travelling down the channel
undergoes a periodic lateral sloshing motion which is largely a linear effect. The envelope remains as a single pulse which does not break down or decrease in amplitude after successive sloshing (recurrence) indicating a special balance of dispersion by nonlinearity. For the case of an incident pulse which is twice as wide as a soliton of the same height, the evolution picture in the oblique channel is considerably more complicated. After an initial transition period, the peak in the center of the group moves forward which is accompanied by the emergence of a smaller hump behind the center. Each of these humps remains in the same group and undergoes transverse sloshing similar to that observed for the last case. There is some, though incomplete, evidence of two-dimensional recurrence in this stage. For the second geometry, we considered a uniform channel which goes through a mild converging section into another channel of narrower constant width. For a plane soliton entering the converging channel along the center-line, the peak amplitude increases considerably together with a longitudinal steepening of the pulse. The subsequent evolution in the downwave channel exhibits elaborate two-dimensional envelope shapes within cycles of recurrence. Thus it appears that recurrence is also a dominant nonlinear feature of two-dimensional modulation.

Despite the efficiency of our modified ADI method, the long time calculation of two-dimensional evolution with no longitudinal boundaries requires a very large amount of computational effort. Consequently, we only solved a few typical examples to study the essential features. More systematic solutions over a range of the relevant parameters should be a useful, though ambitious, project. More importantly, since plane solitons
play such an essential role in the evolution of one-dimensional envelope pulses, a better understanding of the mechanisms of their transverse instability would be useful to the study of diffraction (and two-dimensional evolution) of transient Stokes' waves. Experimental and numerical confirmation of the onset of lateral instability as well as information about the post-instability development and end-state of a one-dimensional soliton is greatly needed.
REFERENCES


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BIOGRAPHY

Dick Kau-Ping Yue was born in Shanghai, China on September 5, 1953. He grew up in Hong Kong where he obtained all his pre-college education. In the fall of 1971, he came to MIT with a full scholarship and began his studies as a freshman, obtaining the S.B. and S.M. degrees in Civil Engineering in 1974 and 1976 respectively.

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Publications:


APPENDIX A

DETAILS OF DERIVATION OF THE TWO-DIMENSIONAL MODULATION EQUATIONS

We write $\phi$ and $\zeta$ in power series of $\varepsilon$:

$$\phi = \sum_{n=1} \varepsilon^n \phi_n$$  \hspace{1cm} (A.1)

$$\zeta = \sum_{n=1} \varepsilon^n \zeta_n$$  \hspace{1cm} (A.2)

and introduce the slow variables (6.5) plus

$$x_2 = \varepsilon^2 x; \quad t_2 = \varepsilon^2 t$$  \hspace{1cm} (A.3)\+

so that

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial x_2} + \varepsilon^2 \frac{\partial}{\partial x_2}$$

$$\frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial y_1}$$

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2} + \varepsilon^2 \frac{\partial}{\partial t_2}$$  \hspace{1cm} (A.4.a,b,c)\+

Note that (A.4.b) follows from the assumption of nearly one-dimensional waves (6.1) so that

$$\phi_n = \phi_n(x, x_1, x_2, y_1, t_1, t_2)$$

$$\zeta_n = \zeta_n(x, x_1, x_2, y_1, t_1, t_2)$$  \hspace{1cm} (A.5.a,b)

and there is no explicit dependence on $y$. Expanding Eqs. (2.1) in Taylor

*\+

Note that for Part I, replace $(x_2, y_1)$ by $(X, Y)$ while all $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2} \approx 0.$
series about $z = 0$ and using (A.1) - (A.4), we obtain the following set
of equations upon collecting terms of the same order in $\varepsilon$:

\[-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_n = F_n, \quad -h < z < 0, \quad n = 1, 2, 3, \ldots \quad (A.6.a)\]

where

\[F_1 = 0 \quad (A.6.b)\]

\[F_2 = 2\phi_1 x_1 \quad (A.6.c)\]

\[F_3 = \phi_1 x_1 x_1 + \phi_1 y_1 y_1 + 2\phi_2 x_1 x_1 + 2\phi_1 x_1 x_2 \quad (A.6.d)\]

\[-\Gamma \phi_n = G_n \quad \text{on } z = 0 \quad n = 1, 2, 3, \ldots \quad (A.7.a)\]

where

\[\Gamma \equiv (g \frac{\partial}{\partial z} + \frac{\partial^2}{\partial t^2}) \quad (A.7.b)\]

\[G_1 = 0 \quad (A.7.c)\]

\[G_2 = \zeta_1 \Gamma \phi_1 + (\phi_1^2 + \phi_2^2) t + 2\phi_1 t t_1 \quad (A.7.d)\]

\[G_3 = \zeta_2 \Gamma \phi_1 + \zeta_1 \Gamma \phi_2 + \frac{1}{2} \zeta_1 T \phi_1 + 2(\phi_1 \phi_2 + \phi_1 \phi_1 t) + 2\phi_2 t t_1 \]

\[+ \zeta_1 (\phi_1^2 + \phi_1^2) t + \frac{1}{2} (\phi_1 \frac{\partial}{\partial x} + \phi_1 \frac{\partial}{\partial z})(\phi_1^2 + \phi_1^2) + 2\phi_2 t t_1 \]

\[+ 2\phi_1 \phi_1 t \phi_1 t + 2\phi_1 \phi_1 t + 2\phi_1 \phi_1 t + 2\phi_1 \phi_1 t + 2\phi_1 \phi_1 t + 2\phi_1 \phi_1 t \]

\[+ 2\phi_1 \phi_1 t \phi_1 t \quad (A.7.d)\]
\[ \phi_{nz} = 0 \quad \text{on } z = -h, \ n = 1,2,3,\ldots \quad (A.8) \]

\[-g\xi_n = H_n \quad \text{on } z = 0, \ n = 1,2,3,\ldots \quad (A.9.a)\]

where

\[ H_1 = \phi_1 t \quad (A.9.b) \]

\[ H_2 = \phi_2 t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + \phi_1 t_1 + \zeta_1 \phi_{1zt} \quad (A.9.c) \]

\[ H_3 = \phi_3 t + \phi_1 \phi_2 x + \phi_1 \phi_2 z + \zeta_1 \phi_{2zt} + \zeta_2 \phi_{1zt} + \frac{1}{2} \zeta_1 \phi_{1zt}^2 \quad (A.9.d) \]

\[ + \frac{1}{2} \zeta_1 (\phi_x^2 + \phi_z^2) z + \phi_{2t_1} + \phi_1 \phi_{1x} + \phi_{1t_2} + \zeta_1 \phi_{1zt_1} \]

We now look for solutions Eqs. (A.6) - (A.9) of the form

\[ [\phi_n, F_n, G_n, H_n] = \sum_{m=-n}^{n} e^{im\psi} [\phi_{mn}, F_{mn}, G_{mn}, H_{mn}] \quad (A.10) \]

where \( \psi \) is as defined in Eq. (2.5.c) and \( (\ )_{mn} \) and \( (\ )_{-mn} \) are complex conjugates so that the left-hand side of Eq. (A.10) remains real. Also the dependences are now

\[ \phi_{mn}, F_{mn} = \text{functions of } (x_1, x_2, y_1, z, t_1) \quad (A.11.a, b) \]

\[ G_{mn}, H_{mn} = \text{functions of } (x_1, x_2, y_1, t_1) \]

If we substitute (A.10) into (A.6) - (A.8) using (A.9), (A.11) we obtain for each \( n \) and \( m \) a boundary value problem in \( z \):

\[ -\left( \frac{\partial^2}{\partial z^2} - m^2 k_0^2 \right) \phi_{mn} = F_{mn} \quad -h < z < 0 \quad (A.12.a) \]
\[-(g \frac{\partial}{\partial z} - m^2 \omega^2)\phi_{mn} = G_{mn} \quad z = 0 \tag{A.12.b}\]

\[\frac{\partial}{\partial z} \phi_{mn} = 0 \quad z = -h \tag{A.12.c}\]

At any order \(n\), Eqs. (A.12) possess non-trivial homogeneous solutions for \(m = 0, \pm 1\) only (on account of the dispersion relationship (2.5.d)). In these cases, then, \(F_{mn}, G_{mn}\) must satisfy consistency equations to avoid secularity. In particular, we require

\[
\frac{1}{g} G_{on} = \int_{-h}^{0} dz F_{on} \quad \text{for } m = 0 \tag{A.13.a}
\]

\[
\frac{1}{g} G_{mn} = \int_{-h}^{0} dz F_{mn} \frac{\cosh Q}{\cosh q} \quad \text{for } m = \pm 1 \tag{A.13.b}
\]

We now proceed with the perturbation analysis. For \(n = 1, F_{m1} = 0\) and \(m = 0\), Eq. (A.12) gives

\[
\frac{\partial^2}{\partial z^2} \phi_{01} = 0 \quad -h < z < 0 \tag{A.14.a}
\]

\[
\frac{\partial}{\partial z} \phi_{01} = 0 \quad \text{on } z = 0, -h \tag{A.14.b}
\]

and the only solution is

\[
\phi_{01} = \phi_{01}(x_1, x_2, y_1, t_1, t_2) \tag{A.15}
\]

For \(n = 1, m = 1\) we obtain a homogeneous solution to (A.12)

\[
\phi_{11} = -\frac{g \cosh Q}{2\omega \cosh q} iA(x_1, x_2, y_1, t_1, t_2) \tag{A.16}
\]

Thus

\[
\phi_1 = \phi_{01} - \frac{g \cosh Q}{2\omega \cosh q} iA + * \tag{A.17.a}
\]

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and from (A.9)
\[ \zeta_1 = \frac{1}{2} A e^{i\psi} + * \]  
(A.17.b)

At the next order, \( n = 2 \), we have using (A.17) in (A.6), (A.7)
\[ \mathcal{F}_2 = \omega \cosh Q \frac{\partial A}{\partial x_1} e^{i\psi} + * \]  
(A.18.a)
\[ \mathcal{G}_2 = \frac{\omega^2}{k_0 \tanh q} \frac{\partial A}{\partial t_1} e^{i\psi} + * + \frac{3\omega^3}{4 \sinh^2 q} iA^2 e^{2i\psi} + * \]  
(A.18.b)

For \( n = 2, m = 0, \phi_{20} \) again satisfies the homogeneous equations (A.14) so that
\[ \phi_{02} = \phi_{02}(x_1, x_2, y_1, t_1, t_2) \]  
(A.19)

For \( n = 2, m = 1 \) we have the solvability condition (A.13.b) which evaluates to give
\[ \frac{\partial A}{\partial t_1} + C_g \frac{\partial A}{\partial x_1} = 0 \]  
(A.20)

which is Eq. (6.8) and \( C_g \) is as given in Eq. (2.39c). With (A.20), (A.12) \( (n = 2, m = 1) \) can be solved to give the homogeneous plus particular solutions:
\[ \phi_{12} = -\frac{g \cosh Q}{2\omega \cosh q} iB(x_1, x_2, y_1, t_1, t_2) \]
\[ \quad - \frac{\omega}{2k_0^2 \sinh q} (Q \sinh Q - q \tanh q \cosh Q) \frac{\partial A}{\partial x_1} \]  
(A.21)

where the second part of (A.21) is written so that in the limit of \( q \to \infty \), \( Q \to q \), \( \phi_{21} \) remains formally bounded. For \( n = 2, m = 2, \) (A.12) yields the particular solution
\[ \phi_{22} = -\frac{3w \cosh 2q}{16 \sinh^4 q} \]

In summary, for \( n = 2 \), we have

\[ \phi_2 = \phi_{02} + (\phi_{12} e^{i\psi} + *) + (\phi_{22} e^{2i\psi} + *) \]  

(A.23.a)

and using (A.9)

\[ \zeta_2 = (-\frac{1}{g} \frac{\partial \phi_{10}}{\partial t_1} - \frac{k_o}{2 \sinh 2q} |A|^2) + \frac{1}{2} \left( B + \frac{i}{\omega} \frac{\partial A}{\partial t_1} \right) e^{i\psi} + * \]

\[ + \frac{k_o \cosh q(2 \cosh^2 q + 1)}{8 \sinh^3 q} A^2 e^{2i\psi} + * \]  

(A.23.b)

Eqs. (A.17), (A.23) combine to give Eqs. (6.3), (6.4) for \( \phi \) and \( \zeta \). At the third order, \( n = 3 \), the evaluation of \( F_3 \) and \( G_3 \) from (A.17), (A.23) is quite tedious but straightforward giving

\[ F_{03} = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}) \phi_{01} \]  

(A.24.a)

\[ G_{03} = \frac{\omega^3}{2k_o \tanh^2 q} \frac{3}{\partial x_1} |A|^2 - \frac{\omega^2}{4 \sinh^2 q} \frac{\partial}{\partial t_1} |A|^2 - \frac{\partial^2}{\partial t_1^2} \phi_{01} \]  

(A.24.b)

\[ F_{13} = \frac{\omega}{k_o \sinh q} \left[ (Q \sinh Q - q \tanh q \cosh Q) + \frac{1}{2} \cosh Q \right] i \frac{\partial^2 A}{\partial x_1^2} - \frac{\omega \cosh Q}{\sinh q} \left( \frac{\partial A}{\partial x_2} + \frac{\partial B}{\partial x_1} - \frac{i}{2k_o} \frac{\partial^2 A}{\partial y_1^2} \right) \]  

(A.24.c)

\[ G_{13} = \frac{\omega^3 k_o \cosh q(\cosh 4q + 8 - 2 \tanh^2 q)}{16 \sinh^5 q} \]  

\[ 1 |A|^2 A \]

\[ - \frac{\omega k_o}{\sinh 2q} \left( \frac{\partial \phi_{10}}{\partial t_1} - \frac{2w \cosh^2 q \phi_{10}}{k_o} \right) dA + \frac{\omega \cosh q}{2k_o \sinh q} \frac{1}{\partial t_1} \frac{\partial^2 A}{\partial t_1^2} \]

\[ + \frac{\omega^2}{k_o \tanh q} \left( \frac{\partial B}{\partial t_1} + \frac{\partial A}{\partial t_2} \right) \]  

(A.24.d)
At $m = 0$, the consistency Eq. (A.13.a) applied to $F_{03}$, $G_{03}$ gives
\[
\frac{\partial^2 \phi_{01}}{\partial t_1^2} - gh(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2})\phi_{01} = \frac{\omega^3}{2k_o \tanh^2 q} \frac{\partial}{\partial x_1} |A|^2
\]
\[- \frac{\omega^2}{4 \sinh^2 q} \frac{\partial}{\partial t_1} |A|^2 \]  
(A.25)

which reduces to Eq. (6.9.a) upon using (A.20). For $m = 2$, Eq. (A.13.b) for $F_{31}$, $G_{31}$ gives

\[
\left(\frac{\partial A}{\partial t_2} + Cg_o \frac{\partial A}{\partial x_2}\right) + \left(\frac{\partial B}{\partial t_1} + Cg_o \frac{\partial B}{\partial x_1}\right) - \frac{Cg_o}{2k_o} i \frac{\partial^2 A}{\partial y_1^2}
\]
\[- \frac{\omega q \tanh q}{k_o \sinh 2q} \left(\sinh q \tanh q - q\right) \frac{\partial^2 A}{\partial x_1^2} + \frac{i}{2 \omega} \frac{\partial^2 A}{\partial t_1^2}
\]
\[- \frac{k_o^2}{2 \omega \cosh^2 q} \left(\frac{\partial \phi_{01}}{\partial t_1} - \frac{2 \omega \cosh^2 q}{k_o} \frac{\partial \phi_{01}}{\partial x_1}\right) i_A
\]
\[+ \frac{\omega k_o^2 (\cosh 4q + 8 - 2 \tanh^2 q)}{16 \sinh^4 q} i |A|^2 = 0 \]  
(A.26)

which is equivalent to Eq. (6.9.b) using (A.20) and writing $\varepsilon x_1$, $\varepsilon t_1$ in favor of $x_2$, $t_2$.

The above derivation is similar in principle to that of Part One, Chapter II, leading to the Schrödinger Eq. (2.38). In particular, if we set all the derivatives with respect to $x_1$, $t_1$ and $t_2$ to zero in (A.26), Eq. (2.38) is exactly recovered.
Appendix B

Sample Computer Programs (in Multima FORTRAN)

B.1 Diffraction of Uniform Stokes' Waves

```
*******

external estimate_bill(descriptors)
complex c1,c1,A(2001),An(2001),LAn(2001)
real eps,tolr,pai,c2,dx,dy,junk1,junk2,sigma
integer jj,jjn,j,l,n,N,None,J,jb(501)
real xmatch(5)
data xmatch/0.43, 1.60, 2.78, 3.95, 5.12/

common/sizes/N,J,dx,dy
common/n/l,n,jj,jjn
common/const/ci,c1,c2,pai,junk1,junk2,K,eps
nmatch=1

pai=3.1415927
ci=(0.,1.)
eps=1.e-6
tolr=.05
ione=1
one=1.

10 print,"ifile,N(=0 to stop),J,dx,dy,K,sigma,ibody=?"
c ibody=1 island; ibody=2 parabola; ibody=3 wedge
input,ifile,N,J,dx,dy,K,sigma,ibody
if(N.le.0) stop
print,"Jend,Nd,Jd=?"
input,Jend,Nd,Jd
Nstored=(N-1)/Nd+1
Jstored=(Jend-1)/Jd+1
ddx=dx#Nd
ddy=dy#Jd
write(ifile) Nstored,Jstored,ddx,ddy,Nd,Jd
write(0,910) ifile,Nstored,Jstored,ddx,ddy
910 format(" > ifile,Nstored,Jstored,ddx,ddy ",3i10,2f10.5)
print,"jfile1,jfile2,jNd,jJd="
input,jfile1,jfile2,jNd,jJd
write(jfile1) N,dx
Nstored=(N-1)/jNd+1
```
Jstored=(J-1)/jJd+1  
ddx=jJd*dx  
ddy=jJd*dy  
write(jfile2) Nstored,Jstored,ddx,ddy,jJd,jNd,jJd  
c1=4.*dy*dy/(ci*dx*sigma)  
c2=K*dy*dy/sigma  
call body(ibody,sigma,jb,S,None)  
initialize  
EO=(J-1)*dy  
do 100  j=1,J  
An(j)=1.  
An2(j)=1.  
100  continue  
write(jfile1) one  
write(jfile2) ione,ione,(An2(j),j=1,J,jJd)  
write(ifile) ione,ione,(An(j),j=1,Jend,Jd)  
N1=N-1  
do 1000  n=1,N1  
n1=n+1  
jj=jb(n1)  
njn=jb(n)  
call setnN(An,LAN,Ab2,An2,S)  
call setdiagN(diag,S,Ab2)  
call setrhsN(rhs,An,LAN,An2)  
call solv3d(diag,rhs,A,work)  
energy  
E=0.  
do 1100  j=jj,J  
An2(j)=abs(A(j))**2  
An(j)=A(j)  
1100  E=E+dy*An2(j)  
E=E-.5*dy*(An2(jj)+An2(J))  
write(jfile1) An2(jj)  
if(abs(n1*dx-xmatch(nmatch)).gt.dx/2.) goto 123  
nmatch=nmatch+1  
write(jfile2) nmatch,jj,(An2(j),j=1,J,jJd)  
123  if(Nd.eq.1.or.mod(n1,Nd).eq.1)  
& write(ifile) n1,jj,(An(j),j=1,Jend,Jd)  
if(abs(E-EO).gt.toir*EO.and.n.gt.10) goto 2000  
1000  continue  
goto 2100  
2000  n=n1  
2100  x=(n-1)*dx  
error=abs(E-EO)/EO  
write(0,900) n,x,E,EO,error  
900  format(" At step n",i5," X=",.f10.4,  
& " E,EO,relative error=".e16.7)  
close(ifile)  
close(jfile1)
close(jfile2)
goto 10
end
subroutine body(ib,sigma,jb,S,n1)
integer jb(1)
real S(1)
common/sizes/N,jk1,dx,dy
common/const/jk2(5),pai,jk3(3),eps
if(ib.eq.2) goto 20
if(ib.eq.3) goto 30

island
10 n1=ifix(1./dx+eps)+1
   do 100 n=1,N
      x=(n-1)*dx
      jb(n)=1
      S(n)=0.
      if(x.gt.1..or.x.le.0.) goto 100
      jb(n)=ifix(sin(pai*x)/dy+eps)+1
      S(n)=pai*cos(pai*x)/sigma
100 continue
   do 110 n=2,N
      if(abs(jb(n)-jb(n-1)).le.1) goto 110
      print,""illegal island $$$$$$$"
      stop
110 continue
   return

parabola
20 n1=N
   do 200 n=1,N
      x=(n-1)*dx
      rx=sqrt(x)
      jb(n)=ifix(rx/dy+eps)+1
      if(n.eq.1) goto 200
      S(n)=.5/(rx*sigma)
200 continue
   S(1)=1./(sqrt(dx)*sigma)
   return

wedge
30 n1=N
   do 300 n=1,N
      x=(n-1)*dx
      jb(n)=ifix(x/dy+eps)+1
300 S(n)=1./sigma
   return
end
subroutine setdiagN(diag,S,Ab2)
complex diag(1),ci,c1
real S(1),Ab2(1),c2
integer J
common/sizes/jk1,J,jk2,dy
common/n/l,n,jj,jk3
common/const/ci,c1,c2,jk4(5)
diag(1)=c1/2.+(1.+ci*dy*S(n+1))+c2*Ab2(jj)/2.
i=1
jj1=jj+1
J1=J-1
do 100 J=jj1,J1
 i=i+1
100 diag(i)=c1+2.+c2*Ab2(j)
i=i+1
diag(i)=c1/2.+1.+c2*Ab2(J)/2.
l=1
return
end
subroutine setrhsN(rhs, An, LAN, An2)
complex rhs(1), An(1), LAN(1), ci, c1
real An2(1), K
integer J
common/sizes/jk1, J, dx, jk2
common/n/jk3, n, jj, jk4
common/const/ci, c1, jk5(4), K, jk6
i=0
do 100 j=jj, J
i=i+1
100 rhs(i)=ci*(An(j)+.5*dx*(LAN(j)-.5*c1*K*An(j)*An2(j)))
rhs(1)=rhs(1)/2.
rhs(i)=rhs(i)/2.
return
end
subroutine setnN(An, LAN, Ab2, An2, S)
complex An(1), LAN(1), cc, ci, c1
real Ab2(1), An2(1), S(1), K
integer J
common/sizes/jk1, J, dx, dy
common/n/jk3, n, jj, jjn
common/const/ci, c1, jk4(4), K, jk5
cc=2./(c1*dx)
LAN(jjn)=2.*cc*(An(jjn+1)-(1.+ci*dy*S(n))*An(jjn))
jjn1=jjn+1
J1=J-1
LAN(J)=2.*cc*(An(J1)-An(J))
do 100 j=jjn1, J1
100 LAN(j)=cc*(An(j-1)-2.*An(j)+An(j+1))
c if(jj.ge.jjn) goto 110
c An(jj)=An(jjn)
c LAN(jj)=2.*cc*(-(1.+ci*dy*S(n))*An(jjn)+An(jjn+1))
c An2(jj)=cabs(An(jj))**2
110 dc 200 j=jj, J
do 200 j=jj, J
200 Ab2(j)=cabs(An(j)+dx*(LAN(j)-.5*ci*K*An2(j)*An(j)))*2
return
end
subroutine solv3d(a,h,x,y)
complex a(1),h(1),x(1),y(1)
common/sizes/Jk3,J,jk4(2)
common/n/l,jk1,jj,Jk2
y(1)=h(1)/a(1)
do 100 j=2,l
a(j)=a(j)-1./a(j-1)
y(j)=(h(j)+y(j-1))/a(j)
x(j)=y(1)
do 200 j=2,l
i,j=1-j+1
ik=i,j+jj-1
x(ik)=y(ij)+x(ik+1)/a(ij)
return
end
B.2 Nonlinear Evolution of a Plane Propagating Wave Front

```plaintext
complex a(1001),p(1001),r(1001),work(1001)
complex ci,ca,cb,c1,c2,c3,ca2
real M

data eps/1.e-5/,M/1.e15/,ione/1/
ci=(0.,1.)

10 print,"N,dt,J,dx,xm,ifile,xt,s"
input,N,dt,J,dx,xm,ifile,xt,s
if(N.1e.0) stop
J1=J-1
ca=ci*dt/(16.*dx*dx)
cb=ci*dt/4.
ca2=ca²

c1=(ca2-1.)/ca
c2=cb/ca
c3=(ca2+1.)/ca

initial condition
ER=0.
do 100 j=1,J
x=xm+(j-1)*dx
a(j)=(1.-tanh(x/s))/2.
100 ER=ER+dx*cabs(a(j))²

write(ifile) J,ione,dx,dx
time=0.
istore=1
write(ifile) time,istore
write(ifile) (a(j),j=1,J)

r(J)=0.
p(1)=M
p(J)=M
do 200 n=2,N
time=(n-1)*dt
r(1)=M*exp(-ci*time/2.)
do 210 j=2,J1
r(j)=a(j-1)+a(j+1)+(c²*cabs(a(j))²-c²)*a(j)
210 p(j)=c1-c²*cabs(ca2*r(j)+a(j))²
call Fsolve(p,r,a,work,J)
if(abs(int(time/xt+eps)-time/xt).gt.eps) goto 200

check error
EL=0.
do 220 j=1,J
220 EL=EL+dx*cabs(a(j))²
teror=abs(EL-ER)/ER
```
istore=istore+1
write(0,900) time,istore,error,abs(cabs(a(2))-1.),cabs(a(J))
format(" t="f7.3," is="i3," err,L,R="e14.7")
write(ifile) time,istore
write(ifile) (a(j),j=1,J)
continue
close(ifile)
goto 10
end
subroutine Fsolv(a,h,x,y,J)
complex a(1),h(1),x(1),y(1)
eps=1.e-12
y(1)=a(1)
x(1)=h(1)/a(1)
do 100 j=2,J
  y(j)=a(j-1)/y(j-1)
x(j)=(h(j)+x(j-1))/y(j)
100 if(cabs(x(j)).lt.eps) x(j)=0.
do 200 j=2,J
  i=j-1
  x(ij)=x(ij)+x(ij+1)/y(ij)
200 if(cabs(x(ij)).lt.eps) x(ij)=0.
return
end
B.3 Diffraction of a Stokes' Wave Packet by a Converging Channel

```fortran
complex z,z1
complex P(201),R(201)
complex ca,ca2,ck,ck2,cs,cs2,ct,ch,cb,cc,cd
complex cc2,cr1,cr2,cr3,cr4,cj,clj,cuj
complex ci,As,zb
real dx,dy,dt,x,y,t,t1,t2,M,epsilon,xK,err,x0,xp,b,B,sigma,LHS
real S(201)
real st(20,2)
integer kk(201)
integer jj(41)
integer N,J,K,N1,J1,K1,n,j,k
common/CC/cr1,cr2,cr3,cr4,ca,ca2,ct,cs,cs2
common/complex_z/J,K,z(201,41)
common/complex_z1/JJ,KK,z1(201,41)
data M/1.e15/,eps/1.e-5/
data r2p/4.5015816e-1/

As(tt)=exp(-ci*epsilon*xK*tt/2.)
&
*coch(2.*sqrt(xK)*(-x0-0 tt/2.))/sigma

ci=(0.,1.)
J=201
JJ=J
K=41
KK=K
print,"J,K=",J,K

c input parameters
10 print,"N,dt,x0,xp,b,B,epsilon,xK,sigma=?"
input,N,dt,x0,xp,b,B,epsilon,xK,sigma

c display & store info
print,"storage info:kfile,sT,ifile/m,mt,mt(st,2)"
input,kfile,sT,ifile
input,mt,((st(i,j),j=1,2),i=1,mt)
st(mt+1,1)=M
st(mt+1,2)=M
mt=1
nsT=int(sT/dt+eps)
ec=b*sigma/(2.*sqrt(xK))
ec1=b/ec
istore=0

c initialize constants
N1=N-1
J1=J-1
```

K1=K-1
K2=K+1
dx=xp/J1
dy=b/K1
call Vbody(jj,kk,S,dx,dy,B,epsilon)
write(kfile) J,K,dx,dy,(jj(k),k=1,K),(kk(j),j=1,J)
KB=int(B/dy+eps)+1
Ks=K-KB+1
write(kfile) J,Ks,dx,dy
c=ci*epsilon*dt/(8.*dy*dy)
ca2=2.*ca
ck=2.+1./ca
ck2=ck/2.
cs=ci*dy
cs2=cs*2.
cr=ca2*cs
ch=1.-ca2
oc=dt/(8.*dx)
cc=ci*epsilon*dt/(16.*dx*dx)
od=ci*epsilon*dt*xK/2.
cc2=cc*2.
cr1=(oc-cc)/ca
cr2=(1.+cc2)/ca
cr3=cd/ca
cr4=-(oc+cc)/ca
cj=1.-cc2
cij=cc-cb
cuj=cc+cb

c
initial condition
do 100 k=1,K
   z(J,k)=0.
   jb=jj(k)
do 100 j=1,jb
100
z1(j,k)=0.
c
time integration
do 200 n=1,N1
t=(n-1)*dt
t1=(n-.5)*dt
t2=t+dt
c
solve y implicit step
   call Vy1(kk,S)
do 300 j=2,J1
   kb=kk(j)
   Kb1=K-kb
   Kb=Kb1+1
c
set up P
do 301 k=2,Kb1
301
   P(k)=ck

224
P(1)=ck2+cs*S(j)
P(Kb)=ck2

c

call V_RK(R,j,kb,Kb)
call csolve3K(P,R,Kb)
do 30 k=1,Kb
   ik=kb+k-1
30   z(j,ik)=R(k)
300  continue

c  add essential b.c. on x -
do 310 k=1,K
   z(1,k)=As(t1)
310  continue

c  solve x implicit step

call Vf(jj,S)
do 400 k1=1,K
   k=K2-k1
   jb=jj(k)
   jb1=jb-1
   call V_RJ(R,k,jb)
   R(1)=M#As(t2)
   R(jb)=0.
   if(jb.lt.j) R(jb)=M*(4.#z(1,jb,k+1)-z1(jb,k+2))/(3.+cs2*S(jb))
   set up P
   P(1)=M
   do 401 j=2,jb1
      zb=R(j)-clj#z(j-1,k)-cuj#z(j+1,k)
      &+(cc2-cd#cabs(z(j,k))**2)#z(j,k)
401   P(j)=cij+cd#cabs(zb)**2
      P(jb)=M
   call csolve3nsK(P,R,cuj,clj,jb)
do 40  j=1,jb
40   z1(j,k)=R(j)
400  continue

c
   xg=x0+t2/2.
   jxg=int(xg/dx+eps)+1
   if(jxg.lt.1) goto 801
   j1=max0(1,jxg-20)
   j2=min0(J,jxg+20)
do 800 k=KB,K
      zmax=0.
do 805 j=j1,j2
      if(cabs(z1(j,k)).le.zmax) goto 805
      zmax=cabs(z1(j,k))
      jmax=j
805  continue
800  R(k)=z1(jmax,k)
go to 802
do 810 k=KB,K
R(k)=1.
write(kfile) t2,(R(k),k=KB,K)
c
if(abs(t2-int(t2+eps)) .gt. eps) goto 870
write(99) t2,J,K,(jj(k),k=1,K),((z1(j,k),j=1,jj(k)),k=1,K)
close(99)
print,"SAVED FILE99 t=" , t2
c
c save plotting data
if(t2.gt.st(mt,2)+eps) goto 850
if(t2.lt.st(mt,1)-eps) goto 860
istore=istore+1
write(ifile) t2,istore
write(ifile) ((z1(j,k),j=1,jj(k)),k=1,K)
write(0,950) ifile,istore,t2
go to 860
t=mt+1
c
if(mod(n,nsT).ne.0) goto 200
c
c check error
ER=sec#(tanh(ec1#(t2/2.+x0))-tanh(ec1#x0))
EL=-0.
pn=0.
RHS=0.
LHS=C.
KB=int(B/dy+eps)
do 1100 k=1,K
if(cabs(z1(2,k)) .gt. LHS) LHS=cabs(z1(2,k))
if(k.4.e.kB) goto 1101
if(cabs(z1(J1,k)) .gt. RHS) RHS=cabs(z1(J1,k))
g=dy
if(k.eq.1.or.k.eq.K) g=g/2.
jb=jj(k)
do 1100 j=1,jb
f=dx
if(j.eq.1.or.j.eq.J) f=f/2.
pn=pn+f*g*cabs(z1(j,k))
do 1100
EL=EL+f*g*cabs(z1(j,k))**2
pn=r2p*pn
err=(ER-EL)/ER
write(0,910) t2,err,ER,EL,pn,cabs(As(t2)),LHS,RHS
format("<" , f8.4," > err,Ee,E,ns=",e12.4,
& 3e13.6/"LHS,E,LHS,RHS=",3e11.4)
continue
close(ifile)
close(kfile)
stop
end
subroutine V_RJ(R,k,jb)
complex R(1),z
complex cjunk,ca,ca2,ch,ct
common/CC/cjunk(4),ca,ca2,ch,ct
common/complex_z/J,K,z(151,41)
jb1=jb-1
k1=k+1
k2=k-1
if(k.eq.1) goto 10
if(k.eq.K) goto 20
   do 300 j=2,jb1
   300 R(j)=ca*(z(j,k1)+z(j,k2))+ch*z(j,k)
goto 99
   10 do 100 j=2,jb1
   100 R(j)=(ch-ct)*z(j,k)+ca2*z(j,k1)
goto 99
   20 do 200 j=2,jb1
   200 R(j)=ch*z(j,k)+ca2*z(j,k2)
   99 return
end

end
subroutine V_Rk(R,j,kb,Kb)
complex R(1),z1
complex cr1,cr2,cr3,cr4
common/CC/cr1,cr2,cr3,cr4
common/complex_z1/J,X,z1(151,41)
  j1=j+1
  j2=j-1
  do 100 k=1,Kb
     ik=k+kb-1
     R(k)=cr1*z1(j2,ik)+cr4*z1(j1,ik)
  &
     + (cr2-cr3*cabs(z1(j,ik))**2)*z1(j,ik)
     R(1)=R(1)/2.
     R(Kb)=R(Kb)/2.
   return
   end
subroutine Vf(jj, S)
complex z, cjunk, cs2
real S(1)
integer jj(1)
common/CC/cjunk(8), cs2
common/complex_z/J, K, z(151, 41)
do 100 k=2, K
k1=k-1
if(jj(k), eq., jj(k1)) goto 100
i1=jj(k1)+1
i2=jj(k)-1
k2=k+1
k3=k+2
k4=k+3
do 110 i=i1, i2
z(i, k1)=z(i, k2)-cs2*S(i)*z(i, k)
110 continue
100 continue
return
end
subroutine Vf1(kk,S)
complex z1,cjunk,cs2
real S(1)
integer kk(1)
common/CC/cjunk(8),cs2
common/complex_z1/J,K,z1(151,41)
do 100 j=2,J
  j1=j-1
  if(kk(j).eq.kk(j1)) goto 100
  kb=kk(j1)
z1(j,kb)=z1(j,kb+2)-cs2*S(j)*z1(j,kb+1)
100 continue
return
end
subroutine Vbody(jj,kk,S,dx,dy,B,eps)
complex z
real S(1)
integer jj(1),kk(1)
common/complex_z/J,K,z(151,41)
data eps/1.e-5/
xl=B/eps
do 100 k=1,K
y=k#dy
if(y.gt.B+eps) goto 110
xb=y/eps
jj(k)=int(xb/dx+eps)
goto 100
110 jj(k)=J
100 continue
kB=int(B/dy+eps)+1
do 200 j=1,J
x=(j-1)*dx
if(x.le.xl-eps) goto 220
S(j)=0.
kk(j)=kB
goto 200
220 S(j)=1.
kk(j)=int(epsilon#x/dy+eps)+1
200 continue
return
end

C
subroutine csolv3nsK(a,h,b,c,n)
complex a(n),h(n),b,c,bc
eps=1.e-12
bc=b/c
h(1)=h(1)/a(1)
do 100 i=2,n
a(i)=a(i)-bc/a(i-1)
h(i)=(h(i)-c*h(i-1))/a(i)
100 if(cabs(h(i)).lt.eps) h(i)=0.
n1=n-1
do 200 i=1,n1
j=n-i
h(j)=h(j)-bc(h(j+1)/a(j)
200 if(cabs(h(j)).lt.eps) h(j)=0.
return
end
subroutine csolv3K(a,h,n)
complex a(n),h(n)
eps=1.e-12
h(1)=h(1)/a(1)
do 100 i=2,n
  a(i)=a(i)-1./a(i-1)
h(i)=(h(i)+h(i-1))/a(i)
  if(cabs(h(i)).lt.eps) h(i)=0.
100  n1=n-1
    do 200 i=1,n1
      j=n-i
      h(j)=h(j)+h(j+1)/a(j)
200    if(cabs(h(j)).lt.eps) h(j)=0.
return
end
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4.6 Contour plots of $\zeta_1(k_0x, k_0y, t=0)$ for a parabola ($\sigma = 1$) with $\epsilon^2 = 0.1$ for (a) linear, (b) nonlinear waves, in uniformly scaled $k_0x, k_0y$ coordinates. The contour increment is 0.25.

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6.2 Definition sketch for a converging channel.

8.1 Evolution of a steeper-than-soliton initial sech pulse (\(\sigma = \frac{1}{2}\)): \(A(\xi, \tau=0) = \text{sech}(2\sqrt{2}\xi)\). The region shown is for \(0 < \xi < 10\) and \(0 < \tau < 20\).

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(a) magnitude: \(|A(\xi, \tau)|\)

(b) phase: \(p(\xi, \tau)\)
Evolution of an initial sech profile with $\sigma = 2$: $A(\xi,0) = \text{sech} \left( \frac{\xi}{\sqrt{2}} \right)$. The region shown is for $0 < \tau < 30, 0 < \xi < 6$.

Centerline amplitude $|A(\xi=0,\tau)|$ for the evolution of an initial sech profile with $\sigma = 2$: $A(\xi,0) = \text{sech} \left( \frac{\xi}{\sqrt{2}} \right)$.

Details of $|A(\xi,\tau)|$ for an initial sech profile with $\sigma = 2$: $A(\xi,0) = \text{sech} \left( \frac{\xi}{\sqrt{2}} \right)$ at the recurring states

\[ a_n: \tau = n \times \tau_o \]
\[ b_o: \tau = (n + 1/4)\tau_o \]
\[ b_1: \tau = (n + 3/4)\tau_o \]
\[ c_n: \tau = (n + 1/2)\tau_o \]

where $\tau_o = 12.6$ for (a) magnitude: $|A(\xi,\tau)|$, (b) phase: $p(\xi,\tau)$.

One period of the evolution of an initial sech profile ($\sigma = 3$): $A(\xi,0) = \text{sech} \left( \frac{\sqrt{2}\xi}{3} \right)$. (a),(b) show the result from two slightly different view angles. The region shown is for $0 < \xi < 6, 0 < \tau < 30$.

Centerline amplitude $|A(\xi=0,\tau)|$ for the evolution of an initial sech profile ($\sigma = 3$): $A(\xi,0) = \text{sech} \left( \frac{\sqrt{2}\xi}{3} \right)$.

Details of $|A(\xi,\tau)|$ for an initial sech profile with $\sigma = 3$: $A(\xi,0) = \text{sech} \left( \frac{\sqrt{2}\xi}{3} \right)$.

(a) magnitude: $|A(\xi,\tau)|$ at recurring states a,b,c
(b) phase: $p(\xi,\tau)$ at a,b,c states.

Evolution of an initial sech profile, $\sigma = 1.5$: $A(\xi,0) = \text{sech} \left( 2\sqrt{2}\xi/3 \right)$. The region shown is for $0 < \tau < 30$ and $0 < \xi < 6$.

Centerline amplitude $|A(\xi=0,\tau)|$ for the evolution of an initial sech profile ($\sigma = 1.5$): $A(\xi,0) = \text{sech} \left( 2\sqrt{2}\xi/3 \right)$.

Details of $A(\xi,\tau)$ for an initial sech profile ($\sigma = 1.5$): $A(\xi,0) = \text{sech} \left( 2\sqrt{2}\xi/3 \right)$ for amplitude $|A(\xi,\tau)|$ and phase $p(\xi,\tau)$ at b,b' and c,c' states and at $\tau = 0,10$.

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9.7

Envelope amplitude $|A|$ following a plane wider-than-soliton pulse entering a bent channel at time intervals of 2. The regions shown are for $0 \leq y \leq b$ and $0 \leq x \leq 10$ for $t \leq 24$ and $x_g - 5 \leq x \leq x_g + 5$ for $t \geq 24$.

10.1

Envelope amplitude $|A|$ of a plane soliton entering a converging channel at time intervals of 2. The domain shown is for $0 \leq x \leq 10$, and the half-channel $y \leq 1$ and $y \geq 0.2x$ (for $x < 5$) and $y \geq 1$ (for $x \geq 5$).

10.2

Transverse profiles of $|A|$ along the envelope peak ($x = x_g$) as a soliton propagates into a converging channel.

10.3

Maximum amplitude along the center of the channel:

$$\max_{x} |A(x,y=1,t)|.$$  

10.4

Wetted profile along the wall of a converging channel $y = \epsilon x$ ($0 \leq x \leq 5$): $|A(x,y=\epsilon x,t)|$.

10.5

Two-dimensional variations of $|A|$ in the uniform channel at selected times. The domains shown are for the full channel width $1 \leq y \leq 3$ and $x_g - 2 \leq x \leq x_g + 2$.

10.6

Time variation of the transverse profiles of $|A|$ along the envelope peak ($x = x_g$)

$$\max_{x} |A(x,y,t)|$$ for

(a) $19.5 \leq t \leq 27.5$

(b) $27.5 \leq t \leq 35.5$

and for the full channel $1 \leq y \leq 3$.

Table No.

3.1

Dependency of $\beta$ on $K$.  

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