FINITE ELEMENT METHODS FOR REDUCTION
OF CONSTRAINTS AND CREEP ANALYSES

by

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ABSTRACT

The objective of the present study is to develop improved finite element
models by using the Hellinger-Reissner principle. Constraints in the
various finite element formulations are discussed for problems involving
plate bending with transverse shear effect and arches considered as a
simple shell. Conditions of zero transverse shear energy and constant
strain states (including rigid body modes) impose severe constraints on
the deformation modes of an assumed displacement model if exact order of
integration is used. The Hellinger-Reissner principle is employed to
derive finite element models with relaxed constraints. The reduced
integration scheme in the assumed displacement method is interpreted as
an application of the Hellinger-Reissner principle. The Hellinger-
Reissner principle is also applied to develop small and large deflection
shell analyses procedures. An eight-node shell element is derived. An
assumed stress hybrid element formulation is provided for creep and
viscoplastic analyses. The resulting system of nonlinear first order ordinary differential equations can be solved by various Runge-Kutta methods. The classical elastoplastic problems are solved by using the viscoplastic formulation. The present formulation is more efficient than the assumed displacement method. Also, an initial exploratory study of various formulations involving a propagating crack has been made.

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SECTION 1
INTRODUCTION

The finite element method has been in wide use and many general purpose computer programs are available for the analysis of engineering structures. Invariably, these programs incorporate the assumed displacement method based on the principle of minimum potential energy or the principle of virtual work which is a one-field variational principle. But there are certain problem areas in which the assumed displacement method does not give adequate answers. For example, the condition of zero transverse shear strain energy in plate bending problems with transverse shear effects and the condition of constant strain states including rigid body modes in shell analysis impose severe constraints on the assumed deformation modes of a finite element, thus resulting in unreliable solutions.

On the other hand, finite element models can be derived from two-field variational principles such as the Hellinger-Reissner principle [22], the modified principle of complementary energy [7] and the modified principle of potential energy [8]. Introduction of one more field variable renders us more freedom in the derivation of finite element models. Furthermore, numerical results indicate that in many occasions finite element models derived from these alternative variational principles give solutions better than those by the conventional assumed displacement elements. For example, the assumed stress hybrid elements derived by the modified principle of complementary energy give calculated stresses more accurate than those by the assumed displacement method. Therefore, for stress-dependent problems such as elastic-plastic problems, the assumed
stress hybrid method can be adopted advantageously as demonstrated by Spilker [62].

The purpose of the present study is two-fold. First, it intends to develop efficient finite element models for plate and shell problems by relaxing the above-mentioned constraints. Second, the assumed stress hybrid method is applied to creep problems which are also stress-dependent. The Hellinger-Reissner principle and the modified Hellinger-Reissner principles are to be employed. More specifically, the present work is organized as follows.

Section 2 deals with the constraints on the assumed deformation modes of finite element models and methods for improving the solution by reducing the constraint conditions. Problems treated include plate bending problems with transverse shear effect and a circular arch as a simple shell.

In Section 3, based on results obtained from plate bending and arch problems, an eight-node shell element is to be derived for linear elastic analysis. Also, an incremental finite element method is to be formulated for large deflection analysis of shell structures.

In Section 4, an assumed stress hybrid finite element formulation is to be derived for creep analysis under the assumption of small displacement. It will be shown that application of the Hellinger-Reissner principle leads to a system of nonlinear first order ordinary differential equations. Also, a procedure to solve elastoplastic problems by means of a viscoplastic formulation will be discussed. Comparison with the assumed displacement method will be made by means of an example problem. 
In Section 5, a preliminary work on the analysis of propagating crack under creep conditions is presented. The purpose of the present work is to establish incremental formulations for structures with a changing boundary.
SECTION 2  
CONSTRAINTS IN FINITE ELEMENT FORMULATIONS

2.1 Introduction

In Subsections 2.1-2.4, it is intended to clarify constraint conditions [1] associated with some finite element models and provide remedies. For this purpose, a review of pertinent finite element models is in order.

For plate bending problems, a large number of assumed displacement elements have been formed under the usual Kirchhoff assumption. In this case, the compatibility requirement is the continuity of the normal slope across the element boundaries. This requirement can be met by using sub-regions in an element or higher order interpolation functions [2,3]. On the other hand, nonconforming elements which do not satisfy the compatibility condition give converging solution if they satisfy the patch test [4]. The continuity requirement can be relaxed by using the hybrid element model or the mixed model. In the mixed formulation [5], both the nodal stresses and displacements appear in the final assembled equation for solution. On the other hand, similar to the conventional assumed displacement model, the assumed stress hybrid method [6,7] and the assumed displacement hybrid method [8,9] have nodal displacements as unknowns in the final assembled equation.

Another approach is to realize that the continuity of the normal slope is not required even in the assumed displacement element models if the Kirchhoff assumption is abandoned. But, inclusion of the transverse shear strain energy in the formulation leads to an excessively stiff element stiffness matrix. Therefore, Wempner et al. [10] introduced the
concept of discrete Kirchhoff assumption in which the Kirchhoff assumption is imposed at selective points in the element and the transverse shear strain energy part is excluded in the formulation.

In thin shell analysis by the finite element method, an element must be able to represent the state of constant strains and rigid body modes as accurately as possible. In the assumed displacement method, the usual displacement assumption by polynomial expansion along the shell coordinates does not satisfy these conditions and leads to a stiffer element stiffness matrix. This was demonstrated by Ashwell and Sabir [11] and Dawe [12] by solving circular arch problems. Haisler and Stricklin [13], Mebane and Stricklin [14] discussed this problem and showed that higher order interpolations improve the performance. Also, Fried [15] deals with the mathematical aspect of the problem by means of arch analysis.

In the cases discussed so far, the reduced integration scheme has been a great success. Zienkiewicz et al. [16], Pawsey and Clough [17] have shown that performance of eight node plate or shell element is greatly improved by reduced integration. Recently, Zienkiewicz and Hinton [18], in dealing with plate bending problems, pointed out that the reduced integration scheme reduces the constraint condition due to zero transverse shear strain energy.

In subsequent sections we consider constraints associated with various finite element models for the cases discussed above. Finite element models to be treated are those based on the assumed displacement method and the Hellinger-Reissner principle. The conventional assumed displacement elements are derived from the principle of minimum potential
energy by assuming displacements in terms of nodal displacements. Minimization of the resulting functional leads to a system of linear equations to be solved for the nodal displacements. The second type of elements is formulated from the Willam's version of the Hellinger-Reissner principle [19]. In his version, instead of stresses, strains appear as independent variables in addition to displacements. Finite elements are derived by assuming independent strains in terms of unknown parameters and displacements in terms of nodal displacement. The assumed strains in an element are independent from those of other elements and can be eliminated on the element level. Thus, the resulting functional is expressed as a function of nodal displacements only. The stationarity condition results in a system of equations to be solved for nodal displacements. This procedure is similar to that of the assumed stress hybrid formulation by Pian [6]. But these elements are to be called "mixed" elements. According to Pian [20,21] a hybrid element is defined as the element model based on a modified variational principle with relaxed compatibility. A historical remark about hybrid elements is also given by Pian [21]. In comparing these mixed elements for plane stress problem, Willam concludes that mixed elements perform better than the corresponding assumed displacement elements. But it is to be noted that mixed elements discussed in this study are different from those initiated by Herrmann [5]. The present mixed elements belong to the matrix displacement method while Herrmann's mixed elements retain nodal stresses in the final assembled equations. We may call Herrmann's formulation mixed model I and the present one mixed model II. It will
be shown that the present mixed formulation can be used to provide element models with relaxed constraints. Also, the reduced integration scheme in the assumed displacement method will be reexamined from a variational point of view by showing its relationship to the present mixed formulation.

2.2 Plate Bending Elements with Transverse Shear Effect

We consider, for simplicity, elastic small displacement bending of a simple beam which has a unit width, a length \( L \) and a depth \( t \). In the conventional assumed displacement method, the element stiffness matrix of a beam with transverse shear effect is derived from the principle of minimum potential energy. In nondimensional form for which the normal deflection \( w \) and the distance \( x \) along the beam are written as

\[
\begin{align*}
x &= Lx' \\
w &= Lw'
\end{align*}
\]  

(2.1)

the functional \( \pi_p \) is

\[
\pi_p = \frac{1}{2} \frac{Et^3}{L} \left[ 0 \int_{1}^{0} \frac{1}{12} \left( \frac{d\phi}{dx} \right)^2 dx' + \frac{Gt}{E} \left( \frac{L}{t} \right)^2 \right. \\
&\left. \int_{0}^{1} \left( \phi + \frac{dw'}{dx} \right)^2 dx' \right] - W
\]

(2.2)
where

\[ \phi = \text{rotation} \]
\[ \beta = \text{form factor} \left( = \frac{5}{6} \text{ for a rectangular cross section} \right) \]
\[ -W = \text{potential energy due to applied loads} \]
\[ E = \text{Young's Modulus} \]
\[ G = \text{shear modulus} \]

The first and second integrals in Eq. (2.2) represent the bending energy and the transverse shear strain energy respectively. As the beam becomes thinner and thinner, the magnitude of the transverse shear strain energy relative to the bending energy becomes smaller. Thus, a finite element must be able to represent a small or zero transverse shear strain energy as accurately as possible. Otherwise any error resulting from inaccurate representation of the transverse shear strain energy will be magnified by \((L/t)^2\) and dominate the bending energy. To be able to represent the state of zero transverse shear strain energy, it is necessary that

\[ \phi + \frac{dw'}{dx'} = 0 \quad (2.3) \]

Equation (2.3) imposes severe constraints on the assumed deformation modes of a finite element. Consider a four degrees of freedom (DOF's) element designated as DB4 with

\[ \phi = a_1 + a_2x' \]
\[ w' = a_3 + a_4x' \quad (2.4) \]
Then from (2.3)

\[ a_1 + a_4 = 0 \] \hspace{1cm} (2.5)
\[ a_2 = 0 \]

Therefore two constraints are present among four degrees of freedom. In actuality, the number of constraints must be measured on an entire structural level. But it is helpful to count the number of constraints on an element level in order to determine relative efficiency of various elements with the same number of degrees of freedom. To demonstrate the effect of the constraints a cantilever beam under tip load \( p \) was analyzed with 20 elements. The calculated nondimensional maximum displacements \( \bar{w} \) are given in Table 1. It is seen that results by DB4 element are very sensitive to the thickness ratios.

We consider next a formulation using the Hellinger-Reissner principle [22] with a functional given in terms of independent stresses \( \sigma_{ij} \) and displacements \( u_i \) as follows:

\[ \Pi_R = \int_V (\sigma_{ij} \varepsilon_{ij} - \frac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl}) dv - \int_G \bar{T}_i u_i ds \] \hspace{1cm} (2.6)

in which

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \]

\[ S_{ijkl} = \text{compliance tensor} \]

\[ \bar{T}_i = \text{applied tractions} \]
The strains $e_{ij}$ are related to the stresses by

$$e_{ij} = S_{ijkl} \sigma_{kl}$$

or

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

(2.7)

Introducing Eq. (2.7) to Eq. (2.6), we obtain

$$\pi_R = \nu \int \left( C_{ijkl} e_{kl} \tilde{e}_{ij} - \frac{1}{2} C_{ijkl} e_{ij} e_{kl} \right) dv$$

$$- s \int_{\Gamma} \bar{u}_1 ds$$

(2.8)

The Hellinger-Reissner functional expressed in Eq. (2.8) appears to be more convenient for thin structures when material nonlinearities such as creep or plasticity effects are included in the analysis. Stresses across the thickness could be highly nonlinear or piecewise continuous in these cases while strains may be considered linear. It means that, if an element is to be formulated using Eq. (2.6), it is necessary to assume a higher order stress field in these cases.

For plate bending problems, the functional in Eq. (2.8) is expressed in matrix form as follows.

$$\pi_R = \int (\kappa^T C_{\kappa\kappa} \kappa - \frac{1}{2} \kappa^T C_{\kappa\kappa} \kappa) dxdy$$

$$+ \int (\gamma^T C_{\gamma\gamma} \gamma - \frac{1}{2} \gamma^T C_{\gamma\gamma} \gamma) dxdy - W$$

(2.9)

in which

$$\kappa = \left\{ \begin{array}{c} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right\}$$

= curvature strain
\[ \gamma = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \text{transverse shear strains} \]

\[ \kappa = \begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \end{Bmatrix} \]

\[ \gamma = \begin{Bmatrix} \phi + \partial \omega/\partial x \\ \psi + \partial \omega/\partial y \end{Bmatrix} \]

with \( \phi, \psi = \text{rotation angles} \)

and \( C_\kappa, C_\gamma \) are appropriate elastic constant matrices. An element is derived by assuming curvature strains and transverse shear strains to be polynomials in \( x \) and \( y \) with unknown parameters \( \beta_i \)'s and \( \alpha_i \)'s respectively which are independent for the element. Rotation angles and normal displacement are assumed in terms of nodal values \( q_i \)'s. Thus,

\[ \kappa = p_\beta \beta \]

\[ \gamma = p_\alpha \alpha \]

\[ u = A q \]

Then

\[ \bar{\kappa} = B_\kappa q \]

\[ \bar{\gamma} = B_\gamma q \] \hspace{1cm} (2.11)

Substituting these into Eq. (2.8)
\[ \pi_R = \sum_n \left( G_{\beta}^T \tilde{q} - \frac{1}{2} \tilde{q}^T H_{\beta} \tilde{q} + G_{\alpha}^T \tilde{q} - \frac{1}{2} \tilde{q}^T H_{\alpha} \tilde{q} - q^T Q \right) \]  \hspace{1cm} (2.12)

with

\[ G_{\beta} = \int_{\tilde{\beta}} P_{\tilde{\beta}}^T \Gamma_{\tilde{\kappa}} \Delta_{\tilde{\kappa}} \, d\tilde{x} \, d\tilde{y} \]
\[ H_{\beta} = \int_{\tilde{\beta}} P_{\tilde{\beta}}^T \Gamma_{\tilde{\kappa}} \Delta_{\tilde{\kappa}} \, d\tilde{x} \, d\tilde{y} \]
\[ G_{\alpha} = \int_{\tilde{\alpha}} P_{\tilde{\alpha}}^T \Gamma_{\tilde{\gamma}} \Delta_{\tilde{\gamma}} \, d\tilde{x} \, d\tilde{y} \]
\[ H_{\alpha} = \int_{\tilde{\alpha}} P_{\tilde{\alpha}}^T \Gamma_{\tilde{\gamma}} \Delta_{\tilde{\gamma}} \, d\tilde{x} \, d\tilde{y} \]
\[ q^T Q = W \]

\[ n = \text{number of elements} \]

By taking \( \delta \pi_R = 0 \) with respect to \( \tilde{\beta} \) and \( \tilde{\alpha} \), we can express \( \tilde{\beta} \) and \( \tilde{\alpha} \) in terms of \( \tilde{q} \),

\[ \tilde{\beta} = H_{\beta}^{-1} G_{\beta} \tilde{q} \]  \hspace{1cm} (2.13)
\[ \tilde{\alpha} = H_{\alpha}^{-1} G_{\alpha} \tilde{q} \]

Substituting Eq. (2.13) into Eq. (2.12),

\[ \pi_R = \sum_n \left( \frac{1}{2} q^T k q - q^T Q \right) \]

with

\[ k = G_{\beta}^T H_{\beta}^{-1} G_{\beta} + G_{\alpha}^T H_{\alpha}^{-1} G_{\alpha} \]  \hspace{1cm} (2.14)
For a thin plate, the constraints on an element are obtained by setting the transverse shear strain energy term to be zero, i.e.

$$\frac{1}{2} q^T G^T \alpha_h^{-1} \alpha G q = 0$$  \hspace{1cm} (2.15)

or

$$\alpha G q = 0$$  \hspace{1cm} (2.16)

Therefore the number of constraints for an element is equal to the rank of the $\alpha G$ matrix. This rank can be controlled by a proper choice of assumed strains.

A part of the Euler equations of the Hellinger-Reissner principle is the following curvature-displacement relation and the relation between transverse shear strains and displacements

$$\kappa = \kappa$$  \hspace{1cm} (2.17)

$$\gamma = \gamma$$  \hspace{1cm} (2.18)

If the curvature-displacement relation in Eq. (2.17) is satisfied exactly in Eq. (2.9), the modified Hellinger-Reissner principle is derived as follows.

$$\pi_{mR} = \frac{1}{2} \int \kappa^T C_k \kappa \; dx \; dy + \int \gamma^T C_\gamma \gamma \; dy$$

$$- \frac{1}{2} \gamma^T C_\gamma \gamma \; dx \; dy - W$$  \hspace{1cm} (2.19)

The curvature strains do not appear as variables in the above expression. The procedure to formulate a finite element is similar to the case of
the Hellinger-Reissner principle. That is, we assume

\[ \gamma = \pi \alpha \]

\[ u = A q \]

or

\[ \kappa = B q \]

(2.20)

Then

\[ \pi_{mR} = \sum_{n=1}^{\infty} \frac{1}{2} q^T k_\beta q + \alpha^T G_\alpha q - \frac{1}{2} \alpha^T H_\alpha \alpha - q^T Q \]

(2.21)

with

\[ k_\beta = \int B^T C B \, dx dy \]

By taking \( \delta \pi_{mR} = 0 \) with respect to \( \alpha \), we obtain

\[ \alpha = H_\alpha^{-1} G_\alpha q \]

(2.22)

Substituting \( q \) in Eq. (2.22) into Eq. (2.21)

\[ \pi_{mR} = \sum_{n=1}^{\infty} \left( \frac{1}{2} \alpha^T k_\beta \alpha + \frac{1}{2} \alpha^T G_\alpha \alpha \right) H_\alpha^{-1} G_\alpha q - q^T Q \]

(2.23)

By setting the transverse shear strain energy equal to zero, we obtain the same constraint as that in Eq. (2.16).

We also observe that if the transverse shear strain-displacement relation in Eq. (2.18) is satisfied exactly, the functional for the principle of minimum potential energy is obtained.
As an example, the same cantilever beam problem was solved by using a new element. The element has 4 DOF's with linear w and φ as in Eq. (2.4) and the transverse shear strain is assumed to be constant, resulting in one constraint for the element. The displacement element has two constraints as mentioned before. If a constant curvature is assumed in the Hellinger-Reissner principle, the resulting element will be the same as that from the modified Hellinger-Reissner principle. We designate the element as RB4. The nondimensional maximum displacement is given in Table 1 in comparison with the results by the displacement elements. Note that only five RB4 elements were used while twenty elements were used for the displacement method. It is seen that the RB4 element is not sensitive to the thickness ratios and gives reliable solutions.

Some Plate Bending Elements

We describe here plate bending elements derived by the formulation discussed above and give solutions for numerical example problems. Three eight-node curved quadrilateral elements with 24 DOF's and three six-node curved triangular elements with 18 DOF's were derived. For description of geometry and displacement isoparametric representation is utilized. Each component of curvature strains and/or transverse shear strains is assumed to have the same distribution in x and y. These elements are listed in Table 2. It is to be noted that elements RL8 and MR18 are identical for a geometry with straight sides and midside nodes. These elements were used to analyze a simply supported square plate under a constant pressure for various thickness ratios. Due to symmetry only a quarter of the plate was discretized by 2x2 rectangular meshes. For six-node triangular elements, 1x1 mesh is made of four
elements. Table 3 lists the nondimensional maximum deflection. It is seen that among the eight-node elements the MR24A element gives good results for the entire range of thickness ratios considered. On the other hand the R24 and MR24 elements tend to be too stiff for very thin plates. All six-node elements give reliable solutions in general. For very thin plates, the MRL8A element with seven integration points appears to yield most accurate solutions. Additional results are listed in Table 4 in which nondimensional maximum deflections of a clamped plate are given for various thickness ratios. The R24 element and MR24A element are compared to each other for 2x2 and 3x3 meshes. Again, the MR24A element gives better results.

2.3 Circular Arch as a Simple Shell

A circular arch is an important structural member in itself and furthermore it exhibits essential characteristics of shell structures. Consider the functional for the principle of minimum potential energy for a circular arch of a unit width and radius \( R \) written as

\[
\Pi_p = \frac{1}{2} \int \left( E_t \frac{du}{ds} + \frac{w}{R} \right)^2 + \frac{1}{12} E_t \frac{1}{R} \left( \frac{du}{ds} - \frac{d^2 w}{ds^2} \right)^2 \right) ds - W
\]  

(2.24)

where

\( u = \) inplane displacement

\( w = \) normal displacement

\( s = \) coordinate along the arch

\( t = \) thickness
If we introduce angle $\phi$ such that $ds = Rd\phi$, then

\[
\pi_p = \frac{E(t)}{2} \left( \frac{t}{R} \right)^3 \int \left\{ (R/t)^2 (du/d\phi + w)^2 + \frac{1}{12} \left( du/d\phi - d^2 w/d\phi^2 \right)^2 \right\} d\phi - W \tag{2.25}
\]

In the above expression, the first integral and the second integral represent the stretching energy and the bending energy respectively.

If displacements $u$ and $w$ are assumed to have a polynomial distribution along $\phi$, the rigid body modes are not explicitly included in the assumed displacement modes. We focus our attention on the stretching part where any error due to inaccurate representation of rigid body modes or constant strain states is to be magnified by the factor $(R/t)^2$. To be able to represent a rigid body mode in the stretching part, it is necessary that

\[
du/d\phi + w = 0 \tag{2.26}
\]

pointwise. Equation (2.26) imposes constraints on the deformation modes of a finite element derived from the principle of minimum potential energy. For example, consider an arch element with linear $u$ and cubic $w$, i.e.

\[
u = a_1 + a_2 \phi \\
w = a_3 + a_4 \phi + a_5 \phi^2 + a_6 \phi^3 \tag{2.27}
\]

Then, according to Eq. (2.26)

\[
a_2 + a_3 = 0 \\
a_4 = a_5 = a_6 = 0 \tag{2.28}
\]
Therefore among 6 DOF's there exist four constraints which will severly limit the performance of the element.

As in the case of plate bending problem, we utilize the Hellinger-Reissner principle whose functional is given as

\[
\tau_R = \int \{ E t c \left( \frac{du}{ds} + \frac{w}{R} \right) - \frac{1}{2} E t e^2 \} \, ds + \int \left\{ \frac{1}{12} E t^3 \kappa \left( \frac{1}{R} \frac{du}{ds} - \frac{d^2 w}{ds^2} \right) \right. \\
- \frac{1}{2} \frac{1}{12} E t^3 \kappa^2 \} ds - W \quad (2.29)
\]

Here the inplane strain \( \varepsilon \) and curvature \( \kappa \) are introduced as additional variables. As usual a finite element is formulated by assuming \( u \) and \( w \) in terms of nodal displacements \( q_i \)'s and \( \varepsilon \) and \( \kappa \) as polynomials in \( \phi \) with unknown parameters \( \alpha_i \)'s and \( \beta_i \)'s respectively, i.e.

\[
\varepsilon = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_T G_\alpha q + \frac{1}{2} \alpha_T H_\alpha \alpha + \beta_T G_\beta q \\
-\frac{1}{2} \beta_T H_\beta \beta - q_T Q \end{pmatrix} \quad (2.30)
\]

Substituting these into Eq. (2.29), \( \tau_R \) is written as function of \( \alpha \), \( \beta \) and \( q \),

\[
\tau_R = \sum_i \left( \alpha_T G_\alpha q - \frac{1}{2} \alpha_T H_\alpha \alpha + \beta_T G_\beta q \right) - \frac{1}{2} \beta_T H_\beta \beta - q_T Q \quad (2.31)
\]
where \( G_\alpha, H_\alpha, G_\beta, H_\beta \) are appropriately defined matrices. Taking \( \delta \pi_R = 0 \) with respect to \( \alpha \) and \( \beta \), we obtain \( \alpha \) and \( \beta \) in terms of \( q \) and thus \( \pi_R \) can be written in terms of \( q \) only as follows,

\[
\pi_R = \sum \frac{1}{2} q^T C_\alpha^{-1} G_\alpha q + \sum \frac{1}{2} q^T C_\beta^{-1} G_\beta q
\]

in which the second part represents the stretching energy. Thus, by setting it to be zero we obtain the constraint equation

\[
G_\alpha q = 0 \tag{2.33}
\]

The number of constraints is again equal to the rank of the \( G_\alpha \) matrix. Here the rank is determined by the number of \( \alpha_i \)'s.

By introducing the curvature-displacement relation

\[
\kappa = \frac{1}{12} Et^3 \left( \frac{1}{R} \frac{du}{ds} - \frac{d^2 w}{ds^2} \right) \tag{2.34}
\]

into the Hellinger-Reissner principle, we obtain a modified Hellinger-Reissner principle written as

\[
\pi_mR = \int \left\{ Et \left[ \frac{du}{ds} + \frac{w}{R} \right] - \frac{1}{2} Et e^2 \right\} ds \\
+ \frac{1}{2} \int \frac{1}{12} Et^3 \left( \frac{1}{R} \frac{du}{ds} - \frac{d^2 w}{ds^2} \right)^2 ds - W \tag{2.35}
\]

The curvature strain \( \kappa \) does not appear as a variable. A finite element is formulated in the usual manner and the constraint on the element is
the same as that in Eq. (2.33).

Several arch elements are listed in Table 5. Two elements were derived by the modified Hellinger-Reissner principle. But it is to be noted that the same element can be derived by the Hellinger-Reissner principle with properly assumed curvature strain. Using these elements a circular ring of radius R pinched at two opposite points was analyzed. Due to symmetry, only a quarter of the ring was discretized for finite element analysis. Results are given in Table 6 and Figure 1. Table 6 gives maximum nondimensional deflection and Figure 1 shows nondimensional inplane stress which is defined as the inplane force divided by the load p. The stresses calculated by the displacement element DA8 show severe fluctuations. This fluctuation grows with increasing R/t ratio. The result of the DA6 element shows similar but even more fluctuations. These result by displacement elements agree with those by Dawe [12] who tested circular arch elements with various displacement assumptions. We observe that RA6 and RA8 elements are not sensitive to R/t ratios and give reliable results.

2.4 Rationalization of the Reduced Integration Scheme

In previous sections, constraints associated with finite element models have been identified and it has been shown that the Hellinger-Reissner principle can be used to reduce these constraints. Also, the reduced integration scheme is a successful means for reducing the conditions of constraints in the assumed displacement element models [18]. Hinton and Campbell [23], in explaining the eight node plane element,
interpreted the reduced integration scheme as an application of the least square method in which a bilinear fit is made to strains calculated from displacements.

The purpose of this section is to rationalize the reduced integration scheme by reexamining it from a variational point of view and by showing its equivalence to the formulation based on the Hellinger-Reissner principle.* First, we consider the Hellinger-Reissner principle with the functional expressed in terms of displacements $u_i$ and independent strains $e_{ij}$:

$$\Pi_R = \int_V \left( C_{ijkl} e_{kl} \tilde{e}_{ij} - \frac{1}{2} C_{ijkl} e_{ij} e_{kl} \right) dV - \int_\partial \tilde{T}_{i1} u_i dS$$  \hspace{1cm} (2.36)

The stresses $\sigma_{ij}$ are related to $e_{ij}$ as

$$\sigma_{ij} = C_{ijkl} e_{kl}$$  \hspace{1cm} (2.37)

A finite element is derived by assuming independent strains $e_{ij}$ in polynomial expansions. The limitation principle by Fraeijs de Veubeke [24] requires that the order of assumed $e_{ij}$ be lower than that of $\tilde{e}_{ij}$. Otherwise it is possible to set

$$e_{ij} = \tilde{e}_{ij}$$  \hspace{1cm} (2.38)

and we recover the functional for the principle of minimum potential energy and

$$\sigma_{ij} = C_{ijkl} \tilde{e}_{kl}$$  \hspace{1cm} (2.39)

*While preparing the final draft of this thesis, the author obtained the unpublished works of Malkus and Hughes [72] and Hughes et al. [73]. A similar result has been obtained in these references.
Stiffness matrices of most finite elements are derived by numerical integration. Thus, the volume integral in Eq. (2.36) is expressed as

$$I_1 = \sum_i \left( \varepsilon^T \mathbf{C} \varepsilon - \frac{1}{2} \varepsilon^T \mathbf{C} \varepsilon \right) W_i$$

(2.40)

where $W_i$ = weighting factor at the $i$th integration point. On the other hand, the stiffness matrix of an assumed displacement model element is derived from the principle of minimum potential energy, i.e.

$$I_2 = \frac{1}{2} \int \varepsilon^T \mathbf{K} \varepsilon = \frac{1}{2} \int \varepsilon^T \mathbf{C} \varepsilon \text{d}V = \frac{1}{2} \sum_i \varepsilon^T \mathbf{C} \varepsilon \quad W_i$$

(2.41)

Since $\varepsilon$ and $\varepsilon$ are of different orders, it is possible to set

$$\varepsilon = \varepsilon$$

(2.42)

at discrete points only. Let us choose the Gaussian points as such points and integrate $I_1$ and $I_2$, then

$$I_1 = \frac{1}{2} \sum_i \varepsilon^T \mathbf{C} \varepsilon \quad W_i = I_2$$

(2.43)

Thus, the stiffness matrices derived from two apparently different variational principles are equal to each other. Invariably in this case the number of Gaussian points is smaller than that required for the exact integration of $I_2$, resulting in an application of a reduced order of integration. On the other hand, $I_1$ can be integrated exactly with the same number of integration points if the polynomials in $\varepsilon$ are of sufficiently low order. One exception occurs for an element with a distorted shape. In this case a lower order integration is employed in practice to
cut computing time. Since $e_{ij} = \tilde{e}_{ij}$ at discrete points only, strictly speaking, we are still using the Hellinger-Reissner principle. Also, it is to be noted that stresses are related to strains not by Eq. (2.39) but by Eq. (2.37). Of course at integration points there is no difference at all. But there could be some difference at other points.

As an illustration, let us consider an eight-node plane stress (or plane strain) element derived from the principle of minimum potential energy and the corresponding element derived from the Hellinger-Reissner principle with the following assumed independent strains in parent coordinates $\xi, \eta$.

$$
\begin{align*}
  e_x &= \beta_1 + \beta_2 \xi + \beta_3 \eta + \beta_4 \xi \eta \\
  e_y &= \beta_5 + \beta_6 \xi + \beta_7 \eta + \beta_8 \xi \eta \\
  e_{xy} &= \beta_9 + \beta_{10} \xi + \beta_{11} \eta + \beta_{12} \xi \eta
\end{align*}
$$

(2.44)

The number of integration point is $2 \times 2 = 4$. Each component of the assumed strains has four $\beta_i$'s and there are four integration points. Therefore it is possible to set

$$
\begin{align*}
  e_x &= \tilde{e}_x \\
  e_y &= \tilde{e}_y \\
  e_{xy} &= \tilde{e}_{xy}
\end{align*}
$$

(2.45)

at Gaussian integration points. The stiffness matrices of two elements are equal to each other.

Another example is the four-node element derived from a modified Hellinger-Reissner principle by Key [25] with the functional
\[
\pi_{mR} = \int_V A(e'_{ij}) \, dV + \sqrt{2k} \int (\sigma_m \bar{e}_m - \frac{1}{2k} \sigma_m^2) \, dV - \int_\Gamma \bar{T}_{ij} \, dS
\]  
(2.46)

where

\[A(e'_{ij}) = \text{deviatoric strain energy in terms of deviatoric}
\]
\[\bar{e}_{ij} = \text{strain}\]
\[\bar{e}_m = \text{dilatational strain}\]
\[\sigma_m = \text{mean stress}\]
\[k = \text{bulk modulus} = \frac{E}{3(1-2\nu)}\]

The strains \(\bar{e}'_{ij}\) and \(\bar{e}_m\) are functions of displacements. The mean stress is assumed constant for the element. We can derive the same element stiffness matrix by using one point integration for the dilatational part of the strain energy term in the principle of minimum potential energy.

Again we note that \(\bar{e}_m = \frac{\sigma_m}{k} = \bar{e}_m\) at the integration point. A similar result has been reported by Hughes [26].

Next we consider beam bending problems with transverse shear effects. The functional in Eq. (2.2) for the principle of minimum potential energy is written as

\[
\pi_p = \frac{1}{2} \left( E t^3 / L \right) \left( \frac{1}{12} \sum_i (d\phi/dx')^2 W_i + (GB/E)(L/t)^2 \right)
\]

\[
\sum_i (\phi + dw'/dx')^2 W_i - W
\]  
(2.47)

where it is assumed that the weighting factor \(W_i > 0\). Now for thin beams, the constraints of zero transverse shear strains are

31
at each integration points. Then the number of constraints for the elements are equal to the number of integration points. Thus, it is beneficial to use a reduced order of integration for the transverse shear strain energy part. Of course, the maximum number of constraints are limited by Eq. (2.3). In Subsection 2.1 it has been shown that a beam element can be derived from the modified Hellinger-Reissner principle with assumed independent transverse shear strain \( \gamma \). Equivalence of a reduced integration element to an assumed strain element can be established if

\[
\gamma = \phi + \frac{dw'}{dx'}
\]  

(2.49)

holds at integration points. For example, two-node (four DOF) assumed displacement element with linear \( \phi \) and \( w' \) requires two integration points for the exact integration of the transverse shear strain energy. But this element does not perform properly for thin beam cases as shown in Subsection 2.1. Much more efficient solutions were obtained by an element with constant assumed transverse shear \( \gamma \). Equivalent reduced integration element is obtained by using one point integration for the transverse shear strain energy. Next, we consider three-node (six DOF) beam element. For an assumed displacement element, three integration points are required for the exact integration of transverse shear strain energy. Here again a reduced integration element with two points are
much more efficient. The reduced integration element is equivalent to
the assumed strain element with

\[ \gamma = \alpha_1 + \alpha_2 x \]  \hspace{1cm} (2.50)

Since \( \gamma \) has two unknown parameters \( \alpha_1 \) and \( \alpha_2 \), it is possible to satisfy
Eq. (2.49) at each integration points. To be consistent with the variational principle, shear \( Q \) is to be calculated by the relation

\[ Q = G\beta \gamma \]  \hspace{1cm} (2.51)

especially at points other than the Gaussian points. One element solu-
tion for a cantilever beam under a tip load \( p \) is given in Table 7 where
values of nondimensional transverse shear are given at two integration
points and at the centroid of the element. It should be noted that
incorrect shear appear at the centroid of the element calculated by the
relation

\[ Q = G\beta t(\phi + dw'/dx') \]  \hspace{1cm} (2.52)

The above discussions are easily extended to plate bending problems.
For example the stiffness matrix of an eight-node displacement element
with 2x2 reduced integration is equal to that of the element with bilinear
assumed curvature strains and transverse shear strains in Subsection 2.1.
Also, the stiffness matrix of a six-node triangular element with linear
assumed strains in Subsection 2.1 is equal to that of the reduced integra-
tion element by Zienkiewicz and Hinton [18] when three point integration
is employed.
Similar discussion can be made on arch analysis (and shells) and equivalence between a displacement element with reduced integration and an assumed strain element can be established.

Another example involves the formation of mass matrices. The kinetic energy in the Hamilton's principle is written as

\[ T = \frac{1}{2} \int_V m \ddot{u}_i \ddot{u}_i \, dV \]  \hspace{1cm} (2.53)

where

\[ m = \text{mass density} \]

\[ u_i = \text{velocity components} \]

We may introduce velocity \( v_i \) (or momentum \( mv_i \)) as additional independent variables and define the complementary kinetic energy \( T^* \) [27],

\[ T^* = \int_V (mv_i \ddot{u}_i - \frac{1}{2}mv_i v_i) \, dV \]  \hspace{1cm} (2.54)

A mass matrix is obtained by assuming \( v_i \) in terms of unknown velocity parameters \( \beta_i \) and \( \dot{u}_i \) in terms of nodal values \( \ddot{q}_i \). The parameters \( \beta_i \) can be eliminated on an element level and \( T^* \) can be expressed in terms of \( \dot{q}_i \) only to yield the mass matrix. The mass matrix is equivalent to the one derived by a reduced integration from \( T \) if we can establish

\[ v_i = \dot{u}_i \]  \hspace{1cm} (2.55)

at the integration points.

In general, the derivation of an element stiffness matrix by the
Hellinger-Reissner principle requires the inversion of a matrix with the size equal to the total number of assumed strain parameters. Therefore, when the reduced integration is employed for the integration of the total strain energy as in the case of the eight-node plane element, the reduced integration scheme is more efficient than the application of the Hellinger-Reissner principle. On the other hand, if the reduced integration is used for the part of the total strain energy as in the case of the four-node plane strain element derived by Key's principle it is more efficient to use the Hellinger-Reissner principle.

While the reduced integration scheme in the assumed displacement method can always find an equivalence through the application of the Hellinger-Reissner principle, the opposite is not always true. For example, the eight-node plate bending element with linear assumed transverse shear strains can not be derived by the assumed displacement method with the reduced integration scheme. The finite element formulation by the Hellinger-Reissner principle provides a more general approach.
SECTION 3

SHELL ANALYSIS

3.1 Introduction

Application of the finite element method to shell analyses has been a major concern in structural mechanics and numerous shell finite elements have appeared to this date. Gallagher [28] has given a good survey of the shell elements in which various elements are classified into the following three groups:

(1) flat faceted elements
(2) shell theory elements
(3) solid elements

In the first approach, a shell is modelled by flat triangular elements. The element behavior is represented by superposing stretching and bending behaviors of a flat plate. Therefore, it excludes the coupling of inplane and normal displacements within the element. Also, flat elements introduce kinks at element junctures which are not present in actual smooth shell surfaces. Spurious moments could arise from these kinks.

In the second approach, a shell element is formulated by using a shell theory such as the Koiter-Sanders theory [29,30]. To avoid complexity of general shell theories, many elements are formed from shallow shell theories. A good documentation can be found in Ref. 31. In the third approach, a shell is treated as a solid thereby avoiding use of complicated shell theories. Twenty-node solid element with 2x2x2 integration can be used for shell analyses and already constitute a part in existing
programs such as ADINA [32]. Also, degenerate elements by Ahmed [33] can be classified into this category.

The finite element method has also been applied to large deflection analyses of shells. Since the problem is nonlinear the formulation is in an incremental form and thus require a step by step and/or an iterative solution procedure. Stricklin et al. [34], Haisler et al. [35] and Boland [36] give reviews pertinent to large deflection analyses. The majority of works deals with the assumed displacement model. The formulation for the assumed stress hybrid model has been provided by Boland. The stationary Lagrangian formulation and updated Lagrangian formulations [37,38,39] are discussed in his work. In the stationary Lagrangian approach, the reference state is the original undeformed state. In the updated Lagrangian formulation the reference state is the current state which is known. Thus, the increments of variables to the next state are referred to the current state.

In this section, we present formulations for small and large deflection analyses of shells. The degenerate element formulation is chosen due to its simplicity in comparison with a general shell theory. An eight-node element is to be developed for linear elastic analysis from the Hellinger-Reissner principle by utilizing what we have learned in the previous section. In the case of elastic large deflection analysis, stationary Lagrangian formulations are presented in a general form. But, for simplicity only a simple arch problem is chosen as an example.
3.2 Eight-Node Shell Element

A shell element can be formulated by the Hellinger-Reissner principle with the same functional as that in Eq. (2.8). But now $C_{ijkl}$, $e_{ij}$ and $\bar{e}_{ij}$ are defined with respect to a local Cartesian coordinates with one coordinate normal to the shell surface and the other two coordinates embedded in the shell midsurface as shown in Fig. 2. The strain $\bar{e}_{ij}$ is obtained by the tensor transformation

$$\bar{e}_{ij} = a_{ik}a_{j1} \bar{E}_{kl}$$

where

$$\bar{E}_{kl} = \frac{1}{2}(U_{k,1} + U_{l,1})$$

with the derivatives of the displacement $U_k$ taken with respect to the global Cartesian Coordinates and $a_{ik}$'s are the direction cosine. This transformation is required at each Gaussian integration point. The geometry of a shell element is defined by assuming Cartesian coordinates of a point in the shell as follows [2,16]

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \sum N_i(\xi,\eta) \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} + \frac{\zeta}{2} \sum N_i(\xi,\eta) t_i \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}$$

where

$$X_i, Y_i, Z_i = \text{global Cartesian coordinates of nodes in the midsurface}$$

$$\zeta = \text{coordinate normal to the midsurface ranging from -1 to +1}$$
\[ t_i = \text{shell thickness at nodal points} \]
\[ a_{3i} = \text{i component of nodal unit vector} \hat{a}_3 \text{ normal to the midsurface} \]
\[ N_i = \text{interpolating functions in terms of curvilinear coordinates} (\xi, \eta) \text{ in the midsurface} \]

The three Cartesian components of the displacement vector are defined as

\[
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix} = \Sigma N_i \begin{bmatrix}
u_i \\
v_i \\
w_i
\end{bmatrix} + \frac{\nu}{2} \Sigma N_i t_i \begin{bmatrix}
a_{11} & a_{21} \\
a_{12} & a_{22} \\
a_{13} & a_{23}
\end{bmatrix} \begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
\]  
(3.4)

where

\[ u_i, v_i, w_i = \text{nodal displacements at node } i \]
\[ a_{1j}, a_{2j} = \text{j component of the nodal unit vectors} \hat{a}_1 \text{ and } \hat{a}_2 \]
\[ \text{tangent to the midsurface} \]
\[ \phi_1, \phi_2 = \text{nodal rotation angles of the normal vector} \hat{a}_3 \]

The direction cosines are interpolated from the nodal orthogonal vectors \( \hat{a}_1, \hat{a}_2 \) and \( \hat{a}_3 \) [40]. With these description of geometry and displacement, we formulate an eight-node quadrilateral element with 40 degrees of freedom. First, consider the following assumed independent strains.
\[
e^x = \beta_1 + \beta_2 \xi + \beta_3 \eta + \beta_4 \xi \eta + \zeta (\beta_5 + \beta_6 \xi + \beta_7 \eta + \beta_8 \xi \eta)
\]
\[
e^y = \beta_9 + \beta_{10} \xi + \beta_{11} \eta + \beta_{12} \xi \eta + \zeta (\beta_{13} + \beta_{14} \xi + \beta_{15} \eta + \beta_{16} \xi \eta)
\]
\[
e^{xy} = \beta_{17} + \beta_{18} \xi + \beta_{19} \eta + \beta_{20} \xi \eta + \zeta (\beta_{21} + \beta_{22} \xi + \beta_{23} \eta + \beta_{24} \xi \eta)
\]

\[\gamma_{xz} = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta + \zeta (\alpha_9 + \alpha_{10} \xi + \alpha_{11} \eta + \alpha_{12} \xi \eta) \]
\[\gamma_{yz} = \alpha_5 + \alpha_6 \xi + \alpha_7 \eta + \alpha_8 \xi \eta + \zeta (\alpha_{13} + \alpha_{14} \xi + \alpha_{15} \eta + \alpha_{16} \xi \eta) \]

It is to be noted that \(x, y\) and \(z\) are components of the local Cartesian coordinates with the \(z\)-axis normal to the shell midsurface. Here we immediately see that the element stiffness matrix derived with the above assumed strain is equivalent to that of the \(2\times2\times2\) reduced integration element by Zienkiewicz et al. [16]. The strains \(\gamma_{xz}\) and \(\gamma_{yz}\) are \(\zeta\)-dependent for curved shells. But, contribution of \(\zeta\)-dependent terms to the total transverse shear strain energy is negligible. For flat plates \(\gamma_{xz}\) and \(\gamma_{yz}\) are independent of \(\zeta\). Therefore, we may drop \(\zeta\)-dependent terms in Eq. (3.6). For the present purpose, we assume the transverse shear strains as follows:

\[
\gamma_{xz} = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta \\
\gamma_{yz} = \alpha_4 + \alpha_5 \xi + \alpha_6 \eta
\]

This approximation is adopted because for plate bending problems, linear assumed transverse shear strains gave better results than bilinear strains. For one-dimensional problems, there is no difference between bilinear strains and linear strains. Symbolically, strains in Eqs. (3.1), (3.5) and (3.7) are written as
\[
\hat{e}' = \begin{cases}
-\hat{e}_x \\
-\hat{e}_y \\
-\hat{e}_{xy}
\end{cases} = Bq
\]  

(3.8)

\[
\hat{e}' = \begin{cases}
e_x \\
e_y \\
e_{xy}
\end{cases} = \frac{p_j}{\varphi}
\]  

(3.9)

\[
\hat{\gamma} = \begin{cases}
-\gamma_{xz} \\
-\gamma_{yz}
\end{cases} = Bq
\]  

(3.10)

\[
\gamma = \begin{cases}
\gamma_{xz} \\
\gamma_{yz}
\end{cases} = \frac{p_j}{\varphi}
\]  

(3.11)

Equations (3.8) through (3.11) are introduced into the following functional for the Hellinger-Reissner principle.

\[
\begin{align*}
\tau_R &= \int (e^T e - \frac{1}{2} e^T C_e e') dV \\
&\quad + \int (\gamma^T C_\gamma \gamma - \frac{1}{2} \gamma^T C_\gamma \gamma) dV - W
\end{align*}
\]  

(3.12)

where \(C_e\) and \(C_\gamma\) are appropriate elastic constant matrices.

The element stiffness matrix can be derived following the usual procedure. Again, by observing that the distribution of the strains \(e_x', e_y', e_{xy}\) in Eq. (3.5) can be effectively taken care-of by the 2x2x2 point reduced integration, we derive an element based on a modified Hellinger-Reissner principle written in matrix form as follows:
\[ \pi_{\text{mR}} = \frac{1}{2} \int_{V} f e'^T \gamma_e e' \, dv + \int_{V} \gamma^T \gamma_{\gamma T} \gamma_{\gamma} \, dv - \frac{1}{2} \gamma^T \gamma \left( \gamma_{\gamma \gamma} \right) \, dv - W \]  

(3.13)

Here we note that only transverse shear strains \( \gamma \) appear as independent variables in addition to displacements. Transverse shear strains are assumed to be linear in \( \xi, \eta \) and 2x2x2 points are used for the integration of the stiffness matrix. With this formulation only an inversion of a 6x6 matrix is required. This element will be called RS40.

**Numerical Examples**

To check the efficiency of the present element, the following example problems were calculated.

(a) **A Pinched Ring**

This is the same problem solved in section 2.3. Eight RS40 elements were used for the solution. The maximum nondimensional deflection and nondimensional inplane stresses are given in Table 6 and Fig. 1 respectively. Similar to RA6 and RA8 arch elements, the present element is insensitive to the radius to thickness ratios considered here.

(b) **A Pinched Cylinder with Free Ends**

The geometry, material constants and mesh pattern are shown in Fig. 3. Due to symmetry only one-eighth of the shell is calculated by a 2x2 mesh. Computed normal deflections at the load point are listed in Table 8 for two different radius to thickness ratios. Since the problem is nearly one-dimensional the differences between results by the present element (RS40) and by the reduced integration element (S40) are
small. They are less than 0.75% using results either by RS40 or S40 as
the reference values. These results are compared with those solved by
Dawe [41] with 54 DOF triangular elements.

(c) A Pinched Cylinder with Diaphragmed Ends

The cylinder has the same shape as in Fig. 3, but both ends
are diaphragmed instead of being free. One-eighth of the shell is
discretized by 5x3 and 6x4 meshes. More meshes are used in the circum-
ferential direction. Results in Table 9 indicates that the present
element is more efficient than the S40 element.

3.3 Incremental Variational Formulation for Large Deflection Analysis

In this section, we develop the Hellinger-Reissner principles for
elastic large displacement analysis and the corresponding incremental
solution procedure. Adopting the stationary Lagrangian formulation, we
start with the principle of virtual work written in Cartesian coordinates.

\[ \int_V S_{ij} \delta \bar{E}_{ij} dV = \delta W \]  \hspace{1cm} (3.14)

where

\[ S_{ij} = \text{2nd Piola-Kirchhoff Stress Tensor} \]

\[ \bar{E}_{ij} = \frac{1}{2} (U_{i,j} + U_{j,i} + U_{k,i} U_{k,j}) \]

\[ = \text{Lagrangian Strain Tensor} \]

We may introduce independent strains \( E_{ij} \) as new variables via Lagrangian
multipliers \( \lambda_{ij} \) such that

\[ \int_V S_{ij} \delta E_{ij} dV + \delta \int_V (\bar{E}_{ij} - E_{ij}) dV = \delta W \]  \hspace{1cm} (3.15)
which is an extended virtual work statement called the Hu-Washizu principle* [74,75]. Here, we immediately see that one of the Euler equations is

\[ \lambda_{ij} = S_{ij} \]

Satisfying Eq. (3.16) exactly,

\[ \int V S_{ij} \delta \tilde{E}_{ij} dV + \int V (\tilde{E}_{ij} - E_{ij}) \delta S_{ij} dV = \delta W \]  \( (3.17) \)

Let

\[ S_{ij} = \ast S_{ij} + \Delta S_{ij} \]
\[ \tilde{E}_{ij} = \ast \tilde{E}_{ij} + \Delta \tilde{E}_{ij} \]
\[ E_{ij} = \ast E_{ij} + \Delta E_{ij} \]  \( (3.13) \)

where \( \ast S_{ij}, \ast \tilde{E}_{ij} \) and \( \ast E_{ij} \) are known quantities and \( \Delta S_{ij}, \Delta \tilde{E}_{ij}, \Delta E_{ij} \) are unknown incremental quantities which are to be determined. Introducing these quantities into Eq. (3.17).

\[ \int V (\ast S_{ij} + \Delta S_{ij}) \delta \Delta \tilde{E}_{ij} dV + \int V (\ast \tilde{E}_{ij} + \Delta \tilde{E}_{ij}) \delta \Delta S_{ij} dV \]
\[ - \int V (\ast E_{ij} + \Delta E_{ij}) \delta \Delta S_{ij} dV = \delta W \]  \( (3.19) \)

\[ \int V (\Delta S_{ij} \delta \Delta \tilde{E}_{ij} + \Delta \tilde{E}_{ij} \delta \Delta S_{ij} - \Delta E_{ij} \delta \Delta S_{ij}) dV \]
\[ + \int V (\ast S_{ij} \delta \Delta \tilde{E}_{ij} dV + \int V (\ast \tilde{E}_{ij} - \ast E_{ij}) \delta \Delta S_{ij} dV = \delta W \]  \( (3.20) \)

*Fraijs de Veubeke [42] also mentioned the possibility of such extended variational statement for linear elasticity problems.
Substituting the incremental stress-strain relation

$$\Delta S_{ij} = C_{ijkl} \Delta E_{kl}$$  \hspace{1cm} (3.21)

$$\delta V \int (C_{ijkl} \Delta E_{kl} \Delta \bar{E}_{ij} - \frac{1}{2} C_{ijkl} \Delta E_{ij} \Delta E_{kl}) dV$$

$$+ V \int S_{ij} \delta \Delta \bar{E}_{ij} dV + V \int C_{ijkl} (\bar{E}_{ij} - \bar{E}_{ij}) \delta \Delta E_{ij} dV = \delta W$$  \hspace{1cm} (3.22)

The strain increment $\Delta \bar{E}_{ij}$ expressed in terms of displacement $U_i$ can be decomposed into the linear and the nonlinear part as follows

$$\Delta \bar{E}_{ij} = \Delta \bar{e}_{ij} + \Delta \bar{n}_{ij}$$  \hspace{1cm} (3.23)

with

$$\Delta \bar{e}_{ij} = \frac{1}{2}(\Delta U_{i,j} + \Delta U_{j,i} + \bar{U}_{k,i} \Delta U_{k,j} + \bar{U}_{k,j} \Delta U_{k,i})$$

$$\Delta \bar{n}_{ij} = \frac{1}{2}(\Delta U_{k,i} \Delta U_{k,j})$$

Substituting Eq. (3.23) into Eq. (3.22),

$$\delta V \int (C_{ijkl} \Delta E_{ij} \Delta \bar{e}_{ij} + C_{ijkl} \Delta E_{ij} \Delta \bar{n}_{ij} - \frac{1}{2} C_{ijkl} \Delta E_{ij} \Delta E_{kl}) dV$$

$$+ V \int S_{ij} \delta \Delta \bar{n}_{ij} dV + V \int C_{ijkl} (\bar{E}_{ij} - \bar{E}_{ij}) \delta \Delta E_{ij} dV$$

$$+ V \int S_{ij} \delta \Delta \bar{e}_{ij} dV - \delta W = 0$$  \hspace{1cm} (3.24)
The second term in the first integral is much smaller than the first term if increments are assumed small and therefore can be neglected. The third integral represents the compatibility check. The equilibrium check is included in the fourth integral. Since $\Delta S_{ij}$ and $\Delta E_{ij}$ are related by Eq. (3.21), the incremental variational statement in Eq. (3.24) is equivalent to that derived by Boland and Pian [43] with stresses and displacements as variables. However, in the latter the stresses satisfy linearized equilibrium equations.

The incremental finite element procedure is formulated as in the previous cases by assuming independent strain increments $\Delta \varepsilon$ in terms of unknown parameters $\Delta \beta$ and displacement increments $\Delta \bar{u}$ in terms of nodal displacement increments $\Delta \bar{q}$ such that

$$\Delta \varepsilon = p \Delta \beta$$

$$\Delta \bar{u} = \Delta \bar{q} + \Delta \bar{e} = B \Delta \bar{q}$$

Substituting these into Eq. (3.24), we obtain by assuming dead loading,

$$\delta \left\{ \Delta \beta^T \Delta \bar{q} - \frac{1}{2} \Delta \beta^T \Delta \beta + \frac{1}{2} \Delta \bar{q}^T K \Delta \bar{q} + \Delta \beta^T \Delta \bar{I} \right\} + \Delta \bar{q}^T \Delta \bar{I} - \Delta \bar{q}^T \Delta \bar{Q} \right\} = 0$$

Relevant matrices are defined as follows:

$$G = V_n \int P^T C B dV$$

$$H = V_n \int P^T C P dV$$
\[
\frac{1}{2} \Delta q_{ij} \Delta q_{ij} = \int_{V_n} S_{ij} \delta \Delta n_{ij} \, dV
\]

\[
R_{IJ} = \int_{V_n} P^T \Sigma (\overline{E} - \bar{E}) \, dV
\]

\[
R_{II} = \int_{V_n} \bar{S} \, dV - Q
\]

and \[
\Delta q^T (\overline{Q} + \Delta Q) = \delta \bar{w}
\]

Taking \(\delta \pi_R = 0\) with respect to \(\Delta \beta\) gives

\[
\Delta q_{\sim} - \Delta \beta_{\sim} + R_{\sim I} = 0
\]

or

\[
\Delta \beta_{\sim} = H^{-1}(\Delta q_{\sim} + R_{\sim I})
\]

Taking \(\delta \pi_R = 0\) with respect to \(\Delta q_{\sim}\),

\[
G^T \Delta \beta_{\sim} + \phi K \Delta q_{\sim} + R_{\sim II} - \Delta q_{\sim} = 0
\]  \hspace{1cm} (3.28)

Substituting Eq. (3.28) into Eq. (3.29) to eliminate \(\Delta \beta_{\sim}\),

\[
(G^{\sim T} H^{-1} G + \phi K) \Delta q = \Delta q_{\sim} - R_{\sim II} - G^{\sim T} H^{-1} R_{\sim I}
\]  \hspace{1cm} (3.30)

In the above expression, \(R_{II}\) and \(G^{\sim T} H^{-1} R_{\sim I}\) represent nodal loads due to the equilibrium check and the compatibility check respectively. Equation (3.30) can be solved for \(\Delta q_{\sim}\) if the current state is known.
Equation (3.30) is valid for large displacement (large strain and large rotation). But for thin shells, strains are assumed to be small. Including transverse shear strains, $\Delta \vec{e}$ and $\Delta \vec{\varepsilon}$ have five independent components defined in local Cartesian coordinates with z-axis normal to the shell midsurface, i.e.,

$$\Delta \vec{e}^T = \begin{bmatrix} \Delta e_x & \Delta e_y & \Delta e_{xy} & \Delta \varepsilon_{xz} & \Delta \varepsilon_{yz} \end{bmatrix}^T$$

$$\Delta \vec{\varepsilon}^T = \begin{bmatrix} \Delta \varepsilon_x & \Delta \varepsilon_y & \Delta \varepsilon_{xy} & \Delta \varepsilon_{xz} & \Delta \varepsilon_{yz} \end{bmatrix}^T$$

In linear shell analysis, an eight-node element was derived by a modified Hellinger-Reissner principle where only transverse shear strains appear as additional independent variables. The corresponding Hellinger-Reissner principle for large deflection problem is formulated by setting

$$\Delta \varepsilon' = \begin{bmatrix} \Delta e_x \\ \Delta e_y \\ \Delta e_{xy} \end{bmatrix} = \begin{bmatrix} \Delta \varepsilon_x \\ \Delta \varepsilon_y \\ \Delta \varepsilon_{xy} \end{bmatrix} = \Delta \vec{\varepsilon}'$$

in Eq. (3.22). Then written in matrix form

$$\delta \left\{ \int_V \left( \frac{1}{2} \Delta \vec{e}'^T \varepsilon_e \Delta \vec{e}' + \Delta \varepsilon'^T \varepsilon_t \Delta \varepsilon' - \frac{1}{2} \Delta \varepsilon'^T \varepsilon_t \Delta \varepsilon' \right) dV \right\}$$

$$+ \int_V \Delta n^T \sigma dV + \int_V \Delta \varepsilon'^T \varepsilon_t \left( \begin{bmatrix} T^T \\ \tau \end{bmatrix} - \begin{bmatrix} T \\ \tau \end{bmatrix} \right) dV + \int_V \Delta \varepsilon^T \sigma dV \right\} - \delta W = 0$$

where $\varepsilon_e$ and $\varepsilon_t$ are appropriate elastic constant matrices.
3.4 Large Deflection of Arches

With the incremental variational formulation given above, it is necessary now to define the kinematics of deformation. For large deflection analyses of thin structures, it is important, in general, for the displacement field to be able to describe large rotations. A proper displacement field for "degenerate" shell elements has been provided by Ramm [44]. The displacement increments were expressed in terms of nodal degrees of freedom $\Delta u$, $\Delta v$, $\Delta w$, $\Delta \phi_1$, and $\Delta \phi_2$. In the following we restrict ourselves to arch problem for simplicity. First, we define the global Cartesian coordinates $X$ and $Z$ as shown in Fig. 4. The unit vector $\hat{n}$ is normal to the arch center line. The angle between $\hat{n}$ and $X$ is denoted as $\theta$. When the arch undergoes large deflection, the vector $\hat{n}$ rotates by the angle $\phi$ to the unit vector $\hat{N}$. Thus, the vector $\hat{N}$ represents the direction of $\hat{n}$ after deformation. Then the displacement vector $\hat{U}$ can be expressed as

$$\hat{U} = \hat{u} + \frac{t}{2} \zeta (\hat{N} - \hat{n}) = U\hat{t} + W\hat{k}$$  \hspace{1cm} (3.34)

where

$\hat{u}$ = displacement of the center line of the arch

t = thickness

$\zeta$ = coordinates normal to the arch ranging from -1 to 1.

Written in components,

$$U = u + \frac{t}{2} \zeta (\cos \phi \cos \theta - \sin \phi \sin \theta - \cos \theta)$$  \hspace{1cm} (3.35)

$$W = w + \frac{t}{2} \zeta (\sin \phi \cos \theta + \cos \phi \sin \theta - \sin \theta)$$
Let

\[ \phi = \phi^* + \Delta \phi \]  \hspace{1cm} (3.36)

\[ u = u^* + \Delta u \]

\[ w = w^* + \Delta w \]

Then

\[ \Delta U = \Delta u - \frac{t}{2} \xi \sin(\phi + \theta) \Delta \phi \]  \hspace{1cm} (3.37)

\[ \Delta W = \Delta w + \frac{t}{2} \xi \cos(\phi + \theta) \Delta \phi \]

The displacements \( \Delta U \) and \( \Delta W \) are assumed in terms of nodal incremental displacements \( \Delta u, \Delta w \) and nodal rotational increments \( \Delta \phi \) such that

\[ \Delta U = \sum_{i=1}^{N} N_i(\xi) \Delta u_i - \frac{1}{2} \xi \sum_{i=1}^{N} N_i(\xi) \sin(\phi + \theta) \Delta \phi_i \]  \hspace{1cm} (3.38)

\[ \Delta W = \sum_{i=1}^{N} N_i(\xi) \Delta w_i + \frac{1}{2} \xi \sum_{i=1}^{N} N_i(\xi) \cos(\phi + \theta) \Delta \phi_i \]

where \(-1 \leq \xi \leq 1\)

Now we consider three-node element with \( N_i(\xi) \) quadratic in \( \xi \) and the following assumed independent strains:

\[ \Delta \varepsilon_{xx} = \beta_1 + \beta_2 \xi + \xi (\beta_3 + \beta_4 \xi) \]  \hspace{1cm} (3.39)

\[ \Delta \varepsilon_{zz} = \beta_5 + \beta_6 \xi \]

Then using the Hellinger-Reissner principle, we arrive at the incremental formulation given in Eq. (3.30). Alternatively, we may use the modified Hellinger-Reissner principle and the following assumed independent transverse shear strain.

\[ \Delta \Gamma_{xz} = \beta_1 + \beta_2 \xi \]  \hspace{1cm} (3.40)
Here a 2-point integration is to be used in $\xi$ direction. Also, an equivalent formulation is obtained by using the conventional assumed displacement formulation and a 2-point integration in $\xi$. As an illustration, an incremental formulation has been implemented and applied to solve large deflection problem of a clamped shallow arch under a point load. The geometry and material constants are given in Fig. 5. One-half of the arch was discretized into five elements. Solutions were obtained by two methods, i.e., by

(1) incrementing the load ($\Delta p = 0.8$ lbs)
and (2) incrementing the displacement of the point where the load is applied ($\Delta W_{\text{max}} = -0.003"$).

The load-displacement curves are given in Fig. 5. Both solutions are close to each other except near the limit load and agree with those by Mallett et al. [46] and by Boland [36]. The limit load $P_{\text{cr}} = 34.18$ lbs was determined by the second method. The equilibrium check and the compatibility check terms were included in the calculation. No iterations were performed within a given load or displacement increment.
SECTION 4
CREEP ANALYSIS BY ASSUMED STRESS
HYBRID FINITE ELEMENTS

4.1 Introduction

Creep and viscoplastic problems are nonlinear in nature and in general require numerical solution methods. The finite element method based on the assumed displacement models has been applied to this type of problem treating creep or viscoplastic strains as initial strains. Percy et al. [47], Greenbaum and Rubinstein [48], Sutherland [49], Chang and Rashid [50], Branca and Boresi [51], Donea and Giuliani [52], etc. solved small displacement creep problems while Cyr and Teter [53] solved large deformation creep problems combined with plastic effect. Numerical stability is very important for all numerical analyses and sufficiently small time steps were used in these works. Cormeau [54] developed a formulation for viscoplastic analysis of solids utilizing the constitutive law proposed by Perzyna [55] and applied it to elastoplastic analysis and to analysis of transient creep under the time hardening law. A theoretical stability criterion has been established in his formulation. Bodner et al. [56,57] proposed an alternative viscoplastic constitutive law and applied it to the finite element method.

An assumed stress hybrid formulation for steady state creep and for transient creep under time hardening law has been given by Pian [58]. In the present work the assumed stress hybrid finite element method for small deformation creep analysis is extended and implemented. An assumed stress
hybrid finite element can be formulated by the modified complementary energy principle or by mixed formulation based on the Hellinger-Reissner principle. For the example problems carried out in this investigation, the Hellinger-Reissner principle is suitable since it is easy to construct compatible displacement fields. The resulting system of nonlinear first order ordinary differential equations can be solved by various Runge-Kutta methods. Finally, by employing Perzyna's constitutive law for viscoplasticity, the time-wise solution procedure can be used for the solution of the elastoplastic problem.

4.2 Constitutive Equations

(a) Creep Law

For the present purpose, phenomenological description of creep will be adopted. In the study of long duration creep the steady state creep law is used and the corresponding uniaxial creep strain rate is expressed by

\[ \dot{\varepsilon}^c = F(\sigma, T) \]  

(4.1)

where \( \sigma \) is the uniaxial stress and \( T \) is the temperature.

If the effects of \( \sigma \) and \( T \) are assumed to be separable, then

\[ \dot{\varepsilon}^c = f(\sigma)g(T) \]  

(4.2)

For the stress-dependent function, we use Norton's power law, i.e.,

\[ f(\sigma) = B\sigma^n \]  

(4.3)

where \( B \) and \( n \) are material constants. For the present study it will be assumed from now on that the temperature remains constant. In the case
of transient creep, the time hardening law and the strain hardening law are available. Under constant temperature and constant stress the uniaxial creep strain may be expressed as

\[ \epsilon^c = A_0^m \frac{k}{t} \]

(4.4)

where \( A, m, k \) are material constants and \( t \) represents time. Differentiating Eq. (4.4) with respect to time leads to

\[ \dot{\epsilon}^c = A_0^m \frac{k-1}{t} \]

(4.5)

The variable \( t \) can be eliminated from Eq. (4.5) by solving Eq. (4.4) for \( t \) and substituting into Eq. (4.5). The result is

\[ \dot{\epsilon}^c = A_0^m \frac{k^l}{\epsilon^c} \left( \frac{1}{k} \right)^{(1-l/k)} \]

(4.6)

Equation (4.5) and (4.6) are called the time hardening law and the strain hardening law respectively. The time hardening law can be written in a form which resembles the steady state creep law by defining a parameter \( \tau = t^k \) and differentiating Eq. (4.4) with respect to \( \tau \) to yield

\[ \frac{d\epsilon^c}{d\tau} = A_0^m \]

(4.7)

Under multiaxial stress condition, creep strain rates are expressed as

\[ \dot{\epsilon}^c_{ij} = \dot{\epsilon}^c \frac{\partial F}{\partial \sigma_{ij}} \]

(4.8)

where \( \dot{\epsilon}^c \) is the equivalent creep strain rate and the yield condition is represented by

\[ F(\sigma_{ij}) = 0 \]

(4.9)

By substituting the uniaxial creep strain rates of Eqs. (4.6) and (4.7)
into Eq. (4.8) for the equivalent creep strain rate and using the equivalent stress $\sigma_e$ in place of the uniaxial stress, we obtain

$$\dot{e}_{ij}^c = A_0^{m} \frac{\partial F}{\partial \sigma_{ij}}$$ \hspace{1cm} (4.10)

for the steady state creep or creep under the time hardening law and

$$\dot{e}_{ij}^c = k \sigma^m \frac{\partial F}{\partial \sigma_{ij}} \left( \frac{e^c}{e} \right)^{1/(1-k)}$$ \hspace{1cm} (4.11)

for creep under the strain hardening law. In Eq. (4.10) the creep strain rate is determined by the present stress values while in Eq. (4.11) it is expressed in terms of the present stresses and the equivalent creep strain or creep strains. If the Mises-Hencky yield criterion is used, then

$$\frac{\partial F}{\partial \sigma_{ij}} = (3/2)(S_{ij}/\sigma_e)$$ \hspace{1cm} (4.12)

where $S_{ij}$ are the deviatoric stresses.

(b) Perzyna's Viscoplastic Law

For materials undergoing viscoplastic process, multiaxial viscoplastic strain rates are expressed as follows

$$\dot{\varepsilon}_{ij}^P = \gamma \phi \frac{\partial F}{\partial \sigma_{ij}}$$ \hspace{1cm} (4.13)

where $F$ is the same yield function as in Eq. (4.9) with uniaxial yield stress $\sigma_y$ and

$$\phi \left\{ \begin{array}{ll}
\sigma - \sigma_y & \sigma_e > \sigma_y \\
0 & \sigma_e < \sigma_y
\end{array} \right.$$ \hspace{1cm} (4.14)
and $\gamma$ is a material constant. If the yield stress is constant, the
viscoplastic strain rates are determined by stresses only. The above
representation has been proposed by Perzyna [55]. It is different from
the classical inviscid plasticity theory in that it permits the stresses
to exceed the yield stress. The viscoplastic formulation can be used to
solve classical plasticity problems. This is achieved by finding a
steady state solution under a given load increment.

(c) Bodner-Partom's Law

In Bodner-Partom's constitutive law inelastic strain rates are
expressed as follows

$$\varepsilon_{i j}^P = f(J_2) \frac{S_{i j}}{J_2} \quad (4.15)$$

where $J_2$ is the second invariant of the deviatoric stresses and the
function $f$ include material constants and plastic work done. No yield
stress is defined and thus inelastic strain components are present all
the time from the moment a load is applied. In the present investigation
we will not use this constitutive law. But it can be treated in the
same manner as creep laws or Perzyna's law.

4.3 Finite Element Formulation

For creep and viscoplastic analyses, the Hellinger-Reissner
principle is extended to include the initial strains. The functional
$\Pi_R$ is written in terms of stress and displacement rates as follows.
\[ \tau_R = \frac{1}{n} \sum_{n} \int_{V_n} \left[ \tilde{\sigma}_{ij} \frac{1}{2} (\dot{\delta}_{ij} + \dot{\delta}_{ji}) - \frac{1}{2} S_{ijkl} \dot{\delta}_{ij} \dot{\delta}_{kl} \right. \\
\left. - \delta_{ij} \ddot{\varepsilon}_{ij} - \frac{\dot{F}_i}{T_i} \dot{u}_i \right] dV - \sum_{n} \int_{S_{\sigma_n}} \frac{\dot{T}_i}{T_i} \dot{u}_i \, ds \]  

(4.16)

where

\[ \dot{\sigma}_{ij} = \text{stress rate tensor} \]
\[ \dot{u}_i = \text{displacement rates} \]
\[ S_{ijkl} = \text{elastic coefficient tensor} \]
\[ \dot{\varepsilon}_{ij} = \text{initial strain rates, i.e., creep strain or viscoplastic strain rates in the present case} \]
\[ \dot{F}_i = \text{applied body force rates per unit volume} \]
\[ \dot{T}_i = \text{applied traction rates} \]
\[ V_n = \text{volume of the element} \]
\[ S_{\sigma_n} = \text{surface of the nth element over which tractions are applied} \]

In the matrix form this functional can be written as

\[ \pi_R = \sum_{n} \int_{V_n} \left[ \tilde{\sigma}^T (\dot{D}u) - \frac{1}{2} S \dot{\sigma}^{\varepsilon} - \dot{\sigma}^{T\varepsilon} \right. \\
\left. - \frac{\dot{F}^T}{T} \dot{u} \right] dV - \sum_{n} \int_{S_{\sigma_n}} \frac{\dot{T}^T}{T} \dot{u} \, ds \]  

(4.17)
The above expression is valid for small deformation. The corresponding variational principle for large deformation has been given by Sanders et al. [59] and Pian [60]. In Eq. (4.16) the displacements and the stresses are independent quantities. For the present finite element implementation, stress and displacement rates are assumed in terms of unknown stress rate parameters \( \dot{\beta} \) and nodal displacement rates \( \dot{q} \) respectively, i.e.,

\[
\dot{\varepsilon} = \frac{\partial \hat{\sigma}}{\partial \hat{\varepsilon}}, \quad \dot{u} = Aq
\]

then

\[
(\vec{D}u) = \ddot{\beta}q
\]

Here the assumed stress rates are made to satisfy the equilibrium condition. Thus \( \ddot{\beta} \) is the homogeneous solution of the equilibrium equations and \( \ddot{\beta}_F \) is a particular solution of the equilibrium equations. Substituting Eq. (4.18) into Eq. (4.17) and performing necessary integrations, we obtain

\[
\pi_R = \sum_n \beta_n \ddot{q}_n + \beta_n \ddot{G}_n - \frac{1}{2} \ddot{H_n} - \ddot{T}_i
\]

\[
\ddot{T}_i = \ddot{G}_n \ddot{F}_n - \ddot{G}_n \ddot{F}_n - \ddot{q}_n \dot{Q}
\]

where

\[
\ddot{G}_n = \int_{V_n} P^T B dV \quad \ddot{G}_n = \int_{V_n} P^T B dV
\]

\[
\ddot{H}_n = \int_{V_n} P^T e_i dV \quad \ddot{H}_n = \int_{V_n} P^T S_P dV
\]

\[
\ddot{H}_n = \int_{V_n} P^T S_P dV \quad \ddot{q}_n = \int_{V_n} \ddot{T}_i dS + \int_{V_n} \ddot{T}_i S dT
\]
Taking $\delta \pi_R = 0$ with respect to $\dot{\beta}$ gives

$$G\dot{q} - \dot{\beta} - H \dot{\beta} - \dot{i} = 0$$

or

$$\dot{\beta} = H^{-1} (G\dot{q} - H \dot{\beta} - \dot{i})$$

(4.21)

Substituting Eq. (4.21) into Eq. (4.19) yields

$$\pi_R = \sum \left[ \frac{1}{2} k_n \dot{q} - q^T \dot{\beta} (Q_n + Q^i_n) \right]$$

(4.22)

where

$$k_n = G^T H^{-1} G = \text{element stiffness matrix}$$

$$\dot{Q}_n = \dot{Q} + (G^T H^{-1} \dot{H} - G^T \dot{F}) \dot{\beta}$$

(4.23)

= rate of applied nodal load

$$\dot{Q}^i_n = G^T H^{-1} \dot{Q}^i = \text{rate of equivalent nodal load due to inelastic strain}$$

The element stiffness matrix and nodal load can be assembled to yield

$$\pi_R = \frac{1}{2} q^T K q - q^T \dot{\beta} (Q + Q^i)$$

(4.24)

and the stationary condition of $\pi_R$ with respect to $\dot{q}$ gives

$$K \dot{q} = \dot{Q} + \dot{Q}^i$$

(4.25)

In Eq. (4.18) the assumed stress $\dot{q}$ includes a particular solution $\hat{p} \dot{\beta}_F$ for the body force terms of the equilibrium equations. An alternative approach is to set $\hat{p} \dot{\beta}_F = 0$. Then the contribution $(G^T H^{-1} \dot{H} - G^T \dot{F}) \dot{\beta}_F$ to the nodal load will disappear from Eq. (4.23).

Solving Eq. (4.25) for $\dot{q}$ and substituting into Eq. (4.21), we obtain

$$\dot{\beta} = H^{-1} (G \dot{q} - G^T H^{-1} \dot{i} - \dot{Q}) \dot{q}^i + H^{-1} \dot{G} \dot{Q} - H^{-1} H \dot{F} \dot{\beta}$$

(4.26)
For problems involving steady state creep, creep under the time hardening law and viscoplastic law with constant yield stress.

\[ \dot{\tilde{\epsilon}} = \tilde{f}(\tilde{\sigma}) = \tilde{f}(\tilde{\beta}) \quad (4.27) \]

and thus from Eq. (4.26)

\[ \dot{\tilde{\beta}} = \tilde{g}(\tilde{\beta}) \quad (4.28) \]

On the other hand, for creep under the strain hardening law or for viscoplasticity law with yield stress dependent on the viscoplastic strains

\[ \dot{\tilde{\beta}} = \tilde{g}_1(\tilde{\beta}, \tilde{\epsilon}) \quad (4.29) \]
\[ \dot{\tilde{\epsilon}} = \tilde{g}_2(\tilde{\beta}, \tilde{\epsilon}) \]

Equations (4.28) and (4.29) are systems of nonlinear first order ordinary differential equations with \( \tilde{\beta} \) and \( \tilde{\beta}, \tilde{\epsilon} \) as unknowns respectively. In the finite element formulation, integrations over an element are performed numerically. Thus, in Eq. (4.29) the vector \( \tilde{\epsilon} \) represents inelastic strains at Gaussian integration points.

In the conventional assumed displacement method based on the principle of virtual work with initial strains, the unknowns are stresses \( \tilde{\sigma} \) at Gaussian integration points instead of \( \tilde{\beta} \) and we have the same type of equations as Eqs. (4.28) and (4.29) with \( \tilde{\beta} \) replaced by \( \tilde{\sigma} \). The amount of computing time depends on the number of unknowns. The assumed stress hybrid formulation will result in a problem with fewer number of unknowns than that by the corresponding assumed displacement method. For example,
consider a four-node plane stress element derived by 2x2 integration rule. An element based on the displacement approach will have three stress components at each Gaussian point and thus has twelve unknown stresses per element. On the other hand, a hybrid stress element with 7 β's has seven unknowns. Similarly, for the assumed stress hybrid model, a four-node axisymmetric element with 8 β's and an eight-node three-dimensional element with 24 β's have fewer number of unknowns than the corresponding elements by the assumed displacement model.

4.4 Solution Method

Equations (4.28) and (4.29) can be solved timewise by numerical methods such as Runge-Kutta methods using elastic solution as initial conditions. All these methods have numerical stability limit. For the integration of Eq. (4.28) the stability criterion developed by Cormeau was utilized. The well known fourth order explicit Runge-Kutta method was used for steady state creep and creep under the time hardening law. In the case of the time hardening law, the integration is performed for the time parameter $\tau = t^k$. The real time is recovered by $t = \tau^{1/k}$. On the other hand, the midpoint Runge-Kutta method is considered more appropriate for elastoplastic problems where a lower order method was chosen due to the possibility of piecewise continuity of stresses.
To solve elastoplastic problems, it is assumed at first that the present state is on the yield surface or in an elastic state. Given a load increment, the material will flow according to Eq. (4.13) if \( \sigma_e > \sigma_y \). But if the structure is stable, the plastic flow will stop after a certain time and a steady state is reached under the given load. This steady state will be the one which is obtainable by the classical plasticity formulation. In actual calculations it is assumed that a steady state has been reached if

\[
\frac{\sigma_e - \sigma_y}{\sigma_y} \leq \varepsilon \tag{4.30}
\]

where \( \varepsilon \) is an arbitrarily assigned value. For the present numerical solution it is chosen as 0.01.

For creep problems under the strain hardening law, the time increment for the numerical integration of Eq. (4.29) has been determined such that

\[
e^C_e \Delta t < \text{total strain at time } t \times a \tag{4.31}
\]

where \( a \) is a preassigned value. The value of \( a = 0.05 \) was used in the present work. It is to be noted that for materials with \( k < 1 \), Eq. (4.11) and hence Eq. (4.29) are singular at time \( t = 0 \) since \( e^C_e = 0 \). Therefore, it is necessary to use a method which does not require evaluation of the creep strain rate at \( t = 0 \). A fourth order implicit Runge-Kutta method developed by Butcher [61] was chosen for this purpose. After one time increment, it is switched to the fourth order explicit method since the explicit method requires less computing time for the same accuracy.
4.5 Numerical Examples

Two assumed stress hybrid elements are used for numerical solution of the example problems. For a four-node quadrilateral plane stress element in Cartesian coordinates x and y the following stress assumptions are made [76]:

\[
\begin{align*}
\sigma_x &= \beta_1 + \beta_4 x + \beta_5 y \\
\sigma_y &= \beta_2 + \beta_6 y + \beta_7 x \\
\sigma_{xy} &= \beta_3 - \beta_4 y - \beta_6 x
\end{align*}
\] (4.32)

For a four-node quadrilateral axisymmetric element with r and z as the axes in the radial and axial directions respectively, the following assumed stresses are used [62]:

\[
\begin{align*}
\sigma_r &= \beta_1 + \beta_2/r + \beta_3 z/r \\
\sigma_\theta &= \beta_4 \\
\sigma_z &= \beta_5 + \beta_6/r - \beta_7 z/r \\
\sigma_{rz} &= -\beta_1 z/r + \beta_4 z/r + \beta_7 + \beta_8/r
\end{align*}
\] (4.33)

The isoparametric representation is used for the displacement assumption. Among the following examples, the first two problems deal with the creep problem and the last one, with an elastic-plastic analysis.

(a) Creep of a Thick Cylinder under Internal Pressure

An infinitely long thick cylinder subjected to internal pressure was solved as a plane strain problem using six axisymmetric elements. 

The geometry, load and material properties are given in Fig. 6. Results are given in Figs. 6, 7, and 8. The time hardening law and the strain hardening law gave almost identical results. All results are close to those obtained by Greenbaum and Rubinstein [48].
(b) **Creep of a Rotating Disk**

This is a problem involving distributed body forces. The geometry and material constants are given in Fig. 9. First, it was analyzed by a row of six axisymmetric elements and then by a row of twelve plane stress elements. In the plane stress element a particular solution of the equilibrium equations was chosen for the averages of the body forces over the volume of the element. For the axisymmetric element the particular solutions were simply set to zero. Results for the time hardening law are shown in Figs. 9 and 10. In Fig. 9, \( h \) and \( m \) represent the thickness and the density of the disk respectively. The plane stress solutions agree with the axisymmetric solutions.

(c) **An Infinitely Long Thick Cylinder of Elastic-Perfectly Plastic Material under Internal Pressure \( p \)**

The geometry and material properties are as follows:

- Inner radius \( a \) = 5 inch
- Outer radius \( b \) = 10 inch
- Young's modulus \( E \) = \( 10^7 \) psi
- Poisson's ratio \( \nu \) = 0.3
- Yield stress \( \sigma_y \) = 20000 psi

Table 10 shows the radial displacement at the inner wall calculated by both the assumed stress hybrid method and the assumed displacement method using eight axisymmetric elements. The 2x2 Gaussian integration points were used for numerical integration. The exact solutions were obtained according to the procedure described in Ref. 63 based on the classical
plasticity theory. The accuracy of the finite element solution decreases as the load approaches collapse value and more fictitious time steps are needed to reach a steady state under the given load increment. In the present case, 6 to 13 iterations were required to satisfy the criteria in Eq. (4.30) for each load increment. But at the collapse load, the steady state was not reached even after 50 iterations and the iteration was terminated. It is seen that the assumed stress hybrid method gives better solution in less computing time than the assumed displacement method.
SECTION 5
INCREMENTAL FORMULATIONS INVOLVING CRACK GROWTH

5.1 Introduction

It has been observed that macroscopic defects such as a crack can exhibit a slow growth under high temperature creep condition. Attempts have been made to correlate the crack growth rate with certain parameters such as the stress intensity factor [64], a path independent energy rate line integral [65], net section stress [66] and the crack opening displacement [67]. But much work has yet to be done. In this section we discuss incremental finite element solution procedures for a solid containing a propagating crack. The crack propagation rate is assumed to have been given. Finite element solutions involving a crack propagation can be found in the works by Andersson [68], Newman and Armen [69]. In such works elastic-plastic analyses of propagating cracks are reported. Andersson uses an incremental approach in conjunction with the tangent stiffness method. On the other hand, Newman and Armen employ a formulation in terms of total displacements. In the following several alternative variational principles and the corresponding finite element formulations are to be described under the assumption of small deformation. A center-cracked panel is chosen as an example problem to check the formulations. The present investigation is exploratory in nature. Also, it should be pointed out that although the present study was motivated by the crack propagation under creep, the formulation developed here can be applied to other areas such as contact problems.

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5.2 Formulations for a Changing Boundary

Let us consider a solid body containing a crack. We define two states. At the current state stresses \( \sigma_{ij} \) and displacements \( u_i \) are known and the increments to the next state \( \Delta \sigma_{ij} \) and \( \Delta u_i \) are to be determined. Since a new boundary appears as the crack propagates it is essential to distinguish the current state and the subsequent state. The tractions which are present at the current state disappear as the crack surface opens up in the subsequent state. In the finite element context this manifest as the disappearance of the current load at the node which is to be released. To take account of this effect, the functional for the Hellinger-Reissner principle with initial strain \( e_{ij}^i \) is written for the subsequent state including the initial (current) stresses \( \sigma_{ij} \) [70]. For small deformation problems with a stationary boundary, the inclusion of \( \sigma_{ij} \) does not affect the final formulation as will be shown later.

\[
\pi_R = \int_V (\sigma_{ij} + \Delta \sigma_{ij})(e_{ij}^i + \Delta e_{ij}^i) dV \\
- \int_V \frac{1}{2} S_{ijkl}(\sigma_{ij} + \Delta \sigma_{ij})(\sigma_{kl} + \Delta \sigma_{kl}) dV \\
- \int_V (\sigma_{ij} + \Delta \sigma_{ij})(e_{ij}^i + \Delta e_{ij}^i) dV \\
- \int_S (\sigma_{ij} + \Delta \sigma_{ij})(u_i + \Delta u_i) dS
\]

or

\[
\pi_R = \int_V (\Delta \sigma_{ij} \Delta e_{ij}^i - \frac{1}{2} S_{ijkl} \Delta \sigma_{ij} \Delta \sigma_{kl} - \Delta \sigma_{ij} \Delta e_{ij}^i) dV \\
- \int_S \sigma_{ij} \Delta u_i dS \\
+ \int_V \sigma_{ij} \Delta e_{ij}^i dV - \int_S \sigma_{ij} \Delta u_i dS
\]

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\[ + f_v (\varepsilon_{ij}^e - S_{ijkl}^e \sigma_{kl}^e - \varepsilon_{ij}^i) \Delta \sigma_{ij} dV \]  \hspace{1cm} (5.2)

The body forces are not considered in the formulation. The assumed stress hybrid method is formulated by assuming stress increments \( \Delta \sigma \) and displacement increments \( \Delta \tilde{u} \) such that

\[ \Delta \sigma \sim = P \Delta \beta \sim \]
\[ \Delta \tilde{u} \sim = A \Delta q \sim \text{ and in turn, } \Delta \tilde{e} \sim = B \Delta q \sim \]  \hspace{1cm} (5.3)

Substituting Eq. (5.3) into Eq. (5.2), we obtain

\[ \pi_R = \sum \left( \Delta \beta \sim \Delta q \sim - \frac{1}{2} \Delta \beta \sim \Delta \tilde{e} \sim - \Delta \beta \sim \Delta q \sim \Delta \tilde{e} \sim \Delta q \sim \right) \]
\[ + \Delta q \sim R_{II} \sim + \Delta \beta \sim R_{I} \sim \]  \hspace{1cm} (5.4)

where

\[ G = \int_{V_n} p^T B dV \sim \]
\[ H = \int_{V_n} p^T S P dV \sim \]
\[ G^i = \int_{V_n} p^T \Delta e^i dV \sim \]  \hspace{1cm} (5.5)
\[ \Delta \tilde{Q} = \int_{S_{\tilde{u}n}} T^T \Delta \tilde{d} S \sim \]
\[ R_{II} \sim = \int_{V_n} B^T \sigma dV \sim - \varepsilon Q \sim \]
\[ \varepsilon Q \sim = \int_{S_{\sigma n}} T^o \tilde{d} S \sim \]
\[ R_I \sim = \int_{V_n} p^T (\varepsilon_{ij}^e - S_{ijkl}^e \sigma_{kl}^e - \varepsilon_{ij}^i) dV \]
and \( \Delta \) takes the values of \( A \) on the element boundaries.

Taking \( \delta \pi_R = 0 \) with respect to \( \Delta \beta \) yields

\[
G\Delta q - H\Delta \beta - G^i + R_I = 0
\]

or

\[
\Delta \beta = H^{-1}(G\Delta q - G^i + R_I)
\]

Taking \( \delta \pi_R = 0 \) with respect to \( \Delta q \)

\[
G^T\Delta \beta - \Delta q + R_{II} = 0
\]

Substituting Eq. (5.7) into Eq. (5.8), we obtain

\[
K\Delta q = \Delta q + G^T H^{-1} G^i - R_{II} - G^T H^{-1} R_I = \Delta q
\]

where

\[
K = G^T H^{-1} G = \text{stiffness matrix}
\]

\[
G^T H^{-1} G^i = \text{equivalent nodal loads due to initial strain increments}
\]

\[
R_{II} = \text{equivalent nodal loads due to the equilibrium check}
\]

\[
G^T H^{-1} R_I = \text{equivalent nodal loads due to the compatibility check}
\]

Barring numerical error, we note that \( R_I = R_{II} = 0 \) for a stationary crack, since equilibrium and compatibility are satisfied exactly for
the current state. Then we obtain the same formulation given in Section 4. For a propagating crack, \( R_1 = 0 \) since compatibility is satisfied. Also, the entries in the \( R_{II} \) vector are zeroes except the one associated with the node which is to be released by crack propagation. In fact, this nonzero value represents the load which is present at the node prior to crack opening and will disappear as the crack advances. In the incremental formulation in Eq. (5.9) it appears as a counterbalancing load \(-R_{II}\). Inclusion of \( o_{ij} \) in Eq. (5.4) leads us to correct evaluation of this load.

Since new degrees of freedom enter Eq. (5.9) as the crack opens up, it is useful to adopt the substructure scheme to save computing time. For this purpose, we set

\[
\Delta q = \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}
\]  \hspace{1cm} (5.10)

and

\[
\Delta Q = \begin{bmatrix} \Delta Q_1 \\ \Delta Q_2 \end{bmatrix}
\]  \hspace{1cm} (5.11)

The vector \( \Delta q_2 \) represents the degrees of freedom near the crack tip including those associated with the newly opening crack surfaces. The vector \( \Delta q_1 \) represents the remaining degrees of freedom. The vectors \( \Delta Q_1 \) and \( \Delta Q_2 \) are nodal load vectors corresponding to \( \Delta q_1 \) and \( \Delta q_2 \) respectively. Rewriting Eq. (5.9) as

\[
\begin{bmatrix} k_{11} \\ k_{12} \\ k_{2}^T \\ k_{22} \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix} = \begin{bmatrix} \Delta Q_1 \\ \Delta Q_2 \end{bmatrix}
\]  \hspace{1cm} (5.12)
or

\[ k_{11} \Delta q_1 + k_{12} \Delta q_2 = \Delta Q_1 \]  
\[ k_{12}^T \Delta q_1 + k_{22} \Delta q_2 = \Delta Q_2 \]

From Eq. (5.13a)

\[ \Delta q_1 = k_{11}^{-1} (\Delta Q_1 - k_{12} \Delta q_2) \]  
\[ \text{(5.14)} \]

Substituting Eq. (5.14) into Eq. (5.13b),

\[ k^* \Delta q_2 = \Delta Q^* \]  
\[ \text{(5.15)} \]

where

\[ k^* = k_{22} - k_{12}^T k_{11}^{-1} k_{12} \]  
\[ \text{(5.16)} \]

\[ \Delta Q^* = \Delta Q_2 - k_{12}^T k_{11}^{-1} \Delta Q_1 \]

The matrix \( k^* \) has to be recalculated at each incremental step. But, the \( k_{11} \) matrix remains the same.

An alternative approach can be formulated by introducing an imaginary cut \( \Sigma \) to the surface in the next state which has yet to be opened (Fig. 11). The displacement on the two sides of the cut are denoted as \( u_i^{(1)} \) and \( u_i^{(2)} \) and it is required that

\[ u_i^{(1)} - u_i^{(2)} = 0 \]  
\[ \text{(5.17)} \]

Introducing the subsidiary condition in Eq. (5.17) into Eq. (5.2) via Lagrangian multipliers \( \lambda_i \).
\[ \pi_{\text{mR}} = \pi_R + \int_{\Sigma} \lambda_i (u_i^{(1)} - u_i^{(2)}) \, dS \]
\[ = \pi_R + \int_{\Sigma} \lambda_i (\circ u_i^{(1)} - \circ u_i^{(2)}) \, dS + \int_{\Sigma} \lambda_i (\Delta u_i^{(1)} - \Delta u_i^{(2)}) \, dS \]  

(5.18)

For finite element implementation, we assume \( \lambda_i \) in terms of unknown parameters \( \alpha_i \) such that

\[ \lambda_i \sim = P_{\sim \alpha_i} \alpha_i \]  

(5.19)

Then the additional terms in Eq. (5.18) can be written as

\[ \int_{\Sigma} \lambda_i (\Delta u_i^{(1)} - \Delta u_i^{(2)}) \, dS = \alpha_i^T (G_{\sim \alpha}^{(1)} \Delta q_i^{(1)} - G_{\sim \alpha}^{(2)} \Delta q_i^{(2)}) \]  

(5.20)

\[ \int_{\Sigma} \lambda_i (\circ u_i^{(1)} - \circ u_i^{(2)}) \, dS = \alpha_i^T (G_{\sim \alpha}^{(1)} \circ q_i^{(1)} - G_{\sim \alpha}^{(2)} \circ q_i^{(2)}) \]

where

\[ G_{\sim \alpha}^{(1)} = \int_{\Sigma} \alpha_i^T \alpha_i L_i^{(1)} \, dS \]
\[ G_{\sim \alpha}^{(2)} = \int_{\Sigma} \alpha_i^T \alpha_i L_i^{(2)} \, dS \]  

(5.21)

Expanding \( G_{\sim \alpha}^{(1)} \) and \( G_{\sim \alpha}^{(2)} \) to \( \alpha_{\sim \alpha} \) which includes terms corresponding to all \( \Delta q \) and \( \circ q \), we can write

\[ \int_{\Sigma} \lambda_i (u_i^{(1)} - u_i^{(2)}) \, dS = \alpha_i^T \alpha_{\sim \alpha} G_{\sim \alpha} \Delta q + \alpha_i^T G_{\sim \alpha} \Delta q \]  

(5.22)

Equation (5.22) is to be added to Eq. (5.4). Then taking \( \delta \pi_{\text{mR}} = 0 \) with respect to \( \Delta \beta \), \( \Delta q \) and \( \alpha_i \), we obtain
\[ G \Delta q - H \Delta \beta - G^i + R^i_I = 0 \]  
(5.23a)

\[ G^T \Delta \beta - \Delta \bar{q} \bar{\alpha} + R^T_{II} + G^T_{\alpha} \bar{\alpha} = 0 \]  
(5.23b)

\[ G_{\alpha} \Delta q = - G_{\alpha} \circ q = R_{III} \]  
(5.23c)

From Eq. (5.23a)

\[ \Delta \beta = H^{-1}(G \Delta q - G^i + R^i_I) \]  
(5.24)

Substituting Eq. (5.24) into Eq. (5.23b)

\[ G^T \Delta q + G^T_{\alpha} \alpha = \Delta \bar{q} + G^T \Delta \bar{q} - R_{II}^I - G^T H^{-1} R_I \]  
(5.25)

or

\[ k \Delta q + G^T_{\alpha} \alpha = \Delta \bar{q} \]  
(5.26)

Equations (5.26) and (5.23c) can be written as

\[
\begin{pmatrix}
k
G^T_{\alpha}
\end{pmatrix}
\begin{pmatrix}
\Delta q
\alpha
\end{pmatrix}
= 
\begin{pmatrix}
\Delta \bar{q}
R_{III}
\end{pmatrix}
\]  
(5.27)

which can be solved for \( \Delta q \) and \( \alpha \). Without numerical error, \( R_{I} = 0 \), \( R_{III} = 0 \) and \( R_{II} = 0 \) except for the newly opened degree of freedom.

The functional for the assumed displacement method is obtained by setting

\[ \Delta \sigma_{ij} = C_{ijkl} \left( \Delta e_{kl} - \Delta e^i_{kl} \right) \]  
(5.28)

and

\[ \circ \sigma_{ij} = C_{ijkl} \left( \circ e_{kl} - \circ e^i_{kl} \right) \]
in Eq. (5.2). Then

\[ \pi = \int_V \left( \frac{1}{2} C_{ijkl} \Delta e_{ij} \Delta e_{kl} - C_{ijkl} \Delta e_{ij} \Delta e_{kl} \right) dV \]

\[ - \int_{\partial \Omega} \Delta t_i \Delta u_i dS + \int_V \sigma_{ij} \Delta e_{ij} - \int_{\partial \Omega} \sigma_{ij} \Delta u_i dS \]  \hspace{1cm} (5.29)

The finite element formulation leads to a system of equations which is of the same form as Eq. (5.9).

5.3 Example Problems

To check the validity of the formulations developed in Subsection 5.2, two example problems were solved for a center-cracked panel under uniform tension.

(a) An Elastic Center-Cracked Panel

The panel is shown in Fig. 12. A quarter of the panel is modelled by 67 four-node 7-b assumed stress hybrid elements and 39 constant strain triangular elements. The total number of degrees of freedom is 214. The mesh pattern around the crack tip is shown in Fig. 12. First, separate elastic solutions were obtained for four different half-crack lengths (a=0.16 in., 0.20 in., 0.24 in., 0.28 in.). Next, solutions were obtained by using the formulations based on the Hellinger-Reissner principle in Eq. (5.27) with the Lagrangian multiplier. The Lagrangian multiplier \( \lambda \) was assumed to be constant along each element boundary. The initial half-crack length was a=0.16 in. The crack was opened along the original crack line by one mesh length (0.04 in.) each time up to a=0.28 in. Figure 13 shows displacement \( v \) along the crack line. The results are in agreement with the separate elastic solutions.
Another set of solutions was also obtained by using the formulation in Eq. (5.9) without the Lagrangian multiplier. As shown in Fig. 13, the calculated displacements \( v \) are in agreement with the separate elastic solutions. These results demonstrate the importance of \( -R \) term.

(b) A Center-Cracked Panel under Creep Condition

The same panel as in Fig. 12 is assumed to undergo creep deformation. Under creep condition, the crack begins to propagate after a certain amount of time called the initiation time. A hypothetical initiation time of 100 hours was used for the present calculation. After 100 hours, the crack was assumed to propagate with the rate of 0.008 in./hr. Figure 14 shows the displacement \( v \) along the crack line before and after the opening of two nodes. Since the panel has been allowed to creep for 100 hours before the opening of nodes, stresses near the crack tip have been relaxed to values lower than elastic solutions. These lower stresses are responsible for displacements smaller than elastic solutions near the released nodes.

In elastic problems, there exists \( 1/\sqrt{R} \) singularity for stresses near the crack tip where \( r \) represents the distance from the tip. This singularity has not been taken into account in the formulation of finite elements used in the present investigation. To improve solutions, it is recommended that special crack tip elements with the proper singularity be employed in the future work. The assumed stress hybrid formulation is particularly convenient for this purpose [71].
SECTION 6

CONCLUSIONS

Based on the results obtained in this work, we can draw the following conclusions.

(1) **Constraints on Finite Elements**

Difficulties associated with finite element models have been identified in terms of constraints on the assumed deformation modes due to zero transverse shear strain energy and the constant strain state including rigid body modes. Two-field variational principles such as various forms of the Hellinger-Reissner principle can be used to form finite element models with relaxed constraints. The reduced integration scheme in the assumed displacement method is shown to be equivalent to an application of the Hellinger-Reissner principle. But, the Hellinger-Reissner principle provides a more general approach in finite element formulation.

(2) **Shell Analysis**

An eight-node shell element has been derived by combining a modified Hellinger-Reissner principle and the reduced integration scheme. The resulting element is more efficient than the corresponding element with reduced integration only. An extension to large deflection analysis of shell structures has been formulated and the basic approach has been verified by means of a large deflection analysis of a shallow arch.

(3) **Creep Analyses by Assumed Stress Hybrid Finite Element**

An assumed stress hybrid finite element formulation has been
derived for viscoplastic and creep analyses. The practicality of the method has been demonstrated by example problems. The classical elasto-plastic problems can be solved by elasto-viscoplastic formulation. An example problem with axisymmetric solid elements indicates that the present formulation is more efficient than the conventional assumed displacement method.

(4) Incremental Formulations Involving Crack Growth

Incremental finite element formulations have been derived for problems involving crack propagation in solids. The validity of formulations has been investigated by solving a center-cracked panel under elastic and creep conditions with ordinary four-node assumed stress hybrid elements and constant strain elements. In the elastic case, results by the incremental formulations with and without the Lagrangian multiplier were in agreement with separate elastic solutions. It is recommended for future research that special crack tip elements with proper singularities be included in the formulation.
REFERENCES


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73. Hughes, T.J.R., Cohen, M. and Haroun, M., "Reduced and Selective Integration Techniques in the Finite Element Analysis of Plates", to be published in Nuclear Eng. and Design.


TABLE 1
MAXIMUM NONDIMENSIONAL DISPLACEMENT $\tilde{w} = \frac{EIw}{pL^3}$
FOR A CANTILEVER BEAM UNDER TIP LOAD $p$

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<thead>
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<th>Length/Thickness</th>
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<td>0.3300</td>
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<td>0.2920x10^{-1}</td>
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TABLE 2
LIST OF PLATE BENDING ELEMENTS

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<td></td>
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<table>
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<td>Modified Hellinger - Reissner</td>
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<td>Hellinger - Reissner</td>
<td>Modified Hellinger - Reissner</td>
<td>Modified Hellinger - Reissner</td>
</tr>
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<td></td>
<td></td>
<td>Linear</td>
<td></td>
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<td>Bilinear</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>1, x+y</td>
</tr>
</tbody>
</table>

No. of $\alpha_i$'s: 8, 8, 6, 6, 6, 4
TABLE 3
NONDIMENSIONAL NORMAL DEFLECTION $\bar{w} = 100 \cdot Dw/\rho a^4$ FOR A SIMPLY
SUPPORTED PLATE UNDER UNIFORM PRESSURE $p$

<table>
<thead>
<tr>
<th>Element Name</th>
<th>R24</th>
<th>MR24</th>
<th>MR24A</th>
<th>R18 &amp; MR18</th>
<th>MR18A</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integration Points</td>
<td>2x2 (exact)</td>
<td>2x2 (exact)</td>
<td>2x2 (exact)</td>
<td>7</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>a/t</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.4270</td>
<td>0.4269</td>
<td>0.4273</td>
<td>0.4272</td>
<td>0.4257</td>
<td>0.4347</td>
</tr>
<tr>
<td>20</td>
<td>0.4111</td>
<td>0.4110</td>
<td>0.4113</td>
<td>0.4105</td>
<td>0.4079</td>
<td>0.4143</td>
</tr>
<tr>
<td>50</td>
<td>0.4059</td>
<td>0.4057</td>
<td>0.4069</td>
<td>0.4049</td>
<td>0.4019</td>
<td>0.4077</td>
</tr>
<tr>
<td>100</td>
<td>0.4020</td>
<td>0.4019</td>
<td>0.4062</td>
<td>0.4040</td>
<td>0.4010</td>
<td>0.4068</td>
</tr>
<tr>
<td>300</td>
<td>0.3655</td>
<td>0.3654</td>
<td>0.4061</td>
<td>0.4042</td>
<td>0.4008</td>
<td>0.4068</td>
</tr>
<tr>
<td>1000</td>
<td>0.1868</td>
<td>0.1867</td>
<td>0.4060</td>
<td>0.4044</td>
<td>0.4008</td>
<td>0.4068</td>
</tr>
</tbody>
</table>

a = length of a side

$\rho$ = thickness

$D$ = bending rigidity
### TABLE 4

**Nondimensional Normal Deflection** \( \bar{w} = 100 \text{ Dw/pa}^4 \) **for a Clamped Square Plate Under Uniform Pressure** \( p \)

<table>
<thead>
<tr>
<th>Element</th>
<th>R24 2x2</th>
<th>R24 3x3</th>
<th>MR24A 2x2</th>
<th>MR24A 3x3</th>
<th>Exact Thin Plate Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh 50</td>
<td>0.1076</td>
<td>0.1261</td>
<td>0.1205</td>
<td>0.1277</td>
<td></td>
</tr>
<tr>
<td>a/t 100</td>
<td>0.0788</td>
<td>0.1210</td>
<td>0.1181</td>
<td>0.1261</td>
<td>0.126</td>
</tr>
<tr>
<td>a/t 300</td>
<td>0.0459</td>
<td>0.09176</td>
<td>0.1171</td>
<td>0.1247</td>
<td></td>
</tr>
<tr>
<td>a/t 1000</td>
<td>0.0391</td>
<td>0.03669</td>
<td>0.1170</td>
<td>0.1245</td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 5

**List of Arch Elements**

<table>
<thead>
<tr>
<th>Element Name</th>
<th>RA6</th>
<th>RA8</th>
<th>DA6</th>
<th>DA8</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of DOF's</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>Assumed u</td>
<td>Linear</td>
<td>Cubic</td>
<td>Linear</td>
<td>Cubic</td>
</tr>
<tr>
<td>Assumed w</td>
<td>Cubic</td>
<td>Cubic</td>
<td>Cubic</td>
<td>Cubic</td>
</tr>
<tr>
<td>Assumed ( \epsilon )</td>
<td>Constant</td>
<td>Linear</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 6

MAXIMUM NONDIMENSIONAL NORMAL DEFORMATION

\[ \bar{w} = \frac{w}{\frac{P(R^3)}{E t}} \]

FOR A PINCHED RING

<table>
<thead>
<tr>
<th>Element Name</th>
<th>No. of Elements</th>
<th>$\frac{R}{t} = 60$</th>
<th>$\frac{R}{t} = 100$</th>
<th>$\frac{R}{t} = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RA6</td>
<td>8</td>
<td>0.8852</td>
<td>0.8852</td>
<td>0.8852</td>
</tr>
<tr>
<td>RA8</td>
<td>4</td>
<td>0.8925</td>
<td>0.8924</td>
<td>0.8924</td>
</tr>
<tr>
<td>DA6</td>
<td>8</td>
<td>0.1989x10^{-1}</td>
<td>0.3681x10^{-2}</td>
<td>0.4931x10^{-3}</td>
</tr>
<tr>
<td>DA8</td>
<td>4</td>
<td>0.8705</td>
<td>0.8322</td>
<td>0.8053</td>
</tr>
<tr>
<td>RS40*</td>
<td>8</td>
<td>0.8985</td>
<td>0.8984</td>
<td>0.8984</td>
</tr>
<tr>
<td><strong>Exact Solution</strong></td>
<td></td>
<td>0.8928</td>
<td>0.8927</td>
<td>0.8927</td>
</tr>
</tbody>
</table>

*Shell Element (See Section 3)

TABLE 7

NONDIMENSIONAL TRANSVERSE SHEAR $\bar{Q} = Q/p$ FOR

A CANTILEVER BEAM UNDER TIP LOAD

<table>
<thead>
<tr>
<th></th>
<th>$\xi$</th>
<th>$\frac{L}{t} = 10$</th>
<th>$\frac{L}{t} = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>-0.57735</td>
<td>-1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>19.83</td>
<td>2082.</td>
</tr>
<tr>
<td></td>
<td>0.57735</td>
<td>-1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>(B)</td>
<td>-0.57735</td>
<td>-1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>-1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td></td>
<td>0.57735</td>
<td>-1.0</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

(A) \[ Q = G\beta t (\phi + \frac{d\phi}{dx}) \]

(B) \[ Q = G\beta t \gamma \]

$-1 \leq \xi \leq 1$
### TABLE 8
NORMAL DEFLECTION AT THE LOAD POINT OF THE PINCHED CYLINDER WITH FREE ENDS

<table>
<thead>
<tr>
<th>Element</th>
<th>S40</th>
<th>RS40</th>
<th>Dawe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh</td>
<td>2x2</td>
<td>2x2</td>
<td>5x5</td>
</tr>
<tr>
<td>t = 0.094&quot;</td>
<td>0.1109</td>
<td>0.1117</td>
<td>0.1136</td>
</tr>
<tr>
<td>p = 100 lbs.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>t = 0.01548</td>
<td>0.2414x10^{-1}</td>
<td>0.2416x10^{-1}</td>
<td>0.2462x10^{-1}</td>
</tr>
<tr>
<td>p = 0.1 lb.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 9
NORMAL DEFLECTION AT THE LOAD POINT OF THE PINCHED CYLINDER WITH DIAPHRAGMED ENDS

<table>
<thead>
<tr>
<th>Element</th>
<th>S 40</th>
<th>RS40</th>
<th>Series Solution [45]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh</td>
<td>5x3</td>
<td>6x4</td>
<td>5x3</td>
</tr>
<tr>
<td>R/t</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.6577x10^{-4}</td>
<td>0.6651x10^{-4}</td>
<td>0.6818x10^{-4}</td>
</tr>
<tr>
<td>100</td>
<td>0.2944x10^{-3}</td>
<td>0.3038x10^{-3}</td>
<td>0.3083x10^{-3}</td>
</tr>
</tbody>
</table>
TABLE 10

RADIAL DISPLACEMENT ($10^{-1}$ IN.) AT THE INNER WALL FOR AN INFINITE CYLINDER UNDER INTERNAL PRESSURE

<table>
<thead>
<tr>
<th>$p/\sigma_y$</th>
<th>Exact Solution</th>
<th>Hybrid Method</th>
<th>Displacement Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5117</td>
<td>0.1002</td>
<td>0.1006</td>
<td>0.0995</td>
</tr>
<tr>
<td>0.5786</td>
<td>0.1206</td>
<td>0.1210</td>
<td>0.1191</td>
</tr>
<tr>
<td>0.6345</td>
<td>0.1435</td>
<td>0.1432</td>
<td>0.1416</td>
</tr>
<tr>
<td>0.6806</td>
<td>0.1689</td>
<td>0.1688</td>
<td>0.1665</td>
</tr>
<tr>
<td>0.7179</td>
<td>0.1966</td>
<td>0.1965</td>
<td>0.1923</td>
</tr>
<tr>
<td>0.7472</td>
<td>0.2264</td>
<td>0.2255</td>
<td>0.2191</td>
</tr>
<tr>
<td>0.7692</td>
<td>0.2583</td>
<td>0.2550</td>
<td>0.2474</td>
</tr>
<tr>
<td>0.7846</td>
<td>0.2922</td>
<td>0.2871</td>
<td>0.2746</td>
</tr>
<tr>
<td>0.7937</td>
<td>0.3278</td>
<td>0.3176</td>
<td>0.2963</td>
</tr>
<tr>
<td>0.7972</td>
<td>0.3650</td>
<td>0.3328</td>
<td>0.3061</td>
</tr>
<tr>
<td>(Collapse)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CPU Time</td>
<td>(sec.)</td>
<td>6.30</td>
<td>9.29</td>
</tr>
<tr>
<td>(IBM 370/165)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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FIG. 1 NONDIMENSIONAL INPLANE STRESS FOR A PINCHED RING
FIG. 2 COORDINATE SYSTEMS AND NODAL DEGREES OF FREEDOM FOR A SHELL ELEMENT
(A) FREE ENDS

\[ E = 1.05 \times 10^7 \text{ PSI} \]
\[ \nu = 0.3125 \]
\[ R = 4.953 \text{ IN.} \]
\[ L = 10.35 \text{ IN.} \]
\[ P = 1 \text{ LB.} \]

(B) DIAPHRAGMED ENDS

\[ E = 1.05 \times 10^7 \text{ PSI} \]
\[ \nu = 0.3 \]
\[ R = 4.953 \text{ IN.} \]
\[ L = 2R \]
\[ P = 1 \text{ LB.} \]

FIG. 3 PINCHED CYLINDER
FIG. 4 DEFINITION OF ANGLES FOR AN ARCH
FIG. 5 LOAD-DEFLECTION CURVE FOR A CLAMPED SHALLOW ARCH UNDER A POINT LOAD

- $E = 10^7$ PSI
- $v = 0$
- $R = 133.114$ IN.
- $\theta = 14.674^\circ$
- $t = 3/16$ IN.

0 BY DISPLACEMENT INCREMENT
($\Delta W_{\text{max}} = -0.003$ IN.)

X BY LOAD INCREMENT
($\Delta P = 0.8$ LBS.)
FIG. 6 RADIAL DISPLACEMENT AT THE INNER WALL VS. TIME FOR AN INFINITE CYLINDER UNDER INTERNAL PRESSURE

A = 0.16 IN.
B = 0.25 IN.
P = 365 PSI
E = 2x10^7 PSI
ν = 0.45

\[ e^c = 6.4\times10^8 \sigma^{4.4} t^{0.7} \]
FIG. 7 HOOP STRESS VS. RADIUS FOR AN INFINITE CYLINDER UNDER INTERNAL PRESSURE
FIG. 8  RADIAL AND NORMAL STRESSES FOR AN INFINITE CYLINDER UNDER INTERNAL PRESSURE
FIG. 9 RADIAL DISPLACEMENT AT THE INNER RIM VS. TIME FOR A ROTATING DISK

\[
\begin{align*}
a &= 1.25 \text{ IN.} \\
b &= 6.0 \text{ IN} \\
h &= 0.5 \text{ IN} \\
E &= 1.8 \times 10^7 \text{ PSI} \\
\nu &= 0.45 \\
e_c &= 1.5 \times 10^{-30} \sigma^6 t^{2/3} \\
m^2 \omega^2 &= 1810 \text{ LBS/IN}^4
\end{align*}
\]
FIG. 10 RADIAL AND HOOP STRESSES VS. RADIUS FOR A ROTATING DISK
FIG. 11  A CRACKED BODY WITH AN IMAGINARY CUT SURFACE $\Sigma$
FINITE ELEMENT MESH PATTERN OF THIS REGION IS SHOWN BELOW

$E = 1.8 \times 10^7$ PSI
$v = 0.3$
$\epsilon^c = 4.4 \times 10^{-2}$

FIG. 12 A CENTER-CRACKED PANEL UNDER TENSION
FIG. 13 COMPARISON OF INCREMENTAL SOLUTIONS AND SEPARATE ELASTIC SOLUTIONS FOR A PANEL WITH DIFFERENT CRACK LENGTHS
FIG. 14

DISPLACEMENT OF THE CRACK SURFACE UNDER CREEP CONDITION

CALCULATED VALUES

\[ E = 1.8 \times 10^7 \text{ psi} \]
\[ v = 0.3 \]
\[ \epsilon_c = 4.41 \times 10^{-3} \]

\[ \epsilon_c = 0.2 \]

DISPLACEMENT OF THE CRACK SURFACE (10^{-3} \text{ in.})
BIOGRAPHY

The author, Sung Won Lee, was born in Korea on December 13, 1944. He graduated from the Department of Aeronautical Engineering of Seoul National University, Seoul, Korea in 1966. He served in the Korean Army from 1967 to 1970 and attended the Graduate School of Seoul National University from 1970 to 1972. He came to MIT in September 1972 and has worked as a research assistant to Professor T.H.H. Pian since then. He received his S.M. Degree in Aeronautics and Astronautics in June 1974. He married in July 1974 to Myung Hee Park.