DIGITAL FREQUENCY-DIVISION MULTIPLEXING USING JOSEPHSON JUNCTIONS

by

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ABSTRACT

The possibility of constructing a digital frequency-division multiplexing system using Josephson junctions is investigated. The ability of Josephson interferometers to act as generators of microwave radiation in the hundred-GHz region (a phenomenon known as "interferometer resonance") is investigated theoretically. Novel results are presented which describe the resonance amplitude as a function of control current, gate voltage, load resistance, and interferometer asymmetry. Experiments were performed to test aspects of this theory, and excellent agreement was found. The theory is likely to be of value not only for designing a microwave multiplexing system, but also for measuring interferometer losses and junction capacitances, and for providing an enhanced understanding of junction resonances in general.

The ability of Josephson interferometers to act as detectors of microwave radiation is also investigated. A theory is developed which calculates the sensitivity of the receiver as a function of frequency. An instability is discovered in the receiver which is detrimental to its performance. The instability is due to nonlinear parametric effects, and is analyzed using techniques from the theory of parametric amplifiers. Good agreement with simulations is found. The use of a low load resistance is recommended to reduce the instability.

Guided by the theories which were developed for the microwave transmitters and receivers, the considerations involved in the design of a multiplexing system are discussed. The configuration of a basic system is described. The important issues which must be considered in the design of a practical multiplexing system for use in a Josephson computer are explained. The significance of coupling techniques, coupling losses, crossing inductance, line attenuation, system gain, transmitter interaction, and channel capacity are discussed.

Finally, experiments were designed to test the basic system concepts. A transmitter-receiver pair, separated by 100 ps of transmission line, was successfully operated, verifying the basic concept of microwave generation and detection of digital signals. Preliminary results indicate that more study is needed of the system gain and channel discrimination.

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Chapter 1 - INTRODUCTION AND BACKGROUND

In this work we investigate, theoretically and experimentally, the possibility of designing an ultrahigh-speed frequency-division multiplexing system using Josephson junctions. The idea of such a system was proposed in 1979 by Calander and Zappe. They showed that Josephson interferometers were capable of acting as narrow-band generators (transmitters) and detectors (receivers) of high-frequency (hundreds of GHz) electromagnetic radiation. They suggested that by placing several transmitters with different frequencies at one end of a transmission line, and a corresponding set of receivers at the other end, a frequency-division multiplexing system could be realized. The purpose of this system would be to carry several digital signals over a single line. More specifically, it could be used to minimize the number of interconnections between integrated circuits in the high-performance Josephson computer proposed by Anacker. Since the interconnections, or "package parts", in this proposed computer involve many discrete components, a multiplexing scheme could result in considerable savings of cost and space.

The reader may wonder why a frequency-division multiplexing system is being investigated, when a time-division multiplexing system would be more "natural" for a computer composed of digital logic. The best answer to this question is that time-division multiplexing imposes stringent timing constraints on the system. The various logic signals which are to be multiplexed will be generated at different times during a single machine cycle. But a time-division multiplexing system requires that the bits be sequenced in a specific order. In particular, the first bit to be transmitted must be available early in the machine cycle. This may not always be possible. It is easier to work with a frequency-division multiplexing system, where timing is not a problem. When a bit is ready to be transmitted, a burst of microwaves at the appropriate frequency (channel) can be sent immediately, independent of whether any other channels are active.
This thesis will be organized as follows. The remainder of this chapter (Chapter 1) will be devoted to basic background information on Josephson devices and on the proposed multiplexing system, background which is essential to understanding the rest of the thesis. Chapter 2 develops a novel theory for the transmitter (also known as the theory of "resonances"). Chapter 3 describes experiments which confirm the theory of Chapter 2 (experiments were performed in collaboration with J. H. Magerlein). Chapter 4 develops the theory of the receiver, including a discussion of its stability. Chapter 5 describes the considerations involved in the design of a multiplexing system using Josephson LSI technology, based on the understanding of the system components developed in Chapters 2-4. Chapter 6 discusses the design and testing of experimental Josephson integrated circuits which were fabricated in order to check the principles developed in Chapters 2-5. The chapter closes with some conclusions and suggestions for further work. We shall now present the basic background necessary for understanding the rest of this thesis.

A Josephson junction\(^3\) (Fig. 1.1) is formed whenever two superconductors are coupled sufficiently tightly that their wavefunctions overlap, allowing pair tunneling to occur. The resulting circuit element is symbolized by a cross, as indicated in Fig. 1.1. The circuit equations (current-voltage relations) for an ideal point junction (one which is infinitesimally small, and which has no parallel capacitance or resistance) are:

\[
I = I_0 \sin \phi , \tag{1.1}
\]

\[
V = \frac{\hbar}{2e} \frac{d\phi}{dt} , \tag{1.2}
\]

where \(\hbar\) is Planck's constant divided by \(2\pi\) and \(e\) is the charge of the electron. \(I_0\) is called the "critical current" of the junction, and represents the maximum current which the junction can sustain. For convenience, we shall define the physical constant \(K \equiv \hbar/2e\), which has the units of flux: \(K \equiv 0.329 \times 10^{-15} \text{ V-sec}\). A more common convention is to define the "flux quantum" \(\Phi_0 \equiv \hbar/2e = 2\pi K\), but we shall generally not use \(\Phi_0\) in our discussions.
Figure 1.1
The parameter $\phi$ is the phase difference of the superconducting order parameter $\Psi$ across the junction ($\Psi$ is essentially a macroscopic wave function). We shall hereafter refer to $\phi$ simply as the "phase". If desired, the phase may be eliminated from Eqs. 1.1 and 1.2 to yield a single circuit equation:

$$I = I_0 \sin \int \frac{V}{K} \, dt.$$ 

In performing circuit analyses, it will be very convenient to define a phase $\phi_i$ for each circuit element, not just for the Josephson junctions. We simply define

$$\phi_i = \int \frac{V_i}{K} \, dt,$$

where $V_i$ is the voltage across the circuit element. We can then write circuit equations for the standard circuit elements in terms of $\phi$ and $I$:

For a Josephson junction, $I = I_0 \sin \phi$.

For an inductance, $I = \int \frac{V}{L} \, dt = K\phi/L$.

For a resistance, $I = V/R = K\dot{\phi}/R$.

For a capacitance, $I = C\ddot{V} = KC\ddot{\phi}$.

Let us briefly review the well-known circuit characteristics of the Josephson point junction. The first obvious characteristic is that the junction is a highly nonlinear circuit element, due to the $\sin \phi$ term. However, if the current through the junction is much less than the critical current ($I << I_0$), then the phase $\phi$ is small ($\phi << 1$), so we can approximate $I = I_0 \sin \phi \approx I_0 \phi$. We see that the ideal Josephson junction behaves simply like an inductance $L_j = K/I_0$, for $I << I_0$. Even if the current through the junction is not small compared to $I_0$, we can still linearize for small oscillations about the operating point. That is, let $I$ be the average current through the junction. Then $\phi = \sin^{-1} \frac{I}{I_0}$, and
\[ dl = I_0 \cos \phi \, d\phi = I_0 \sqrt{1 - (I/I_0)^2} \, d\phi \]. Thus the junction behaves like an inductance

\[ L = \frac{L_J}{\cos \phi} = \frac{L_J}{\sqrt{1 - (I/I_0)^2}} \geq L_J \]

with respect to perturbations about the operating current.

Another characteristic of the Josephson junction is that it can sustain a dc current \( I < I_0 \) without any voltage across it, i.e. it can act like a superconductor. This is consistent with the fact that the junction behaves like an nonlinear inductance with no series resistance, as discussed above. The Josephson current \( I_0 \sin \phi \) can be explained as due to the quantum-mechanical tunneling of "Cooper pairs", which are the carriers of supercurrents. This effect is known as the dc Josephson effect.

If a constant voltage \( V_0 \) is applied across a Josephson junction, we will have \( \phi = \omega_0 t \), where \( \omega_0 = V_0 / K \); the phase increases linearly with time. Then \( I = I_0 \sin \omega_0 t \), so we have generated a monochromatic alternating current from a constant voltage source. This is known as the ac Josephson effect. This voltage-to-frequency conversion will be used in our work to generate microwaves for digital signal transmission.

Note that the ideal Josephson point junction (as defined by Eqs. 1.1 and 1.2) is a lossless, or conservative device. That is, it does not dissipate any power. Specifically, the power supplied to the junction is:

\[ \text{Power} = \frac{dE}{dt} = IV \]

so \( dE = IV \, dt = IK \, d\phi = KI_0 \sin \phi \, d\phi \),


\[ \text{hence } E = -KI_0 \cos \phi . \]

Thus the energy \( E \) of a Josephson junction is a function only of the state variable \( \phi \). At its
lowest energy state, \( \phi = 0 \); at its highest energy state, \( \phi = \pi \).

A real Josephson junction is generally slightly more complicated than the ideal circuit of Fig. 1.1. Since a Josephson junction is usually fabricated as a thin (≈40 Å) oxide barrier between two superconductors, we expect the junction to have some parallel capacitance \( C \) in addition to the Josephson tunneling term. Furthermore, since the absolute temperature of the superconductors is nonzero, the electrodes will contain thermally generated "normal electrons" (really "quasiparticles") as well as "superconducting electrons" (Cooper pairs), in accordance with the two-fluid model of superconductivity. These normal electrons, or quasiparticles, will tunnel across the barrier in response to an applied voltage. Thus they create a conductance in parallel with the junction. This conductance will be nonlinear (voltage-dependent), and is determined by the quasiparticle density of states in the two superconducting electrodes. Fig. 1.2 is a photograph of a typical current-voltage curve for a Josephson tunnel junction fabricated at IBM. We see that the quasiparticle tunneling current is much smaller than the Josephson critical current \( I_0 \) until the voltage approaches the "gap voltage" \( V_{\text{gap}} = \frac{\Delta_1 + \Delta_2}{e} \) (2.8 mV for our metallurgy), where \( \Delta_1 \) and \( \Delta_2 \) are the energy gaps of the two superconducting electrodes. At the gap voltage, the quasiparticle current rises sharply; the height of this current step is denoted \( \Delta I \). Although the quasiparticle current should properly be modelled as a nonlinear conductance, we shall be operating below the gap voltage in most of our work, where the conductance is small. In the work that follows we shall either model it by a linear conductance \( G_j \) or neglect it entirely, depending on circumstances.

We are now led to a simple model for a real Josephson junction, consisting of an ideal Josephson element (Eqs. 1.1 and 1.2) in parallel with a linear capacitance \( C \) and resistance \( R_j = G_j^{-1} \) (Fig. 1.3a). This model, developed independently by McCumber and by Stewart, is often called the "resistively-and-capacitively-shunted junction model", or simply the RCSJ model. It has been remarkably successful in explaining junction phenomena of interest to circuit designers. We shall therefore use the RCSJ model without any further elaboration.
Figure 1.2
Figure 1.3
The RCSJ model is already sufficiently complicated to show an amazing wealth of nonlinear phenomena. We shall briefly describe the important effects. We begin by writing down Kirchhoff's current law for the circuit model in Fig. 1.3a:

\[ I = C \dot{V} + G_j V + I_0 \sin \int (V/K) \, dt , \]

where \( \dot{V} \) denotes the time-derivative of \( V \). As discussed before, it will be more convenient to deal with the phase \( \phi = \int (V/K) \, dt \):

\[ I = CK \ddot{\phi} + G_j K \dot{\phi} + I_0 \sin \phi . \tag{1.3} \]

This nonlinear differential equation is the circuit equation for the RCSJ junction model. It is not an unfamiliar equation, for it has the same form as the equation of motion of a simple pendulum.\(^7\) To see the analogy, we write here the pendulum equation of motion:

\[ \tau = I \ddot{\theta} + \eta \dot{\theta} + Wl \sin \theta , \]

where \( \tau \) is the applied torque on the pendulum, \( \theta \) is its angle measured from the vertical, \( \tilde{I} \) is its moment of inertia, \( Wl \) is its turning moment (\( W \)=weight, \( l \)=moment arm), and \( \eta \) is a damping or viscosity factor. Thus the dc Josephson effect is analogous to the pendulum at rest with a finite torque \( \tau \) applied to it \((\tau<Wl)\). The ac Josephson effect corresponds to a spinning pendulum, in which a constant angular velocity (corresponding to voltage) implies a sinusoidally varying torque (corresponding to current) on the pendulum. The pendulum model is often useful for obtaining insight into the dynamics of Josephson junctions.

When the junction phase \( \phi \) is small, it is possible to linearize the Josephson term and replace it by an inductance \( L_j = K/I_0 \), as discussed previously. Then the RCSJ model reduces to a parallel RLC resonant circuit. The circuit equation is then linear, and Eq. 1.3 can be written as follows:

\[ \frac{I}{I_0} = \omega_p^{-2} \dot{\phi} + \gamma_j \omega_p^{-1} \dot{\phi} + \sin \phi , \tag{1.4} \]
where \( \omega_p = \frac{1}{\sqrt{L_jC}} \) and \( \gamma_j = \sqrt{\frac{K}{I_0 R_j^2 C}} \). This is simply the equation of a driven, damped harmonic oscillator having resonant frequency \( \omega_p \) and quality factor \( Q = \frac{1}{\gamma_j} \). The resonant frequency \( \omega_p \) is known as the "plasma frequency". The small oscillations which occur at this frequency when the junction is momentarily excited are called "plasma oscillations". They correspond to the swings of a pendulum about its equilibrium position.

The quality of an RCSJ circuit is often expressed as a dimensionless parameter \( \beta_c \), defined by McCumber as \( \beta_c = \frac{I_0 R_j^2 C}{K} \). Referring to Eq. 1.4, we see that \( \beta_c = Q^2 = \gamma_j^{-2} \), where \( Q \) is the quality factor of the plasma resonance.

We can now understand how a high- \( \beta_c \) junction may be used as a bistable switch. If the supply current to a junction momentarily exceeds \( I_0 \), the Josephson current \( I_0 \sin \phi \) will not be able to accommodate all the current. Thus some of the supply current will begin to charge up the capacitance \( C \), resulting in a nonzero voltage \( V \) across the junction. This voltage will cause the phase \( \phi \) to begin increasing, because \( \dot{\phi} = \frac{V}{K} \). When \( \phi \) exceeds approximately \( \pi/2 \), the junction will rapidly begin transferring current into the capacitance, which increases the junction voltage even more. From then on, the junction capacitance will maintain a nonzero voltage \( V \) across the junction, which will result in an ac Josephson current of frequency \( \omega = \frac{V}{K} \) in the junction. This ac current will flow primarily through the junction capacitance \( C \). Since the ac current has amplitude \( I_0 \), the resultant ac voltage across the capacitance will have an amplitude of approximately \( I_0/\omega C = KI_0/CV \). If this ac voltage is much smaller than \( V \) (that is, \( V >> \sqrt{KI_0/C} \)), then the junction voltage is essentially unaffected by the ac Josephson current; it is as if the Josephson term is not there anymore. The junction is said to be "switched" to the "voltage state" from its initial "zero-voltage state", or "superconducting state".

Since the Josephson term is no longer very important once the junction has switched, the junction behaves like an RC circuit. For a dc supply current \( I \), the junction voltage will
stabilize at \( V \approx IR_j \). As long as this voltage is much greater than \( \sqrt{K I_0/C} \), the junction will remain in the voltage state. But \( IR_j \geq \sqrt{K I_0/C} \) implies that \( I/I_0 \geq \sqrt{K/I_0 R_j^2 C} = \gamma_j = \beta_c^{-1/2} \). Thus for a high- \( \beta_c \) junction, a very small amount of current (relative to \( I_0 \)) is required to keep the junction in the voltage state. Whereas a current pulse in excess of \( I_0 \) was necessary to switch the junction to that state, the junction will stay switched until the supply current is reduced below approximately \( I_0 \beta_c^{-1/2} \), at which time the junction will "reset" to the zero-voltage state.

For low- \( \beta_c \) junctions (\( \beta_c \leq 1 \)), the situation is different. Here the current required to maintain the device in the voltage state exceeds \( I_0 \), hence the junction is no longer bistable. In other words, the junction must be held in the voltage state by a supply current \( I > I_0 \); as soon as \( I \) is reduced below \( I_0 \) the device will reset to the zero-voltage state. Thus low- \( \beta_c \) junctions are "nonlatching" devices, and high- \( \beta_c \) junctions are "latching" devices. Consequently the I-V curve of a high- \( \beta_c \) junction appears to have two branches (a zero-voltage branch and a resistive branch, as in Fig. 1.3c), whereas the I-V curve of a low- \( \beta_c \) junction is a single, continuous curve (Fig. 1.3b).

It should be evident by now that the properties of a Josephson junction, such as its I-V curve, depend strongly upon what kind of circuit the junction is connected to. For example, adding a low load resistance in parallel with a latching, high- \( \beta_c \) junction may make it nonlatching. Connecting a resonant circuit to a junction will result in current steps in the I-V curve at voltages corresponding to multiples of the resonant frequency. Connecting an inductance across a junction can result in a memory cell having several stable states.

It is possible to design digital logic circuits using single junctions as gates. For example, Fig. 1.4 shows an AND gate, in which two input currents \( I_a \) and \( I_b \) and a power supply current \( I_{\text{supply}} \) are injected into a high- \( \beta_c \) junction. The current levels are such that neither input alone will exceed the critical current \( I_0 \) of the junction, but together they will
switch the junction \((I_a + I_b + I_{\text{supply}} > I_0)\). When the junction switches, it transfers most of its current into the load resistance \(R_L\). The output current \(I_{\text{out}}\) can in turn drive the input of another gate. The same circuit could function as an OR gate by adjusting the current levels so that either input alone is sufficient to switch the device.

The logic gate described above was intended merely as an illustration. In fact, that circuit is quite impractical for use in an LSI (Large-Scale-Integration) technology, primarily because it gives poor isolation between input and output signals and because it requires close tolerances ("margins") on the current levels. The standard approach to logic design is now to make gates out of several Josephson junctions, connected by superconducting loops. The loops introduce a constraint on the junction phases, causing them to interact or "interfere" with each other. Such structures were first proposed for use in logic by Zappe,\(^8\) and are called "interferometers", or "SQUIDs" (Superconducting Quantum Interference Devices).

Fig. 1.5 is a diagram of a two-junction interferometer, and Fig. 1.6 is its associated circuit model. Here we have two junctions imbedded in a single superconducting loop which is modelled by an inductance \(2L\). The damping resistance \(R_d\) across the inductance may or may not be physically present in the device, but there will always be some intrinsic, frequency-dependent damping in the superconducting loop at nonzero frequencies (this is discussed further in Chapter 3). The current source \(I_g\) which feeds the two junctions is known as the "gate current". An insulated "control line" carrying a "control current" \(I_c\) passes over the device and magnetically couples flux into the loop inductance. It is also possible to directly inject the control current, if electrical isolation between the gate and control lines is not required. Even if the control current is only magnetically coupled to the loop, it is often a good approximation to model it as being directly injected into the device (more on this in Chap. 6). If \(I_c=0\), then the gate current flow is symmetric and splits evenly between the junctions. In that case, the interferometer behaves much like a single junction with critical current \(I_{m0}=2I_0\). If \(I_c\neq0\), however, then a phase difference will be introduced between the
Figure 1.6
junctions which will result in a lower maximum gate current $I_m$ ($I_m$ is the maximum current which the interferometer can sustain in the zero-voltage state; if $I_g$ exceeds $I_m$, the device will switch to the voltage state).

The function $I_m(I_c)$ is known as the interferometer "threshold curve".\textsuperscript{9} It defines the boundary beyond which the device cannot remain in the zero-voltage state. Since the resistances and capacitances are not drawing any current when the interferometer is in the zero-voltage state, the threshold curve will depend only upon the junction critical current and on the loop inductance $2L$. In fact, if we normalize all currents in terms of $I_0$, we find that the shape of the threshold curve depends only on the dimensionless parameter $\chi = K/LI_0$. Fig. 1.7 shows threshold curves for various values of $\chi$. We see that as $\chi$ is reduced, the sensitivity of $I_m$ to $I_c$ increases; this means increased "gain" for the logic circuits. On the other hand, the minimum value of $I_m$, defined as $I_x = I_m(I_c = \Phi_0/4L)$, also increases with decreasing $\chi$. Thus the amount of "modulation" of $I_m$ diminishes as the gain is increased. There is therefore a tradeoff between gain and operating margins in a Josephson interferometer. The details of the tradeoff are very important to logic designers, but will not be discussed further here.

Although the dynamic behavior of an interferometer resembles that of a single junction, there is one particularly important difference: the phenomenon of "resonance", which occurs in interferometers\textsuperscript{10} but not in point junctions. The interferometer circuit of Fig. 1.6 contains a series resonant circuit, consisting of a loop inductance $2L$ in series with the two junction capacitances $C$. The resonant frequency is $\omega_r = 1/\sqrt{(2L)(C/2)} = 1/\sqrt{LC}$. We therefore expect that the resonant circuit will be excited when the gate voltage $V_g$ is equal to $V_r = K\omega_r$, for at this voltage the junctions will be generating ac currents at the resonant frequency. This is in fact the case, and oscillations build up in the resonant circuit. These oscillations dissipate power in the damping resistance $R_d$ and in $R_j$, power which must be supplied by the gate current supply. Thus the gate current at the "resonant voltage" $V_r$ will
Figure 1.7
be considerably higher than expected, because of this extra power needed to feed the resonance. If the resonant circuit quality factor $Q = \sqrt{C/L/(R_d^{-1} + R_j^{-1})}$ is sufficiently large, the resonance will appear as a sharp current step or spike in the I-V curve, as shown in Fig. 1.8.

Although resonances are most noticeable in terms of their effect on the I-V curves, we are more interested in the internal ac oscillations which result from them. If the damping resistance $R_d$ is really a transmission line of impedance $Z_o$, then RF radiation will be transmitted into this line. The frequency can be quite high: for a resonance at half the gap voltage ($V_r = \frac{1}{2}V_{gap} = 1.4$ mV), we would have $\nu = \frac{V_r}{\Phi_0} \approx 680$ GHz. (This is the highest practical operating frequency, for above this frequency the superconductors are very lossy). In 1979 Calander and Zappe\textsuperscript{1} showed that an interferometer could also be used as a detector (receiver) of microwave radiation. They then proposed the idea of a Josephson frequency-division multiplexing system, as discussed previously.

It should be noted that the concept of using Josephson devices as generators and detectors of microwave (mm-wave) radiation was recognized as early as 1965 by Langenberg and Scalapino.\textsuperscript{11} They discussed the idea in terms of the "long" Josephson junction, which can be thought of as an interferometer containing infinitely many point junctions. Unfortunately the design of a multiplexing system using long junctions appears impractical. There are two main reasons for this. First, ordinary rectangular long junctions do not have as many adjustable design parameters as interferometers (a rectangular long junction is characterized by its length, width, and current density $j_1$). Second, the mathematical analysis of a long junction is exceedingly difficult, as it involves a nonlinear partial differential equation. In contrast, the analysis of a two-junction interferometer involves only 2 or 3 ordinary differential equations. We shall see that a large number of simple theoretical results are obtainable when we work with interferometers.
Figure 1.8
In the subsequent chapters we discuss the theory, design, and experiments for a digital frequency-division multiplexing system composed of interferometers.
Chapter 2 - THEORY OF INTERFEROMETER RESONANCES

In this chapter we develop the theory of the proposed microwave transmitter. The generation of RF radiation by Josephson interferometers is a well-known phenomenon and generally goes under the name of "resonances." As the existing theories of resonances will not be adequate for our purposes, this chapter is primarily devoted to the exposition of an improved theory of resonances in Josephson interferometers. The following identities will be used frequently in the theoretical work which follows.

\[ 2 \cos x \cos y = \cos (x - y) + \cos (x + y) \]  

(2.1)

\[ 2 \sin x \cos y = \sin (x - y) + \sin (x + y) \]  

(2.2)

\[ 2 \sin x \sin y = \cos (x - y) - \cos (x + y) \]  

(2.3)

\[ \sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2} \]  

(2.4)

\[ \sin (\delta \sin \theta) = 2J_1(\delta) \sin \theta + 2J_3(\delta) \sin 3\theta + 2J_5(\delta) \sin 5\theta + ... \]  

(2.5)

\[ \cos (\delta \sin \theta) = J_0(\delta) + 2J_2(\delta) \cos 2\theta + 2J_4(\delta) \cos 4\theta + ... \]  

(2.6)

\[ J_{n-1}(\delta) + J_{n+1}(\delta) = 2nJ_n(\delta)/\delta \]  

(2.7)

The expressions \( J_n(\delta) \) are the Bessel functions of the first kind of order \( n \). Fig. 2.1 is a graph of the first few orders.

Let us begin by reviewing the state of the art in Josephson device resonances as it was understood in June 1978. "Resonances" occur whenever a Josephson device is coupled to a resonant circuit. Consider the circuit in Fig. 2.2. Here we have an ideal point junction of critical current \( I_0 \), powered by a current source, and connected to a resonant circuit. The resonator has a low impedance except near the resonance frequency \( \omega_r \), where it has a high impedance. A parallel R-L-C circuit would have such properties, as would a transmission-line
Figure 2.1
Figure 2.2
stub. The effect of the resonant circuit is to nearly short-circuit all frequencies except those near \( \omega_r \). We shall assume that the voltage \( V \) across the junction is periodic, hence it follows that only a single frequency \( \omega \) is likely to be present with significant amplitude, provided that \( \omega \) is near \( \omega_r \). All harmonics or subharmonics would be shorted out. Thus we are led to the "sinusoidal approximation" that
\[
V(t) \approx V_0 + V_1 \cos \omega t .
\] (2.8)

This can be integrated to give the junction phase:
\[
\phi(t) = \phi_0 + \omega_0 t + \delta \sin \omega t
\]

where \( \omega_0 = V_0/K \) and \( \delta = V_1/K \omega \). We further assume that there is some rational relationship between \( \omega_0 \) and \( \omega \), i.e. \( \omega_0 = r \omega \), where \( r \) is a rational number. Now the current through the junction is
\[
I_J(t) = I_0 \sin \phi(t) = I_0 \sin (\phi_0 + r \omega t)[J_0(\delta) + 2J_2(\delta) \cos 2\omega t + ...]
\]
\[
+ I_0 \cos (\phi_0 + r \omega t)[2J_1(\delta) \sin \omega t + 2J_3(\delta) \sin 3\omega t + ...].
\]

The resonant cavity will nearly short out all oscillations which are not at the frequency \( \omega \) (assumed to be near \( \omega_r \)). That is, the impedance will be so low at any other frequency that negligible voltage is produced, in accordance with our assumptions (Eq. 2.8). We therefore need only examine the component of \( I_J \) at frequency \( \omega \). But we see by inspection that in order to get such a component, we need \( r \) to be an integer. Thus there can only be resonances near voltages \( V_0 = nK \omega_r \), where \( n \) is an integer.

As an example, let us assume \( n = 1 \). Then the dc component of \( I_J/I_0 \) created by the resonance is, by inspection,
\[
\{I_J/I_0\}_{dc} = \{(- \sin \phi_0 \sin \omega t)(2J_1(\delta) \sin \omega t)\}_{dc} = -J_1(\delta) \sin \phi_0 .
\]

The component of \( I_J/I_0 \) at frequency \( \omega \) is:
\[ \{I_j/I_0\}_{\omega} = \{ \sin(\phi_0 + \omega t)[J_0(\delta) + 2J_2(\delta) \cos 2\omega t]\}_{\omega} = \]

\[ J_0(\delta)[ \sin \phi_0 \cos \omega t + \cos \phi_0 \sin \omega t] + J_2(\delta)[ \sin \phi_0 \cos \omega t - \cos \phi_0 \sin \omega t] = \]

\[ [J_0(\delta) + J_2(\delta)] \sin \phi_0 \cos \omega t + [J_0(\delta) - J_2(\delta)] \cos \phi_0 \sin \omega t. \]

This can also be compactly written as:

\[ \{I_j/I_0\}_{\omega} = J_0(\delta) \sin(\omega t + \phi_0) - J_2(\delta) \sin(\omega t - \phi_0) = \text{Re}[-j(J_0(\delta)e^{j\phi_0} - J_2(\delta)e^{-j\phi_0})e^{j\omega t}] . \]

To have self-consistency at this frequency, we require that \( \{I_j\}_{\omega} = \text{Re}[Y(\omega)V_1e^{j\omega t}], \) where \( Y(\omega) \) is the complex admittance of the filter at frequency \( \omega. \) This gives:

\[ [J_0(\delta)e^{j\phi_0} - J_2(\delta)e^{-j\phi_0}]/j\delta = K\omega/I_0Z(\omega) , \]

where \( Z(\omega) = Y(\omega)^{-1} . \) This result is general, depending only upon the "sinusoidal approximation" Eq. 2.8. The quantity on the right-hand side of Eq. 2.9 is a complex number (determined by the frequency \( \omega), \) and the quantity on the left-hand side is a complex function of \( \delta \) and \( \phi. \) Eq. 2.9 was essentially derived by Werthamer in 1966.\textsuperscript{12} We have written it here in a somewhat more general form by assuming a general impedance \( Z(\omega), \) rather than insisting upon a Lorentzian form for the resonant circuit.

Eq. 2.9 can in principle be solved for \( \delta \) and \( \phi, \) given the impedance \( Z(\omega). \) The amplitude \( V_1 \) of the ac voltage across the junction is then calculated from the relation \( V_1 = K\omega\delta. \) The larger the magnitude of \( Z(\omega), \) the larger \( \delta \) will be. It is particularly convenient to choose \( \omega = \omega_r, \) in which case we have a real resonator impedance \( R \equiv Z(\omega_r). \) In this case, we have \( \phi_0 = \pi/2, \) whence

\[ [J_0(\delta) + J_2(\delta)]/\delta = \Gamma^{-1}, \]

where we have defined \( \Gamma \equiv I_0R/K\omega_r = I_0R/V_r, \) a parameter used by Werthamer. Eq. 2.10 was first correctly solved by Guéret.\textsuperscript{13} (For reasons which we shall discuss later, Werthamer had assumed that \( \delta \) could not exceed 1.84, the first zero of \( J_0(\delta) - J_2(\delta). \) Fig. 2.3 is a plot
Figure 2.3

Figure 2.4
of \( \delta \) as a function of \( \Gamma \). Since \( \delta \) approaches 3.84 for high-\( \Gamma \) devices, we see that our dc supply voltage \( V_0 \) can generate an ac voltage \( V_1 \) of considerably larger amplitude \( (V_1 = \delta V_0) \).

This generation of ac power from a dc supply manifests itself as current steps of amplitude \( I_r \) in the static I-V curve of the Josephson junction. Specifically, enough dc power must be supplied from the current source to account for the power dissipated in the resonator. This fact has been known for a long time, but was not explicitly used to calculate the current until Zappe and Landman's work\(^{10} \) in 1978. We rederive the result here. Let \( I_r \equiv [I_0]_{dc} \) be the excess dc current supplied to the junction, due to the resonance. The excess power drawn from the battery is \( P_{dc} = I_r V_0 \), and this power must be dissipated in the ac resonance. The time-averaged power \( <P_{ac}> \) is

\[
<P_{ac}> = \frac{1}{\hbar} (\text{Re}[Y(\omega)]) V_1^2.
\]

At the resonant frequency \( \omega = \omega_c \), this power is simply \( \frac{1}{\hbar} V_1^2 / R \). Power-conservation implies that \( P_{dc} = <P_{ac}> \), or

\[
\frac{I_r}{I_m} = \frac{V_1^2}{2I_0 RV_0} = \frac{\delta^2}{2\Gamma},
\]

(2.11)

where \( I_m = I_0 \) in this example. Using the values of \( \delta \) obtained from Eq. 2.10 or Fig. 2.3, this gives \( I_r \) as a function of \( \Gamma \), as shown in Fig. 2.4. We see that the junction current (and hence power) reaches a maximum when \( \Gamma = \Gamma_c \approx 2.9 \), which we shall call the "critical value" of \( \Gamma \). Fig. 2.4 was first computed correctly by Zappe and Landman from simulations, and later solved analytically by Guéret.

Now that the basic phenomenon of junction resonances has been discussed, we turn to the case of interest to us: the use of an interferometer as a generator of RF radiation. We are interested in the circuit model of Fig. 2.5. Although Werthamer has studied the theory of single-junction resonances, there has been relatively little work on resonances in interferometers. Zappe and Landman\(^{10} \) recently developed a complete theory for resonances in low-\( \Gamma \)
devices \((\Gamma \leq 1)\), and also some approximations for the high-\(\Gamma\) region. Guéret\textsuperscript{13} obtained an analytical theory of resonances for all values of \(\Gamma\), but only for the specific case where \(V_g = V_r\) and \(I_c = \Phi_0/4L\). The goal of this chapter is to extend the works of Zappe and Landman and of Guéret so as to obtain an accurate (within a few \%), analytical theory of interferometer resonances for all values of damping parameter \(\Gamma\), voltage \(V_g\), and control current \(I_c\). Such a theory will be important for designing transmitter circuits.

Before beginning our analysis, it is worth mentioning why a theory of resonances is needed, when the differential equations governing the circuit are known. One could argue, after all, that numerical integration (simulations) of the node equations are all that is required. The answer is, of course, that simulations provide very little insight into the nature of resonances. Moreover they are quite inefficient because one must integrate over many oscillation periods before the resonance amplitude stabilizes. Hence a simple (although approximate) analytical theory of interferometer resonances (especially high-\(\Gamma\) ones) would be very desirable.

The obvious way to begin the circuit analysis for Fig. 2.5 is to write down the node equations. As discussed in Chapter 1, it is very convenient to work with phases, which are the time integrals of voltages:

\[
\phi_i = \int \frac{V_i}{K} dt , \text{ where } K = \Phi_0/2\pi .
\]

We have designated the node common to the two junctions as a ground node. The other three nodes have been labeled \(V_1\), \(V_2\), and \(V_g\), corresponding to the left, right, and center (gate) node voltages. Writing Kirchoff's current law at each node, we obtain:

**Node 1:** \(KC\dddot{\phi}_1 + KG_j\ddot{\phi}_1 + I_0 \sin \phi_1 + KL^{-1}(\phi_1 - \phi_g) + \frac{1}{2}KG_d(\dot{\phi}_1 - \dot{\phi}_2) = I_c .\) \(2.12\)

**Node 2:** \(KC\dddot{\phi}_2 + KG_j\ddot{\phi}_2 + I_0 \sin \phi_2 + KL^{-1}(\phi_2 - \phi_g) + \frac{1}{2}KG_d(\dot{\phi}_2 - \dot{\phi}_1) = -I_c .\) \(2.13\)
Node 3: \( KL^{-1}(\phi_g-\phi_1) + KL^{-1}(\phi_g-\phi_2) + KG_L\phi_g = I_g \) \hspace{1cm} (2.14)

We have used the symbol \( G \) to denote a conductance, e.g. \( G_j = R_j^{-1} \).

These node equations were the basis for Guéret's work on interferometer resonances. However, it is cumbersome and somewhat unnatural to use this set of equations. We shall instead use a change of variables proposed by Harris:14

\[
\phi_- = \frac{1}{2}(\phi_1 - \phi_2) \quad \phi_+ = \frac{1}{2}(\phi_1 + \phi_2) \quad \phi_L = \phi_g - \phi_+ .
\]

The physical motivation for this change of variables is the symmetry of the interferometer. It is well known that the differential equations describing a linear system can be decoupled by a suitable change of variables. The new variables represent "normal modes" of the system; each normal mode has its own natural frequency. Often the normal modes of a system can be deduced from symmetry considerations. If the Josephson terms are removed from the interferometer circuit of Fig. 2.5, or replaced by linear inductances \( L_j \), we are left with a linear circuit which will have symmetric and antisymmetric normal modes represented by \( \phi_+ \) and \( \phi_- \), respectively. \( \phi_+ \) is the average, or "common mode" phase of the two junctions, whereas \( \phi_- \) is the internal, relative, or "difference-mode" phase.

Let us now rewrite the node equations in terms of our new variables. Subtracting Eq. 2.13 from Eq. 2.12 and dividing by 2, we get:

\[
KC\ddot{\phi}_- + KG_j\dot{\phi}_- + \frac{1}{2}I_0(\sin \phi_1 - \sin \phi_2) + KL^{-1}\phi_- + KG_d\dot{\phi}_- = I_c .
\]

Dividing by \( K \) and applying Eq. 2.4, this can be written as:

\[
C\ddot{\phi}_- + G_i\dot{\phi}_- + L^{-1}\phi_- + \frac{I_0}{K} \sin \phi_- \cos \phi_+ = \frac{I_c}{K}, \hspace{1cm} (2.15)
\]

where we have defined \( G_i \equiv R_i^{-1} \equiv G_d + G_j \) for convenience. Adding Eq. 2.12 to Eq. 2.13 and dividing by 2, we get:
\[ KC\ddot{\phi}_+ + KG_j\dot{\phi}_+ + \frac{1}{2}I_0(\sin \phi_1 + \sin \phi_2) + \frac{1}{2}KL^{-1}(\phi_1 + \phi_2 - 2\phi_g) = 0 . \]

Using Eq. 2.4 and the definition of \( \phi_L \), this gives:

\[ C\ddot{\phi}_+ + G_j\dot{\phi}_+ + \frac{I_0}{K} \sin \phi_+ \cos \phi_- = \phi_L/L . \]  

(2.16)

Finally, we can immediately rewrite Eq. 2.14 using our new variables:

\[ 2\phi_L/L + G_L\dot{\phi}_g = I_g/K , \]  

or

\[ 2\phi_L/L = I_g/K - G_L(\dot{\phi}_+ + \dot{\phi}_L) . \]  

(2.17)

This can be combined with Eq. 2.16 to yield:

\[ C\ddot{\phi}_+ + (G_j + \frac{1}{2}G_L)\dot{\phi}_+ + \frac{I_0}{K} \sin \phi_+ \cos \phi_- = \frac{1}{2}(I_g/K - G_L\dot{\phi}_L) . \]  

(2.18)

Eqs. 2.15, 2.17, and 2.18 are our fundamental circuit equations for the interferometer, and all our analyses will be based on them.

For now, we shall neglect the effects of the load resistance \( R_L \), i.e. we set \( G_L = 0 \). This has the considerable advantage of leaving only two differential equations. For Eq. 2.17 reduces simply to \( \phi_L = LI_g/2K \), which is decoupled from Eqs. 2.15 and 2.18. We shall call Eq. 2.15 the "internal" circuit equation, and Eq. 2.18 the "external" circuit equation.

Before proceeding with our analysis, it will be very convenient to introduce some normalized units. Specifically, it would be helpful to work with a set of units in which the resonant frequency is normalized to 1. This can be done without loss of generality by defining a
normalized time $\tau \equiv \omega_f = t/\sqrt{LC}$. Eqs. 2.15 and 2.18 then become:

$$
\chi \frac{d^2 \phi_-}{d\tau^2} + \gamma \frac{d\phi_-}{d\tau} + \chi \phi_- + \sin \phi_- \cos \phi_+ = I_c/I_0,
$$

(2.19)

$$
\chi \frac{d^2 \phi_+}{d\tau^2} + \gamma_j \frac{d\phi_+}{d\tau} + \sin \phi_+ \cos \phi_- = I_g/2I_0,
$$

(2.20)

where $\chi = K/LI_0$, $\gamma = \Gamma^{-1} = V_r/I_0 R_i = \frac{K(G_d + G_j)}{I_0 \sqrt{LC}}$, and $\gamma_j = V_r/I_0 R_j$.

Note that after normalization, there are only two dimensionless parameters which characterize the internal equation of the interferometer: $\gamma$ (or $\Gamma$) and $\chi$. Note also that if we ignore the Josephson term $\sin \phi_- \cos \phi_+$, Eq. 2.19 is simply the differential equation of a resonant cavity with quality factor $Q = \chi/\gamma = \chi \Gamma$. Furthermore, the "importance" of the Josephson term is determined by the values of $\chi$ and $\gamma$. For example, if they are both much greater than one, then the nonlinear junction term is contributing very little to the equation and may be treated as a perturbation, as in Zappe and Landman's work.

We are now in a position to attempt an analysis of interferometer resonances. Let us begin by trying to reproduce the results of Gue\'ret, i.e. solve for the case where $V_g = V_r$ and $I_c = \Phi_0/4L$. Gue\'ret ended up with the following approximate forms for the junction phases:

$$
\phi_1 \approx \pi/2 + \omega_f \tau - \delta \sin \omega_f \tau,
$$

(2.21)

$$
\phi_2 \approx -\pi/2 + \omega_f \tau + \delta \sin \omega_f \tau,
$$

(2.22)

where $\omega_f = 1/\sqrt{LC}$. This works out very nicely in our normalized sum and difference variables:

$$
\phi_- = \frac{1}{2}(\phi_1 - \phi_2) \approx \pi/2 - \delta \sin \tau,
$$

(2.23)

$$
\phi_+ = \frac{1}{2}(\phi_1 + \phi_2) \approx \tau.
$$

(2.24)
This says that the resonance is simply an internal sinusoidal oscillation \( \phi_- \) of amplitude \( \delta \).

Substituting Eqs. 2.23 and 2.24 into Eq. 2.19 gives:

\[
\chi_\delta \sin \tau - \gamma_\delta \cos \tau + \chi(\pi/2-\delta \sin \tau) + \sin (\pi/2-\delta \sin \tau) \cos \tau \approx \Phi_0/4LI_0 .
\]

Using the definition of \( \chi = K/LI_0 = \Phi_0/2\pi LI_0 \) and cancelling terms, this reduces to:

\[
[ \cos (\delta \sin \tau) ] \cos \tau \approx \gamma_\delta \cos \tau . \tag{2.25}
\]

The left-hand side can be expanded in a Fourier series using Eqs. 2.6 and 2.1:

\[
\cos (\delta \sin \tau) \cos \tau = [J_0(\delta) + 2 \sum_{n=2,4,...}^\infty J_n(\delta) \cos n\tau] \cos \tau = [J_0(\delta) + J_2(\delta)] \cos \tau + [J_2(\delta) + J_4(\delta)] \cos 3\tau + ...
\]

This Fourier series has frequency components at all odd harmonics of \( \omega_r \). However the right-hand side of Eq. 2.25 has only the fundamental frequency. Thus the theory is not entirely self-consistent, or in other words, the assumption of a sinusoidal voltage (Eqs. 2.23, 2.24) is only an approximation. But as we shall see, it is a remarkably good approximation.

We shall make Eq. 2.25 hold as well as we can by equating the fundamental frequency components:

\[
J_0(\delta) + J_2(\delta) = \gamma_\delta , \text{ or } \\
[J_0(\delta) + J_2(\delta)]/\delta = \Gamma^{-1} .
\]

This is Guéret's result (Eq. 2.10).

To obtain the excess gate current \( I_r \) due to the resonance, there are two possible courses. We can follow Zappe and Landman and invoke time-averaged power conservation,
which leads to:

\[ \frac{I_r}{2I_0} = \frac{\delta^2}{2\Gamma}, \]

as in Eq. 2.11. The other approach is to substitute the approximations for \( \phi_+ \) and \( \phi_- \) (Eqs. 2.23, 2.24) into the normalized external circuit equation (Eq. 2.20), which up to now has not been used. This gives:

\[ \gamma_j + (\sin \tau) \sin (\delta \sin \tau) = I_g/2I_0. \]

The left-hand side can again be expanded as a Fourier series:

\[ \gamma_j + (\sin \tau) \sin (\delta \sin \tau) = \gamma_j + (\sin \tau) \left[ 2 \sum_{n=1,3,\ldots}^{\infty} J_n(\delta) \sin n\tau \right] = \]

\[ \gamma_j + J_1(\delta) + [J_3(\delta) - J_1(\delta)] \cos 2\tau + [J_5(\delta) - J_3(\delta)] \cos 4\tau + \ldots \]

Now \( I_g \) is a constant dc current, hence all other Fourier components (the even harmonics of \( \omega_r \)) are not self-consistent. We nonetheless collect the dc term:

\[ \frac{I_g}{2I_0} = \gamma_j + J_1(\delta). \]

\( \gamma_j \) is simply the current through the junction resistance \( R_j \), and has nothing to do with the resonance. Subtracting it out, we have:

\[ \frac{I_r}{2I_0} = J_1(\delta), \]

(2.26)

where \( I_r \) is the excess gate current due to the resonance. This was the result obtained by Guéret.

An important question is whether Eq. 2.26, obtained through Fourier analysis (Guéret's approach), is consistent with Eq. 2.11, obtained from power conservation (Zappe and Landman's approach). This issue has not been addressed in the literature. From Eq. 2.7, we have:

\[ J_1(\delta) = \frac{\delta}{2} [J_0(\delta) + J_2(\delta)]. \]

But Eq. 2.10 gives \([J_0(\delta) + J_2(\delta)] = \delta/\Gamma\), whence we get \( J_1(\delta) = \delta^2/2\Gamma \). Thus Eq. 2.11 and Eq. 2.26 give identical results. We shall use the
power conservation approach from now on, because it will be less cumbersome when we extend the theory.

We have seen how all of Guéret's results can be derived in a very quick and natural way, using a simple change of variables. This motivates us to go beyond Guéret's results by removing the requirements that \( V_g = V_r \) and \( I_c = \Phi_0/4L \). Furthermore, we would also like to extend the theory to include the resonance steps which are observed at multiples of the voltage \( V_r \) (Fig. 3.5).

We shall now try to extend Guéret's approach to the case of a general value of control current \( I_c \). Recall our original assumptions for the form of \( \phi_+ \) and \( \phi_- \) (Eqs. 2.23 and 2.24):

\[
\phi_- \approx \pi/2 - \delta \sin \tau , \quad \phi_+ \approx \tau .
\]

The \( \pi/2 \) term in \( \phi_- \) is due to the control current \( I_c \), because \( I_c \) creates flux \( \Phi = 2LI_c \) in the inductor, which results in a phase difference of \( \phi_1 - \phi_2 = 2\pi\Phi/\Phi_0 \) between the two junctions. Thus it seems likely that we can account for the control-current dependence simply by making the dc component of \( \phi_- \) a parameter \( \phi_c \). That is, we assume:

\[
\phi_- \approx \phi_c - \delta \sin \tau , \quad \phi_+ \approx \tau .
\]

As before, we substitute these into the normalized internal circuit equation (Eq. 2.19). This gives:

\[
-\gamma \delta \cos \tau + \chi \phi_c + \sin (\phi_c - \delta \sin \tau) \cos \tau \approx I_c/I_0 .
\] (2.27)

The Josephson term is then expanded as a Fourier series:

\[
\sin (\phi_c - \delta \sin \tau) = \sin \phi_c [\cos (\delta \sin \tau)] - \cos \phi_c [\sin (\delta \sin \tau)] =
\]

\[
\sin \phi_c [J_0(\delta) + 2J_2(\delta) \cos 2\tau + ...] - \cos \phi_c [2J_1(\delta) \sin \tau + 2J_3(\delta) \sin 3\tau + ...] .
\]

When the right-hand term (the one multiplied by \( \cos \phi_c \)) is multiplied by \( \cos \tau \), there will be no dc term because \( \{ \sin \tau \cos \tau \}_d = 0 \); only even harmonics of \( \omega_r \) are present. The
left-hand term (the one multiplied by \( \sin \phi_c \)) also has no dc component when multiplied by \( \cos \tau \), but it will have a fundamental frequency component which works out to be \( \sin \phi_c [J_0(\delta) + J_2(\delta)] \cos \tau \). Thus equating the dc components of Eq. 2.27 gives simply \( \chi \phi_c = I_c/I_0 \), or \( \phi_c = LI_c/K \), as expected. Equating the fundamental ac components gives:

\[
-\gamma \delta \cos \tau + \sin \phi_c [J_0(\delta) + J_2(\delta)] \cos \tau = 0 \quad \text{or}
\]

\[
[J_0(\delta) + J_2(\delta)]/\delta = (\Gamma \sin \phi_c)^{-1}.
\]  

(2.28)

This equation bears a remarkable similarity to Eq. 2.10. The only difference is that here \( \Gamma \) has been replaced by \( \Gamma \sin \phi_c \), where \( \phi_c = LI_c/K \). Thus we have found an exceedingly simple approximate theory for the control-current dependence of the resonance amplitude. To obtain the gate current, we can use time-averaged power conservation (Eq. 2.11), as before.

Zappe and Landman\(^{10}\) have previously proposed some approximations for the control-current dependence of resonances. Specifically, they concluded that the resonance current \( I_r \) should be proportional to \( \sin^2 LI_c/K \) for small \( \Gamma \), but should be nearly independent of \( I_c \) for large \( \Gamma \). We can now confirm these approximations using our analytic theory. For a low-\( \Gamma \) device (\( \Gamma \leq 1 \)), \( \delta \) will be small (Fig. 2.3). Thus we can approximate \( J_0(\delta) + J_2(\delta) \approx 1 \), whence from Eq. 2.28, we have \( \delta \approx \Gamma \sin \phi_c \). From Eq. 2.11, this gives

\[
I_r/2I_0 \approx \frac{(\Gamma \sin \phi_c)^2}{2\Gamma} = \frac{1}{2} \Gamma \sin^2 (LI_c/K),
\]

which was the result derived by Zappe and Landman. For large \( \Gamma \), Fig. 2.3 shows that \( \delta \) is nearly independent of \( \Gamma \), hence also of \( \Gamma \sin \phi_c \). This leads to a nearly constant \( I_r \), as predicted by Zappe and Landman.

We can also develop a new approximation, not treated by Zappe and Landman, which is valid for the medium-\( \Gamma \) (\( \Gamma \approx 1 \) to 10) region. This is based on Zappe and Landman's observation that \( I_r/2I_0 \) is nearly constant at about 0.6 in this region (Fig. 2.4). Thus \( \delta^2/2\Gamma \approx 0.6 \), or \( \delta \approx \sqrt{1.2\gamma} \). For a variable control current, we would have \( \delta \approx \sqrt{1.2\gamma \sin \phi_c} \),
and therefore we have a novel approximation for the control-current dependence of the resonance in the medium-\( \Gamma \) region:

\[
\frac{I_r}{2I_0} \approx 0.6 \sin \left( \frac{LI_c}{K} \right).
\]

In order to verify the accuracy of our theory of control-current dependence, computer simulations were performed. The simulations were performed as follows: The interferometer was biased with a control current \( I_c = \Phi_0/4L \). The gate current was then slowly (quasistatically) ramped up. First the interferometer would switch out of the zero-voltage state when \( I_g \) exceeded \( I_x \), the critical current of the interferometer with \( I_c = \Phi_0/4L \). If the maximum resonance current exceeded \( I_x \), the interferometer would switch into the resonance. \( I_g \) is continually ramped up until the interferometer switches out of the resonance; this determines the maximum resonance current. Except for very low-\( Q \) devices, this maximum current occurs at the resonant voltage \( V_r \), hence it is the \( I_r \) that we want. Fig. 2.6 shows typical simulated waveforms produced by such a technique. Fig. 2.6a shows the internal phase \( \phi_- \), and Fig. 2.6b shows the gate voltage (the time-derivative of the external phase \( \phi_+ \)). Fig. 2.7 compares our theory with simulations for devices with \( \Gamma = 2/3, \Gamma = 2, \Gamma = 6, \) and \( \Gamma = 20; \chi = 8 \) and \( \beta = 16 \) for all devices. The agreement is excellent in all cases. Thus we can be confident in our theory up to this point.

The next important generalization we shall make is to let the gate voltage vary, i.e. we remove the restriction that \( V_g = V_r \). That is, we shall allow the normalized frequency \( \Omega \equiv \omega/\omega_r \neq 1 \). We will obviously need to introduce yet another parameter into our assumed solutions for \( \phi_+ \) and \( \phi_- \). To see what is needed, we look at Eq. 2.19 and move the Josephson term over to the right-hand side. Assuming \( \phi_+ = \Omega \tau \), we get:

\[
\frac{d^2 \phi_-}{d\tau^2} + \gamma \frac{d\phi_-}{d\tau} + \chi \phi_- = \frac{I_r}{I_0} \sin \phi_- \cos \Omega \tau.
\]  

(2.29)
Figure 2.6a
Figure 2.7
As we have remarked earlier, this is precisely the equation of a damped, driven RLC circuit, except that the driving term \( \sin \phi_- \cos \Omega \tau \) is dependent on the amplitude of the oscillation \( \phi_- \). When we had \( \phi_- = \phi_c - \delta \sin \tau \), the terms \( \chi \frac{d^2 \phi_-}{d\tau^2} \) and \( \chi \phi_- \) cancelled each other ac-wise, leaving only a \( \cos \tau \) term due to \( \gamma \frac{d\phi_-}{d\tau} \) on the left-hand side. With a general choice of normalized frequency \( \Omega \), there will now be a \( \sin \tau \) term as well as a \( \cos \tau \) term on the left-hand side, which we must balance by a similar term on the right-hand side. In other words, the junction currents and the resonant circuit voltage are no longer in phase when \( \omega \neq \omega_r \) (\( \Omega \neq 1 \)). An obvious way to remedy this is to introduce a phase factor \( \theta \) into \( \phi_+ \), i.e. define

\[
\phi_+ \approx \Omega \tau - \theta . \tag{2.30}
\]

The same effect could also be achieved by leaving \( \phi_+ = \Omega \tau \) and instead writing \( \phi_- = \phi_c - \delta \sin (\Omega \tau + \theta) \), but the mathematics is more cumbersome when done this way.

We proceed with the analysis. Substituting our definitions of \( \phi_+ \approx \Omega \tau - \theta \) and \( \phi_- \approx \phi_c - \delta \sin \Omega \tau \) into Eq. 2.19, we have:

\[
(\Omega^2 - 1) \chi \delta \sin \Omega \tau - \Omega \gamma \delta \cos \Omega \tau + \chi \phi_c + \left[ \sin (\phi_c - \delta \sin \Omega \tau) \right] \cos (\Omega \tau - \theta) \approx I_c / I_0 \tag{2.31}
\]

In the same manner as before, we can Fourier-analyze the junction term and extract the dc and fundamental frequency components. Equating the dc components of Eq. 2.31 gives, after much algebra:

\[
\chi \phi_c - J_1(\delta) \sin \theta \cos \phi_c = I_c / I_0 . \tag{2.32}
\]

Equating the \( \sin \Omega \tau \) components gives:

\[
(\Omega^2 - 1) \chi \delta + \left[ J_0(\delta) - J_2(\delta) \right] \sin \theta \sin \phi_c = 0 . \tag{2.33}
\]

Equating \( \cos \Omega \tau \) components gives:

\[
-\Omega \gamma \delta + \left[ J_0(\delta) + J_2(\delta) \right] \cos \theta \sin \phi_c = 0 . \tag{2.34}
\]

Eqs. 2.32-2.34 must be solved for the three unknowns \( \theta \), \( \phi_c \), and \( \delta \). The gate current due to
the resonance is then obtained from time-averaged power conservation:

\[
\frac{I_r}{2I_0} = \Omega \frac{\delta^2}{2\Gamma} .
\]  

(2.35)

This is simply a generalization of Eq. 2.11 to voltages other than \( V_r \) (\( \Omega \neq 1 \)).

The coupled equations 2.32-2.34 are rather difficult to solve for a general choice of \( I_c \) and \( V_g \), mainly because of the nonlinear term in Eq. 2.32. However, this term can be eliminated and we can solve for the \( I_r \)-vs.-\( V_g \) curve if we set \( \phi_c = \pi/2 \), which can be achieved by fixing \( I_c = \Phi_0/4L \). In that case, Eq. 2.32 is satisfied, and the other two equations reduce to:

\[
\chi(1-\Omega^2)\delta = [J_0(\delta)-J_2(\delta)] \sin \theta ,
\]  

(2.36)

\[
\Omega \gamma \delta = [J_0(\delta) + J_2(\delta)] \cos \theta .
\]  

(2.37)

If desired, \( \theta \) may be eliminated by squaring both sides, to yield a single equation in \( \delta \):

\[
\left[ \frac{\chi(1-\Omega^2)\delta}{J_0(\delta)-J_2(\delta)} \right]^2 + \left[ \frac{\gamma \Omega \delta}{J_0(\delta) + J_2(\delta)} \right]^2 = 1 .
\]  

(2.38)

This equation must be solved numerically for \( \delta \) as a function of \( \Omega \), in order to obtain the I-V curve.

Let us investigate the nature of these equations more carefully. Fig. 2.8 is a plot of \( J_0(\delta)-J_2(\delta) \) and \( J_0(\delta) + J_2(\delta) \). The first obvious question is whether or not \( \delta \) can equal \( \delta_c = 1.8413 \), the first zero of \( J_0(\delta)-J_2(\delta) \). Suppose it can; then from Eq. 2.36 we require that \( \Omega = 1 \) (\( V = V_r \)), whence from Eq. 2.37 we have \( 1/(\Gamma \cos \theta) = [J_0(\delta_c) + J_2(\delta_c)]/\delta_c = 0.3432 \), or \( \Gamma \cos \theta = 2.914 \). It follows that \( \Gamma \geq \Gamma_c = 2.914 \) is required if \( \delta \) is to reach \( \delta_c \). We see that two solutions exist for \( \Omega = 1 \) when \( \Gamma > \Gamma_c \); either \( \delta = \delta_c \) or \( \theta = 0 \); in the latter case \( \delta \) equals the value which solves \( \delta/\Gamma = J_0(\delta) + J_2(\delta) \). This is a rather remarkable result; two possible resonance amplitudes for the same voltage \( V_r \) ! To investigate further, we can solve
for the complete I-V curve exactly using the following procedure: Assume a specific value of \( \delta \), and substitute it into Eq. 2.38, to get a quadratic equation in \( \Omega^2 \). This equation will generally have two solutions: one giving \( \Omega > 1 \), and one giving \( \Omega < 1 \). For each value of \( \Omega \), the gate current \( I_g \) can then be obtained from Eq. 2.35. Thus two points on the I-V curve have been obtained (note that \( V_g = \Omega V_r \)). The complete I-V curve is then obtained by repeating this procedure for many values of \( \delta \). The results of this process are plotted as solid curves in Fig. 2.9 for a variety of values of \( \Gamma \) and for two different values of \( \chi \) (\( \chi = 1 \) and \( \chi = 2 \)). We see that the theoretical I-V curve has a remarkable "looped" shape for \( \Gamma > \Gamma_c \); as the current is increased, the voltage overshoots \( V_r \) and then decreases as the current approaches the maximum value. The loop is most pronounced for large \( \Gamma \) and small \( \chi \). Only half of the loop is usually observable. This is because as the current is increased beyond the peak, the device switches out of the resonance. It is possible to obtain part of the other half of the loop by voltage-biasing the device (Fig. 2.10 shows some points obtained by this technique), but a voltage-biased interferometer is not physically interesting.

The results of computer simulations of the I-V curve (using Eqs. 2.12-2.14 or, equivalently, Eqs. 2.15, 2.17, and 2.18) are plotted as circles in Fig. 2.9. The agreement with theory is generally excellent, and tends to be best for the larger values of \( \chi \). The region near the peak of the resonance shows a definite negative resistance for \( \Gamma > \Gamma_c \), as predicted. To our knowledge, such regions of negative resistance have not been previously reported.

It is interesting to note that Werthamer, who studied resonances in a single junction coupled to a resonant cavity, obtained equations similar (although not identical) to Eqs. 2.36 and 2.37. However, he did not obtain any multiple-valued solutions or loops. This is because he incorrectly assumed that \( \delta \) saturates at \( \delta_c = 1.84 \). It is clear how such an error could be made; when Werthamer found that \( \delta_c = 1.84 \) is a solution to his equations at \( V = V_r \), he probably assumed that this was the peak of the resonance. In fact there is a second, higher
Figure 2.10
value of \( \delta \) which solves Eq. 2.38 and represents the true resonance peak (the one observed by Zappe and Landman and by Guéret).

It would be interesting to see whether "Fiske steps",\(^{15}\) which are resonances in long tunnel junctions, can also show this negative-resistance region in the I-V curves. It seems likely that they can, because the theory is rather similar. However, there appears to be no mention of this issue in the literature, and we shall not pursue it further in this work.

It has been observed\(^{10}\) that high-\( \Gamma \) interferometers exhibit resonances at integral multiples of \( V_r \) (see, for example, Fig. 3.5). There has been almost no treatment of these in the literature; let us see whether our theory can explain them. Physically, we still expect the internal phase oscillation to be sinusoidal at a frequency \( \omega \) near \( \omega_r \), but the gate voltage can be at an integral multiple of \( K \omega / n K \omega \). Thus we assume \( \phi_\pm = \phi_c - \delta \sin \Omega \tau \) (where \( \Omega = \omega / \omega_r \approx 1 \)) as before, but now we have \( \phi_+ = n \Omega \tau - \theta \). Then:

\[
\sin \phi_- = \sin \phi_c [J_0(\delta) + 2J_2(\delta) \cos 2\Omega \tau + \ldots] - \cos \phi_c [2J_1(\delta) \sin \Omega \tau + 2J_3 \sin 3\Omega \tau + \ldots]
\]

and

\[
\cos \phi_+ = \cos \theta \cos n \Omega \tau + \sin \theta \sin n \Omega \tau.
\]

We are interested in the dc component and the fundamental ac components of \( \sin \phi_- \cos \phi_+ \). After much work, the results can be summarized in this table:
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<tr>
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<td>(\sin \phi_c \sin \phi_g J_0 (\delta))</td>
<td>(-\cos \phi_c \cos \phi_g J_1 (\delta))</td>
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<td>(\cos \phi_c \cos \phi_g J_3 (\delta))</td>
<td>(-\sin \phi_c \sin \phi_g J_4 (\delta))</td>
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<td>(\cos \omega t:)</td>
<td>(\sin \phi_c \cos \phi_g J_0 (\delta))</td>
<td>(\cos \phi_c \sin \phi_g J_1 (\delta))</td>
<td>(\sin \phi_c \cos \phi_g J_2 (\delta))</td>
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<td></td>
<td>(\sin \phi_c \cos \phi_g J_2 (\delta))</td>
<td>(\cos \phi_c \sin \phi_g J_3 (\delta))</td>
<td>(\sin \phi_c \cos \phi_g J_4 (\delta))</td>
</tr>
</tbody>
</table>

It would be very convenient if we could manipulate these terms so that all the trigonometric functions looked alike, independent of the parity of \(n\). As it turns out, this can be achieved by replacing \(\phi_c\) by \(\phi_c - \pi/2\) and \(\theta\) by \(\theta - \pi/2\), if \(n\) is even. To phrase it differently, we define:

\[
\phi_- = \phi'_c - \delta \sin \Omega \tau ,
\]

\[
\phi_+ = n\Omega \tau - \theta' ,
\]

where \(\phi'_c = \phi_c\) and \(\theta' = \theta\) if \(n\) is odd, and \(\phi'_c = \phi_c - \pi/2\) and \(\theta' = \theta - \pi/2\) if \(n\) is even. With these substitutions, Eqs. 2.32-2.34 generalize to:

\[
\chi \phi'_c J_n(\delta) \sin \theta \cos \phi_c = I_c / I_0 ,
\]

\[
(\Omega^2-1)\chi \delta + [J_{n-1}(\delta) - J_{n+1}(\delta)] \sin \theta \sin \phi_c = 0 ,
\]

\[
-\Omega \gamma \delta + [J_{n-1}(\delta) + J_{n+1}(\delta)] \cos \theta \sin \phi_c = 0 .
\]
The control-current dependence and I-V curves can be obtained precisely the same way as for the case \( n=1 \) which we analyzed earlier; one sets \( \phi_c=\pi/2 \) or \( \theta=0 \), respectively. One finds that for \( n=2 \), there is no nontrivial (\( \delta \neq 0 \)) solution if \( \Gamma < 2 \); furthermore, the resonance I-V curve for \( \Gamma > 2 \) has sharp cutoff voltages, beyond which no resonance is observed. For \( n>2 \), even higher values of \( \Gamma \) are required in order to have nontrivial solutions. The shapes of the I-V curves are quite interesting; Fig. 2.11 shows such curves for \( n=1, 2, \) and 3.

Another interesting property of the higher harmonics is that whereas for odd \( n \) the resonance is maximized when \( I_c = \Phi_0 / 4L \), for even \( n \) it is maximized when \( I_c = 0 \). One might wonder how a resonance can appear with \( I_c = 0 \), for by inspection of Eq. 2.19 we see that if \( \phi_- = 0 \) initially, it will remain zero for all time. The answer is that the solution \( \phi_-(t) = 0 \) is not stable; the slightest amount of noise will set up oscillations which ultimately lead to a steady-state resonance.

To check the accuracy of our theory (Eqs. 2.39-2.41), the maximum amplitudes of these higher harmonic resonances were compared with theory for various values of \( \Gamma \). As an example, Fig. 2.12 shows a comparison of the predicted peak resonance amplitudes for \( n=2 \) with simulations. The agreement is, as usual, excellent.

The harmonic steps may possibly prove more useful than the fundamental steps as generators of RF radiation. We find that as \( \Gamma \to \infty \), \( \delta \) approaches the first nontrivial zero of \( J_{n-1}(\delta) + J_{n+1}(\delta) \). For \( n=1, \delta \to 3.83 \); for \( n=2, \delta \to 5.14 \); for \( n=3, \delta \to 6.38 \); etc. (The internal oscillations are still at the same fundamental frequency \( \omega_r \).) The larger values of \( \delta \) obtainable with \( n>1 \) imply greater output power, which may be helpful in a Josephson microwave multiplexing system. It may therefore be beneficial to operate a transmitter on one of the higher harmonic resonance steps.
Figure 2.11
Figure 2.12
To summarize, a simple analytical theory has been developed which explains the amplitude of resonances in symmetric, current-biased two-junction interferometers as a function of voltage, control current, and damping. The theory agrees very well with simulations for $\chi \geq 1$, and it should be adequate for our design work. However, there remain some obvious questions regarding what happens when the interferometer has a nonzero gate conductance $G_L$, when the device is made asymmetric, or when extra circuit elements are added to the interferometer circuit of Fig. 2.5. We shall now develop some simple extensions to our theory so as to understand these cases, even though they are not of direct importance to our designs.

We begin by considering the effects of a load resistance $R_L$. It has been observed in simulations that for low-resistance loads ($\beta_c \equiv I_m R^2 C/K \approx 1$), the control-current dependence of the gate current is not as strong as predicted from our theory. In particular, the $n=1$ resonance current has the correct value for $I_c = \phi_0 / 4L$, but does not decrease with $I_c$ as fast as predicted from Eq. 2.28, and in fact does not go to zero as $I_c \to 0$. Simulations clearly indicate that no oscillation in $\phi_-$ is present when $I_c = 0$, hence the excess current can only be due to oscillations in the external phase $\phi_+$. Such an effect is not, strictly speaking, a "resonance", for it does not involve an internal oscillation in the LC circuit. Nonetheless, it shows up as an excess gate current in the I-V curve and cannot be distinguished from a resonance current when performing dc measurements.

Before developing an analytic theory, let us first understand the phenomenon physically. When $I_c = 0$, the two junctions are in phase and the interferometer behaves very much like a point junction having a critical current $2I_0$ (in fact, as $\chi \to \infty$ or $L \to 0$, it does become a point junction). It is well known that the I-V curve of a resistively-shunted point junction (Fig. 2.13) exhibits a dc current $\{I_j\}_{dc}$ in excess of the resistive current $V/R_j$. This excess current is due to a time-average dc
component of the Josephson supercurrent, i.e. \( \{ I_J \}_{dc} = \{ I_0 \sin \phi(t) \}_{dc} \). It can also be interpreted as a consequence of energy conservation: a low-resistance junction exhibits relatively large voltage oscillations across it, which cause ac power dissipation in the load resistance. This ac power must come from excess dc power \( I_J V \) supplied to the junction.

The exact I-V curve of a single junction has been analyzed numerically in great detail in the literature\(^5\),\(^6\), but these results will not be repeated here. We shall instead use our own approach, which involves a sinusoidal approximation, to approximate the I-V curve of a single junction. This result will then be applied to an interferometer, which is the case of interest to us.

Consider a resistively-shunted junction (Fig. 2.13). The circuit equation is

\[
CK\dot{\phi} + G_J\dot{\phi} + I_0 \sin \phi = I .
\]

This can be rewritten to contain only one dimensionless parameter:

\[
\frac{d^2 \phi}{d\tau^2} + \eta \frac{d\phi}{d\tau} + \sin \phi = \frac{I}{I_0} ,
\]

(2.42)

where \( \eta = \beta_c^{-1/2} = (I_0 R^2 C/K)^{-1/2} \) and \( \tau = \omega_p t = t / \sqrt{KC/I_0} \). Let \( I \) be a dc supply current. Then for any value of \( I \), the junction will exhibit some nonzero average voltage \( V \), if it is in the voltage state. Thus \( \phi(\tau) = \Omega \tau + f(\tau) \) where \( f(\tau) \) is a periodic function with normalized frequency \( \Omega = V/K \omega_p \). For our calculations, we shall approximate \( f(\tau) \) by a sinusoid:

\[
\phi(\tau) \approx \Omega \tau + \delta \sin (\Omega \tau + \theta) .
\]

(2.43)

This approximation is good unless the junction is very close to resetting, as pointed out
Figure 2.13
by Zappe.\textsuperscript{16} Substituting Eq. 2.43 into Eq. 2.42, we get:

\[
\frac{I}{I_0} \approx -\Omega^2 \delta \sin(\Omega_T + \theta) + \Omega \eta \delta \cos(\Omega_T + \theta) + \\
+ \Omega \eta + \sin[\Omega_T + \delta \sin(\Omega_T + \theta)] .
\]  

(2.44)

The last term, denoted \( I_f \), can be Fourier-analyzed:

\[
I_f = \sin[\Omega_T + \delta \sin(\Omega_T + \theta)] = \sin \Omega_T [J_0(\delta) + 2J_2(\delta) \cos 2(\Omega_T + \theta) + \ldots] + \\
\cos \Omega_T [2J_1(\delta) \sin(\Omega_T + \theta) + 2J_3(\delta) \sin 3(\Omega_T + \theta) + \ldots] .
\]

The dc component of \( I_f \) is

\[
\{I_f\}_{dc} = \{2J_1(\delta) \cos \Omega_T \sin(\Omega_T + \theta)\}_{dc} = J_1(\delta) \sin \theta .
\]

The fundamental ac component is:

\[
\{J_0(\delta) \sin \Omega_T + 2J_2(\delta) \sin \Omega_T \cos 2(\Omega_T + \theta)\}_\Omega = J_0(\delta) \sin \Omega_T - J_2(\delta) \sin(\Omega_T + 2\theta)
\]

Because \( \delta \) will generally be small, we shall assume that \( J_2(\delta) \) is small and neglect it.

The dc components of Eq. 2.44 are:

\[
\Omega \eta + J_1(\delta) \sin \theta = I/I_0 .
\]  

(2.45)

The \( \sin \Omega_T \) components are:

\[
-\Omega^2 \delta \cos \theta - \Omega \eta \delta \sin \theta + J_0(\delta) = 0 .
\]  

(2.46)

The \( \cos \Omega_T \) components are:

\[
-\Omega^2 \delta \sin \theta + \Omega \eta \delta \cos \theta = 0 .
\]  

(2.47)

From Eq. 2.47, we have \( \tan \theta = \eta/\Omega \). Substituting this into Eq. 2.46 gives:

\[
J_0(\delta)/\delta = \Omega \sqrt{\Omega^2 + \eta^2} .
\]  

(2.48)

Eq. 2.48 can be solved numerically for \( \delta \). An easy way to do this is to write
Eq. 2.48 in a recursive form:

\[
\delta_{n+1} = \frac{J_0(\delta_n)}{\Omega \sqrt{\Omega^2 + \eta^2}}.
\]

After a few iterations, \(\delta_n\) will converge to the desired solution \(\delta\). The supply current \(I\) may then be evaluated using Eq. 2.45:

\[
\frac{I}{I_0} = \Omega \eta + \frac{\eta J_1(\delta)}{\sqrt{\Omega^2 + \eta^2}}.
\] (2.49)

In Table 1, the second term of Eq. 2.49 (the dc current due to the junction) is compared with simulations for various values of \(\Omega\) and \(\beta_c = \eta^{-2}\). The agreement is clearly very good. In particular, it appears that for \(\Omega > 1\) (in unnormalized units, \(V > K \omega_p = \sqrt{K I_0/C}\)), the error is at most 1% of \(I_0\). This should be quite adequate for design work. Note that the dc supercurrent contribution is quite small unless both \(\Omega \leq 2\) and \(\beta_c \approx 1\). The dc component is zero if \(\beta_c = \infty\) (a current-biased junction) or \(\beta_c = 0\) (a voltage-biased junction), because there is no dissipative element to draw power from the supply.

How do the above results, derived for a point junction, apply to interferometers? Well, if \(\chi\) is large (the junctions are tightly coupled), then an interferometer in the voltage state may be thought of simply as a point junction with an effective critical current of \(2I_0 \cos \phi_c\). This is clear from examining the external circuit equation Eq. 2.20; we see that it looks like the equation of a point junction having critical current \(2I_0 \cos \phi_c\). We shall therefore assume that the contribution of the internal resonance (computed previously) and the nonresonant contribution (as computed above, but using an effective critical current \(2I_0 \cos \phi_c\)) may simply be summed to obtain the total gate current.
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This technique for accounting for the effects of a load resistance across the gate is more rigorous than it may have appeared in the preceding discussion. Consider our original resonance calculations. There, we assumed that $\phi_-$ was the important variable for analyzing resonances (it was responsible for the power dissipation), and hence we included terms up through the fundamental: $\phi_- = \phi_c - \delta \sin \Omega \tau$; a first-order approximation. On the other hand, $\phi_+$ was considered to have only secondary importance, so we used a zeroth-order approximation, consisting only of a constant dc voltage: $\phi_+ = \Omega \tau - \theta$. In contrast, the excess gate current observed for low values of $I_c$ in low-$\beta_c$ interferometers is primarily due to the external phase $\phi_+$ dissipating power in the load resistance. Thus $\phi_+$ is the important variable for this phenomenon, so we include terms up through the fundamental frequency: $\phi_+ = \Omega \tau + \delta \sin (\Omega \tau + \theta)$. On the other hand, $\phi_-$ is of secondary importance to this effect and hence only a zeroth-order approximation is used: $\phi_- \approx \phi_c$. Substituting these approximations into the external circuit equation gives the results obtained above (Eqs. 2.48 and 2.49).

When $I_c$ is near $\Phi_0/4L$, the resonant contribution is large and the nonresonant contribution is small. Conversely, when $I_c$ is near zero, the resonant current is small but the nonresonant current can be important. It is fortunate that the resonant current (due to oscillations in $\phi_-$) and the nonresonant current (due to oscillations in $\phi_+$) are never simultaneously large; otherwise we would be obliged to use simultaneous first-order approximations for both $\phi_+$ and $\phi_-$, which would lead to a very complicated set of nonlinear coupled equations.

To verify our theory of resonances in low-$\beta_c$ interferometers, simulations were performed on an interferometer with $\Gamma=1$, $\chi=2$, and $\beta_c = 1$. The gate current $I_g$ due to the junctions is plotted as a function of control current $I_c$ in Fig. 2.14. Note that our original resonance theory would have computed only $I_r$, which fits very poorly to the data. Our new theory, however, gives a nonresonant contribution $I_{nr}$ which when
Figure 2.14
added to $I_r$ gives virtually perfect agreement. Thus we can conclude that we have developed an adequate theory of resonances in interferometers with low-resistance loads ($\beta_c \approx 1$).

Asymmetric interferometers have attracted attention for use in Josephson memory designs. It has been observed that the resonances are lower in asymmetric devices than in symmetric devices, relative to the maximum critical current $I_{m0} \equiv I_{01} + I_{02}$. However, to date no theory has been published which even approximates the resonance amplitude for asymmetric devices. An exhaustive treatment of the problem is exceedingly difficult. For example, a general asymmetric interferometer (two different junctions, with an asymmetric gate feed point) can in principle be analyzed using a sinusoidal approximation for each junction phase:

$$\phi_1 \approx \omega t + \delta_1 \sin(\omega t + \theta_1) + \phi_{01},$$

$$\phi_2 \approx \omega t + \delta_2 \sin(\omega t + \theta_2) + \phi_{02}.$$  

Substituting these into the node equations and collecting Fourier components up through the fundamental frequency, we get seven coupled transcendental equations which provide absolutely no insight into the problem. Rather than pursue this approach, we shall instead develop a very simple, very approximate theory of resonances in current-biased asymmetric interferometers. We do not have a general theory, but our approach does give helpful insights into the problem.

Fig. 2.15 shows a current-biased (no load resistance) asymmetric interferometer. We have neglected the junction resistances $R_j$, and we assume further that $K/L_1I_{01} = K/L_2I_{02} \equiv X$, and that $C_1/I_{01} = C_2/I_{02} \equiv C/I_0$. Under these assumptions, we can get some useful equations by working with the internal and external phases.
Figure 2.15
The node equations are as follows:

**Node 1:** \[ C_1 \ddot{\phi}_1 + \frac{K(\dot{\phi}_1 - \dot{\phi}_2)}{2R_d} + \frac{K(\phi_1 - \phi_2)}{L_1} + I_{01} \sin \phi_1 = I_c. \] (2.50)

**Node 2:** \[ C_2 \ddot{\phi}_2 + \frac{K(\dot{\phi}_2 - \dot{\phi}_1)}{2R_d} + \frac{K(\phi_2 - \phi_1)}{L_2} + I_{02} \sin \phi_2 = -I_c. \] (2.51)

**Node 3:** \[ \frac{K(\phi_g - \phi_1)}{L_1} + \frac{K(\phi_g - \phi_2)}{L_2} = I_g. \] (2.52)

Dividing Eq. 2.50 by 2I_{01} and Eq. 2.51 by 2I_{02}, and then subtracting, we get:

\[ K \frac{C}{I_0} \dddot{\phi}_- + \frac{K(I_{01}^{-1} + I_{02}^{-1})}{2R_d} \ddot{\phi}_- + \chi \phi_- + \sin \phi_- \cos \phi_+ = \frac{\chi I_c}{2} (I_{01}^{-1} + I_{02}^{-1}). \] (2.53)

If we remove the Josephson terms, we have a resonant circuit with a single resonant frequency, as before:

\[ \omega_r = (L_1 C_1)^{-1/2} = (L_2 C_2)^{-1/2}. \]

We again define a normalized time: \( \tau \equiv \omega_r t \). It is also convenient to define the harmonic mean \( \tilde{I}_0 \) of \( I_{01} \) and \( I_{02} \): \( \tilde{I}_0^{-1} = \frac{1}{2} (I_{01}^{-1} + I_{02}^{-1}) \). Then Eq. 2.53 can be rewritten:

\[ \frac{\chi}{\omega_r} \frac{d^2 \phi_-}{d\tau^2} + \gamma \frac{d\phi_-}{d\tau} + \chi \phi_- + \sin \phi_- \cos \phi_+ = I_c / \tilde{I}_0, \]

where \( \gamma = \Gamma^{-1} \equiv \left( \frac{I_0 R_d}{K \omega_r} \right)^{-1} \). Remarkably, this internal circuit equation has the same form as for the symmetric interferometer (Eq. 2.19). The external equation, however, is somewhat different.

Since the internal circuit equation is the same as before, this suggests that for \( V_g = V_r \), we can use the same approximations for \( \phi_- \) and \( \phi_+ \) as before, i.e.
\[ \phi_- = \phi_c - \delta \sin \tau, \text{ and } \phi_+ = \tau. \] Then we must get the same equation for \( \delta \):

\[ [J_0(\delta) + J_2(\delta)]/\delta = (1 - \sin \phi_c)^{-1}, \]

where \( \phi_c = I_c/xI_0 = (L_1 + L_2)/2K \). The resonant current \( I_r \) can be obtained from power conservation:

\[ I_r V_r = \frac{1}{2} \frac{(2\delta V_r)^2}{2R_d}, \text{ so } \]

\[ \frac{I_r}{2I_0} = \frac{\delta^2 V_r}{2I_0 R_d} = \frac{\delta^2}{2\Gamma}. \] (2.54)

Eq. 2.54 is exactly like Eq. 2.11, except that the relevant ratio is now \( I_r/2I_0 \) rather than \( I_r/I_m = I_r/(I_{01} + I_{02}) \). Note that

\[ \frac{2\tilde{I}_0}{I_{m0}} = \frac{2\tilde{I}_0}{I_{01} + I_{02}} = \frac{4}{(1 + I_{01}/I_{02})(1 + I_{02}/I_{01})} \leq 1, \] (2.55)

where \( I_{m0} = I_{01} + I_{02} \) is the maximum possible gate current. We see that the resonance peak amplitude relative to \( I_{m0} \) in a current-biased asymmetric interferometer is predicted to be less than for a symmetric interferometer by the factor \( 4/(1 + I_{01}/I_{02})(1 + I_{02}/I_{01}) \). This simple result may be important for circuit designs which involve asymmetric interferometers.

It turns out that the requirement that \( L_1I_{01} = L_2I_{02} \) is not necessary for Eq. 2.54 to be a good approximation. If the gate current is injected at some other feed point in the loop inductance, the result will be to couple some net flux into the inductance, which changes \( \phi_c \) (the average value of \( \phi_- \)). But there will still be some value of \( I_c \), no longer equal to \( \Phi_0/(L_1 + L_2) \), which will result in \( \phi_c = \pi/2 \) (half a flux-quantum in the inductor), and hence the maximum resonance amplitude is still given by Eq. 2.54.
It should be emphasized that this theory is very dependent on the fact that the device is current-biased ($G_L = 0$), much more so than for symmetric interferometers. For example, if a load resistance is placed across the device at any feed point other than the point where $L_1 I_{01} = L_2 I_{02}$, then the circuit will have two resonant frequencies and things get very complicated. In the extreme case where the device is voltage-biased, we get two distinct resonances at $\omega_1 = (L_1 C_1)^{-1/2}$ and $\omega_2 = (L_2 C_2)^{-1/2}$.

The above theory was checked against simulations of asymmetric interferometers. Fig. 2.16 compares theory with simulations as a function of $\Gamma$ for asymmetry factors $I_{01}/I_{02} = 2$ and $3$ ($\chi = 4$ and $\beta_c > 1000$ for all cases). The agreement is fairly good.

One final issue which we should address is the effect of including extra circuit elements in the interferometer. For example, Fig. 2.17 shows an interferometer with a series inductance $L_d$ in the damping resistor, as often occurs in damped interferometers. Without going through the labor of writing down circuit equations, it is clear what is going to happen when our approximations for $\phi_+$ and $\phi_-$ are substituted into the internal circuit equation. We can rewrite Eqs. 2.33 and 2.34 in a more general form:

$$-\text{Re}[j\Omega Y(\Omega)]\delta + [J_0(\delta) - J_2(\delta)] \sin \theta \sin \phi_c = 0,$$

$$-\text{Im}[j\Omega Y(\Omega)]\delta + [J_0(\delta) + J_2(\delta)] \cos \theta \sin \phi_c = 0,$$

where $Y(\Omega)$ is the normalized admittance of the internal (difference-mode) resonant circuit. For the standard interferometer of Fig. 2.5, this internal admittance is $Y(\Omega) = \gamma + j(\chi \Omega - \chi / \Omega)$; it contains terms corresponding to the interferometer conductance, capacitance, and inductance. For the circuit in Fig. 2.17, which has an additional inductance $L_d$ in series with the damping resistance $R_d$, we would instead
Figure 2.16
Figure 2.17
have

\[ Y(\Omega) = \left[ \gamma^{-1} + (-jx_d/\Omega)^{-1} \right]^{-1} + j(x\Omega - x/\Omega) \] .

The first term has now been modified to include the normalized series inverse inductance \( x_d = K/L_d I_0 \). Similar modifications may be made to account for any other linear circuit elements which would affect the internal (difference-mode) impedance of the interferometer. This technique will be important when we discuss the coupling of resonances out to a transmission line in Chapter 5.

Let us summarize what has been accomplished. Using a very natural change of variables, we have extended the interferometer resonance theories of Zappe and Landman and of Guéret to describe variation in control current and voltage, the harmonic steps, the effects of a low load resistance, the resonant properties of asymmetric current-biased interferometers, and the effects of additional linear circuit elements in the interferometer loop. Our analytical theory will be particularly useful in the design of a microwave multiplexing system, because we can now predict the amplitude \( \delta \) of the internal RF oscillations which will carry the digital information.
Chapter 3 - EXPERIMENTAL TEST OF RESONANCE THEORY

This chapter discusses experiments which were performed in order to test the theory of resonances described in Chapter 2. In a sense the theory has already been verified by the comparisons with simulations in Chapter 2. However, there is no guarantee that the simple model we used in Fig. 2.5 is accurate at the high frequencies (hundreds of GHz) involved. Furthermore, the results of our theory are so novel and interesting that they invite experimental confirmation.

It was not possible to design a Josephson integrated circuit just to study resonances, within the prevailing time constraints. However, J. H. Magerlein of IBM already had such an integrated circuit which he had designed for junction capacitance measurements. The purpose of the test vehicle was to measure the resonant voltage \( V_r \), and hence compute the junction capacitances \( C \) from knowledge of the loop inductance \( 2L \), using the relationship \( V_r = K/\sqrt{LC} \). These circuits were ideally suited for testing the resonance theory, including the shape of the I-V curve and the control-current dependence of resonance amplitudes. The experiments described in this chapter were done in collaboration with Magerlein; highlights of these experiments are included in a publication by Tuckerman and Magerlein.\(^{17}\)

To begin with, we shall briefly describe the contents of the Josephson circuit chip. The chip consisted of a collection of 2-junction interferometers having various junction sizes (from 2.5 \( \mu \)m to 10 \( \mu \)m in diameter) and various loop inductances (from 0.5 pH to 32 pH). Fig. 3.1 is a photomicrograph of a typical interferometer having two 2.5-\( \mu \)m circular junctions and a 2 pH loop inductance. Each interferometer consisted of a Pb-In-Au base electrode and a Pb-Bi counter electrode, fabricated by a process described by Greiner et al.\(^{18}\) The control line was directly connected to the counter electrode.

The devices were fabricated without a ground plane. This was done so that the gate and control lines which connect to the interferometer would have a relatively high characteris-
tic impedance (hundreds of Ohms). This is particularly important for the control line, for its impedance $Z$ is in parallel with $R_d$. If $Z$ is comparable to $R_d$, then a significant amount of RF power due to the resonance will escape into the control line. Furthermore the control line would likely have very complicated resonances of its own which would cause additional resonant structure in the I-V curve. The fact that no such structures were observed is a sign that the control line did indeed have a very high impedance. It is less important that the gate supply have a high impedance, but it is helpful because the theory of resonances is simplest when the gate is current-biased ($G_L = 0$).

The two-junction interferometers are, for the most part, straightforwardly modeled by the circuit of Fig. 2.5. Although the loop inductance is in reality distributed between the two electrodes, lumping it into a single inductance $2L$ produces an equivalent circuit. The "damping resistance" $R_d$ in Fig. 2.5, however, does not correspond to any physical resistor in our experimental interferometers. Rather, $R_d^{-1}$ should be interpreted as the real part of the admittance $Y(\omega)$ of the resonant circuit, when at or near the resonant frequency. This admittance is presumably due to quasiparticle losses in the superconductors which occur at non-zero frequencies.\textsuperscript{19} Confusion may be caused by the fact that superconductor losses are generally expressed as a frequency-dependent "surface resistance" $R_s(\omega)$ in series with the inductance $L$. (Of course $R_s(0) = 0$ because the electrodes are superconducting). Thus the interferometer can be more accurately modeled as in Fig. 3.2. The internal admittance for this model is, neglecting $R_j$,

$$Y(\omega) = j\omega C + [j\omega L + R_s(\omega)]^{-1} = j\omega C + [j\omega L]^{-1} / [1 - jR_s(\omega)/\omega L] \approx$$


Using the definition $Q = \omega L/R_s(\omega_r)$, the real part of the admittance at $\omega = \omega_r$ can be written as

$$\text{Re}[Y(\omega_r)] = R_s(\omega_r)/[\omega_r L]^2 = 1/R_s(\omega)Q^2 = R_d^{-1}.$$
Figure 3.2
Thus we have $R_d = Q^2 R_r(\omega_r)$. Note that $\Gamma = I_0 R_d / V_r$ is therefore dependent on the value of $R_r$, which is not easy to predict. Hence $\Gamma$ will be an unknown parameter in our experiments, which can be determined only by fitting the theory to the data.

We shall now describe the experimental techniques. The Josephson circuits were operated in a liquid helium bath (4.2 K). The devices were connected through a sample holder to standard room-temperature test equipment. We measured threshold curves, which show the device critical current $I_m$ as a function of control current $I_c$. Fig. 3.3 shows typical threshold curves. The relatively flat region between the major lobes of the curves is the peak of the fundamental resonance $I_r$; it is visible only when $I_m < I_r$, because otherwise the device switches over the resonance along its horizontal load line. We generally looked only at interferometers with low $L I_0$ products (large values of $\chi$) so that $I_m$ could be reduced almost to zero when $I_c = \Phi_0 / 4L$. Thus almost the entire resonance $I$-$V$ curve was visible. Fig. 3.4 shows a typical resonance $I$-$V$ curve taken with $I_c = \Phi_0 / 4L$. In this example, the negative-resistance region is clearly visible near the peak.

One of the problems we had with measuring resonance peaks and interferometer critical currents accurately was noise. It is well known that the critical current $I_m$ of a Josephson junction or interferometer is proportional to the amplitude $\Delta I$ of the current step which occurs at the gap voltage. In our junctions, the relationship is $I_m = 0.7 \Delta I$, obtained from measurements of high-$I_m$ junctions.\textsuperscript{20} In order to obtain low $L I_0$ products, however, it was necessary for us to use interferometers with rather low values of $I_m$ (\approx 200 \mu A). For such low-current, hysteretic junctions, it has been observed\textsuperscript{21} that the measured $I_m$ tends to be approximately 10 \mu A less than $0.7 \Delta I$, independent of the value of $I_m$. This effect is generally attributed to noise, although the origin of the noise is unclear. The idea is that a hysteretic junction or interferometer biased very near to $I_m$ will switch into the voltage state simply due to noise currents. Thus the measured $I_m$ will appear to be less than the "true" $I_m$.\textsuperscript{22}
Figure 3.4
The same effect occurs when trying to measure a resonance amplitude. The resonance current will reach roughly 10 \( \mu A \) below its expected amplitude before switching to the full voltage state. **Thus we conclude that a direct measurement of the resonance peak amplitude gives unreliable data.** This is especially true when we are trying to determine \( \Gamma \) from the resonance amplitude when \( \Gamma \approx 3 \). In this region, the peak amplitude is very insensitive to \( \Gamma \) (Fig. 2.4), hence \( \Gamma \) cannot be determined accurately. These problems with noise were generally avoided simply by not using any direct measurement of critical currents or resonance peak heights, whenever possible.

Our first measurements for each interferometer were made in order to characterize the device, i.e. determine what the device parameters are. There are basically 4 parameters which must be determined in order to compare the theory with the experiment: \( \chi, \Gamma, I_0, \) and \( V_r \). We determined \( I_0 \) from the relationship \( I_m = 2I_0 = 0.7\Delta I \), because a direct measurement of \( I_m \) would be affected by noise as discussed earlier. We determined \( L \) from the lobe-lobe separation of the threshold curve;\(^{22}\) a simple correction was required to account for the uncoupled inductance in the lower electrode.\(^{23}\) Knowing \( L \) and \( I_0 \), we immediately obtain \( \chi = K/LI_0 \). As for the remaining two parameters, \( \Gamma \) and \( V_r \), there was no way to directly measure them. \( V_r \) was not directly measurable because the peak of the resonance was cut off by noise). We therefore treated \( \Gamma \) and \( V_r \) as adjustable parameters, choosing them so as to obtain the best agreement between the theoretical and experimental resonance I-V curve.

As discussed previously, our use of large values of \( \chi \) made it easy to obtain I-V curves of the fundamental \( (n=1) \) resonances. Observing the harmonic steps \( (n>1) \) was not nearly so simple. The problem is that if the \( n \)th resonance has a lower amplitude than one of the lower-order resonances, it will be completely bypassed when the device switches out of that preceding resonance. The problem is particularly severe for the \( n=2 \) resonance. We know that this step is maximized when \( I_c = 0 \). Unfortunately when \( I_c = 0 \), the critical current \( I_{m0} \) will
exceed the resonance amplitude. Thus the second harmonic step (or any other even-harmonic step, for that matter) will be invisible.

To circumvent this problem, we have developed a technique which allows such hidden resonances to be displayed. When an I-V curve is traced, one normally uses a current generator which produces a sine wave. Instead, we synthesize a more complex current which goes negative only every other cycle. In the interleaved cycles, the current has a local minimum which is adjusted to be just below the resonance that we want to view, but not so low that the device resets to the zero-voltage state. Then as the current increases, the resonance is displayed on the oscilloscope. Using this technique, we have been able to observe all the harmonics through \( n = 5 \). Fig. 3.5 is a multiple exposure (somewhat overexposed) which shows all the resonances from \( n = 1 \) to \( n = 5 \). The difficulty in viewing these higher resonances without our special technique may explain why they have received hardly any attention in the literature.

Now that the experimental techniques have been described, we shall discuss our results. Numerous I-V curves of the fundamental resonance were photographed and compared with the theory. The unknown parameters \( \Gamma \) and \( V_r \) were adjusted to give the best fit. In all but a few cases, the agreement was excellent. Fig. 3.6 shows typical fits for two different devices: for one of them, \( \Gamma = 2.7 \), and for the other one (the looped one), \( \Gamma = 12.87 \). The estimated measurement errors are approximately equal to the size of the symbols. Note that for both curves, the top 10 \( \mu A \) or so is not observed, presumably because of the noise effect discussed earlier. The amount cut off appears different in the two cases because the currents are normalized to \( I_{m0} \), which was substantially different for the two devices. The quasiparticle current \( V/R_j \) was of course subtracted before plotting the data.

We also investigated the control-current dependence of the various harmonics (at \( V = n V_r \)). Here we measured only peak heights, hence the presumed effects of noise were a problem in our measurements. We have assumed that the effects of noise can be accounted
Figure 3.6
for by adding 10 $\mu$A to all of our measurements. Fig. 3.7 compares theory with experiment for two devices having different values of $\Gamma$; for the larger $\Gamma$, both $n=1$ and $n=2$ are plotted. $\Gamma$ was determined independently from I-V curve measurements, hence there are no adjustable parameters in this figure! In view of this, the agreement between theory and experiment is remarkable.

For devices with large junctions and small loop inductances, the critical current $I_0$ of the individual junctions was reduced by the control current. This effect had to be corrected for before comparing with the theory. This was done by measuring $I_m(I_c)$ at $I_c=0$ and at $I_c=2(\Phi_0/4L)$, where $\Phi_0/4L$ was determined by locating the minimum of $I_m$ in the threshold curve. We then assumed that

$$I_0(I_c) = I_0 \frac{\sin I_c/I_{cl}}{I_c/I_{cl}}$$

(3.1)

in accordance with the usual "diffraction pattern" dependence of the critical current, which is valid for junctions which are short compared to the Josephson penetration depth $\lambda_J = \sqrt{\lambda/\mu d j_1}$. ($d$ is the barrier thickness plus the sum of the electrode penetration depths, and $j_1$ is the junction current density). The value of $I_{cl}$ was determined by requiring that Eq. 3.1 hold at $I_c=2(\Phi_0/4L)$. Eq. 3.1 was derived for rectangular junctions rather than circular ones, but the error is not likely to be large since we rarely exceeded a 10% reduction in $I_0$, so that $I_0(I_c)$ is well approximated by a parabola. This procedure works quite well, as is indicated by Fig. 3.8, in which theory and experiment are compared for an interferometer which has this effect. Whereas the agreement is initially poor without the correction (upper curve), it becomes excellent after accounting for the suppression of $I_m$ (lower curve).

As a final set of experiments, we varied the damping ($\Gamma$) of an interferometer by raising the operating temperature $T$. The idea here is that as the circuit temperature is raised above 4.2 K, the superconductors will become increasingly lossy at high frequencies. In fact the surface resistance $R_s(\omega)$ approaches that of a normal metal as $T$ approaches $T_c$. We were
Figure 3.8
not attempting to predict $\Gamma$ as a function of $T$, but rather we just wanted to measure the qualitative effect on the resonance of decreasing $\Gamma$.

Fig. 3.9 shows experimental resonance I-V curves for a sequence of temperatures from 4.2 $K$ to about 6.4 $K$. The initial value of $\Gamma$ is 12.9, and it becomes steadily lower. Comparing with Fig. 2.4, we see that the resonance peak amplitude $I_r$ has the correct qualitative dependence on damping. That is, the resonance amplitude increases as $\Gamma$ is reduced, reaches a maximum of about $0.6 \times I_m$ when $\Gamma \approx 3$, and then begins to decrease again as $\Gamma \approx 0$. Furthermore the broadness of the resonance I-V curves clearly show that $Q \approx \chi \Gamma$ is decreasing. Fig. 3.9 also shows a comparison of each I-V curve with the theory; $\Gamma$ and $V_r$ were determined by adjusting for the best fit. The agreement is in all cases excellent. The estimated experimental errors are $\pm 1$ $\mu$A and $\pm 1$ $\mu$V for $I_r$ and $V_r$, respectively. The experimental errors are simply our estimate of the accuracy with which we can read the oscilloscope photographs.

It should be noted that $R_j$ will decrease with increasing temperature, which will contribute to the decline in $\Gamma$. But the decrease in $R_d$ (or increase in surface resistance $R_s$) is the dominant cause of the increased damping.

In addition to verifying the resonance theory, our experimental work has shown that it is rather easy to obtain $\Gamma$ and $V_r$ by making a two-parameter fit to the experimental resonance I-V curves. This capability is likely to be very important to future experimenters. $V_r$ is an important quantity because the junction capacitance $C$ can be directly deduced from it; there is no other reliable method for finding $C$. It is clear that simply measuring the voltage at the peak of the resonance, as had been done in the past, does not give an accurate value of $V_r$. For example, in the case of a low-$\chi$, high-$\Gamma$ interferometer, the looped I-V curve could reach a voltage of $\approx 1.1V_r$ before bending back to $V_r$ (Fig. 2.9). If the device has a low critical current $I_m$, then the backward-bending, negative-resistance region might be completely cut off by the 10 $\mu$A of noise. Thus the experimenter might erroneously conclude that $V_r$ is 10%
Figure 3.9
higher than it really is, and hence that C is 20% lower than it really is. This problem does not occur if one uses our new theory to fit to the entire I-V curve.

Γ is also an important quantity to know, for it gives information on the high-frequency losses (surface resistance) in the superconductors which comprise the interferometer loop. To give an example of the kind of information that can be obtained, Fig. 3.10 shows Q as a function of temperature for the sequence in Fig. 3.9. A careful comparison of this data with theory has not been made, mainly because it is not clear what theory of surface resistance is appropriate for our superconductors.

It should be mentioned that under certain conditions (larger values of V_r and n), significant deviations from our theory were observed. We can attribute these effects as being due to the nonlinearity of R_j, the quasiparticle tunnelling resistance, which was not accounted for in our model. Recall that the quasiparticle current rises very sharply near the gap voltage \( V_g \approx 2.8 \text{ mV} \). (Furthermore the current is really not linearly dependent on voltage even below \( V_g \).) When a device is biased at voltage \( nV_r \), we see from our theory that the voltage across each junction is

\[
V_j(t) = nV_r \pm K\omega \delta \cos \omega t = V_r(n \pm \delta \cos \omega t).
\]

Thus the magnitude of the instantaneous voltage can far exceed the average voltage \( nV_r \), reaching as much as \( (n + \delta)V_r \). If this voltage is close to or greater than \( V_g \), we expect our theory to be inaccurate. As an example, suppose \( n=3 \); then for large \( \Gamma \), \( \delta \approx 6.4 \), so \( V_j(t) \) can reach \( 9.4V_r \). If \( V_r > 0.3 \text{ mV} \), as is often the case, \( 9.4V_r \) will exceed the gap voltage of 2.8 mV. Even the fundamental resonance can do this: if \( n=1 \), \( \delta \) can be as large as 3.8, whence if \( V > 0.58 \text{ mV} \) the oscillations will reach the gap in a high-\( \Gamma \) device.

What are the expected consequences of the nonlinearity in \( R_j \)? We expect that it would result in more damping (lower \( \Gamma \)) for the larger values of \( \delta \). This corresponds to a higher resonance, in accordance with the formula \( I_r/I_{mo} = \delta^2/2\Gamma \). Thus we might expect the
Figure 3.10
base of the resonance I-V curve to appear correct (δ is small here), but the peak of the resonance will be higher than expected. This effect has been observed in some of our experimental I-V curves, and has also been seen in simulations.

For the same reason, deviations from the theoretical control-current dependence have been observed. Fig. 3.11 shows a few data points for the control-current dependence of a 3rd harmonic resonance. That data are compared with theory in the figure, using the value of \( \Gamma \) determined from I-V curve measurements of the \( n=1 \) resonance. The result is that the points are too high, but approach the theoretical value as \( I_c \rightarrow 0 \). This makes sense because \( \delta \) is smaller for the lower control currents, hence the resonance should see a more linear \( R_j \) than for the higher control currents.

It should be noted that the deviations from theory due to the nonlinear \( R_j \) are not as large as might be expected, because \( R_d \) is also an important contributor to the damping. Unlike \( R_j \), it is presumably linear. In fact the damping due to \( R_j \) was always less than that due to \( R_d \) in our experiments, and these were "undamped" interferometers. If we were to include an external damping resistor, the relative contribution of \( R_j \) would become negligible. Similarly, in the design of a microwave multiplexing system, we would want most of the resonance power to couple out to a transmission line of impedance \( Z_0 \). We would therefore design so that \( Z_0 \ll R_j \). Thus the deviations from theory discussed above will not be important in most practical situations.

Let us summarize what has been accomplished. The resonance theory of Chapter 2 was tested experimentally and was found to agree very well with the theory. In the few instances where deviations were found, they were satisfactorily explained by the nonlinearity of \( R_j \), which had been assumed to be linear in the theory. Accurate fits to the I-V curves and verification of the control-current dependence of the various resonances were obtained. The theory is likely to be very useful for determining the interferometer parameters \( V_r \) and \( \Gamma \);
Figure 3.11
these are needed for calculating the junction capacitance $C$ and the superconductor surface resistance $R_s$, respectively.
Chapter 4 - RECEIVER THEORY

In this chapter we discuss the theory of the proposed microwave receiver. To begin with, let us briefly review the prior art pertaining to the detection of microwave radiation using Josephson devices. The idea dates back to the work of Shapiro\textsuperscript{24}, who demonstrated that the I-V curve of a nonhysteretic resistively-shunted Josephson junction, irradiated with microwave power at frequency $\omega$, exhibits current steps at integral multiples of the voltage $V = \hbar \omega / 2e = K \omega$.

Here is a simple explanation of this effect, commonly known as "Shapiro Steps". Fig. 4.1 shows a junction biased with a DC voltage source $V_0 = n K \omega$, and with an RF signal $V_{RF} = V_1 \cos \omega t$ impressed on it, which we also model as a voltage source. The junction phase will be

$$\phi = \int \frac{V}{K} dt = \phi_0 + n \omega t + \left( \frac{V_1}{K \omega} \right) \sin \omega t.$$ 

Thus the junction current is

$$I_J = I_0 \sin \left[ \phi_0 + n \omega t + \delta \sin \omega t \right] =$$

$$I_0 \left( \sin \left[ \phi_0 + n \omega t \right] \cos \left[ \delta \sin \omega t \right] + \cos \left[ \phi_0 + n \omega t \right] \sin \left[ \delta \sin \omega t \right] \right),$$

where $\delta \equiv V_1 / K \omega$. We are only interested in the dc component of $I_J$, which is

$$\{I_J \}_d = \left( -1 \right)^n I_0 I_n(\delta) \sin \phi_0.$$

Since any value of $\phi_0$ can be achieved, we can have the dc component of the junction current as large as $\pm I_0 I_n(\delta)$. In practice the RF generator and the dc bias voltage have a finite impedance (they are not perfect voltage sources), so these equations do not strictly apply. But the junction will nonetheless "lock on" to the applied frequency and exhibit current steps in the I-V curve with peak-to-peak extent of $2I_0 I_n(\delta)$. Fig. 4.2 shows a typical I-V curve for a resistively-shunted junction irradiated with microwaves.
In 1966, Langenberg and Scalapino\textsuperscript{11} pointed out that such "Shapiro steps" could be useful for detection of RF radiation. In particular, if we are biased near $V = K\omega$, the junction current will respond linearly to the applied RF, because $J_1(\delta) \approx \frac{1}{2} \delta$ is approximately linear for small $\delta$.

Another way in which a junction could be used as a detector is to bias it in the zero-voltage state, near the critical current $I_0$. Then application of an RF voltage of amplitude $V_1 = K\omega\delta$ will reduce the critical current to $I_0 J_0(\delta) < I_0$, which will result in the junction switching to the voltage state. It is not necessary that the junction be nonhysteretic for this application.

In 1979, Calander and Zappe\textsuperscript{1} proposed a microwave detector which works on this principle of critical-current reduction, except that they coupled the junction to a resonant circuit which shorts out all incident frequencies that are not within its passband. The particular structure which they proposed was a two-junction interferometer (Fig. 4.3). The idea here is that an RF signal at the resonant frequency $\omega_r$, coupled in as a control current, will reduce the critical current $I_m$. Thus if the device is initially biased near the critical current, the RF control current will cause it to switch, transferring current into the load resistance $R_L$. Although Calander and Zappe stated a formula for the receiver sensitivity, we shall see that their formula is valid only for $\Gamma << 1$, which is not a very useful operating region. They also mentioned an "asymmetry" which occurred in analog computations\textsuperscript{25} of the receiver threshold curve. Actually this effect was probably caused by the onset of an instability in the receiver, which we shall discuss later.

We shall now analyze the frequency response of the receiver, i.e calculate $I_m$ as a function of $\omega$ and of RF input $I_{co}$. We shall use the same analytical techniques which proved so successful in our analysis of the transmitter (Chapters 2 and 3). We begin by recalling the
\[ I_c = I_{co} \sin \omega t \]

Figure 4.3
normalized interferometer circuit equations:

\[
\chi \frac{d^2 \phi_-}{d\tau^2} + \gamma \frac{d\phi_-}{d\tau} + \chi \phi_- + \sin \phi_- \cos \phi_+ = \frac{I_c}{I_0}, \tag{4.1}
\]

\[
\chi \frac{d^2 \phi_+}{d\tau^2} + \gamma_+ \frac{d\phi_+}{d\tau} + \sin \phi_+ \cos \phi_- = \frac{I_g}{2I_0}. \tag{4.2}
\]

These equations are identical to the internal and external equations of Chapter 2 (Eqs. 2.20 and 2.21), except that we have included for the first time the load resistance \( R_L = G_L^{-1} \).

This appears in the term \( \gamma_+ = (\gamma G_L + G_J)V_r/I_0 \). The load resistance is included now because it will become important in our later discussion of receiver instability. For our analysis, we shall assume that the control current is a pure sinusoid, i.e. \( I_c(\tau) = I_{co} \cos \Omega \tau \).

We shall begin by assuming certain forms for \( \phi_+ \) and \( \phi_- \) when the interferometer is in the zero-voltage state. It is clear that \( \phi_- \) must have a sinusoidal component, in response to the oscillating control current. The amplitude and phase of \( \phi_- \) relative to \( I_c(\tau) \) are unknown quantities. Thus a logical assumption is

\[
\phi_- = \delta \sin (\Omega \tau + \theta). \tag{4.3}
\]

If the Josephson term \( \sin \phi_- \cos \phi_+ \) were removed, Eq. 4.1 would reduce to an externally driven damped harmonic oscillator, and Eq. 4.3 would be the exact, general solution to the problem. We shall assume that it is a good approximation even in the presence of the nonlinear Josephson term. As for \( \phi_+ \), the simplest assumption we can make is to approximate it by a constant:

\[
\phi_+ = \phi_g. \tag{4.4}
\]

\( \phi_g \) will depend on the gate current and on the magnitude and frequency of the control current. Substituting our approximations for \( \phi_+ \) and \( \phi_- \) (Eqs. 4.3 and 4.4) into Eq. 4.1, we get:

\[
\chi(1-\Omega^2)\delta \sin (\Omega \tau + \theta) + \gamma \Omega \delta \cos (\Omega \tau + \theta) + \sin \left[ \delta \sin (\Omega \tau + \theta) \right] \cos \phi_g \approx \frac{I_{co}}{I_0} \cos \Omega \tau.
\]
The fundamental frequency component of the Josephson term is $2J_1(\delta) \sin (\Omega \tau + \theta)$. Collecting $\sin (\Omega \tau + \theta)$ and $\cos (\Omega \tau + \theta)$ terms separately, we get:

$$\chi(1-\Omega^2)\delta + 2J_1(\delta) \cos \phi_g = \frac{I_{co}}{I_0} \sin \theta ,$$  \hspace{1cm} (4.5)

$$\gamma \Omega \delta = \frac{I_{co}}{I_0} \cos \theta .$$  \hspace{1cm} (4.6)

Substituting our approximations for $\phi_-$ and $\phi_+$ into Eq. 4.2, we get:

$$\sin \phi_g \cos [\delta \sin (\Omega \tau + \theta)] = \frac{I_g}{2I_0} .$$

Clearly only the dc component of this equation can be self-consistent, and this gives:

$$J_0(\delta) \sin \phi_g = \frac{I_g}{2I_0} .$$  \hspace{1cm} (4.7)

Eqs. 4.5, 4.6, and 4.7 must be solved for the three unknowns $\delta, \phi_g$, and $\theta$; the critical current $I_m$ is defined as the maximum value of $I_g$ for which solutions exist to these equations. In general the determination of $I_m$ must be done numerically. However, it is educational to first consider the case where $\phi_g = \pi/2$. Then the term $2J_1(\delta) \cos \phi_g$ in Eq. 4.5 vanishes, and we get:

$$\frac{I_g}{2I_0} = J_0(\delta) = J_0 \left[ \frac{I_{co}/I_0}{\{\chi(1-\Omega^2)^2 + [\gamma \Omega]^2\}^{1/2}} \right] .$$  \hspace{1cm} (4.8)

This is the formula which Calander and Zappe\(^1\) stated (without proof) in their work, except that they claimed that this formula gives the maximum gate current $I_m/2I_0$. Actually, $I_m/2I_0$ is generally larger than this value. Eq. 4.8 is a valid solution to Eqs. 4.5-4.7, but there exist solutions for larger gate currents $I_g$. Thus Calander and Zappe's formula overestimates the receiver sensitivity.

Fig. 4.4 shows the theoretical $I_m$ (solid line) as a function of $\Omega$ for various values of $\Gamma$ and $I_{co}/I_0$, as determined from numerical solutions of Eqs. 4.5-4.7 and maximization of $I_g$. \(\chi\)
was chosen so that $Q=8$ in all cases. The Calander-Zappe formula (dashed lines) is also shown on these graphs. We see that their formula agrees well with ours only for low-$\Gamma$ devices ($\Gamma \leq 1$), which is not the region in which we are likely to be designing.

Note that the sensitivity of the receiver can also be described in terms of the maximum RF amplitude $I_{c0}$ which the receiver can withstand without switching to the voltage state, when biased with a constant gate current $I_g$. That is, we can talk about the maximum RF control current for a given gate current, rather than talking about the maximum gate current for a given RF control current. Although the latter approach was used above, the former approach is equivalent and is more convenient to use in the subsequent theory.

Let us now see how well our theory compares with computer simulations. The simulations were performed by biasing the receiver with a constant gate current $I_g$, then applying an RF control current $I_c(\tau) = I_{c0} \sin \Omega \tau$. The amplitude $I_{c0}$ of this RF was slowly increased until the device switched, at which point we have found the maximum control current amplitude $I_{c0}$ for the frequency $\Omega$. Fig. 4.5 shows the results of these simulations for a current-biased ($G_L = 0$) receiver with $\Gamma = 2$, $\chi = 4$, and $I_g/I_{m0} = 0.7$. It is a plot of $I_{c0}$ as a function of $\Omega$. We see that although the agreement is very good in certain regions, there is a large band of normalized frequencies (in this example, $1.0 \leq \Omega \leq 1.4$) for which the simulated maximum RF current appears lower than predicted. This discrepancy between theory and simulations warrants a closer investigation. Fig. 4.6 shows what a simulation of $\phi_+$ looks like for one of these anomalous frequencies. The RF control current amplitude has been ramped up and leveled off at a value which is greater than the simulated maximum current, but less than the theoretical current. We see that the receiver first appears to be in a steady state. After a while, however, an instability manifests itself. $\phi_+$, which is normally constant, begins to oscillate like a growing sinusoid. This is a clear indication that the interferometer is not dynamically stable in the zero-voltage state. Although our assumed solutions (Eqs. 4.3 and 4.4) are generally quite accurate, they evidently do not represent stable solutions.
Figure 4.4a

Figure 4.4b
Gamma=2, Q=8, Ig/I_{mo}=.7

Figure 4.5
It is not obvious at first that this instability is a bad thing. After all, according to Fig. 4.5, the receiver appears to be more sensitive (switching at a lower control current) than expected! The problem is, we cannot count on using the instability to switch the receiver in a well-defined time interval. Whereas exceeding the theoretical control current amplitude \( I_{c0} \) will switch the device almost immediately, one can wait for many cycles before the receiver will switch due to the instability if the theoretical threshold has not been exceeded, as in Fig. 4.6. The exact time required to switch will depend on the amount of noise present, initial conditions, etc. This is not a satisfactory way to design a high speed receiver. We therefore must use the higher, theoretical value of \( I_{c0} \) for designing a receiver which is tuned to the signal frequency \( \omega_r \). We cannot take advantage of the receiver's "extra sensitivity" near \( \Omega_r \). Furthermore, the instability degrades the receiver's discrimination, or selectivity. If a channel at frequency \( \Omega = 1.2\Omega_r \) is transmitting, the instability may appear, causing the receiver to switch even though the RF signal is below the theoretical threshold. Thus the instability is harmful; it degrades the receiver's discrimination and does not increase its sensitivity in any useful way.

In view of the above considerations, it seems that we should theoretically investigate the cause of the instability, so that we can then attempt to eliminate it. Our theoretical approach will be to assume that our approximate solutions for \( \phi_- \) and \( \phi_+ \) (Eqs. 4.3 and 4.4) are good "unperturbed" solutions, but that they may not be stable. We shall therefore examine the dynamic behavior of the interferometer in response to perturbations from these solutions. By linearizing the circuit equations and examining the natural frequencies, we can then determine whether or not the receiver is stable under the assumed conditions.

We begin by defining small perturbations \( d\phi_- (\tau) \) and \( d\phi_+ (\tau) \). Specifically, we have:

\[
\phi_- \approx \delta \sin (\Omega \tau + \theta) + d\phi_-
\]

\[
\phi_+ \approx \phi_e + d\phi_+
\]
We assume that $\delta, \phi_g$, and $\theta$ are chosen so that Eq. 4.1 is self-consistent up to the first harmonic, and Eq. 4.2 is self-consistent in the dc term, as we discussed before. We are particularly interested in the effects of $d\phi_-$ and $d\phi_+$ on the nonlinear terms of Eqs. 4.1 and 4.2. Taking the differential of $\sin\phi_- \cos\phi_+$ gives:

$$
\frac{d}{d\tau} (\sin\phi_- \cos\phi_+) = -\sin\phi_- \sin\phi_+ d\phi_+ + \cos\phi_- \cos\phi_+ d\phi_- = 
$$

$$
- \sin[\delta \sin(\Omega \tau + \theta)] \sin\phi_g d\phi_+ + \cos[\delta \sin(\Omega \tau + \theta)] \cos\phi_g d\phi_- = 
$$

$$
- 2J_1(\delta) \sin\phi_g \sin(\Omega \tau + \theta) d\phi_+ + J_0(\delta) \cos\phi_g d\phi_- .
$$

Note that whereas $d\phi_-$ has a constant coefficient, $d\phi_+$ has a time-varying (sinusoidal) coefficient. We similarly differentiate $\sin\phi_+ \cos\phi_-$:

$$
\frac{d}{d\tau} (\sin\phi_+ \cos\phi_-) = -\sin\phi_+ \sin\phi_- d\phi_- + \cos\phi_+ \cos\phi_- d\phi_+ = 
$$

$$
- \sin\phi_g \sin[\delta \sin(\Omega \tau + \theta)] d\phi_- + \cos\phi_g \cos[\delta \sin(\Omega \tau + \theta)] d\phi_+ = 
$$

$$
- 2J_1(\delta) \sin\phi_g \sin(\Omega \tau + \theta) d\phi_- + J_0(\delta) \cos\phi_g d\phi_+ .
$$

It is convenient to define the constants $\alpha \equiv J_0(\delta) \cos\phi_g$ and $\beta \equiv J_1(\delta) \sin\phi_g$. $\alpha$ is simply the inverse of the "Josephson inductance", whereas $\beta$ is a coupling coefficient which provides the key to understanding the instability. It is also convenient to set $\theta = \pi/2$. This can be done without loss of generality because $\theta$ is only defined relative to the control current $I_c$, which will not appear in the linearized circuit equations. Taking the differential of Eq. 4.1, we then have:

$$
\frac{d^2(d\phi_-)}{d\tau^2} + \frac{d(d\phi_-)}{d\tau} + (\chi + \alpha) d\phi_- - 2\beta \cos\Omega \tau d\phi_+ = 0 .
$$

Taking the differential of Eq. 4.2, we have:

$$
\frac{d^2(d\phi_+)}{d\tau^2} + \frac{d(d\phi_+)}{d\tau} + \alpha d\phi_+ - 2\beta \cos\Omega \tau d\phi_- = 0.
$$
Eqs. 4.9 and 4.10 are a pair of homogeneous coupled linear differential equations in $d_{\phi-}$ and $d_{\phi+}$, having time-varying coefficients. We wish to examine the stability of their solutions.

We must make some assumptions about the form of $d_{\phi-}(\tau)$ and $d_{\phi+}(\tau)$ in order to proceed with a stability analysis. Judging from the simulations of Fig. 4.6, it seems likely that $d_{\phi+}$ can be approximated by a single complex exponential function of time. That is, we assume

$$d_{\phi+} = \text{Re}[\tilde{\phi}_+ e^{s\tau}], \quad (4.11)$$

where $s$ and $\tilde{\phi}_+$ are complex numbers. $d_{\phi-}$, on the other hand, appears to have a sinusoidal oscillation $\sin \Omega \tau$, modulated by $e^{s\tau}$. Thus we assume

$$d_{\phi-} = \text{Re}[\tilde{\phi}_+^+ e^{(s+j\Omega)\tau} + \tilde{\phi}_-^+ e^{(s-j\Omega)\tau}], \quad (4.12)$$

where $\tilde{\phi}_+^+$ and $\tilde{\phi}_-^+$ are unknown complex coefficients. It should be noted that the simultaneous use of several different complex frequencies (in this case, $s$ and $s \pm j\Omega$) is a standard technique in the analysis of nonlinear microwave systems such as parametric amplifiers. We will want to substitute our formulas for $d_{\phi-}$ and $d_{\phi+}$ into Eqs. 4.9 and 4.10. In order to do this, we need to calculate $(\cos \Omega \tau) d_{\phi+}$ and $(\cos \Omega \tau) d_{\phi-}$:

$$\cos \Omega \tau \ d_{\phi+} = \text{Re}[\tilde{\phi}_+^+ e^{s\tau}] \text{Re}[e^{j\Omega \tau}] = \frac{1}{2} \text{Re}[\tilde{\phi}_+^+ (e^{(s+j\Omega)\tau} + e^{(s-j\Omega)\tau})],$$

$$\cos \Omega \tau \ d_{\phi-} = \text{Re}[\tilde{\phi}_-^+ e^{(s+j\Omega)\tau} + \tilde{\phi}_-^- e^{(s-j\Omega)\tau}] \text{Re}[e^{j\Omega \tau}] = \frac{1}{2} \text{Re}[\tilde{\phi}_-^+ + \tilde{\phi}_-^-] e^{s\tau}].$$

We have neglected terms in the latter equation which would give second harmonics of $\Omega$, i.e. terms containing $e^{(s \pm 2j\Omega)\tau}$. We can now substitute Eqs. 4.11 and 4.12 into the linearized internal circuit equation (Eq. 4.9). After doing this and collecting terms in $e^{(s+j\Omega)\tau}$ and
\[ e^{(s-j\Omega)\tau}, \text{ we get:} \]
\[
[(\chi + \alpha) + \gamma(s + j\Omega) + \chi(s + j\Omega)^2] \tilde{\phi}_- e^{(s+j\Omega)\tau} - \beta \tilde{\phi}_+ e^{(s+j\Omega)\tau} = 0 ,
\]
\[
[(\chi + \alpha) + \gamma(s-j\Omega) + \chi(s-j\Omega)^2] \tilde{\phi}_- e^{(s-j\Omega)\tau} - \beta \tilde{\phi}_+ e^{(s-j\Omega)\tau} = 0 .
\]

Similarly, we can substitute into the linearized external circuit equation (Eq. 4.10) and collect terms in \(e^{\tau}\) :
\[
(\alpha + \gamma_+ s + \chi s^2) \tilde{\phi}_+ e^{\tau} - \beta (\tilde{\phi}_+ + \tilde{\phi}_-) e^{\tau} = 0 .
\]

These equations are conveniently expressed in matrix form:
\[
M \tilde{\phi} = \begin{bmatrix}
\mathcal{Y}_-(s+j\Omega) & 0 & -\beta & \tilde{\phi}_+ \\
0 & \mathcal{Y}_-(s-j\Omega) & -\beta & \tilde{\phi}_- \\
-\beta & -\beta & \mathcal{Y}_+(s) & \tilde{\phi}_+
\end{bmatrix} = 0
\quad (4.13)
\]
where we have defined
\[
\mathcal{Y}_-(s) \equiv \chi s^2 + \gamma s + (\chi + \alpha) \quad \text{and} \quad \mathcal{Y}_+(s) \equiv \chi s^2 + \gamma_+ s + \alpha .
\]

\(\mathcal{Y}_-(s)/s\) and \(\mathcal{Y}_+(s)/s\) are essentially the normalized internal and external admittances of the interferometer. \(\alpha = J_0(\delta) \cos \phi_g\) represents the inverse Josephson inductance; \(\beta = J_1(\delta) \sin \phi_g\) is a parameter which represents the amount of coupling between the internal and external phases due to the junctions. Note that this coupling appears as a mixing, or parametric term: a perturbation of \(\phi_+\) is effectively multiplied by \(\beta \cos \Omega \tau\) and then perturbs \(\phi_-\), and vice versa.

The possible values of \(s\) can be determined by setting the determinant of the admittance matrix \(M\) in Eq. 4.13 to zero. If there are no internal oscillations \((\delta=0)\), or there is no applied gate current \((\phi_g=0)\), then \(\beta=0\) and \(M\) has no off-diagonal terms. The internal and external modes are then decoupled, and we have separate equations \(\mathcal{Y}_-(s+j\Omega) = 0\),
\[ \mathcal{V}_-(s-j\Omega) = 0 \text{, or } \mathcal{V}_+(s) = 0 . \] But since \( \mathcal{V}_+ \) and \( \mathcal{V}_- \) are simply the admitances of passive, linear RLC circuits, they must have only stable natural frequencies, i.e. \( \text{Re}[s]<0 \). Thus no instability can exist. We therefore conclude that any receiver instability must be due to the coupling \( \beta \) between the internal and external modes of the interferometer. This coupling, caused by the Josephson junctions, can only be present when \( \delta \neq 0 \) and \( \phi_g \neq 0 \); that is, when the receiver has a dc gate current applied and is oscillating in response to an RF control signal.

In order to test our theory, we need a technique for calculating the zeros of the determinant of the matrix \( M \). This determinant is:

\[ \mathcal{P}(s) \equiv \det M = \mathcal{V}_-(s+j\Omega) \mathcal{V}_-(s-j\Omega) \mathcal{V}_+(s) - \beta^2(\mathcal{V}_-(s+j\Omega) + \mathcal{V}_-(s-j\Omega)) = 0 . \]

This is clearly going to result in a 6th degree polynomial equation \( \mathcal{P}(s) = 0 \), where \( \mathcal{P}(s) \) has complex coefficients. We want to know whether \( \mathcal{P}(s) \) has any zeros in the right half of the \( s \)-plane. It is a long and tedious numerical computation to find all the complex zeros of \( \mathcal{P}(s) \). Instead, we would like to use a simple stability test, like the well-known Routh-Hurwitz criterion for finding the zeros of real polynomials. It turns out that there exists a little-known extension of the Routh-Hurwitz test which works on polynomials with complex coefficients.\(^{26}\) For the reader's benefit, we shall briefly describe this test (the source from which we are paraphrasing was in German).

Let \( f(s) \) be a polynomial with complex coefficients. Let \( s = -ih \), then multiply \( f(s) \) by a complex constant so that the highest power of \( h \) is real. Denote the real part of the resultant polynomial by \( f_0(h) \), and the imaginary part by \( f_1(h) \). We then divide \( f_0(h) \) by \(-f_1(h)\) if \( f_1(h) \) has the same degree as \( f_0(h) \), or by \(-hf_1(h)\) if \( f_1(h) \) is one lower degree than \( f_0(h) \). The quotient will be a real number (denoted \( q_0 \)), and the remainder polynomial will be denoted \( f_2(h) \). Now we divide \( f_1(h) \) by \(-f_2(h)\) or \(-hf_2(h)\) to obtain the quotient \( q_1 \), and the remainder is denoted \( f_3(h) \). This process is repeated until \( f_n(h) \) is a constant (degree zero). Then the signs of the real parts of the zeros of \( f(s) \) are in one-to-one correspondence with the
signs of the quotients of even index \((q_0, q_2, q_4, \text{ etc.})\) In particular, if all the \(q_{2i}\) are negative, then \(f(s)\) is the characteristic equation of a stable system.

This extended Routh-Hurwitz stability test has been programmed on a computer in conjunction with the receiver analysis. The idea is first to solve for \(\delta, \theta, \text{ and } \phi_g\), given \(I_g\) and \(I_{c0}\). We then test each solution for stability to find the largest value of \(I_{c0}\) for which stable solutions exist. This usually results in a lower RF amplitude than before. Fig. 4.5 compares the results of this procedure with simulations. We see that the agreement is quite good, especially in view of the approximations and linearizations which were required to obtain this simple stability theory. All the qualitative features are correct, and the quantitative discrepancies are small. We can therefore conclude that we have a satisfactory theory for the receiver instability.

Let us briefly review the physical nature of the receiver instability, so that we can find a way to reduce it. The instability arises from the coupling between two resonant cavities: the internal interferometer resonance \((\phi_-)\) at frequency \(\omega_r = 1/\sqrt{LC}\), and the external "plasma" resonance \((\phi_+)\) at frequency \(\omega_p \approx 1/\sqrt{KC/I_0\alpha}\). Whereas the internal resonance is primarily due to the loop inductance and the junction capacitances, the external resonance is due to the Josephson junction inductance \(L_J/\alpha = K/I_0\alpha\) and the junction capacitances. These two resonant modes are initially uncoupled, and hence are both stable in the zero-voltage state, although they may both be high-\(Q\) (poles near the \(j\Omega\) axis). However, when an RF control current \(I_c = I_{c0}\cos \Omega \tau\) is applied, the junctions behave like mixers or parametric amplifiers with respect to perturbations, and they couple the internal and external modes together. A perturbation in \(\phi_-\) will be mixed with \(\cos \Omega \tau\) to perturb \(\phi_+\), and vice versa. If the coupling parameter \(\beta = J_1(\delta) \sin \phi_g\) is sufficiently large, the interaction can counteract the loss terms in Eqs. 4.1 and 4.2 \((\gamma \text{ and } \gamma_+)\), causing the poles to move over from the left half-plane to the right half-plane. In some sense the receiver is behaving like a parametric amplifier which is pumped into a region of instability by the RF control current.
Now that we understand the receiver instability theoretically and to some extent intuitively, we can suggest a way to reduce it. We could damp the device resonances heavily enough so that the poles cannot get over to the right half-plane, even with a significant coupling factor $\beta$. While it is not desirable to damp the internal resonance (this degrades the selectivity of the receiver), it is certainly feasible to damp the external "plasma" resonance by making the load resistance $R_L$ small. This makes the plasma resonance quality factor $Q = \sqrt{\beta_c}$ low, where $\beta_c = (2I_0)R_L^2/(2C)/K$ is the usual definition of the McCumber $\beta_c$. Fig. 4.7 shows a plot of the maximum RF control current $I_{co}(\Omega)$ (using the stability analysis) for various values of $\beta_c$. Clearly, reducing $\beta_c$ helps considerably, and we recommend doing this in the microwave receiver design. (It does not completely eliminate the instability, however). One must note that if $\beta_c \leq 1$, the receiver will be nonhysteretic ("nonlatching"). The only other way we know of to reduce the range of instability is to design with $\Gamma$ small (and $\chi$ large, in order to get a satisfactory value of $Q = \chi \Gamma$). However, this unfortunately is inconsistent with the requirement that the transmitter-receiver combination have "gain", as we shall see later.

Let us summarize what has been accomplished. Using analytical techniques similar to those of Chapter 2, we have developed an approximate theory for the Josephson microwave receiver proposed by Calander and Zappe. Calander and Zappe's formula for the receiver sensitivity was found to be accurate only for $\Gamma \leq 1$. Our theory agrees well with simulations except in certain frequency ranges, where an instability appears which is detrimental to the receiver operation. An approximate theory has been developed to explain this instability; it too agrees well with simulations. The deleterious effects of the instability can be reduced by using a low load resistance $R_L$. 

Chapter 5 - SYSTEM CONSIDERATIONS

In this chapter we discuss the various considerations involved in the design of a frequency-division multiplexing system. Whereas in previous chapters we have concentrated on the theory of the individual system components (transmitters and receivers), we now discuss the issues which arise from trying to interconnect these components. We shall begin by discussing how to couple the transmitters to a transmission line. We then discuss the inclusion of receivers, the overall system gain and margins, and the potential number of channels in such a system.

The first question which we must address in the design of a multiplexing system is: how shall we couple several transmitters to a single transmission line? Remember that the oscillations appear only differentially across the interferometer inductance. In Calander and Zappe's work,\(^1\) a direct coupling scheme was proposed in which the transmitters were directly connected to a transmission line, using odd-integer quarter-wave stubs to provide isolation between the different transmitter frequencies. A similar coupling scheme is shown in Fig. 5.1, in which series LC circuits (rather than stubs) provide the desired isolation. (The use of stubs would introduce extra resonances which would interfere with the system operation).

Unfortunately, these direct-coupling techniques have a major disadvantage which makes them unattractive for an LSI communications system. Specifically, they require series resonant circuits (either stubs or LC circuits) with resonant frequencies which are precisely matched to the transmitter frequencies. In the case of a series LC circuit, this would mean that \(L_{\text{series}} = Q^2/L\) and \(C_{\text{series}} = C/Q^2\), where \(Q\) is the desired quality factor for the series resonator. We see that for \(Q \gg 1\), these values of series inductance and capacitance differ markedly from the values found in the interferometer loop. In fact, they are not even physically realizable in our technology. But the main point is that it is difficult to match the resonant frequency of a series circuit or stub to the resonant frequency of the parallel resonant circuit contained in the transmitter interferometer loop, because the two resonators are
Figure 5.1

Figure 5.2
physically very different. Although one could conceive of special techniques to "fine-tune" resonant circuits on a Josephson IC, it is generally not practical to do so in large-scale integrated circuits.

It is clear from the above discussion that we should avoid introducing any extra resonators into our coupling scheme, because of the difficulty in matching the resonant frequencies. We therefore propose that the transmitter loop inductances be coupled in series, rather than in parallel. Since each interferometer contains a parallel resonant circuit which shorts out off-resonant frequencies, the signals from other transmitters will presumably not affect an individual transmitter very much.

It would be very convenient for the interferometers to be isolated from each other dc-wise. This would make it easy to operate the devices at different resonant voltages \( V_r \), which is necessary in order to have different operating frequencies. We therefore propose to transformer-couple the interferometers to the transmission line, as is customary in Josephson interferometer logic circuits. Fig. 5.2 is a circuit diagram of the proposed series coupling scheme. The RF signal which is generated across each loop inductance will couple out bidirectionally to the transmission line. The line must therefore be properly terminated at both ends, to prevent reflections.

Let us outline how to analyze the resonant behavior of a single transmitter which is magnetically coupled to a transmission line (Fig. 5.3). If the coupling is simply an ideal 1:1 transformer, then the difference-mode transmission line impedance \( 2Z_0 \) effectively appears across the loop inductance \( 2L \). Then the circuit can be analyzed using the resonance theory of Chapter 2 (\( R_d \) is replaced by \( Z_0 \)). In practice, the magnetic coupling will not behave like an ideal transformer. The general model\(^{23}\) is shown in Fig. 5.3, where we have a "crossing inductance" \( 2L_c \) in series with the transmission line, and a mutual inductance \( 2M \) which couples the crossing inductance to the interferometer loop inductance. In general, \( M \leq L \) and \( M \leq L_c \); when they are all equal, the circuit reduces to an ideal 1:1 transformer. In order to
solve for the resonant properties, it is necessary to calculate the difference-mode (internal) impedance seen by the junctions at the resonant frequency. We readily see from Fig. 5.3 that:

\[ V_c = 2L_c \dot{I}_c + 2\dot{M}I = -2Z_0 I_c , \]

\[ V = 2L\dot{I} + 2\dot{M}I_c . \]

At a complex frequency \( s \), this gives:

\[ V_c = 2L_c s I_c + 2MsI = -2Z_0 I_c , \quad (5.1a) \]

\[ V = 2LsI + 2MsI_c . \quad (5.1b) \]

Thus \( I_c = -MsI/(L_c s + Z_0) \), whence \( V = 2LsI - \frac{2M^2 s^2}{L_c s + Z_0} I \). Thus the "internal impedance" is

\[ Z(s) = \frac{V}{I} = Ls - \frac{M^2 s^2}{L_c s + Z_0} . \]

Replacing \( s \) by \( j\omega \) and inverting, we can write the internal admittance:

\[ Y(\omega) = \left[ j\omega L + \frac{\omega^2 M^2}{j\omega L_c + Z_0} \right]^{-1} + j\omega C , \]

where we have now included the junction capacitances. If we assume that \( Q = Z_0/\omega L >> 1 \), then we can approximate

\[ \left[ j\omega L + \frac{\omega^2 M^2}{j\omega L_c + Z_0} \right]^{-1} = \frac{1}{j\omega L} \left[ 1 - \frac{j\omega M^2/L}{j\omega L_c + Z_0} \right]^{-1} \approx \]

\[ \frac{1}{j\omega L} \left[ 1 + \frac{j\omega M^2/L}{j\omega L_c + Z_0} \right] = \frac{1}{j\omega L} + \frac{M^2/L^2}{j\omega L_c + Z_0} . \]

Thus \( Y(\omega) \approx \frac{1}{j\omega L} + \frac{\xi^2}{j\omega L_c + Z_0} + j\omega C \), where \( \xi = M/L \leq 1 \). At the resonant frequency
\[ \omega_r = \frac{1}{\sqrt{LC}}, \] this gives:

\[ Y(\omega_r) \approx \frac{\xi^2}{j\omega_r L_c + Z_0}. \]

Thus in place of the internal damping resistance \( R_d \), we have an impedance \( \xi^{-2}(Z_0 + j\omega_r L_c) \). Using this, the resonance RF amplitude \( \delta \) can be computed using the techniques described in Chapter 2.

Once \( \delta \) is known, we must then determine how much of the oscillation couples out to the transmission line. Referring to equations 5.1a and 5.1b, we find:

\[ \frac{V_c}{V} = \frac{\frac{2L_c \delta I_c + 2Ms}{2LsI_c + 2MsI_c} = \frac{-L_c[Ms/(L_c \delta + Z_0)] + M}{L - M[Ms/(L_c \delta + Z_0)]} = \frac{MZ_0}{(LL_c - M^2)s + LZ_0}. \]

At frequency \( s = j\omega_r \), this gives:

\[ \frac{V_c}{V} = \frac{\xi}{\xi^2 - j(\omega_r L/Z_0)(L_c/L - \xi^2)}. \]

The term \( L_c/L \) is likely to be substantially larger than \( \xi^2 \), so

\[ \frac{V_c}{V} \approx \frac{\xi}{1 + j\omega_r L_c/Z_0}. \]

This will always be less than 1 in magnitude, so as expected, not all of the signal gets coupled out to the transmission line.

The preceding discussion shows that there are two parameters (in addition to \( \Gamma \)) which are important in determining the amplitude of the transmitter output. One is the parameter \( \xi \), which is always less than one and is the same "coupling loss" which is found in logic interferometers. The other parameter is \( \omega_r L_c/Z_0 \), which is the reactance of the crossing inductance relative to the transmission line impedance. Only when \( L_c = 0 \) and \( \xi = 1 \) do we get perfect coupling out to the line.
We now discuss the issues involved in coupling several transmitters to a transmission line. First note that when we couple our transmitters in series to the line, RF power will be radiated bidirectionally. Since it is probably useful only to transmit data in one direction, the other end of the line must be terminated with a matched resistance. Unfortunately this throws away half of the transmitter power, but this seems unavoidable.

In our proposed multiplexing system, the various transmitters must have different resonant frequencies. To achieve this, we require accurate adjustment of the interferometer LC product. In general the capacitance is the much more difficult parameter to control, for it depends on the precise area and thickness of the small Josephson tunnel barrier. The inductance, on the other hand, can be more carefully controlled, as it is determined by large electrode areas and thick oxides than on the absolute area. The most practical course is therefore to use identical junctions in all the transmitters. This presumably results in identical capacitances, so the inductance L determines the transmitter frequency.

Note that when we couple several transmitters to the same transmission line, the crossing inductance $L_c$ of each transmitter will cause discontinuities in the line, creating reflections and hence attenuation of the signal. To minimize this effect, the quantities $\omega L_c/Z_0$ should be much less than 1 for all channel frequencies $\omega$.

How realistic are the conditions that we require for our transmitters ($\xi \approx 1$ and $\omega L_c/Z_0 \approx 0$)? In the case of $\xi$, the condition is a feasible one. By making the thickness of the base electrode much greater than its penetration depth $\lambda$, and by using a wide control line, $\xi$ can be 0.9 or larger (this is a routine consideration for logic designers). In the case of $\omega L_c/Z_0$, the problem is not so simple. Suppose our highest-frequency transmitter operates at frequency $\omega_h$, and the lowest one transmits at $\omega_l$. In the worst case, a signal at frequency $\omega_h$ must be able to pass over the transmitter of frequency $\omega_l$ with little or no attenuation. If we
define $Q_l$ to be the quality factor of the transmitter at frequency $\omega_l$, we have:

$$\frac{\omega_h L_c}{Z_0} = \left( \frac{\omega_h}{\omega_l} \right) \left( \frac{L_{cl}}{L_l} \right) Q_l^{-1}.$$

Now $\frac{\omega_h}{\omega_l}$ would probably be about 2 in a multichannel communications system, and $\frac{L_{cl}}{L_l}$ (the ratio of crossing inductance to interferometer loop inductance) would be at least 2.5 in any practical interferometer structure. If we want only a 10% loss in signal, we require that $\frac{1}{|1 + j\omega L_{cl} / Z_0|} > 0.9$, or $\omega_h L_c / Z_0 < 0.5$, hence $Q_l \geq 10$. Thus we have a lower bound on the $Q$ required for the lowest-frequency transmitter. High values of $Q$ would be required in any case if one wishes to have many channels.

We now consider the problem of connecting the receivers to the transmission line. Again, we propose to connect the devices in series with the line. The only problem with such an arrangement is that the receivers will reflect incident power when in the steady state. This is because the impedance of a receiver is high at its resonant frequency, thus the transmission line will appear to be terminated by an open circuit. The reflections can conceivably bounce back and forth along the line, eventually interfering with the data being transmitted on the following machine cycle. To prevent this, we propose including an additional terminating resistance $Z_o$ immediately in front of the receivers. Then when a receiver has become an "open circuit" (its oscillations have built up), the line is then properly terminated and little power is reflected in the steady state. Fig. 5.4 is a circuit diagram of the proposed receiver coupling arrangement.

Now that the coupling of the transmitters and receivers to the transmission line has been described, we are in a position to calculate the "gain" of the system. That is, we can find how much the receiver critical current $I_m$ is reduced in response to the transmitter output. We shall assume that the transmitter and receiver are identical devices. Assuming zero coupling loss for both devices, perfect frequency matching between devices, and negligible attenuation along the transmission line, we find that $I_m$ is a function only of the transmitter
damping parameter $\Gamma$. Fig. 5.5 shows $I_m(\Gamma)$ for our proposed coupling scheme. We see that the suppression is negligible for $\Gamma<<1$ or $\Gamma>>10$. The suppression is a maximum when $\Gamma \approx 3$; for this value, $I_m/I_m0 \approx 0.51$. This is more than sufficient to give adequate operating margins (it is customary to bias logic interferometers at currents of $0.7I_m0$ or higher). Thus we see that the proposed multiplexing system is in principle feasible. All the transmitters must be designed so that their damping parameters $\Gamma$ are in the vicinity of 3.

It should be noted that our "gain curve" in Fig. 5.5 represents ideal conditions. In practice, numerous effects will cause the gain to degrade. For example, both the transmitter and receiver have coupling losses and crossing inductance, as discussed earlier. Furthermore, the superconducting transmission line will have some attenuation at the high frequencies which we are operating; the longer the line and the higher the frequency, the greater the attenuation. Finally, the transmitter operating frequency might not be precisely matched to the receiver frequency (for example, because of the looped resonance I-V curve), and it may not be operating at full power (in fact it should not be biased too close to maximum power, because noise might switch it out of the resonance). All these factors should be taken into account if one wishes to design a practical multiplexing system.

If the system gain and margins are unacceptable after accounting for these various effects, there are several possible ways to improve the situation. One is the use of parametric amplification. This has been investigated, and the author is of the opinion that it is not likely to be useful in the context of a multiplexing system. There are two main reasons for this. First, to get sufficient gain from the amplifier, one must operate it very near to an unstable condition. The amplifier would therefore require careful process control to achieve satisfactory operation, which cancels any benefits obtained from its gain. A second, more fundamental problem, is that the proposed amplifier is of the negative-resistance type and hence it would be bidirectional, sending as much power back along the input terminals as it sends forward to its output. This will result in severe problems with standing waves in the input line. This
Figure 5.5
problem could be eliminated if a "circulator" were available, but circulators are not compatible with the low temperatures and high operating frequencies required for our system.

Another way in which the gain could be improved is to use junctions with lower critical currents in the receivers. This can give impressive improvements in the operating margins. Unfortunately it is not possible to change the critical current without also changing the junction capacitance, which in turn changes the transmitter frequency unless the loop inductance is adjusted to compensate. In the present Josephson technology it would be difficult to design interferometers having different critical currents but identical resonant frequencies. However, one can effectively reduce $I_0$ by applying a magnetic field (via a control wire) in a direction orthogonal to the ordinary control line (so that no flux is coupled into the loop inductance). This would have the effect of allowing $I_0$ to vary without changing $C$.

If transmission line losses are reducing the gain, this can be corrected by operating at a lower frequency. Unfortunately this is not as easy as one might think. Recalling that $Q \geq 10$ and $\Gamma \approx 3$ for our transmitters, we have $\chi = Q/\Gamma \geq 3$ which means that $\omega_r = \omega_p \sqrt{\chi} \geq 1.7 \omega_p$ where $\omega_p = \sqrt{KC/I_0}$. In IBM's Josephson logic technology, $C/I_0 \approx 4.2$ pF/mA. Thus $\omega_p \geq 2$ Terahertz/sec, or $\nu \geq 320$ GHz. The lines may be too lossy at this frequency. The only way to avoid this problem while maintaining compatibility with the Josephson logic circuits is to add external capacitances to the interferometer. The easiest way to do this is to build the interferometer over a thin ($\approx 350\text{Å}$) layer of Nb$_2$O$_5$ formed by anodic oxidation of the Nb ground plane.$^{18}$ This can easily reduce the resonant frequency by a factor of two. It has the additional advantage of lessening the relative importance of the junction capacitance; the latter is harder to predict and control.

One remaining consideration in the design of a multiplexing system is the question of transmitter interaction. When two transmitters are simultaneously operating at nearby frequencies, there is a possibility that the transmitters will lock together on a single intermediate frequency. This phenomenon of "frequency pulling" is characteristic of systems containing
coupled nonlinear oscillators (such as lasers). In our design there is no solution except to space the channel frequencies far enough apart so that they cannot interact. Computer simulations have indicated that a spacing of $\Delta \omega \geq 2\omega/Q$ is adequate to prevent this effect.

Another kind of transmitter interaction can occur, in which the higher-frequency transmitter causes the lower-frequency one to switch out of its resonant state. This effect, which we call "frequency knockout", appears to be analogous to the receiver instability discussed in Chapter 4. There are two solutions to this problem: use low load resistances, as was done with the receivers, and/or maintain adequate channel spacing (again, $\Delta \omega \geq 2\omega/Q$ appears to suffice).

It should be clear from the above discussions that the design of a multichannel multiplexing system is a formidable job, involving many more theoretical and technological issues than are found in logic design. The design of a practical system is not attempted in this work. However, the author has obtained some feeling for the potential number of channels which such a system could reasonably accommodate. Briefly, the design of a 2-channel system should be easy. The channel spacing can be extremely large (1.5:1 or larger), placing the receivers out of the instability region and allowing excellent discrimination. Furthermore, having only two devices minimizes the attenuating effects of crossing inductances. The design of a 4-channel system seems feasible, but much design work (including many simulations) as well as good process control will be required, primarily because the gain is not spectacular. The design of a system with more than 4 channels would most likely require some very fundamental improvements in the whole approach (more sensitive receivers with no instabilities, etc.).
Chapter 6 - EXPERIMENTS

In this chapter we briefly discuss some experiments which were designed to test the multiplexing system concepts developed in Chapters 2-5. Several experimental circuits were designed and fabricated as parts of two different Josephson integrated circuits. Because of time constraints, there is at present only limited experimental data from these circuits. Nonetheless, sufficient experimental evidence was obtained to confirm the basic idea of microwave communications.

Experiment 1 was designed to test the simultaneous operation of several transmitters at different frequencies. Several transmitters were connected in series to a single 5-Ohm transmission line. Control current was magnetically coupled to the interferometers, and also directly injected into some, allowing a direct measurement of the coupling loss factor $\xi$. A low-current Josephson junction was placed at each end of the line to provide an analysis of the frequency spectrum in the line (by looking at the Shapiro steps). Different line lengths were used in order to measure line attenuations.

At present, only limited results have been obtained from this experiment. Shapiro steps have been observed (Fig. 6.1), indicating that the transmitters can send RF down 100 ps of transmission line. However, the amplitude of the RF was much lower than expected. This can be explained by a combination of three effects. First, the coupling parameter $\xi$ was measured and found to be only about 60% of what was expected. Second, the junction critical currents were substantially lower than their design values, resulting in reduced gain and a lower signal-to-noise ratio. Third, the amount of crossing inductance had been underestimated in the design. All these effects are well-defined, caused either by design errors or processing problems. They can be remedied with more careful design and processing. There was no indication of frequency pulling or other transmitter interaction, which is good.
Experiment 2 was designed to test the microwave receiver sensitivity and discrimination. A single transmitter sent microwaves down 100 ps of transmission line. At the end of the line were three receivers; one was tuned to the transmitter frequency, and the other two were tuned to frequencies 20% lower or higher than the signal frequency. This experiment was successful; the operating transmitter did indeed reduce the receiver critical current to about $0.88 I_{m0}$. When the receiver is biased in in the zero-voltage state with a gate current exceeding $0.88 I_{m0}$, the transmitted RF would switch the receiver to the voltage state. Fig. 6.2 is a photograph of this effect. The upper trace shows a transmitter power pulse, followed by a small reflection which indicates that the transmitter has received the power. The lower trace is a double exposure showing the receiver output. After being powered up to above $0.88 I_{m0}$ (the initial large pulse), we see from the latter part of the trace that the receiver either switches (upper part of trace) or doesn't switch (lower part of trace) according to whether the transmitter is enabled (control current applied) or not. Various tests were made to verify that this effect was not due to crosstalk or other spurious processes.

The validity of the concept of generating microwaves, transmitting them down a line, and detecting them again, has been confirmed by Experiment 2. There are two presumed reasons why $I_m$ was reduced only by 12%. First, the magnetic coupling losses were considerably larger than expected, as in Experiment 1. Second, the critical currents (and hence $\Gamma$) were lower than the designed values. Again, these are not fundamental problems.

The receivers which were detuned by 20% from the transmitters were then examined to see whether they discriminated against the signal (they should). The receiver which was tuned to a higher frequency showed excellent discrimination; the reduction in $I_m$ was at most 1%. However the other receiver, tuned to a lower frequency, had its critical current $I_m$ reduced to as low as $0.89 I_{m0}$. This is believed to be due to the instability discussed in Chapter 4; the effect occurs in the expected frequency range (signal frequency > receiver resonant frequency). However, the situation was not noticeably affected by the inclusion of a load.
resistance across the receiver. This is probably because the load resistance was not low enough. In any case, this is not a satisfactory amount of discrimination; hopefully further testing will clear up this matter.

Experiment 3 was designed to test a practical Josephson parametric amplifier; it has not yet been fully tested. Experiment 4 investigated high-resolution sampling techniques, including measurements of receiver switching speed; it has not yet been tested. Experiment 5, also as yet untested, contains a complete two-channel multiplexing system (two transmitters and two receivers).

To summarize, the experiments involving the multiplexing system worked as expected, except that the coupling loss and receiver instability (lack of discrimination) were larger than expected. The higher coupling loss is now realized to be due to the low device inductance, and to the fact that the control lines did not cover the entire loop inductance. In the future the device geometry should be designed more favorably. The receiver discrimination problem is presumably due to the instability discussed in Chapter 2, but the effect was larger than expected. A possible explanation is that the transmitter frequency was higher than the design value, which would result in less sensitivity for the main receiver and more sensitivity for the higher-frequency receiver. There has not yet been an opportunity to check out this explanation.

We now conclude this work with some suggestions for further research, in order of importance. To begin with, the characteristics of the receiver instability ought to be studied further. It would be very helpful if some technique could be found to eliminate the instability altogether (other than using a zero gate resistance!). Another issue that needs more attention is the attenuation in the superconductors; there is no definite experimental data for the frequencies of interest. Efforts should be made to improve the gain of the system (this might involve the use of 3-junction interferometers). The effects of transmitter interaction (in particular, the phenomenon of frequency pulling) should be understood theoretically; simple
empirical results may not be adequate for systems with many transmitters. Finally, some study should be made of the reflections produced by microwaves incident on the receivers, for these reflections may potentially interfere with the system operation during subsequent machine cycles.
REFERENCES


