June 1980

LIDS-TH-1004
OSP.No. 87049

MODELLING AND ANALYSIS OF UNRELIABLE MANUFACTURING ASSEMBLY NETWORKS WITH FINITE STORAGES

by

Mostafa Hamed Ammar

This report is based on the unaltered thesis of Mostafa Hamed Ammar, submitted in partial fulfillment of the requirements for the degree of Master of Science at the Massachusetts Institute of Technology in May, 1980. The research was carried out at the Laboratory for Information and Decision Systems with partial support extended by the National Science Foundation under grant DAR78-17826.

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Submitted to the Department of Electrical Engineering
and Computer Science on May 23, 1980
in partial fulfillment of the requirements for the
Degree of Master of Science.

ABSTRACT

A Markov chain, queueing theory model of an assembly network is presented, of which the transfer line is a special case. Machines are unreliable and buffers have finite capacities. The aim of the research is to calculate performance measures of such systems.

Fundamental equivalence properties, which include transfer line reversibility, are stated and proved. These properties group networks with different structures into equivalence classes. The relationship among the performance measures of members of the same equivalence class are discussed.

A method for obtaining measures of performance of the networks is presented. This is a systematized and slightly modified version of the one that appears in Gershwin and Schick(1980). The solution is complete for two- and three-machine systems, and conjectures are made into how it extends to larger systems.

Thesis Supervisor: Dr. Stanley B. Gershwin
Title: Lecturer, Department of Electrical Engineering and Computer Science; Principal Research Scientist, Laboratory for Information and Decision Systems.

DEDICATION

To My Parents

LEILA and HAMED

May 1980

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to Dr. Stanley Gershwin who supervised this thesis. His interest and enthusiasm for the research have been the source for a great amount of inspiration.

In addition, I am greatly indebted to all members of the "Manufacturing Group" at LIDS. In particular, Ms. Ellen Hahne, and Mr. Majid Ibrahim deserve special thanks for being always available and interested when I needed to discuss my ideas.

I would also like to thank Mrs. Marjorie Paradis for typing this thesis, and Mr. Norman Darling for drawing the figures.

This research was carried out at the M.I.T. Laboratory for Information and Decision Systems with support extended by the National Science Foundation grant DAR78-17826. Any opinions, findings, and conclusions or recommendations expressed in this thesis are those of the author, and do not necessarily reflect the views of the National Science Foundation.

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Chapter 1

Introduction

1.1 General Remarks

In this thesis we are concerned with studying manufacturing systems with the goal of improving their performance. We use the tool of mathematical modelling of such systems to reach that goal. In this chapter simple examples of manufacturing systems are given, and some major issues involved in the design and operation of these systems are discussed. We also survey past research that investigated models of manufacturing systems. The contributions of this work are also discussed in this chapter.

1.2 Manufacturing Systems - An Overview

In many manufacturing processes a final product is composed of a number of smaller subassemblies. To manufacture such a product it is necessary to first produce each of its components. These parts are then assembled into larger subassemblies and so on until the final product emerges from the system.

We will call all operations in such a process "assembly" operations, although some may not involve the physical assembly of several products. That is, unitary operations such as drilling a single hole on a single piece are, for the purposes of this research, treated as a special case of assembly operations. All assemblies are carried out in work stations which are also referred to as machines. This terminology is not meant to preclude operations done manually.

In manufacturing processes work stations or machines are subject to failure. The term "failure" here should be interpreted in the broadest sense to include all cases where a machine is incapable of stand-alone operation. In the case of a manual process, this could include the operator taking a break. We say a machine is up if it is operational, and down if it has failed.

As an example, we consider a highly abstracted process that manufactures Formica table tops. A schematic of such a process appears in Figure 1.1.

In this plant the Formica is cut to the desired size by one machine. A second machine cuts the wood base to size. The Formica and the wood base are then pasted together to produce the table top. The latter operation is a physical assembly.

Suppose that for example, the pasting machine fails. This implies that the other machines in the system have to stop working although they are perfectly capable of performing their functions. This is because they have no place to put their output. This is clearly a waste in the productivity of the system. Similar wastage is incurred when one of the cutting machines is down. In this situation the pasting machine and the other cutting machine are forced to stop working. This is because the pasting machine is not being supplied the necessary parts by the failed cutting machine, and because the other cutting machine, as a consequence, has no place to put its output.

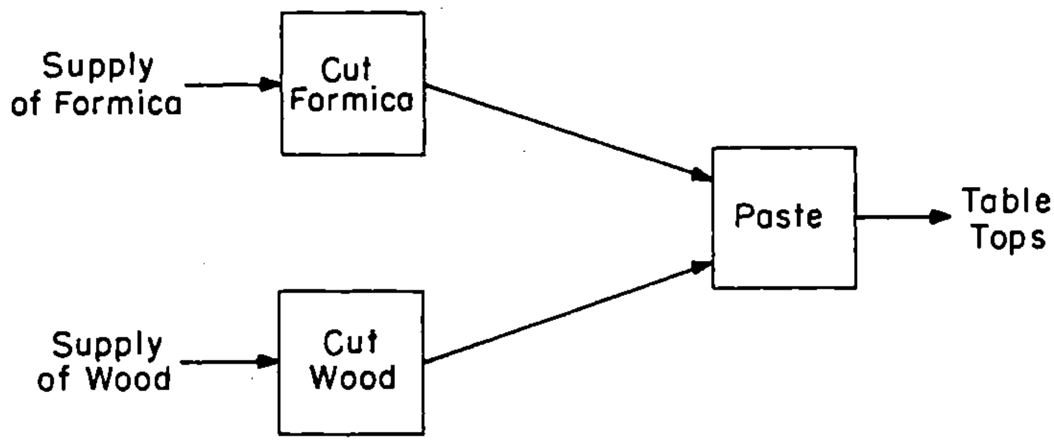


Figure 1.1 The Formica Plant

To reduce the strong coupling effect, a storage capability is introduced between machines. These storages serve a dual purpose. First, they provide space into which the output of a machine can go even when the machine that is to accept that output is down. Second, they provide a backlog of subassemblies, which keep an assembly machine operating even though one or more of the machines producing the subassemblies is down. It is also clear that larger storage sizes decrease machine coupling and hence provide for better productivity. It is exactly this interplay between the sizes of storages or buffers, the reliability of the machines and the productivity of the system that is the subject under study.

A schematic of a manufacturing network of the type we are dealing with appears in Figure 1.2. The squares represent the machines and the circles represent the storage devices or buffers. We call these networks "assembly merge networks" (AMN's). An important special case of the assembly merge network is where no assembly takes place. This is called a transfer line (see Figure 1.3). Transfer lines have become one of the most highly utilized ways of manufacturing large quantities of standardized items at a low cost (Koenigsberg (1959)). For this reason they have been the subject of a great deal of study (see Section 1.4).

The study of AMN's is a significant step towards understanding flexible manufacturing systems. However, other issues such as routing of parts through networks and scheduling that are not addressed here

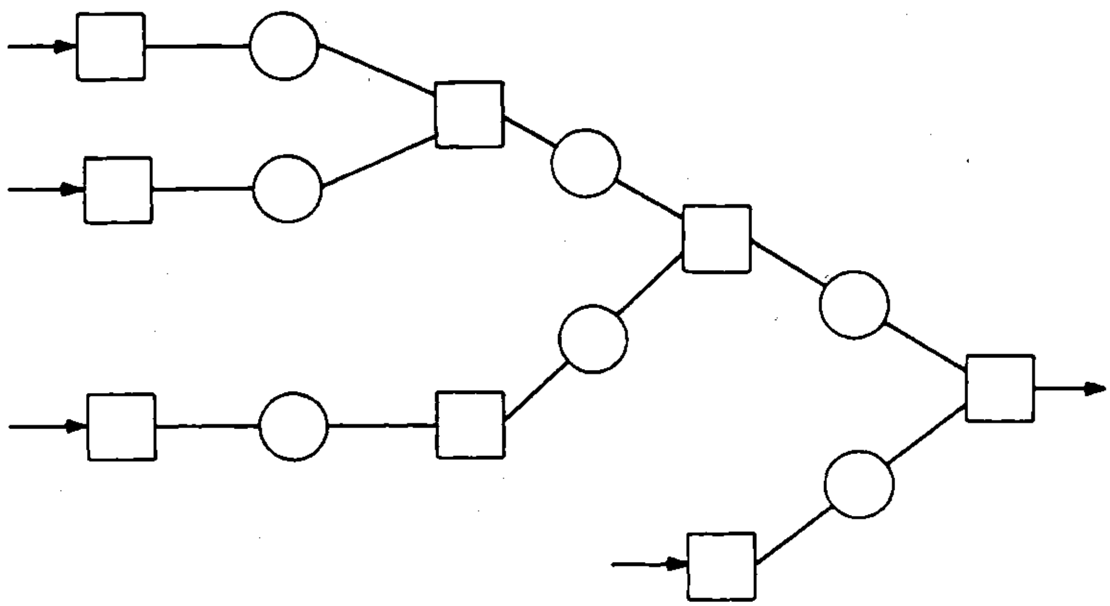


Figure 1.2   Assembly Merge Network

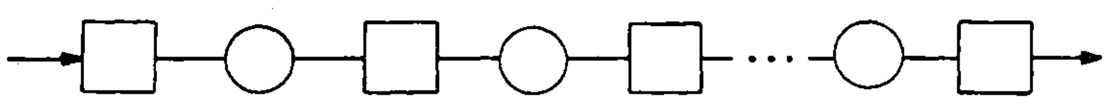


Figure 1.3   Transfer Line

have to also be studied to provide a complete understanding of flexible manufacturing.

1.3 Measures of Performance

In designing a manufacturing process two performance measures are of great interest. The first measure is the production rate which is the average number of completed parts that the system produces per unit time. The second measure is the average in-process inventory, which is the expected number of pieces in each buffer. That measure coupled with the knowledge of relevant cost information, such as cost/piece at each buffer, can provide an average in-process inventory cost.

In the design as well as the operation of a manufacturing network, one is interested in how performance measures are affected by changes in the system. Such changes include improvements to the reliability of machines, increases in the sizes of buffers, and possibly changes in the network configuration.

1.4 Previous Work

Numerous authors have looked at the problem of modelling of production systems. In this section a survey of some of the work relevant to this thesis is presented.

Several papers survey the issues involved in the design and operation of production systems. Those papers include Buzacott and Hanifin (1978a and 1978b), Hillier and Boling (1966), Koenigsberg (1959), and

Soyster and Toof (1976). Most of the production systems dealt with in these papers are of the transfer line variety. Koenigsberg (1959) mentions more complicated assembly networks which use the transfer line as a building block.

Most of the literature, however, dealt with specific types of production systems. All the studies to be discussed deal with movement of discrete parts through the system. Because of the economic importance of this problem, many simulation studies have been performed, such as Anderson and Moodie (1969) and Kay (1972). However, these studies are not discussed here, since the work to be reported is analytic.

One classification of the papers is by how they treat the three most important issues described in our model, namely storage size, reliability, and processing times.

First, there is the issue of the use and size of storages to be placed in a production system. Avi-Itzhak and Yadim (1965), Hunt (1956), Muth (1973), and Buzacott (1968) analyze systems with no buffer storage. Other authors including Goode and Saltzman (1962), and Hunt (1956) discuss systems in which the storage sizes are infinite. In fact Jackson networks are analyzed under the infinite-buffer assumption (see Jackson (1963), Disney (1975)).

Assuming that the buffers have finite capacity seems to complicate matters considerably. Several authors including Artamanov (1977), Avi-itzhak (1965), Buzacott (1971, 1967, and 1972), Gordon and Newell (1967), Hatcher (1969), Hillier and Boling (1966), Sheskin (1974), Gershwin and Schick (1978, 1979 and 1980), Schick and Gershwin (1978), Gershwin and Berman (1978), and Gershwin and Ammar (1979) have considered the effect of placing finite capacity storages in production systems. Most have been able to analyze two-stage systems with no success in generalizing to more complex configurations. Notable exceptions are Buzacott (1967) in which the author uses approximations to solve a three stage system, Sheskin (1974) in which a numerical solution to three and four stage systems is obtained. However, Sheskin had to make a less than satisfactory assumption on the reliability of machines. As far as this author is aware, Gershwin and Schick (1979, 1980) present the only complete solution to a three-stage production system with finite buffers made under plausible assumptions. The work reported here is an immediate outgrowth from that work.

The second important issue in production systems is reliability. Several authors analyze such systems under the assumption that the machines are totally reliable. Those include Avi-Itzhak (1965), Goode and Saltzman (1962), Gordon and Newell (1967), Hillier and Boling (1966), Hunt (1956), Muth (1973), and Neuts (1965). Artamanov (1977), Buzacott (1967, 1971, and 1972), Sheskin (1974), Gershwin and Schick (1979, 1980) and Gershwin and Ammar (1979) consider unreliable machines in the formulation of their models.

The third issue on which authors differ is the modelling of the processing times of the stages in the production system. The standard queueing theory assumption of exponential service time is made in the Jackson network literature. Other authors have chosen to use this model in the context of production systems such as Buzacott (1972), Gordon and Newell (1967), Hillier and Boling (1966), Hunt (1956), Muth (1973), Neuts (1965), and Gershwin and Berman (1978). Some papers deal with more general service time distributions such as Erlang. These include Gershwin and Berman (1978), Hillier and Boling (1967), and Berman (1979).

In the model presented here we use the regular or deterministic processing time assumption. Authors which have chosen to use this type of processing time include Avi-itzhak (1965), Artamanov (1977), Buzacott (1967), Goode and Saltzman (1962), Sheskin (1974), Schick and Gershwin (1978), Gershwin and Schick (1978, 1979, 1980) and Gershwin and Ammar (1979).

We are aware of only one work, Harrison (1973), that describes a queueing model of an assembly operation. The assembly machine is allowed to have a general service time distribution. A general arrival process is assumed. Harrison's work, however, has a different goal from the one sought in this thesis.

Also not included in the above discussion are works that deal with the qualitative behavior of the models of manufacturing systems. One such issue is the reversibility of transfer lines, that is, how the order of machines in a transfer line affects performance. Significant results in this area are by Muth (1979), and Dattatreya (1978).

In these two works it is proved that production rate of a series of work stations (a transfer line) remains unchanged when the order of the stations is reversed. Hillier and Boling (1977) conjecture that this result is true for their model.

1.5 Contributions of this Thesis

This thesis is a significant step towards understanding production systems as well as queueing networks with finite waiting room. Specifically the contributions fall into three main categories:

1 - Modelling: The formulation of a discrete state, discrete-time model for a general assembly system.

2 - Qualitative Analysis: Equivalence properties of assembly merge networks are established in this thesis. Specifically it is shown that there exists equivalence classes of AMN's. All members of the same equivalence class have related performance measures. Thus one need only solve for the measures of performance of a single member of a given equivalence class. These properties are proven in the context of the model developed. However there is evidence to the effect that such ideas are extendable to more general models.

3 - Solution Technique: We analyze the solution procedure developed in Gershwin and Schick (1979, 1980) for two-and three-machine transfer lines. The aim of the analysis is to relate the two-machine solution to the three-machine one, and to emphasize features that will extend the solution to more complicated systems.

1.6 Thesis Outline

In the subsequent chapters, a formal mathematical model of assembly merge networks is presented, equivalence properties of such networks are discussed and a method of analysis is proposed.

The model description is carried out in Chapter 2. In Chapter 3 theorems and corollaries relating to transfer line reversibility are proven, and in Chapter 4 we extend the reversibility ideas to more general AMN's. Chapter 5 is an overview of a solution technique for two- and three-machine AMN's and conjectures on how it might extend to more complex AMN's. Appendices I, and II contain proofs of some of the theorems presented in the main text. In Appendix I we prove theorems presented in Chapter 2 relating to the performance measures of AMN's. Appendix II has the proof of an equivalence theorem presented in Chapter 4. In Appendix III we derive the analysis result of Chapter 5.

Chapter 2

The Model

2.1 General Remarks

In this chapter a formal description of an assembly merge network is presented. As was mentioned in Chapter 1, a very important special case of an assembly merge network is a transfer line, where no assembly takes place. A model for a K machine transfer line has been formulated in Schick and Gershwin (1979) and Gershwin and Schick (1980). The model described in this chapter is an extension of that model and includes it as a special case. Therefore, it must be emphasized that the term "assembly" is used here in a general sense, and includes the case where a machine operates on a single item and thus does no physical assembly. For example, in this sense all machines in a transfer line are assembly machines. Also the term "part" (or "piece") refers to items flowing through an assembly network. That includes subassemblies and assemblies.

At the end of this chapter a discussion of the disassembly operation is included. The need for this discussion will be apparent in the ideas presented in Chapter 4.

2.2 Model Assumptions

An assembly merge network (AMN) consists of K machines. Machine i can be fed by, i.e. receives parts from, a set L(i) of buffers called the upstream buffers of machine i. Machine i in turn feeds exactly one buffer, D(i), called the downstream buffer of machine i (Figure 2.1). A machine takes a specified number of parts from each of its buffers simultaneously, assembles them, and produces a single part. That part

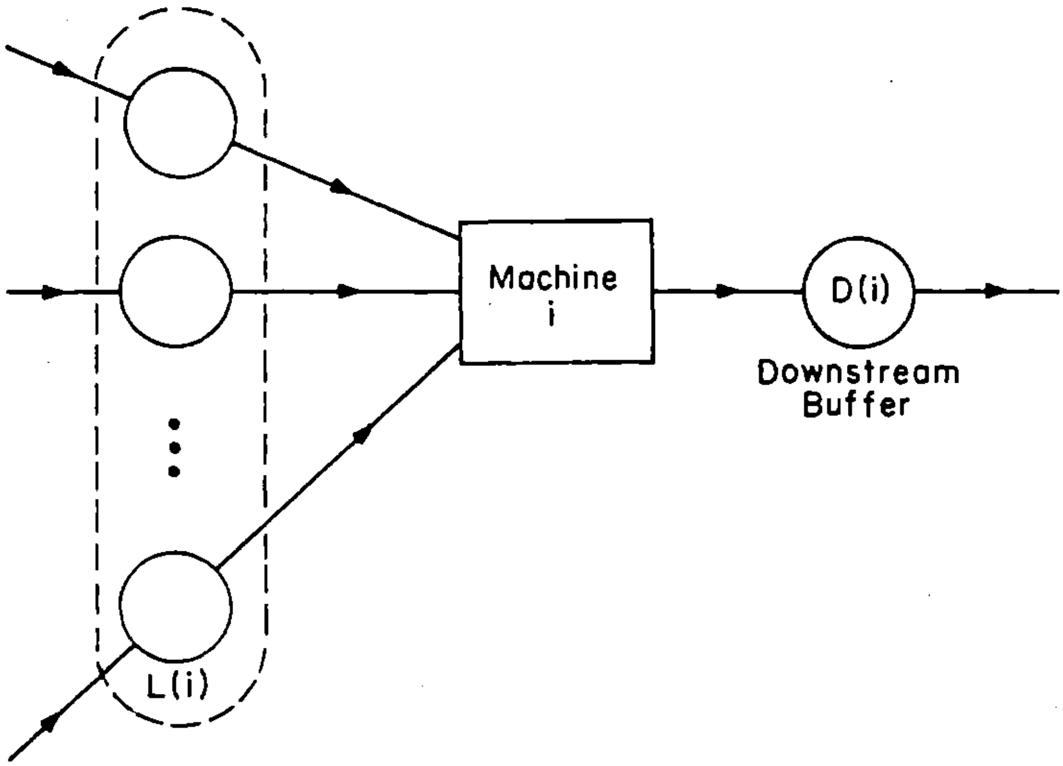


Figure 2.1 Upstream and Downstream Buffers of Machine i

is then put into the downstream buffer.

There are two special cases of machines. An <u>input machine</u> is one that does the first processing on the raw material entering the system. It is assumed that buffers upstream of input machines contain an unlimited supply of the required raw material. An <u>output machine</u> is the one from which the final assembled product emerges. It is assumed that the buffer downstream of the output machine has infinite capacity. In general an AMN is assumed to have several input machines, but only one output machine. (Note that this is equivalent to saying that the network is connected.) Also buffers upstream of input machines and downstream of the output machine are considered to be outside the system of study.

To be able to operate, machine i requires γ_j pieces from each of its upstream buffers $j \in L(i)$. If the number of parts in buffer j is less than γ_j, the machine is said to be <u>starved</u>. Also a machine needs to have room for one assembly in its downstream buffer to accommodate its output. If not, the machine is said to be <u>blocked</u>. Because of the assumptions on the buffers upstream of input machines and downstream of the output machine, an input machine is never starved and an output machine is never blocked.

In an AMN with K machines there are exactly K-1 buffers. This is because every machine except the output machine is followed by exactly one buffer. Buffer i feeds exactly one machine, its <u>downstream machine</u> which is labelled d_i. Buffer i is fed by exactly one machine, its <u>upstream machine</u> labelled ℓ_i (see Figure 2.2). Buffers are assumed to have <u>finite</u> capacity. That is, buffer i can hold no more than N_i pieces (or assemblies),

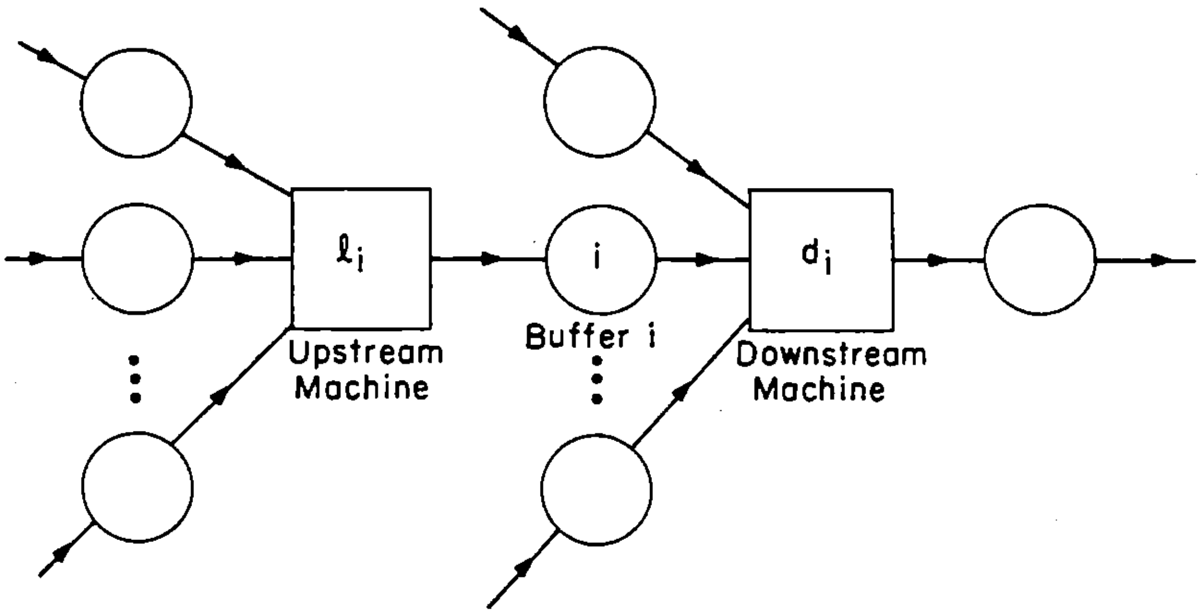


Figure 2.2 Upstream and Downstream Machines of Buffer i

where N_i is a finite number. It is precisely this assumption that leads to blockage of machines, and makes the analysis of such systems difficult. Nevertheless, the finite-capacity buffers assumption is crucial in making such a model realistic.

Also assumed here is that all machines have the same deterministic processing time. This is taken to be the basic unit of time throughout the following analysis.

Machines are modelled as unreliable, with geometrically distributed times between failures (TBF), and times to repair (TTR). This implies that the TBF and the TTR are integer multiples of the basic time unit. Also implied is that the probability of failure (or repair) during a given time unit, given that the machine is up (or down) in the previous time unit, is the reciprocal of the mean TBF (or mean TTR). The TBF (and TTR) is measured during times when a machine is operating (or down).

The repair process (or a failed machine) takes place regardless of the state of any other machine or buffer in the system. In particular, it is unaffected by the states of its adjacent (upstream and downstream) buffers. However, it is assumed that the failure of a machine can take place only when the machine is operating on a piece. Machines do not operate, and thus cannot fail, when they are starved or blocked.

2.3 Describing the structure of an AMN

Consider an AMN with K machines and K-1 buffers. Without loss of generality one can impose a labelling scheme as follows:

- Label the output machine as machine K;

- Arbitrarily label the rest of the machines with integers between 1 and K-1, not repeating any label.

- Label each buffer with the label of its upstream machine.

Figure 2.3 provides an example of the above labelling scheme for an AMN with seven machines.

For the special case of a transfer line we establish the convention that machines are labelled in ascending order from input to output. Thus the input machine is labelled 1 and the second machine has label 2, and so on until the output machine which has label K (for a K machine transfer line).

One can describe the structure of an AMN completely by specifying the list of upstream buffers of each machine. Recall that L(i) is the list of upstream buffers of machine i. To describe an AMN with K machines we need a list of the following form:

$$(L(1),\ L(2),\ \ldots,\ L(K))$$

Where if machine i is an input machine, $L(i)=\phi$, the empty set. For example, to describe the network in Figure 2.3 we write:

$$(\phi,\ \{1,6\},\ \phi,\ \phi,\ \{3\},\ \{2,5,4\})$$

Two other examples appear in Figure 2.4.

An alternative way of describing the network labelled in the above manner is by listing the downstream machine of each buffer as follows:

$$(d_1,\ d_2,\ \ldots,\ d_{k-1}).$$

For example, to describe the AMN in Figure 2.4 we write:

$$(2, 7, 5, 7, 7, 2).$$

The latter dexcription is more compact than the former. However, it has the disadvantage of not having the sets L(i) immediately available. As will be seen in Chapter 5, these sets are important in the analysis of

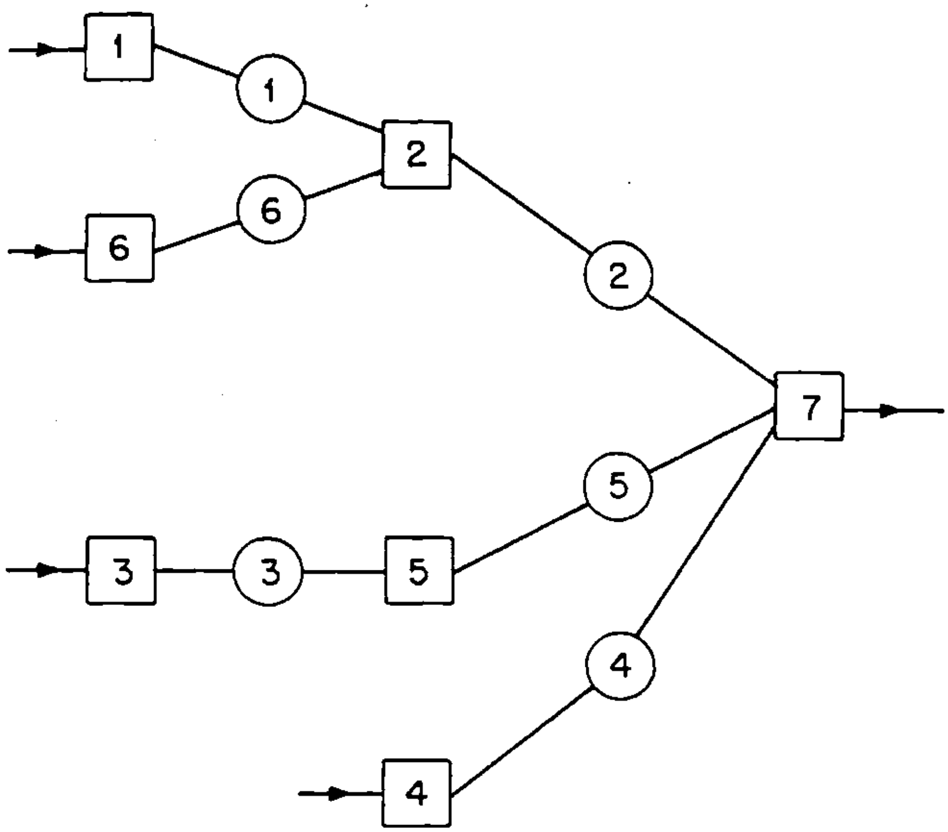


Figure 2.3 A Seven-Machine AMN

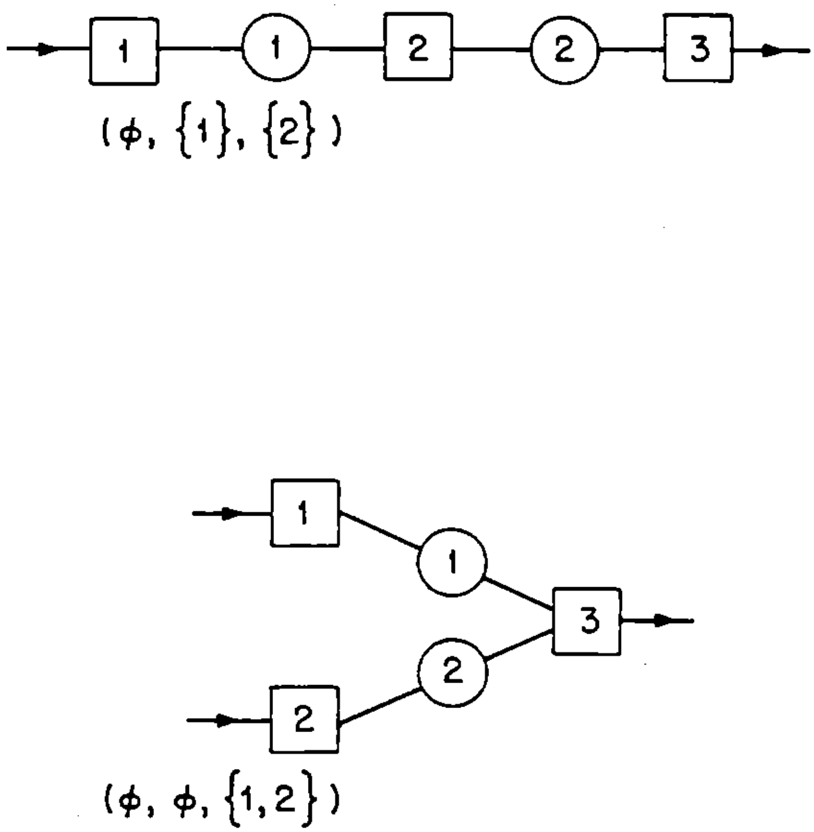


Figure 2.4 Three-Machine AMN's

the model.

2.4 System Parameters

The following parameters are needed to describe the system: (Note that a complete system description requires the topology of the network in addition to system parameters.)

For each machine i,

p_i = probability that a machine i fails in the next time period given that it is operational and neither starved nor blocked in the present period.

r_i = probability that machine i is repaired in the next time period given that it is down in this time period.

For each buffer i,

N_i = capacity of buffer i.

γ_i = number of parts that machine d_i takes from buffer i in one time unit.

2.6 State Space Formulation

The state s(t), of an AMN with K machines at time t, is described by:

$$s(t) = (\underline{n}(t), \underline{\alpha}(t)) ,$$

Where $\underline{n}(t) = (n_1,(t), \ldots, n_{K-1}(t))$,

and $\underline{\alpha}(t) = (\alpha_1(t), \ldots, \alpha_K(t))$.

In the above

$n_i(t)$ = number of parts in buffer i at time t.

$$0 \leq n_i(t) \leq N_i \qquad \text{for all } i=1, \ldots, K-1.$$

Also

$$\alpha_i(t) = \begin{cases} 0 & \text{if machine i is \underline{down} at time t} \\ 1 & \text{if machine i is \underline{up} at time t} \end{cases}$$

for all i=1, ..., K.

The system is modelled as a discrete state, discrete time Markov process. This is because the probability of being in a particular state at time t + 1 depends only on the state of the system at time t.

It is assumed that state transitions occur in the following manner;

- Machines change state from $\underline{\alpha}(t)$ to $\underline{\alpha}(t+1)$ depending on information provided in $(\underline{n}(t), \underline{\alpha}(t))$. In particular, whether a transition is possible depends on whether or not a machine is starved or blocked.

- Buffers undergo their change of state from $\underline{n}(t)$ to $\underline{n}(t+1)$ depending on $\underline{\alpha}(t+1)$ and $\underline{n}(t)$.

Physically this means that the machines undergo transitions at the beginning of a period, while buffers change state at the end. This assumption is made for mathematical convenience, as evidenced in Chapter 5.

An important consequence of this assumption is that a machine cannot process a part if it is starved even though the upstream machines can produce the needed parts. Similarly, a blocked machine cannot start processing even if the downstream machine is ready to take in a piece.

Table 2.1 lists the machine state transition probabilities. For example, the probability that machine i is down at time t+1 given that it is up and neither starved nor blocked at time t is equal to p_i. Also if machine i is blocked and operational, it cannot fail. Thus the transition

$n_j(t)$ $j \in L(i)$	$n_i(t)$	$\alpha_i(t)$	$\alpha_i(t+1)$	PROBABILITY
-	-	0	0	$1 - r_i$
-	-	0	0	r_i
-	N_i	1	0	0
-	N_i	1	1	1
0 for any j	-	1	0	0
0 for any j	-	1	1	1
$\geq \gamma_j$ for all j	$< N_i$	1	0	p_i
$\geq \gamma_j$ for all j	$< N_i$	1	1	$1 - p_i$

Table 2.1 Machine Transitions

Prob $[\alpha_i(t+1) | \alpha_i(t), n_i(t), n_j(t), j \in L(i)]$

between $\alpha_i(t)=1$ to $\alpha_i(t+1) = 0$ has zero probability if $n_i(t) = N_i$. In Table 2.1 the index j in the first column refers to a buffer upstream of machine i. The index i in the second column refers to the buffer downstream of machine i.

Table 2.2 describes how the buffers change state after the machines change state. Recall that the buffer state at time t+1 depends only on buffer states at time t and machine states at time t+1. Therefore the transitions described in Table 2.2 are of probability one, and all other events have a zero probability.

In order to understand Table 2.2 consider Figure 2.5. Buffer i is affected by the buffers in L(i) and $L(d_i)$, as well as by buffer d_i. It is also affected by machines i and d_i. Machine i is starved if for some $j \in L(i)$ buffer j contains less than γ_j parts. Similarly, machine d_i is starved if for some $j \in L(d_i)$ buffer j contains less than γ_j parts. Machine i is blocked if buffer i is full and machine d_i is blocked if buffer d_i is full.

We now consider four cases in Table 2.2 as examples:

Case 1: Here neither machine i nor machine d_i is starved or blocked. Then if machine i is up at time t+1 it deposits a part in buffer i. If it is down no parts are added to buffer i. Similarly if machine d_i is up at time t+1 it takes γ_i parts from buffer i. Otherwise no parts are taken from buffer i. Thus the value of $n_i(t+1)$ can be obtained from $n_i(t)$ by:

$$n_i(t+1) = n_i(t) - \gamma_i \, \alpha_{d_i}(t+1) + \alpha_i(t+1). \qquad (2.1)$$

case	$n_j(t)$ $j \in L(i)$	$n_m(t)$ $m \in L(d_i)$	$n_{d_i}(t)$	$n_i(t+1)$
1	$\geq \gamma_j$	$\geq \gamma_m, m \neq i$ $\geq \gamma_i, < N_i$	$< N_{d_i}$	$n_i(t) - \gamma_i \alpha_{d_i}(t+1) + \alpha_i(t+1)$
2	$\geq \gamma_j$	$-$, $m \neq i$ $< N_i$	N_{d_i}	$n_i(t) + \alpha_i(t+1)$
3	$\geq \gamma_j$	0 for any m $< N_i$	-	$n_i(t) + \alpha_i(t+1)$
4	-	$\geq \gamma_m, m \neq i$ N_i	$< N_{d_i}$	$n_i(t) - \gamma_i \alpha_{d_i}(t+1)$
5	0 for any j	$\geq \gamma_m$ for all m	$< N_{d_i}$	$n_i(t) - \gamma_i \alpha_{d_i}(t+1)$
6	-	$-$, $m \neq i$ N_i	N_{d_i}	$n_i(t)$
7	0 for any j	-	N_{d_i}	$n_i(t)$
8	0 for any j	0 for any m	-	$n_i(t)$
9	-	0 for any $m \neq i$ N_i	-	$n_i(t)$

Table 2.2 Possible Buffer Transitions. For these transitions $p[n_i(t+1) \mid n_j(t), j \in L(i), n_m(t), m \in L(d_i), n_{d_i}(t), \alpha_i(t+1), \alpha_{d_i}(t+1)] = 1$. For all other transitions this probability is zero.

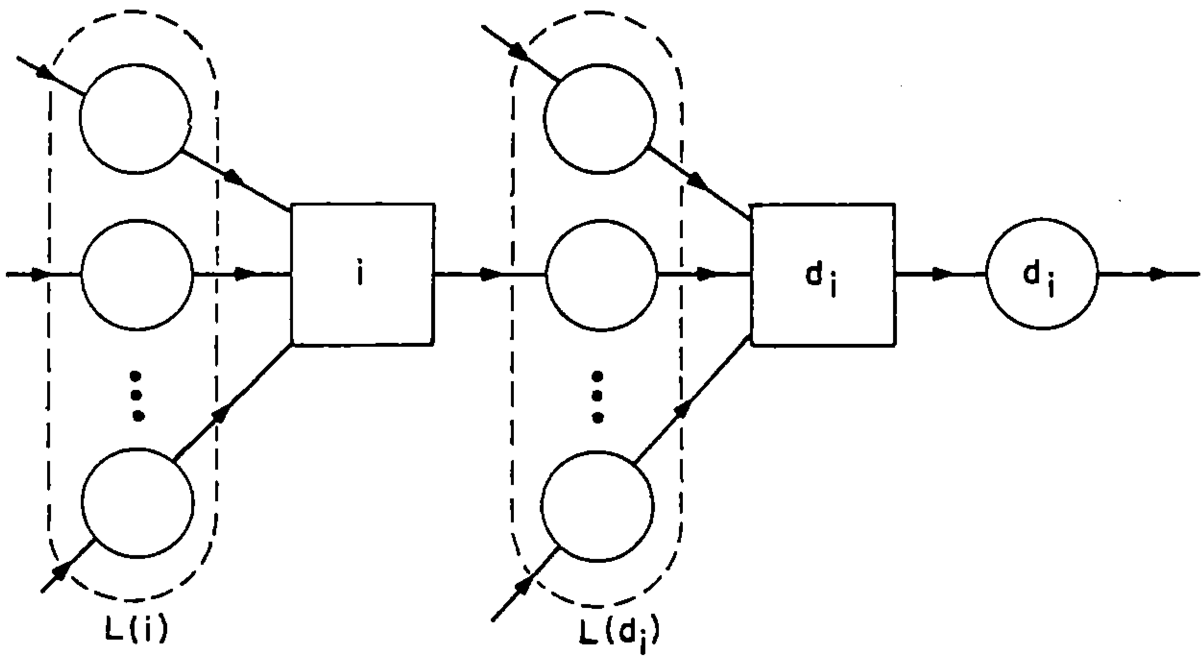


Figure 2.5   A Portion of an AMN

Case 2: In this case machine i is neither starved nor blocked, but machine d_i is blocked. Therefore machine d_i cannot operate and buffer i has no parts removed from it at time t+1. Machine i can add a piece if it is up at time t+1. Hence

$$n_i\ (t+1) = n_i\ (t) + \alpha_i\ (t+1). \tag{2.2}$$

Case 5: Here machine i is starved while machine d_i is neither starved or blocked. Thus machine i is not operational and cannot add a piece to buffer i. However, machine d_i can take γ_i parts from buffer d_i if it is up. Hence

$$n_i\ (t+1) = n_i\ (t) - \gamma_i\ \alpha_{d_i}\ (t+1) \tag{2.3}$$

Case 7: In this case machine i is starved and machine d_i is blocked. Hence neither machine can add parts to or take parts from buffer i. Thus the level of buffer i remains unchanged from time t to t+1. For this case

$$n_i\ (t+1) = n_i\ (t) \tag{2.4}$$

All other cases are treated in the same manner.

In summary if either of the machines i or d_i is starved or blocked at time t it does not contribute any change to the level of buffer i at time t+1. However, if machine i is neither starved nor blocked at time t and is up at time t+1 then it adds a part to buffer i at time t+1. If machine d_i is neither starved nor blocked at time t and is up at time t+1 then it removes γ_i parts from buffer i at time t+1.

For all the cases in Table 2.2 the following is a summary of the machine conditions (starvation or blockage)

Case 1 - Neither machine i nor machine d_i is starved or blocked.

Case 2 - Machine d_i is blocked.

Case 3 - Machine d_i is starved.

Case 4 - Machine i is blocked.

Case 5 - Machine i is starved.

Case 6 - Both machines, i and d_i, are blocked.

Case 7 - Machine i is starved and machine d_i is blocked.

Case 8 - Both machines are starved.

Case 9 - Machine i is blocked and machine d_i is starved.

Note that if machine i is an input machine, Cases 5, 7, and 8 are not applicable. Also if machine d_i is an output machine, Case 2, 6, and 7 do not arise. Case 9 is not applicable when $L(d_i)$ contains only one element.

2.6 Markov Process Formulation

We are now concerned with using Tables 2.1 and 2.2 to construct a discrete time, discrete state Markov chain. We define the transition probability to state S_2 at time t+1 given that the system is in state S, at time t as

$$T(s_2, s_1) = \text{Prob}\left[s(t+1) = s_2 \mid s(t) = s_1\right] \qquad (2.5)$$

$$= \text{Prob}\left[s(t+1) = (\underline{n}(t+1), \underline{\alpha}(t+1)) \mid s(t) = (\underline{n}(t), \underline{\alpha}(t))\right] \qquad (2.6)$$

$$= \prod_{i=1}^{K-1} \text{Prob}\left[n_i(t+1) \mid n_j(t),\ j \in L(i),\ n_m(t),\ m \in L(d_i),\ n_{d_i}(t),\ \alpha_i(t+1), \alpha_{d_i}(t+1)\right]$$

$$\cdot \prod_{i=1}^{K} \text{Prob}\left[\alpha_i(t+1) \mid n_j(t),\ j \in L(i),\ n_i(t),\ \alpha_i(t)\right] \qquad (2.7)$$

The value of expressions (2.7) can be determined from Tables 2.1 and 2.2 as follows:

$$\text{Prob}\left[n_i(t+1) \mid n_j(t)\ ,\ j \in L(i),\ n_m(t), m \in L(d_i),\ n_{d_i}(t),\ \alpha_i(t+1),\ \alpha_{d_i}(t+1)\right]$$

$$= \begin{cases} 1 & \text{if the quantities } n_i(t+1),\ n_j(t),\ n_m(t), \\ & \alpha_i(t+1),\ \alpha_{d_i}(t+1) \text{ conforms to one of the cases} \\ & \text{in Table 2.2} \\ 0 & \text{Otherwise} \end{cases} \tag{2.8}$$

A product of such expressions as in (2.8) is 0 or 1. The value of

$\text{Prob}\ [\alpha_i\ (t+1) \mid n_j\ (t)\ ,\ j \in L(i),\ n_i(t)\ ,\ \alpha_i\ (t)]$

can be obtained from the last column of Table 2.1. Thus

$$T\ (s_2,s_1) = \text{①} \prod_{i=1}^{K} \text{Prob}\left[\alpha_i(t+1) \mid n_j(t)\ ,\ j \in L(i)\ ,\ n_i(t), \alpha_i(t)\right]$$

or

$$\text{②} \quad 0 \tag{2.9}$$

The pair (s_2,s_1) is in case ① of (2.9) if

$$\text{Prob}\ \left[n_i(t+1) \mid n_j(t),\ j \in L(i), n_m(t),\ m \in L(d_i),\ n_{d_i}(t), \alpha_i(t+1),\ \alpha_{d_i}(t+1)\right]$$

$$= 1 \quad , \quad \forall\ i = 1, \ldots, K-1 \tag{2.10}$$

The value of $T(s_2, s_1)$ forms the transition matrix T. The matrix is square with dimensions

$$M = 2^K \prod_{i=1}^{K-1} (N_i+1) \tag{2.11}$$

For this to be a valid Markov process we need to show that T is a stochastic matrix, that is that T possesses two properties. First all its elements need to be non-negative. This is obvious from (2.9). The second property of stochastic matrices is that

$$\sum_{s_2} T(s_2, s_1) = 1 \tag{2.12}$$

From 2.9 we see that

$$\sum_{s_2} T(s_2, s_1) = \sum_{\alpha_1(t+1)=0}^{1} \cdots \sum_{\alpha_K(t+1)=0}^{1} \prod_{i=1}^{K} \text{Prob}[\alpha_i(t+1) | n_j(t), j \in L(i), n_i(t), \alpha_i(t)] \tag{2.13}$$

Equation (2.13) can be rewritten as (See Gershwin and Schick (1979)

$$\sum_{s_2} T(s_2, s_1) = \prod_{i=1}^{K} \sum_{\alpha_i(t+1)=0}^{1} \text{Prob}[\alpha_i(t+1) | n_j(t), j \in L(i), n_i(t), \alpha_i(t)] \tag{2.14}$$

From Table 2.1 we have

$$\text{Prob}[\alpha_i(t+1) = 0 | n_j(t), j \in L(i), n_i(t), \alpha_i(t)] + \text{Prob}[\alpha_i(t+1) = 1 | n_j(t), j \in L(i), n_i(t), \alpha_i(t)] = 1 \tag{2.15}$$

Thus (2.14) becomes

$$\sum_{s_2} T(s_2, s_1) = 1 \tag{2.16}$$

We have now shown that matrix T is a stochastic matrix. This proof is similar to the one provided in Gershwin and Schick (1979) for the special case of a transfer line. In the same work it is shown that the Markov process under consideration is ergodic. For the more general Markov chain describing an AMN it can be argued similarly that the process is ergodic. For the proof the reader is referred to Gershwin and Schick (1979).

2.6.1 Transient Behaviour

As mentioned before the state of an AMN at time is defined by

$$s(t) = (\underline{n}(t), \quad \underline{\alpha}(t)) \tag{2.17}$$

We say

$$p_i(t) = \text{Prob} \; (s(t) = s_i). \tag{2.18}$$

That is $p_i(t)$ is the probability of the system being at state s_i at time t . Also let

$$\underline{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_M(t) \end{bmatrix} \tag{2.19}$$

where M is the number of states (see equation (2.11)). The probability vector $\underline{p}(t)$ is given by:

$$\underline{p}(t+1) = T \, \underline{p}(t) \tag{2.20}$$

and the normalization equation

$$\sum_{i=1}^{M} p_i(t) = 1 \tag{2.21}$$

From (2.25) we can obtain the following relation

$$\underline{p}(t) = T^t \; \underline{p}(0) \tag{2.22}$$

2.6.2 Steady State Behaviour

In analyzing this system it is assumed that steady state has been reached. That is, all the effects of the starting conditions have disappeared. This is appropriate for systems that have been running for a "long" period of time relative to the characteristic system times (processing failure and repair times). The steady state assumption is a good one for manufacturing systems if one is interested in their behaviour after a sufficiently long time has elapsed since start-up.

The aim of the analysis is to calculate the performance measures discussed in Chapter 1. These calculations require the values of the probability of the system being in each state. The steady state probability of being in state $s = (\underline{n}, \underline{\alpha})$ is denoted by

$$p(s) = p(\underline{n},\underline{\alpha}) = p(n_1,\ldots n_{k-1},\alpha_1,\ldots,\alpha_k). \tag{2.23}$$

Ergodicity for a discrete time, discrete state Markov chain implies that the steady state probabilities (2.23) exist. In particular,

$$p(s_i) = \lim_{t\to\infty} \text{Prob}\ [s(t) = s_i] \tag{2.24}$$

The limiting steady state probabilities, $p(s_i)$ can be determined by the matrix equation:

$$T\ \underline{p} = \underline{p} \tag{2.25}$$

where $$\underline{p} = \begin{bmatrix} p(s_1) \\ \vdots \\ p(s_M) \end{bmatrix} \tag{2.26}$$

in conjunction with

$$\sum_{i=1}^{M} p(s_i) = 1. \tag{2.27}$$

Equations (2.25) and (2.27) can be determined by taking the limit, as $t\to\infty$, of equations (2.20) and (2.21).

We can rewrite equation (2.25) as

$$p(s_j) = \sum_{i=1}^{M} T(s_j, s_i)\ p(s_i) \qquad j=1, \ldots, M \tag{2.28}$$

Equations (2.28) are called the steady state transition equations describing the Markov chain.
(See Bharucha-Reid (1960), Feller (1966) and Howard (1971))

For the systems at hand the number of states can be quite large (see equation (2.11)). This makes a direct solution of the steady state transition equations by standard simultaneous linear equations techniques impractical. A method of solution which circumvents this difficulty, utilizing the special structure of this problem is presented in Chapter 5.

For the purpose of reducing complexity only the special case where $\gamma_i = 1$ for all i is studied here. This simplifies the analysis while retaining many of the important features of the model.

2.7 Performance Measures from Steady State Probabilities

In this section we show how the performance measures, discussed in Chapter 1, can be calculated directly from steady state probabilities. Only the important results are shown in this section. The detailed proofs of these results are contained in Appendix I.

First we focus on the production rate of an AMN. Define $R_K(t)$ as the number of parts released from machine K (the output machine) in the time interval $[0,t]$. We then define production rate as follows:

$$R = \lim_{t \to \infty} \frac{R_K(t)}{t} . \tag{2.29}$$

Proposition 2.1

$$R = \lim_{t \to \infty} \frac{R_K(t)}{t} = \text{Prob} \left[\alpha_K(t) = 1,\ n_j(t-1) > 0, \quad \forall_j \in L(k) \right] \tag{2.30}$$

The proof of proposition 2.1 is presented in Appendix I.

Proposition 2.1 says that the production rate is the same as the probability of the output machine producing a part at time t . The difficulty with expression (2.30) is that it depends on the state at two time instants and thus cannot be calculated from steady state probabilities. Proposition 2.2 resolves this difficulty.

Proposition 2.2

$$\begin{aligned} R &= \text{Prob} \left[\alpha_K(t) = 1,\ n_j(t-1) > 0,\ \forall_j \in L(K) \right] \\ &= \text{Prob} \left[\alpha_K(t) = 1,\ n_j(t) > 0,\ \forall_j \in L(K) \right] \end{aligned} \tag{2.31}$$

The proof of proposition 2.2 is in Appendix I.

Expression (2.31) allows for a straightforward calculation of production rate as follows:

$$R = \text{Prob} \left[\alpha_K(t) = 1,\ n_j(t) > 0,\ \forall_j \in L(K) \right]$$

$$= \sum_{\alpha_1=0}^{1} \cdots \sum_{\alpha_{K-1}=0}^{1} \cdots \sum_{\substack{n_i=0 \\ i \in L(K)}}^{N_i} \cdots \sum_{\substack{n_j=1 \\ j \notin L(K)}}^{N_j} p\,(n_1, \ldots, n_{K-1}, \alpha_1, \ldots, \alpha_{K-1}, 1) \tag{2.32}$$

We now define a related measure of performance, the rate at which an input machine takes parts. Let $D_j(t)$ be the number of parts taken in by input machine j in the interval $[0,t]$. Define input rate:

$$D_j = \lim_{t\to\infty} \frac{D_j(t)}{t} \tag{2.33}$$

For the quantity D_j the following is proven in Appendix I.

Proposition 2.3

For j an input machine to an AMN.

$$D_j = \lim_{t\to\infty} \frac{D_j(t)}{t} = \text{Prob}\,[\alpha_j(t) = 1,\ n_j(t-1) < N_j] \tag{2.34}$$

Proposition 2.4

For input machines j to an AMN

$$D_j = \text{Prob}\,[\alpha_j(t) = 1,\ n_j(t-1) < N_j]$$

$$= \text{Prob}\,[\alpha_j(t) = 1,\ n_j(t\ \ ) < N_j] \tag{2.35}$$

In a manner similar to the expansion of R, the production rate as a sum or probabilities, (2.35) allows us to write:

$$D_j = \sum_{\substack{i=0\\ i\neq j}}^{1} \cdots \sum_{n_j=0}^{N_j-1} \cdots \sum_{\substack{n_i=0\\ i\neq j}}^{N_i} p\,(n_1, \ldots, n_{k-1}, \alpha_1, \ldots, \alpha_{j-1}, 1, \alpha_{j+1}, \ldots, \alpha_k) \tag{2.36}$$

We now state a conservation of flow theorem. Conservation of flow certainly holds for a real finite capacity production system, provided there is no mechanism for part creation or destruction. The following theorem asserts that conservation of flow does indeed hold for the present AMN model.

Theorem 2.1 Conservation of flow

For a k-machine AMN

$$D_j = R \quad \text{For all input machines } j \tag{2.37}$$

Proof: Theorem 2.1 is proven in Appendix I.

Theorem 2.1 states one possible version of conservation of flow: The rate of input into input machine j is the same as the production rate of the system. Note that the intention of the conservation of flow theorem here is not to show that mass flow is conserved. What we are concerned with here is that the input rate of each part type, in parts per unit time, is equal to the output rate in parts per unit time.

The second performance measure described in Chapter 1 is the mean in-process inventory at buffer i; we denote this by $\bar{n}_i$. The value of $\bar{n}_i$ can be calculated in a straightforward manner from the steady state probabilities.

$$\bar{n}_i = \sum_{\text{all } s} n_i \, p(s) \tag{2.38}$$

2.8 The Disassembly Operation

Assembly merge networks contain machines that perform one type of operation, namely: an assembly. Recall that "assembly" here includes single part operations as well as physical assembly operations.

In this section we describe briefly another type of operation: disassembly. This is needed for the discussion in Chapter 4 to be complete. Furthermore, the AMN model in this Chapter can be generalized to include machines that perform disassembly.

Consider the following variation on the Formica plant example described in Chapter 1. Assume that the ultimate product is not the assembled table top but two tables of different sizes. After the assembled top emerges out of the pasting machine it passes through a sawing machine that cuts the top into two tops of the desired sizes.

Each then goes to a separate machine where the tables are manufactured. (See Figure 2.6). The machine where the sawing takes place is an example of a disassembly machine. (If the pasting and cutting operations are performed in one machine, then the machine is an Assembly-Disassembly machine). Generally speaking a disassembly operation involves breaking up a single part into several smaller parts. Thus disassembly machine i is fed a piece by its upstream buffer $L(i)$ and as a result of the operation it puts a predetermined number of pieces in each of its downstream buffers $j \in D(i)$ (See Figure 2.7).

Note that the disassembly operation is basically a reversed assembly operation. This idea is expounded in Chapter 4.

A machine that combines the assembly and disassembly capabilities is one that has a set of upstream buffers, $L(i)$, and downstream buffers $D(i)$. Parts are taken from buffers in $L(i)$, and assembled into one piece. The machine then takes the assembled product and disassembles it (hopefully in a manner that does not just reverse the assembly operation just performed). The output goes to the downstream buffers. Figure 2.8a is a schematic of an Assembly-disassembly machine (ADM). Figure 2.8b is a typical network constructed using the ADM as a building block. It is conjectured that such networks are amenable to the same kind of analysis that is presented in this thesis.

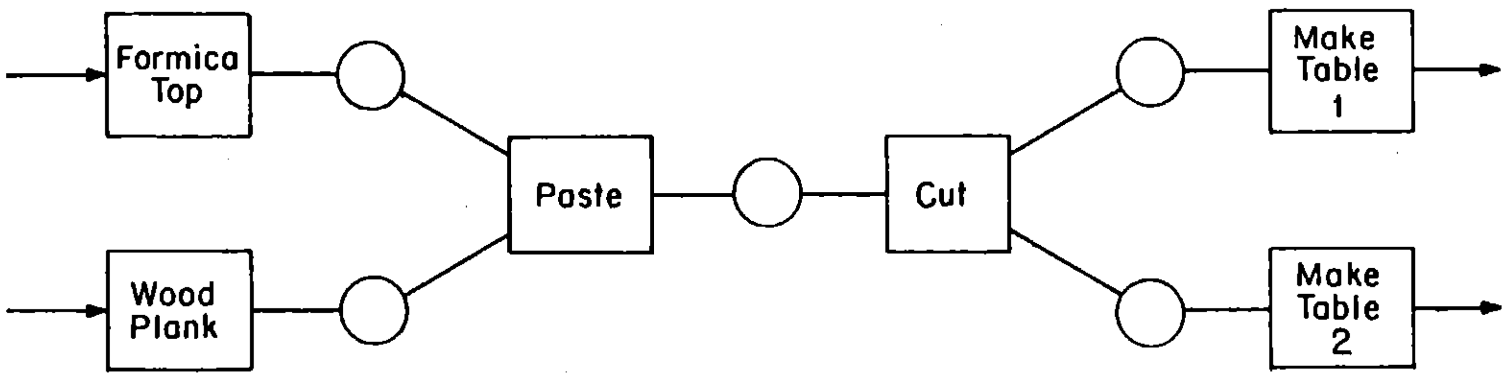


Figure 2.6 New Formica Plant

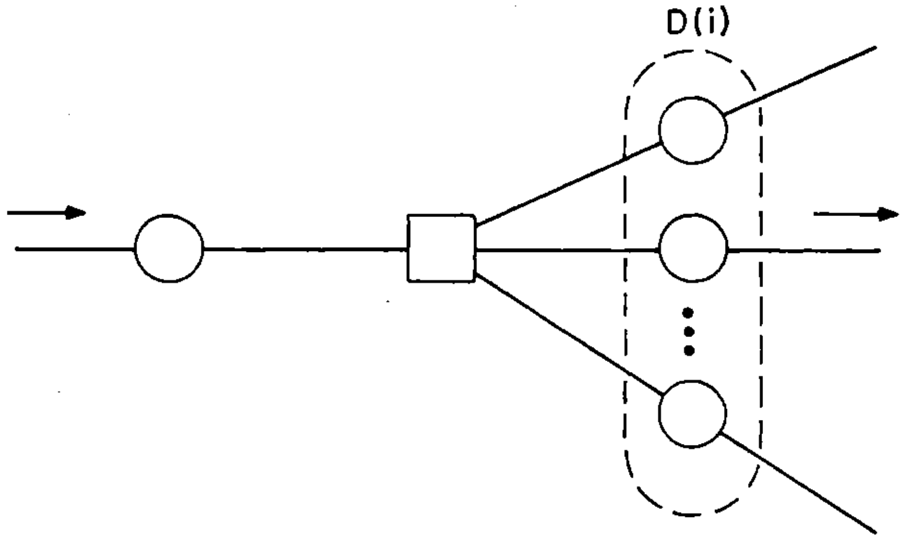


Figure 2.7   A Disassembly Machine

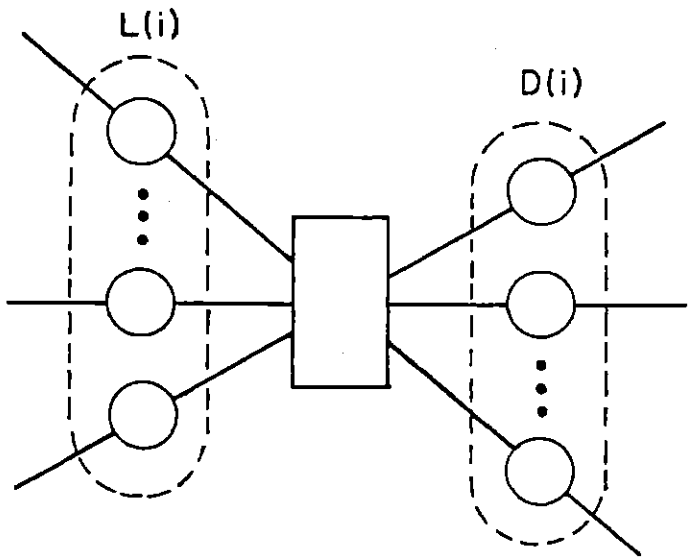


Figure 2.8a) Assembly-Disassembly Machine

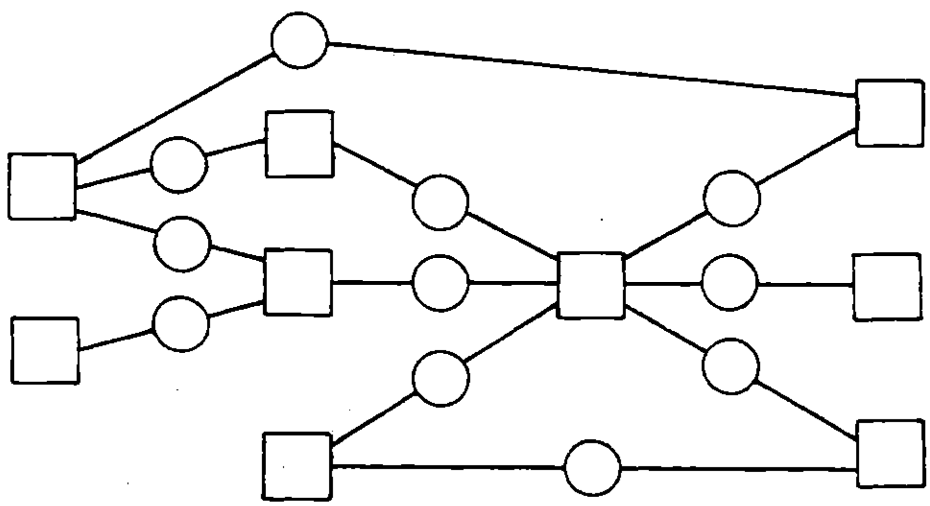


Figure 2.8b) Assembly-Disassembly Network

Chapter 3

Transfer Line Reversibility

3.1 General Remarks

Studies on transfer line reversibility are concerned with investigating the effect of reversing the order of machines in a transfer line on performance measures. Several works in the literature have dealt with this issue, (e.g. Muth (1979), and Dattatreya (1978)) . The basic result of the reversibility studies that prevails in the literature is that when the order of machines of a transfer line is reversed the production rate is unchanged.

We show that this is indeed true for our transfer line model (a special case of the AMN model). We also show that the other performance measure of concern, namely mean in-process inventory, does change. The latter result, as far as this author is aware, does not appear in the literature. One clear implication is that for transfer lines that can be modelled using our AMN model, reversing the order of machines <u>can</u> change the performance of the system.

Our basic tool for arguing the above results is the notion of a hole, or empty space, in a transfer line. (We define this more formally later). We find it useful to consider the motion of the empty spaces (instead of parts) in the transfer line. We note that Newell (1979) also considered hole movement through a queueing network. However, that notion is not put to the same use as is done in this chapter.

It should also be noted that some of the Theorems and Corollaries stated in this chapter are motivated by numerical experience obtained by Pomerance (1979) for three machine transfer lines.

3.2 Part-Hole Duality

In analyzing production systems (AMN's being an example) one focuses on the motion of parts through the stages of the production process. Alternatively one can just as easily analyze the motion of empty spaces in the system. That is there is a one-to-one correspondence between the number of empty spaces and the number of parts in a system. For example consider a finite buffer of capactiy N. If at any time t the number of parts in the buffer in n(t), the number of empty space n'(t) is determined uniquely by:

$$n'(t) = N - n(t) \tag{3.1}$$

We call the empty spaces, "holes", (borrowing terminology from semi-conductor physics).

Consider the behavior of the AMN model when one analyzes the motion of holes instead of parts. When a machine produces a part it also, at the same time, takes a part from each of its upstream buffers. By depositing a part in its upstream buffer, the machine decreases the number of holes in that buffer by one. Also by taking a part from each of its upstream buffer it increases the number of holes in each of those buffers by one. Thus a machine could be thought of as taking a hole from its downstream buffer and performing a disassembly operation on it, as a result of which a hole is deposited into each of its upstream buffers.

It has been argued that an event of part production always corresponds to an event of hole production, i.e. adding a hole to an upstream buffer. Also, in the AMN model, when a machine is down it produces no parts. Thus it cannot produce holes. Also a machine cannot fail when any of its upstream buffers is empty because it has no parts to take in to produce the assembly. However by (3.1) an empty buffer is full of holes. Therefore a machine that is starved of parts is blocked by holes and can neither operate nor fail.

Using the same line of argument one can also conclude that a machine blocked by parts is starved by holes and hence can neither operate or fail. Therefore we have shown that failure and repair play the same role in our model whether one considers the motion of parts or the motion of holes. However, blockage and starvation exchange roles when one considers the flow of holes instead of parts.

Finally, if one has an infinite storage for parts (as there is downstream of the output machine), then there is an infinite number of empty spaces, or alternatively infinite supply of holes. Similarly, an infinite supply of parts (which appears upstream of each input machine) corresponds to an infinite storage capacity for holes.

To summarize, we have shown a well defined duality between parts and holes in our model for an AMN. See Table 3.1. This duality suggests that looking at the motion of parts or the motion of holes in a system should not result in any new information.

Parts	Holes
n	N-n
∞ room	∞ supply
∞ supply	∞ room
Blockage	Starvation
Starvation	Blockage
Failure	Failure
Repair	Repair

Table 3.1 Part-Hole Duality

3.3 The Strong Reversibility Property

In this section we present and prove several properties related to transfer line reversibility. This is used as an example of how the hole concept can be utilized to construct equivalent systems. Recall that a transfer line is a special case of an AMN in Chapter 2. (See Figure 1.2).

Let the two AMN's F and R be defined as follows:

$$F = (\phi , \{1\}, \{2\},\ldots,\{K-1\}) \tag{3.2}$$

$$R = (\phi , \{1\}, \{2\},\ldots,\{K-1\}) \tag{3.3}$$

Both systems are transfer lines.

Let r_i^F, p_i^F, N_i^F and r_i^R, p_i^R, N_i^R be the system parameters for F and R respectively.

We say R is the <u>reverse</u> of F if and only if

$$r_i^F = r_{K-i+1}^R \tag{3.4}$$

$$p_i^F = p_{K-i+1}^R \tag{3.5}$$

$$N_i^F = N_{K-i}^R \tag{3.6}$$

In other words, R is the reverse of F, if the machines and buffers in R are arranged in the opposite order to the arrangement in F.

To motivate the theorem proving the strong reversibility property, consider the motion of holes in system F as defined above. Holes in system F enter the system through machine K, which is identical to machine 1 in R, where parts enter that system. The holes in F then proceed through the system until they exit via machine 1, which is identical to machine K in R where parts exit that system. At any time t if buffer i contains $n_i(t)$ parts, it has $N_i^F - n_i(t)$ holes. This is the same as the number of parts in buffer K-i of system R at time t. It appears that holes go through system F in the same manner as parts go through system R. The above argument suggests the following theorem.

<u>Theorem 3.1</u>: <u>Strong Reversibility Property</u>

For $i = 1, \ldots, K-1$, $j=1, \ldots, K$

For all n_i, $0 \leqslant n_i \leqslant N_i$ and for all $\alpha_j = 0, 1$

$$p^F (n_1, \ldots, n_{K-1}, \alpha_1, \ldots, \alpha_K)$$

$$= p^R (n_1', \ldots, n_{K-1}', \alpha_1', \ldots, \alpha_K') \tag{3.7}$$

when

$$n_i' = N_{K-i}^F - n_{K-i} = N_i^R - n_{K-i} \tag{3.8}$$

and

$$\alpha_i' = \alpha_{K-i+1} \tag{3.9}$$

<u>Proof</u>:

Consider Tables 2.1 and 2.2 as specialized to transfer lines.

Table 3.2a contains the transfer line machine transition table, and Table 3.3a is the transfer line buffer transition table. (These tables appear in a less compact form in Schick and Gershwin (1978).

In Table 3.2b we modify Table 3.2a by conditioning the $\alpha_i(t)$ - to - α_i (t+1) transition on $N_i^F-1 - n_{i-1}(t)$ and $N_i^F - n_i(t)$ instead of on $n_{i-1}(t)$ and $n_i(t)$. For example where Table 3.1a states that $n_{i-1}(t) = 0$ we have $N_{i-1}^F - n_{i-1}(t) = N_{i-1}^F$ in Table 3.1b. Consistently with (3.8) and (3.9) we can define

$$n'_{K-i+1}(t) = N_{i-1}^F - n_{i-1}(t), \tag{3.10}$$

$$n'_{K-i}(t) = N_i^F - n_i(t), \tag{3.11}$$

and

$$\alpha'_{K-i+1}(t) = \alpha_i(t). \tag{3.12}$$

Also from the definitions of F and R we have

$$r_i^F = r_{K-i+1}^R,$$

and

$$p_i^F = p_{K-i+1}^F.$$

From the substitutions (3.10) through (3.14) we can obtain Table 3.2c from Table 3.2b. Note that Table 3.2c is the machine transition table for system R. (To see this clearly K-i+1 by j in Table 3.2c.)

We now manipulate Table 3.3a in a similar manner. First find the value of $N_i^F - n_i(t+1)$ conditioned on the values of $N_{i-1}^F - n_{i-1}(t)$,

$n_{i-1}(t)$	$n_i(t)$	$\alpha_i(t)$	$\alpha_i(t+1)$	PROBABILITY
-	-	0	0	$1 - r_i^F$
-	-	0	1	r_i^F
0	-	1	0	0
0	-	1	1	1
-	N_i^F	1	0	0
-	N_i^F	1	1	1
> 0	$< N_i^F$	1	0	p_i^F
> 0	$< N_i^F$	1	1	$1 - p_i^F$

Table 3.2a Machine Transition Probabilities for a Transfer Line

$N^F_{i-1} - n_{i-1}(t)$	$N^F_i - n_i(t)$	$\alpha_i(t)$	$\alpha_i(t+1)$	PROBABILITIES
-	-	0	0	$1 - r^F_i$
-	-	0	1	r^F_i
N^F_{i-1}	-	1	0	0
N^F_{i-1}	-	1	1	1
-	0	1	0	0
-	0	1	1	1
$< N^F_{i-1}$	> 0	1	0	p^F_i
$< N^F_{i-1}$	> 0	1	1	$1 - p^F_i$

Table 3.2b Modification of Table 3.1a

$n'_{K-i+1}(t)$	$n'_{K-i}(t)$	$\alpha_{K-i+1}(t)$	$\alpha_{K-i+1}(t+1)$	PROBABILITY
-	-	0	0	$1 - r^R_{K-i+1}$
-	-	0	1	r^R_{K-i+1}
N^R_{K-i+1}	-	1	0	0
N^R_{K-i+1}	-	1	1	1
-	0	1	0	0
-	0	1	1	1
$< N^R_{K-i+1}$	>0	1	0	p^R_{K-i+1}
$< N^R_{K-i+1}$	>0	1	1	$1 - p^R_{K-i+1}$

Table 3.2c Machine Transition Table for R, the Reversed Line

$n_{i-1}(t)$	$n_i(t)$	$n_{i+1}(t)$	$n_i(t+1)$
>0	$>0, <N_i^F$	$<N_{i+1}^F$	$n_i(t) + \alpha_i(t+1) - \alpha_{i+1}(t+1)$
>0	$<N_i^F$	N_{i+1}^F	$n_i(t) + \alpha_i(t+1)$
>0	0	-	$n_i(t) + \alpha_i(t+1)$
-	N_i^F	$<N_{i+1}^F$	$n_i(t) - \alpha_{i+1}(t+1)$
0	>0	$<N_{i+1}^F$	$n_i(t) - \alpha_{i+1}(t+1)$
-	N_i^F	N_{i+1}^F	$n_i(t)$
0	-	N_{i+1}^F	$n_i(t)$
0	0	-	$n_i(t)$

Table 3.3a Possible Buffer Transitions for a Transfer Line

$N_i^F - n_i(t)$, and $N_{i+1}^F - n_{i+1}(t)$. This yields Table 3.3b. Then we use equations (3.8) and (3.9) to give us

$$n'_{K-i+1}(t) = N_{i-1}^F - n_{i-1}(t) \tag{3.13}$$

$$n'_{K-i}(t) = N_i^F - n_i(t) \tag{3.14}$$

$$n'_{K-i-1}(t) = N_{i+1}^F - n_{i+1}(t), \tag{3.15}$$

$$\alpha'_{K-i}(t) = \alpha_i(t) \tag{3.16}$$

and

$$\alpha'_{K-i+1}(t) = \alpha_{i+1}(t). \tag{3.17}$$

Also from the definitions of F and R we have

$$N_i^F = N_{K-i}^R , \tag{3.18}$$

and

$$N_{i+1}^F = N_{K-i+1}^R. \tag{3.19}$$

Using equations (3.13) through (3.19) on Table 3.3b yields Table 3.2c. This is exactly the buffer transition table for system R.

We have so far shown that by applying equations (3.8) and (3.9) on Tables (3.2a) and (3.3a) for system F, we get Tables (3.2c) and (3.3c) which are the tables for system R. In Chapter 2 it is shown how the probability of transition from any state to any other state can be calculated from the tables. Thus by constructing the tables for

$N^F_{i-1} - n_{i-1}(t)$	$N^F_i - n_i(t)$	$N^F_{i+1} - n_{i+1}(t)$	$N^F_i - n_i(t+1)$
$< N^F_{i-1}$	$>0, < N^F_i$	>0	$N^F_i - n_i(t) - \alpha_i(t+1) + \alpha_{i+1}(t+1)$
$< N^F_{i-1}$	>0	0	$N^F_i - n_i(t) - \alpha_i(t+1)$
$< N^F_{i-1}$	N^F_i	-	$N^F_i - n_i(t) - \alpha_i(t+1)$
-	0	>0	$N^F_i - n_i(t) + \alpha_{i+1}(t+1)$
N^F_{i-1}	$< N^F_i$	>0	$N^F_i - n_i(t) + \alpha_{i+1}(t+1)$
-	0	0	$N^F_i - n_i(t)$
N^F_{i-1}	-	0	$N^F_i - n_i(t)$
N^F_{i-1}	N^F_i	-	$N^F_i - n_i(t)$

Table 3.3b Modification of Table 3.3a

$n'_{K-i+1}(t)$	$n'_{K-i}(t)$	$n'_{K-i-1}(t)$	$n'_{K-i}(t+1)$
$< N^R_{K-i+1}$	$>0, < N^R_{K-i}$	>0	$n'_{K-i}(t) - \alpha'_{K-i+1}(t+1) + \alpha'_{K-i}(t+1)$
$< N^R_{K-i+1}$	>0	0	$n'_{K-i}(t) - \alpha'_{K-i+1}(t+1)$
$< N^R_{K-i+1}$	N^R_{K-i}	-	$n'_{K-i}(t) - \alpha'_{K-i+1}(t+1)$
-	0	>0	$n'_{K-i}(t) + \alpha'_{K-i}(t+1)$
N^R_{K-i+1}	$< N^R_{K-i}$	>0	$n'_{K-i}(t) + \alpha'_{K-i}(t+1)$
-	0	0	$n'_{K-i}(t)$
N^R_{K-i+1}	-	0	$n'_{K-i}(t)$
N^R_{K-i+1}	N^R_{K-i}	-	$n'_{K-i}(t)$

Table 3.3c Possible Buffer Transitions for R, the Reversed Line

system R from the tables for system F by using (3.8) and (3.9) we have shown that

$$T^F(s_2, s_1) = T^R(s_2', s_1'). \tag{3.20}$$

Where $T^F(s_2, s_1)$ is the probability of transition from state s_1 to state s_2 in system F. $T^R(s_2', s_1')$ is defined similarly. Relation (3.20) is true provided states s_2 and s_2' are related by (3.8) and (3.9). Also states s_1 and s_1' have to be related in the same way.

Let $p_i^F(t)$ = Prob (State of system F at time t = s_i),

and $p_i^R(t)$ = Prob (State of system R at time t = s_i').

Where states s_i and s_i' are related by (3.8) and (3.9).

Also let

$$\underline{p}^F(t) = \begin{bmatrix} p_1^F(t) \\ \vdots \\ p_M^F(t) \end{bmatrix}$$

and

$$\underline{p}^R(t) = \begin{bmatrix} p_1^R(t) \\ \vdots \\ p_M^R(t) \end{bmatrix}$$

Equation (3.20) indicates that by relabelling the states of system F according to (3.8) and (3.9) one can construct the transition matrix for system R. This matrix is the same as that for system F. One implication of this is that if any time t $\underline{p}^F(t) = \underline{p}^R(t)$, then $\underline{p}^F(t+1) = T\underline{p}^F(t) = T\underline{p}^R(t) = \underline{p}^R(t+1)$. Where T is the transition matrix.

In particular, the steady state probabilities are related as follows:

$$p^F(s) = p^R(s') \qquad (3.21)$$

if s' is the new label of s. The proof of Theorem 1 is now complete.

Consider the following example of an application of Theorem 2.1

Define the following two systems:

$$F3 = (\phi \; , \; \{1\}, \; \{2\} \;),$$

$$R3 = (\phi \; , \; \{1\}, \; \{2\} \;),$$

where

$$r_1^{F3} = r_3^{F3}, \quad r_2^{F3} = r_2^{R3}, \quad r_3^{F3} = r_1^{R3}$$

$$p_1^{F3} = p_3^{R3}, \quad p_2^{F3} = p_2^{R3}, \quad p_3^{F3} = p_1^{R3},$$

and $\quad N_1^{F3} = N_2^{R3}, \quad N_2^{F3} = N_1^{R3}.$

Systems F3 and R3 are three-machine transfer lines and R3 is the reverse of F3.

By Theorem 2.1 we have

$$p^{F3}\,(n_1,\, n_2, \alpha_1, \alpha_2, \alpha_3) = p^{R3}\,(n_1',\, n_2', \alpha_1', \alpha_2', \alpha_3')$$

when

$$n_1' = N_2^{F3} - n_2 \;\; = \;\; N_1^{R3} - n_2,$$

$$n_2' = N_1^{F3} - n_1 \;\; = \;\; N_2^{R3} - n_1,$$

$$\alpha_1' = \alpha_3,$$

$$\alpha_2' = \alpha_2,$$

and $$\alpha_3' = \alpha_3.$$

For example if

$$N_1^{F3} = 10,$$

and

$$N_2^{F3} = 10.$$

then

$$p^{F3}(3,2,1,0,0) = p^{R3}(8,7,0,0,1).$$

3.4 Consequences of Strong Reversibility

3.4.1 Measures of performance

Recall that we are actually interested in calculating measures of performance for AMN's. Two such measures where identified in Section 2.6, the production rate R and the mean in-process inventory in buffer i, $\overline{n}_i$. Another measure was introduced in Section 2.6 namely D_j, the input rate through input machine j. It is proved in Appendix 1 that for each input machine j, conservation of flow holds. That is

$$D_j = R. \tag{3.21}$$

In the special case of transfer lines there is only one input machine (machine 1). Thus for a transfer line (3.21) becomes

$$D_1 = R. \tag{3.22}$$

Theorem 3.1 is now used to prove that the production rates of systems F and R are equal. We call this the weak reversibility property for transfer lines. Define R^F and R^R as the production rates of systems F and R, respectively.

Corollary 3.1.1 Weak Reversibility Property

For the two systems F and R

$$R^F = R^R. \tag{3.23}$$

Proof:

Consider the formula for R^F presented in Section 2.6

$$R^F = \sum_{\alpha_1=0}^{1} \sum_{\alpha_{K-1}=0}^{1} \sum_{n_1=0}^{N_1^F} \sum_{n_{K-2}=0}^{N_{K-2}^F} \sum_{n_{K-1}=1}^{N_{K-1}^F} p^F(n_1,\ldots,n_{K-1},\alpha_1,\ldots\alpha_{K-2}, 1) \tag{3.24}$$

From Theorem 3.1 we know

$$p^F(n_1, n_2,\ldots, n_{K-1},\alpha_1,\alpha_2,\ldots, 1)$$

$$= p^R(n_1', n_2',\ldots, n_{K-1}', 1, \alpha_2', \alpha_3',\ldots,\alpha_K') \tag{3.25}$$

where $n_i' = N_{K-i}^F - n_{K-i} \qquad i=1,\ldots,K-1 \qquad (3.26)$

$\qquad = N_i^R - n_{K-i}$

and $\alpha_i' = \alpha_{K-i+1} \qquad i=2,\ldots,K \qquad (3.27)$

Thus (3.23) becomes

$$R^F = \sum_{\alpha_2'=0}^{1} \cdots \sum_{\alpha_K'=0}^{1} \sum_{n_1'=0}^{N_1^R-1} \sum_{n_2'=0}^{N_2^R} \cdots \sum_{n_{K-1}'=0}^{N_{K-1}^R}$$

$$p^R(n_1', n_2', \ldots, n_{K-1}', 1, \alpha_2', \alpha_3', \ldots, \alpha_K') \qquad (3.28)$$

But from (2.23) the right hand side of (3.28) is equal to D_1^R where D_1^R is the input rate for system R.

Hence

$$R^F = D_1^R \qquad (3.29)$$

But from (3.22) (conservation of flow)

$$D_1^R = R^R \qquad (3.30)$$

Thus the proof of the corollary is complete. It must be noted here that Muth (1979), and Dattatreya (1978) prove weak reversibility for transfer lines under more general assumptions.

In the next corollary a relationship between the mean in-process inventories of the two systems is established. Let $\bar{n}_i^F$ and $\bar{n}_i^R$ be the mean in-process inventories at buffer i in systems F and R.

Corollary 3.1.2

For the systems F and R

$$\bar{n}_i^R = N_{K-i}^F - \bar{n}_{K-i}^F = N_i^R - \bar{n}_{K-i}^F \qquad (3.31)$$

For all $i = 1, \ldots, K-1$.

Proof:

Using the formula for $\bar{n}_i^F$

$$\bar{n}_i^{\,F} = \sum_{\substack{\text{all } \alpha_i = 0,1 \\ \text{all } n_i = 0,\ldots,N_i^F}} n_i \quad p^F\,(n_1,\ldots,n_{K-1},\alpha_1,\ldots,\alpha_K)$$

$$= \sum_{\substack{\text{all } \alpha_i = 0,1 \\ \text{all } N_i^F - n_i = 0,\ldots,N_i^F}} n_i\; p^F\,(N_1^F - n_i,\ldots,N_{K-1}^F - n_{K-1},\ {}_1,\ldots,\ {}_K) \tag{3.32}$$

By Theorem 2.1

$$p^F(n_1,\ldots,n_{K-1},\ {}_1,\ldots,\ {}_K) = p^R(n'_1,\ldots,n'_{K-1},\ {}'_1,\ldots,\ {}'_K) \tag{3.33}$$

Whenever

$$n'_i = N_{K-i}^F - n_{K-i} \qquad i=1,\ldots,K-1 \tag{3.34}$$

$$\alpha'_i = \alpha_{K-i+1} \qquad i=1,\ldots,K \tag{3.35}$$

Thus (3.32) becomes

$$\bar{n}_i^F = \sum_{\substack{\text{all } \alpha'_i = 0,1 \\ \text{all } n'_i = 0,\ldots,N_i^R}} (N_{K-i}^R - n'_{K-i})\; p^R(n'_1,\ldots,n'_{K-1}\alpha'_1,\ldots,\alpha'_K) \tag{3.36}$$

$$\bar{n}_i^F = \overline{N_{K-i}^R - n'_{K-i}} = N_{K-i}^R - \bar{n}^R_{K-i}. \tag{3.37}$$

We now investigate the usefulness of corollaries 3.1.1 and 3.1.2 in the design of transfer lines. Suppose a manufacturing process requires

K stages to complete and suppose that parts can go through the stages of the process in any order. One problem the designer of such a system is faced with is determining the order in which operations are to be performed. Corollary 3.1.1 tells the designer that, once an arrangement has been found, reversing the order of operations will not change the performance of the system as far as production rate is concerned. However, Corollary 3.1.2 reveals that if the designer is concerned with keeping a low in-process inventory, reversing the order of operations might help in that regard. Note, also, that there are other rearrangements for which we can say nothing as far as their effect on performance.

3.4.2 Symmetric Transfer Lines

A symmetric transfer line is one which is identical to its reverse. Formally, System F is symmetric if:

$$N_i^F = N_i^R, \qquad i = 1,\dots,K-1 \tag{3.38}$$

$$r_i^F = r_i^R \qquad i = 1,\dots,K \tag{3.39}$$

and

$$p_i^F = p_i^R \qquad i = 1,\dots,K \tag{3.40}$$

From the definitions of system R we have

$$N_i^F = N_{K-i}^R, \tag{3.41}$$

$$r_i^F = r_{K-i+1}^R, \tag{3.42}$$

and

$$p_i^F = p_{K-i+1}^R. \tag{3.43}$$

Thus for a symmetric system F:

$$N_i^F = N_{K-i}^F \qquad i = 1,\ldots,K-1 \tag{3.44}$$

$$r_i^F = r_{K-i+1}^F \qquad i = 1,\ldots,K \tag{3.45}$$

$$p_i^F = p_{K-i+1}^F \qquad i = 1,\ldots,K \tag{3.46}$$

Symmetric lines have appeared in the literature as results of optimization problems (See Ho et al (1979), Hillier and Boling (1979)). An example of an optimization for a model of transfer lines is maximizing production rate given a constraint on the total amount of buffer storage available e.g.

$$\sum_{i=1}^{K-1} N_i \leqslant M \tag{3.47}$$

where M is a constant.

Such an optimization problem has not been attempted for one transfer line model, but it is suspected that a constraint such as (3.47) under the conditions (3.45) and (3.46) will yield condition (3.44).

We now proceed to state and prove results based on Theorem 3.1, relating to symmetric transfer lines.

<u>Corollary 3.1.3</u>

If system F is a symmetric transfer line then

$$p^F(n_1,\ldots,n_{K-1},\alpha_1,\ldots,\alpha_K)$$

$$= p^F(n_1',\ldots,n_{K-1}',\alpha_1',\ldots,\alpha_K') \quad (3.48)$$

when $\quad n_i' = N_{K-i}^F - n_{K-i} \qquad i = 1,\ldots,K-1 \quad (3.49)$

and $\quad \alpha_i' = \alpha_{K-i+1} \qquad i = 1,\ldots,K \quad (3.50)$

Proof:

From Theorem 3.1 we know

$$p^F(n_1,\ldots,n_{K-1},\alpha_1,\ldots,\alpha_K)$$

$$= p^R(n_1',\ldots,n_{K-1}',\alpha_1',\ldots,\alpha_K') \quad (3.51)$$

Whenever

$$n_i' = N_{K-i}^F - n_{K-i} \qquad i = 1,\ldots,K-1 \quad (3.52)$$

$$\alpha_i' = \alpha_{K-i+1} \qquad i = 1,\ldots,K \quad (3.53)$$

However from the symmetry of the transfer line (F and R are identical) (3.51) is also true whenever

$$n_i' = n_i \qquad i = 1,\ldots,K-1, \quad (3.54)$$

and

$$\alpha_i' = \alpha_i \qquad i = 1,\ldots,K, \quad (3.55)$$

Thus we have

$$p^R\ (n_1,\ldots,n_{K-1}, \alpha_1,\ldots, \alpha_K)$$

$$= p^R\ (n_1',\ldots,n_{K-1}', \alpha_1',\ldots, \alpha_K') \qquad (3.56)$$

Whenever

$$n_i' = N_{K-i}^F - n_{K-i} = N_{K-i}^R - n_{K-i} \qquad i=1,\ldots,K-1 \qquad (3.57)$$

and

$$\alpha_i' = \alpha_{K-i+1} \qquad i = 1,\ldots,K \qquad (3.58)$$

Since R and F are identical systems the R superscripts on the probabilities can be replaced by F and thus the Corollary is proven.

One implication of Corollary 3.1.3 is that if one is solving for the probability distribution of a symmetric transfer, one only need to solve for about half the probabilities. The number is not exactly half because (3.48) will cometimes yield a trivial identity indicating that

$$p^F(s) = p^F(s) \qquad (3.59)$$

The next corollary states a result for the mean in-process inventory of a symmetric transfer line.

<u>Corollary 3.1.4</u>

For a symmetric transfer line F

$$\overline{n}_i^F = N_{K-i}^F - \overline{n}_{K-i}^F \qquad (3.60)$$

Proof:

$$\bar{n}_i^F = \sum_{\substack{\text{all } \alpha_i=0,1 \\ \text{all } n_i=0,\ldots,N_i^F}} n_i \; p^F(n_1,\ldots, n_{K-1} \;, \alpha_1,\ldots, \alpha_K) \tag{3.61}$$

From Corollary 3.1.3 we have

$$p^F(n_1,\ldots, n_{K-1}, \alpha_1,\ldots, \alpha_K)$$

$$= p^F(n_1',\ldots, n_{K-1}', \alpha_1',\ldots, \alpha_K') \tag{3.62}$$

Whenever

$$n_i' = N_{K-i}^F - n_i \qquad i = 1,\ldots,K-1 \tag{3.63}$$

$$\alpha_i' = \alpha_{K-i+1} \qquad i = 1,\ldots,K \tag{3.64}$$

Therefore (3.61) becomes by a change of variables:

$$\bar{n}_i^F = \sum_{\substack{\text{all} \alpha_i=0,1 \\ \text{all } n_i=0,\ldots,N_{K-i}^F}} (N_{K-i}^F - n_{K-i}') \; p^F(n_1',\ldots, n_{K-1}', \alpha' \;,\ldots \alpha_K') \tag{3.65}$$

$$\bar{n}_i^F = \overline{N_{K-i}^F - n_{K-i}} = N_{K-i}^F - \bar{n}_{K-i}^F \tag{3.66}$$

Hence the corollary is proven.

One interesting consequence of the above corollary is that if a transfer line is symmetric around a buffer, i.e. if $\frac{K}{2}$ is an integer, the following relationship holds.

$$\bar{n}^F_{\frac{K}{2}} = N^F_{K-\frac{K}{2}} - \bar{n}^F_{K-\frac{K}{2}} \tag{3.67}$$

$$= N^F_{\frac{K}{2}} - \bar{n}^F_{\frac{K}{2}} \tag{3.68}$$

Thus

$$\bar{n}^F_{\frac{K}{2}} = \frac{1}{2} \quad N^F_{\frac{K}{2}} \tag{3.69}$$

Equation (3.69) implies that the expected in-process inventory in the middle buffer $(\frac{K}{2})$ is always half of its capacity. This is independent of all system parameters.

3.5 Reversibility for other Transfer Line Models

In this section we present evidence that reversibility holds for other transfer line and queueing models.

1 - The M/M/1/K Queue Kleinrock (1975)

This system has poisson arrivals with rate λ ,exponential service times with rate μ , one server, and a finite queueing capacity for K customers. If an arrival finds the system full it is lost forever. We denote the steady state probability of finding k customers in the system by $p^F(k)$. From Kleinrock (1975)

$$p^F(k) = \begin{cases} \dfrac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{K+1}} \left(\dfrac{\lambda}{\mu}\right)^k & 0 \leqslant k \leqslant K \\ 0 & \text{Otherwise} \end{cases} \tag{3.70}$$

Define the reversed M/M/1/K system as one with arrival rate and service rate , and let $p^R(k)$ be the steady state probability of finding k customers in the reversed systems. Thus

$$p^R(k) = \begin{cases} \dfrac{1 - \mu/\lambda}{1 - (\mu/\lambda)^{K+1}} \left(\dfrac{\mu}{\lambda}\right)^k & 0 \leqslant k \leqslant K \\ 0 & \text{Otherwise} \end{cases} \tag{3.71}$$

By simple algebraic manipulations it can be shown that:

$$p^F(k) = p^R(K-k) \tag{3.72}$$

This is analogous to the strong reversibility property proven for our transfer line model. (Theorem 3.1).

Note that the M/M/1/K system is a model for a reliable two-machine transfer line. "Reliable" indicates that the machines are not prone to failure. The arrival rate is the processing rate of the first machine in the line. The queue capacity is the interstage buffer of capacity k. From this point of view (3.72) describes the relation between the steady state probability distribution of a transfer line and its reverse.

There is further evidence that when queues of the above type are placed in tandem, strong reversibility properties similar to (3.72) can be established.

2 - Muth's Proof of Weak Reversibility

Muth (1979) proves that under rather general assumptions the production rate of a transfer line is equal to that of its reverse. For the case where each part entering the line has deterministic (and in general, different) processing times on each machine. Muth requires time reversibility in addition to line reversibility. Time reversibility means that parts have to be fed to the reversed line in the reverse order in which they are fed to the forward line. This restriction is not relevant to our model since machines cannot distinguish between parts, i.e. all parts take the same time to process in all machines.

3.6 Summary and Conclusions

In this Chapter we prove a strong reversibility property for transfer lines. Implications of this property are stated and proved for both general transfer lines, and the special case of symmetric lines. We conjecture that such properties will hold for more general models of transfer lines.

These results, in addition to being significant in their own right, provide the ground work for even more striking results and strong conjectures regarding more general AMN's. By "reversing" only portions of AMN's, in the same manner entire transfer lines are reversed, we are able to show equivalences among systems that seem to bear no relationship to each other at first glance. This is done in the next chapter.

Chapter 4

Equivalence Concepts for General AMN's

4.1 General Remarks

In this chapter we use the part-hole duality ideas introduced in Chapter 3 to argue equivalence properties for AMN's. These are a generalization of the transfer line reversibility properties of Chapter 3, to more general AMN configurations. The main result is that AMN's can be grouped into equivalence classes where performance measures of the members of an equivalence class are closely related.

4.2 Some Definitions

Two AMN's are said to be structurally equivalent if:

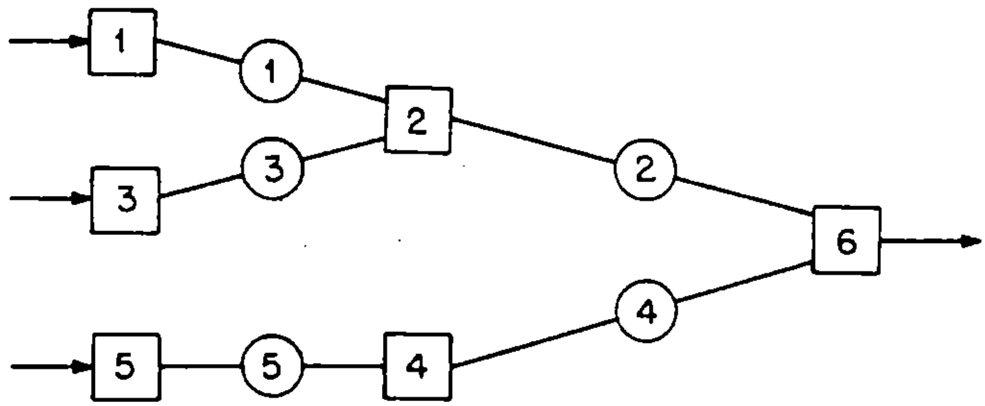
1 - There exists a correspondence between machines of each and buffers of each so that corresponding machines have the same r and p parameters, and corresponding buffers have the same capacities.

2 - Corresponding buffers are connected to corresponding machines, although parts do not necessarily flow in the same direction.

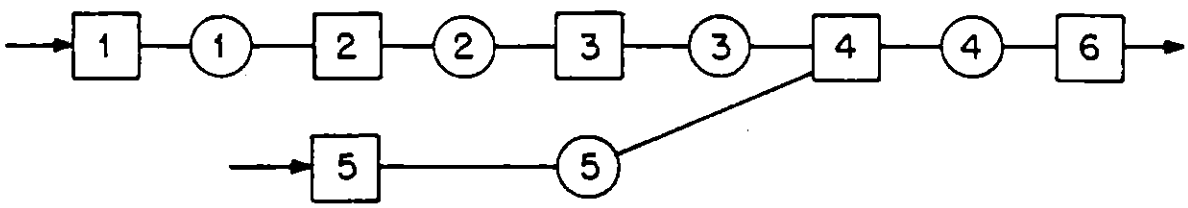
An example of two structurally equivalent systems appears in Figure 4.1, where the tables define the machine and buffer correspondences. So if we use the superscripts I and II to identify parameters of AMN I and AMN II respectively we have:

$$N_4^I = N_2^{II},\ N_1^I = N_4^{II},\ N_2^I = N_3^{II},\ N_3^I = N_5^{II},\ N_5^I = N_1^{II}$$

$$r_1^I = r_6^{II},\ r_2^I = r_4^{II},\ r_3^I = r_5^{II},\ r_4^I = r_2^{II},\ r_5^I = r_1^{II},\ r_6^I = r_3^{II}$$



AMN I



AMN II

CORRESPONDENCE

Buffers		Machines	
I	II	I	II
1	4	1	6
2	3	2	4
3	5	3	5
4	2	4	2
5	1	5	1
		6	3

Figure 4.1 Two Structurally Equivalent AMN's

and

$$p_1^I = p_6^{II},\ p_2^I = p_4^{II},\ p_3^I = p_5^{II},\ p_4^I = p_2^{II},\ p_5^I = p_1^{II},\ p_6^I = p_3^{II}$$

Also we say that a transfer line and its reverse are structurally equivalent, by definition.

Two AMN's are probabilistically equivalent when there exists a one-to-one correspondence between states of each, such that corresponding states have the same probabilities.

Following the above definitions, and using the results of Chapter 3 we can immediately state the following theorem:

Theorem 4.1

A transfer line and its reverse are probabilistically equivalent.

This is merely a restatement of Theorem 3.1.

Also above we argued that a transfer line and its reverse are structurally equivalent. Thus we have shown that the following corollary is true.

Corollary 4.1.1

For transfer lines structural equivalence implies probabilistic equivalence.

This follows immediately from theorem 4.1 and the definition of the structural equivalence.

Theorem 4.1 and Corollary 4.1.1 illustrate the concepts of structural and probabilistic equivalence. Also Corollary 4.1.1 motivates the classification of AMN's into equivalence classes. We say two AMN's belong to the same <u>equivalence class</u> if they are probabilistically and structurally equivalent (or just <u>equivalent</u> for short).

In this Chapter we use the part-hole duality ideas introduced in Chapter 3 to construct equivalence classes. We will also argue that members of the same equivalence class have the same production rates and related mean in-process inventories. These results are only in the form of conjectures for AMN's with more than three machines.

4.3 The Two-Machine Equivalence Class

The only possible configuration of two machines, within the confines of our AMN model, is that of a transfer line. (See Figure 4.2) Thus by Theorem 4.1 and Corollary 4.1.1, we can trivially construct for any given set of parameters a two-machine equivalence class. This will consist of a two-machine transfer line and its reverse.

Formally, define

$$F2 = (\phi, \{1\})$$

$$R2 = (\phi, \{1\})$$

where

$$N_1^{F2} = N_1^{R2} \tag{4.1}$$

$$r_1^{F2} = r_2^{R2}, \quad r_2^{F2} = r_1^{R2} \tag{4.2}$$

and

$$p_1^{F2} = p_2^{R2}, \quad p_2^{F2} = p_1^{R2} \tag{4.3}$$

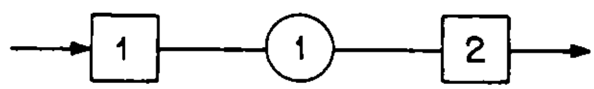


Figure 4.2 Two-Machine Transfer Line

Systems F2 and R2 are structurally equivalent. Theorem 4.1 states that they are also probabilistically equivalent, and thus by definition F2 and R2 belong to the same equivalence class.

4.4 The Three-Machine Equivalence Class

In this section we construct, for any given set of parameters, a three-machine equivalence class. The fact that there is only one such class is not immediately obvious. In fact the proof that for a given set of parameters, there is only one three-machine equivalence class is the major task to be carried out in this section.

We start by considering the three-machine transfer line. Following the argument in Section 4.1 we know that a three-machine transfer line and its reverse belong to the same equivalence class.

In other words define

$$F3 = (\phi, \{1\}, \{2\})$$

$$R3 = (\phi, \{1\}, \{2\})$$

where

$$N_1^{F3} = N_2^{R3}, \quad N_2^{F3} = N_1^{R3} \tag{4.4}$$

$$r_1^{F3} = r_3^{R3}, \quad r_2^{F3} = r_2^{R3}, \quad r_3^{F3} = r_1^{R3} \tag{4.5}$$

$$p_1^{F3} = p_3^{R3}, \quad p_2^{F3} = p_2^{R3}, \quad p_3^{F3} = p_1^{R3} \tag{4.6}$$

Thus R3 is the reverse of F3. We know that R3 and F3 belong to the same equivalence class.

4.4.1 The Strong Equivalence Property

Now consider the only other possible AMN configuration of three machines and two buffers. This is where machines 1 and 2 produce parts to be assembled into a final product by machine 3. (See Figure 4.3). We call this a simple assembly system.

Define a simple assembly system that is structurally equivalent to system F3 (and thus R3) as follows:

$$A3 = (\phi, \phi, \{1,2\})$$

wherem

$$N_1^{A3} = N_1^{F3},\ N_2^{A3} = N_2^{F3} \tag{4.7}$$

$$r_1^{A3} = r_1^{F3},\ r_2^{A3} = r_3^{F3},\ r_3^{A3} = r_2^{F3} \tag{4.8}$$

and

$$p_1^{A3} = p_1^{F3},\ p_2^{A3} = p_3^{F3},\ p_3^{A3} = p_2^{F3} \tag{4.9}$$

The question to be answered now is whether systems A3 and F3 (and R3) are also probabilistically equivalent.

Claim: Systems A3, F3, and R3 are probabilistically equivalent, and hence belong to the same equivalence class.

Before presenting a formal proof of this claim, we use part-hole duality to argue that the assembly system (A3) resembles the three-machine transfer line (F3). This is done by focusing on the motion of holes (or empty spaces) instead of parts in the subsystem of A3 consisting of machines 2 and 3 and buffer 2.

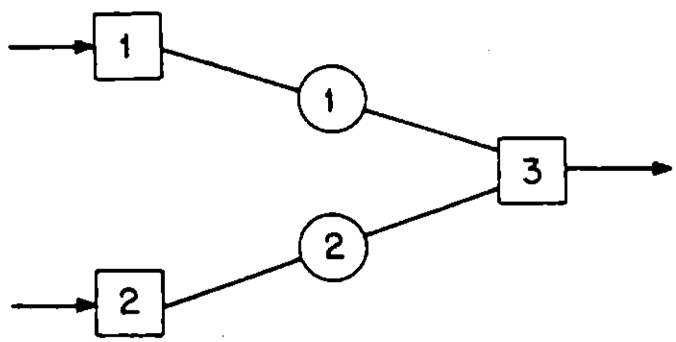


Figure 4.3 Three-Machine Assembly System

Machine 3 in System A3 takes one part from buffer 1 and one part from buffer 2 at the same time. But taking a part from a buffer is equivalent to depositing a hole in the same buffer. Thus machine 3 can be viewed as taking a part from buffer 1 and at the same time depositing a hole in buffer 2. Holes that accumulate in buffer 2 exit the system via machine 2.

Let us now follow the progress of a "work unit" through System A3. The work unit is a part in the section of the system consisting of machines 1 and 3, and buffer 1. It then becomes a hole in the section consisting of machines 3, and 2, and buffer 2. Whether the work unit is a part or a hole the machines fail and get repaired in the same manner, (see Table 3.1). This work unit enters the system (A3) via machine 1, it gets processed by machine 3, and leaves through machine 2. Thus machines 1 and 2 look like the input machine and the output machine respectively of a three-machine transfer line.

This argument suggests the statement of a strong equivalence property for the two systems A3 and F3.

Theorem 4.2 Strong Equivalence Property

For the two Systems A3 and F3

$$p^{F3}\ (n_1,\ n_2,\ \alpha_1,\ \alpha_2,\ \alpha_3) = p^{A3}\ (n_1',\ n_2',\ \alpha_1',\ \alpha_2',\ \alpha_3') \qquad (4.10)$$

Whenever $n_1' = n_1$ (4.11)

$$n_2' = N_2^{F3} - n_2 \qquad (4.12)$$

$$\alpha_1' = \alpha_1 \tag{4.13}$$

$$\alpha_2' = \alpha_3 \tag{4.14}$$

$$\alpha_3' = \alpha_2 \tag{4.15}$$

The proof of Theorem 4.2 follows along the same lines as that for Theorem 3.1. Because of the excessive number of tables involved in the proof, it has been relegated to Appendix II. Also note that we can make statements about the transient behavior of the probability distribution in exactly the same manner as was discussed in the proof of Theorem 3.1.

One important consequence of Theorem 4.2 is that solving for the steady state probability distribution of Systems A3 and F3 are identical problems. For example if one has a program that generates the probability distribution of System F3, only modifications as to how the results are interpreted are needed to make that same program workable for System A3. Such a program is actually available (Gershwin and Schick (1979)). In the next section we show how it can be used to determine the performance measures of System A3 as well as the steady state probability distribution.

Theorem 4.2 shows that Systems F3 and A3 are probabilistically equivalent. By construction A3 is also structurally equivalent to F3. Thus F3 and A3 belong to the same equivalence class. Note that R3 also belongs to the same class.

We conjecture that a three-machine disassembly system (Figure 4.4) that is structurally equivalent to Systems A3, F3, and R3 is also probabilistically equivalent, and hence belongs to the same equivalence class. To argue this one can focus on the motion of holes in System A3. This hole motion is the same as part potion in a disassembly system. A formal proof of this is beyond the scope of this thesis, since no formal disassembly model has been introduced.

4.4.2 Performance Measures

Theorem 4.2 leads one to suspect that it is possible to calculate the performance measures of System A3 directly from those of System F3. In fact this is true. We now state and prove corollaries to Theorem 4.2 that establish the relationships between the performance measures of System F3 and those of System A3.

Corollary 4.2.1 Weak Equivalence Property

For Systems A3 and F3

$$R^{A3} = R^{F3} \tag{4.16}$$

where R^{A3} and R^{F3} are the production rates for Systems A3 and F3, respectively.

Proof:

We know that

$$R^{F3} = \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \sum_{n_1=0}^{N_1^{F3}} \sum_{n_2=1}^{N_2^{F3}} p^{F3} (n_1, n_2, \alpha_1, \alpha_2, 1) \tag{4.17}$$

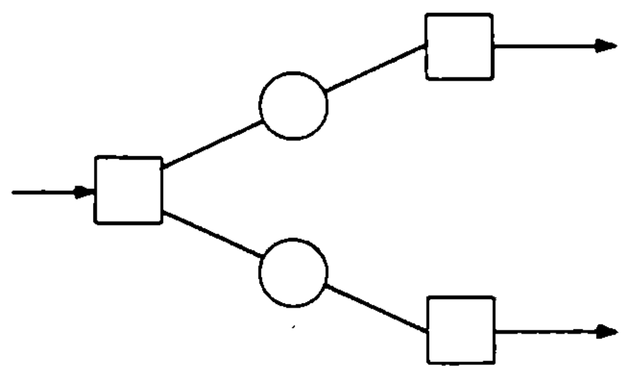


Figure 4.4 Three-Machine Disassembly System

However by Theorem 4.1 we have

$$P^{F3}(n_1,n_2,\alpha_1,\alpha_2,\alpha_3) = P^{A3}(n_1',n_2',\alpha_1',\alpha_2',\alpha_3') \tag{4.18}$$

Whenever $n_1' = n_1$ (4.19)

$$n_2' = N_2^{F3} - n_2 \tag{4.20}$$

$$\alpha_1' = \alpha_1 \tag{4.21}$$

$$\alpha_2' = \alpha_3 \tag{4.22}$$

$$\alpha_3' = \alpha_2 \tag{4.23}$$

Hence

$$R^{F3} = \sum_{\alpha_1'=0}^{1} \sum_{\alpha_3'=0}^{1} \sum_{n_1'=0}^{N_1^{F3}} \sum_{n_2=N_2^{F3}-1}^{0} P^{A3}(n_1',n_2',\alpha_1',1,\alpha_3') \tag{4.24}$$

But since $N_i^{F3} = N_i^{A3}$ $\quad i = 1,2$ (4.25)

we have

$$R^{F3} = \sum_{\alpha_1'=0}^{1} \sum_{\alpha_3'=0}^{1} \sum_{n_1'=0}^{N_1^{A3}} \sum_{n_2'=0}^{N_2^{A3}-1} P^{A3}(n_1',n_2',\alpha_1',1,\alpha_3') \tag{4.26}$$

The expression on the right hand side of (4.26) is exactly that for the input rate into System A3 via machine 2, D_2^{A3} (equation (2.36))

Therefore

$$R^{F3} = D_2^{A3} \tag{4.27}$$

However by conservation of flow (Theorem 2.1)

$$D_2^{A3} = R^{A3} \tag{4.28}$$

Hence

$$R^{F3} = R^{A3} \tag{4.29}$$

and the corollary is proven.

The next corollary deals with relating the mean in-process inventories for Systems A3 and F3.

<u>Corollary 4.2.2</u>

Let $\bar{n}_i^{A3}$, and $\bar{n}_i^{F3}$, i=1,2, be the mean in-process inventories at buffer i for Systems A3, and F3, respectively. Then

$$\bar{n}_1^{A3} = \bar{n}_1^{F3} \tag{4.30}$$

$$\bar{n}_2^{A3} = N_2^{F3} - \bar{n}_2^{F3} \tag{4.31}$$

<u>Proof</u>:

We know

$$\bar{n}_i^{A3} = \sum_{\alpha_1'=0}^{1} \sum_{\alpha_2'=0}^{1} \sum_{\alpha_3'=0}^{1} \sum_{n_1'=0}^{N_1^{A3}} \sum_{n_2'=0}^{N_2^{A3}} n_i P^{A3}(n_1', n_2', \alpha_1', \alpha_2', \alpha_3') \tag{4.32}$$

By Theorem 4.2

$$P^{F3}(n_1, n_2, \alpha_1, \alpha_2, \alpha_3) = P^{A3}(n_1', n_2', \alpha_1', \alpha_2', \alpha_3') \tag{4.33}$$

whenever

$$n_1' = n_1, \; n_2' = N_2^{F3} - n_2 \tag{4.34}$$

$$\alpha_1' = \alpha_1, \; \alpha_2' = \alpha_3, \; \alpha_3' = \alpha_2 \tag{4.35}$$

Thus by a change of variables (4.33) becomes

$$\bar{n}_i^{A3} = \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \sum_{\alpha_3=0}^{1} \sum_{n_1=0}^{N_1^{A3}} \sum_{n_2=0}^{N_2^{A3}} n_i P^{F3} (n_1, n_2, \alpha_1, \alpha_2, \alpha_3) \tag{4.36}$$

Where n_i' is related to the summation variables by (4.34).

For i=1 we have $n_1'=n_1$ and since $N_i^{A3} = N_i^{F3}$, i=1,2.

Equation (4.37) becomes

$$\bar{n}_1^{A3} = \bar{n}_1^{F3} \tag{4.37}$$

For i=2 we have $n_2' = N_2^{F3} - n_2$,

therefore (4.36) becomes

$$\bar{n}_2^{A3} = \overline{N_2^{F3} - n_2} = N_2^{F3} - \bar{n}_2^{F3} \tag{4.38}$$

Thus the corollary is proven.

4.4.3 Symmetric Three-Machine Assembly Systems

In this section we prove certain statements regarding symmetric three machine assembly systems. Define System A3 to be symmetric when

$$N_1^{A3} = N_2^{A3} \tag{4.39}$$

$$r_1^{A3} = r_2^{A3} \tag{4.40}$$

and

$$P_1^{A3} = P_2^{A3} \tag{4.41}$$

A symmetric A3 system belongs to the same equivalence class as a symmetric three-machine transfer line with the same parameters for corresponding machines and buffers. This can be easily shown by the following argument.

Consider the three-machine transfer line F3 that belongs to the same equivalence class as the symmetric system A3 defined above. By (4.7), (4.8), and (4.9) we have

$$N_1^{F3} = N_1^{A3} = N_2^{A3} = N_2^{F3} \tag{4.42}$$

$$r_1^{F3} = r_1^{A3} = r_2^{A3} = r_3^{F3} \tag{4.43}$$

$$p_1^{F3} = p_1^{A3} = p_2^{A3} = p_3^{F3} \tag{4.44}$$

Relations (4.42), (4.43), and (4.44) establish that F3 is indeed a symmetric transfer line. We use this to prove the following corollary.

<u>Corollary 4.2.3</u>

For a Symmetric A3 System we have

a) $$p^{A3}(n_1,n_2,\alpha_1,\alpha_2,\alpha_3) = p^{A3}(n_1',n_2',\alpha_1',\alpha_2',\alpha_3') \tag{4.45}$$

whenever

$$n_1'=n_2,\ n_2'=n_1 \tag{4.46}$$

$$\alpha_1'=\alpha_2,\ \alpha_2'=\alpha_1,\ \alpha_3'=\alpha_3 \tag{4.47}$$

b) $$\bar{n}_1^{A3} = \bar{n}_2^{A3} \tag{4.48}$$

The results stated in Corollary 4.2.2 are rather intuitive and can be argued from the symmetry of the system. Thus, they do not need formal proof. However, a proof is provided here mainly to check on the consistency of the results obtained so far regarding reversibility of transfer lines and equivalence of A3 and F3 systems.

Proof of Corollary 4.2.3

a) We know by Theorem 4.2

$$P^{A3}(n_1, n_2, \alpha_1, \alpha_1, \alpha_3) = P^{F3}(n_1', n_2', \alpha_1', \alpha_2', \alpha_3') \qquad (4.49)$$

whenever $n_1' = n_1, \quad n_2' = N_2^{F3} - n_2$ (4.50)

and $\alpha_1' = \alpha_1, \ \alpha_2' = \alpha_3, \ \alpha_3' = \alpha_2$ (4.51

But we know that F3 is a symmetric transfer line and hence by Corollary 3.1.3

$$P^{F3}(n_1', n_2', \alpha_1', \alpha_2', \alpha_3') = P^{F3}(n_1'', n_2'', \alpha_1'', \alpha_2'', \alpha_3'') \qquad (4.52)$$

whenever

$$n_1'' = N_2^{F3} - n_2', \quad n_2'' = N_1^{F3} - n_1' \qquad (4.53)$$

$$\alpha_1'' = \alpha_3', \quad \alpha_2'' = \alpha_2', \quad \alpha_3'' = \alpha_1' \qquad (4.54)$$

Thus from (4.49) through (4.54)

$$P^{A3}(n_1, n_2, \alpha_1, \alpha_2, \alpha_3) = P^{F3}(n_1'', n_2'', \alpha_1'', \alpha_2'', \alpha_3'') \qquad (4.55)$$

whenever

$$n_1'' = N_2^{F3} - n_2 = N_2^{F3} - N_2^{F3} + n_2 = n_2 \qquad (4.56)$$

$$n_2'' = N_1^{F3} - n_1' = N_1^{F3} - n_1 \qquad (4.57)$$

$$\alpha_1'' = \alpha_3' = \alpha_2 \qquad (4.58)$$

$$\alpha_2'' = \alpha_2' = \alpha_3$$

and

$$\alpha_3'' = \alpha_1' = \alpha_1 \qquad (4.60)$$

However by Theorem 4 we have

$$P^{F3}(n_1'', n_2'', \alpha_1'', \alpha_2'', \alpha_3'') = P^{A3}(n_1''', n_2''', \alpha_1''', \alpha_2''', \alpha_3''') \qquad (4.61)$$

whenever

$$n_1''' = n_1'' \qquad (4.62)$$

$$n_2''' = N_2^{F3} - n_2'' \qquad (4.63)$$

$$\alpha_1''' = \alpha_1'' \qquad (4.64)$$

$$\alpha_2''' = \alpha_3'' \qquad (4.65)$$

$$\alpha_3''' = \alpha_2'' \qquad (4.66)$$

From (4.55) through (4.66) we have

$$P^{A3}(n_1, n_2, \alpha_1, \alpha_2, \alpha_3) = P^{A3}(n_1''', n_2''', \alpha_1''', \alpha_2''', \alpha_3''') \qquad (4.67)$$

whenever

$$n_1''' = n_1'' = n_2 \qquad (4.68$$

$$n_2''' = N_2^{F3} - n_2'' = N_2^{F3} - N_1^{F3} + n_1 = n_1 \quad (4.69)$$

$$(N_2^{F3} = N_1^{F3} \text{ by } (4.43))$$

$$\alpha_1''' = \alpha_1'' = \alpha_2 \quad (4.70)$$

$$\alpha_2''' = \alpha_3'' = \alpha_1 \quad (4.71)$$

$$\alpha_3''' = \alpha_2'' = \alpha_3 \quad (4.72)$$

Thus part a) of the corollary is proven.

b) From Corollary 4.2.2 we have

$$\bar{n}_1^{A3} = \bar{n}_1^{F3} \quad (4.73)$$

However since F3 is a symmetric transfer line we have from Corollary (3.1.4)

$$\bar{n}_1^{F3} = N_2^{F3} - \bar{n}_2^{F3} \quad (4.74)$$

However from Corollary 4.2.2 we know

$$\bar{n}_2^{A3} = N_2^{F3} = \bar{n}_2^{F3} \quad (4.75)$$

or

$$\bar{n}_2^{F3} = N_2^{F3} - \bar{n}_2^{A3} \quad (4.76)$$

Thus (4.74) becomes

$$\bar{n}_1^{F3} = N_2^{F3} - (N_2^{F3} - \bar{n}_2^{A3}) = \bar{n}_2^{A3} \quad (4.77)$$

Hence

$$\bar{n}_1^{A3} = \bar{n}_1^{F3} = \bar{n}_2^{A3} \tag{4.78}$$

and part b) of the corollary is proven.

4.5 K-Machine AMN Equivalence Classes

We have shown two examples of AMN equivalence classes. In this section we informally describe how a general K-machine AMN equivalence class may be constructed. The discussion is based on the intuitive idea of part-hole duality, and no formal proof is attempted.

For simplicity we take a departure from our usual AMN labelling convention introduced in Chapter 2. Here two machines from two different members of an equivalence class have the same label if they are identical (i.e. have the same failure and repair probabilities). Similarly buffers with the same label within an equivalence class have the same capacity.

As an example consider constructing, for some set of parameter, all four machine structural equivalence (SE) classes, where a SE class is one whose members are structurally equivalent. We claim that there are two such classes shown in Figures 4.5 and 4.6. In Figure 4.5 the class which has the four-machine transfer line and its reverse as members is illustrated. The systems in 4.5 c) and 4.5 d) can be constructed from the forward transfer line by considering hole motion in the appropriate portion of the transfer line. (Note that we can informally add two disassembly systems to this class. They are the reverse of systems 4.5 (c) and (d)).

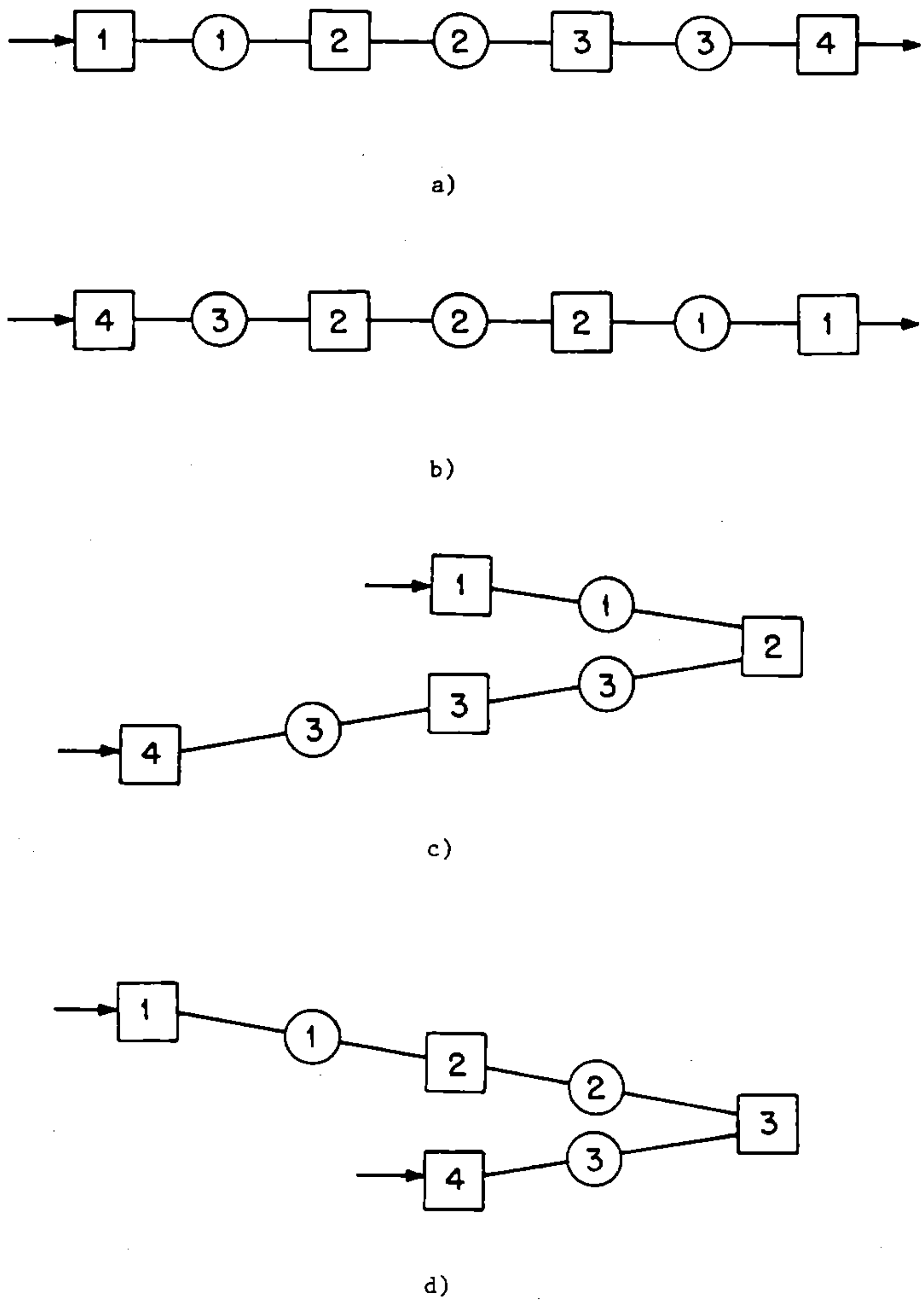


Figure 4.5 One Four-Machine Equivalence Class

The second four-machine SE class is illustrated in Figure 4.6. The system in Figure 4.6a), the subassemblies (or parts) are produced by machines 1, 2, and 3. They are then assembled by the last machine to produce the final product. Consider the hole motion in one of the branches that produce the subassemblies, say machine 1, and buffer 1. The holes leave machine 4 and are then deposited in buffer 1 and leave the system via machine 1. By making machine 1 the output machine we construct the structurally equivalent system in Figure 4.6.b). The systems in Figures 4.6c) and d) are constructed similarly. (We can also add to this class four disassembly systems which are the "reverses" of systems a), b), c) and d).) We now conjecture that the members of each SE class are also probabilistically equivalent. We already know this is true for the case of a transfer line and its reverse, and suspect that for others table proofs similar to the ones for Theorems 3.1 and 4.2 can be developed.

4.6 Summary and Conclusions

In this Chapter we have shown how two AMN's that are structurally equivalent can be shown to be probabilistically equivalent. This was done in a rigorous manner for two and three machine systems. Also an informal discussion of how one can extend this to larger systems was carried out. In essence, we have shown that by solving for the probability distribution and performance measures of one AMN, one is solving for the same quantities of several other AMN's. Systems related in this

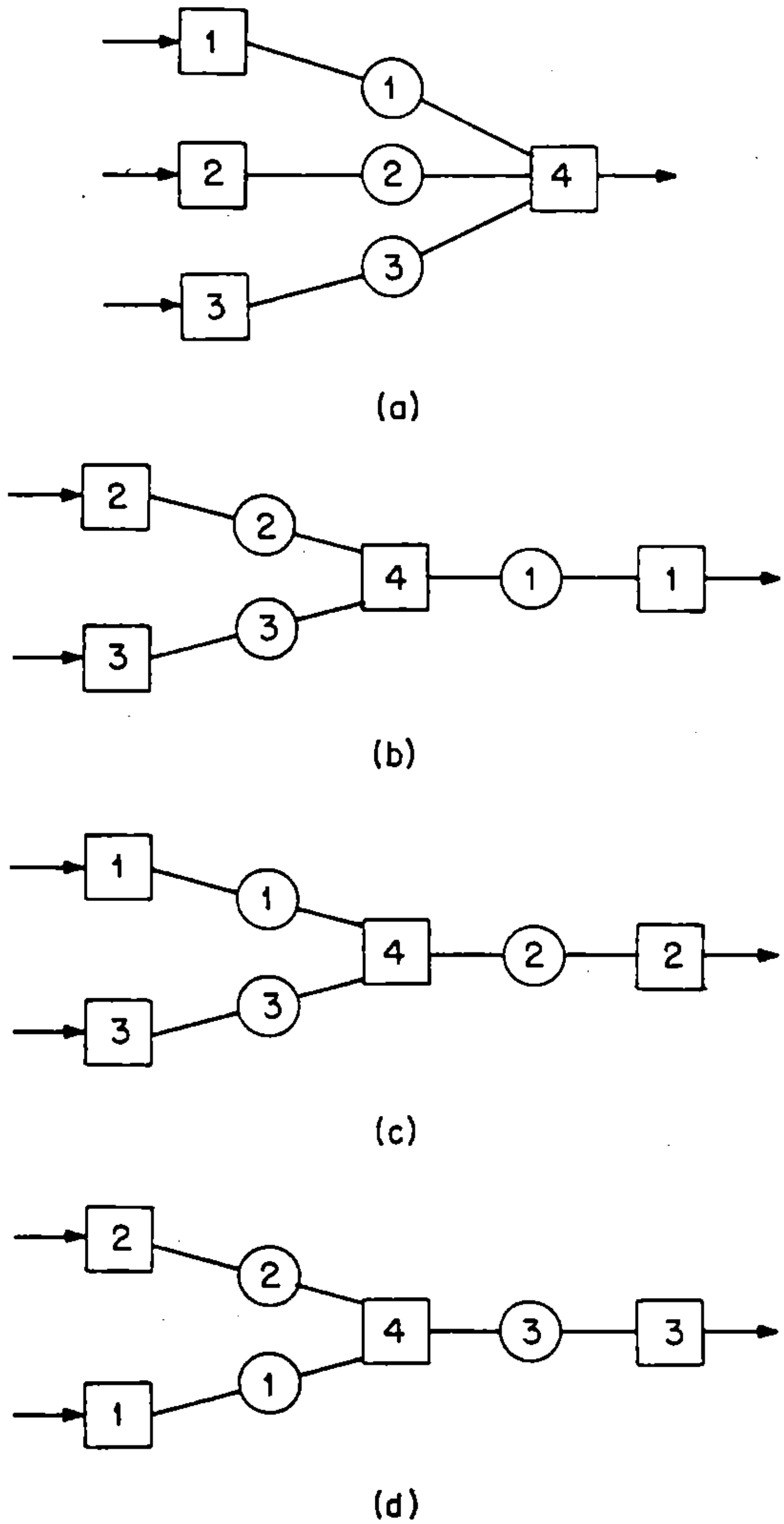


Figure 4.6 Another Four-Machine Equivalence Class

manner are said to belong to the same equivalence class.

To put this result in perspective, consider that one of the tasks that was to be performed for this thesis was a complete solution to the three machine assembly systems (Figure 4.3). It was thought at the outset that we had to devise a solution procedure similar to that in Gershwin and Schick (1979,1980) for the three-machine transfer line. Instead we were able to prove Theorem 4.2 relating the three-machine transfer line and assembly system. All that is needed to obtain the derived quantities in a three machine assembly system from the program in Gershwin and Schick (1979) is a slightly different interpretation of the output.

Similar savings of effort are implied by the conjectured results pertaining to larger AMN's. The immediate usefulness of such results is hampered by the fact that by present methods we are only partially able to extend the solution procedure to systems containing more than three machines.

Chapter 5

Solution Technique

5.1 General Remarks

In this section we present a technique for obtaining the values of the steady state probability distribution for AMN's. These probabilities are to be used to calculate the performance measures discussed in Chapter 2.

The most conceptually straightforward method of obtaining the steady state probabilities is by solving M linear equations for M unknowns, where M is given by (2.11). The equations are M-1 of the M transition equations (2.28) and the normalization equation (2.27). However, for any moderate size problem, the number of equations is prohibitively large, thus making a solution of the system of equations by standard linear equation techniques, impractical.

A method of solution is presented here which circumvents some of these difficulties. The method, unfortunately is only complete for two- and three- machine AMN's. It should be noted that the technique to be presented here is basically the same one presented in Gershwin and Schick (1979, 1980). The contribution of this thesis to the solution method is threefold. First we present a complete internal analysis for general AMN's, the results of which have been conjectured in Gershwin and Ammar (1979). Second, the results of the boundary analysis for three-machine systems are related to those found for two-machine systems in Gershwin and Schick (1980), and Schick and Gershwin (1979). Furthermore conjectures are presented on how the three-machine boundary results can be extended to larger, more complex systems. Third, we use the results of Chapters 3 and 4 to recognize that by solving for the performance measures of a three-machine transfer line, one really is solving for the measures of all members of a three-machine equivalence classes (See Chapter 4).

5.2 Internal Analysis

5.2.1 Some Definitions

Internal states $s=(n_1,\ldots, n_{K-1}, \alpha_1,\ldots,\alpha_K)$ are defined as those that in which $2 \leqslant n_i \leqslant N_i-2$ for all $i \in \{1 \ldots, K-1\}$.

Internal transition equations are defined to be those involving only internal states, i.e. the subset of equations (2.28) in chich s_j is internal and, for all s_i such that $T(s_j, s_i) = 0$, s_i is internal. When all states are internal, each operational machine can take parts from its upstream buffers, and can put a part in its downstream buffer.

Internal states are those that conform to Case 1 of Table 2.2. Hence

$$n_i(t+1) = n_i(t) - \alpha_{d_i}(t+1) + \alpha_i(t+1) \qquad (5.1)$$

for all $i = 1,\ldots, K-1$. Recall that γ_i is assumed to be 1.

5.2.2 The Internal Transition Equations

The general form of any transition equation is the following:

$$p(s(t+1)) = \sum_{\text{all } s(t)} T\,(s(t+1), s(t))\; p(s(t)) \qquad (5.2)$$

This is the same as equation (2.28) with s_j replaced by $s(t+1)$ and s_i

replaced by s(t).

From the discussion in Chapter 2 we have

$$T(s(t+1),\ s(t)) =$$

$$\begin{cases} \prod_{i=1}^{K} \text{Prob}\left[\alpha_i(t+1) \mid n_i(t),\ \alpha_i(t),\ n_j(t),\ j \in L(i)\right] & \text{if (5.1) is satisfied for all } i \\ 0 & \text{Otherwise} \end{cases} \tag{5.3}$$

For each i we can write (from Table 2.1)

$$\text{Prob}\left[\alpha_i\ (t+1) \mid n_i(t),\ \alpha_i(t),\ n_j(t),\ j \in\ L\ (i)\right]$$

$$= \left[(1-r_i)^{1-\alpha_i(t+1)}\ r_i^{\alpha_i(t+1)}\right]^{1-\alpha_i(t)}$$

$$\cdot \left[(1-p_i)^{\alpha_i(t+1)}\ p_i^{1-\alpha_i(t+1)}\right]^{\alpha_i(t)} \tag{5.4}$$

Using (5.4), (5.2) can be rewritten as

$$p(\underline{n}\ (t+1),\ \ \underline{\alpha}\ (t+1)) =$$

$$\sum_{\alpha_1(t)=0}^{1} \cdots \sum_{\alpha_K(t)=0}^{1} \prod_{i=1}^{K} \left[(1-r_i)^{1-\alpha_i(t+1)}\ r_i^{\alpha_i(t+1)}\right]^{1-\alpha_i(t)}$$

$$\cdot \left[(1-p_i)^{\alpha_i(t+1)}\ p_i^{1-\alpha_i(t+1)}\right]^{\alpha_i(t)}$$

$$p(\underline{n}(t)\ ,\ \ \underline{\alpha}\ (t)) \tag{5.5}$$

where $\underline{n}$ (t) satisfies (5.1).

5.2.3 The Sum of Products Solution Forms

It is assumed that the steady state probabilities for internal states have the following form:

$$p(s) = \sum_{j=1}^{\ell} c_j \xi(s, U_j) \tag{5.6}$$

where

$$s = (n_1, \ldots, n_{K-1}, \alpha_1, \ldots, \alpha_K) \tag{5.7}$$

$$U_j = (X_{1j}, \ldots, X_{K-1,j}, Y_{1j}, \ldots Y_{Kj}), \tag{5.8}$$

and

$$\xi(s, U_j) = \prod_{i=1}^{K-1} X_{ij}^{n_i} \prod_{i=1}^{K} Y_{ij}^{\alpha_i} \tag{5.9}$$

for s internal.

It is also assumed that each term in (5.9) by itself must satisfy the internal transition equations (5.5). For justification of these assumptions see Gershwin and Schick (1980), and Gershwin and Berman (1978).

We now substitute one term of the solution form (5.6) into equations (5.5) to obtain: (note that the second subscript on X_{ij} and Y_{ij} is ommitted for clarity).

$$\prod_{i=1}^{K-1} X_i^{n_i(t+1)} \prod_{i=1}^{K} Y_i^{\alpha_i(t+1)} =$$

$$\sum_{\alpha_1(t)=0}^{1} \cdots \sum_{\alpha_K(t)=0}^{1} \prod_{i=1}^{K} \left[(1-r_i)^{1-\alpha_i(t+1)} r_i^{\alpha_i(t+1)} \right]^{1-\alpha_i(t)}$$

$$\cdot \left[(1-p_i)^{\alpha_i(t+1)} p_i^{1-\alpha_i(t+1)} \right]^{\alpha_i(t)} \prod_{i=1}^{K-1} X_i^{n_i(t)} \prod_{i=1}^{K} Y_i^{\alpha_i(t)} \tag{5.10}$$

where $n_i(t)$ and $n_i(t)$ are related by (5.1).

By performing algebraic manipulations, the details of which are in Appendix III the following relations among the X_i's and the Y_i's are obtained (See Gershwin and Schick (1980)).

$$\prod_{i=1}^{K} (1-r_i + p_i Y_i)=1 \tag{5.11}$$

$$\frac{X_j}{\prod_{i \in L(j)} X_i} = \frac{r_j + (1-p_j)\, Y_j}{(1-r_j + p_j\, Y_j)\, Y_j} \qquad j=1,\ldots,K \tag{5.12}$$

where $X_K=1$, and for $L(j) = \phi$, as in an input machine, $\prod_{i \in L(j)} X_i = 1$.

These relationships are called <u>parametric</u> <u>equations.</u> They are K+1 non-linear equations in 2K-1 unknowns. Except for the case where K=2, they have an infinite number of solutions. For any $U = (X_1, \ldots X_{K-1}, Y_1, \ldots, Y_K)$ that is a solution to the parametric equations, each $\xi(s, U_j)$ of the form (5.9) satisfies the internal transition equations. Thus any linear combination of the form (5.6), where U_j satisfies (5.11) and (5.12) for each j, also satisfies the transition equations.

In equations (5.12) we define

$$Q_j = \frac{X_j}{\prod_{i \in L(j)} X_i} \tag{5.13}$$

In the parametric equations only the Q_j's contain information about the AMN connectivity properties. For each machine j the numerator of (5.13) represents the X of the downstream buffer, and the denominator is the product of the X's of the upstream buffers. Equation (5.11) and the right hand sides of equations (5.12) summarize the reliability information of the AMN.

Lemma 5.1

$$\prod_{j=1}^{K} Q_j = 1 \tag{5.14}$$

Proof:

$$\prod_{j=1}^{K} Q_j = \prod_{j=1}^{K} \frac{X_j}{\prod_{i \in L(j)} X_i} \tag{5.15}$$

$$= \frac{\prod_{j=1}^{K} X_j}{\prod_{j=1}^{K} \prod_{i \in L(j)} X_i} \tag{5.16}$$

Since each buffer feeds exactly one machine, we have:

$$L(i) \cap L(j) = \phi \text{ , for } i \neq j. \tag{5.17}$$

Also since each buffer must feed some machine we have

$$\bigcup_{i=1}^{K} L(i) = \{ 1, \ldots, K-1 \} \tag{5.18}$$

Thus L(i), i=1,..., K are mutually exclusive and collectively exhaustive sets.

Hence

$$\prod_{j=1}^{K} \prod_{i \in L(j)} X_i = \prod_{j=1}^{K-1} X_j = \prod_{j=1}^{K} X_j \tag{5.19}$$

since $X_K = 1$.

Therefore from (5.16) and (5.19)

$$\prod_{j=1}^{K} Q_j = 1. \tag{5.20}$$

and the Lemma 5.1 proven.

Lemma 5.1 is useful because it is used as an identity throughout the analysis of the model.

We note that the results in this section have been conjectured in Gershwin and Ammar (1979). Also notice that the same results provide a complete internal analysis for general AMN's.

5.3 Boundary Analysis

5.3.1 Some Definitions

Boundary States are states $(\underline{n}, \underline{\alpha})$ for which at least one n_i is equal to 0, 1, N_i-1, or N_i for $i \in \{1, \ldots, K-1\}$. They are further classified into inner and outer boundary states. Inner boundary states are states that have at least one n_i=1, or N_i-1 for $i \in \{1, \ldots, K-1\}$, and for no $i \in \{1, \ldots, K-1\}$ is n_i equal to 0 or N_i. Outer boundary states are all other boundary states.

Transition equations that are not internal are defined as boundary transition equations. These are in turn classified into inner boundary transition equations, i.e. those involving only inner boundary states and internal states, and outer boundary transition equations which are defined to be all others.

5.3.2 Solution Form

For boundary states, it seems reasonable to assume that the solution for (5.6) applies. However, for s not internal $\xi(s, U_j)$ may not be of the form (5.9). This is due to the fact that transition equations involving boundary states have forms that are different from the internal transition equations (5.5). Appropriate forms must be determined for each boundary state s.

Ideally the set of $\xi(s, U_j)$ would satisfy all transition equations for some U_j. This would imply that the summation (5.6) has only one term. However, experience has shown that this is not possible. Therefore, more than one term is needed, i.e. $\ell > 1$, and the c_j's are to be determined by satisfying all the equations not satisfied by the individual terms $\xi(s, U_j)$.

In the following sections the boundary analysis is carried out for simple assembly merge networks consisting of two and three machines.

5.4 The Two-Machine Transfer Line

As was seen in Chapter 4 the only possible two-machine AMN is of the transfer line type. In this case the parametric equations become three non-linear equations in three unknowns. It is found in Schick and Gershwin (1978) and Gershwin and Schick (1980), that there are two sets of solutions satisfying the three equations. Thus the solution is found to be of the following form:

$$p(s) = \sum_{j=1}^{2} c_j \xi(s, U_j). \tag{5.21}$$

Using boundary transition equations, expressions $\xi(s, U_j)$ are found for boundary states. It is also found that the constant c_j associated with one of the $\xi(s, U_j)$'s is zero, and thus the solution has only one term in the summation.

A complete analysis of the two-machine AMN (or transfer line) appears in Schick and Gershwin (1978) and Gershwin and Schick (1980). In addition to being significant in its own right, this solution serves as a benchmark with which solutions to more complex AMN's are to be compared. Specifically, any solution obtained to the general AMN should reduce to the two-machine solution. After we get solutions for the three-machine AMN's we will return to show that the solutions obtained do in fact reduce to the two-machine solution.

5.5 The Three-Machine AMN's

It is shown in Chapter 4 that there are two possible configurations of three machines and two buffers that are consistent with the definitions of an assembly merge network. The two networks are shown in Fugure 5.1.

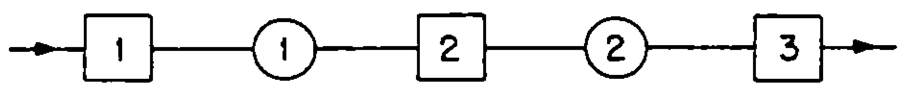


Figure 5.1a) Three-Machine Transfer Line

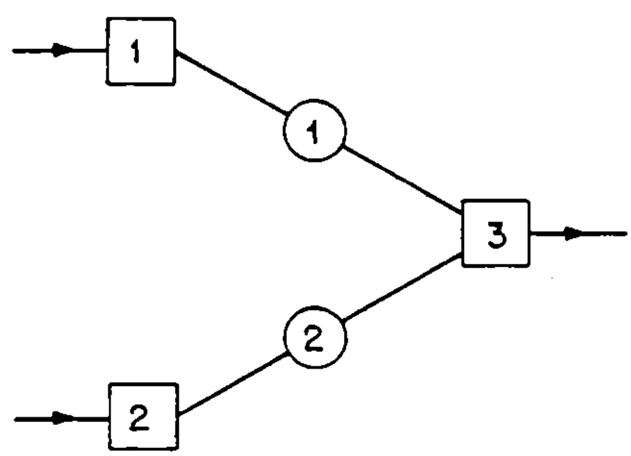


Figure 5.1b) Three-Machine Simple Assembly System

From the results of Chapter 4 we know that the above two systems belong to the same equivalence class. This implies that we only need to provide a solution technique for one of the systems. We choose to concentrate on the three-machine transfer line, mainly because it is the case already solved by Gershwin and Schick (1980).

We should remark here that Gershwin and Schick (1980) go about finding the expressions $\xi(s,U)$ for boundary states s in a haphazard manner. The main goal there is to satisfy as many of the transition equations as possible using one term expressions. The solution method as it is presented here is a systematized version, and our main goal is to produce solution steps that are extendible to larger AMN's.

In the context of the three-machine AMN configurations the following definitions are made:

Edge states are boundary states (inner or outer) that have only one i for which $n_i = 0, 1, N_i-1$, or N_i, for all $i = 1,\ldots,K$.

Corner states are all other boundary states.

Thus for example, state (1, 1, 1, 1, 0) is an inner corner state, (0, 1, 0, 1, 1) is an outer corner state, (0, 2, 0, 1, 1) is an outer edge state, and finally (2, N_2-1, 1, 1, 0) is an inner edge state.

5.5.1 Transient States

Transient states are those that have zero steady state probability. The difficulty here is clearly not in finding $\xi(s,U)$ for s transient, but in finding rules for determining the transient states. (For s transient we choose $\xi(s,U) = 0$.) Such rules are presented, for the transfer line case, in Gershwin and Schick (1979). We use these rules as specialized to three-machine transfer lines to determine the transient states.

We remark here that the problem of determining transient states for general AMN's has not been solved yet. It has been observed, however, that if a state s is a transient state, it must be that s is a boundary state. (This is one of the factors that motivated the classification of the state space into internal and boundary states).

5.5.2 Inner Boundary Analysis

Using the inner boundary transition equations, expressions for inner boundary states are found for the three machine transfer line in Gershwin and Schick (1980). It is observed that all inner boundary transition equations can be satisfied by these expressions. Here we investigate the form of these expressions with the aim of gaining insight that will suggest generalization. Inner edge state expressions obtained in Gershwin and Schick (1980) are of the following form:

$$\xi(s,U) = \xi^I(s,U) \quad \xi^B(s,U) \tag{5.22}$$

where
$$\xi^I(s,U) = X_i^{n_1} \; X_2^{n_2} \; Y_1^{\alpha_1} \; Y_2^{\alpha_2} \; Y_3^{\alpha_3} \tag{5.23}$$

and
$$\xi^B(s,U) = \left(\frac{Z_i}{P_i Y_i}\right)^{\alpha_j} \tag{5.24}$$

where
$$Z_i = 1 - r_i + P_i Y_i \tag{5.25}$$

Note that form (5.22) only applies to non-transient states. The superscripts I and B refer to Internal and Boundary, respectively. We say the machine downstream of a buffer with one part is almost starved, and the machine upstream of a buffer with one empty location almost blocked. The cause of an almost starved machine is the upstream machine of the buffer with one part, while the cause of an almost blocked machine is the downstream machine of the buffer with one empty slot. Hence in (5.24) i is the index of the almost starved or blocked machine, and j is the index of the causing machine.

The reason for this terminology, i.e. the "cause" machine comes from the fact that a machine is blocked (or almost blocked) due to a failure in its downstream machine. Similarly a machine is starved (or almost starved) due to a failure in its upstream machine.

For example, the state (1,3,1,1,1) is an inner edge state, provided $N_2 \geqslant 5$. The expression for this state is found in Gershwin and Schick (1980) to be

$$\xi(1,3,1,1,1,U) = X_1 X_2^3 Y_1 \frac{Z_2}{P_2} Y_3 \qquad (5.26)$$

or

$$\xi^I(1,3,1,1,1,U) = X_1 X_2^3 Y_1 Y_2 Y_3 \qquad (5.27)$$

and

$$\xi^B(1,3,1,1,1,U) = \frac{Z_2}{P_2 Y_2} \qquad (5.28)$$

For this state machine 2 is almost starved and machine 1 is the cause.

Also consider the inner boundary state (1,3,0,1,1). It is found in Gershwin and Schick (1980) that $\xi(1,3,0,1,1,U)$ is of internal form i.e.

$$\xi(1,3,0,1,1) = X_1 X_2^3 Y_2 Y_3. \tag{5.29}$$

Note that this conforms to the form established in (5.22) with the coupling term

$$\xi^B(1,3,0,1,1,U) = \left(\frac{Z_2}{p_2 Y_2}\right)^0 = 1 \tag{5.30}$$

For inner corner states the expressions conform to (5.22). However $B_{(s,U)}$ for these states take on different forms. We now discuss four different cases for $B_{(s,U)}$ for an inner corner state.

<u>Case 1</u> $s = (N_1\text{-}1,\ 1,\ \alpha_1,\ \alpha_2,\ \alpha_3)$

For non-transient states of this type machine 1 is almost blocked and machine 3 is almost starved. For both machine 2 is the cause. So we expect

$$\xi^B(s,U) = \left(\frac{Z_1}{p_1 Y_1}\right)^{\alpha_2} \left(\frac{Z_3}{p_3 Y_3}\right)^{\alpha_2}. \tag{5.31}$$

This does indeed conform to the expressions for these states found in Gershwin and Schick (1980).

<u>Case 2</u> $s = (1, N_2-1,\ \alpha_1,\ \alpha_2,\ \alpha_3)$

Here machine 2 is simultaneously almost starved and almost blocked. The cause of starvation is machine 1 and the cause of blockage is machine 3. For this case

$$\xi^B(s,U) = \left(\frac{Z_2}{p_2Y_2}\right)^{\max(\alpha_1,\ \alpha_3)} \tag{5.32}$$

Here also the results conform to the expressions found in Gershwin and Schick (1980).

Case 3 $\quad s = (N_1-1,\ N_2-1,\ \alpha_1, \alpha_2, \alpha_3)$

In this case machines 1 and 2 are both almost blocked. The cause for machine 1 blockage is machine 2, and for machine 2 blockage is machine 3. Here we have a chain of events propagating from machine 3 to machine 1. We say that the coupling factor in a chain of this type is summarized by the last link, i.e. machine 2 almost blocking machine 1. Thus

$$\xi^B(s,U) = \left(\frac{Z_1}{p_1Y_1}\right)^{\alpha_2} \tag{5.33}$$

These expressions also conform to the ones in Gershwin and Schick (1980).

For these states, as well as the ones in Case 4, we encounter an anomaly. The expression for state $(N_1-1,\ N_2-1,\ 0,\ 1,\ 1)$ defies the form (5.22). This state, in conjunction with its counterpart in Case 4, has the only two unexplained expressions for the inner boundary.

Case 4 $\quad s = (1,1,\alpha_1,\alpha_2,\alpha_3)$

Here machines 2, and 3 are almost starved. The cause for machine 2 starvation is machine 1, and for machine 3 starvation is machine 1.

Thus we have the same propagation of events as in Case 3, from machine 1 to machine 3. The last link of this chain of events is machine 2 almost starving machine 3. Hence

$$\xi^{B}(s,U) = \left(\frac{Z_3}{p_3 Y_3}\right)^{\alpha_2} \qquad (5.34)$$

The anomalous state here is (1,1,1,1,0). Its expression does not conform to (5.22), and remains unexplained.

The form of the expressions in Case 4 are not the same as those found in Gershwin and Schick (1980). Specifically the expression for state (1,1,1,1,1) has to be changed. In their derivation of other expressions using the transition equations Gershwin and Schick (1980) used $\xi(1,1,1,1,1,U)$ to obtain the expressions for other states. Hence any change in the expression of state (1,1,1,1,1) necessitates changes in other expressions.

A complete list of the changed expressions is in Table 5.1. We have shown how one might explain the forms of the expressions for the inner boundary of the three-machine transfer line. These explanations can be used, as is done later in this chapter, to form conjectures on boundary expressions for larger AMN's.

5.5.3 Outer Boundary Analysis

Recall that in deriving boundary expressions Gershwin and Schick (1980) attempt to satisfy as many transition equations as possible. It so happens that the expressions they derive for outer boundary states satisfy all the outer boundary transition equations that describe

$$\xi(1,1,1,1,1,U) = X_1X_2Y_1Y_2 \frac{Z_3}{p_3}$$

$$\xi(1,1,1,1,0,U) = X_1X_2Y_1 \frac{(1 - r_2)}{p_2} \frac{Z_3}{(1 - r_3)}$$

$$\xi(0,1,0,1,0,U) = X_1X_2Y_1 \frac{(1 - r_1)}{r_1} \frac{(1 - r_2)}{p_2} \frac{Z_3}{(1 - r_3)}$$

$$\xi(0,1,0,1,1,U) = X_1X_2Y_1 \frac{(1 - r_2)}{r_1p_2} \frac{Z_3}{p_3(1 - r_3)} (r_1 + r_3 - r_1r_3)$$

$$\xi(0,0,0,1,1,U) = X_1X_2Y_1 \frac{(1 - r_1)}{r_1^2} \frac{(1 - r_2)}{p_2} \frac{Z_3}{p_3(1 - r_3)} (r_1+r_3-r_1r_3-p_3r_1)$$

$$\xi(1,0,1,1,1,U) = X_1X_2Y_1 \frac{(1 - r_2)}{r_1p_2} \frac{Z_3}{p_3(1 - r_3)} (r_1 + r_3 - r_1r_3 - p_3r_1)$$

$$\xi(1,0,0,0,1,U) = X_1X_2 \frac{\left[1-p_3 - (1-r_3-p_3)(1-r_2)(1-r_1) + \dfrac{r_3Z_3 \ (1-r_2) \ p_1Y_1}{(1 - r_3) \ r_1} \right]}{p_3 \ (r_2 + r_1 - r_1r_2)}$$

Table 5.1 New Expressions $\xi(s,u)$ for the Three-Machine Transfer Line transitions into inner edge states. The results of such an analysis for the three-machine transfer line is presented in Gershwin and Schick (1980). In this section we look at these expressions in the same manner as was done for inner boundary expressions. Some insight is gained into the form of the expressions which helps in making conjectures for larger systems.

For non-transient states on the outer edge, expressions satisfy (5.22).

The coupling term is now

$$\xi^{B}(s,U) = f(i,j) \tag{5.35}$$

where

$$f(i,j) = \frac{Y_j}{Q_i Y_i p_i r_j} \left[(1-r_i)\, Q_i Z_i Q_j Z_j - (1-p_j)\,(1-r_i p_i)\right] \tag{5.36}$$

The indices i and j correspond, respectively, to the starved or blocked machine and the machine that is the cause. Recall that the cause of an empty buffer is the upstream machine, and the cause of a full buffer is its downstream machine.

Outer corner states remain the least understood. Expressions for those states have been obtained for the three-machine transfer line in Gershwin and Schick (1980).

5.5.4 Some Remarks on Boundary Analysis

In deriving expressions for boundary states, inner and outer, we have been trying for two objectives. First, to have a rational, systematic, generalizeable set of expressions. Second, to minimize the number of unsatisfied transition equations. As the solution stands now that number is linear in N_1 and N_2.

Gershwin and Schick (1980) had only the second goal in mind while deriving their expressions. With a few changes to some expressions, we have shown that there really is some underlying logic to the way they approached the solution.

5.5.5 Analysis of Unsatisfied Transition Equations

Recall that the steady state probabilities are assumed to be of the form (5.6). Given the way the $\xi(s,U)$ expressions are constructed, probabilities of the form (5.6) satisfy most of the transition equations. This is true regardless of the choice of the c_j's and . The set U_j can also be chosen arbitrarily as long as U_j satisfies the parametric equations for each j. This freedom is to be utilized to satisfy all unsatisfied equations.

The error at state s, g(s,U) is defined as

$$g(s,U) = -\xi(s,U) + \sum_{\text{all } s'} T(s,s')\,\xi(s',U) \qquad (5.37)$$

For most states s the transition equations describing transitions into state s are satisfied by the $\xi(s,U)$ expressions. Thus for these states the error g(s,U) is identically zero. Other states are called odd states. These states occur on boundaries only and are thus divided into odd edge states and odd corner states. A list of odd states for the three machine transfer line appears in Gershwin and Schick (1980). This list is unchanged by the changes in expressions in Table 5.1.

Table 5.2 contains a list of odd edge states. Those states are of special interest because we have been able to obtain closed form expressions for their errors. This is done using the appropriate equations of the form (5.37). The following relations have been found to hold for the errors at odd edge states:

$$g(n_1,0,1,0,1,U) = Y_1 g(n_1,0,0,0,1,U) \tag{5.38}$$

$$g(n_1,N_2,1,1,0,U) = Y_1 g(n_1,N_2,0,1,0,U) \tag{5.39}$$

$$g(n\ ,n_2,0,1,1,U) = Y_3 g(0,n_2,0,1,0,U) \tag{5.40}$$

$$g(N_1,n_2,1,0,1,U) = Y_3 g(N_1,n_2,1,0,0,U) \tag{5.41}$$

Also

$$g(n_1,0,0,0,1,U) = X_1^{n_1} q(2,3) \tag{5.42}$$

$$g(n_1,N_2,0,1,0,U) = X_1^{n_1} X_2^{N_2} q(3,2) \tag{5.43}$$

$$g(0,n_2,0,1,0,U) = X_2^{n_2} q(1,2) \tag{5.44}$$

$$g(N_1,n_2,1,0,0,U) = X_1^{N_1} X_2^{n_2} q(2,1) \tag{5.45}$$

where

$$q(i,j) = Y_i + \frac{Y_j}{Q_i p_i r_j} \Big[(1-r_i-p_i) \quad (1-p_j) - \frac{(1-r_j-p_j)}{Z_i Z_j} + Q_i Q_j (1-r_i)(1-r_j-Z_i Z_j) \Big] \tag{5.46}$$

EDGE #	ODD STATE
1	$(n_1, 0, 0, 0, 1)$
	$(n_1, 0, 1, 0, 1)$
2	$(n_1, N_2, 0, 1, 0)$
	$(n_1, N_2, 1, 1, 0)$
3	$(0, n_2, 0, 1, 0)$
	$(0, n_2, 0, 1, 1)$
4	$(N_1, n_2, 1, 0, 0)$
	$(N_1, n_2, 1, 0, 1)$

$2 \leqslant n_1 \leqslant N_1-2$

$2 \leqslant n_2 \leqslant N_2-2$

Table 5.2 Odd Edge States for the Three-Machine Transfer Line

Note that i and j represent, respectively, the labels of the starved or blocked machine, and the cause of blockage or starvation.

For all odd states, the set $\{c_j\}$, $j=1,\ldots,\ell$ in (5.6) is to be chosen such that

$$\sum_{j=1}^{\ell} c_j \, g(s,U_j) = 0. \tag{5.47}$$

One straightforward method of dealing with conditions (5.47) is presented in Gershwin and Schick (1980). The number ℓ is assumed to be equal to the number of odd states. Thus, for a certain choice of $\{U_j\}$, $j=1,\ldots,\ell$ that satisfy the parametric equations, there are linear equations in $c_1,\ldots,c_\ell$. One can then solve the linear system

$$G \, \underline{C} = \underline{0} \tag{5.48}$$

where $\underline{C} = (c_1,\ldots,c_\ell)$ and G is the matrix of errors $g(s,U_j)$.

In Gershwin and Schick (1979) the structure of the matrix G is investigated in detail. It is shown, there, that under certain assumptions that G has rank ℓ-1, which implies that (5.48) has a non-zero solution. The c_j's can then be determined to within a multiplicative constant. That constant can be found using the normalization equation (2.27).

When implementing this method one runs into numerical difficulties which are caused by limits on computer precision. This bad behavior seems to improve with certain choices of the U_j's. Another major difficulty with this method is that the size of the system (5.48) is linear in the two storage sizes. This restricts the storage sizes that can be handled by this method and also limits its applicability.

5.6 Conjectures on Solution Features for more Complex AMN's

In this section we make conjectives on the form of the expressions. $\xi(s,U)$ for general K-machine AMN's. Recall that this form of the expressions for internal states has been shown for general AMN's in Section 5.2.

5.6.1 Inner Boundary

Define a decoupled inner boundary state as one where no machine is connected (upstream or downstream) to more than one buffer which has one part or one empty slot. Coupled inner boundary states are all others.

For example an inner edge state for a three machine system is decoupled, while, an inner corner state is a coupled state.

For decoupled inner boundary states each buffer i has associated with it a pair of labels (i,d_i), indicating the upstream and downstream machines. One of the machines is either almost starved or almost blocked. The other is the cause.

Conjecture 1 For all non-transient decoupled inner boundary states s

$$\xi(s,U) = \prod_{i=1}^{K-1} X_i^{n_i} \prod_{i=1}^{K} Y_i^{\alpha_i} \prod_{i;n_i=1} \left(\frac{Z_{d_i}}{p_{d_i} Y_{d_i}}\right)^{\alpha_i} \prod_{i;n_i=N_i-1} \left(\frac{Z_i}{p_i Y_i}\right)^{\alpha_{d_i}} \tag{5.49}$$

The expressions for coupled inner boundary states will depend very much on the nature of the coupling. In Section 5.5.2 we have investigated four types of coupling in a three-machine transfer line. It is clear that the varieties of coupling will increase tremendously as we go to larger systems. All that can be said for coupled inner boundary state expressions is that they will behave in a manner similar to those found for the three machine case.

5.6.2 Outer Boundary

Define a decoupled outer boundary state as one where no machine is connected to more than one empty or full buffer. Coupled outer boundary states are all others. As an example, outer edge states for three machine systems are decoupled while outer corner states are not.

For decoupled outer boundary states each empty or full buffer i has a pair of labels (i, d_i) indicating its upstream and downstream machines respectively. One machine is either starved or blocked, and the other is the cause.

Conjecture 2 For all non-transient decoupled outer-boundary states s

$$\xi(s,U) = \prod_{i=1}^{K-1} X_i^{n_i} \prod_{i=1}^{K} Y_i^{\alpha_i} \prod_{i;\, n_i=0} f(i,d_i) \prod_{i;\, n_i=N_i} f(d_i,i) \tag{5.50}$$

where $f(i,j)$ is given by (5.36). No conjectures can be made at this time for coupled outer boundary states.

Note that the three-machine transfer line expressions obtained in Gershwin and Schick (1980) and mofified earlier in this chapter conform to the conjectures.

5.7 Relating the Two-Machine Solutions to the General Conjectures

In this section we show that the solution for the two-machine transfer line in Gershwin and Schick (1980) does indeed conform to our general conjectures in the last section. Note that all boundary states in a two machine system are decoupled. This is due to the fact that there is only one buffer.

5.7.1 Inner Boundary

There are six non-transient inner boundary states for a two machine system; namely, (1,0,0) (1,0,1), (1,1,1), (N_1-1,0,0), (N_1-1,1,1) and (N_1-1,1,0). According to Conjecture 1, the expressions for these states should be as follows:

$$\xi(1,0,0) = X_1 \tag{5.51}$$

$$\xi(1,0,1) = X_1 Y_2 \tag{5.52}$$

$$\xi(1,1,1) = X_1 Y_1 Y_2 \frac{Z_2}{p_2 Y_2} \tag{5.53}$$

$$\xi(N_1-1,0,0) = X_1^{N_1-1} \tag{5.54}$$

$$\xi(N_1-1,1,0) = X_1^{N_1-1} Y_1 \tag{5.55}$$

$$\xi(N_1-1,1,1) = X_1^{N_1-1} Y_1 Y_2 \frac{Z_1}{p_1 Y_1} \tag{5.56}$$

Equations (5.51), (5.52), (5.54), and (5.55) conform to the solution form found in Gershwin and Schick (1980). We now show that the remaining two expressions (5.53), and (5.56) indeed are the same as those found in the same paper.

$$\xi(1,1,1) = X_1 Y_1 Y_2 \quad \frac{Z_2}{p_2 Y_2} \tag{5.57}$$

$$= X_1 Y_1 \frac{Z_2}{p_2} \tag{5.58}$$

From the solutions to the two-machine parametric equations we have

$$Y_1 = \frac{r_1 + r_2 - r_1 r_2 - p_2 r_1}{p_1 + p_2 - p_1 p_2 - p_1 r_2} \tag{5.59}$$

$$Y_2 = \frac{r_1 + r_2 - r_1 r_2 - p_1 r_2}{p_1 + p_2 - p_1 p_2 - p_1 r_2} \tag{5.60}$$

and

$$X_1 = \frac{Y_2}{Y_1} \tag{5.61}$$

Hence $Y_1 Z_2 = \dfrac{r_1 + r_2 - r_1 r_2 - p_2 r_1}{p_1 + p_2 - p_1 p_2 - r_1 p_2}$ (5.62)

Thus $\xi(1,1,1) = \dfrac{X_1 (r_1 + r_2 - r_1 r_2 - p_2 r_1)}{p_2 (p_1 + p_2 - p_1 p_2 - r_1 p_2)}$ (5.63)

Expression (5.63) is the one found for $\xi(1,1,1,U)$ in Gershwin and Schick (1980). Using similar manipulations we can show that expression found for state $(N_1-1,1,1)$ is the same as the one in Gershwin and Schick (1980).

5.7.1 Outer Boundary

For the two machine systems there are two non-transient outer boundary states (0,0,1) and N,1,0), both are decoupled because there is only one buffer. Thus according to Conjecture 2:

$$\xi(0,0,1) = Y_2 \; f(2,1) \tag{5.64}$$

and

$$\xi(N,1,0) = X_1^{N_1} Y_1 \; f\,(1,2) \tag{5.65}$$

where f(i,j) is given by (5.36).

We now show that (5.64) and (5.65) are the same as those found in Gershwin and Schick (1980).

Recall that for the two machine systems

$$Z_1 Z_2 = 1 \tag{5.66}$$

and

$$Q_1 Q_2 = 1 \tag{5.67}$$

Equation (5.66) is the parametric equation (5.11), and (5.66) is true by Lemma 5.1.

We have

$$\xi(0,0,1) = Y_2 \; \frac{Y_1}{Q_2 Y_2 p_2 r_1} \; [(1-r_2)Q_2 Z_2 Q_1 Z_1 - (1-p_1)\;(1-r_2-p_2)] \tag{5.68}$$

$$= \frac{Y_1}{Q_2 p_2 r_1} \; [(1-r_2) - (1-p_1)\;(1-r_2-p_2)] \tag{5.69}$$

Also recall that

$$Q_2 = \frac{1}{X_1} \tag{5.70}$$

Thus

$$\xi(0,0,1) = \frac{X_1}{p_2 r_1} \quad Y_1 \quad [p_1+p_2-p_1r_2-p_2p_1] \tag{5.71}$$

$$= \frac{X_1}{p_2 r_1} \quad [r_1+r_2-r_1r_2-p_2r_1] \tag{5.72}$$

Expression (5.72) is the same as the one found in Gershwin and Schick (1980). We can handle (5.65) in a similar manner to show that it is the same as the one found by Gershwin and Schick (1980).

5.8 Summary and Conclusions

In this Chapter we have shown how one might go about finding the steady state probabilities for two-and three-machine systems. The details of the method of solution are in Gershwin and Schick (1980). Emphasis, here, is put on understanding the form of the solution with the aim of generalizing it. Minor changes are found to be necessary.

Two conjectures are made on how the solution extends to systems with more than three machines.

Chapter 6

Summary and Suggestions for Future Research

6.1 General Remarks

In this chapter we summarize the work presented in this thesis. Also directions for future research are suggested.

6.2 Thesis Summary

In this thesis we present a discrete time, discrete state Markov chain model for assembly merge networks (Chapter 2). The model is intended to contain some of the important features of a manufacturing network.

Some fundamental equivalence results for the model are presented. In Chapter 3 we deal strictly with transfer lines. Specifically we prove a strong reversibility property (Theorem 3.1) regarding the equivalence of the probability distributions of a transfer line and its reverse. This is used to state and prove how the performance measures of a transfer line and its reverse are related. Chapter 4 extends the ideas on transfer line reversibility to three-machine assembly merge networks (AMN's). We conclude that the solution of the three-machine assembly system (Figure 5.1b) is identical to that of the three-machine transfer line (Figure 5.1a). We also conjecture on how the three-machine equivalence results extend to larger systems.

In Chapter 5 we briefly discuss a solution method for AMN's. The solution is only complete for two-and three-machine systems. Note that this is basically the same method as the one presented in Gershwin and Schick (1980). However, the discussion in this thesis systematizes the steps of the solution, and attempts to explain some of the boundary expressions obtained. Based on these explanations, conjectures are made as to how the solution might extend to AMN's with more than three machines. Also using the insight gained into the solution, we are able to relate the two-and three-machine solutions.

In Appendices I, II, and III we include proofs and derivations that are too cumbersome to include in the main text. Appendix I contains the proof of the conservation of flow theorem, and other propositions relating to the performance measures. Appendix II has the proof of the strong equivalence property for three-machine systems (Theorem 3.1). In Appendix III we derive the parametric equations (5.11), and (5.12) of Chapter 5.

6.3 Future Research Directions

In this section we suggest future research that can be based on the work in this thesis. The new research directions fall into these categories: modelling, extension of equivalence results, and the solution method.

6.3.1 Modelling

The formulation of a disassembly machine and assembly-disassembly network models are an immediate extension of the material presented in Chapter 2. Also continuous time models of assembly and/or disassembly networks can be formulated based on Chapter 2 and the two-machine transfer line model presented in Gershwin and Berman (1978), and Gershwin and Ammar (1979).

6.3.2 Extension of Equivalence Results

The results of Chapter 3 and 4 can be extended in several directions. The conjectures for four-machine equivalence classes in Chapter 4 have to be proven. This could conceivably be in the context of an equivalence result for general AMN's or general assembly-disassembly networks. It is suspected that the idea of focusing on hole motion in parts of the networks will play an important role in such extensions.

As was mentioned in Chapter 3, there is evidence that the reversibility results hold for carefully formulated continuous time models. An important question to be answered is: What common features of these models make reversibility and equivalence results hold?

6.3.3 Solution Method

In Chapter 5 we make conjectures on how boundary expressions for three-machine systems might extend to larger systems. One obvious task that awaits future researchers is to prove or disprove these conjectures. However, the most crucial step is to find some method of solution that circumvents the difficulties of the technique in Chapter 5.

Appendix I

Conservation of Flow and Other Propositions

Relating to Performance Measures

In this appendix we prove the theorems, and propositions in Section 2.6 relating to performance measures of an AMN.

We now prove the theorem stating conservation of flow. Here we state and prove a slightly more general version of Theorem 2.1. First we introduce some definitions that are needed for the proof.

Definitions

For all $i = 1, \ldots, k$

$$e_i(t) = \begin{cases} 1 & \text{if a piece } \underline{\text{leaves}} \text{ machine } i \text{ at time } t, \\ 0 & \text{otherwise} \end{cases} \tag{I.1}$$

For all input machines j

$$d_j(t) = \begin{cases} 1 & \text{if a piece enters machine } j \text{ at time } t, \\ 0 & \text{otherwise} \end{cases} \tag{I.2}$$

For all $i = 1, \ldots, k$

$$R_i(t) = \text{number of parts leaving machine } i \text{ in } [0,t]$$

$$= \sum_{\tau=1}^{t} e_i(\tau), \tag{I.3}$$

and

$$R_i = \lim_{t \to \infty} \frac{R_i(t)}{t} \tag{I.4}$$

For all <u>input machines</u>, j

$$D_j(t) = \text{number of parts entering machine } j \text{ in } [0,t]$$

$$= \sum_{\tau=1}^{t} d_j(\tau). \quad (I.5)$$

and

$$D_j = \lim_{t \to \infty} \frac{D_j(t)}{t}. \quad (I.6)$$

<u>Branch (j,i)</u> of an AMN is the path of machines and buffers, traced by a part that enters the system at input machine j until it leaves machine i. For examples see Figure I.1. It must be pointed out that a branch (j,i) is completely determined by j and i. Also there exists some j, and i combinations that do not define branches. For example in Figure I.1 branch (2,3) is non-existent.

For all branches (j,i)

$$m_{(j,i)}(t) = \text{number of parts in branch } (j,i) \text{ at time } t$$

$$= \sum_{k,\ k \text{ on } (j,i)} n_k(t). \quad (I.7)$$

$N_{(j,i)}$ = capacity of branch (j,i), i.e. the maximum number of parts that can be held in branch (j,i) at any time

$$= \sum_{k,\ k \text{ on } (j,i)} N_k \quad (I.8)$$

We now need to prove certain lemmas before proceeding with the main theorem.

<u>Lemma 1</u>

For all input machines j

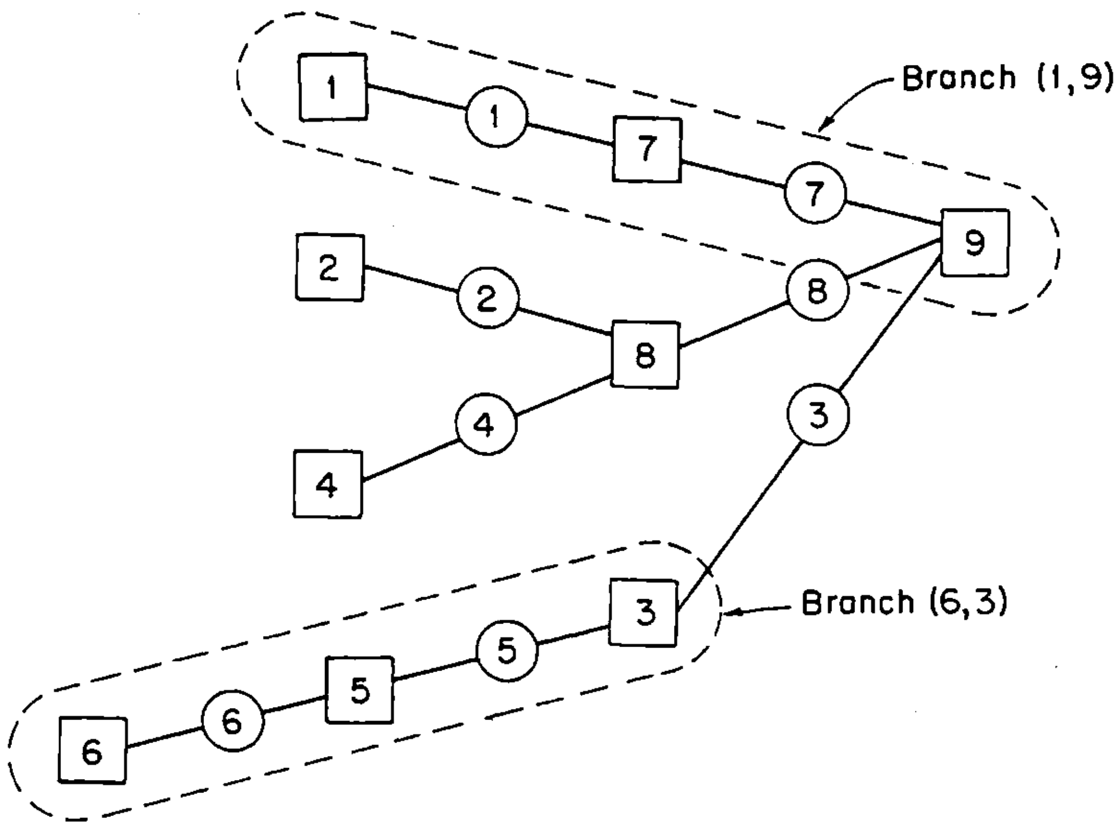


Figure I.1   Branch Examples

$$d_j(t) = e_j(t) \tag{I.9}$$

Proof:

This lemma states the assumption that a machine takes a part in whenever a part leaves. This assumption is indeed reflected in Table 2.2. As an example consider Case 3 of Table 2.2. In this case buffer i has a part added to it whenever machine i is up at time t +1, and is not starved or blocked. We note that if a machine is up and neither starved nor blocked it takes a part from its upstream buffer (Recall $\gamma_i = 1, \forall i$), as witnessed by Case 1. Considering the other examples in Table 2.2 equation (I.9) is confirmed.

Lemma 2

For all input machines j

$$D_j(t) = R_j(t) \tag{I.10}$$

and

$$D_j = R_j \tag{I.11}$$

Proof:

This follows immediately from the definitions (I.5) and (I.6) and also Lemma 1.

Lemma 3

For all branches (j,i) and all $t \geqslant 0$:

$$0 \leqslant m_{(j,i)}(t) \leqslant N_{(j,i)} \tag{I.12}$$

Proof:

From (I.7)

$$m_{(j,i)}(t) = \sum_{k:k \text{ on } (j,i)} n_k(t). \tag{I.13}$$

However, we have

$$0 \leqslant n_k(t) \leqslant N_k \quad \text{for all } k, \text{ and } \quad 0. \qquad (I.14)$$

Thus

$$0 \leqslant m_{(j,i)}(t) = \sum_{k,k \text{ on } (j,i)} n_k(t) \leqslant \sum_{k,k \text{ on } (j,i)} N_k = N_{(j,i)} \qquad (I.15)$$

which implies

$$0 \leqslant m_{(j,i)}(t) \leqslant N_{(j,i)} \quad \text{for all } t \geqslant 0 \qquad (I.16)$$

and the lemma is proven.

<u>Lemma 4</u>

For all i=1,...,k

$$e_i(t) = \begin{cases} \alpha_i(t) & \text{if machine i is not starved or blocked at time } t-1, \text{ or } n_j(t-1) > 0, \; \forall j \in L(i) \; \underline{\text{and}} \; n_i(t-1) < N_i \\ 0 & \text{Otherwise} \end{cases} \qquad (I.17)$$

<u>Proof</u>:

The lemma is true because if machine i is starved or blocked at time $t-1$ it does not produce a piece at time t, thus $e_i(t) = 0$. However if machine i is neither starved nor blocked at time $t-1$, it produces a piece if it is up at time t ($\alpha_i(t) = 1$) and if it is down ($\alpha_i(t) = 0$) no parts are produced. Note that Table 2.2 was constructed according to this assumption.

Lemma 5

For all $i = 1, \ldots, k-1$

$$n_i(t+1) = n_i(t) + e_i(t+1) - e_{d_i}(t+1) \qquad \text{(I.18)}$$

Where machine d_i is the downstream machine of buffer i.

Proof:

This Lemma is proven by considering Table 2.2. The reader is reminded that we are dealing only with the case where $\gamma_i = 1, \forall i$.

Case 1 - Machine i is not blocked or starved.

Therefore $e_i(t+1) = \alpha_i(t+1)$. (I.19)

Machine d_i is not blocked or starved, thus

$$e_{d_i}(t+1) = \alpha_{d_i}(t+1) \qquad \text{(I.20)}$$

Hence (I.18) holds.

Case 2 - Only machine d_i is blocked

thus $e_i(t+1) = \alpha_i(t+1)$ (I.21)

and

$$e_{d_i}(t+1) = 0. \qquad \text{(I.22)}$$

and relation (I.18) holds.

Case 3 - Only machine d_i is starved. Equations (I.21) and (I.22) apply here, so (I.18) holds.

Case 4 - Machine i is blocked and machine d_i is neither starved nor blocked thus

$$e_i(t+1) = 0 \tag{I.23}$$

and $$e_{d_i}(t+1) = \alpha_{d_i}(t+1) \tag{I.24}$$

and hence (I.18) holds.

Case 5 - Machine i is starved, and machine d_i is neither starved nor blocked. Hence

$$e_i(t+1) = 0 \tag{I.25}$$

and $$e_{d_i}(t+1) = \alpha_{d_i}(t+1) \quad \text{, and (I.18) holds.} \tag{I.26}$$

Cases 6, 7, 8, 9 - In all these cases both machines are not operating (either starved or blocked or both).

Thus $$e_i(t+1) = 0 \tag{I.27}$$

and $$e_{d_i}(t+1) = 0 \tag{I.28}$$

and (I.18) holds. Thus we have proven that (I.18) is consistent with Table 2.2.

Lemma 6

For all branches (j,i) of an AMN

$$m(j,i)(t+1) = m_{(j,i)}(t) + e_j(t+1) - e_i(t+1) \tag{I.29}$$

Proof:

From the definition (I.7) we have

$$m_{(j,i)}(t+1) = \sum_{k,k \text{ on } (j,i)} n_k(t+1) \tag{I.30}$$

From Lemma 5 we have

$$m_{(j,i)}(t+1) = \sum_{k,k \text{ on } (j,i)} n_k(t) + e_k(t+1) - e_{d_k}(t+1) \tag{I.31}$$

$$= \sum_k n_k(t) + \sum_k e_k(t+1) - e_{d_k}(t+1) \qquad (I.32)$$

$$= m_{(j,i)}(t) + e_j(t+1) - e_i(t+1) \qquad (I.33)$$

Hence the Lemma is proven.

We now state and prove the conservation of flow theorem. The proof uses Lemmas 1 through 6.

Theorem

For all branches (j,i) of an AMN

$$D_j = R_i \qquad (I.34)$$

That is, the steady state input rate to the branch (D_j) is equal to the steady state output rate from the same branch (R_i).

Proof:

From Lemma 6 we can write:

$$m_{(j,i)}(t+1) - m_{(j,i)}(t) = e_j(t+1) - e_i(t+1) \qquad (I.35)$$

summing both sides of (I.35)

$$\sum_{\tau=0}^{t} [m_{(j,i)}(\tau+1) - m_{(j,i)}(\tau)] = \sum_{\tau=0}^{t} [e_j(\tau+1) - e_i(\tau+1)] \qquad (I.37)$$

or

$$m_{(j,i)}(t+1) - m_{(j,i)}(0) = R_j(t+1) - R_i(t+1) \qquad (I.38)$$

By Lemma 3 we have

$$0 \leqslant m_{(j,i)}(t+1) - m_{(j,i)}(0) \leqslant N_{(j,i)} < \infty \qquad (I.39)$$

Therefore

$$0 \leqslant R_j (t+1) - R_i(t+1) \leqslant N_{(j,i)} < \infty \qquad (I.40)$$

and thus dividing by t+1 and taking the limit as $t \to \infty$ we have

$$0 \leqslant \lim_{t\to\infty} \frac{R_j (t+1) - R_i(t+1)}{t+1} \leqslant \lim_{t\to\infty} \frac{N_{(j,i)}}{t+1} \qquad (I.41)$$

but

$$\lim_{t\to\infty} \frac{N_{(j,i)}}{t+1} = 0 \qquad (I.42)$$

Hence

$$\lim_{t\to\infty} \frac{R_j(t+1) - R_i(t+1)}{t+1} = 0 \qquad (I.43)$$

Therefore

$$\lim_{t\to\infty} \frac{R_j(t+1)}{t+1} = \lim_{t\to\infty} \frac{R_i(t+1)}{t+1} \qquad (I.44)$$

since both limits exist.

Hence

$$R_j = R_i \qquad (I.45)$$

But from Lemma 2 we have

$$D_j = R_j \quad \text{for all input machines } j \qquad (I.46)$$

Hence

$$D_j = R_i \quad \text{for all branches } (j,i) \qquad (I.47)$$

and the theorem is proven.

In particular (I.47) states that

$$D_j = R_k \quad \forall \text{ input machines } j \qquad (I.48)$$

Note that (I.48) is the statement of Theorem 2.1.

We now proceed to prove the propositions of Chapter 2.

Proposition 2.1

$$R_i = \lim_{t\to\infty} \frac{R_i(t)}{t} = \text{Prob}\left[\alpha_i(t) = 1,\ n_j(t-1) > 0,\ \forall_j \in L(i),\ n_i(t-1) < N_i\right] \quad (I.49)$$

Note that this is a generalization of the statement of the proposition in Chapter 2.

Proof:

$$R_i = \lim_{t\to\infty} \frac{\sum_{\tau=0}^{t} e_i(\tau)}{t} \quad (I.50)$$

However applying the law of large numbers.

$$R_i = E\left[e_i(t)\right] \quad (I.51)$$

Where $E\left[\cdot\right]$ is the expectation operator.

Thus $$R_i = 1 \,.\, \text{Prob}\left[e_i(t) = 1\right] + 0.\text{Prob}\left[e_i(t) = 0\right] \quad (I.52)$$

or

$$R_i = \text{Prob}\left[e_i(t) = 1\right] \quad (I.53)$$

Where Prob $[A]$ is the steady state probability of event A.

From Lemma 4 we have

$$e_i(t) = \begin{cases} 1 & \text{when } \alpha_i(t) = \text{ and } n_j(t-1) > 0, \\ & \quad {}_j \in L(i),\ n_i(t-1) < N_i \\ 0 & \text{Otherwise} \end{cases} \quad (I.54)$$

Thus

$$R = \text{Prob } [e_i(t) = 1]$$

$$= \text{Prob } [\alpha_i(t) = 1 \text{ and } n_j(t-1) > 0, \quad \forall_j \in L(i), \text{ and } n_i(t) < N_i] \tag{I.55}$$

and the proposition is proven.

We now prove a Lemma needed for the proof of the next propositions.

<u>Lemma 7</u>

For all i = 1,..., K

$$r_i \text{ Prob } [\alpha_i(t) = 0; n_j(t) > 0, \forall_j \in L(i); n_i(t) < N_i]$$
$$= p_i \text{ Prob} [\alpha_i(t) = 1, n_j(t) > 0, \forall_j \in L(i), n_i(t) < N_i] \tag{I.56}$$

<u>Proof</u>:

$$\text{Let event } A = \{ n_j(t) \geq 1, \forall_j \in L(i); n_i(t) < N_i \} \tag{I.57}$$

and consider the sets of states

$$\Omega_{0i} = \{ s \mid \alpha_i(t) = 0, A \} \tag{I.58}$$

$$\Omega_{1i} = \{ s \mid \alpha_i(t) = 1, A \} \tag{I.59}$$

The system can leave states in Ω_{0i} only by the repair of a machine i. This is because machine i is down and cannot produce any parts until it is repaired. Note that this is consistent with the construction of Tables 2.1, and 2.2.

Therefore the probability of leaving set Ω_{oi} is:

$$r_i \ \text{Prob}[\alpha_i(t) = 0, A] \tag{I.60}$$

States in set Ω_{oi} can be reached from outside states only from set Ω_{ii}, by a failure in machine i. This is because the model assumptions as reflected in Tables 2.1 and 2.2 prohibit the failure of a machine if it is starved or blocked. The probability of entering states in Ω_{oi} from outside is thus

$$p_i \ \text{Prob}\left[\alpha_i(t) = 1, A\right] \tag{I.61}$$

and by the steady state assumption we have

$$r_i \ \text{Prob}[\alpha_i(t) = 0, A] = p_i \ \text{Prob}[\alpha_i(t) = 1, A] \tag{I.62}$$

and the Lemma is proven.

Proposition 2.2

For all i = 1,..., k

$$R_i = \text{Prob}\left[\alpha_i(t) = 1; \ n_j(t) > 0, \ \forall_j \in L(i); \ n_i(t) < N_i\right] \tag{I.63}$$

Note: This is a slight generalization of proposition 2.2 of Chapter 2. Here R_i is the production rate of any machine in the network instead of just the output machine k.

Proof:

Let event $A = \{n_j(t) \geq 1, \ \forall_j \in L(i); \ n_i(t) < N_i\}$ (I.64)

Hence from Proposition 2.1

$$R_i = \text{Prob}\left[\alpha_i(t+1) = 1, A\right] \quad (I.65)$$

or

$$R_i = \text{Prob}\left[\alpha_i(t+1) = 1 \mid \alpha_i(t) = 0, A\right] \text{Prob}\left[\alpha_i(t) = 0, A\right] + \text{Prob}\left[\alpha_i(t+1) = 1 \mid \alpha_i(t) = 1, A\right] \text{Prob}\left[\alpha_i(t) = 0, A\right] \quad (I.66)$$

Using Table 2.1 (I.53) becomes

$$R_i = r_i \text{Prob}\left[\alpha_i(t) = 0, A\right] + (1-p_i) \text{Prob}\left[\alpha_i(t) = 1, A\right] \quad (I.67)$$

Let $S_i = \text{Prob}\left[\alpha_i(t) = 1, A\right]$ (I.68)

We need to prove that

$$R_i = S_i \quad (I.69)$$

or

$$R_i - S_i = r_i \text{Prob}\left[\alpha_i(t) = 0, A\right] - p_i \text{Prob}\left[\alpha_i(t) = 1, A\right] = 0 \quad (I.70)$$

However by Lemma 7

$$r_i \text{Prob}\left[\alpha_i(t) = 0, A\right] = p_i \text{Prob}\left[\alpha_i(t) = 1, A\right] \quad (I.71)$$

Hence $R_i = S_i$

$$= \text{Prob}\left[\alpha_i(t);\ n_j(t) > 0,\ \forall_j \in L(i);\ n_i(t) < N_i\right] \quad (I.72)$$

and the proposition is proven.

Proposition 2.3

For input machines j

$$D_j = \lim_{t\to\infty} \frac{D_j(t)}{t}$$

$$= \text{Prob } [\alpha_j(t) = 1,\ n_j(t-1) < N_j] \qquad (I.73)$$

Proof:

Using the same steps as in the proof of proposition 2.1 (I.49 through I.53) we can show

$$D_j = \text{Prob } [d_j(t) = 1] \qquad (I.74)$$

However by Lemma 1 we have

$$d_j(t) = e_j(t) \qquad (I.75)$$

Thus

$$D_j = \text{Prob } [e_j(t) = 1] \qquad (I.76)$$

$$= R_j \qquad (I.77)$$

Hence by proposition 2.1 and since an input machine is never starved we have

$$D_j = \text{Prob } [\alpha_j(t),\ \ n_j(t-1) < N_j] \qquad (I.78)$$

and the proposition is proven.

Proposition 2.4

For input machines j

$$D_j = \text{Prob } [\alpha_j(t) = 1,\ n_j(t) < N_j] \qquad (I.79)$$

Proof:

This proof follows along the same lines as the proof of proposition 2.2.

Here let event $A_j = \{ n_j(t) < N_j \}$ (I.80)

Then it can be shown that

$$D_j = r_j \text{ Prob } [\alpha_j(t) = 0, A_j] + (1-p_j) \text{ Prob}[\alpha_j(t) = 1, A_j] \quad \text{(I.81)}$$

Let

$$F_j = \text{Prob } [\alpha_j(t) = 1, A_j] \quad \text{(I.82)}$$

We need to show

$$D_j = F_j. \quad \text{(I.83)}$$

By the same elements as (I.70), and (I.71) we arrive at the conclusion that we need to show that

$$r_j \text{ Prob}[\alpha_j(t) = 0, A_j] = p_j \text{ Prob } [\alpha_j(t) = 1, A_j] \quad \text{(I.84)}$$

Equation (I.84) holds by Lemma 7.

Hence

$$D_j = F_j = \text{Prob } [\alpha_j(t) = 1, A_j]$$

$$= \text{Prob } [\alpha_j(t) = 1, n_j(t) < N_j] \quad \text{(I.85)}$$

and proposition 2.4 is proven.

Appendix II

Proof of the Strong Equivalence Property

In this appendix the strong equivalence property of Systems F3 and A3 is proven.

Recall,

$$F3 = (\phi, \{1\}, \{2\})$$

and $$A3 = (\phi, \phi, \{1,2\})$$

and

$$N_1^{F3} = N_1^{A3},\ N_2^{F3} = N_2^{A3} \quad (II.1)$$

$$r_1^{F3} = r_1^{A3},\ r_2^{F3} = r_3^{A3},\ r_3^{F3} = r_2^{A3} \quad (II.2)$$

$$P_1^{F3} = P_1^{A3},\ P_2^{F3} = P_3^{A3},\ P_3^{F3} = P_2^{A3} \quad (II.3)$$

Theorem 4.1 Strong equivalence property

for Systems A3 and F3

$$P^{F3}(n_1, n_2, \alpha_1, \alpha_2, \alpha_3) = P^{A3}(n_1', n_2', \alpha_1', \alpha_2', \alpha_3') \quad (II.4)$$

whenever

$$n_1' = n_1,\ n_2' = N_2^{F3} - n_2 \quad (II.5)$$

$$\alpha_1' = \alpha_1,\ \alpha_2' = \alpha_3,\ \alpha_3' = \alpha_2 \quad (II.6)$$

Proof:

The purpose of this proof is to show that if states of System F3 are relabeled according to (II.5) and (II.6) the new transition matrix is

$n_1(t)$	$\alpha_1(t)$	$\alpha_1(t+1)$	PROBABILITY
-	0	0	$1 - r_1^{F3}$
-	0	1	r_1^{F3}
N_1^{F3}	1	0	0
N_1^{F3}	1	1	1
$< N_1^{F3}$	1	0	p_1^{F3}
$< N_1^{F3}$	1	1	$1 - p_1^{F3}$

a)

$n_1(t)$	$n_2(t)$	$\alpha_2(t)$	$\alpha_2(t+1)$	PROBABILITY
-	-	0	0	$1 - r_2^{F3}$
-	-	0	1	r_2^{F3}
-	N_2^{F3}	1	0	0
-	N_2^{F3}	1	1	1
0	-	1	0	0
0	-	1	1	1
> 0	$< N_2^{F3}$	1	0	p_2^{F3}
> 0	$< N_2^{F3}$	1	1	$1 - p_2^{F3}$

b)

$n_2(t)$	$\alpha_3(t)$	$\alpha_3(t+1)$	PROBABILITY
-	0	0	$1 - r_3^{F3}$
-	0	1	r_3^{F3}
0	1	0	0
0	1	1	1
> 0	1	0	p_3^{F3}
> 0	1	1	$1 - p_3^{F3}$

c)

Table II.1
Machine Transition
Tables for F3

$n'_1(t)=n_1(t)$	$\alpha'_1(t)=\alpha_1(t)$	$\alpha'_1(t+1)=\alpha_1(t+1)$	PROBABILITY
-	0	0	$1 - r_1^{F3}$
-	0	1	r_1^{F3}
N_1^{F3}	1	0	0
N_1^{F3}	1	1	1
$<N_1^{F3}$	1	0	p_1^{F3}
$<N_1^{F3}$	1	1	$1 - p_1^{F3}$

a)

$n'_1(t)=n_1(t)$	$n'_2(t)=N_2^{F3}-n_2(t)$	$\alpha'_3(t)=\alpha_2(t)$	$\alpha'_3(t+1)=\alpha_2(t+1)$	PROBABILITY
-	-	0	0	$1 - r_2^{F3}$
-	-	0	1	r_2^{F3}
-	0	1	0	0
-	0	1	1	1
0	-	1	0	0
0	-	1	1	1
>0	>0	1	0	p_2^{F3}
>0	>0	1	1	$1 - p_2^{F3}$

b)

$n'_2(t)=N_2^{F3}-n_2(t)$	$\alpha'_2(t)=\alpha_3(t)$	$\alpha'_2(t+1)=\alpha'_3(t+1)$	PROBABILITY
-	0	0	$1 - r_3^{F3}$
-	0	1	r_3^{F3}
N_2^{F3}	1	0	0
N_2^{F3}	1	1	1
$<N_2^{F3}$	1	0	p_3^{F3}
$<N_2^{F3}$	1	1	$1 - p_3^{F3}$

c)

Table II.2
Modification
of II.1

$n_1'(t)$	$\alpha_1'(t)$	$\alpha_1'(t+1)$	PROBABILITY
-	0	0	$1 - r_1^{A3}$
-	0	1	r_1^{A3}
N_1^{A3}	1	0	0
N_1^{A3}	1	1	1
$< N_1^{A3}$	1	0	p_1^{A3}
$< N_1^{A3}$	1	1	$1 - p_1^{A3}$

a)

$n_i'(t), i=1,2$	$\alpha_3'(t)$	$\alpha_3'(t+1)$	PROBABILITY
-	0	0	$1 - r_3^{A3}$
-	0	1	r_3^{A3}
0 for any i	1	0	0
0 for any i	1	1	1
>0 for all i	1	0	p_3^{A3}
>0 for all i	1	1	$1 - p_3^{A3}$

b)

$n_2'(t)$	$\alpha_2'(t)$	$\alpha_3'(t+1)$	PROBABILITY
-	0	0	$1 - r_2^{A3}$
-	0	1	r_2^{A3}
N_2^{A3}	1	0	0
N_2^{A3}	1	1	1
$< N_2^{A3}$	1	0	p_2^{A3}
$< N_2^{A3}$	1	1	$1 - p_2^{A3}$

c)

Table II.3 Machine Transition Tables for A3

$n_1(t)$	$n_2(t)$	$n_1(t+1)$
$>0, <N_1^{F3}$	$<N_2^{F3}$	$n_1(t) + \alpha_1(t+1) - \alpha_2(t+1)$
N_1^{F3}	N_2^{F3}	$n_1(t) + \alpha_1(t+1)$
0	-	$n_1(t) + \alpha_1(t+1)$
N_1^{F3}	$<N_2^{F3}$	$n_1(t) - \alpha_2(t+1)$
N_1^{F3}	N_2^{F3}	$n_1(t)$

a)

$n_1(t)$	$n_2(t)$	$n_2(t+1)$
>0	$>0, <N_2^{F3}$	$n_2(t) + \alpha_2(t+1) - \alpha_3(t+1)$
>0	0	$n_2(t) + \alpha_2(t+1)$
-	N_2^{F3}	$n_2(t) - \alpha_3(t+1)$
0	>0	$n_2(t) - \alpha_3(t+1)$
0	0	$n_2(t)$

b)

Table II.4 Buffer Transitions for F3

$n_1'(t)=n_1(t)$	$n_2'(t)=N_2^{F3}-n_2(t)$	$n_1'(t+1)=n_1(t+1)$
$>0, <N_1^{F3}$	>0	$n_1(t) + \alpha_1(t+1) - \alpha_2(t+1)$
N_1^{F3}	0	$n_1(t) + \alpha_1(t+1)$
0	-	$n_1(t) + \alpha_1(t+1)$
N_1^{F3}	>0	$n_1(t) - \alpha_2(t+1)$
N_1^{F3}	0	$n_1(t)$

a)

$n_1'(t)=n_1(t)$	$n_2'(t)=N_2^{F3}-n_2(t)$	$n_2'(t+1)=N_2^{F3}-n_2(t+1)$
>0	$>0, <N_2^{F3}$	$N_2^{F3}-n_2(t)-\alpha_2(t+1)+\alpha_3(t+1)$
>0	N_2^{F3}	$N_2^{F3}-n_2(t)-\alpha_2(t+1)$
-	0	$N_2^{F3}-n_2(t)+\alpha_3(t+1)$
0	$<N_2^{F3}$	$N_2^{F3}-n_2(t)+\alpha_3(t+1)$
0	N_2^{F3}	$N_2^{F3}-n_2(t)$

b)

Table II.5 Modification of Table II.4

$n_1'(t)$	$n_2'(t)$	$n_1'(t+1)$
$>0, <N_1^{A3}$	>0	$n_1'(t) + \alpha_1'(t+1) - \alpha_3'(t+1)$
$<N_1^{A3}$	0	$n_1'(t) + \alpha_1'(t+1)$
0	-	$n_1'(t) + \alpha_1'(t+1)$
N_1^{A3}	>0	$n_1'(t) - \alpha_3'(t+1)$
N_1^{A3}	0	$n_1'(t)$

a)

$n_1'(t)$	$n_2'(t)$	$n_2'(t+1)$
>0	$>0, <N_2^{A3}$	$n_2'(t) - \alpha_3'(t+1) + \alpha_2'(t+1)$
>0	N_2^{A3}	$n_2'(t) - \alpha_3'(t+1)$
-	0	$n_2'(t) + \alpha_2'(t+1)$
0	$<N_2^{A3}$	$n_2'(t) + \alpha_2'(t+1)$
0	N_2^{A3}	$n_2'(t)$

b)

Table II.6 Buffer Transitions for A3

exactly that for System A3. In other words if we let state s_i' be the relabeled state s_i we need to show that

$$T^{F3}(s_2,s_1) = T^{A3}(s_2',s_1') \qquad (II.7)$$

Equation (I.7) says that the transition probability between states s_1 and s_2 in System F3 is the same as the transition probability between states s_1' and s_2' in System A3. This can be shown by showing that the relabelling implied by (II.5) and (II.6) reduces the tables (2.1, and 2.2) for Systems F3 to those of System A3.

We start by considering Table 2.1 as it specializes to the three machine transfer line. This is shown in Tables II.1. Note that in Tables II.1 a), and c) only one buffer capacity is relevant since machine 1 is never starved and machine 3 is never blocked.

On these tables we now make the transformation implied by (II.5), and (II.6). The new tables are shown in Tables II.2. We now use the relationships (II.1) through to (II.3) to produce a final set of Tables II.3. Upon close examination it can be seen that Tables II.3 are indeed the derived machine transition tables for System A3.

We now focus our attention on the buffer transitions tables. For System F3 there are two such tables; one for each buffer. These are shown in Tables II.4. We now apply the transformations implied by (II.5) and (II.6) to Tables II.4. This yields the new set of tables in Tables II.5. We now apply the relationship (II.1) to obtain Tables II.6. These under close examination are indeed the buffer transition tables for System A3. (Compare with Table 2.2 for general AMN). Thus the Theorem is proven.

Appendix III

Deriving the Parametric Equations for AMN's

In this appendix we derive the general AMN parametric equations (5.11), and (5.12). This derivation follows that of Gershwin and Schick (1979), (1980) for transfer lines closely. We start with the form of the internal equation given by (5.10). Recall that in this equation $n_i(t+1)$ and $n_i(t)$ are related by (5.1).

Equation (5.10) can be rewritten as follows:

$$\prod_{i=1}^{K-1} X_i^{\alpha_i(t+1) - \alpha_{d_i}(t+1)} \prod_{i=1}^{K} Y_i^{\alpha_i(t+1)}$$

$$= \sum_{\alpha_1(t)=0}^{1} \cdots \sum_{\alpha_K(t)=0}^{1} \prod_{i=1}^{K} \left[(1-r_i)^{1-\alpha_i(t+1)} \, r_i^{\alpha_i(t+1)} \right]^{1-\alpha_i(t)} \cdot$$

$$\left[(1-P_i)^{\alpha_i(t+1)} \, P_i^{1-\alpha_i(t+1)} \, Y_i \right]^{\alpha_i(t)} \tag{III.1}$$

Where d_i is the index of the machine downstream of buffer i. We can divide both sides of (III.1) by

$$\left[(1-r_i)^{1-\alpha_i(t+1)} \, r_i^{\alpha_i(t+1)} \right]^{1-\alpha_i(t)} \tag{III.2}$$

This yields

$$\prod_{i=1}^{K} \left[\frac{X_i^{\alpha_i(t+1)-\alpha_{d_i}(t+1)} \; Y_i^{\alpha_i(t+1)}}{(i-r_i)^{1-\alpha_i(t+1)} \; r_i^{\alpha_i(t+1)}} \right] =$$

$$\sum_{\alpha_1(t)=1}^{1} \cdots \sum_{\alpha_K(t)=1}^{1} \prod_{i=1}^{K} \left[\frac{(1-P_i)^{\alpha_i(t+1)} \; P_i^{1-\alpha_i(t+1)} \; Y_i}{(1-r_i)^{1-\alpha_i(t+1)} \; r_i^{\alpha_i(t+1)}} \right]^{\alpha_i(t)} \qquad \text{(III.3)}$$

Where $X_K = 1$. Recall that K is the index of the output machine. We now make use of Lemma 3.1 in Gershwin and Schick (1979). This states that for all sets of real numbers $A_1, \ldots, A_K$, ,

$$\sum_{\alpha_1=0}^{1} \cdots \sum_{\alpha_K=0}^{1} \prod_{i=1}^{K} A_i^{\alpha_i} = \prod_{i=1}^{K} (1+A_i) \qquad \text{(III.4)}$$

We now use (III.4) to write the right hand side of (III.3) as

$$\prod_{i=1}^{K} \left[1 + \frac{(1-P_i)^{\alpha_i(t+1)} \; P_i^{1-\alpha_i(t+1)} \; Y_i}{(1-r_i)^{1-\alpha_i(t+1)} \; r_i^{\alpha_i(t+1)}} \right] \quad . \qquad \text{(III.5)}$$

When (III.5) is substituted in (III.3) the argument t disappears and thus for simplicity we write α_i instead of $\alpha_i(t+1)$.

$$\prod_{i=1}^{K}\left[\frac{X_i^{\alpha_i-\alpha_{d_i}}\,Y_i^{\alpha_i}}{(1-r_i)^{1-\alpha_i}\,r_i^{\alpha_i}}\right] =$$

$$\prod_{i=1}^{K}\left[1+\frac{(1-P_i)^{\alpha_i}\,P_i^{1-\alpha_i}\,Y_i}{(1-r_i)^{1-\alpha_i}\,r_i^{\alpha_i}}\right] \qquad \text{(III.6)}$$

Equation (III.6) can be simplified as

$$\prod_{i=1}^{K}\left[X_i^{\alpha_i-\alpha_{d_i}}\,Y_i^{\alpha_i}\right] =$$

$$\prod_{i=1}^{K}\left[(1-r_i)^{1-\alpha_i}\,r_i^{\alpha_i} + (1-P_i)^{\alpha_i}\,P_i^{1-\alpha_i}\,Y_i\right] \qquad \text{(III.7)}$$

Relation (III.7) has been obtained with no condition on α_i, and thus holds for all values of α_i.

If $\alpha_i=0$, for all $i=1,\ldots,K$, (III.7) becomes

$$1 = \prod_{i=1}^{K}\left[1-r_i+P_iY_i\right] \quad . \qquad \text{(III.8)}$$

If $\alpha_j=1$, and $\alpha_i=0$, for $i \neq j$, $i=1,\ldots,K$. We have $\alpha_j=1$, then $0=\alpha_{d_i} \neq \alpha_j$. Thus the factors on the left hand side of (III.7) are either $X_j\,Y_j$ or $\frac{1}{X_i}$

where the latter appears if i is a buffer upstream of machine j. Thus (III.7) reduces to

$$\frac{X_j Y_j}{\prod_{i\in L(j)} X_i} = \prod_{\substack{i=1 \\ i\neq j}}^{K} [1-r_i+P_iY_i] \quad [r_j+(1-P_j)Y_j] \quad , \qquad j=1,\ldots,K \tag{III.9}$$

We now use (III.8) to reduce (III.9) further, obtaining:

$$\frac{X_j Y_j}{\prod_{i\in L(j)} X_i} = \frac{r_j + (1-P_j)\, Y_j}{1-r_j + P_j\, Y_j} \quad , \qquad j=1,\ldots,K \tag{III.10}$$

Equations (III.8) and (III.10) are the desired parametric equations.

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