IDENTIFICATION OF ELECTROLYTIC CELL PARAMETERS

USING A SELF-TUNING PREDICTOR

by

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ABSTRACT

Electrode measurements of bioelectric signals are corrupted by noise, offset, and parameter variations. The confidence on observer has that the measurements accurately reflect the bioelectric potential determines the extent to which analysis may be conducted. A self-tuning predictor is proposed to remove electrode-induced noise and offset from the observation signal, taking into account effects of parameter variations in the biological medium and at the electrode junctions.

Thesis Supervisor: Timothy L. Johnson

Title: Associate Professor of Electrical Engineering
I take this opportunity to express my gratitude to my advisor, Professor Tim Johnson, working with him has been a rewarding and educational experience.

Particular thanks go to Wolf Kohn. Though we worked together for a short time he was most helpful in introducing me to MIT's adage computer facility.

Lou Dadok was extremely helpful in preparing the electrolytic cell and obtaining the impedance vs. frequency plots.

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DEDICATION

To my mother and step-father

Lillian and Joseph Goff
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1.1 INTRODUCTION

Galvani demonstrated, through his third experiment on contraction the existence of a bioelectric potential [1]. Since this discovery during the last decade of the eighteenth century, the use of electrodes to measure bioelectric events has progressed significantly. Until recently, careful attention to electrode selection and application has been the means of minimizing electrode artifacts [2]. As experimental situations become more demanding, requiring greater refinement of the measured signal, this empirical method, basic band-pass filtering methods, and fixed parameter linear filters, are ineffective in selectively removing electrode effects. The adaptive predictor, introduced by Wittenmark and Astrom [4],[8] takes advantage of modern statistical filtering theory to directly estimate parameters of a minimum mean square error predictor from input-output data. The tuned parameters are then utilized for the prediction of future observations. The optimal predictor parameters are not independent of the plant parameters: we investigate the possibility of deriving the plant parameters from the estimated predictor parameters.

The usual experiment involves placement of metallic electrodes in contact with an electrolyte (electrode paste or biological material such as the skin) in order to make measurements of bioelectric potential either in response to some underlying bioelectric event or to a stimulating current pulse. The potential actually measured at the terminals of the electrodes is a result of the stimulating event and electrode-induced distor-
tions. Geddes points out [2] that even when procedural precautions are taken to minimize the electrochemical potential difference between electrodes, there often exists a residual potential difference which may be unstable and randomly varying. The origin of this residual voltage is hypothesized as slight differences in electrode metal or surface contamination of the electrodes. There are other sources of disturbance; for example, mechanical vibrations in the electrode-electrolyte interface generate electrical artifacts in the frequency spectrum of the bioelectric source so that simple filtering can not be employed without loss of the desired signal component. Environmental changes such as temperature concentration and pressure may result in parameter drift that would be difficult to remove with a fixed parameter filter. The self-tuning predictor learns from experience and is capable of tracking these types of parameter fluctuations.

1.2 SUMMARY OF RESULTS

The self-tuning predictor has been applied to a simulated plant, generating input-output data digitally, and to data obtained from experiments with an electrolytic cell. The self-tuning predictor is compared to various non-adaptive predictors, including the optimal predictors.

The results of the simulation studies presented in tables one through four, lead us to the conclusion that the self-tuning predictor is very good at reducing the prediction error to a minimum. However there exist parameter sets, beside the calculated optimal sets, that are comparable in their prediction capabilities.

The experimental results utilizing the cell, presented in tables 5
and 6, indicate the estimated parameters of the predictor converge to values that upon implementation in a non-adaptive predictor have prediction capabilities comparable to the self-tuning predictor.

In conclusion, the self-tuning predictor is an appropriate means of filtering random noise from observation signals, with the added feature of convergence to parameter sets that are nearly optimal. It is also shown that the self-tuning predictor is capable of following trend variations in the plant parameters.

1.3 ORGANIZATION OF THE THESIS

This thesis deals with the applicability of a particular filtering scheme to the problem of estimating the true plant output given noisy observations, and the identifcation of the underlying plant parameters. In the following chapter on modelling, a continuous time state space model is developed from an equivalent circuit model of the electrode-subject system. The continuous time model of the plant is then discretized and expressed in auto-regressive form. The optimal predictor assuming known and constant plant parameters is derived to obtain the minimum square error. A type of extended Kalman filter for the predictor gains is then implemented for the case when the plant parameters are not a priori known or include some time variation.

The following two chapters describe the implementation of the self-tuning predictor. A simulation study was conducted to help monitor proper coding of the algorithm along with presenting the opportunity to study the filter response in a controlled environment.

The fourth chapter concerns the electrolytic cell study. A cell simi-
lar to the one studied by Johnson and Salzsieder [3] was used. The electrodes in this application performed both the stimulation and recording functions. Time series data was processed to determine estimates of the predictor parameters. The estimates are compared to those derived manually from impedance measurements on the cell.

The last chapter concludes the thesis with a summary and suggestions for further research.
CHAPTER 2
MODELLING

2.1 INTRODUCTION

In order to apply modern filtering theory it is necessary to compose the problem in an appropriate manner. This chapter is intended to form a bridge between the empirical knowledge that exists concerning application of electrodes and the structured form of the theory. We begin with a linear circuit representation of the electrodes and their environment, generalize for variation in the environment and then develop a discrete model whose output may be considered an observation of the underlying system parameters.

2.2 THE MODEL

To facilitate conceptualization of the relationship between the bio-electric source, medium, and the electrodes refer to Figure 1a. Electrodes are placed on the surface of a subject in order to make observations of either the underlying source signal or response to stimulation from the electrodes themselves. A slight modification of Geddes' [2] approximate equivalent circuit for the electrode arrangement of Figure 1a is shown in Figure 1b.

Johnson and Salzsieder [3] have investigated the effects of variation of certain silver/silver cloride cell parameters on cell impedance. Johnson [9] has suggested the generalization of the circuit model of Figure 1b by incorporating functional dependences of the model elements on the cell parameters. Let \( p \) be the set of time varying parameters:

\[
\begin{align*}
  p_1 &= \text{temperature of the medium} \\
  p_2 &= \text{bulk solution concentration of ion(s) participating}
\end{align*}
\]
in the electrode reaction

\[ p_3 = \text{bulk solution concentration of non-participating ions} \]

\[ p = [p_1, p_2, p_3]^T \]

The following functional dependencies may be derived using experimental data obtained by methods presented by Geddes [1].

\[ R_1 = R_1(p_1, p_2, p_3) \]
\[ v_{ei} = v_{ei}(p_1, p_2, p_3, v_s), \quad i=1,2 \]
\[ R_2 = R_2(p_1, p_2, v_s) \]
\[ C = C(p_1, p_2, v_s) \]
\[ R_3 = \text{constant} \]

Analysis of the circuit of Figure 1b results in the following set of equations:

\[ \dot{x}_1(t) = ax_1(t) + \beta(v_s(t) + v_e(t)) \quad 1a \]
\[ y(t) = \gamma x_1(t) + \xi(v_s(t) + v_e(t)) \quad 1b \]
\[ v_e(t) = v_{e1}(t) + v_{e2}(t) \quad 1c \]

\[ \alpha = -(2(R_1 + R_2) + R_3)/CR_2(2R_1 + R_3) \]
\[ \beta = 1/(C(2R_1 + R_3)) \]
\[ \gamma = -2R_3/(2R_1 + R_3) \]
\[ \xi = R_3/(2R_1 + R_3) \]

The total junction potential, \( v_e(t) \), may be decomposed as:

\[ v_e(t) = \beta^*(p)v_s(t) + v_e^O(p) \]

The first term on the right hand side represents fluctuations in the half cell potential due to the driving source voltage. The second
term shows the implicit dependence of the half cell potential on the parameters $p$.

Substituting for $v_e(t)$ in equations 1a,b:

$$\dot{x}_1(t) = \alpha x_1(t) + \beta (1 + \beta^e(p)) v_s(t) + \beta v_e^0(p) \quad 2a$$

$$y(t) = \gamma x_1(t) + \xi (1 + \beta^e(p)) v_s(t) + \xi v_e^0(p) \quad 2b$$

In general, $p$ is dependent on time, but substantial simplification results when we assume parameter variation is significantly slower than the response time of the biological system. We then make the approximation:

$$p(t) = p^0$$

and

$$v_e^0(p) = v_e^0(p^0) \quad \text{for all } t.$$ 

Recognizing

$$\frac{dv_e^0(p)}{dt} = 0$$

we can augment the state equation 2a setting $x_2 = v_e^0(p^0)$ and $x_2 \dot{} = 0$:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \beta (1 + \beta^e(p)) \\ 0 \end{bmatrix} v_s(t) \quad 3a$$

and the observation equation is:

$$y(t) = [\gamma \ \xi] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \xi (1 + \beta^e(p^0)) v_s(t) \quad 3b$$

The system equations are now in the matrix form:
\[ \dot{x}(t) = A \dot{x}(t) + B v_s(t) \quad \text{4a} \]
\[ y(t) = C \dot{x}(t) + D v_s(t) \quad \text{4b} \]

where

- \( y(t) \) and \( v_s(t) \) are scalars
- \( x(t) \) is a (2x1) vector
- \( A \) is a (2x2) matrix
- \( B \) is a (2x1) matrix
- \( C \) is a (1x2) matrix
- \( D \) is a (1x1) matrix

The auto-regressive representation of the system may be derived by first obtaining the sampled data equivalent to equations 4a, b.

\[ x(t_{i+1}) = \hat{A}x(t_i) + \hat{B}u(t_i) \quad \text{5a} \]
\[ y(t_i) = \hat{C}x(t_i) + Du(t_{i-1}) \quad \text{5b} \]

where

- \( \hat{A} = \exp(A\Delta) \)
- \( \Delta = t_{i+1} - t_i \)
- \( \hat{B} = \int_{t_i}^{t_{i+1}} \exp[A(t_{i+1} - \tau)]Bd\tau \)
- \( \hat{C} = C \)
- \( \hat{D} = D \)

\[ u(t_i) = v_s(t = t_i) \quad \text{for} \quad t_i < t_i < t_{i+1} \]

\( u(t_i) \) is the piecewise constant sampled data equivalent of the continuous input \( v_s(t) \).

The following notational convention has been assumed:
\[ t = \text{continuous time} \]
\[ t_i = \text{discrete instant of time representing the } i^{th} \text{ sampling instant.} \]

For constant sampling interval \( \Delta \), \( t_i = i \Delta \).

The sampled data transfer function matrix is:
\[ \hat{G}(z) = \hat{G}(z) = \frac{Y(z)}{U(z)} \]

\( \hat{G}(z) = Y(z)/U(z) \)

\( \hat{G}(z) \) may be written as \( G'(z^{-1}) = Y(z)/U(z) \). \( G(z^{-1}) \) is a ratio of polynomials in \( z^{-1} \). Taking \( z^{-1} \) to be the backward shift (in time) operator [5] the system equations may be expressed in the autoregressive form:

\[ y(t_i) = \sum_{i=0}^{n} a_i y(t_{i-i}) + \sum_{j=0}^{m} b_j u(t_{i-j}) \]

The coefficients in this equation inherit the dependence on \( p \), which is assumed to vary slowly with time. We will incorporate effects of nonlinearities and uncertainty in the source \( u(t_i) \) as a single disturbance term \( v(t_i) \) acting on the output \( y(t_i) \) [8]. \( v(t_i) \) is taken to be stationary, white and zero mean, with covariance \( E[v(t_i) v(t_j)] = \delta(t_i-t_j) \).

For convenience with later equation manipulation the autoregressive equation is represented in operator form:

\[ A(z^{-1}) y(t_i) = B(z^{-1}) u(t_i) + C(z^{-1}) v(t) \]
where:

\[ A(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_n z^{-n} \]
\[ B(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m} \]
\[ C(z^{-1}) = 1 + c_1 z^{-1} + \ldots + c_n z^{-n} \]
\[ z^{-1} = \text{The Backward Shift Operator} \]

We will concentrate on the case \( C(z^{-1}) = 1 \).

2.3 THE OPTIMAL PREDICTOR

Consider the stochastic process introduced in the last section:

\[ A(z^{-1}) Y(t_i) = B(z^{-1}) U(t_i) + C(z^{-1}) V(t_i) \]

where

\[ [Y(t_i), i = 0,1,2,\ldots] \] is the output sequence
\[ [U(t_i), i = 0,1,2,\ldots] \] is the input sequence, and
\[ [V(t_i), i = 0,1,2,\ldots] \] is the noise sequence.

For the moment \( A(z^{-1}), B(z^{-1}), C(z^{-1}) \) are assumed to be known and constant. The k step-ahead prediction of the output signal, given observations through time \( t_i \) is denoted \( \hat{y}(t_{i+k}/t_i) \). The predictor error is defined as:

\[ \varepsilon(t_{i+k}) = y(t_{i+k}) - \hat{y}(t_{i+k}/t_i). \]

The loss function is taken to be:

\[ L = E[E^2(t_{i+k})]. \]
We now wish to find the predictor which minimizes the loss function. Astrom [8] has solved the prediction problem using the identity:

\[ C(z^{-1}) = A(z^{-1}) F(z^{-1}) + z^{-k} G(z^{-1}) \]  

where

\[ F(z^{-1}) = 1 + f_1 z^{-1} + \ldots + f_k z^{-k+1} \]
\[ G(z^{-1}) = g_0 + g_1 z^{-1} + \ldots + g_{n-1} z^{-n+1} \]

The process (8) may be written as:

\[ y(t_i) = [B(z^{-1})/A(z^{-1})] u(t_i) + [C(z^{-1})/A(z^{-1})] v(t_i) \]  

The prediction error equation (9) becomes:

\[ e(t_{i+k}) = z^k (B(z^{-1})/A(z^{-1})) u(t_i) + z^k (C(z^{-1})/A(z^{-1})) v(t_i) - y(t_{i+k}/t_i) \]

It can be shown that

\[ y(t_{i+k}/t_i) = \frac{G(z^{-1})}{C(z^{-1})} y(t_i) + (z^k - \frac{G(z^{-1})}{C(z^{-1})} \frac{B(z^{-1})}{A(z^{-1})}) u(t_i) \]

reduces the prediction error (9) to

\[ e(t_{i+k}) = \frac{1}{A(z^{-1})} (C(z^{-1})z^k - G(z^{-1})) v(t_i) \]

Applying the identity (11) in the form:

\[ z^k C(z^{-1}) = z^k A(z^{-1}) F(z^{-1}) + G(z^{-1}) \]

(9) becomes
\[ \varepsilon(t_{1+k}) = z^k F(z^{-1}) v(t_1). \]

Thus the predictor (13) makes the error, \( \varepsilon \), a moving average of \( v \). Since there is no "energy storage" this gives a minimum variance.

The \( k \)-step ahead prediction (13) is a linear combination of past predictions, present and past observations, and future, present and past inputs. The inclusion of future input values may present a problem depending on the experimental situation. If the electrodes are used for both stimulating and recording it is likely the experimenter knows the input series a priori or can implement the identification off line with measured values of the input signal. However, where the electrodes are employed for recording only; and the underlying source signal is not known, estimation of the input series must be conducted. This thesis focuses on the experimental configuration using one pair of electrodes for stimulation and recording.

2.4 THE SELF-TUNING PREDICTOR

The coefficients in (8) are taken to be time varying and imperfectly known as a means of approximating the effect variation of \( p \) has on the equivalent model elements. Considering that the relation between the coefficients and \( p \) is very complicated we make the assumption that the coefficients are themselves a random process generated by:

\[ \phi(t_{i+1}) = \phi(t_i) + \gamma(t_i) \]

where \( \phi(\cdot) = [a_1 a_2 \ldots a_n b_1 b_2 \ldots b_m c_1 c_2 \ldots c_n]^T \) and \( \gamma(\cdot) = [\gamma_1 \gamma_2 \ldots \ldots \ldots \gamma_{2n+m}]^T \)

\( \{\gamma(t_i)\} \) is a zero-mean white noise sequence.
Since $A(z^{-1}) B(z^{-1})$ and $C(z^{-1})$ are now time varying the performance of the non-adaptive filter presented in the previous section will be suboptimal. The filter can be made adaptive by recognizing (8) as an observation on the process generating $\phi(\cdot)$. Kalman filtering can be applied to obtain minimum-variance estimates of $\phi(t_i)$ given observations through time $t_i$. The new estimates of $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are then used to derive the optimal predictor. This procedure requires three steps: parameter estimation and then calculation of the optimal predictor parameters via the factorization (11) and solving of the predictor equation (13).

These latter steps are time consuming and thus not conducive to on-line implementation. It has been suggested in the literature [4,9,10] that the form of the predictor (13) may be adopted, allowing the parameters to be identified directly, as follows.

For $C(z^{-1})=1$, the optimal predictor takes one of the following two forms:

(i) Multiplication of both sides of equation (13) by the term $A(z^{-1})$:

$$A(z^{-1})\hat{\gamma}(t_{i+k}/t_i) = G(z^{-1})A(z^{-1})\gamma(t_i) + (z^k - G(z^{-1})) B(z^{-1}) u(t_i); \quad (15)$$

(ii) Recognize that the identity (11) may be expressed in the form

$$(z^k - G(z^{-1})) = z^k A(z^{-1}) F(z^{-1})$$

the predictor may be written as:

$$\hat{\gamma}(t_{i+k}/t_i) = G(z^{-1})\gamma(t_i) + z^k F(z^{-1}) B(z^{-1}) u(t_i) \quad (16)$$

In the case $C(z^{-1})\neq 1$, (15) will be the optimal predictor. For either form of the predictor in the previous case, the predictor equation may be
written explicitly as:

\[
\hat{y}(t_{i+k}/t_{i}) = -q_1 \hat{y}(t_{i+k-1}/t_{i-1}) - \cdots - q_n \hat{y}(t_{i+k-n}/t_{i-n})
\]

\[
+ q_{n+1} y(t_i) + \cdots + q_{3n} y(t_{i-2n+1}) + q_{3n+1} u(t_{i+k-1})
\]

\[
+ \cdots + q_{4n+m+k-1} u(t_{i-m-n+1}).
\]  

(17)

This equation may be factored into the following form:

\[
\hat{y}(t_{i+k}/t_{i}) = H(t_i)Q
\]

where

\[
Q = \begin{bmatrix} q_1 & q_2 & q_{4n+m+k-1} \end{bmatrix}^T
\]

and

\[
H(t_i) = \begin{bmatrix} -\hat{y}(t_{i+k-1}/t_{i-1}) & -\cdots & -\hat{y}(t_{i+k-n}/t_{i-n}) & y(t_i) & \cdots & y(t_{i-2n+1}) & u(t_{i+k-1}) & \cdots & u(t_{i-m-n+1}) \end{bmatrix}
\]

Each element of the vector \(Q\) is derived from the plant parameters by applying (11) and (13). As stated earlier, if the plant parameters are known and constant the predictor parameters derived in this manner will represent the optimal predictor; in the case where the plant parameters are unknown or time varying the predictor derived from our initial estimates of the plant parameters will be suboptimal.

The time variation of the plant parameters and thus the predictor parameters is due to the dependence of the equivalent model elements on \(p\). The assumption of slow time variation of \(p\) in relation to the response time of the subject, biological preparation, and electrodes, allows the assumption that the unknown coefficients \(Q\) are generated by the random process:
\[
Q(t_i) = Q(t_{i-1}) + w(t_i), \quad q(0) = Q^0
\]  

(19)

where \(w(t_i)\) is zero mean, white sequence with correlation matrix

\[
E[w(t_j)w(t_m)] = \omega \delta(t_j - t_m).
\]

We interpret (19) as the state equation of a linear dynamical system with the unknown predictor parameters as state variables. We view the observation given by (18), as an observation of the parameters, \(Q\), where the observation matrix \(H(\cdot)\), is time varying, consisting of past predictions, and measured input and output values.

In order to apply the well known method of Kalman filtering it is necessary to make general statistical assumptions. We assume the initial estimate of \(Q\), the state vector, is a Gaussian random variable:

\[
Q(0) \sim \mathcal{N}(Q^0, P(0/-1)).
\]

We have already stated the assumptions concerning the state driving noise and the observation noise.

The Kalman filter equations may be given as:

\[
\hat{Q}(t_i) = \hat{Q}(t_{i-1}) + K(t_i)(y(t_i) - \hat{y}(t_i/t_{i-k}))
\]  

(20a)

\[
K(t_i) = P(t_i/t_{i-1})H^T(t_{i-k})(V + H(t_{i-k})P(t_i/t_{i-1})H^T(t_{i-k}))^{-1}
\]  

(20b)

\[
P(t_{i+1}/t_i) = P(t_i/t_{i-1}) + W - K(t_i)(V + H(t_{i-k})P(t_i/t_{i-1})H^T(t_{i-k}))K^T(t_i)
\]  

(20c)

\[
\hat{y}(t_i/t_{i-k}) = H(t_{i-k})\hat{Q}(t_{i-k})
\]
A block diagram for this filter is presented in figure (3). The state vector $\hat{Q}(t_i)$, is interpreted as the parameter estimate incorporating observations through time $t_i$. The prediction $\hat{y}(t_i/t_{i-k})$ is the predicted plant output using the observations and parameter update through time $t_{i-k}$.
CHAPTER 3

SIMULATION STUDY

3.1 INTRODUCTION

The simulation phase of this research was beneficial in that it allowed controlled examination of the filter characteristics. The tests were designed to permit explicit determination of the capabilities of the self-tuning predictor when initial conditions, noise level, and design parameters were varied. The plant structure was also variable. Two cases are included here: constant plant parameters, and time-varying random parameters.

3.2 IMPLEMENTATION

The simulation study was implemented as follows. A plant of the form (7) was selected. The choice was based on consideration of stability and computational simplicity. The input time series was previously determined and stored on disk. The noise sequence was generated digitally to allow for variability in the starting "seed" value of the pseudorandom noise generator. After certain initial conditions were established the observations were calculated recursively by application of the auto-regressive equation (7). With each new observation the self tuning predictor parameters were updated by application of the filter (20). The simulation ends when either the input series is exhausted or a specified number of iterations has occurred.

We now elaborate upon each of the above steps. Two plant structures were used for simulation. They differed in that one involved constant plant parameters, and the other a time series trend superimposed upon a
nominal value. In general, the plant was of the following form:

\[ y(t_i) = -a_1 y(t_{i-1}) - a_2 y(t_{i-2}) + b_1 u(t_{i-1}) + v(t_i) \]  

(21)

The first set of tests involved constant plant parameters. In particular they were:

\[
\begin{align*}
a_1 &= .25 \\
a_2 &= .5 \\
b_1 &= 1.0
\end{align*}
\]

The operators are identified to be

\[
\begin{align*}
A(z^{-1}) &= 1 + .25z^{-1} + .5z^{-2} \\
B(z^{-1}) &= z^{-1} \\
C(z^{-1}) &= 1
\end{align*}
\]

The stability of this plant is determined by ascertaining the location on the complex plane of the roots of the characteristics equation. The characteristic equation is:

\[ s^2 + .25s + .5 = 0 \]

Applying the quadratic formula the roots, a complex conjugate pair are obtained:

\[
\begin{align*}
s_1 &= -.125 + .696j \\
s_2 &= -.125 - .696j
\end{align*}
\]

\[ |s_{1,2}| = .707 \]
Since $|s_{i}|<1$ for $i=1,2$ the plant is stable. [5]

The second set of tests involved trend variation in the plant parameters. In this case the initial $a_j's$ were taken to be the same as above. The $a_j's$ would then vary in time according to the following:

$$a_j(t_{i+1}) = a_j(t_i) + \delta$$

where $\delta$ was chosen to be a constant, .0001. The "poles" of this time-varying system move with time. Initially they are the same as in the case with constant plant parameters. At the end of these tests $a_1=.458$ and $a_2=.708$, and the new poles are

$$s_{1,2} = -.23 \pm .81j$$

$$|s_{1,2}| = .842$$

The system is still stable.

The input series was taken as a sum of sinusoids. Mehra [12] presents an algorithm for calculating optimal design inputs. We have chosen the input series as

$$u(t_i) = \sum_{j=1}^{NF} v_j \sin(2\pi f_j t_i)$$

where

$t_i = i\Delta$, $i=1,2,...$

$\Delta$ = sampling period

$v_j$ = magnitude of $j^{th}$ sinusoid

$f_j$ = frequency of $j^{th}$ sinusoid

$j = 1,2,...NF$

$NF$ = number of frequencies superimposed.
Although different input groups were employed in the course of the research, one group was selected to impart some standardization to the results presented here. The input series was generated by recursive application of the following.

\[ u(t_i) = 0.6\sin(100\pi t_i\Delta) - 0.5\sin(150\pi t_i\Delta) + 0.2\sin(196\pi t_i\Delta) \]

with a sampling period of \( \Delta = 0.00117 \) seconds.

A random number generator was designed exclusively for this project. Because of some difficulty in obtaining a sequence with a variance equal to the one desired, a standard design variance was chosen. For the random number sequence generated, a mean and variance were calculated. Correction for bias was unnecessary since the mean was zero to 5 decimal places. To obtain sequences with various magnitudes and variances a scale factor was introduced.

Let \( r \) be a random number sequence with mean \( \mu \) and variance \( \sigma^2 \). The multiplication of \( r \) by a constant \( \alpha \) results in a new random variable \( r' = \alpha r \). It is well known the mean of \( r' \) is \( \alpha \mu \) and the variance of \( r' \) is \( \alpha^2 \sigma^2 \). The distribution, mean, and variance for the standard noise sequence is presented in figure (4).

The system was assumed to be initially at rest. Specifically, \( u(t_i) = 0 \) for \( t_i < 0 \) and \( y(t_i) = 0 \) for \( t_i < 0 \). The initial guess of the predictor parameters was designated \( Q(-1) \).

The initial parameter estimate correlation matrix, \( P(0/-1) \), the correlation matrix for the noise driving the predictor parameters, \( W \), and the covariance of the observation noise, \( V \), were taken as filter design parameters. This seems reasonable since we may have difficulty in quantifying
1) How accurately we feel our initial estimate of the predictor parameters truly reflect the optimal parameters; 2) the noise process that is driving the plant parameters and; 3) the characteristics of the noise signal corrupting the observations.

If the plant parameters are known and constant the optimal predictor (13) for the simulated plant (21) may be determined by first applying the identity (11).

For \( k = 1 \),

\[
1 = (1 + a_1 z^{-1} + a_2 z^{-2})(1) + z^{-1}(g_0 + g_1 z^{-1})
\]

\[
G(z^{-1}) = -a_1 - a_2 z^{-1}
\]

\[
F(z^{-1}) = 1.
\]

Since \( C(z^{-1}) = 1 \) there are two forms to the optimal one step-ahead predictor. We specify a predictor by its parameter vector \( Q \). These vectors for the optimal predictor are designated \( Q_{opt1} \) and \( Q_{opt2} \).

Solving (13) the optimal predictor takes the form

\[
y(t_{i+1}/t_i) = H(t_i)Q_{optj} \quad j = 1, 2
\]

with

\[
Q_{opt1} = [0 \quad 0 \quad -a_1 \quad -a_2 \quad 0 \quad 0 \quad b_1 \quad 0 \quad 0]^T \quad (22a)
\]

\[
Q_{opt2} = [a_1 \quad a_2 \quad -a_1 \quad -(a_1^2 + a_2) \quad -a_1 a_2 \quad a_2^2 \quad b_1 \quad a_1 b_1 \quad a_2 b_1]^T \quad (22b)
\]

and \( H = \{-\hat{y}(t_i/t_{i-1}) - \hat{y}(t_{i}/t_{i-2})y(t_i)y(t_{i-1})y(t_{i-2})y(t_{i-3})u(t_i) \}

\[
u(t_{i-1})u(t_{i-2})\}.
\]
The above equations show the explicit dependence of the optimal predictor parameters on the plant parameters. It is clear that if the plant parameters vary the desired optimal predictor parameters will change. The non-adaptive filter will not be able to follow this change and will thus be sub-optimal.

In the case where the plant parameters are constant and a priori known, the prediction error for the optimal predictor is given by (14). For \( k=1 \), this becomes:

\[
e(t_{i+1}) = A F(z^{-1}) v(t_i)
\]

Since \( F(z^{-1})=1 \), for our simulated plant

\[
e(t_{i+1}) = v(t_{i+1}).
\]

We expect the optimal predictor to work very well under these conditions of perfect knowledge of the plant parameters.

3.3 ANALYSIS OF SIMULATION RESULTS

We will evaluate two capabilities of the self-tuning predictor: First, the proficiency of the self-tuning predictor as a filter; Second, our ability to infer information about the plant from the parameter estimate vector \( \hat{Q}(\cdot) \). For comparison of prediction capability a non-adaptive predictor is used. The optimal predictor, for a plant with constant parameters, is a non-adaptive predictor whose parameters are those given in equations 22a or b. Non-optimal parameter sets are also implemented. The criteria for comparing prediction capabilities will be:

\[
\text{Accumulated Absolute Error} = \sum_{i=1}^{t_f} |e(t_i)|
\]
Accumulated Square Error = \[ \sum_{i=1}^{t_f} e_i^2(t_i). \]

Test results are summarized in Tables 1, 2, 3, and 4. All tests utilized the same input series and initial conditions. The noise sequence differed by a multiplication factor thus resulting in different noise variances. The accumulated loss and error is over a period of 2081 iterations.

Analysis of the results leads to the following conclusions concerning predictor capabilities. When the plant is known a priori and the parameters are known to be constant, the non-adaptive optimal predictor is excellent. However, if the predictor parameters are poorly known, the self-tuning predictor learns very quickly. For example, Test 25 on Table 2 involves an initial parameter estimate that is grossly distant from the optimal set, Figure (5) shows both the accumulated loss for the self-tuning predictor (curve II) and the accumulated loss for the optimal predictor (curve I).

From the graph one sees that most of the loss is incurred within the first twenty iterations of the filter. In fact, the average loss per iteration over the last half of the test is almost identical for the self-tuning predictor and the optimal predictor. Figures 6, 7, 8, plot the parameter estimates for this same test. The estimates approximate the optimal values rather quickly, but final adjustments take a considerable amount of time. In fact, it is well-known that the estimates may be biased and may not converge to the optimal values.

One may also deduce from the results presented in Tables 1 and 2 that although \( Q_{\text{opt1}} \) and \( Q_{\text{opt2}} \) are the theoretical optimal parameters, there are other sets that offer comparable prediction capability. In particular, a
parameter set whose elements were determined by taking the numerical average of the respective optimal elements was comparable in prediction capability to the two optimal sets. Applied in a non-adaptive predictor the accumulated loss for this new parameter set (Tests 3, 7, and 11) is almost identical to the accumulated loss of either optimal set (Tests 1, 2, 5, 6, 9, 10). When this parameter set is taken as an initial estimate for the self-tuning predictor (Tests 15, 20 and 24) there is no tendency for the parameter updates to diverge from their initial values. We ask the immediate question of uniqueness. Although $Q_{\text{opt1}}$ and $Q_{\text{opt2}}$ are derived from the unique $F(z^{-1})$ and $G(z^{-1})$ that solve (11), it is not clear that (11) is the only means of reducing the prediction error to a moving average of the noise. This possibility has not been fully explored in this work, but represents an area for future investigation.

Tables 3 and 4 summarize the tests involving a time series trend in the simulated plant parameters. In all cases the self-tuning predictor performed superior to the non-adaptive predictor. There is a significant effect upon prediction and estimation of the filter design parameter $W$. In the case $W=0$ (Tests 44, 46) the predictor parameter estimates diverge from the proper set. However when $W$ is non zero (Tests 45, 47) the prediction and estimation performance is enhanced. This is partially explained by the integration of $W$ via the estimate covariance equation (20c) and its effect on the Kalman gain $K(\cdot)$.

The simulation tests that were conducted were not meant to be exhaustive in their examination of the self-tuning predictor. They were chosen to be representative of the type of situations to which the predictor would be applied.
CHAPTER 4

ELECTRODE STUDIES

4.1 INTRODUCTION

The application phase of the research focuses on indentification of electrolytic cell impedance parameters. In general, the theory of the self-tuning predictor presented in this thesis is applicable to any single-input, single-output system that can be characterized by continuous or discrete state-space models or an auto-regressive equation. We have chosen a pair of silver-silver chloride electrodes in a saline solution as the plant to be identified. Two silver disc electrodes form the end pieces of a cylindrical volume containing sodium chloride solution.

Johnson and Salzsieder [3] have conducted extensive testing to determine the dependence of this cell's impedance on various experimental variables (e.g. temperature, concentration). We have applied the self-tuning predictor to cells containing different concentrations of salt in order to determine the adaptability of the predictor.

4.2 IMPLEMENTATION

Figures 9, 10 present the circuit and flow chart for experimental data collection. A single pair of electrodes is used both to "stimulate" the medium and record the response. A previously calculated input sequence is selected and read into memory from the disk. This input is applied as a piecewise constant voltage via the digital to analogue converter. The two analogue-to-digital channels sample the voltage across the known resistance $R_s$. For identification purposes the cell current is taken as the stimulus...
and the cell voltage is the response. The current may be calculated from the time series voltage measurements as follows:

$$u(t_{i-1}) = \frac{c(t_i) - d(t_i)}{R_s}$$

The different time subscripts are the result of our sampling the voltages $c(\cdot)$ and $d(\cdot)$ at the end of each sampling interval; $u(t_{i-1})$ is the current applied to the cell over the time interval $(t_{i-1}, t_i]$. The cell voltage response, $y(t_i)$, is of course $d(t_i)$. The time series measurements are stored on disk.

The identification routine is applied to the stored data. Figure presents a flow chart for the identification procedure. Filter design parameters and the initial estimates of the predictor parameters are the program inputs. The stimulus and response series are selected from those stored on disk and read into memory. The routine then iterates through the 2080 sample pairs. The block diagram for the filter is presented in figure 3. If the sampling interval was extended or a more efficient implementation of the algorithm were used it would be possible to implement the self-tuning predictor in real time.

The optimal predictor parameters are derived from estimated plant parameters (see the appendix). The plant parameters are assessed by matching a transfer function of the form:

$$Z(s) = \frac{\left(\frac{s}{z_1} + 1\right)\left(\frac{s}{z_2} + 1\right)\ldots\left(\frac{s}{z_n} + 1\right)}{\left(\frac{s}{p_1} + 1\right)\left(\frac{s}{p_2} + 1\right)\ldots\left(\frac{s}{p_n} + 1\right)}$$

(23)

to the log impedance vs log frequency plot for the cell (Figure 11 a,b)
The auto regressive form for the stimulus-response relationship is obtained by determining a continuous time state-space model realization for $Z(s)$. This model is discretized via the transformation equations (6); and the sampled data transfer function is obtained. The auto regressive form is the result of taking $z^{-1}$ as the backward shift (in time) operator. The optimal parameters are found by solving the identity (11) for $G(z^{-1})$ and then (13) for the predictor parameters. These parameter sets are utilized in the non-adaptive "optimal" predictors and as initial estimator for the self-tuning predictor. The estimate covariance matrix $P(0/-1)$ was chosen to reflect our uncertainty in these estimates. $W$ and $V$ were varied as filter design parameters.

4.2 ANALYSIS OF RESULTS

The results of selected tests are presented in Tables 5 and 6 for saline solutions of .01 and .1 normality respectively. As in the simulation study we wish to analyze the self-tuning predictor as both a filter and a parameter estimator.

Strict comparison of the accumulated square error can be misleading. Both tables offer examples of wide difference in accumulated square error, but closer examination indicates the predictors to be almost identical. Tests 2 and 4 of Table 5 and Tests 2 and 3 of Table 6 are indicative of this situation. Both pairs of tests begin with different initial estimates of the predictor parameters. In both cases the self-tuning predictor parameters converge to similar values. Calculations of the average square error over the last 208 iterations indicates that the predictors are almost identical in performance. This is evidence of the robustness of the algor-
ithm and the predictors' ability to learn to track the cell parameters.

For the .01 normal cell initial predictor parameters estimator were derived from plant estimates assuming a transfer function with one pole and one zero. The plot of log impedance vs log frequency for the .01 N cell is presented in Figure 11a. The asymptotic Bode plot of the proposed transfer function is superimposed over the laboratory results. The optimal predictor for a plant with one pole and one zero is of order 5 and is derived in the appendix. In Table 5 we compare two 5th order self-tuning predictors, one of which takes the calculated optimal predictor as its initial estimate (test 1) and the other begins with zero's (test 3). The estimates of the predictor parameters converged in both cases. We compare the average square error of these self-tuning predictors with the average square error of the calculated "optimal" predictor (test 5) and the non-adaptive implementation of the converged parameters (test 6). The "optimal" predictor has an average square error over the entire test on the order of 3 times greater than the self-tuning predictors. The non-adaptive implementation of the converged parameters results in an average square error over the entire test that is approximately 1.6 times as great as the self-tuning predictor.

Similar results were obtained for tests 2, 4, 7, and 8 of Table 5. The order of the self-tuning predictor was increased to nine to investigate the effect of increasing the degree's of freedom of the predictors. Initial predictor gain estimates were again the "optimal" set (assuming one pole and one zero in the transfer function) based on prior experimental data and the zero set. The estimates again converge but to values that are slightly different from those of the lower order predictor. Prediction via
the converged parameter set (test 7) is significantly better than the calculated optimal predictor and not quite as good as the self-tuning predictor.

The cell containing the .1 Normal solution was hypothesised to be of higher order than the cell with the lower concentration of ions. A transfer function with two poles and two zeros was used to match the data points. The derivation of the optimal predictor is in the appendix. Table 6 presents the results of selective tests. Three self-tuning predictors were applied, one of order 5 and the other two of order eleven. The initial estimates are presented in the table under the column $Q_0$. The two eleventh order predictors were initiated with parameter estimates of the zero vector (test 2) and the calculated "optimal" values (test 3). The estimated parameters for both predictors converge to similar values. The time series data indicates the parameters themselves are not constant, but the two predictors converge to this time-varying parameters set within 500 iterations. The time variation of the cell parameters is believed to be a result of the input series applied to the cell. The time series data for the test results of the .01 normal cell show a convergence to constant parameters. The non-adaptive predictors performed similarly to those of table 5. The converged parameters sets were best, with the fifth order predictor having a lower accumulated square error but a slightly higher average over the last 208 iterations. The calculated "optimal" predictor performed significantly worse.

The question of deciding which parameter set to invert to obtain estimated plant parameters is non-trivial. One possible procedure involves implementing self-tuning predictors of various orders to obtain estimated
parameters, use these parameters in the non-adaptive model and choose the set that minimizes the accumulated square error. Attempts to obtain the plant parameters via inversion of the procedure in Section 2.3 may prove futile; in particular we can not solve for a $G(z^{-1})$ with the parameters from test 7 table 5. However we can solve for the plant parameters for the predictor of test 6, table 5 and test 6 table 6. The inversion is simple and the auto regressive equations for the plants are

$$y(t_i) = .62y(t_{i-1}) - 240u(t_{i-1}) + 440u(t_{i-2})$$

(24)

$$y(t_i) = .238y(t_{i}) + .02y(t_{i-2}) - 180u(t_{i-1}) + 240u(t_{i-2})$$

(25)

for test 6 and table 5 and test 6 and table 6 respectively.

In the above equations $y(\cdot)$ represents the response voltage across the cell and $u(\cdot)$ is the stimulating current. The coefficients of the last two terms on the right hand side thus acquire the physical dimension of resistance. The results are consistent with our qualitative expectation that the cell with the higher concentration should have the lower resistance.

The test results have shown that the self-tuning predictor does very well in reducing the prediction error of the aprior - "optimal" predictor. We have also found that the self-tuning predictors initialized with different parameter sets converge to the same values. This set of parameters when implemented as a non-adaptive predictor performs comparably to the self-tuning predictor and significantly better than the apriori - "optimal" predictor. The self-tuning predictor does not always reduce the accumulated square error, however it does reduce the average square error after an initial period of tuning.
CHAPTER 5

SUMMARY AND CONCLUSIONS

5.1 SUMMARY AND CONCLUSIONS

We have applied the self-tuning predictor to simulated data and to an actual laboratory application. The simulation results, for conditions of known and constant plant parameters, indicate that the prior calculated optimal predictor does in fact perform best. However there are sets of parameters other than the optimal sets that perform almost identically. The application results show that parameter estimates converge asymptotically to nearly identical (but generally time-varying) values for different initializations. Since the parameters describing the cell may not be constant, we do not expect the predictor parameters to converge to some constant value, and the time series data indicates that this is the case. The time dependence of the plant parameters is more of a factor in the cell with the Normal solution than with the cell containing the solution with the lower concentration.

To explain the above results we hypothesize the existence of points in parameter space representing the optimal parameter sets. (Refer to figure 13). If the underlying plant parameters are constant these optimal points are also constant; if the plant parameters are time-dependent then the optimal parameters will follow some time dependent trajectory through parameter space. For the simulated plant with constant parameters we calculated two distinct "optimal" predictors. These predictors represent distinct points in the parameter space. There exists a region connecting these points where the capability of non-adaptive predictors with parameter sets from this
region will be nearly optimal. Self-tuning predictors initialized with these parameter sets will show little if any tendency to converge to optimal values in the face of plant driving noise. Self-tuning predictors initialized with parameter sets outside this region will converge to apriori optimal parameter sets. Further investigation is necessary to determine whether the apriori "optimal" parameter sets represent extreme points within the "nearly optimal" set or central values of an "optimal" volume. The square error accumulated by the predictors will depend on some measure of their deviation from the optimal. Experiments have shown that most of the square error accumulated by the self-tuning predictor is incurred during the first few iterations of the algorithm. The self-tuning predictor learns quickly to reduce the prediction error to a minimum.

In conclusion, the self-tuning predictor is well suited for applications of the nature described herein. It has been shown that the self-tuning predictor reduces mean square error to a minimum, and its parameters converge to sets that are nearly optimal.

The inversion of the relationships between the plant parameters and the predictor parameters is not always possible. However we can always reduce the order of the predictor such that the inversion is simple.

5.2 AREAS FOR FUTURE RESEARCH

During the course of this research many areas of investigation had to be curtailed due to time constraints. Specifically we were unable to determine the effect different input groups would have on the cell parameters and on the identification of predictor parameters. In the interest of standardization for comparative purposes we restricted ourselves to one standard
input group (actually two different input groups were used; one for simulation and another for the electrode studies). The cell parameters are dependent on the input applied. In fact the dynamics of the input may cause time variation in the underlying plant parameters. Present theory on optimal input synthesis for identification is inadequate under these circumstances.

We studied the effect of a step change in the concentration of the salt solution in the cell. Future researchers may wish to study the system under dynamic variation, such as continuous change of solution concentration, or temperature.

The experimental configuration used to collect data employed the electrodes to both stimulate and record. Other arrangements, more closely realizing the pattern in figure one should also be considered. Three cells could be placed in series. The stimulating signal would be applied to the electrodes of the middle cell; and the response measured at the terminal of the outside cells.

The self-tuning predictor could also be used for identifying parameters of the Hall cell. Proper control of electrolytic reduction of aluminum, for instance, could be enhanced with identification of impedance parameters.
In this appendix we derive the optimal predictors for electrolytic cells with salt solutions of .01 and .1 normality given the impedance vs. frequency plots. We first derive optimal predictor for the electrolytic cell containing the .01 normal solution.

The first step is to match a transfer function of the form (23) to the data points. Figure 11a illustrates the match of the asymptotic Bode plot for a transfer function with one pole and one zero. The pole and zero frequencies of $Z(s)$ are:

$$f_p = 1 \text{Hz}$$
$$f_z = 4 \text{Hz}$$

$$Z(s) = \frac{\frac{s}{25} + 1}{\frac{s}{6.3} + 1} \quad K = 2000$$

The transfer function, $Z(s)$, is the ratio of the Laplace transforms of the voltage response to the stimulating current.

$$\frac{Y(s)}{U(s)} = Z(s)$$

The Laplace transform of the differential equation representing this system is

$$\left(\frac{s}{6.3} + 1\right)Y(s) = 2000\left(\frac{s}{25} + 1\right)U(s)$$

or

$$(s + 6.3)Y(s) = (500s + 12500)U(s)$$
A state space representation may be obtained using the procedure proposed by DeRouss [13].

\[ \dot{x}(t) = 6.3x(t) + 9350u(t) \]  
\[ y(t) = x(t) + 500u(t) \]  

(A3a)  

(A3b)

The sampled data version of (A3) is obtained by the transformation (6).

\[ x(n+1) = \hat{A}x(n) + \hat{B}u(n-1) \]  
\[ y(n) = \hat{C}x(n) + \hat{D}u(n-1) \]  

(A4)

Note: \( u(n-1) \) is used since the current calculated at sample time \( n \) is the current that was applied to the cell over the interval \([t_{i-1}, t_i]\).

\[ \hat{A} = e^{AT} = 1.0074 \]

\[ \hat{B} = \int_0^T e^{AT}B\,d\tau = 11 \]

\[ \hat{C} = C \]

\[ \hat{D} = D \]

\[ T = .0012 \]

\[ G(z) = \hat{C}(zI-\hat{A})^{-1}\hat{B} + \hat{D} \]

\[ = (z - 1.0074)^{-1}(11) + 500 \]

\[ \frac{y(n)}{u(n-1)} = \left[ \frac{11}{z-1.0074} + 500 \right] \frac{z^{-1}}{z^{-1}} = \hat{G}(z^{-1}) \]

\[ = \frac{11z^{-1}}{1-1.0074z^{-1}} + 500 \]
The estimated auto-regressive representation of the plant is:

\[ y(n)[1-1.0074z^{-1}] = 11z^{-1} + 500[1-1.0074z^{-1}] u(n-1) + v(n) \]

where \( v \) is added to incorporate the observation noise.

\[ y(n) = 1.0074y(n-1) + 500u(n-1) - 500u(n-2) + \epsilon(n) \quad (A4) \]

\[ A(z^{-1}) = (1-1.0074z^{-1}) \]
\[ B(z^{-1}) = (500z^{-1} - 500z^{-2}) \]
\[ C(z^{-1}) = 1 \]

Since \( k = 1 \)

\[ F(z^{-1}) = 1 \]

The identity (11) is approximately

\[ 1 = (1-z^{-1}) + z^{-1} g_0 \quad g_0 = 1 \]

The optimal predictor is given by (13):

\[ \hat{y}(t_{i+1}/t_i) = y(t_i) + (z-1) \frac{500z^{-1} - 500z^{-2}}{1-1z^{-1}} u(t_i) \]

Cancelling common factors of the last term on the right hand side:

\[ \hat{y}(t_{i+1}/t_i) = y(t_i) + 500u(t_i) - 500u(t_{i-1}) \]

The \( H \) vector for the one step predictor is

\[ H(t_i) = [-\hat{y}(t_i/t_{i-1}) y(t_i) y(t_{i-1}) u(t_i) u(t_{i-1})] . \]

The optimal \( Q \) vector based on the prior experimental data is then:
Calculation of the optimal predictor for the cell with \( \) normal solution follows the same procedure. \( \mathcal{Z}(s) \) in this case is proposed to have two poles and two zeros:

\[
\begin{align*}
\mathcal{f}_p &= .127 \text{ Hz} \\
\mathcal{f}_{z_2} &= 8 \text{ Hz} \\
\mathcal{f}_p &= 35 \text{ Hz} \\
\mathcal{f}_{z_2} &= 40 \text{ Hz}
\end{align*}
\]

\[
\mathcal{Z}(s) = K \frac{(\frac{s}{50} + 1)(\frac{s}{251} + 1)}{(\frac{s}{8} + 1)(\frac{s}{220} + 1)} \quad K=5620
\]

The match of the asymptotic Bode plot, for \( \mathcal{Z}(s) \), to the data points is illustrated in figure 11b.

The state space representation is derived to be:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1100 & -225 \end{bmatrix} x(t) + \begin{bmatrix} 4.16 \times 10^4 \\ -3.7 \times 10^6 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + 495 u(t) + v(t)
\end{align*}
\]

Since the ranks of \([B \ AB] \) and \([C \ CA] \) are both two the system is both controllable and observable.

The sampled data version of (6) is obtained by the transformation (6).
\[
\hat{G}(z) = \hat{C}(z I - \hat{A})^{-1} \hat{B} + \hat{Y}
\]

\[
= [1, 0] \begin{bmatrix}
  z^{-1} & -0.0012 \\
  1.16 & z^{-1} - 0.773
\end{bmatrix}^{-1} \begin{bmatrix}
  -13 \\
  -3.43 \times 10^{-3}
\end{bmatrix} + [495]
\]

This can be reduced to

\[
\hat{G}(z) = \frac{y(z)}{u(z)} = \frac{495 z^2 - 877 z + 396}{z^2 - 1.773 z + 0.8}
\]

and in terms of \(z^{-1}\)

\[
\hat{G}(z^{-1}) = \frac{y(z^{-1})}{u(z^{-1})} = \frac{495 - 877 z^{-1} + 396 z^{-2}}{1 - 1.773 z^{-1} + 0.8 z^{-2}}.
\]

The form \(A(z^{-1}) y(z^{-1}) = B(z^{-1}) u(z^{-1}) + \nu(z^{-1})\) is obtained recognizing

\[
A(z^{-1}) = (1 - 1.773 z^{-1} + 0.8 z^{-2})
\]

\[
B(z^{-1}) = 495 - 877 z^{-1} + 396 z^{-2}
\]

and incorporating an additive noise term, \(C(z^{-1}) = 1\).

Solving the identity (11)

\[
F(z^{-1}) = 1
\]

\[
G(z^{-1}) 1.773 - 0.8 z^{-1}
\]

The optimal predictor is now derived by solving (13).
\[ \hat{y}(t_{i+1}/t_i) = H(t_i)Q \]

where

\[ H(t_i) = \begin{bmatrix}
-\hat{y}(t_i/t_{i-1}) - \hat{y}(t_{i-1}/t_{i-2}) y(t_i) y(t_{i-1}) y(t_{i-2}) y(t_{i-3}) \\
u(t_i) u(t_{i-1}) u(t_{i-2}) u(t_{i-3}) u(t_{i-4})
\end{bmatrix} \]

and
\[ Q = [0 \ 0 \ 1.8 \ -1.8 \ 0 \ 0 \ 495 \ -877 \ 400 \ 0 \ 0]^T \]
subject

bioelectric source $v_s(t)$

electrodes

observations, $y(t)$

biological transmission medium

Figure 1a

Figure 1b

$R_1$: resistance of biological medium
$R_2$: resistance of electrode junction
$C$: capacitance of electrode junction
$R_3$: amplifier input resistance
$v_s(t)$: bioelectric source signal
$v_{e1}(t), v_{e2}(t)$: half-cell potentials due to electrode-electrolyte interface
$y(t)$: measured voltage
System Block Diagram

![System Block Diagram](image)

Filter Block Diagram

![Filter Block Diagram](image)

Figure 2

Figure 3
Figure 4
Distribution of Standard Random Noise Sequence

SAMPLE SIZE: 2000
SAMPLE MEAN: 0.00000
SAMPLE VARIANCE: 0.00008
NOTE: LOSS OF OPTIMAL PREDICTOR IS IDENTICAL TO THE SUMATION OF THE SQUARE OF THE NOISE

\[
\frac{1}{1040} \sum_{1040}^{2080} v^2(i) = 8.11 \times 10^{-3}
\]

\[
\frac{1}{1040} \sum_{1040}^{2080} \epsilon^2(i) = 8.12 \times 10^{-3}
\]

Figure 5
Accumulated Square Error
Test 25, Table II
Figure 6
Parameter Estimates vs. Time
Test 25, Table II
Figure 7
Parameter Estimates vs. Time
Test 25, Table II
Figure 8
Parameter Estimates vs. Time
Test 25, Table II
Figure 9
Electrolytic Cell Implementation Circuit
READ INPUT SERIES FROM DISK TO MEMORY, \( i = 1 \)

GET \( \phi(t_i) \) AND APPLY TO D/A CHANNEL

SAMPLE A/D CHANNELS \( c(t_i); d(t_i) \)

CALCULATE STIMULUS
\[
\nu(t_{i-1}) = \frac{c(t_i) - d(t_i)}{R_s}
\]

\( i = i + 1 \)

NO \( \{ i > 2080 \} \) YES

WRITE STIMULUS SERIES \( u(\cdot) \) AND RESPONSE SERIES \( y(\cdot) = d(\cdot) \) ON DISK

Figure 10
Electrolytic Cell Implementation Flow Chart
Figure 11
Log Impedence vs. Log Frequency

(a) 0.01 Normal Solution

(b) 0.1 Normal Solution
INITIALIZE FILTER DESIGN PARAMETERS, ESTIMATE OF PREDICTOR PARAMETERS, i = 1

READ STIMULUS AND RESPONSE SERIES FROM DISK TO MEMORY

CALCULATE PREDICTOR
\[ Y = \hat{y}(t_{i+1}/t_i) = H(t_i)Q(t_i) \]

Y = y(t_{i+1})
U = u(t_{i+1})

CALL IDENTIFICATION ROUTINE
ESTIMATE \( \hat{Q}(t_{i+1}) \)
UPDATE \( H(t_i) \)

i = i + 1

NO

i > 2080

YES

END

Figure 12
Identification Block Diagram
Q1 and Q2 are optimal parameter sets assuming constant plant parameters.

Q3 falls within region of near optimal performance, no convergence to optimal sets.

Q4 initial estimate outside region of near optimal performance... estimates of parameters converge to an optimal set.

Figure 13
Relation of predictor parameters in parameter space.
<table>
<thead>
<tr>
<th>Test Number</th>
<th>Actual Observation Noise Variance</th>
<th>Non-Adaptive Predictor Parameters</th>
<th>Accumulated Absolute Error</th>
<th>Accumulated Square Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>[0 0 -.25 -.5 0 0 1 0 0]</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>[.25 .5 -.25 -.563 -.25 -.25 1 0 .25 .5]</td>
<td>0.00052</td>
<td>0.00000</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>[.125 .25 -.25 -.532 -.125 -.125 1 .125 .25]</td>
<td>0.13353</td>
<td>0.00001</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>[1 1 1 1 1 1 1 1 1 1]</td>
<td>1228.7</td>
<td>1090.4</td>
</tr>
<tr>
<td>5</td>
<td>$8 \times 10^{-7}$</td>
<td>[0 0 -.25 -.5 0 0 1 0 0]</td>
<td>1.55089</td>
<td>0.00171</td>
</tr>
<tr>
<td>6</td>
<td>$8 \times 10^{-7}$</td>
<td>[.25 .5 -.25 -.563 -.25 -.25 1 .25 .5]</td>
<td>1.55091</td>
<td>0.00171</td>
</tr>
<tr>
<td>7</td>
<td>$8 \times 10^{-7}$</td>
<td>[.125 .25 -.25 -.532 -.125 -.125 1 .125 .25]</td>
<td>1.55441</td>
<td>0.00172</td>
</tr>
<tr>
<td>8</td>
<td>$8 \times 10^{-7}$</td>
<td>[1 1 1 1 1 1 1 1 1 1]</td>
<td>1228.8</td>
<td>1090.7</td>
</tr>
<tr>
<td>9</td>
<td>.008</td>
<td>[0 0 -.25 -.5 0 0 1 0 0]</td>
<td>155.1</td>
<td>17.13</td>
</tr>
<tr>
<td>10</td>
<td>.008</td>
<td>[.25 .5 -.25 -.563 -.25 -.25 1 .25 .5]</td>
<td>155.1</td>
<td>17.13</td>
</tr>
<tr>
<td>11</td>
<td>.008</td>
<td>[.125 .25 -.25 -.532 -.125 -.125 1 .125 .25]</td>
<td>155.1</td>
<td>17.133</td>
</tr>
<tr>
<td>12</td>
<td>.008</td>
<td>[1 1 1 1 1 1 1 1 1 1]</td>
<td>1510.6</td>
<td>1678.4</td>
</tr>
</tbody>
</table>

Note:

<table>
<thead>
<tr>
<th>Accumulated Absolute Noise</th>
<th>Accumulated Square Noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tests 1-4</td>
<td>0</td>
</tr>
<tr>
<td>Tests 5-8</td>
<td>1.551</td>
</tr>
<tr>
<td>Tests 9-12</td>
<td>155.1</td>
</tr>
</tbody>
</table>

Table I
Non Adaptive Predictor
Simulated Plant/Constant Parameters
### Table II

**Self-Tuning Predictor**

**Simulated Plant/Constant Parameters**

<table>
<thead>
<tr>
<th>Test Number</th>
<th>Actual Observation Noise Variance</th>
<th>Initial Estimate of Self-Tuning Predictor Parameters</th>
<th>Final Estimate of Parameters</th>
<th>Accumulated Absolute Error</th>
<th>Accumulated Square Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>0.0</td>
<td>[0 0 .25 .5 0 0 1 0 0]</td>
<td>[0 0 .25 .5 0 0 1 0 0]</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>14</td>
<td>0.0</td>
<td>[0 .25 .5 .25 .25 .25 .25 1 .25 .5]</td>
<td>[.25 .5 .25 .25 .25 .25 1 .25 .5]</td>
<td>.00095</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>0.0</td>
<td>[.125 .25 .25 .25 .25 .25 .25 1 .25 .5]</td>
<td>[.125 .125 .125 .125 .125 1 .125 .5]</td>
<td>.00125</td>
<td>0.0</td>
</tr>
<tr>
<td>16</td>
<td>0.0</td>
<td>[1 1 1 1 1 1 1 1 1]</td>
<td>[.013 .014 .167 .083 .319 .235 .999 .402 .470]</td>
<td>1.50167</td>
<td>236/4</td>
</tr>
<tr>
<td>17**</td>
<td>0.0</td>
<td>[1 1 1 1 1 1 1 1 1]</td>
<td>[.073 .078 .258 .069 .322 .238 .951 .315 .486]</td>
<td>17.02</td>
<td>4.5</td>
</tr>
<tr>
<td>18</td>
<td>$8 \times 10^{-7}$</td>
<td>[0 0 .25 .5 0 0 1 0 0]</td>
<td>[.005 .005 .256 .512 .062 .003 1 .002 .006]</td>
<td>1.57371</td>
<td>.00176</td>
</tr>
<tr>
<td>19</td>
<td>$8 \times 10^{-7}$</td>
<td>[.25 .5 .25 .25 .25 .25 .25 1 .25 .5]</td>
<td>[.238 .488 .299 .576 .240 .251 1 .246 .501]</td>
<td>1.57073</td>
<td>.00172</td>
</tr>
<tr>
<td>20</td>
<td>$8 \times 10^{-7}$</td>
<td>[.125 .25 .25 .25 .25 .25 .25 1 .25 .5]</td>
<td>[.117 .241 .257 .544 .125 .127 1 .125 .25 ]</td>
<td>1.5698</td>
<td>.00176</td>
</tr>
<tr>
<td>21</td>
<td>$8 \times 10^{-7}$</td>
<td>[1 1 1 1 1 1 1 1 1]</td>
<td>[.011 .012 .059 .082 .217 .169 .997 .790 .343]</td>
<td>3.08</td>
<td>2377</td>
</tr>
<tr>
<td>22</td>
<td>.008</td>
<td>[0 0 .25 .5 0 0 1 0 0]</td>
<td>[.064 .057 .267 .517 .069 .012 .975 .016 .015]</td>
<td>156.47</td>
<td>17.42</td>
</tr>
<tr>
<td>23</td>
<td>.008</td>
<td>[.25 .5 .25 .25 .25 .25 .25 1 .25 .5]</td>
<td>[.128 .6 .269 .259 .576 .325 .303 .378 .183 .617]</td>
<td>155.79</td>
<td>17.29</td>
</tr>
<tr>
<td>24</td>
<td>.008</td>
<td>[.125 .25 .25 .25 .25 .25 .25 1 .25 .5]</td>
<td>[.034 .310 .268 .545 .127 .148 .975 .106 .305]</td>
<td>156.28</td>
<td>17.35</td>
</tr>
<tr>
<td>25</td>
<td>.008</td>
<td>[1 1 1 1 1 1 1 1 1]</td>
<td>[.018 .064 .254 .318 .017 .002 .964 .107 .003]</td>
<td>163.39</td>
<td>24.83</td>
</tr>
<tr>
<td>26**</td>
<td>.008</td>
<td>[0 0 .25 .5 0 0 1 0 0]</td>
<td>[.039 .009 .263 .523 .01 .008 .983 .024 .001]</td>
<td>155.92</td>
<td>17.25</td>
</tr>
</tbody>
</table>

**Notes:**

1. Tests run with following design parameters:
   - $W = [0.0]$
   - $P = [0.51]$
   - $V = 0.09$
   - $A = 0.05$

2. For accumulated noise loss see note Table I
<table>
<thead>
<tr>
<th>Test Number</th>
<th>Actual Observation Noise Variance</th>
<th>Non Adaptive Predictor Parameters</th>
<th>Accumulated Absolute Error</th>
<th>Accumulated Square Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.0</td>
<td>([0 \ 0 \ -0.25 \ -0.5 \ 0 \ 0 \ 1 \ 0 \ 0 ])</td>
<td>110.67</td>
<td>11.44</td>
</tr>
<tr>
<td>41</td>
<td>0.0</td>
<td>([0.25 \ 0.5 \ -0.25 \ -0.563 \ -0.25 \ -0.25 \ 1 \ 0.25 \ 0.5 ])</td>
<td>110.67</td>
<td>11.44</td>
</tr>
<tr>
<td>42</td>
<td>0.0</td>
<td>([0.125 \ 0.25 \ -0.25 \ -0.562 \ -0.125 \ -0.125 \ 1 \ 0.125 \ 0.5 ])</td>
<td>111.64</td>
<td>11.50</td>
</tr>
<tr>
<td>48</td>
<td>(8 \times 10^{-7})</td>
<td>([0 \ 0 \ -0.25 \ -0.5 \ 0 \ 0 \ 1 \ 0 \ 0 ])</td>
<td>110.72</td>
<td>11.44</td>
</tr>
<tr>
<td>43</td>
<td>0.008</td>
<td>([0 \ 0 \ -0.25 \ -0.5 \ 0 \ 0 \ 1 \ 0 \ 0 ])</td>
<td>198.73</td>
<td>29.43</td>
</tr>
</tbody>
</table>

**TABLE III**

Non Adaptive Predictor
Simulated Plant/Time Series Trend
<table>
<thead>
<tr>
<th>Test Number</th>
<th>Test Observation Noise Variance</th>
<th>Initial Estimate of Self-Tuning Predictor Parameters</th>
<th>Final Estimate of Parameters</th>
<th>Accumulated Absolute Error</th>
<th>Accumulated Square Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>46: 0</td>
<td>0.0</td>
<td>[0 0 -0.25 -0.5 0 0 1 0 0]</td>
<td>1.25 0.204 2.39 -0.838 0.601 -0.467 0.917 -1.145 0.83</td>
<td>9.295</td>
<td>0.00003</td>
</tr>
<tr>
<td>45: 0</td>
<td>0.0</td>
<td>[0 0 -0.25 -0.5 0 0 1 0 0]</td>
<td>0.07 0.07 -0.32 -0.57 0.05 0.02 1.0 -0.07 -0.03</td>
<td>0.275</td>
<td>0.00005</td>
</tr>
<tr>
<td>46: 8 x 10^-7</td>
<td>0.0</td>
<td>[0 0 -0.25 -0.5 0 0 1 0 0]</td>
<td>0.02 0.01 -0.49 -0.67 -0.03 -0.03 1.0 -0.04 -0.03</td>
<td>2.07256</td>
<td>0.00035</td>
</tr>
</tbody>
</table>

Note:
1. For design parameters see Table 2.

Table IV
Self-Tuning Predictor
Simulated Plant/Time Series Trend
Table V
Electrode Studies
.01 Normal Solution
Table VI
Electrode Studies
.1 Normal Solution
REFERENCES


York, 1967.


