SIMPLE FLOW-REVERSIBLE MODELS FOR DYNAMICS
AND CONTROL OF HEAT EXCHANGERS

by

RAHMATALLAH SHOURESHI

B.S., Arya-Mehr University of Technology
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ABSTRACT

Simple models are required to handle nonlinear effects in
heat exchangers due to flow rate changes, and especially to
flow reversal transients, such as occur in loss-of-coolant
accidents. This thesis presents very simple models for the
dynamic behavior of such systems. Exact linear distributed
models have been presented by Takahashi (2), Paynter and
Takahashi (1), and Hsu and Gilbert (9). Approximate methods for
simplifying these models include Friedly (4) who presented an
asymptotic approximation which satisfies dynamic response of
such systems at both low and high frequencies; however, this
technique involves an infinite order model using a distributed
system.

The simple models of the present paper employ finite
state models or "Lumped Models"; and two, three, and four lump
heat exchanger models are discussed. Both dynamic and static
behaviors of these models are compared with exact results. In
addition, some results for flow reversals are shown.

Studying the results for these simple models shows that
to get the best agreement with exact solutions, a linear
combination of the intermediate outputs should be used. The
benefits of such an improvement are shown.
Finally, the monotonic parameters are calculated directly and they are compared with graphical results which are obtained by technique due to Paynter (8).

Thesis Supervisor: Henry M. Paynter
Title: Professor of Mechanical Engineering
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NOMENCLATURE

A  Heat transfer area (m²)

A_s  Cross sectional area for each fluid (m²)

\( \frac{U \cdot A}{M \cdot C} \) (dimensionless)

\( a_1 \)

\( \frac{U \cdot A}{M \cdot C} \) (dimensionless)

\( a_2 \)

C  Specific heat transfer at constant pressure (\( \frac{\text{cal}}{\text{kg} \cdot \text{sec}} \))

G(s)  Transfer function (dimensionless)

\( |G| \)  Gain (dimensionless)

\( |G|_N \)  Normalized gain (dimensionless)

\( j \)  \( \sqrt{-1} \)

K  Gain, used for temperature in improved case (dimensionless)

L  Total length of heat exchanger's tubes or shell (m)

L_n  Length of nth section (m)

M  Amount of mass accumulated in each section (kgm)

M'  Mass flow rate (kgm/sec)

N  Number of Lumped-Models (dimensionless)

Q'  Total heat transfer per unit time (cal/sec)

r  Ratio of velocities of two fluids (dimensionless)

s  Laplace transform variable with respect to \( \Theta_s \) (dimensionless)

T_i  Temperature of ith fluid (°C), (i=1,2,...)
$T^n$  Temperature of nth section (°C)

$t$  Time (sec)

$t_n$  Time during that fluid goes through nth section (sec)

$U$  Overall heat transfer coefficient (cal/m$^2$.sec.C)

$V$  Velocity of fluid (m/sec)

$W_{sh}^*$  Shear work per unit time (joul/sec)

$\alpha$  Temperature factor, in stirred tank assumption (dimensionless)

$\theta$  Phase angle (deg.)

$\theta_0$  Dimensionless time

$\rho$  Density of fluid (kg/m$^3$)

$\varphi$  Distribution coefficient for heat transfer (dimensionless)

$\gamma$  $\varphi.a_1$ or $\varphi.a_2$

$\omega$  Frequency (dimensionless)

**Subscripts:**

1  Supply flow (control agent), tube side

2  Demand flow (controlled medium), shell side

$c$  Cold fluid

$H$  Hot fluid

$i$  Inlet

$n$  nth section

$o$  Outlet
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SECTION I

INTRODUCTION

One of the problems which recently arisen is accounting for thermal lag in fluid systems; in the general case, such systems contain heat capacitance, which produce significant effects in many situations such as mechanical, chemical, and aeronautical applications where precise temperature control is very important. Temperature-control systems in air-craft are subject to extreme environmental variation. The controls must be designed to adjust quickly to these changes in ambient conditions so as to deliver an air stream without excessive temperature fluctuation. The choice among proposed control systems which achieve a required steady state is based on their transient operation. It is then desirable that the analysis of systems be carried quite far with paper and pencil alone leaving a minimum of adjustment to be made on a working model. However, at present, temperature control problems are solved mainly through costly experimentation on models of proposed systems.

To obtain design information without experiment, equations describing the transient operation of the separate parts of a
proposed system must be solved. A major difficulty is that the equations for many of these parts are so complicated that only very rough approximate methods have been available for their solutions. One such general system which is used in almost all areas is the heat exchanger. Upon changing physical or chemical conditions of the working fluids flows in such a system, it is then important to know what is happening in different parts of industrial or power plants, etc., containing such heat exchangers. Therefore, it would be very useful to determine sensitivity of heat exchangers response upon changing different parameters.

Finally, it should be mentioned that the engineer frequently must simplify the basic scientific picture to make it more useful for practical application. Calculations, which render approximate results but at the same time allow a rapid survey over a wide range of conditions and assumptions, are important in the approach to an engineering problem. They serve as timesavers in that they confine the more detailed investigations to a smaller numerical range. The purpose of this thesis is to explore the task of finding simpler forms of the exact equations for dynamic response of heat-exchangers.
A PICTURE OF PRECEDING ATTEMPTS FOR THIS PROBLEM:

In the literature, dynamic performance of many heat exchangers of various configurations have been explored in very great detail. Takahashi (2) presented transfer function analysis of heat exchangers processes in 1952, then Paynter and Takahashi (1) gave a new method of evaluating dynamic response of heat exchangers, Hsu and Gilbert represented the same results of Takahashi in 1966. Wen-Jei Yang (17) has produced an analysis of transient heat transfer in a vapor-heated heat exchanger with arbitrary time-wise variant flow perturbation. Myers and others (16),(18) analysed the transient response of cross-flow heat exchangers, evaporators and condensers. Rea and Ablow (13) presented a model for transient air temperatures in a duct. They investigated experimentally and theoretically a thin-walled duct carrying heated air, and they found the duct wall is shown to be an important heat reservoir. Rizika (14) produced a method to find the thermal lag in systems such as heat exchanger and pipes. Dusinbere (19) showed a numerical methods for calculation of transient temperatures in pipes and heat exchangers. Finally Friedly (4) presented an asymptotic approximation for exact solution of heat exchangers. His method is useful both at high and low frequencies. All
of these researches and evaluations are based on Profos(7) method of evaluating such systems, and his operator is still used for this problem.
Takahashi (2) used the Profos(7) operator to solve the heat exchanger problem as follow:

\[
T + \frac{\partial T}{\partial t} dt + \frac{\partial T}{\partial x} dx
\]

From energy balance (applied for above element) following formula is gained:

\[
\frac{\gamma T}{\gamma t} + \sigma \frac{\partial T}{\partial x} = A (\varphi - T)
\]

Where \( \varphi \) is the surface temperature of solid.

By applying this result for the heat exchanger of Fig. (2-2), the following results are produced:
These results are based on the assumption that both fluids are unmixed, solid capacities are neglected, and system parameters are constant.

By assuming that the "hot fluid" (tube side) of temperature $T_1$ is the control agent (supply side), and the "cold fluid" (shell side), fluid of temperature $T_2$ is the controlled medium (demand side), the transfer function is:

\[
\begin{align*}
\frac{dT_1}{dt} + \frac{T_1}{X} &= a_1(T_2 - T_1) \\
\frac{dT_2}{dt} + \frac{T_2}{X} &= a_2(T_1 - T_2)
\end{align*}
\]

[*Plus sign is for parallel flow, minus sign is for counter-flow.*]
Then the transfer functions are in the following form:

i- For the parallel flow:

\[
G(j\omega) = \frac{g_2}{P_1-P_2} \left( e^{P_1} - e^{P_2} \right)
\]  (2-1)

Where

\[
P_1, P_2 = \left[ \frac{-(f_1+f_2) \pm \sqrt{(f_1-f_2)^2+4g_1g_2}}{2} \right]
\]  (2-2)

ii- For the counter flow:

\[
G(j\omega) = \frac{g_2(1-e^{P_2})}{-(P_1-P_2 e^{P_2}) + \frac{P_2}{2} (1-e^{P_2})}
\]  (2-3)

Where

\[
P_1, P_2 = \left[ \frac{-(f_1-f_2) \pm \sqrt{(f_1+f_2)^2-4g_1g_2}}{2} \right]
\]

The parameters are defined as follow:

\[
f_1 = a_1 + j\Omega
\]

\[
f_2 = a_2 + jr\Omega
\]

\[
g_1 = a_1
\]

\[
g_2 = a_2
\]

Friedly's approximate method is based on Schöen's\(^{(4)}\) approach to the dynamics of double-pipe heat exchangers, which is in the general form of:

\[
\tilde{G}(s) = \frac{m+Ts}{1+Ts} \times \frac{1-e^{(\alpha_1+\beta_1)s}}{\alpha_2+\beta_2s}
\]  (2-5)

Where \(\alpha_1, \beta_1\) are constants which result from the high-frequency limit of the exact transfer function and \(m\) results from the low frequency limit. The time constant \(T\) is arbitrary and
adjusted to match the exact frequency response as well as possible. Friedly retained the delay time but eliminated the adjustable time constant. Then his approximate form is:

\[ \tilde{G}(s) = \frac{K}{1 + \frac{T}{\frac{1}{\beta_2}}} \left[ 1 - e^{-\left(\alpha_1 + \beta_2 s\right)} \right] \quad (2-6) \]

Where:

\[ \frac{K}{T} = \frac{1}{\beta_2}, \quad K = \frac{m}{\alpha_2} \]

In other words, he expanded the denominator of exact solution and neglected the terms of order two and more; therefore he got:

\[ G(x, s \to \infty) \rightarrow \frac{a_2 e^{a_2(x-1)}}{(1+r)s} \left[ e^{r(x-1)s} - e^{-(a_1 + a_2)}e^{-(a_1 + a_2)x} + \ldots \right] \]

\[ G(x, 0) = a_2 \frac{e^{(a_2 - a_1)x} - 1}{a_1 e^{(a_2 - a_1)x} - a_1} \]

Therefore his approximate transfer function is:

\[ \tilde{G}(x, s) = \frac{K}{1 + TS} \left[ e^{-r(x-1)s} - e^{-(a_1 + a_2)x} \right] \quad (2-7) \]

Where:

\[ \frac{K}{T} = \frac{a_2}{1+r} e^{a_2(x-1)} \]

\[ K = \frac{a_2}{1 - e^{-(a_1 + a_2)x}} \times \frac{e^{(a_2 - a_1)x} - 1}{a_2 e^{a_2} - a_1} \]

But as the results show, this method is infinite order and Friedly used a distributed system model.
It should be mentioned that after presenting the exact solution, Paynter and Takahashi (1) created another method for evaluating which based on monotonic systems that Paynter (3) presented. As Paynter says: The Laplace transform solution of a monotone process can be written as:

\[ G(S) = e^{-T_m S + \frac{T_1}{2} S^2 - \frac{T_2}{3} S^3 + \ldots} \]

Where the parameters \( \xi, T_m, T_1, T_2 \) are given in terms of system constants, the symbol \( S \) is the complex variable of the Laplace transformation. In summary, \( \xi \) measures the steady state amplitude ratio between response and disturbance, \( T_m \) measures the mean time delay between response and disturbance, \( T_1 \) defines the dispersion or attenuation, and \( T_2 \) the asymmetry or phase nonlinearity. This characterization is very efficient for any physical process where the step response is monotonic and nondecreasing in time.

If the Laplace transformation of system is in the form of

\[ G(S) = \frac{1}{1 + a_1 S + a_2 S^2 + a_3 S^3 + \ldots} = \frac{1}{\sum_{k=0}^{\infty} a_k S^k} \]  

(2-8)

Where the denominator polynomial may be either finite or infinite, then the monotone parameters can be obtained directly by following relationships:
These monotonic parameters are obtained by this method for simple model of this present paper.
SECTION III

ANALYSIS AND FORMATION OF HEAT EXCHANGER PROBLEM:

The dynamic performance of a cocenteric pipe heat exchanger could be characterized by solution of four simultaneous nonlinear partial differential equations, the assumptions for this analysis are:

1-The heat flow and temperature distribution are functions of time and axial distance from tube inlet.

2-Both the inner radius and the outer radius of the tube are assumed constant.

3-The tube material is homogeneous and isentropic, the density and the specific heat are constant.

4-The thermal conductivity of tube material is zero in the axial direction; the thermal conductivity of the tube material is considered infinite in the radial direction, this condition valid for thin metal walls.

5-There is no energy source within the tube material itself.

6-The thermal conductivity in the outer wall in the longitudinal direction is zero and in the transverse direction is finite, a condition valid for thick insulated walls.

7-The film coefficients of heat transfer between the fluid and tube material are uniform and constant over the inner
and outer tube surfaces.

8- The specific heats at constant pressure of both the inner and outer fluids are assumed to be constant.

9- The fluid pressure at any section is independent of time for both inner and outer fluids.

Nonlinear partial differential equations are:

Heat balance, for inner fluid:

$$\frac{\partial T_i}{\partial t} + \frac{V_i}{\frac{\partial T_i}{\partial x}} = \frac{U_i A_i}{M_i C_i} (T_i - T_{in})$$

Heat balance, inner wall:

$$\frac{\partial T_{in}}{\partial t} = \frac{U_i A_i}{M_i C_i} (T_i - T_{in}) + \frac{U_2 A_2}{M_2 C_2} (T_o - T_{in})$$

Heat balance, outer fluid:

$$\frac{M_i}{M_2} \left( \frac{\partial T_o}{\partial t} + \frac{V_o}{\frac{\partial T_o}{\partial x}} \right) = \frac{U_2 A_2}{M_2 C_2} (T_o - T_{in}) + \frac{V_o A_2}{M_2 C_2} (T_o - T_o)$$

Heat balance, outer wall:

$$\frac{\partial T_o}{\partial t} = \frac{\alpha_s \theta_i}{r^2} x \frac{\partial^2 T_o}{\partial y^2}$$

[*Positive sign is for parallel-flow, negative sign is for counter-flow.]

These results can be gotten by Profos(7) operator, in
appendix(a) there are the manners of getting these formulas. These formulas are for general cases that the thermal storage of walls aren't negligible.

Analysis the problem consists of two parts: Static case, and Dynamic response, as follow:

**STATIC CASE OF HEAT EXCHANGERS:**

Based on basic assumptions and neglecting the thermal storage of walls, for steady state case, from energy balance (appendix-a), the results are:

for counter-flow

\[
\begin{align*}
M_1 C_1 \frac{dT_1}{dX} &= +P \\
M_2 C_2 \frac{dT_2}{dX} &= +P
\end{align*}
\]  

(3-1)

FIGURE(3-1): Counter Flow Heat Exchanger
A model for this counter-flow case is shown in figure (3-2):

\[ M_2 C_2 \frac{dT_b}{d\chi} = -P \]
\[ M_1 C_1 \frac{dT_c}{d\chi} = -P \]

\[ P = AU(T_b - T_c) \]

FIGURE (3-2): Model for Counter-Flow Heat Exchanger

Since the objective of this thesis is to reduce the mathematical model to a set of simultaneous ordinary differential equations, the system will be "finite differenced" or "lumped".

The exchanger is divided into \( N \) sections (Fig.3-3) with each section on tube side corresponding to a section on the shell side.

FIGURE (3-3): Lumped Heat Exchanger
For each lump the Profos(7) operator is used then:

\[ T_k s \bar{T}_{hk} = \frac{T_{hk-1} - T_{hk}}{s} - \gamma_{hk} (\bar{T}_{hk} - \bar{T}_{cj}) \]  

**Hot Side**

\[ T_j s \bar{T}_{cj} = \frac{T_{cj-1} - T_{cj}}{s} + \gamma_{cj} (\bar{T}_{hk} - \bar{T}_{cj}) \]  

**Cold Side**

Calculations to get these results are in appendix(c). For static case (steady-state) they are:

\[
\begin{align*}
T_{hk-1} - T_{hk} - \gamma_{hk} (T_{hk} - T_{cj}) &= 0 \\
T_{cj-1} - T_{cj} + \gamma_{cj} (T_{hk} - T_{cj}) &= 0
\end{align*}
\]

(3-2)

Therefore, it depends on number of lumps, and it is clear that by increasing this number better results will be gotten.

For \( N=2 \):

- **Parallel Flow Heat Exchanger**:

![Diagram of a Parallel Flow Heat Exchanger](image)

**FIGURE(3-4): Two Lumped Models Heat Exchanger (Parallel)**

Calculations are in appendix(c) which give following
results by disturbing only inlet temperature of hot side fluid:

\[
\frac{T_{c3}}{T_{h1}} = \frac{\varphi_2 a_2(1 + \varphi_1 a_1) + \varphi_2 a_2(1 + \varphi_1 a_1)}{(1 + \varphi_1 a_1 + \varphi_2 a_2)(1 + \varphi_2 a_1 + \varphi_2 a_2)}
\]

\[
\frac{T_{h2}}{T_{h1}} = \frac{(1 + \varphi_2 a_2)(1 + \varphi_1 a_2) + \varphi_1 a_2 a_2}{(1 + \varphi_1 a_1 + \varphi_2 a_2)(1 + \varphi_2 a_1 + \varphi_2 a_2)}
\]

\[
\frac{T_{b2}}{T_{h1}} = \frac{1 + \varphi_2 a_2}{(1 + \varphi_1 a_1)(1 + \varphi_2 a_2) - \varphi_1^2 a_1 a_2}
\]

\[
\frac{T_{c2}}{T_{h1}} = \frac{\varphi_1 a_2}{(1 + \varphi_1 a_1)(1 + \varphi_2 a_2) - \varphi_1^2 a_1 a_2}
\]

**ii- Counter Flow Heat Exchanger:**

![Diagram of Counter Flow Heat Exchanger](image)

**FIGURE(3-5): Tow Lumped Models Heat Exchanger (counter-flow)**

The beauty of this model is that for counter-flow only one of the fluids is reversed and everything is the same:

\[
\frac{T_{c3}}{T_{h1}} = \frac{\varphi_2 a_2(1 + \varphi_1 a_1 + \varphi_2 a_2) + \varphi_2 a_2}{(1 + \varphi_1 a_1 + \varphi_2 a_2)(1 + \varphi_2 a_1 + \varphi_2 a_2) - \varphi_1^2 a_1 a_2}
\]
Now, for different values of $a_1$, $a_2$ we can compare the results of this model and results of exact solution, but by changing $q_1$ and $q_2$ the optimum results can be gained. Table (3-1) through (3-4) contain the results of different values of $a_1$, $a_2$, $q_1$, and $q_2$.

For $N=3$:

i-Parallel Flow: As shown following, for this case there are six equations for six outlet temperatures as shown in Fig. (3-6):

Equations are:

\[
\begin{align*}
\frac{T_{w3}}{T_{w1}} &= \frac{(1+q_1 a_2)(1+q_2 a_2)}{(1+q_1 a_1+q_2 a_2)(1+q_2 a_1+q_2 a_2)-q_2 a_2 a_2} \\
\frac{T_{c2}}{T_{w1}} &= \frac{q_2 a_2 (1+q_2 a_2)}{(1+q_1 a_1+q_2 a_2)(1+q_2 a_1+q_2 a_2)-q_2 a_2 a_2} \\
\frac{T_{w2}}{T_{w1}} &= \frac{(1+q_1 a_2)(1+q_2 a_1+q_2 a_2)}{(1+q_1 a_1+q_2 a_2)(1+q_2 a_1+q_2 a_2)-q_2 a_2 a_2}
\end{align*}
\]
Therefore, results for disturbing only inlet temperature of hot fluid \( T_{H1} \) are:

\[
\begin{align*}
\frac{T_{C4}}{T_{H1}} &= \frac{q_3 a_3 (1 + q_3 a_3) (1 + q_3 a_3) + q_2 (1 + q_2 a_2) (q + q_2 a_2 + q_3 a_3 + q_4 a_4)}{\prod_{i=1}^{3} (1 + q_i a_i + q_i a_i)} \\
\frac{T_{H3}}{T_{H1}} &= \frac{\prod_{i=1}^{3} (1 + q_i a_i + q_i a_i) + q_3 a_3 (q_2 + q_3 a_3 + q_4 a_4 + q_5 a_5) + q_2 a_2 a_2 (1 + q_3 a_3)}{\prod_{i=1}^{3} (1 + q_i a_i + q_i a_i)}
\end{align*}
\]
\[\frac{T_{C3}}{T_{H1}} = \frac{\phi_2 (\phi_1 + \phi_2 + \phi_2 a_1 + \phi_2 a_2)}{(1 + \phi_2 a_1 + \phi_2 a_2)(1 + \phi_2 a_1 + \phi_2 a_2)}\]

\[\frac{T_{H3}}{T_{H1}} = \frac{(1 + \phi_1 a_2)(1 + \phi_2 a_2) + \phi_1 \phi_2 a_2 a_2}{(1 + \phi_2 a_1 + \phi_2 a_2)(1 + \phi_2 a_1 + \phi_2 a_2)}\]

\[\frac{T_{C2}}{T_{H1}} = \frac{\phi_1 a_2}{1 + \phi_1 a_1 + \phi_1 a_2}\]

\[\frac{T_{H2}}{T_{H1}} = \frac{1 + \phi_1 a_2}{1 + \phi_1 a_1 + \phi_1 a_2}\]

ii- For Counter-Flow (N=3): As shown in following, these six equations are for this case (see Fig.3-7):

\[\text{FIGURE(3-7): Three Lumped Models Heat Exchanger(counter-flow)}\]
Results for disturbing \( T_{H1} \) are:

\[
\begin{align*}
\frac{T_{E5}}{T_{H1}} &= \frac{\varphi A_2 (1 + \varphi_2 A_2) (1 + \varphi_2 A_1) + (1 + \varphi_2 A_2) (1 + \varphi_2 A_3) + (1 - \varphi_2 A_2) (1 + \varphi_2 A_4) + 1}{(1 + \varphi_2 A_2 + \varphi_4 A_2) (1 + \varphi_2 A_1 + \varphi_4 A_1) (1 + \varphi_2 A_3 + \varphi_4 A_3) (1 + \varphi_2 A_4 + \varphi_4 A_4)} \times \frac{1}{a_2} \frac{1}{a_2} \left( \frac{1}{\varphi_2 A_5 + \varphi_4 A_5 + \varphi_6 A_6 + \varphi_8 A_8} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{T_{H4}}{T_{H1}} &= \frac{(1 + \varphi_2 A_2 - \varphi_2 A_2)}{(1 + \varphi_2 A_2 + \varphi_4 A_2) (1 + \varphi_2 A_1 + \varphi_4 A_1) (1 + \varphi_2 A_3 + \varphi_4 A_3) (1 + \varphi_2 A_4 + \varphi_4 A_4)} \times \frac{1}{a_2} \frac{1}{a_2} \left( \frac{1}{\varphi_2 A_5 + \varphi_4 A_5 + \varphi_6 A_6 + \varphi_8 A_8} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{T_{C3}}{T_{H1}} &= \frac{A_2 (1 + \varphi_2 A_2) (1 + \varphi_2 A_3) (1 + \varphi_2 A_4)}{(1 + \varphi_2 A_2 + \varphi_4 A_2) (1 + \varphi_2 A_1 + \varphi_4 A_1) (1 + \varphi_2 A_3 + \varphi_4 A_3) (1 + \varphi_2 A_4 + \varphi_4 A_4)} \times \frac{1}{a_2} \frac{1}{a_2} \left( \frac{1}{\varphi_2 A_5 + \varphi_4 A_5 + \varphi_6 A_6 + \varphi_8 A_8} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{T_{H3}}{T_{H1}} &= \frac{(1 + \varphi_2 A_2) (1 + \varphi_2 A_3) (1 + \varphi_2 A_4 + \varphi_3 A_4)}{(1 + \varphi_2 A_2 + \varphi_4 A_2) (1 + \varphi_2 A_1 + \varphi_4 A_1) (1 + \varphi_2 A_3 + \varphi_4 A_3) (1 + \varphi_2 A_4 + \varphi_4 A_4)} \times \frac{1}{a_2} \frac{1}{a_2} \left( \frac{1}{\varphi_2 A_5 + \varphi_4 A_5 + \varphi_6 A_6 + \varphi_8 A_8} \right)
\end{align*}
\]
\[
\frac{T_{C2}}{T_{H1}} = \frac{\alpha_3 a_2 (1 + \eta_1 a_2) (1 + \eta_2 a_2) (1 + \eta_3 a_2) (1 + \eta_4 a_2)}{(1 + \eta_1 a_2 + \eta_2 a_2) (1 + \eta_3 a_2 + \eta_4 a_2) (1 + \eta_5 a_2) - a_2 \sum_{i=1}^{4} (\eta_2 \eta_3 \eta_4 + \eta_3 \eta_4 \eta_5 + \eta_4 \eta_5 \eta_1) a_2}
\]

\[
\frac{T_{H2}}{T_{H1}} = \frac{(1 + \eta_1 a_2 + \eta_2 a_2) (1 + \eta_3 a_2) (1 + \eta_4 a_2) (1 + \eta_5 a_2) - a_2 \sum_{i=1}^{4} (\eta_2 \eta_3 \eta_4 + \eta_3 \eta_4 \eta_5 + \eta_4 \eta_5 \eta_1) a_2}{(1 + \eta_1 a_2 + \eta_2 a_2) (1 + \eta_3 a_2 + \eta_4 a_2) (1 + \eta_5 a_2) - a_2 \sum_{i=1}^{4} (\eta_2 \eta_3 \eta_4 + \eta_3 \eta_4 \eta_5 + \eta_4 \eta_5 \eta_1) a_2}
\]

The results for different values of \(a_1, a_2, \eta_1, \eta_2, \) and \(\eta_3\) are in tables (3-1) through (3-4).

For \(N=4:\)

i- Parallel flow heat exchanger, for eight outlet temperatures as shown in Fig. (3-8) there are eight equations which can be obtained by using the energy balance for each section.

\[
\begin{cases}
T_{H1} - (1 + \eta_1 a_2) T_{H2} + \eta_2 a_2 T_{C2} = 0 \\
\eta_3 a_2 T_{H2} + T_{C1} - (1 + \eta_4 a_2) T_{C2} = 0
\end{cases}
\]

FIGURE(3-8): Four Lumped Model Heat-Exchanger(parallel-flow)
\[
\begin{align*}
T_{c4} &= \frac{a_2^3 \prod_{i=1}^{2} [(q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2 + (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2 + (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2 + (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2] + a_1 a_2 a_3 a_4 (1+q_{a_1}+q_{a_2})}{\prod_{i=1}^{4} (1+q_{a_i}+q_{a_2})} \\
T_{c5} &= \frac{a_2 (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4}) + a_2 (1+q_{a_i}+q_{a_2} + q_{a_3} + q_{a_4}) + q_{a_i} q_{a_2} (1+q_{a_2})}{\prod_{i=1}^{3} (1+q_{a_i}+q_{a_2})} \\
T_{c6} &= \frac{a_2 (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})}{\prod_{i=1}^{4} (1+q_{a_i}+q_{a_2})} \\
T_{c7} &= \frac{a_2 (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})}{\prod_{i=1}^{4} (1+q_{a_i}+q_{a_2})} \\
T_{c8} &= \frac{a_2 (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})}{\prod_{i=1}^{4} (1+q_{a_i}+q_{a_2})}
\end{align*}
\]

The results for disturbing \( T_{II} \) are:

\[
\begin{align*}
\frac{T_{c5}}{T_{II}} &= \frac{a_2 \prod_{i=1}^{2} [(q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2 + (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2 + (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2 + (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})^2] + a_1 a_2 a_3 a_4 (1+q_{a_1}+q_{a_2})}{\prod_{i=1}^{4} (1+q_{a_i}+q_{a_2})} \\
\frac{T_{c6}}{T_{II}} &= \frac{a_2 (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4}) + a_2 (1+q_{a_i}+q_{a_2} + q_{a_3} + q_{a_4}) + q_{a_i} q_{a_2} (1+q_{a_2})}{\prod_{i=1}^{3} (1+q_{a_i}+q_{a_2})} \\
\frac{T_{c7}}{T_{II}} &= \frac{a_2 (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})}{\prod_{i=1}^{4} (1+q_{a_i}+q_{a_2})} \\
\frac{T_{c8}}{T_{II}} &= \frac{a_2 (q_{a_i} + q_{a_2} + q_{a_3} + q_{a_4})}{\prod_{i=1}^{4} (1+q_{a_i}+q_{a_2})}
\end{align*}
\]
ii- For Counter-Flow ($N=4$): from appendix(e) which shows how to get the equations, and Fig. (3-9) the following results obtained:

\[
\frac{T_{C2}}{T_{H1}} = \frac{q_1 a_2}{1 + q_1 a_1 + q_1 a_2} \\
\frac{T_{H2}}{T_{H1}} = \frac{1 + q_2 a_2}{1 + q_2 a_1 + q_2 a_2}
\]

\[
\begin{align*}
T_{H1} - (1 + q_1 a_1) T_{H2} + q_1 a_1 T_{C5} &= 0 \\
q_1 a_2 T_{H2} + T_{C4} - (1 + q_2 a_2) T_{C5} &= 0 \\
T_{H2} - (1 + q_2 a_1) T_{H3} + q_2 a_1 T_{C4} &= 0 \\
q_2 T_{H3} a_2 + T_{C3} - (1 + q_2 a_2) T_{C4} &= 0 \\
T_{H3} - (1 + q_3 a_1) T_{H4} + q_3 a_1 T_{C3} &= 0 \\
q_3 a_2 T_{H4} + T_{C2} - (1 + q_3 a_2) T_{C3} &= 0
\end{align*}
\]

FIGURE (3-9): Four Lumped Models Heat-Exchanger (Counter-Flow)
Therefore, the outlet temperatures by disturbing $T$ are:

\[
\frac{T_{c5}}{T_{n1}} = \frac{q_2 a_2 (1 + q_2 a_1)}{1 + q_2 a_1} + \frac{1}{1 + q_2 a_1} \frac{T_{c5}}{T_{n1}}
\]

\[
\frac{T_{c4}}{T_{n1}} = \frac{1 - q_2 a_2 (1 + q_2 a_1)}{1 + q_2 a_1} + \frac{3}{1 + q_2 a_1} \frac{T_{c5}}{T_{n1}}
\]

\[
\frac{T_{c3}}{T_{n1}} = \frac{1 - q_2 a_2 + q_2 a_1}{(1 + q_2 a_1)(1 + q_2 a_1)} + \frac{1}{(1 + q_2 a_1)(1 + q_2 a_1)} \frac{T_{c5}}{T_{n1}}
\]

\[
\frac{T_{c2}}{T_{n1}} = \frac{1}{1 + q_2 a_1} + \frac{q_2 a_1}{1 + q_2 a_1} \frac{T_{c5}}{T_{n1}}
\]
Adding the number of Lumped elements gives a better solution compare with exact results, but if only a simple model is desired then only two, three, and four Lumped elements need to be used, as shown later, by choosing good values for the distribution coefficients \( T_f \) & \( T_a \) results will be very close to exact results.

The restrictive assumptions which are shown at the beginning of this analysis may be omitted and the solution will appear for the more general case. This solution can be realized by virtue of the Lumped Model approach via appropriate choice of parameters. Quantitive effects of variable heat transfer coefficients, existance of heat capacitors, heat interaction with ambient or any other cases can be practically obtained only by getting the results of a few equations in a very simple form.

The results of steady state(static) case are tabulated and plotted. In appendix(a) the exact solutions are presented, for steady state case, and in Fig.(3-12a) through(3-13b) the results of exact method and several Lumped Models are shown, for the static case.

Because of differences between exact results and the Lumped Models, fitting the Lumped Model parameters via the
exact solution yields substantial improvement. First the parallel flow case is fitted and its effects on counter flow are shown, then the results of fitting counter flow and its effects on parallel flow are discussed.

The exact equations for static cases (as they are presented in appendix a) are:

**i- For Parallel Flow:**

\[
\frac{T_{10}(t)}{T_{ii}} = \frac{a_2 \left[ 1 - e^{-(a_1+a_2)x} \right]}{a_1 + a_2} \quad (3-3)
\]

\[
\frac{T_{ii}(t)}{T_{ii}} = \frac{a_2 + a_1 e^{-(a_1+a_2)x}}{a_1 + a_2} \quad (3-4)
\]

**ii- For Counter Flow:**

\[
\frac{T_{ii}(t)}{T_{ii}} = \frac{a_1 e^{-(a_1-a_2)x} - a_2 e^{-(a_1-a_2)x}}{a_1 - a_2 e^{-(a_1-a_2)x}} \quad (3-5)
\]
Now, for two special cases, the results of static case are computed and plotted:

1) For case: \( a_1 = 4 \), \( a_2 = 1 \)

\[ T_2 = T_C \text{ (cold fluid)} \quad T_1 = T_H \text{ (hot fluid)} \]

Parallel Flow

\[
\frac{T_{c(x)}}{T_{Hc}} = \frac{1 - e^{-5x}}{5} \quad (3-7)
\]

Counter Flow

\[
\frac{T_{H(x)}}{T_{Hc}} = \frac{1 + 4e^{-5x}}{5} \quad (3-8)
\]

\[
\frac{T_{c(x)}}{T_{Hc}} = \frac{4e^{3x} - e^3}{4 - e^3}
\]

\[
\frac{T_{H(x)}}{T_{Hc}} = \frac{e^{3x} - e^3}{4 - e^3}
\]
2) For case: \(a_1 = 1, a_2 = 1\)

\[
\begin{align*}
\frac{T_{c1}(\alpha)}{T_{Wc}} &= \frac{1}{2} \left( 1 - e^{2\alpha} \right) \\
\frac{T_{in}(\alpha)}{T_{Wc}} &= \frac{1}{2} \left( 1 + e^{2\alpha} \right) \\
\frac{T_{c2}(\alpha)}{T_{Wc}} &= \frac{1}{2} \left( 1 - \alpha \right) \\
\frac{T_{in}(\alpha)}{T_{Wc}} &= 1 - \frac{1}{2} \alpha
\end{align*}
\] (3-9) (3-10)

The results of these two special cases are plotted in Fig. (3-12a) through (3-13b), as they indicated these results bear a very good agreement with Lumped Model results.

For making clear and easier comparison, the results of static cases for outlet temperatures are plotted respect to number of Lumped sections. For determining the approximate slopes of these results, they are plotted respect to \(N^m\), that \(N\) is number of Lumped elements, and \(m\) is an exponent value which makes the relationship linear, and Fig. (3-14) through (3-17) show these results, indicating for each case the minimum
number of Lumped elements which gives the best results, and they are shown on the figures (3-14) through (3-17).

For preceding special cases, the outlet temperatures are fitted as follows:

a) Fitting Parallel Flow: Tables (3-1) through (3-4) contain the results of Lumped Models—outlet temperatures for each value of $\varphi$, by calling those data, and putting $x=0$ or $x=1$ in equations (3-7) through (3-10), the results are:

For case $a_1=4$, $a_2=1$

Parallel Flow: $\frac{T_{oc}}{T_{in}} = 0.1986$, $\frac{T_{ok}}{T_{in}} = 0.2054$

Lumped Models:

Counter Flow:

Therefore, by fitting the parallel flow with exact results, the counter flow gets the 90% value of the exact results, in appendix (d) calculations for other cases (different $\varphi$) are presented and their results are in tables (3-5) & (3-6).
b) Fitting Counter-Flow: The results of Lumped Models and exact results make following calculations, they are for whole values of \( \Phi_B \), in appendix(d):

For \( a_1 = 4 \), \( a_2 = 1 \)

Counter Flow:

\[
\Phi_B = \frac{1}{2} \quad \Rightarrow \quad \begin{cases} 
\frac{T_{oc}}{T_{ch}} = 0.2 \\
\frac{T_{on}}{T_{ch}} = 0.2 
\end{cases} \quad \Rightarrow \quad \left| \frac{T_{oc}}{T_{oc}} \right|_{counter} = 1.2
\]

Parallel Flow:

\[
\Phi_B = \frac{1}{2} \quad \Rightarrow \quad \begin{cases} 
\frac{T_{oc}}{T_{ch}} = 1.837 \\
\frac{T_{on}}{T_{ch}} = 2.663 
\end{cases}
\]

Therefore,

\[
\left| \frac{T_{oc}}{T_{oc}} \right|_{parallel} = 1.837 \times 1.2 = 2.20 
\Rightarrow \quad \left| \frac{T_{oc}}{T_{oc}} \right|_{parallel} = 1.108
\]

Then by fitting the counter flow with exact results, the parallel flow gets the 10.8% higher of exact value. Therefore, as the results show it is better to fit the parallel flow and improve the counter flow. Figures (3-14) through (3-17) show the fitting results for different flow, and different cases.

This improvement of static case, by fitting the outlet
temperature, is useful for dynamic and transient responses, because the Lumped Model matches the exact solution at low frequency (by fitting) and it improves the results for high frequency or in early transient time, as they are shown later.
In the first part of this thesis, the steady state case was discussed and equations were obtained showing the temperature gradient as a function of location of each Lumped section in the heat-exchanger. The more general case of unsteady-state operation can now be assumed so that the dependent temperature variables are functions of the independent variables. Since there is more than one independent variable, the relationship between temperature, time, and distance can be stated as a partial differential equation, as it is shown in (1), (4), (6). However, in keeping with the objective of this thesis, this partial differential equation can in turn be written as a set of simultaneous ordinary differential equations. For this staged system it is always possible to use the canonical transformation to uncouple a large number of simultaneous equations and write them in terms of single first and second order systems. However, there are cases where it is both simpler and faster to make a direct attack on the problem. A direct approach is particularly advantageous when the coefficient
matrix is bi- or tri-diagonal, since for these cases the eigenvalues, or characteristic roots, can be obtained analytically.

Lumped Formulation

As Fig. (4-1) shows, a heat exchanger is divided by N Lumped, from first law of thermodynamics (energy equation), there is following relationship for each Lumped:

Rate of Heat Accumulation = Heat Flow in - Heat Flow out ± Heat Transferred

FIGURE (4-1): Lumped Heat Exchanger

FIGURE (4-2): Element of Heat Exchanger
For analysis the problem one of the sections should be formulated for energy balance and then it can be developed for other sections, that section analysis can be the same as Profos operator (7). By using above relationship for each section (as shown in fig. 4-2) the results are:

For hot side:

Rate of heat accumulation = \( M_H \cdot \frac{dT_H}{dt} = M_H C_H \frac{dT_H}{dt} \)

Heat flow in = \( M_H \rho_{h_{n-1}} \)

Heat flow out = \( M_H \rho_{h_n} \)

Heat transferred = \( \dot{E} = UA(T_H^n - T_C^n) \)

Therefore, energy balance is:

\[
M_H C_H \frac{dT_H^n}{dt} = M_H C_H T_{h_{n-1}} - M_H C_H T_{h_n} - UA_n(T_{h_n} - T_{h_n})
\]

or

\[
\frac{dT_H^n}{dt} = \frac{M_H}{M_H C_H} (T_{h_{n-1}} - T_{h_n}) - \frac{UA_n}{M_H C_H} (T_{h_n} - T_{h_n})
\]  

(4-1)

An used for area because all of the sections don't need to be equal, only hot and cold sections in the same number should be the same.

\[ M_{H} = \rho A_{s} l_{n} \quad A_{s} l_{n} = \text{volume of section } n \]

\[ M_{H} = \rho A_{s} V_{h} \]

By assumption of constant density and constant geometry for both sides:

continuity (mass conservation) - \( \rho A_{s} V_{1} = \rho A_{s} V_{2} \)

or \( V = \text{const.} \)
If $T_n$ defined as the time which fluid going through section n then:

$$V = \frac{L_n}{T_n}$$

Therefore, 

$$M^*_n = \rho \ A_s \ \frac{L_n}{T_n}$$

and 

$$M^*_n = \frac{1}{T_n}$$

If the total length of tube is L and $\Phi_n$ is the fraction or ratio of the length section n and total length, and defined as total time which hot fluid takes to flow through tube, then:

$$\Phi_n L_n = L_n \implies T_n = \Phi_n \ell$$  \hspace{1cm} (4-2)

Defined $\alpha_n$ as:

$$\alpha_n = \frac{U.A}{M^*_n \cdot C_d} = \frac{U \cdot P \cdot L}{M^*_n \cdot C_d} = \frac{U \cdot P \ell}{\Phi_n \cdot M^*_n \cdot C_d}$$

Therefore:

$$\frac{U \cdot A_n}{M^*_n \cdot C_d} = \frac{\Phi_n \alpha_n}{T_n} \hspace{1cm} (4-3)$$

By substituting (4-2)&(4-3) in (4-1):

$$\frac{dT_n^m}{dt} = \frac{1}{\Phi_n} (T_{n-1} - T_n) - \frac{\Phi_n \alpha_n}{T_n} (T_n^m - T_c) \hspace{1cm} (4-4)$$

If $\Theta_e = \frac{T}{C}$ \hspace{1cm} $\implies dt = \frac{T_n}{\Phi_n} d\Theta_e$

"using equation(4-2)"

Therefore, equation (4-5) becomes as:

$$\Phi_n \frac{dT_n^m}{d\Theta_e} = (T_{n-1} - T_n) - \Phi_n \alpha_n (T_n^m - T_c)$$

**Hot Side**

The same equation can be derived for cold side flow, only
r which is velocity ratio of two fluids is added to equation:

$$\frac{d}{d\theta} \left( T_{e} - T_{c} \right) + \frac{q_{n} a_{2}}{\Delta h} \left( T_{w} - T_{e} \right)$$

(4-7)

There are some changes because of counter flow, and for such case the equation is:

$$\frac{d}{d\theta} \left( T_{c} - T_{c_{n}} \right) + \frac{q_{n} a_{2}}{\Delta h} \left( T_{w} - T_{c} \right)$$

(4-8)

Before any further step, the temperature of each stage should be defined. As it was used in steady state case and also it is shown in appendix(c), the assumption of stirred tank is used for each lumped as in following figure:

**FIGURE (4-3): A Model of Lumped Section**

Therefore, the exit temperature of each lumped(sections) is used for the temperature of whole fluid in that section,
then as one of the sections picked up as an element for evaluating, it looks like Fig. (4-4):

\[ \text{FIGURE (4-4): nth Lumped Model} \]

Therefore, by using this assumption and substitute the outlet temperature for section temperature, equations (4-6) through (4-8) become:

\[ \begin{align*}
\text{Parallel Flow} & \quad \left\{ \begin{array}{l}
\phi_n \frac{dT_{\text{in}}}{d\theta} = (T_{\text{in}} - T_n) - \phi_n \alpha_1 (T_{\text{in}} - T_n) \\
N_n \frac{dT_n}{d\theta} = (T_n - T_{n-1}) - \phi_n \alpha_2 (T_n - T_{n-1})
\end{array} \right. \\
\text{Counter Flow} & \quad \left\{ \begin{array}{l}
N_n \frac{dT_{\text{in}}}{d\theta} = (T_{\text{in}} - T_n) - \phi_n \alpha_1 (T_{\text{in}} - T_{n+1}) \\
N_n \frac{dT_{n+1}}{d\theta} = (T_{n+1} - T_n) - \phi_n \alpha_2 (T_{n+1} - T_n)
\end{array} \right.
\]

Because solving the equations by Laplace transform is easier; therefore, equations (4-9) & (4-10), by using zero cond-
ition at the inlet or \( t = 0 \) are:

\[
\begin{align*}
\text{Parallel Flow} & \\
& \begin{cases}
(q_n s + q_n a_{t+1}) \overline{T}_{n+1} = \overline{T}_{wn+1} + q_n a_t \overline{T}_{cn} \\
(r q_n s + q_n a_{2+1}) \overline{T}_{cn} = \overline{T}_{cn-1} + q_n a_2 \overline{T}_{hn}
\end{cases} \\
\text{Counter Flow} & \\
& \begin{cases}
(q_n s + q_n a_{t+1}) \overline{T}_{hn} = \overline{T}_{hn-1} + q_n a_t \overline{T}_{cn+1} \\
(r q_n s + q_n a_{2+1}) \overline{T}_{cn+1} = \overline{T}_{cn-1} + q_n a_2 \overline{T}_{hn}
\end{cases}
\end{align*}
\]

As equations (4-11) & (4-12) show, if the number of Lumped sections are given, these equations can be extended and their solutions give the temperature at any section. The same as steady state case, here for \( N \) equals to 2, 3, and 4 it will be discussed, also we can see by putting a large value for time \((t \to \infty)\) the results of static case can be obtained. Following is a schematic of this evaluation:
A simplest problem which is close to heat exchanger problem of dynamic response is one where there are a number of first-order systems arranged in series. Letting the time constant of the nth stage be $\tau_n$, denoting the dependent variable leaving stage by the stage number, and including the possibility of being able to control each stage, a balance on the nth stage gives the equation:

$$\tau_n \frac{dy_n}{dt} + y_n = \alpha_n y_{n-1} + \beta_n u$$  \hspace{1cm} (4-13)
and the same equation will apply to every stage. Fig. (4-6) shows this series of first order systems.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4-6.png}
\caption{Series of First-Order Systems}
\end{figure}

If we assume that the system is originally operating at some steady state conditions (as it discussed for heat exchanger is last part), and that the dependent variable represents deviations from steady state, the Laplace transform of equation (4-13) gives the result:

\[(s \alpha + 1) \overline{y}_n = \alpha_n \overline{y}_{n-1} + \beta_n \overline{u} \tag{4-14}\]

Thus the complete set of equations can be written as:

\[\begin{align*}
(s \alpha + 1) \overline{y}_1 &= \alpha_1 \overline{y}_0 + \beta_1 \overline{u} \\
(s \alpha + 1) \overline{y}_2 &= \alpha_2 \overline{y}_1 + \beta_2 \overline{u} \\
&\quad \vdots \\
(s \alpha + 1) \overline{y}_n &= \alpha_n \overline{y}_{n-1} + \beta_n \overline{u}
\end{align*}\]

Now, the first equation can be solved in the set for \(\overline{y}_1\) in terms of \(\overline{y}_0\) and \(\overline{u}\) and then use this result to eliminate \(\overline{y}_1\).
from the second equation. This procedure can be repeated until we obtain an explicit expression for \( \overline{y}_n \) in terms of \( \overline{y}_0 \) and \( \overline{u} \), the system inputs. Hence

\[
\overline{y}_n = \frac{\alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1 \overline{y}_0}{(c_n s+1)(c_{n-1}s+1) \cdots (c_2 s+1)} + \left[ \frac{\beta_n (c_n s+1)(c_{n-1}s+1) \cdots (c_2 s+1)}{(c_1 s+1)(c_2 s+1) \cdots (c_n s+1)} + \cdots \right.
\]

\[
+ \alpha_n \frac{\beta_{n-1} (c_{n-1}s+1)(c_{n-2}s+1) \cdots (c_1 s+1)}{(c_n s+1)(c_{n-1}s+1) \cdots (c_2 s+1)} + \cdots
\]

\[
+ \frac{\beta_1 \alpha_1 \alpha_2 \cdots \alpha_{n-1}}{(c_n s+1)(c_{n-1}s+1) \cdots (c_1 s+1)} \overline{u}
\]

or, for a case where all of the time constants and system gains are equal, the transfer function for inlet disturbances is

\[
\frac{\overline{y}_n}{\overline{y}_0} = \frac{\alpha^n}{(c s+1)^n}
\]

and the transfer function for control variable changes is:

\[
\frac{\overline{y}_n}{\overline{u}} = \beta \frac{\alpha^{n-1} \sum_{i=0}^{n-1} \alpha^i (c s+1)^i}{(c s+1)^n}
\]
SPECIAL CASES FOR DYNAMIC RESPONSE:

For two lumped models (N=2):

i-Parallel Flow:

From equation (4-11), for N=2 the results are:

\[
\left\{
\begin{align*}
\left(\varphi_1 s + \varphi_1 a_1 + 1\right) \bar{T}_{H2} &= \bar{T}_{H1} + \varphi_1 a_1 \bar{T}_{C2} \\
\left(\varphi_2 s + \varphi_2 a_2 + 1\right) \bar{T}_{C2} &= \bar{T}_{C1} + \varphi_2 a_2 \bar{T}_{H2} \\
\left(\varphi_2 s + \varphi_1 a_2 + 1\right) \bar{T}_{H3} &= \bar{T}_{H2} + \varphi_2 a_1 \bar{T}_{C3} \\
\left(\varphi_2 s + \varphi_2 a_2 + 1\right) \bar{T}_{C3} &= \bar{T}_{C2} + \varphi_2 a_2 \bar{T}_{H3}
\end{align*}
\]

The results of these equations, by disturbing only inlet temperature of hot fluid, are:

\[
\frac{\bar{T}_{C3}}{\bar{T}_{H1}} = \frac{\varphi_1 a_2}{(\varphi_1 s + \varphi_1 a_1 + 1)(\varphi_2 s + \varphi_2 a_2 + 1) - \varphi_1^2 a_1 a_2}
\]

\[
\frac{\bar{T}_{H3}}{\bar{T}_{H1}} = \frac{\varphi_2 a_2 + 1}{(\varphi_2 s + \varphi_1 a_1 + 1)(\varphi_2 s + \varphi_2 a_2 + 1) - \varphi_2^2 a_1 a_2}
\]

(4-15a-0)
\[
\begin{align*}
\frac{\Delta T_{H3}}{\Delta T_{H1}} &= \frac{(\theta_{R1}+\theta_{R2}+1)(\theta_{R1}+\theta_{R2}+1)+\theta_{R2}a_1a_2}{\prod_{i=1}^{n}[(\theta_{R1}+\theta_{R2}+1)(\theta_{R1}+\theta_{R2}+1) - \theta_{R2}a_2]} \quad (4-15a) \\
\frac{\Delta T_{C3}}{\Delta T_{H1}} &= \frac{\theta_{R2}a_2(\theta_{R1}+\theta_{R2}+1)+\theta_{R2}a_1(\theta_{R1}+\theta_{R2}+1)}{\prod_{i=1}^{n}[(\theta_{R1}+\theta_{R2}+1)(\theta_{R1}+\theta_{R2}+1) - \theta_{R2}a_2]} \quad (4-15b)
\end{align*}
\]

ii- Counter Flow:

From equation (4-12), and as Fig. (4-8) shows the results are:

![Counter Flow Diagram](counter_flow_diagram.png)

**FIGURE (4-8): Two Lumped Models of Counter Flow Heat-Exchanger**

\[
\begin{align*}
(\theta_{R1}+\theta_{R2}+1) \bar{T}_{H2} &= \bar{T}_{H1} + \theta_{R1}a_1 \bar{T}_{C3} \\
(\theta_{R1}+\theta_{R2}+1) \bar{T}_{C1} &= \bar{T}_{C2} + \theta_{R2}a_2 \bar{T}_{H1} \\
(\theta_{R1}+\theta_{R2}+1) \bar{T}_{H1} &= \bar{T}_{H2} + \theta_{R1}a_1 \bar{T}_{C2} \\
(\theta_{R1}+\theta_{R2}+1) \bar{T}_{C2} &= \bar{T}_{C1} + \theta_{R2}a_2 \bar{T}_{H1}
\end{align*}
\]

For disturbing only inlet temperature of hot fluid, the solutions of above equations are:
It is clear that by putting $s=0$, the result of steady-state case can be gotten.

For three lumped models ($N=3$):

i- Parallel Flow:

From equation (4-11) for $N=3$ the results for temperature labeled on Fig. (4-9) are:
There are three θ's, which can be equal or different, but the sum of them should be unit.

\[
\begin{align*}
(\varphi_1 \varphi_2 + \varphi_1 \varphi_3 + 1) \frac{\bar{\theta}_1}{\bar{\theta}_2} &= \frac{\varphi_1 \bar{\theta}_2}{(\varphi_2 \varphi_3 + \varphi_1 \varphi_3 + 1)(\varphi_2 \varphi_3 + \varphi_1 \varphi_3 + 1) - \varphi_1 \bar{\theta}_2} \\
(\varphi_1 \varphi_2 + \varphi_2 \varphi_3 + 1) \frac{\bar{\theta}_2}{\bar{\theta}_1} &= \frac{\varphi_2 \varphi_3 + \varphi_2 \bar{\theta}_1 + 1}{(\varphi_2 \varphi_3 + \varphi_2 \bar{\theta}_1 + 1)(\varphi_2 \varphi_3 + \varphi_2 \bar{\theta}_1 + 1) - \varphi_2 \varphi_3 - \varphi_2 \bar{\theta}_1} \\
(\varphi_1 \varphi_2 + \varphi_3 \varphi_1 + 1) \frac{\bar{\theta}_3}{\bar{\theta}_1} &= \frac{\varphi_1 \varphi_3 \varphi_1 + \varphi_3 \varphi_2 \varphi_1 + 1}{(\varphi_1 \varphi_3 + \varphi_2 \varphi_1 + 1)(\varphi_1 \varphi_3 + \varphi_2 \varphi_1 + 1) - \varphi_1 \varphi_3 - \varphi_2 \varphi_1} \\
(\varphi_1 \varphi_2 + \varphi_3 \varphi_2 + 1) \frac{\bar{\theta}_4}{\bar{\theta}_1} &= \frac{\varphi_1 \varphi_2 + \varphi_2 \varphi_3 + 1}{(\varphi_1 \varphi_2 + \varphi_2 \varphi_3 + 1)(\varphi_1 \varphi_2 + \varphi_2 \varphi_3 + 1) - \varphi_1 \varphi_2 + \varphi_2 \varphi_3}
\end{align*}
\]

And the solutions for changing only inlet hot temperature fluid are:

\[
\begin{align*}
\frac{\bar{\theta}_2}{\bar{\theta}_1} &= \frac{\varphi_1 \varphi_2}{(\varphi_2 \varphi_3 + \varphi_1 \varphi_3 + 1)(\varphi_2 \varphi_3 + \varphi_1 \varphi_3 + 1) - \varphi_1 \bar{\theta}_2} \\
\frac{\bar{\theta}_1}{\bar{\theta}_2} &= \frac{\varphi_2 \varphi_3 + \varphi_2 \bar{\theta}_1 + 1}{(\varphi_2 \varphi_3 + \varphi_2 \bar{\theta}_1 + 1)(\varphi_2 \varphi_3 + \varphi_2 \bar{\theta}_1 + 1) - \varphi_2 \varphi_3 - \varphi_2 \bar{\theta}_1} \\
\frac{\bar{\theta}_3}{\bar{\theta}_1} &= \frac{\varphi_1 \varphi_3 \varphi_1 + \varphi_3 \varphi_2 \varphi_1 + 1}{(\varphi_1 \varphi_3 + \varphi_2 \varphi_1 + 1)(\varphi_1 \varphi_3 + \varphi_2 \varphi_1 + 1) - \varphi_1 \varphi_3 - \varphi_2 \varphi_1} \\
\frac{\bar{\theta}_4}{\bar{\theta}_1} &= \frac{\varphi_1 \varphi_2 + \varphi_2 \varphi_3 + 1}{(\varphi_1 \varphi_2 + \varphi_2 \varphi_3 + 1)(\varphi_1 \varphi_2 + \varphi_2 \varphi_3 + 1) - \varphi_1 \varphi_2 + \varphi_2 \varphi_3}
\end{align*}
\]
ii- Counter Flow:

Using equation (4-12) for temperature labeled on Fig. (4-10), they give six equations and six unknown temperatures as follows:

\[
\begin{align*}
\frac{T_{C4}}{T_{W1}} &= \frac{\varphi_2 a_2 \sum_{i=1}^{2} \left( \varphi_i r_s + r_{i+1} \right) + \varphi_2 a_2^2 + a_2 (\varphi_2 r_s + r_{2+1})}{\sum_{i=1}^{2} \left[ (\varphi_2 r_s + r_{2+1})(\varphi_i r_s + r_{i+1}) - \varphi_i a_2 a_2 \right]} \\
\frac{T_{W4}}{T_{W1}} &= \frac{\sum_{i=1}^{3} \left( \varphi_i r_s + r_{i+1} \right) + \varphi_3 a_2 (\varphi_3 r_s + r_{3+1}) + \varphi_2 a_2 \sum_{i=2}^{3} \left( \varphi_i r_s + r_{i+1} \right)}{\sum_{i=1}^{3} \left[ (\varphi_3 r_s + r_{3+1})(\varphi_i r_s + r_{i+1}) - \varphi_i a_2 a_2 \right]} 
\end{align*}
\]

\[
\begin{align*}
(\varphi_1 r_s + r_{1+1}) T_{W2} &= T_{W1} + \varphi_1 a_1 T_{C4} \\
(\varphi_2 r_s + r_{2+1}) T_{C2} &= T_{C3} + \varphi_2 a_2 T_{W2} \\
(\varphi_3 r_s + r_{3+1}) T_{W3} &= T_{W1} + \varphi_3 a_1 T_{C3} \\
(\varphi_4 r_s + r_{4+1}) T_{C1} &= T_{C3} + \varphi_3 a_2 T_{W3} \\
(\varphi_5 r_s + r_{5+1}) T_{W4} &= T_{W1} + \varphi_5 a_1 T_{C2} \\
(\varphi_6 r_s + r_{6+1}) T_{C2} &= T_{C3} + \varphi_5 a_2 T_{W4} 
\end{align*}
\]
Solution for getting outlet temperatures of each section based on disturbing only hot inlet fluid temperature are in the next page but instead of these long answers by putting the numerical values of parameters (namely $\varphi_1, \varphi_2, \varphi_3,$ and $a_1, a_2, a_3, r$) the results will be very simple as they are shown for special values of these parameters.
For four lumped models (N=4):

**i- Parallel Flow:**

By using N=4, there are four \( \psi \)'s, and equation (4-11) is extended to following equations:

\[
\begin{align*}
(\psi_1 + \psi_4 a_1 + 1) \frac{T_{he}}{T_h} &= T_h + \psi_1 a_1 \frac{T_{c2}}{T_c} \\
(\psi_1 + \psi_2 a_1 + 1) \frac{T_{c2}}{T_c} &= T_c + \psi_2 a_2 \frac{T_{w2}}{T_w} \\
(\psi_2 + \psi_4 a_1 + 1) \frac{T_{w2}}{T_w} &= T_w + \psi_2 a_2 \frac{T_{c3}}{T_c} \\
(\psi_3 + \psi_2 a_2 + 1) \frac{T_{c3}}{T_c} &= T_c + \psi_3 a_2 \frac{T_{w3}}{T_w} \\
(\psi_3 + \psi_4 a_1 + 1) \frac{T_{w3}}{T_w} &= T_w + \psi_3 a_2 \frac{T_{c4}}{T_c} \\
(\psi_3 + \psi_2 a_2 + 1) \frac{T_{c4}}{T_c} &= T_c + \psi_3 a_2 \frac{T_{w4}}{T_w} \\
(\psi_4 + \psi_4 a_1 + 1) \frac{T_{w4}}{T_w} &= T_w + \psi_4 a_3 \frac{T_{c5}}{T_c} \\
(\psi_4 + \psi_2 a_2 + 1) \frac{T_{c5}}{T_c} &= T_c + \psi_4 a_2 \frac{T_{w5}}{T_w} \\
\end{align*}
\]
The solutions can be obtained in the same manner of other cases, but they obtained only for special cases that are in the next part.

**ii- Counter Flow:**

Using equation (4-12) for $N=4$ and four different $\psi$'s, it is extended to the following equations:

\[
\begin{align*}
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{H2} & = T_{H1} + \psi_1 a_1 T_{C5} \\
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{C5} & = T_{C4} + \psi_2 a_2 T_{H2} \\
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{H3} & = T_{H2} + \psi_2 a_2 T_{C4} \\
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{C4} & = T_{C3} + \psi_2 a_2 T_{H3} \\
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{H4} & = T_{H3} + \psi_3 a_3 T_{C3} \\
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{C3} & = T_{C2} + \psi_3 a_3 T_{H4} \\
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{H5} & = T_{H4} + \psi_4 a_4 T_{C2} \\
\left(\psi_1 + \psi_{4} + \psi_{2} + \psi_{3} \right) T_{C2} & = T_{C1} + \psi_4 a_4 T_{H5}
\end{align*}
\]
Solutions for cases which are discussed in next part are presented. We can develop and get better results by putting more or larger $N$. 
RESULTS OF TRANSIENT RESPONSE FOR THREE CASES:

In the following cases, solutions for outlet temperatures of last section for each fluid are found and compared with the exact results(2) and results of Friedly's model(4).

The exact solution and Friedly's results, as shown in appendix(a), can be obtained by putting x=1 in them:

Exact solution for counter flow:

$$G(j\omega) = \frac{T_{oc}}{t_{ih}} = \frac{g_2(1-e^{-A_1})}{-(P_1-P_2 e^{-B^*})+P_2(1-e^{-A_1})}$$

Where,

$$P_1, P_2 = \left[-(f_1-f_2) \pm \sqrt{(f_1+f_2)^2 - 4g_1g_2}\right]/2$$

Exact solution for parallel flow:

$$G(j\omega) = \frac{T_{oc}}{t_{ih}} = \frac{g_2}{P_1-P_2} (e^{-P_2}-e^{-P_1})$$

Where,

$$P_1, P_2 = \left[-(f_1+f_2) \pm \sqrt{(f_1-f_2)^2 + 4g_1g_2}\right]/2$$

$$f_1 = a_1 + j\omega, \quad f_2 = a_2 + rj\omega, \quad g_1 = a_1, \quad g_2 = a_2$$

Friedly's solution for counter flow:

$$\bar{Z}(s) = \frac{K}{s} \left[ 1 - e^{-(a_1+a_2)} \cdot e^{(r+1)s} \right]$$

Where,

$$\frac{K}{s} = \frac{a_2}{1+r} \text{ and, } \quad K = \frac{a_2}{1-e^{(a_1+a_2)}} \times \frac{e^{-a_2}-1}{a_2 e^{-a_2} - a_1}$$

Then for cases that $a_1, a_2$, and $r$ are specified the above equations appear as follows:

1- Exact:

For case $a_1 = 4, \quad a_2 = 1, \quad r = 1$
-Counter Flow

\[
G(\Omega) = \frac{2\sqrt{2.25 - \Omega^2 + 5j\Omega}}{1 - e^{2\sqrt{2.25 - \Omega^2 + 5j\Omega}}} - \frac{1 - e^{4j\Omega}}{(2.5 + j\Omega + \sqrt{2.25 - \Omega^2 + 5j\Omega}) e^{\sqrt{2.25 - \Omega^2 + 5j\Omega}}} - \frac{e^{2j\Omega}}{2.5 + j\Omega + \sqrt{2.25 - \Omega^2 + 5j\Omega}} - \frac{e^{4j\Omega}}{2.5 + j\Omega + \sqrt{2.25 - \Omega^2 + 5j\Omega}}
\]

(4-19)

-Parallel Flow

\[
G(\Omega) = \frac{1 - e^{\Omega}}{e^\Omega}
\]

(4-20)

For case: \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

-Counter Flow

\[
G(\Omega) = \frac{1 - e^{\sqrt{2\Omega^2 - \Omega^2}}}{1 - e^{\sqrt{2\Omega^2 - \Omega^2}}} - \frac{1 - e^{3j\Omega}}{(1 + j\Omega - \sqrt{2\Omega^2 - \Omega^2}) - (1 + j\Omega + \sqrt{2\Omega^2 - \Omega^2}) e^{\sqrt{2\Omega^2 - \Omega^2}}}
\]

(4-21)

-Parallel Flow

\[
G(\Omega) = \frac{1 - e^{\Omega^2}}{2}
\]

(4-22)

For case: \( a_1 = 2 \), \( a_2 = 1 \), \( r = 2 \)

-Counter Flow

\[
G(\Omega) = \frac{e^{2\sqrt{25 - \Omega^2 + 45j\Omega}}}{(1.5 + 1.5j\Omega - \sqrt{25 - 225 - 45j\Omega}) - (1.5 + 1.5j\Omega + \sqrt{25 - 225 - 45j\Omega}) e^{\sqrt{25 - 225 - 45j\Omega}}}
\]

(4-23)

-Parallel Flow

\[
G(\Omega) = \frac{e^{-15(1+j\Omega)}}{\sqrt{9 - \Omega^2 - 2j\Omega}} (e^{e^{\sqrt{25 - 225 - 45j\Omega}}} - e^{-e^{\sqrt{25 - 225 - 45j\Omega}}})
\]

(4-24)

As it is shown in reference(4), Friedly discussed only counter-flow heat exchanger, and its results for preceding cases are:
ii- Friedly's Results:

For \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

-Counter Flow

\[
\bar{G}(s) = \frac{K}{1 + 2K^2} \left[ 1 - e^{5-2s} \right] \tag{4-25}
\]

Where,

\[
K = \frac{1 - e^3}{(4 - e^3)(6 - e^3)}
\]

For \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

-Counter Flow

\[
\bar{G}(s) = \frac{K}{1 + 2K^2} \left[ 1 - e^{2-2s} \right] \tag{4-26}
\]

Where,

\[
K = \frac{1}{2(1 - e^3)}
\]

For \( a_1 = 2 \), \( a_2 = 1 \), \( r = 2 \)

-Counter Flow

\[
\bar{G}(s) = \frac{K}{1 + 3K^2} \left[ 1 - e^{3-3s} \right] \tag{4-27}
\]

Where,

\[
K = \frac{e - 1}{(2e - 1)(6 - e^3)}
\]

iii- Lumped Model Results:

First, for simplify the calculations all of \( \Psi \)'s assumed to be equal for all cases.

For \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

a-Two Lumped Models:

\( \Psi_1 = \Psi_2 = 1/2 \) from equations (4-15) & (4-16):
Counter Flow

\begin{align*}
G_1(s) &= \frac{T_{\text{oc}}}{T_{\text{in}}} = \frac{5}{25s^2 + 2.25s + 2.5} \\
G_2(s) &= \frac{T_{\text{lw}}}{T_{\text{in}}} = \frac{5s + 1.5}{(5s+3)(25s^2 + 25s + 25)}
\end{align*}

Parallel Flow

\begin{align*}
G_1(s) &= \frac{T_{\text{oc}}}{T_{\text{in}}} = \frac{5(s^2 + 4.5)}{(5s+3)^2(5s+1)^2} \\
G_2(s) &= \frac{T_{\text{lw}}}{T_{\text{in}}} = \frac{25s^2 + 15s + 3.25}{(5s+3.5)^2(5s+1)^2}
\end{align*}

b- Three Lumped Models:

\( \Phi_1 = \Phi_2 = \Phi_3 = 1/3 \) using equations (4-17) & (4-18)

Counter Flow

\begin{align*}
G_1(s) &= \frac{T_{\text{oc}}}{T_{\text{in}}} = \frac{35s^4 + 16s^3 + 534s^2 + 188s + 2511}{s^6 + 33s^5 + 435s^4 + 2315s^3 + 10318s^2 + 18216s + 11772} \\
G_2(s) &= \frac{T_{\text{lw}}}{T_{\text{in}}} = \frac{27(s^3 + 3)}{s^6 + 33s^5 + 435s^4 + 2315s^3 + 10318s^2 + 18216s + 11772}
\end{align*}

Parallel Flow

\begin{align*}
G_1(s) &= \frac{T_{\text{oc}}}{T_{\text{in}}} = \frac{27[(s^2 + 7s^2 + 6 + 7s^2 + 4)]}{(s^2 + 12s + 24)^3} \\
G_2(s) &= \frac{T_{\text{lw}}}{T_{\text{in}}} = \frac{27[(s^2 + 4)^3 + 12s + 60]}{(s^2 + 12s + 24)^3}
\end{align*}
c- Four Lumped Models:

\[ q_1 = q_2 = q_3 = q_4 = 1/4 \]

using equations (A-15)\&(A-16), (A-21)\&(A-22) in the appendix(e), solution of four lumped are presented for special cases only.

\[
G_1(s) = \frac{T_{oc}}{T_{ch}} = \frac{256(4s^3 + 78s^2 + 532s + 1261)}{(s+9)^4(s+4)^4}
\]

Parallel Flow

\[
G_2(s) = \frac{T_{oc}}{T_{ch}} = \frac{256[(s+5)^4 + 2(s+5)^3 + 8(s+8)(s+5) + 4(s+8)^2 + 16]}{(s+9)^4(s+4)^4}
\]

Parallel Flow

There are following results for counter flow case:

\[
G_1(s) = \frac{T_{oc}}{T_{ch}} = \frac{4(s^4 + 39s^3 + 635s^2 + 5421s^2 + 26124s^2 + 67184s^2 + 74048)}{s^6 + 58s^5 + 1158s^4 + 1440s^3 + 10915s^2 + 51352s^1 + 1452448s^0 + 2294776s^0 + 344728}
\]

Counter Flow

\[
G_2(s) = \frac{T_{oc}}{T_{ch}} = \frac{256(8+5)(s^4 + 39s^3 + 635s^2 + 5421s^2 + 26124s^2 + 67184s^2 + 74048)}{(s+8)(s^4 + 58s^3 + 1158s^2 + 1440s^3 + 10915s^2 + 51352s^1 + 1452448s^0 + 2294776s^0 + 344728}
\]

Counter Flow

For \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

a- Two Lumped Models:

Using \( q_1 = q_2 = 1/2 \) in equations (4-15)\&(4-16), gives:

\[
G_1(s) = \frac{T_{oc}}{T_{ch}} = \frac{2(s^4 + 6s + 12)}{(s^2 + 6s + 1)(s^2 + 6s + 10)}
\]

Counter Flow

\[
G_2(s) = \frac{T_{oc}}{T_{ch}} = \frac{4(s^3 + 3)}{(s^2 + 6s + 1)(s^2 + 6s + 10)}
\]
Parallel Flow

\[
\begin{align*}
G_1(s) &= \frac{T_{oc}}{T_{in}} = \frac{8(s+3)}{(s^2+6s+8)^2} \\
G_2(s) &= \frac{T_{on}}{T_{in}} = \frac{4(s+3)^2+4}{(s^2+6s+8)^2}
\end{align*}
\]

b- Three Lumped Models:

Using three equal \( \phi \)'s \( \phi_1 = \phi_2 = \phi_3 = 1/3 \), it means the heat exchanger has been divided into three equal sections; then, by applying equations (4-17) & (4-18), the results are:

Counter Flow

\[
\begin{align*}
G_1(s) &= \frac{T_{oc}}{T_{in}} = \frac{3(s+4)^4+21(s+4)^2+392}{(s^3+13s^2+55s+84)(s^2+11s^2+39s+94)} \\
G_2(s) &= \frac{T_{on}}{T_{in}} = \frac{27(s+4)^3}{(s^3+13s^2+55s+84)(s^2+11s^2+39s+94)}
\end{align*}
\]

Parallel Flow

\[
\begin{align*}
G_1(s) &= \frac{T_{oc}}{T_{in}} = \frac{81(s+4)^2+27}{(s+3)^3(s+5)^3} \\
G_2(s) &= \frac{T_{on}}{T_{in}} = \frac{27(s+4)^3+81(s+4)}{(s+3)^3(s+5)^3}
\end{align*}
\]
c- Four Lumped Models:

Applying \( q'_1 = q'_2 = q'_3 = q'_4 = 1/4 \) in the results which are presented in appendix(e), give following equations:

\[
\begin{align*}
G_1(s) &= \frac{\tau_{bc}}{\tau_{lh}} = \frac{1024(s+5)(s^2+10s+26)}{(s+6)^4(s+4)^4} \\
G_2(s) &= \frac{\tau_{bc}}{\tau_{lh}} = \frac{256(s+5)^4 + 1536(s+5)^2 + 256}{(s+6)^4(s+4)^4}
\end{align*}
\]

Parallel Flow

\[
\begin{align*}
G(s) &= \frac{\tau_{bc}}{\tau_{lh}} = \frac{4(s+4)(s+5)^2(s^2+10s+39) + 83(s^6+893s^5+34400)}{8^4 + 65s^9 + 8689s^8 + 482985s^7 + 1649910s^6 + 1391680s^5 + 528060s^4 + 228000}
\end{align*}
\]

Counter Flow

\[
\begin{align*}
G_1(s) &= \frac{\tau_{bc}}{\tau_{lh}} = \frac{4s(s+4)(s+5)^2(s^2+10s+39) + 83(s^6+893s^5+34400)}{8^4 + 65s^9 + 8689s^8 + 482985s^7 + 1649910s^6 + 1391680s^5 + 528060s^4 + 228000}
\end{align*}
\]

For \( a_1 = 2 \), \( a_2 = 1 \), \( r = 2 \)

a- Two Lumped Models:

Using \( q'_1 = q'_2 = 1/2 \) in equations(4-15)&(4-16), gives following results:

\[
\begin{align*}
G_1(s) &= \frac{\tau_{bc}}{\tau_{lh}} = \frac{s^2 + 5.5s^2 + 7}{(s^2 + 5.5s + 5)^2 - 2} \\
G_2(s) &= \frac{\tau_{bc}}{\tau_{lh}} = \frac{(s^2 + 15)^2}{(s^2 + 5.5s + 5)^2 - 2}
\end{align*}
\]

Counter Flow
Parallel Flow

\[
G_1(s) = \frac{T_{ec}}{T_{ic}} = \frac{3s + 7}{(s^2 + 5.5s + 5)^2}
\]

\[
G_2(s) = \frac{T_{in}}{T_{ic}} = \frac{4s^4 + 12s^3 + 11}{(s^2 + 5.5s + 5)^2}
\]

b- Three Lumped Models:

By using \( \phi_1 = \phi_2 = \phi_3 = 1/3 \) and equations (4-17)&(4-18), there are:

Counter Flow

\[
G_1(s) = \frac{T_{ec}}{T_{ic}} = \frac{12(s+5)^2(s+2)^2 + 30(s+5)(s+2) + 147}{8(s+5)^3(s+2)^3 - 24(s+5)^2(s+2)^2 - 48(s+5)(s+2)^2 - 98}
\]

\[
G_2(s) = \frac{T_{in}}{T_{ic}} = \frac{216(s+2)^3}{8(s+5)^3(s+2)^3 - 24(s+5)^2(s+2)^2 - 48(s+5)(s+2)^2 - 98}
\]

Parallel Flow

\[
G_1(s) = \frac{T_{ec}}{T_{ic}} = \frac{108(s+2)^8 + 54(s+2)(s+5) + 27(s+5)^2 + 54}{8(s^2 + 7s + 9)^3}
\]

\[
G_2(s) = \frac{T_{in}}{T_{ic}} = \frac{27(s+2)(2s^2 + 8s + 11)}{2(s^2 + 7s + 9)^3}
\]

c- Four Lumped Models:

By using \( \phi_1 = \phi_2 = \phi_3 = \phi_4 = 1/4 \) and the result of solution for four lumped in appendix(e), following results are obtained:
\[
G(s) = \frac{\tau_c}{\tau_{Ell}} = \frac{16(\tau^3 + 5\tau^2 + 16\tau + 16)(\tau^6 + 24\tau^5 + 144\tau^4 + 64\tau^3 + 16\tau^2 + \tau)}{(s^2 + 8.5s + 14)^4}
\]

Parallel Flow

\[
G(s) = \frac{\tau_c}{\tau_{Ell}} = \frac{16(\tau^3 + 3\tau^2 + \tau^2 + 16(\tau^6 + 24\tau^5 + 144\tau^4 + 64\tau^3 + 16\tau^2 + \tau)}{(s^2 + 8.5s + 14)^4}
\]

Counter Flow

\[
G(s) = \frac{2(s^2 + 25.5s + 267.75s^4 + 1445^2 + 44725s^2 + 73708s + 5368)}{s^2 + 34s^2 + 4335s^4 + 32865s^2 + 18510s^2 + 73975s^2 + 27418s^2 + 8646s^2 + 31472}
\]

\[
G(s) = \frac{(s^2 + 25.5s + 267.75s^2 + 1445^2 + 44725s^2 + 73708s + 5368)}{s^2 + 34s^2 + 4335s^2 + 32865s^2 + 18510s^2 + 73975s^2 + 27418s^2 + 8646s^2 + 31472}
\]

All of above cases are normal situations of heat-exchangers, for comparing the results of Lumped Models, exact solution, and Friedly's approximate method all of above cases are plotted in frequency domain, and the results of step responses for two Lumped Models (the simplest form) are plotted.

One of the beauties of the Lumped Models is finding the Monotone parameters directly by the method which H.M. Paynter presented in (8), and these coefficients can be compared with the same parameters which are obtained by simple technique that again Paynter presented which are gained by using the probability paper. All of these evaluations are in next parts.
RESULTS OF FREQUENCY RESPONSE:

I- Counter Flow Heat-Exchanger

i- Exact Solution:

In preceding section, transfer functions for transient response were discussed and equations for special cases were derived, now by recalling those equations the numerical values of gains and phase angles can be calculated and plotted, here is for exact solution; recall equation (4-19):

For case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

\[
G(j\Omega) = \frac{1 - e^{j2\Omega - \frac{\omega_1}{r}}}{\left(-0.5 + j\Omega - \sqrt{2.25 - \frac{\omega_1^2}{r^2}}\right) - \left(2.5 + j\Omega + \sqrt{2.25 - \frac{\omega_1^2}{r^2}}\right) e^{j2\Omega - \frac{\omega_1}{r}}} 
\]

For \( \Omega = 0 \) \( |G| = \frac{1 - 0.5}{1 - 0.5} = 2.405 \), \( \theta = \angle G(j\Omega) = 0^\circ \)

For \( \Omega \to \infty \) \( |G| \to 0 \), \( \theta \to -90^\circ \)

Table (4-1) contains the value of gain and phase angle for different frequencies.

For case: \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

\[
G(j\Omega) = \frac{1 - e^{j2\Omega - \frac{\omega_1}{r}}}{\left(1 + 3\Omega - \sqrt{2.25 - \omega_1^2}\right) - \left(1 + 3\Omega + \sqrt{2.25 - \omega_1^2}\right) e^{j2\Omega - \frac{\omega_1}{r}}} 
\]

For \( \Omega = 0 \) \( |G| = 0.5 \), \( \theta = 0^\circ \)

For \( \Omega \to \infty \) \( |G| \to 0 \), \( \theta \to -90^\circ \)
In this case $|G|$ is obtained by using Hopital's rule. In table (4-5) there are numerical values of $|G|$ and $\theta$ for different frequencies.

For case: $a_1=2$, $a_2=1$, $r=2$

$$G(j\Omega) = \frac{1 - e^{\frac{2\sqrt{25-25a_1^2+45\Omega}}{(1.5+1.5i\Omega-\sqrt{25-25a_1^2+45\Omega})-(1.5+1.5i\Omega+\sqrt{25-25a_1^2+45\Omega})e^{\frac{2\sqrt{25-25a_1^2+45\Omega}}{2}}}}}{1-e^{\frac{2\sqrt{25-25a_1^2+45\Omega}}{(1.5+1.5i\Omega-\sqrt{25-25a_1^2+45\Omega})-(1.5+1.5i\Omega+\sqrt{25-25a_1^2+45\Omega})e^{\frac{2\sqrt{25-25a_1^2+45\Omega}}{2}}}}}$$

For $\Omega=0$, $|G|=.3873$, $\theta=0^\circ$

For $\Omega \to \infty$, $|G|\to 0$, $\theta\to -90^\circ$

Results for other values of $\Omega$ are presented in table (4-9).

ii- Friedly's Solution:

For case $a_1=4$, $a_2=1$, $r=1$

$$G(j\Omega) = \frac{K}{1+2K\Omega} \left[1 - e^{\frac{5-2\Omega}{2}}\right] \text{ where } K=2422$$

For $\Omega=0$, $|G|=.2405$, $\theta=0^\circ$

For $\Omega \to \infty$, $|G|\to 0$, $\theta\to -90^\circ$

Table (4-1) contains all numerical values for $|G|$ and $\theta$, for different frequencies.

For case $a_1=1$, $a_2=1$, $r=1$

Recall equation (4-26):

$$G(j\Omega) = \frac{K}{1+2K\Omega} \left[1 - e^{\frac{2-2\Omega}{2}}\right] \text{ where } K=578$$
For $\Omega = 0$ \quad $|G| = 0.5$ \quad , \quad $\theta = 0^\circ$
For $\Omega \to \infty$ \quad $|G| \to 0$ \quad , \quad $\theta \to -90^\circ$

For other values of $\Omega$ the results are presented in table (4-5).

For case $a_1 = 2$ , $a_2 = 1$ , $r = 2$

By using equation (4-27) from preceding section, following results can be gotten:

\[
G(\omega) = \frac{K}{1 + j \omega K} \left[ 1 - e^{-3 \omega} \right] \quad \text{where} \quad K = 0.4076
\]

For $\Omega = 0$ \quad $|G| = 0.873$ \quad , \quad $\theta = 0^\circ$
For $\Omega \to \infty$ \quad $|G| \to 0$ \quad , \quad $\theta \to -90^\circ$

Results for other values of frequency are in table (4-9).

iii- Lumped Models Results:

The more interested outlet temperature is usually the outlet temperature of cold fluid; therefore, the Lumped Models cold fluid results are discussed.

For case $a_1 = 4$ , $a_2 = 1$ , $r = 1$

a- Two Lumped Models:

\[
G(\omega) = \frac{0.5}{2.5 - 25 \omega^2 + 125 \omega^4}
\]

For $\Omega = 0$ \quad $|G| = 0.2$ \quad , \quad $\theta = 0^\circ$
For $\Omega \to \infty$ \quad $|G| \to 0$ \quad , \quad $\theta \to -180^\circ$

As in static case discussed, for getting better results we can fit the Lumped Models results with the exact results at low frequency; therefore, for two Lumped Models all of results
are multiplied by a factor:

\[ \text{Factor} = \frac{\frac{\gamma}{\delta}}{\frac{\gamma}{\delta}} = \frac{24.05}{0.2} = 1.202 \]

As it is shown in table (4-2) in addition for absolute results of Lumped Models there is a column for fitted results.

b- Three Lumped Models:

\[ G(j\Omega) = \frac{(8511 + 23.5 \omega^2 - 584 \omega^2) + \frac{1}{3}(1881 \omega^2 - 66 \omega^3)}{(1772 + 635 \omega^4 - 103.5 \omega^5 - \omega^6) + \frac{1}{3}(18216 \omega + 33 \omega^5 - 291.5 \omega^3)} \]

For \( \Omega = 0 \) \[ |G| = 0.2133 \quad , \quad \theta = 0^\circ \]

For \( \Omega \rightarrow \infty \) \[ |G| \rightarrow 0 \quad , \quad \theta \rightarrow -180^\circ \]

factor for fitting \( \frac{24.05}{0.2133} = 1.1275 \)

The results for other values of frequency are shown in table (4-3).

c- Four Lumped Models:

\[ G(j\Omega) = \frac{4(74048 + 231 \omega^2 - 261 \omega^2 \omega^6) + \frac{1}{4}\left(67184 \omega + 33 \omega^5 - 5421\omega^3\right)}{(1344 + 973 \omega^4 + 39.5 \omega^9 - 981 \omega^4 + 145.8 \omega^2) + \frac{1}{4}\left(221970 \omega + 144 \omega^6 - 51.7 \omega^5 - 5205 \omega^3\right)} \]

For \( \Omega = 0 \) \[ |G| = 0.22025 \quad , \quad \theta = 0^\circ \]

For \( \Omega \rightarrow \infty \) \[ |G| \rightarrow 0 \quad , \quad \theta \rightarrow -180^\circ \]

factor for fitting \( \frac{24.05}{0.22025} = 1.092 \)

Results for other values of frequency are represented in table (4-4).

For case \( a_1 = 1 \quad , \quad a_2 = 1 \quad , \quad r = 1 \)

a- Two Lumped Models:

\[ G(j\Omega) = \frac{24 - 25 \omega^2 + 12j \omega}{(6 + \omega^2 - 520 \omega^4) + j(9\omega^2 - 12 \omega^3)} \]
For $\Omega = 0 \quad |G| = 0.4 \quad , \quad \theta = 0^\circ$

For $\Omega \rightarrow \infty \quad |G| \rightarrow 0 \quad \theta \rightarrow -180^\circ$

fitting factor $= \frac{5}{4} = 1.25$

Table (4-6) shows these results for other values of $\Omega$.

b- Three Lumped Models:

$$G(\omega) = \frac{3(432 + \omega^4 - 16\omega^2) + 3j(312\omega - 16\omega^3)}{(84 - 13\omega^2) + j(58\omega - \omega^3)}[(36 - 11\omega^2) + j(39\omega - \omega^3)]$$

For $\Omega = 0 \quad |G| = 0.4286 \quad , \quad \theta = 0^\circ$

For $\Omega \rightarrow \infty \quad |G| \rightarrow 0 \quad \theta \rightarrow -180^\circ$

fitting factor $= \frac{5}{4.286} = 1.166$

The whole results are shown in table (4-7).

c- Four Lumped Models

$$G(\omega) = \frac{4(22000 + 2880\omega^2 - 1520\omega^2 - \omega^6) + 4j(27944.5\omega + 365^2 - 276\omega^2)}{(28000 + 4228.5\omega^2 + 86\omega^4 - 350\omega^2) + j(5295\omega + 11800\omega - 640\omega^3)}$$

For $\Omega = 0 \quad |G| = 0.4444 \quad , \quad \theta = 0^\circ$

For $\Omega \rightarrow \infty \quad |G| \rightarrow 0 \quad \theta \rightarrow -180^\circ$

fitting factor $= \frac{5000}{4444} = 1.125$

Results for other values of frequency are shown in table (4-8).

For case: $a_1 = 2 \quad , \quad a_2 = 1 \quad , \quad r = 2$

a- Two Lumped Models:

$$G(\omega) = \frac{(7.5\omega^2 + 5.5j\omega}{(\omega^2 + 420\omega^2 + j(55\omega - \omega^3)}$$

For $\Omega = 0 \quad |G| = 3.043 \quad , \quad \theta = 0^\circ$

For $\Omega \rightarrow \infty \quad |G| \rightarrow 0 \quad \theta \rightarrow -180^\circ$
fitting factor $= \frac{2.3873}{3.3043} = 1.8725$

Other results for different values of frequency are in table (4-10).

b- Three Lumped Models:

$$ G(j\Omega) = \frac{3((74.5 + 2\Omega^4 - 243.5i) + 3j(315.5 - 28.5\Omega^3))}{(2511 + 6166\Omega^4 - 4.95\Omega - 228.5\Omega^2) + j(6832 + 84\Omega^5 - 2854\Omega^3)} $$

For $\Omega = 0$ \hspace{1cm} $|G| = 0.328$, \hspace{1cm} $\theta = 0^\circ$

For $\Omega \to \infty$ \hspace{1cm} $|G| \to 0$, \hspace{1cm} $\theta \to -180^\circ$

fitting factor $= \frac{2.3873}{3.328} = 1.8$

Results are presented in table (4-11)

c- Four Lumped Models:

$$ G(j\Omega) = \frac{2((44725.5i + 5318 + 26750.5i + 86) + 2j(73105\Omega + 25.5\Omega^3 - 1464.5\Omega^2))}{(31472 + 10048\Omega^2 + 32505\Omega^4 - 334\Omega^6) + j(8446\Omega + 3865.5\Omega^2 + 34.5\Omega^4 - 5397.5\Omega^6)} $$

For $\Omega = 0$ \hspace{1cm} $|G| = 0.3411$, \hspace{1cm} $\theta = 0^\circ$

For $\Omega \to \infty$ \hspace{1cm} $|G| \to 0$, \hspace{1cm} $\theta \to -180^\circ$

fitting factor $= \frac{2.3873}{3.411} = 1.135$

Table (4-12) shows the numerical values of gains and phase angle for other amount of frequency.

In all tables there is a column for normalized gain which means ratio of gain for every frequency and gain for zero frequency.

$$ |G|_n = \frac{|G|_\Omega}{|G|_0} $$
II- PARALLEL FLOW HEAT EXCHANGER

i- The Exact Solution:

From equation (4-20) we have following transfer function for transient response in parallel flow heat exchanger:

For case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

\[
G(j\Omega) = \frac{1-e^{-5}}{5} \cdot e^{j\Omega}
\]

As it shows, it is a complete circle with radius \( R = 0.1986 \) and the phase angle is \( \theta = \Omega \). Table(4-13) contains the value of phase angles for different value of \( \Omega \).

For case: \( a_1 = 2 \), \( a_2 = 1 \), \( r = 2 \)

\[
G(j\Omega) = \frac{e^{-15(1+j\Omega)}}{2\sqrt{2.25-25\Omega^2-5j\Omega}} \cdot \left[ e^{\sqrt{2.25-25\Omega^2-5j\Omega}} - e^{\sqrt{2.25-25\Omega^2+5j\Omega}} \right]
\]

For \( \Omega = 0 \) \( |G| = 3.167 \), \( \theta = 0^\circ \)

For \( \Omega \rightarrow \infty \) \( |G| \rightarrow 0 \), \( \theta \rightarrow -\infty \)

Table(4-17) contains the results for other values of \( \Omega \).

ii- The Lumped Model Results:

Because the interested outlet temperature is the outlet temperature of cold side, then the following results are for that temperature, also in following there is a fitting factor for each case which defined as:

\[
\text{fitting factor} = \frac{|G(j\Omega)_{\text{exact}}|}{|G(j\Omega)_{\text{lumped}}|}
\]
For case: $a_1 = 4$, $a_2 = 1$, $r = 1$

a- Two Lumped Models:

\[ G(\Omega) = \frac{-5(4.5 + \Omega)}{(3.5 + 3.5 \Omega)^2 (1 + 5.5 \Omega)^2} \]

For $\Omega = 0$ \[ |G| = 1.837, \quad \theta = 0^\circ \]

For $\Omega \to \infty$ \[ |G| \to 0, \quad \theta \to -270^\circ \]

fitting factor $= \frac{1.986}{1.837} = 1.081$

Table (4-14) contains the other values of $|G|$ and $\theta$ for different amount of $\Omega$, also it has a column for $|G|_\infty$, which is the ratio of gain for each $\Omega$ and gain of exact solution at zero frequency.

b- Three Lumped Models:

\[ G(\Omega) = \frac{27 (27 - 3 \Omega^2 + 33 \Omega)}{(24 - \Omega^2 + 11 \Omega)^2} \]

For $\Omega = 0$ \[ |G| = 1.974, \quad \theta = 0^\circ \]

For $\Omega \to \infty$ \[ |G| \to 0, \quad \theta \to -360^\circ \]

fitting factor $= \frac{1.986}{1.974} = 1.0483$

The results for other values of frequency are in table (4-15).

c- Four Lumped Models:

\[ G(\Omega) = \frac{256 (1261 - 785 \Omega^2) + 256 \Omega (5251 - 45 \Omega^3)}{(36 - \Omega^2 + 13 \Omega^4)^4} \]

For $\Omega = 0$ \[ |G| = 1.922, \quad \theta = 0^\circ \]

For $\Omega \to \infty$ \[ |G| \to 0, \quad \theta \to -450^\circ \]

fitting factor $= \frac{1.986}{1.922} = 1.0333$
Results for other values of frequency are presented in table (4-16).

For case: $a_1 = 2$, $a_2 = 1$, $r = 2$

a- Two Lumped Models:

$$G(j\Omega) = \frac{7 + 3j\Omega}{(5 - \Omega^2 + 5j\Omega)\Omega}$$

For $\Omega = 0$ \quad $|G| = 0.28$, $\theta = 0^\circ$

For $\Omega \to \infty$ \quad $|G| \to 0$, $\theta \to -270^\circ$

fitting factor $= \frac{0.28}{0.28} = 1.00$

Table (4-18) contains the value of gain and phase angle for different amount of frequency.

b- Three Lumped Models:

$$G(j\Omega) = \frac{1701 - 189\Omega^2 + 1080j\Omega}{8(9 - \Omega^2 + 7j\Omega)\Omega}$$

For $\Omega = 0$ \quad $|G| = 0.292$, $\theta = 0^\circ$

For $\Omega \to \infty$ \quad $|G| \to 0$, $\theta \to -240^\circ$

fitting factor $= \frac{0.292}{0.292} = 1.00$

Table (4-19) contains all of the results.

c- Four Lumped Models:

$$G(j\Omega) = \frac{11440 - 2416\Omega^2 + 3(8752\Omega - 2405\Omega^2)}{(14 - \Omega^2 + 8.5\Omega)\Omega^2}$$

For $\Omega = 0$ \quad $|G| = 0.298$, $\theta = 0^\circ$

For $\Omega \to \infty$ \quad $|G| \to 0$, $\theta \to -450^\circ$

fitting factor $= \frac{0.298}{0.298} = 1.00$
Results for other values of gain and phase angle for different amount of frequency are presented in table(4-20).

To compare the exact solution and simple Lumped Models results, all of results are plotted in Fig.(4-19)&(4-20).
SECTION V

EFFECT OF USING NONEQUAL $\Phi$'s:

As it was mentioned before, existance of $\Phi$'s in all of results of Lumped Model systems makes it flexible to adjust the results with exact values. All of preceding calculations and results are based on using equal $\Phi$'s in the whole Lumped Models, here it is shown that for one case($a_1=4$, $a_2=1$, $r=1$) in two Lumped Models what is the effect of using different $\Phi$'s.

Counter Flow Heat Exchanger, two Lumped Models:

Recalling equation(4-16a) and substitute the $a_1=4$, $a_2=1$, $r=1$, and using $\Phi_1=1/3$, $\Phi_2=2/3$, following transfer function is gotten:

\[
G(s) = \frac{\prod_{k=1}^{n}}{\prod_{k=1}^{m}} = \frac{3(57-4.5^2+32.3\Omega)}{(864+4.5^2-487.2\Omega)+j(117.5^2-76.2^2)}
\]

For $\Omega=0$ \[|G|=1.979, \quad \theta=0^\circ\]

For $\Omega \to \infty$ \[|G| \to 0, \quad \theta \to -180^\circ\]

fitting factor $= \frac{0.2405}{1.979} = 1.2152$

Table(5-1) shows the results of this case for different values of frequency, also Fig.(5-1) shows this results comparing to the exact value and results of using equal $\Phi$'s.
IMPROVEMENT OF LUMPED MODELS TO GET THE BEST RESULTS:

In the preceding section the results of various Lumped-Models and exact solutions were compared, via tabulations and plots. The results for counter-flow heat exchangers, Fig. (4-3) through (4-18), show that a Lumped Model can yield a good agreement with exact results at low frequency, but at high frequency both gains and phase angles are characteristically different. If one looks very carefully at these results, one finds in all cases the discrepancy between the Lumped-Model and the exact solution is always the same, i.e. the phase angles always have 90 degree difference and gains are about half of the exact values; therefore, one may conclude that some feature is characteristically the same for all of them. It is clear that thus 90 degree difference between phase angles means if a zero is added to the transfer function of the Lumped Model, then the phase angles at high frequency will match the exact results very well.

Since the purpose of this thesis is to find simple models which well match the exact results, evaluating the transfer functions which were found for different Lumped Models showed that two-lump models have a very simple form to handle; therefore, one should look for a way to add a zero to results of
two Lump Models to get better agreement.

Recalling the outlet temperature results of two-lump models for counter-flow heat exchangers, they show that the transfer function of the first lump has one zero more than the transfer function of outlet temperature of the second lump; therefore, if the outlet temperature of the second lump of the cold fluid is combined with outlet temperature of the first lump of hot fluid, then the overall result has one zero more than the zeroes of the outlet temperature of the cold fluid.

Therefore, from above discussion we conclude that to get the best agreement for lumped model approximations, we should employ in general a linear combination of the outlet temperatures of the various lump, and in particular we have the combination for two lump models as shown in Fig.(5-2).

Now, it is very interesting to determine whether there is any value for $K_1$ and $K_2$ which yields a good match between Lumped Model results and exact solutions for both gain and phase angle. The positive answer to this question is explored for the various cases already discussed. The results are expressed in terms of the values of $K_1$ and $K_2$ for different cases. Finally one may ask whether these values are the same or different for different cases and if the parameters are different...
how great is the range of their deviation?

![Diagram of Heat Exchanger](image)

**FIGURE (5-2):** Linear Combination for Two Lumped Models Counter-Flow Heat Exchanger

$K_1$ and $K_2$ for case: $a_1 = 4$, $a_2 = 1$, $r = 1$

Counter-Flow Heat Exchanger

By substituting the parameters in equations (4-15a) & (4-15a-0) and using equal $\varphi$'s i.e. $\varphi = 1/2$, following transfer functions are gotten:

\[
\frac{T_{OC}}{T_{NI}} = \frac{2}{s^2 + 9s + 10}
\]

\[
\frac{T_{C3}}{T_{NI}} = \frac{2(s^2 + 9s + 14)}{(s^2 + 9s + 10)(s + 6)}
\]
Therefore, the new outlet temperature of cold fluid is:
\[
\frac{T_{ec}}{T_{ih}} = \frac{2K_2}{s^2 + 9s + 10} + \frac{2(c_1^2 + 9s + 14)K_1}{(s^2 + 9s + 10)(s + 6)}
\]
or,
\[
G(s) = \frac{2(K_1s^2 + 9K_1s + K_2s + 14K_1 + 6K_2)}{(s^2 + 9s + 10)(s + 6)}
\]
\[
G(j\Omega) = \frac{2(14K_1 + 6K_2 - K_1\Omega^2) + 2j(9K_1\Omega + K_2\Omega)}{(10 - \Omega^2 + 9j\Omega)(6 + j\Omega)}
\]

To decide how should the gains $K_1$ and $K_2$ be determined, we have to see the effect of high values of frequency, also we can't adjust $K_1$ and $K_2$ exactly for high frequency because we thereby preclude the adjustment at low frequency. Therefore, it is clear that adjustments must begin at middle frequencies. For value of $\Omega = 2$, the amount of gain and phase angle are set equal to the exact values and the necessary $K_1$ and $K_2$ calculated. Then the results for other values of frequency are compared.

The results show a very satisfactory agreement at most values of frequency except at very lowest frequencies; therefore, by optimizing $K_1$ and $K_2$ around the values that was gotten for $\Omega = 2$, two values for $K$'s are gotten which they are:

$K_1 = 1.10$, $K_2 = 2.44$

Therefore, transfer function will be:
For $\Omega = 0$, $|G|_\nu = 1 \quad \theta = 0^\circ$

For $\Omega \to \infty$, $|G|_\nu \to 0 \quad \theta \to -90^\circ$

This transfer function is for normalized gain, and the results for other values of frequency are in table (5-2).

As we compare these results with those for the exact solution, we conclude that this is the best agreement for whole range of frequency in a very simple closed form.

Fig.(5-3) shows the plotted results and it is shown that the results of the simple Lumped Model is more close to exact results than Friedly's model, with advantage of having a very easy a simple form which is very useful for control of heat-exchangers.

$K_1$ and $K_2$ for case: $a_1 = 2$, $a_2 = 1$, $r = 2$

Counter-Flow Heat Exchanger-

Recalling equations (4-15a)&(4-15a-0) and substituting given numbers, the results are:

$$\frac{T_{C3}}{T_{W}} = \frac{s^2 + 55s + 7}{(s^2 + 55s + 5)^2 - 2}$$

$$\frac{T_{H1}}{T_{H1}} = \frac{2(s+15)(s^2 + 55s + 5)}{(s^2 + 55s + 5)^2 - 2}$$

Then, the new outlet temperature for cold fluid is:
\[
\frac{T_{oc}}{T_{in}} = \frac{K_1(2s^3 + 14s^2 + 21s + 15) + K_2(s^2 + 5s + 7)}{(s^2 + 5s + 5)^2 - 2}
\]

or,

\[
G(s) = \frac{T_{oc}}{T_{in}} = \frac{(7K_1s^2 + 15K_1s - 14K_2s^2 - 14K_2s^3) + j(5K_2s^2 + 21K_2s - 2K_2s^3)}{(23 + 2s^4 - 6s^2 + 5s^2) + (5s^2 - 11s^2)j}
\]

To get the proper values for \(K_1, K_2\), the same manner which used for first case is used here and the results are:

\[K_1 = 0.4, \quad K_2 = 2.55\]

Table (5-3) shows the results of improvement case and it compares them with the exact results. In fig. (5-4) the exact solution and improvement of Lumped Model results are plotted, and it is clear that how this simple model predicts the transient response of heat exchanger so close to exact solution.

In third case which \(a_1 = a_2 = r = 1\), we do the same as what we have done so forth, but for fluctuation part we can use average values for them and adjust the \(K\)'s with those values.
STEP AND IMPULSE RESPONSES:

One of the advantages of Lumped Models system is that it describes the transfer functions in a very simple form which it is easy to find the response of heat exchanger respect to any kind of disturbance like step function or impulse disturbance or periodic perturbation and so forth. Specially the expression of the improved case has a very simple form because it works only with two stirred tank. Here the results of step and impulse responses are presented for improved case and is compared with exact responses. The method is the same for every case; therefore, only for one case the results are presented.

a) - Step response for case: \( a_1=4 \), \( a_2=1 \) \( r=1 \)

In reference(4), the results of step response for exact solution is presented, and they are used here and they are tabulated in table(5-4).

For Friedly's Method:

\[
\tilde{G}(x,s) = \frac{K}{1+Ts} \left[ e^{-Ts(\theta-x)} - e^{-(a_1+a_2)x} \cdot e^{s(x+r)} \right]
\]

Where:

\[
\frac{K}{T} = \frac{a_2}{1+r} e^{a_1(x-r)} \quad , \quad K = \frac{a_2}{a_1[1-e^{a_1(x+r)}]} \left( \frac{1-e^{-(a_1-a_2)x}}{1-a_1 e^{a_1(x-r)}} \right)
\]

For step response it will be:

\[
\frac{\Delta T}{\Delta T_u} = \frac{K}{s(1+Ts)} \left[ e^{-s(\theta-x)} - e^{-(a_1+a_2)x} \cdot e^{s(x+r)} \right]
\]
Therefore, for \( a_1 = 4 \), \( a_2 = r = 1 \) and \( x = 1 \):

\[
\frac{T_{oc}}{T_{sh}} = \frac{K}{T} \begin{cases} 
0 & t \leq 0 \\
T \left( 1 - e^{-r(0-x)} \right) & t \geq r(0-x) 
\end{cases} - \frac{K}{T} e^{-r(0-x)} x \begin{cases} 
0 & t \leq r(0-x) \\
T \left( 1 - e^{-r(0-x)} \right) & t \geq r(0-x) 
\end{cases} 
\]

(5-1)

For Lumped Model:

\[ G(s) = \frac{T_{oc}}{T_{sh}} = \frac{2 (s + 1.15 + 12.34 + 30)}{(s^2 + 9s + 10)(s + 6)} \]

For Step Response:

\[
\frac{T_{oc}}{T_{sh}} = \frac{2 (1.15^2 + 12.34s + 30)}{2s(s + 6)(2s + 4.5 + \sqrt{10.85})(s + 4.5 - \sqrt{10.85})} 
\]

\[
\frac{T_{oc}}{T_{sh}} = \mathcal{L}^{-1} \left[ \frac{2 (1.15^2 + 12.34s + 30)}{2s(s + 6)(2s + 4.5 + \sqrt{10.85})(s + 4.5 - \sqrt{10.85})} \right] 
\]

(5-2)

Then the result is:

\[ \frac{T_{oc}}{T_{sh}} = -2.405 \left( 1 - 1.95 e^{-0.05t} - 0.05 e^{-0.772t} - 0.81 e^{-1.298t} \right) \]

Table(5-4) shows the result of step response for three methods which are Lumped Models, Exact and Friedly's approximation method. Fig.(5-5) presents the plotted comparison bet-
ween these three methods. It shows clearly that the Lumped-
Model results are closer to exact results than Friedly's meth-

od.

b)- Impulse Response for Case: a =4 , a =1 , r=1

The exact results for impulse response were presented in reference(10), they also are shown in table (5-5).

Friedly's Method:

Because the impulse response is derivative of step res-
ponse therefore, by taking the derivative of equation (5-1)

following results are obtained:

\[
\frac{T_{oc}}{T_{in}} = \begin{cases} 
0 & t \leq 0 \\
2.065 e^{-2.065t} & 0 < t \leq 2 \\
1.199 e^{-2.065t} & t > 2 
\end{cases}
\]

Results for different values of time (normalized time) 
are presented in table (5-5).

Lumped Model:

To get the impulse response we take only the derivative 
of equation which is for step response; therefore, its results 
are:

\[
\frac{T_{oc}}{T_{in}} = 1.11 e^{6t} + 0.038 e^{-2.77t} + 1.051 e^{-1.698t}
\]

Table (5-5) contains the results of impulse response for 
each solution and Friedly's method and Lumped Model system.
In Fig.(5-6) these results are plotted for comparing the accuracy of two approximate methods, and it is clear that in Lumped-Model method the error is almost zero for all values of time but we can't say for Friedly's method the error is near zero.
H.M. Paynter (3) presented a new evaluation method for dynamic response including that for counter-flow and parallel-flow heat exchangers. He showed that the Laplace transformation solutions for heat exchangers can be written as:

\[ G(s) = e^{\frac{\epsilon - T_m s + \frac{T_s^2}{2} s^2 - \frac{T_a^3}{6} s^3}{2}} \ldots \]

Where, \( \epsilon \) measures the steady state amplitude ratio between response and disturbance, \( T_m \) measures the mean time delay between response and disturbance, \( T_s \) defines the dispersion or attenuation, and \( T_a \) is the asymmetry or phase nonlinearity. As it is shown in his paper, he concluded that for a given Laplace transformation solution of the form:

\[ G(s) = \frac{N(s)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots} \]

and for special case \( N(s) = 1 \) the monotonic parameters can be written directly by following relationships:

\[ T_m = a_1 \]
\[ T_s^2 = a_2^2 - 2a_2 \]
\[ T_a^3 = 2a_3^3 - 6a_2 a_2 + 6a_3 \]

If \( N(s) \) is a function of \( s \) then the results of parameters should be diminished by the same amount which are obtained from \( N(s) \).
Therefore, since the results of Lumped Model method are in the form of \( G(s) = \frac{N(s)}{D(s)} \), then the monotonic parameters can be calculated very easily.

To compare the calculated parameter with exact monotonic parameters H.M. Paynter presented following technique(8):

Simple Technique for determination of Monotonic Parameters:

By using a commercially available "Probability Paper", a Gaussian distribution can be plotted linearly. If a step response of any dynamic behavior is plotted on such paper (as shown in Fig.5-6) by following method the monotonic parameters are obtained.

FIGURE(5-6): Use of Probability Paper for Monotone Parameters
PROCEDURE:

1- Points A-A are determined where the step response curve crosses the 16% and 84% recovery levels.

2- Point C is on the response curve vertically above point B.

3- The vertical distance BC set off upwards from the 16% and 84% levels determines the points D-D.

4- The intersection of line DD with the 84% level fixes point E.

5- Point F is vertically below E at the 50% level.

Then for determine the parameters we have:

$T_m$: - The time interval from the origin to point B gives the mean delay, $T_m$.

$T_s$: - The time interval from point B to point F gives the dispersion time, $T_s$.

$\alpha$: - The ratio of distance BC to distance FE gives one-sixth the skew coefficient, $\alpha$.

$T_a$: - Then by determination of $\alpha$ we have: $T_a^3 = \alpha T_s^3$

Now, by plotting the results of step responses of exact solution, Lumped Model, and Friedly's method, on probability paper we can get the Monotone parameters, also these parameters can be obtained directly from Laplace transformation solution.
for Lumped Model Method.

Fig. (5-7) through (5-9) show results of plotting the step responses to exact solution, Lumped Model, and Friedly's method, the above technique is used and from those plotting the following results are gained:

For Exact Solution
\[
\begin{align*}
T_m &= 0.65 \\
T_s &= 0.81 \\
T_a &= 1.08 
\end{align*}
\]

For Lumped Model
\[
\begin{align*}
T_m &= 0.66 \\
T_s &= 0.87 \\
T_a &= 1.19 
\end{align*}
\]

For Friedly's Method
\[
\begin{align*}
T_m &= 0.48 \\
T_s &= 0.51 \\
T_a &= 0.62 
\end{align*}
\]

As the results of Paynter's method for determination of Monotonic parameters show, the values which obtained by Lumped Model are very close to exact values, but we can't see this accuracy from Friedly's method, and this is one of the other advantage of Lumped Model.

Monotone parameters are calculated by the direct method for Lumped Model and written as follow:
Therefore:

\[ G(s) = \frac{9.2s^2 + 24.6s + 60}{s^3 + 15s^2 + 64s + 60} = \frac{1 + 0.115s + 0.0367s^2}{1 + 1.06s + 25s^2 + 0.0167s^3} \]

\[ T_{m1} = n = 1.06, \quad T_{m2} = a = .411 \quad \Rightarrow \quad T_m = 1.06 - 0.41 = 0.65 \]

\[ \frac{T_m}{T_n} = \sqrt{a_n^2 - 2a_n} = 0.80, \quad T_{m2} = \sqrt{a_n^2 - 2a_n} = 0.3 \quad \Rightarrow \quad T_k = \sqrt{0.8^2 - 0.3^2} = 0.74 \]

\[ \frac{T_m}{T_n} = \sqrt{a_n^2 - 2a_n + a_n} = 1.0, \quad T_{m2} = \sqrt{a_n^2 - 2a_n + a_n} = 0.3 \quad \Rightarrow \quad T_k = \sqrt{1 - 0.3^2} = 0.99 \]

The difference between these results and graphical results are because of approximation in graphical method.
The essential part of this thesis has been devoted to determination of very simple conceptual and physical models that would describe heat-exchangers in some of the most important practical cases that one expects to encounter, particularly under conditions where the flow rates are varied. The low frequency behavior for shell-and-tube exchangers has been matched as carefully as possible by use of a method due to Professor H.M. Paynter. It has appeared reasonable to take into account at least qualitatively the very high frequency behavior to complement the low frequency information. To this end, Friedly presented an approximate method which works for low and high frequencies, but which is infinite order; therefore, a simple low-order model should be found. For certain applications (e.g. optimal control) since lump models constitute the most linear way to get Simple results, such Lumped Model have been used in present work.

Analysis of Lumped Models has illustrated that it is possible to approximate the dynamics of a variety of plug flow processes with very simple and basic transfer functions.
As the discussed examples show, the simple Lumped Models even without improvement demonstrate very good agreement at low frequency or for long-duration fluctuations. As those results show, there is no sensible difference between two lump-Models and four-lumps ones; in other words, the rate of convergence of the Lumped Model to the exact solution is quite small with respect to changes in the number of sections. Therefore, from the stand-point of significant improvement only two-lump models were discussed and results show that they sensibly agree with exact solution over the whole range of frequency.

Since by this technique the outlet temperature of each section is determined by a very simple transfer function, this model gives not only the exit temperature of the heat-exchanger but also the temperatures for all intermediate points at each section along the heat-exchanger, the special cases, which are included, show that both the frequency and time response are well approximated over their entire range by a transfer function derived only from a two-lump model.

There is no evidence limitation for this model, especially in the improved case, because it includes some adjustable parameters which make it flexible for all situations; therefore, many assumptions can be removed by changing those particular
parameters to achieve a satisfactory solution for most cases.

Another important aspect of this thesis is that the Lumped Model is capable of directly predicting certain Flow Reversal effects. The results for steady state behavior show, thus the outlet temperature of Lumped-Models is very close to exact solution. To obtain the exact values the Lumped-Models results can be adjusted by calibration. Thus, it is important to know that if the parallel-flow case (or alternatively, the counter-flow case) is so fitted, what will then happen if flow is reversed. Fitting the parallel-flow situation with exact value is shown to yield better results in such flow-reversal cases.

Comparison of Friedly's approximate method and the technique of Lumped-Models shows that the use of Lumped-Models has many advantages with respect to Friedly's model which some of them are:

1- Basically Friedly's method is an infinite order model which is complicated from the control point of view while Lumped-Models are of finite order which are easy to handle and work with.

2- Friedly's method doesn't yield good agreement in all cases (as it is shown in reference 4) but the Lumped-Models can be used in a variety of cases with overall better results than Friedly's model.
3- Friedly's model has the limitations indicated in reference (4) but as previously mentioned, Lumped-Models provide flexibility which can be expanded for many cases.

Since the present work gives very useful results for the discussed cases it would appear to be worthwhile to further develop the method for more general cases. Therefore, following efforts are recommended:

I- The improvement technique is physically acceptable because Two Lumped-Models for whole heat exchanger yield too low an exit temperature. Therefore, a combination of an intermediate temperature and an exit temperature gives the best results, but it should be proved mathematically why such a compensatory technique is true and why a linear combination is sufficient.

II- It is interesting to know the effectiveness and sensitivity of the parameters $\alpha$ and $\varphi$, where $\alpha$ is used in temperature of each section $[T=\alpha T_0+\varphi]$ -because in the present work $\alpha$ was assumed to be zero which clearly isn't precisely correct for any real case. Also for one special case different $\varphi$'s are used to improve the results, Therefore, this parameter has been shown to have significant influence.

III- Lumped-Model could be used (especially the improvement case) to eliminate all of the common simplifying assum-
ption such as: including the wall heat capacities, changing overall heat transfer coefficient between two fluids along the heat-exchanger, boundary layer effects, and eliminating the plug flow assumption.

IV- Heat-exchangers involving gases and vapors are more complicated to deal with, since the momentum and continuity equations are now coupled with the thermal equations. Therefore, it will be interesting to consider Lumped-Models for such two phase situations.
REFERENCES:


**a-THE EXACT SOLUTION:**

Exact solution of dynamic responses of heat exchangers can be obtained by using the energy balance (First law of thermodynamics) for Fig. (A-1):

\[
\dot{E} - \dot{W}_{th} = m_i(h_1 + \frac{I^2}{2} + g_3)_{out} - m_i'(h_1 + \frac{I^2}{2} + g_3)_{in} + M_o E^i
\]

Mass conservation:

\[ \rho AV = \text{Const.} \]
1- Assume constant geometry \( A = \text{Const.} \) \\
\( V = \text{Const.} \)

2- Assume constant fluid properties \( \rho = \text{Const.} \)

Therefore, there is no change in kinematic energy. There is no shaft work transfer; therefore, we get:

\[
Q' = M'(h_{\text{out}} - h_{\text{in}}) + ME'
\]
\[
ME' = Mh + Q' - \mu h_{\text{out}}
\]

By using the Profos operator (7) and equation we get:

\[
\begin{align*}
\frac{\partial \theta_1}{\partial t} + \frac{\partial \theta_1}{\partial x} &= a_1(\theta_2 - \theta_1) \\
r \frac{\partial \theta_2}{\partial t} \pm \frac{\partial \theta_2}{\partial x} &= a_2(\theta_1 - \theta_2)
\end{align*}
\]

Where \( r \) is ratio of cold flow rate and hot flow rate. Negative sign is for counter-flow and positive sign is for parallel-flow case.

Initial conditions:
\[
\begin{align*}
\theta_1(x,0) &= 0 \\
\theta_2(x,0) &= 0
\end{align*}
\]

Boundary Conditions:

counter flow
\[
\begin{align*}
\theta_1(1,t) &= 0 \\
\theta_1(0,t) &= T_{\text{in}}
\end{align*}
\]
Parallel Flow

\[
\begin{align*}
\Theta(0, t) &= T_{in} \\
\Theta_2(0, t) &= 0
\end{align*}
\]

The easiest way to solve partial differential equations is using Laplace transform, therefore by taking Laplace-transform of equation:

i- for Counter Flow:

\[
\begin{align*}
S\tilde{\Theta}_i + \frac{d\tilde{\Theta}_i}{dx} &= a_{11} \tilde{\Theta}_2 - a_{12} \tilde{\Theta}_1 \\
rs\tilde{\Theta}_1 - \frac{d\tilde{\Theta}_1}{dx} &= a_{21} \tilde{\Theta}_2 - a_{22} \tilde{\Theta}_2
\end{align*}
\]

or

\[
\begin{align*}
\tilde{\Theta}_2 &= \tilde{\Theta}_1 + \frac{S}{a_{11}} \tilde{\Theta}_1 + \frac{1}{a_{11}} \frac{d\tilde{\Theta}_1}{dx} \\
(rs+a_2)(\tilde{\Theta}_1 + \frac{s}{a_{21}} \tilde{\Theta}_1 + \frac{1}{a_{21}} \frac{d\tilde{\Theta}_1}{dx}) - d\left(\tilde{\Theta}_1 + \frac{s}{a_{21}} \tilde{\Theta}_1 + \frac{1}{a_{21}} \frac{d\tilde{\Theta}_1}{dx}\right) &= a_2 \tilde{\Theta}_2
\end{align*}
\]

then:

\[
\frac{d^2\tilde{\Theta}_1}{dx^2} - [s(r-1)+(a_2-a_1)] \frac{d\tilde{\Theta}_1}{dx} - s(rs+ra_1+a_2) \tilde{\Theta}_1 = 0
\]

using \(\tilde{\Theta}_1 = e^{\rho x}\), it gives:

\[
P^2 - [s(r-1)+(a_2-a_1)]P - s(rs+ra_1+a_2) = 0 \quad (A-I)
\]

Equation (A-I) gives two values \(P_1, P_2\), then by boundary condition we have:

\[
\begin{align*}
\tilde{\Theta}_1 &= A e^{P_1 x} + B e^{P_2 x} \\
\tilde{\Theta}_2 &= \left(\frac{S+a_2}{a_1}\right)\left(A e^{P_1 x} + B e^{P_2 x}\right) + \frac{1}{a_1}(P_1 e^{P_1 x} + P_2 e^{P_2 x})
\end{align*}
\]
Inserting boundary condition:

\[ A + B = \frac{\overline{T}}{T_{in}} \]

\[ A(s + a_1 + p_1) e^{\frac{p_1}{t}} + B(s + a_1 + p_2) e^{\frac{p_2}{t}} = 0 \]

which gives:

\[ \frac{\overline{\theta}_1}{\overline{T}} = \frac{(s + a_1 + p_2) e^{\frac{p_1}{s}} - (s + a_1 + p_1) e^{\frac{p_2}{s}}}{s + a_1 + p_2 - (s + a_1 + p_1) e^{s - p_2}} \]  \hspace{1cm} (A-2)

\[ \frac{\overline{\theta}_2}{\overline{T}} = \frac{(s + a_1 + p_2)(s + a_1 + a_2)(e^{\frac{p_1}{s}} - e^{\frac{p_2}{s}})}{a_1 [s + a_1 + p_2 - (s + a_1 + p_1) e^{s - p_2}]} \]  \hspace{1cm} (A-3)

Putting \( x = 1 \) and using proper parameters equations (A-2), (A-3) becomes the same as equation (2-3).

ii-Parallel flow: using Laplace transform:

\[
\left\{ \begin{array}{l}
\frac{d}{dx} \overline{\theta}_1 + \frac{d}{dx} \overline{\theta}_2 = a_1 \overline{\theta}_1 - a_2 \overline{\theta}_2 \\
rs \overline{\theta}_2 + \frac{d}{dx} \overline{\theta}_2 = a_2 \overline{\theta}_1 - a_6 \overline{\theta}_2
\end{array} \right. 
\]

By continue in the same manner of counter-flow, we get:

\[ \frac{\overline{\theta}_1}{\overline{T}} = \frac{(s + a_1 + p_1) e^{\frac{p_1}{s}} - (s + a_1 + p_2) e^{\frac{p_2}{s}}}{p_2 - p_1} \]  \hspace{1cm} (A-4)

\[ \frac{\overline{\theta}_2}{\overline{T}} = \frac{(s + a_1 + p_1)(s + a_1 + p_2)(e^{\frac{p_1}{s}} - e^{\frac{p_2}{s}})}{a_1 (p_2 - p_1)} \]  \hspace{1cm} (A-5)

Where:

\[ p_1, p_2 = -\left( \frac{a_1 + a_2}{2} \right) - \frac{s(s+1)}{2} \pm \sqrt{\left( \frac{a_1 + a_2 + r s + s^2}{2} \right)^2 - 4s (r s + a_1 r + a_2)} \]
FOR STATIC CASE:

To get the Static (steady state) case, S should be zero, then:

\[
\frac{\Theta_1}{T_{in}} = \frac{(P_2 + a_1) e^{P_1} - (P_1 + a_1) e^{P_2}}{(P_2 + a_1 - (P_1 + a_1) e^{P_2})} \frac{P_2}{P_1}
\]

counter flow

\[
\frac{\Theta_2}{T_{in}} = \frac{(P_1 + a_1)(P_2 + a_1)(e^{P_1} - e^{P_2})}{a_1 (P_2 + a_1 - (P_1 + a_1) e^{P_2})} \frac{P_2}{P_1}
\]

parallel flow

By using the values of \( P_1, P_2 \) in each case the summary results are:

\[
\frac{\Theta_1(\omega)}{T_{in}} = \frac{a_2}{a_1 + a_2} \left[ 1 - e^{-\frac{(a_1 + a_2)\omega}{a_1 + a_2}} \right]
\]

parallel flow

\[
\frac{\Theta_2(\omega)}{T_{in}} = \frac{a_2 + a_1 e^{-(a_1 + a_2)\omega}}{a_1 + a_2}
\]
Counter Flow

\[
\begin{align*}
\frac{\theta_2(x)}{T_{i\infty}} &= \frac{a_1 e^{(\alpha_1 - \alpha_2)x} - e^{(\alpha_1 - \alpha_2)x}}{a_1 - a_2 e^{(\alpha_1 - \alpha_2)x}} \\
\end{align*}
\]

\[
\begin{align*}
\frac{\theta_1(x)}{T_{i\infty}} &= \frac{a_2 e^{(\alpha_1 - \alpha_2)x} - e^{(\alpha_1 - \alpha_2)x}}{a_1 - a_2 e^{(\alpha_1 - \alpha_2)x}} \\
\end{align*}
\]
b-TEMPERATURE OF EACH SECTION IN LUMPED MODELS:

For each section in Lumped Models, we assumed that they work as a stirred tank, as it is shown in Fig. (A-2); therefore, the temperature of tank is a linear combination of its inlet and outlet temperatures say:

\[ T = \alpha T_i + (1-\alpha) T_o \]  

(A-7)

FIGURE (A-2): Stirred Tank

For tank as a control volume, the first law of thermodynamics is:

\[ \dot{M} \frac{\partial u}{\partial t} = \dot{M}_{in} (h + \frac{v^2}{2} + gz)_{in} - \dot{M}_{out} (h + \frac{v^2}{2} + gz)_{out} \]

There is no mass accumulation and outlet and inlet pipes have the same geometry; therefore, by continuity we have:

\[ \dot{V}_{in} = \dot{V}_{out} \]

In the steady state case it is shown that: \( \dot{M} = \dot{M}_x ? \)
where $M$ is mass of tank and $m'$ is mass flow rate through control volume; therefore:

$$T_c c_p \frac{dT}{dt} = c_p (T_i - T_o)$$

or

$$T c \frac{dT}{dt} = T_i - T_o$$

(A-8)

Combination of equations (A-7) & (A-8) gives:

$$T_o = \frac{1 - \alpha^2 s}{1 + (-\alpha) 2s} T_i, \quad T = \frac{1}{-\alpha c s + 1}$$

(A-9)

Because $\alpha$ is very small and $c$ is a small number; therefore, for perfect stirred tank $\alpha$ goes to zero and from equation (A-9) we get:

$$T \approx T_{out}$$

This is why, in all of cases (steady or dynamic) the temperature of each section is put the outlet temperature.

---

* $\alpha$ is small because fluid which leaves the tank has property of the fluid in the tank, but fluid which comes to tank has properties of outside the tank; therefore, the temperature of tank has more tendency to outlet temperature than inlet temperature, which means $\alpha$ is a small number.*
c-ENERGY EQUATION FOR EACH SECTION OF LUMPED MODEL

For each section of Lumped Model we can use the Profos (7) operator as shown for Kth section:

\[
\Phi = \frac{\partial}{\partial t} \left( \rho \left( \mu + \frac{V_i^2}{2} + g_z \right) \right) + \rho \left( \mu + \frac{V_i^2}{2} + g_z \right) \nabla \cdot \mathbf{V}
\]

FIGURE(A-3): Kth Section of Lumped Model

Applying first law of thermodynamics for each of control volumes:

\[
\Phi - W_g = \frac{\partial}{\partial t} \left( \rho \left( \mu + \frac{V_i^2}{2} + g_z \right) \right) + \rho \left( \mu + \frac{V_i^2}{2} + g_z \right) \nabla \cdot \mathbf{V}
\]

Where \( M \) is mass of fluid in side the control volume, \( U \) is internal energy of fluid inside the control volume and \( M' \) is mass flow rate which comes in and out.

Geometry of control volume is constant (means rigid boundary) then from continuity: \( \rho_{in} V_{in} = \rho_{out} V_{out} \Rightarrow V_{in} = V_{out} \)
Because $w_h = 0$ (no shaft work), and the internal energy of fluid and its enthalpy can be written as temperature, then we get:

$$\dot{E} = MC \frac{dT}{dt} + M_h^c (T_{out} - T_{in})$$

Therefore, for control volume of fig. (A-3) the results are:

$$c_v I \implies \dot{E} = M_h^c \frac{dT}{dt} + M_h^c (T_{out} - T_{in})$$

(A-10)

But $Q$ can be written as:

$$\dot{E} = -UA_h (T_{out}' - T_{in}')$$

and for relation between $M_h$ and $M_h'$, they can be written as:

$$M_h = \rho A_s L_k$$

$$M_h' = \rho A_s V_h$$

Where $A_s$ is cross-section area and $L_k$ is length of section then if $T_k$ defined as the time which fluid needs to go through section; therefore:

$$V_h = V_{in} = V_{out} = \frac{L_k}{L_k}$$

Then:

$$M_h' = \rho A_s \frac{L_k}{L_k}$$

or

$$\frac{M_h}{M_h'} = T_k$$

By definition $\Omega_k = \frac{L_k}{L}$, where $L$ is the total length of tube, it concludes that the time which fluid takes to go through
whole of the heat-exchanger is:

\[ Q = \frac{\dot{q}_k}{q_k} \]

Therefore, equation (A-10) becomes:

\[-UA_k (T^{(x)}_H - T^{(x)}_c) = M_H \dot{q}_k c \frac{dT^{(x)}_H}{dt} + M_H \dot{c} (T^{(x)}_{H_{k-1}} - T^{(x)}_H)\]

or

\[ \dot{q}_k \frac{dT^{(x)}_H}{dt} = T^{(x)}_H - T^{(x)}_{H_{k-1}} - \frac{UA_k}{M_H \dot{c}} (T^{(x)}_H - T^{(x)}_c) \]

(A-11)

Define:

\[ \alpha_1 = \frac{UA_k}{M_H \dot{c}} = \frac{U}{C} \cdot \frac{\dot{q}_k}{q_k} \times \frac{1}{M_H} \]

\[ \therefore \alpha_1 = \frac{UA_k}{M_H \dot{c}} \times \frac{1}{q_k} \]

Define:

\[ \theta_o = \frac{t}{C} \quad \Rightarrow \quad dt = C \, d\theta_o \]

\[ \therefore \, \, dt = \frac{q_k}{\dot{q}_k} \, d\theta_o \]

Therefore, equation (A-11) becomes as:

\[ q_k \frac{dT^{(x)}_H}{d\theta_o} = T^{(x)}_H - T^{(x)}_{H_{k+1}} - q_k \alpha_1 (T^{(x)}_H - T^{(x)}_c) \]

(A-12)

By taking Laplace transform of equation (A-12):
Using assumption of perfect stirred tank:

\[ q_k S \frac{dT_{w1}}{dt} = T_{w1} - T_{w1} - q_k a_1 (T_{w1}^{(i)} - T_c^{(i)}) \]

By the same manner, for second control volume, the result become as:

\[ r q_k S \frac{dT_{w2}}{dt} = T_{w2} - q_k a_2 (T_{w2}^{(i)} - T_c^{(i)}) - T_{w2}^{(i)} \quad (A-13) \]

Where \( r \) is velocity ratio of two hot and cold fluid putting \( s=0 \), gives the results of steady-state case. For parallel-flow the policy is the same and results has been shown in both static and dynamic cases.

As an example for results of each section treatments, here is calculations for case \( N=2 \):

Counter Flow:

From preceding calculations, there are four equation for outlet temperatures as follows:
Solving above equations gives following results:

\[
\frac{T_{c3}}{T_{h1}} = \frac{q_1 a_2 (1+q_1 a_1+q_2 a_2)+q_2 a_2}{(1+q_1 a_1+q_2 a_2)(1+q_2 a_1+q_2 a_2)-q_1 q_2 a_2}
\]

\[
\frac{T_{h3}}{T_{h1}} = \frac{(1+q_1 a_2)(1+q_2 a_2)}{(1+q_1 a_1+q_2 a_2)(1+q_2 a_1+q_2 a_2)-q_1 q_2 a_2}
\]
CALCULATIONS ABOUT FITTING STATIC CASE WITH EXACT VALUE:

i- Fitting Parallel-Flow:

case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

For this case the exact results are:

\[
\frac{T_{oc}}{T_{in}} = 0.1986 \quad \frac{T_{on}}{T_{in}} = 0.2054
\]

Results of Lumped Models are:

"Parallel Flow"

\[
\varphi = \frac{1}{2}
\]

\[
\begin{align*}
\frac{T_{oc}}{T_{in}} & = 0.1937 \\
\frac{T_{on}}{T_{in}} & = 0.2153
\end{align*}
\]

Which gives: \( \text{Fitting Factor} = 1.0811 \)

Therefore, all of results of parallel and counter-flow are multiplied by 1.0811, then:

"Counter-Flow"

\[
\begin{align*}
\varphi & = \frac{1}{2} \\
\frac{T_{oc}}{T_{in}} & = 0.2
\end{align*}
\]

\[
\therefore \frac{T_{oc} \text{F-L}}{T_{in}} = 0.2 \times 1.0811 = 0.2162 < 0.24 = \frac{T_{oc}\text{L}}{T_{in}} \bigg|_{\text{Counter}}
\]

As it shows, if we fit parallel-flow then we diminish the error in counter-flow by 6.75%. Then the new error is: 9.9%.
\[ \phi = \frac{1}{2} \]

\[ \frac{T_{oc}}{T_{en}} \bigg|_{\text{Parallel}} = 0.375 \quad \Rightarrow \quad \frac{T_{oc}}{T_{en}} = \frac{0.432}{0.375} = 1.152 \]

The error is decreased by 12\% and new error is 7.8\%.

Case: \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

\[ \phi = \frac{1}{2} \]

\[ \frac{T_{oc}}{T_{en}} \bigg|_{\text{Parallel}} = 0.392 \quad \Rightarrow \quad \frac{T_{oc}}{T_{en}} = \frac{0.432}{0.392} = 1.102 \]

\[ \phi = \frac{1}{4} \]

\[ \bigg| \frac{T_{oc}}{T_{en}} \bigg|_{\text{Counter-F}} = 0.22 \times 1.033 = 0.227 \]

The error is decreased by 3\% and new error is 5.4\%.
The error is decreased by 7%, and new error is 5.6%.

\[ \varphi = \frac{1}{4} \left\{ \begin{array}{l}
\frac{T_{oc}}{T_{in}} \bigg|_{\text{rem}} = 0.401 \quad \Rightarrow \quad \frac{T_{oc}E}{T_{oc}L} = 0.402 \\
\frac{T_{oc}}{T_{in}} \bigg|_{\text{counter}} = 0.444 \times 1.077 = 0.479
\end{array} \right. 
\]

The error is decreased by 7% and new error is 4.2%.

ii-Fitting Counter-Flow:

Case: \( a = 4 \), \( a = 1 \), \( r = 1 \)

\[ \varphi = \frac{1}{2} \left\{ \begin{array}{l}
\frac{T_{oc}}{T_{in}} \bigg|_{\text{counter}} = 0.2 \quad \Rightarrow \quad \frac{T_{oc}E}{T_{oc}L} = \frac{0.24}{0.2} = 1.2
\end{array} \right. 
\]

Fitting Factor = 1.2

\[ \frac{T_{oc}}{T_{in}} \bigg|_{\text{counter}} = 1.2 \times 1.837 = 2.22 \quad 0.1936 = \frac{T_{oc}}{T_{in}} \bigg|_{\text{exact}} \]

Its value is more than exact value, but it is also farther to exact value than unfitted, and it increases the error by 3.2%, and new error is 10.7%.

\[ \varphi = \frac{1}{3} \left\{ \begin{array}{l}
\frac{T_{oc}}{T_{in}} \bigg|_{\text{counter}} = 0.213 \quad \Rightarrow \quad \frac{T_{oc}E}{T_{oc}L} = \frac{0.24}{0.213} = 1.126
\end{array} \right. 
\]

\[ \frac{T_{oc}}{T_{in}} \bigg|_{\text{rem}} = 1.034 \times 1.126 = 1.1936 \]
It increases the error by 2.6% , and new error is 7.25% .

\[
\varphi = \frac{1}{4}
\]  
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ConvEy}}}^{-} \frac{.22}{.22} = 1.09\]
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ParEy}}}^{-} \frac{.1922 \times 1.09 = .209 \times .986}\]

It increases the error by 2% , and new error is 5.2% .

For case: \(a_1 = a_2 = 1\)

\[
\varphi = \frac{1}{2}
\]  
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ConvEy}}}^{-} \frac{.4}{.4} = 1.25\]
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ParEy}}}^{-} \frac{.375 \times 1.25 = .4687 > .432}\]

Because the new value is closer to exact value but higher then it decreases the error by 4.7% , and new error is 8.5% .

\[
\varphi = \frac{1}{3}
\]  
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ConvEy}}}^{-} \frac{.4286}{.4286} = 1.16\]
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ParEy}}}^{-} \frac{.392 \times 1.16 = .457 > .432}\]

It decreases the error by 3.5% , and new error is 5.8% .

\[
\varphi = \frac{1}{4}
\]  
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ConvEy}}}^{-} \frac{.444}{.444} = 1.125\]
\[\frac{T_{oc}}{T_{ch}} \rightarrow_{{\text{ParEy}}}^{-} \frac{.401 \times 1.125 = .451 > .432}\]
It decreases the error by 2.8%, and new error is 4.4%.
The general form of energy balance for each section of Lumped Models gives following equations for heat exchangers of Fig. (A-5)&(A-6).

I- Parallel-Flow:

\[
\begin{align*}
(s+8) \bar{T}_{H2} &= 4 \bar{T}_{H1} + 4 \bar{T}_{C2} \\
(s+5) \bar{T}_{C2} &= 4 \bar{T}_{C1} + \bar{T}_{H2} \\
(s+8) \bar{T}_{H3} &= 4 \bar{T}_{H2} + 4 \bar{T}_{C3} \\
(s+5) \bar{T}_{C3} &= 4 \bar{T}_{C2} + \bar{T}_{H3} \\
(s+8) \bar{T}_{H4} &= 4 \bar{T}_{C4} + 4 \bar{T}_{H3}
\end{align*}
\]

For case \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

**Figure (A-5): Four Sections Heat-Exchanger**
\[
\begin{align*}
(s+5) \, T_{c4} &= 4 \, T_{c3} + T_{m4} \\
(s+8) \, T_{m5} &= 4 \, T_{m4} + 4 \, T_{c5} \\
(s+5) \, T_{c6} &= 4 \, T_{c4} + T_{m5}
\end{align*}
\]

Solution of these equations gives following results:

\[
G_1(s) = \frac{T_{c5}}{T_{m4}} = \frac{256(4s^3 + 78s^2 + 532s + 1281)}{(s+9)^4(s+4)^4} \tag{A-15}
\]

\[
G_2(s) = \frac{T_{m5}}{T_{m4}} = \frac{256[(s+5)^4 + 12(s+5)^2 + 8(s+8)(s+5) + 4(s+8)^2 + 16]}{(s+9)^4(s+4)^4} \tag{A-16}
\]

For case: \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

Equations for this case are:

\[
\begin{align*}
(s+5) \, T_{m2} &= 4 \, T_{m1} + T_{c2} \\
(s+5) \, T_{c2} &= 4 \, T_{c1} + T_{m2} \\
(s+5) \, T_{m3} &= 4 \, T_{m2} + T_{c3} \\
(s+5) \, T_{c3} &= 4 \, T_{c2} + T_{m3} \\
(s+5) \, T_{m4} &= 4 \, T_{m3} + T_{c4} \\
(s+5) \, T_{c4} &= 4 \, T_{c3} + T_{m4} \\
(s+5) \, T_{m5} &= 4 \, T_{m4} + T_{c5} \\
(s+5) \, T_{c5} &= 4 \, T_{c4} + T_{m5}
\end{align*}
\]

The solution of above equations gives following results:

\[ G_1(s) = \frac{T_{C5}}{T_{H1}} = \frac{1024(s+5)(s^2+10s+26)}{(s+6)^4(s+4)^4} \] (A-17)

\[ G_2(s) = \frac{T_{H5}}{T_{H1}} = \frac{256(s+5)^4+1536(s+5)^3256}{(s+6)^4(s+4)^4} \] (A-18)

For case: \( a_1 = 2 \), \( a_2 = 1 \), \( r = 2 \)

\[ q_1 = q_2 = q_3 = q_4 = \frac{1}{4} \]

The equations for each Lumped Model give following results:

\[
\begin{align*}
(5+6) \ T_{H2} &= 4 \ T_{H1} + 2 \ T_{C2} \\
(5+5) \ T_{C2} &= 4 \ T_{C1} + \ T_{H2} \\
(5+6) \ T_{H3} &= 4 \ T_{H2} + 2 \ T_{C3} \\
(5+5) \ T_{C3} &= 4 \ T_{C2} + \ T_{H3} \\
(5+6) \ T_{H4} &= 4 \ T_{H3} + 2 \ T_{C4} \\
(5+5) \ T_{C4} &= 4 \ T_{C3} + \ T_{H4} \\
(5+6) \ T_{H5} &= 4 \ T_{H4} + 2 \ T_{C5} \\
(5+5) \ T_{C5} &= 4 \ T_{C4} + \ T_{H5}
\end{align*}
\]

And by solving the above equations, following results are gained:

\[ G_1(s) = \frac{T_{C5}}{T_{H1}} = \frac{16(28+5)^3+16(s+6)^3+16(28+5)(s+6)(s+11)+64(38+11)}{(s^2+8.5s+14)^4} \] (A-19)
\[ C_2(s) = \frac{T_{hs}}{T_{ni}} = \frac{16(s+5)^3 + 96(s+5)^2 + 64(s+6)(s+5) + 32(s+6)^2 + 64}{(s^2 + 8.5s + 14)^4} \] (A-20)
II- Counter-Flow Heat-Exchanger:

Fig. (A-6) shows the counter flow case for four-Lumped Models.

For case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

\[
q_1 = q_2 = q_3 = q_4 = \frac{1}{4}
\]

Using equation of each section for these eight sections, results are:

\[
\begin{align*}
(5+8) & \quad T_{H2} = 4 \overline{T_{H1}} + 4 \overline{T_{C5}} \\
(5+5) & \quad T_{C5} = 4 \overline{T_{C4}} + \overline{T_{H2}} \\
(5+8) & \quad T_{H3} = 4 \overline{T_{H2}} + 4 \overline{T_{C4}} \\
(5+5) & \quad T_{C4} = 4 \overline{T_{C3}} + \overline{T_{H3}} \\
(5+8) & \quad T_{H4} = 4 \overline{T_{H3}} + 4 \overline{T_{C3}} \\
(5+5) & \quad T_{C3} = 4 \overline{T_{C2}} + \overline{T_{H4}}
\end{align*}
\]
There are eight equations and eight unknowns which
solutions of them gives following results:

\[
G(\varphi) = \frac{4(r^6 + 29r^5 + 631r^4 + 5421r^3 + 26124r^2 + 67185r + 74048)}{s^8 + 52s^7 + 1158s^6 + 14404s^5 + 109153s^4 + 513528s^3 + 14524485s^2 + 21477285s + 134217728} \quad (A-21)
\]

\[
G(\varphi) = \frac{256(s+5)(r^6 + 39r^5 + 631r^4 + 5317r^3 + 25080r^2 + 62405r + 6400)}{(s+5)(s^6 + 52s^5 + 1158s^4 + 14404s^3 + 109153s^2 + 513528s^3 + 14524485s^2 + 21477285s + 134217728)} \quad (A-22)
\]

For case: \( a_1 = 1, \quad a_2 = 1, \quad r = 1 \)

\[
q_1 = q_2 = q_3 = q_4 = \frac{1}{4}
\]

There are following equations:

\[
\begin{align*}
(s+5) \quad \bar{T}_{n2} &= 4 \bar{T}_{n1} + \bar{T}_{c5} \\
(s+5) \quad \bar{T}_{c5} &= 4 \bar{T}_{c4} + \bar{T}_{n2} \\
(s+5) \quad \bar{T}_{n3} &= 4 \bar{T}_{n2} + \bar{T}_{c4} \\
(s+5) \quad \bar{T}_{c4} &= 4 \bar{T}_{c3} + \bar{T}_{n3} \\
(s+5) \quad \bar{T}_{n4} &= 4 \bar{T}_{n3} + \bar{T}_{c3} \\
(s+5) \quad \bar{T}_{c3} &= 4 \bar{T}_{c2} + \bar{T}_{n4} \\
(s+5) \quad \bar{T}_{n5} &= 4 \bar{T}_{n4} + \bar{T}_{c2} \\
(s+5) \quad \bar{T}_{c2} &= 4 \bar{T}_{c1} + \bar{T}_{n5}
\end{align*}
\]
And the results of solution are:

\[ G(\beta) = \frac{4(s+4)(s+6)(s+5)^2(s^2+10s+39) + 8s^2+8936s+34400}{s^9+40s^7+69s^6+1880s^5+42208s^4+64600s^3+391680s^2+520960s+288000} \] (A-23)

\[ G(\alpha) = \frac{64(s^6+64s^4+1194s^3+5750s^2+69480s^1+271071s^0+633160s^0+361200s^0+288000}{(s+4)(s+6)(s+5)^9(s^6+40s^4+69s^6+1880s^5+42208s^4+64600s^3+391680s^2+520960s+288000) \} (A-24) \]

For case: \( a_1 = 2, \ a_2 = 1, \ r = 2 \)

Equations for this case are:

\[
\begin{align*}
(5+6) \overline{T_{H2}} &= 4 \overline{T_{H1}} + 2 \overline{T_{C5}} \\
(2s+5) \overline{T_{C5}} &= 4 \overline{T_{C4}} + \overline{T_{H2}} \\
(5+6) \overline{T_{H3}} &= 4 \overline{T_{H2}} + 2 \overline{T_{C4}} \\
(2s+5) \overline{T_{C4}} &= 4 \overline{T_{C3}} + \overline{T_{H3}} \\
(5+6) \overline{T_{H4}} &= 4 \overline{T_{H3}} + 2 \overline{T_{C3}} \\
(2s+5) \overline{T_{C3}} &= 4 \overline{T_{C2}} + \overline{T_{H4}} \\
(5+6) \overline{T_{H5}} &= 4 \overline{T_{H4}} + 2 \overline{T_{C2}} \\
(2s+5) \overline{T_{C2}} &= 4 \overline{T_{C1}} + \overline{T_{H5}}
\end{align*}
\]

Solution of these equations gives following results:
\[ G(\theta) = \frac{2 \left( s^6 + 25.5 s^5 + 246.75 s^4 + 116.5 s^3 + 44(7.5 s^2 + 73)/s + 53/3 \right)}{s^6 + 34.5 s^5 + 438.5 s^4 + 385 s^3 + 1850 s^2 + 5397 s^1 + 9340 s^0 + 8640s + 31472} \]  

(A-25)

\[ G(\theta) = \frac{(2s+5) \left[ (s^4 + 8.5 s^3 + 14) + 32(s^4 - 14) (s^4 - 8.5 s^3 + 14) + 512 + 14 - 172 \right]}{2(s+4) \left[ s^4 + 34.5 s^3 + 438.5 s^2 + 385 s + 1850 s^1 + 5397 s^0 + 9340 s^0 + 8640 s^0 + 31472 \right]} \]  

(A-26)
TABLE (3-1): Results of Steady State for Parallel Flow

For case: $a_1 = 4$, $a_2 = 1$

"Two Lumped-Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{C_2}}{T_{H_1}}$</th>
<th>$\frac{T_{C_3}}{T_{H_1}}$</th>
<th>$\frac{T_{H_2}}{T_{H_1}}$</th>
<th>$\frac{T_{H_3}}{T_{H_1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.184</td>
<td>.199</td>
<td>.266</td>
<td>.205</td>
</tr>
<tr>
<td>Lump</td>
<td>.143</td>
<td>.184</td>
<td>.428</td>
<td>.265</td>
</tr>
</tbody>
</table>

"Three Lumped-Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{C_1}}{T_{H_1}}$</th>
<th>$\frac{T_{C_3}}{T_{H_1}}$</th>
<th>$\frac{T_{C_4}}{T_{H_1}}$</th>
<th>$\frac{T_{H_2}}{T_{H_1}}$</th>
<th>$\frac{T_{H_3}}{T_{H_1}}$</th>
<th>$\frac{T_{H_4}}{T_{H_1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.162</td>
<td>.193</td>
<td>.199</td>
<td>.351</td>
<td>.228</td>
<td>.205</td>
</tr>
<tr>
<td>Lump</td>
<td>.125</td>
<td>.172</td>
<td>.189</td>
<td>.500</td>
<td>.312</td>
<td>.242</td>
</tr>
</tbody>
</table>

"Four Lumped-Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{C_1}}{T_{H_1}}$</th>
<th>$\frac{T_{C_3}}{T_{H_1}}$</th>
<th>$\frac{T_{C_4}}{T_{H_1}}$</th>
<th>$\frac{T_{C_5}}{T_{H_1}}$</th>
<th>$\frac{T_{H_2}}{T_{H_1}}$</th>
<th>$\frac{T_{H_3}}{T_{H_1}}$</th>
<th>$\frac{T_{H_4}}{T_{H_1}}$</th>
<th>$\frac{T_{H_5}}{T_{H_1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.143</td>
<td>.184</td>
<td>.195</td>
<td>.199</td>
<td>.43</td>
<td>.266</td>
<td>.219</td>
<td>.205</td>
</tr>
<tr>
<td>Lump</td>
<td>.111</td>
<td>.160</td>
<td>.182</td>
<td>.192</td>
<td>.555</td>
<td>.358</td>
<td>.270</td>
<td>.231</td>
</tr>
</tbody>
</table>
TABLE(3-2): Results of Steady State for Counter Flow

For case: \( a_1 = 4 \), \( a_2 = 1 \)

"Two Lumped-Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>( \frac{T_{C2}}{T_{H1}} )</th>
<th>( \frac{T_{C3}}{T_{H1}} )</th>
<th>( \frac{T_{C3}}{T_{H1}} )</th>
<th>( \frac{T_{H2}}{T_{H1}} )</th>
<th>( \frac{T_{H3}}{T_{H1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.044</td>
<td>.240</td>
<td>.213</td>
<td>.038</td>
<td></td>
</tr>
<tr>
<td>Lump</td>
<td>.067</td>
<td>.200</td>
<td>.467</td>
<td>.200</td>
<td></td>
</tr>
</tbody>
</table>

"Three Lumped-Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>( \frac{T_{C2}}{T_{H1}} )</th>
<th>( \frac{T_{C3}}{T_{H1}} )</th>
<th>( \frac{T_{C4}}{T_{H1}} )</th>
<th>( \frac{T_{H2}}{T_{H1}} )</th>
<th>( \frac{T_{H3}}{T_{H1}} )</th>
<th>( \frac{T_{H4}}{T_{H1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.022</td>
<td>.080</td>
<td>.240</td>
<td>.360</td>
<td>.124</td>
<td>.038</td>
</tr>
<tr>
<td>Lump</td>
<td>.037</td>
<td>.100</td>
<td>.213</td>
<td>.550</td>
<td>.293</td>
<td>.147</td>
</tr>
</tbody>
</table>

"Four Lumped-Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>( \frac{T_{C2}}{T_{H1}} )</th>
<th>( \frac{T_{C3}}{T_{H1}} )</th>
<th>( \frac{T_{C4}}{T_{H1}} )</th>
<th>( \frac{T_{C5}}{T_{H1}} )</th>
<th>( \frac{T_{H2}}{T_{H1}} )</th>
<th>( \frac{T_{H3}}{T_{H1}} )</th>
<th>( \frac{T_{H4}}{T_{H1}} )</th>
<th>( \frac{T_{H5}}{T_{H1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.014</td>
<td>.044</td>
<td>.107</td>
<td>.240</td>
<td>.466</td>
<td>.213</td>
<td>.094</td>
<td>.038</td>
</tr>
<tr>
<td>Lump</td>
<td>.023</td>
<td>.062</td>
<td>.122</td>
<td>.220</td>
<td>.612</td>
<td>.370</td>
<td>.214</td>
<td>.118</td>
</tr>
</tbody>
</table>
TABLE (3-3): Results of Steady State for Parallel Flow

For case: $a_1 = 1$, $a_2 = 1$

"Two Lumped Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{c2}}{T_{H1}}$</th>
<th>$\frac{T_{c3}}{T_{H1}}$</th>
<th>$\frac{T_{H2}}{T_{H1}}$</th>
<th>$\frac{T_{H3}}{T_{H1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.316</td>
<td>.432</td>
<td>.684</td>
<td>.568</td>
</tr>
<tr>
<td>Lump</td>
<td>.250</td>
<td>.375</td>
<td>.750</td>
<td>.625</td>
</tr>
</tbody>
</table>

"Three Lumped Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{c2}}{T_{H1}}$</th>
<th>$\frac{T_{c3}}{T_{H1}}$</th>
<th>$\frac{T_{c4}}{T_{H1}}$</th>
<th>$\frac{T_{H2}}{T_{H1}}$</th>
<th>$\frac{T_{H3}}{T_{H1}}$</th>
<th>$\frac{T_{H4}}{T_{H1}}$</th>
<th>$\frac{T_{H5}}{T_{H1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.243</td>
<td>.368</td>
<td>.432</td>
<td>.757</td>
<td>.632</td>
<td>.568</td>
<td></td>
</tr>
<tr>
<td>Lump</td>
<td>.200</td>
<td>.320</td>
<td>.392</td>
<td>.800</td>
<td>.680</td>
<td>.608</td>
<td></td>
</tr>
</tbody>
</table>

"Four Lumped Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{c2}}{T_{H1}}$</th>
<th>$\frac{T_{c3}}{T_{H1}}$</th>
<th>$\frac{T_{c4}}{T_{H1}}$</th>
<th>$\frac{T_{c5}}{T_{H1}}$</th>
<th>$\frac{T_{H2}}{T_{H1}}$</th>
<th>$\frac{T_{H3}}{T_{H1}}$</th>
<th>$\frac{T_{H4}}{T_{H1}}$</th>
<th>$\frac{T_{H5}}{T_{H1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.197</td>
<td>.316</td>
<td>.388</td>
<td>.432</td>
<td>.803</td>
<td>.684</td>
<td>.612</td>
<td>.568</td>
</tr>
<tr>
<td>Lump</td>
<td>.167</td>
<td>.278</td>
<td>.352</td>
<td>.400</td>
<td>.833</td>
<td>.722</td>
<td>.648</td>
<td>.600</td>
</tr>
</tbody>
</table>
TABLE (3-4): Results of Steady State for Counter-Flow

For case: $a_4 = 1$, $a_2 = 1$

"Two Lumped Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{C2}}{T_{H1}}$</th>
<th>$\frac{T_{C3}}{T_{H1}}$</th>
<th>$\frac{T_{H2}}{T_{H1}}$</th>
<th>$\frac{T_{H3}}{T_{H1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.25</td>
<td>.50</td>
<td>.75</td>
<td>.50</td>
</tr>
<tr>
<td>Lump</td>
<td>.20</td>
<td>.40</td>
<td>.80</td>
<td>.60</td>
</tr>
</tbody>
</table>

"Three Lumped Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{C2}}{T_{H4}}$</th>
<th>$\frac{T_{C3}}{T_{H4}}$</th>
<th>$\frac{T_{C4}}{T_{H4}}$</th>
<th>$\frac{T_{H2}}{T_{H4}}$</th>
<th>$\frac{T_{H3}}{T_{H4}}$</th>
<th>$\frac{T_{H4}}{T_{H4}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.167</td>
<td>.333</td>
<td>.50</td>
<td>.833</td>
<td>.667</td>
<td>.50</td>
</tr>
<tr>
<td>Lump</td>
<td>.142</td>
<td>.289</td>
<td>.429</td>
<td>.857</td>
<td>.714</td>
<td>.571</td>
</tr>
</tbody>
</table>

"Four Lumped Models"

<table>
<thead>
<tr>
<th>Temp. Ratio</th>
<th>$\frac{T_{C2}}{T_{H5}}$</th>
<th>$\frac{T_{C3}}{T_{H5}}$</th>
<th>$\frac{T_{C4}}{T_{H5}}$</th>
<th>$\frac{T_{C5}}{T_{H5}}$</th>
<th>$\frac{T_{H2}}{T_{H5}}$</th>
<th>$\frac{T_{H3}}{T_{H5}}$</th>
<th>$\frac{T_{H4}}{T_{H5}}$</th>
<th>$\frac{T_{H5}}{T_{H5}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.125</td>
<td>.25</td>
<td>.375</td>
<td>.50</td>
<td>.875</td>
<td>.75</td>
<td>.625</td>
<td>.50</td>
</tr>
<tr>
<td>Lump</td>
<td>.111</td>
<td>.222</td>
<td>.333</td>
<td>.444</td>
<td>.888</td>
<td>.777</td>
<td>.666</td>
<td>.555</td>
</tr>
</tbody>
</table>
TABLE (3-5): Results of Fitting Parallel-Flow "Static Case"

For case: \( a_1 = 4 \), \( a_2 = 1 \)

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \frac{T_{dc}}{T_{nm}} \mid L-P )</th>
<th>( \frac{T_{dc}}{T_{nm}} \mid L-C )</th>
<th>Unfitted Error</th>
<th>Fitted Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>.199</td>
<td>.216</td>
<td>.240</td>
<td>16.7%</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>.199</td>
<td>.224</td>
<td>.240</td>
<td>11.25%</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>.199</td>
<td>.228</td>
<td>.240</td>
<td>8.3%</td>
</tr>
</tbody>
</table>

TABLE (3-6): Results of Fitting Parallel-Flow "Static Case"

For case: \( a_1 = 1 \), \( a_2 = 1 \)

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \frac{T_{dc}}{T_{nm}} \mid L-C )</th>
<th>Unfitted Error</th>
<th>Fitted Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>.432</td>
<td>.461</td>
<td>.50</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>.432</td>
<td>.473</td>
<td>.50</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>.432</td>
<td>.480</td>
<td>.50</td>
</tr>
</tbody>
</table>
TABLE (3-7): Results of Fitting Counter-Flow "Static Case"

For case: \( a_1 = 4 \), \( a_2 = 1 \)

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \frac{T_{oc}}{T_{ih}} \mid L-C )</th>
<th>( \frac{T_{oc}}{T_{ih}} \mid L-P )</th>
<th>( \frac{T_{oc}}{T_{ih}} \mid E-P )</th>
<th>Unfitted Error</th>
<th>Fitted Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>.240</td>
<td>.220</td>
<td>.199</td>
<td>7.5%</td>
<td>10.9%</td>
</tr>
<tr>
<td>1/3</td>
<td>.240</td>
<td>.213</td>
<td>.199</td>
<td>5.0%</td>
<td>7.0%</td>
</tr>
<tr>
<td>1/4</td>
<td>.240</td>
<td>.209</td>
<td>.199</td>
<td>3.5%</td>
<td>5.2%</td>
</tr>
</tbody>
</table>

TABLE (3-8): Results of Fitting Counter-Flow "Static Case"

For case: \( a_1 = 1 \), \( a_2 = 1 \)

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \frac{T_{oc}}{T_{ih}} \mid L-C )</th>
<th>( \frac{T_{oc}}{T_{ih}} \mid L-P )</th>
<th>( \frac{T_{oc}}{T_{ih}} \mid E-P )</th>
<th>Unfitted Error</th>
<th>Fitted Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>.50</td>
<td>.469</td>
<td>.432</td>
<td>13.2%</td>
<td>8.6%</td>
</tr>
<tr>
<td>1/3</td>
<td>.50</td>
<td>.457</td>
<td>.432</td>
<td>9.2%</td>
<td>5.8%</td>
</tr>
<tr>
<td>1/4</td>
<td>.50</td>
<td>.450</td>
<td>.432</td>
<td>7.4%</td>
<td>4.2%</td>
</tr>
</tbody>
</table>
TABLE(4-1): Gain and Phase Angle
For Case: $a_1=4$, $a_2=1$, $r=1$

"Exact Results"  "Friedly's Results"

| $\Omega$ | $|G|_m$  | $\theta^\circ$ | $|G|_m$ | $\theta^\circ$ |
|---------|---------|---------------|---------|---------------|
| 0.0    | 1.0     | 0.0           | 1.0     | 0.0           |
| 0.1    | 0.990   | -3.88         | 0.990   | -2.8          |
| 0.3    | 0.970   | -11.2         | 0.984   | -8.04         |
| 0.7    | 0.880   | -22.8         | 0.946   | -20.8         |
| 1.0    | 0.820   | -29.4         | 0.900   | -25.5         |
| 2.0    | 0.630   | -45.0         | 0.707   | -45.0         |
| 4.0    | 0.416   | -60.1         | 0.458   | -62.7         |
| 6.0    | 0.300   | -69.1         | 0.325   | -71.0         |
| 10.0   | 0.200   | -76.0         | 0.202   | -78.3         |
| 20.0   | 0.100   | -82.7         | 0.102   | -84.1         |
| 50.0   | 0.041   | -86.7         | 0.041   | -87.6         |
| 80.0   | 0.026   | -88.1         | 0.026   | -88.5         |
TABLE (4-2): Gain and Phase Angle

For Case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

"Two Lumped Models"

| \( \Omega \) | \(|G|\) | \(|G|/\|\) | \(\Theta^\circ\) | \(|G|_{\text{fitted}}\) | \(|G|_{\text{\|fitted|}}\) |
|---|---|---|---|---|---|
| 0.0 | 0.2 | 0.832 | 0.0 | 0.2405 | 1.000 |
| 0.1 | 0.199 | 0.831 | -5.1 | 0.240 | 0.997 |
| 0.3 | 0.195 | 0.812 | -15.2 | 0.234 | 0.974 |
| 0.7 | 0.175 | 0.73 | -33.5 | 0.210 | 0.880 |
| 1.0 | 0.157 | 0.655 | -45.0 | 0.189 | 0.780 |
| 2.0 | 0.106 | 0.440 | -71.5 | 0.127 | 0.536 |
| 4.0 | 0.055 | 0.230 | -99.5 | 0.066 | 0.268 |
| 6.0 | 0.034 | 0.140 | -115.5 | 0.040 | 0.169 |
| 10.0 | 0.016 | 0.065 | -135 | 0.020 | 0.078 |
| 20.0 | 0.005 | 0.020 | -155 | 0.006 | 0.024 |
| 50.0 | 0.001 | 0.003 | -169.7 | 0.001 | 0.004 |
| 80.0 | 0.0003 | 0.001 | -173.6 | 0.0003 | 0.001 |
TABLE(4-3): Gain and Phase Angle

For Case: $a_1=4$ , $a_2=1$ , $r=1$

"Three Lumped Models"

| $\Omega$ | $|G|$ | $|G|_M$ | $\theta^\circ$ | $|G|_{\text{fitted}}$ | $|G|_{M_f\text{fitted}}$ |
|----------|------|--------|----------------|---------------------|---------------------|
| 0.0      | 0.2133 | 0.890  | 0.0            | 0.2405              | 1.00                |
| 0.1      | 0.213  | 0.886  | -4.6           | 0.240               | 0.997               |
| 0.3      | 0.209  | 0.870  | -13.5          | 0.236               | 0.980               |
| 0.7      | 0.192  | 0.800  | -30            | 0.216               | 0.900               |
| 1.0      | 0.174  | 0.725  | -40.6          | 0.196               | 0.816               |
| 2.0      | 0.122  | 0.506  | -65.5          | 0.137               | 0.570               |
| 4.0      | 0.067  | 0.280  | -91.7          | 0.076               | 0.315               |
| 6.0      | 0.043  | 0.180  | -107.5         | 0.048               | 0.202               |
| 10.0     | 0.021  | 0.090  | -127.6         | 0.024               | 0.101               |
| 20.0     | 0.007  | 0.030  | -147.5         | 0.008               | 0.034               |
| 50.0     | 0.001  | 0.005  | -167.5         | 0.001               | 0.006               |
| 80.0     | 0.0003 | 0.001  | -171.6         | 0.0003              | 0.001               |
TABLE (4-4): Gain and Phase Angle

For Case \( a_1=4 \), \( a_2=1 \), \( r=1 \)

"Four Lumped Models"

| \( \Omega \) | \( |G| \) | \( |G|_M \) | \( \theta^\circ \) | \( |G|_{\text{fitted}} \) | \( |G|_{M,\text{fitted}} \) |
|---|---|---|---|---|---|
| 0.0 | 0.2202 | 0.918 | 0.0 | 0.2405 | 1.0 |
| 0.1 | 0.2200 | 0.916 | -4.2 | 0.240 | 0.998 |
| 0.3 | 0.2160 | 0.900 | -12.6 | 0.236 | 0.981 |
| 0.7 | 0.1990 | 0.830 | -28.0 | 0.218 | 0.905 |
| 1.0 | 0.1830 | 0.761 | -38.0 | 0.200 | 0.830 |
| 2.0 | 0.1310 | 0.543 | -61.8 | 0.142 | 0.592 |
| 4.0 | 0.0750 | 0.312 | -86.0 | 0.082 | 0.340 |
| 6.0 | 0.050 | 0.210 | -101.3 | 0.055 | 0.230 |
| 10.0 | 0.026 | 0.110 | -121.4 | 0.029 | 0.120 |
| 20.0 | 0.010 | 0.040 | -145.6 | 0.010 | 0.043 |
| 50.0 | 0.001 | 0.006 | -165.3 | 0.002 | 0.007 |
| 80.0 | - | - | -170 | - | - |
TABLE(4-5): Gain and Phase Angle

For Case: $a_1 = 1$, $a_2 = 1$, $r = 1$

"Exact Results" "Friedly's Results"

**COUNTER-FLOW**

| $\Omega$ | $|G|_N$ | $\phi$ | $\Omega$ | $|G|_N$ | $\phi$ |
|----------|---------|--------|----------|---------|--------|
| 0.0      | 1.00    | 0.0    | 0.0      | 1.00    | 0.0    |
| 0.1      | 0.997   | -4.9   | 0.1      | 0.999   | -6.6   |
| 0.3      | 0.980   | -14.2  | 0.3      | 0.980   | -6.6   |
| 0.7      | 0.850   | -37.8  | 0.7      | 0.880   | -33    |
| 1.0      | 0.814   | -46    | 1.0      | 0.822   | -44    |
| 2.0      | 0.460   | -73.3  | 2.0      | 0.523   | -66    |
| 4.0      | 0.254   | -69    | 4.0      | 0.250   | -75.5  |
| 6.0      | 0.142   | -84.3  | 6.0      | 0.143   | -80    |
| 10.0     | 0.096   | -76.5  | 10.0     | 0.087   | -82    |
| 20.0     | 0.055   | -81.6  | 20.0     | 0.045   | -82    |
| 50.0     | 0.018   | -93.4  | 50.0     | 0.020   | -81.6  |
| 80.0     | 0.0141  | -88.6  | 80.0     | 0.014   | -87.0  |
TABLE(4-6): Gain and Phase Angle

For case: $a_1=1, a_2=1, r=1$

"Two Lumped Models"

| $\Omega$ | $|G|$  | $|G|_N$ | $\theta^\circ$ | $|G|_{\text{Fitted}}$ | $|G|_{N, \text{Fitted}}$ |
|----------|-------|--------|----------------|-------------------------|-------------------------|
| 0.0      | 0.400 | 0.800  | 0.0            | 0.50                    | 1.00                    |
| 0.1      | 0.398 | 0.797  | -8.0           | 0.497                   | 0.995                   |
| 0.3      | 0.386 | 0.772  | -18.8          | 0.483                   | 0.965                   |
| 0.7      | 0.338 | 0.676  | -40.5          | 0.422                   | 0.845                   |
| 1.0      | 0.297 | 0.593  | -55.2          | 0.371                   | 0.741                   |
| 2.0      | 0.177 | 0.353  | -87.7          | 0.221                   | 0.442                   |
| 4.0      | 0.076 | 0.151  | -117.2         | 0.094                   | 0.190                   |
| 6.0      | 0.041 | 0.083  | -131.9         | 0.052                   | 0.104                   |
| 10.0     | 0.018 | 0.035  | -148.0         | 0.022                   | 0.044                   |
| 20.0     | 0.005 | 0.010  | -162.7         | 0.006                   | 0.012                   |
| 50.0     | 0.0008| 0.0016 | -173.0         | 0.001                   | 0.002                   |
| 80.0     | 0.0003| 0.0006 | -175.7         | 0.0004                  | 0.0008                  |
TABLE(4-7): Gain and Phase Angle

For case: \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

"Three Lumped Models"

| \( \Omega \) | \( |G| \) | \( |G|_W \) | \( \theta^\circ \) | \( |G|_{\text{fitted}} \) | \( |G|_W_{\text{fitted}} \) |
|-----------|------|--------|----------|-----------------|-----------------|
| 0.0       | 0.4286 | 0.8672 | 0.0      | 0.500           | 1.000           |
| 0.1       | 0.4272 | 0.8544 | -5.8     | 0.498           | 0.996           |
| 0.3       | 0.4165 | 0.8330 | -17.3    | 0.486           | 0.971           |
| 0.7       | 0.3720 | 0.7430 | -38.4    | 0.433           | 0.867           |
| 1.0       | 0.3280 | 0.6550 | -52.0    | 0.382           | 0.764           |
| 2.0       | 0.1980 | 0.3960 | -83.16   | 0.231           | 0.462           |
| 4.0       | 0.0886 | 0.1770 | -108.3   | 0.103           | 0.210           |
| 6.0       | 0.0527 | 0.1054 | -122.0   | 0.061           | 0.123           |
| 10.0      | 0.0243 | 0.0486 | -140.6   | 0.0283          | 0.057           |
| 20.0      | 0.0070 | 0.0141 | -158.0   | 0.0080          | 0.016           |
| 50.0      | 0.0012 | 0.0024 | -171.0   | 0.001           | 0.003           |
| 80.0      | 0.0004 | 0.0009 | -174.6   | 0.0005          | 0.001           |
TABLE (4-8): Gain and Phase Angle

For case: \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)

"Four Lumped Models"

| \( \Omega \) | \( |G| \) | \( |G|_N \) | \( \theta^\circ \) | \( |G|_{\text{filtered}} \) | \( |G|_{N,\text{filtered}} \) |
|-----------|--------|--------|--------|----------------|----------------|
| 0.0       | 0.444  | 0.888  | 0.0    | 0.500          | 1.000          |
| 0.1       | 0.443  | 0.886  | -5.5   | 0.499          | 0.997          |
| 0.3       | 0.434  | 0.868  | -16.3  | 0.488          | 0.976          |
| 0.7       | 0.392  | 0.785  | -36.7  | 0.441          | 0.883          |
| 1.0       | 0.350  | 0.700  | -58.0  | 0.393          | 0.787          |
| 2.0       | 0.212  | 0.414  | -81.25 | 0.239          | 0.477          |
| 4.0       | 0.094  | 0.189  | -102   | 0.106          | 0.212          |
| 6.0       | 0.060  | 0.120  | -114.5 | 0.067          | 0.134          |
| 10.0      | 0.030  | 0.060  | -132.75| 0.033          | 0.066          |
| 20.0      | 0.009  | 0.018  | -153   | 0.010          | 0.020          |
| 50.0      | 0.0016 | 0.0032 | -166   | 0.0018         | 0.0036         |
| 80.0      | 0.0006 | 0.0012 | -173   | 0.0007         | 0.0014         |
TABLE (4-9): Gain and Phase Angle

For Case: $a_4 = 2$, $a_2 = 1$, $r = 2$

"Exact Results"            "Friedly's Results"

COUNTER-FLOW

| $\Omega$ | $|G|_n$ | $\theta^\circ$ | $\Omega$ | $|G|_n$ | $\theta^\circ$ |
|----------|---------|----------------|----------|---------|----------------|
| 0.0      | 1.00    | 0.0            | 0.0      | 1.00    | 0.0            |
| 0.1      | 0.996   | -6.7           | 0.1      | 0.999   | -7.0           |
| 0.3      | 0.954   | -49.3          | 0.3      | 0.939   | -20.0          |
| 0.7      | 0.790   | -41.0          | 0.7      | 0.760   | -40.5          |
| 1.0      | 0.642   | -53.4          | 1.0      | 0.633   | -50.5          |
| 2.0      | 0.370   | -72.2          | 2.0      | 0.378   | -67.5          |
| 4.0      | 0.196   | -77.1          | 4.0      | 0.203   | -74.2          |
| 6.0      | 0.135   | -82.6          | 6.0      | 0.134   | -76.0          |
| 10.0     | 0.086   | -87.1          | 10.0     | 0.080   | -84.3          |
| 20.0     | 0.045   | -87.55         | 20.0     | 0.042   | -85.0          |
| 50.0     | 0.018   | -87.1          | 50.0     | 0.018   | -87.6          |
| 80.0     | 0.011   | -89.87         | 80.0     | 0.011   | 89.80          |
TABLE 4-10: Gain and Phase Angle
For Case: $a_1=2$, $a_2=1$, $r=2$
"Two Lumped Models"

| $\Omega$ | $|G|$  | $|G_H|$ | $\theta^\circ$ | $|G|_{fitted}$ | $|G|_{H_fitted}$ |
|---------|-------|---------|----------------|----------------|-----------------|
| 0.0     | 0.304 | 0.786   | 0.0            | 0.387          | 1.00            |
| 0.1     | 0.301 | 0.778   | -9.16          | 0.384          | 0.990           |
| 0.3     | 0.281 | 0.726   | -25.5          | 0.358          | 0.924           |
| 0.7     | 0.217 | 0.560   | -53.7          | 0.276          | 0.712           |
| 1.0     | 0.173 | 0.448   | -68.1          | 0.220          | 0.570           |
| 2.0     | 0.092 | 0.237   | -95.0          | 0.117          | 0.300           |
| 4.0     | 0.040 | 0.101   | -120.5         | 0.050          | 0.129           |
| 6.0     | 0.021 | 0.055   | -134.4         | 0.027          | 0.070           |
| 10.0    | 0.009 | 0.023   | -150.5         | 0.011          | 0.030           |
| 20.0    | 0.0024| 0.0063  | -164.5         | 0.0027         | 0.007           |
| 50.0    | 0.0004| 0.001   | -173.75        | 0.0005         | 0.0013          |
| 80.0    | 0.0002| 0.0004  | -176.0         | 0.0002         | 0.0005          |
TABLE(4-11): Gain and Phase Angle

For case: $a_1=2$, $a_2=1$, $r=2$

"Three Lumped Models"

| $\Omega$ | $|G|$ | $|G|_N$ | $\theta^\circ$ | $|G|_{Fitted}$ | $|G|_{H,Fitted}$ |
|----------|-------|---------|---------------|----------------|----------------|
| 0.0      | 0.328 | 0.847   | 0.0           | 0.387          | 1.00           |
| 0.1      | 0.324 | 0.837   | -8.3          | 0.383          | 0.988          |
| 0.3      | 0.307 | 0.792   | -24.25        | 0.362          | 0.934          |
| 0.7      | 0.243 | 0.627   | -50.0         | 0.286          | 0.740          |
| 1.0      | 0.197 | 0.510   | -63.75        | 0.232          | 0.600          |
| 2.0      | 0.107 | 0.276   | -89.0         | 0.126          | 0.325          |
| 4.0      | 0.050 | 0.127   | -112.3        | 0.060          | 0.150          |
| 6.0      | 0.028 | 0.074   | -126.7        | 0.034          | 0.087          |
| 10.0     | 0.013 | 0.033   | -143.7        | 0.015          | 0.038          |
| 20.0     | 0.004 | 0.009   | -160.5        | 0.004          | 0.011          |
| 50.0     | 0.0006| 0.0015  | -172          | 0.0007         | 0.0018         |
| 80.0     | 0.0002| 0.0006  | -175          | 0.0003         | 0.0007         |
TABLE (4-12): Gain and Phase Angle

For case: \( a_1 = 2 \), \( a_2 = 1 \), \( r = 2 \)

"Four Lumped Models"

| \( \Omega \) | \( |G| \) | \( |G|_N \) | \( \theta^\circ \) | \( |G|_{\text{fitted}} \) | \( |G|_{N, \text{fitted}} \) |
|-------------|---------|---------|-------------|----------------|----------------|
| 0.0         | .341    | .881    | 0.0         | .387           | 1.000          |
| 0.1         | .339    | .874    | -7.7        | .384           | .993           |
| 0.3         | .321    | .828    | -22.0       | .364           | .940           |
| 0.7         | .257    | .665    | -48.0       | .292           | .755           |
| 1.0         | .210    | .542    | -61.5       | .238           | .616           |
| 2.0         | .116    | .300    | -76.0       | .132           | .340           |
| 4.0         | .050    | .124    | -81.7       | .055           | .160           |
| 6.0         | .033    | .085    | -104.8      | .037           | .097           |
| 10.0        | .021    | .055    | -135.0      | .024           | .062           |
| 20.0        | .004    | .012    | -160.0      | .005           | .014           |
| 50.0        | .0008   | .002    | -171.1      | .0009          | .0022          |
| 80.0        | .0003   | .0008   | -174.2      | .0003          | .0009          |
TABLE (4-13): Gain and Phase Angle

For Case $a_1 = 4$, $a_2 = 1$, $r = 1$

"Exact Results for Parallel Flow"

| $\Omega$ | $|G|$ | $|G|_w$ | $\theta^o$ |
|----------|------|---------|-----------|
| 0.0      | 0.1986 | 1.0     | -0.0     |
| 0.1      | 0.1986 | 1.0     | -5.7     |
| 0.3      | 0.1986 | 1.0     | -17.2    |
| 0.7      | 0.1986 | 1.0     | -40.3    |
| 1.0      | 0.1986 | 1.0     | -57.6    |
| 2.0      | 0.1986 | 1.0     | -114.6   |
| 4.0      | 0.1986 | 1.0     | -229.2   |
| 6.0      | 0.1986 | 1.0     | -343.7   |
| 10.0     | 0.1986 | 1.0     | -573     |
TABLE (4-14): Gain and phase Angle

For case: $a_1 = 4$, $a_2 = 1$, $r = 1$

"Two Lumped Models"

| $\Omega$ | $|G|$ | $|G|_W$ | $\theta^\circ$ | $|G|_{\text{Fitted}}$ | $|G_{W,\text{Fitted}}$ |
|----------|------|--------|--------------|-----------------|------------------|
| 0        | .1837| .925   | 0            | .1986           | 1.0              |
| .1       | .183 | .922   | -6.1         | .198            | .997             |
| .3       | .180 | .905   | -18          | .194            | .978             |
| .7       | .164 | .825   | -41          | .177            | .893             |
| 1.0      | .147 | .743   | -56.8        | .159            | .803             |
| 2.0      | .093 | .468   | -98          | .101            | .506             |
| 4.0      | .037 | .186   | -144.7       | .040            | .202             |
| 6.0      | .018 | .089   | -171.2       | .019            | .096             |
| 10.0     | .006 | .03    | -201.6       | .006            | .030             |
TABLE (4-15): Gain and phase Angle

For Case: $a_1 = 4$, $a_2 = 1$, $r = 1$

"Three Lumped Models"

| $\Omega$ | $|G|$ | $|G|_N$ | $\phi$ | $|G|_{\text{Fitted}}$ | $|G|_{N,\text{Fitted}}$ |
|---------|------|--------|-------|----------------|----------------|
| 0.0     | 0.1894 | 0.954  | 0     | 0.1986         | 1.0            |
| 0.1     | 0.189  | 0.952  | -5.9  | 0.198          | 0.998          |
| 0.3     | 0.187  | 0.940  | -17.7 | 0.196          | 0.985          |
| 0.7     | 0.175  | 0.882  | -40.8 | 0.184          | 0.925          |
| 1.0     | 0.162  | 0.817  | -57.3 | 0.170          | 0.857          |
| 2.0     | 0.110  | 0.557  | -105.3| 0.116          | 0.583          |
| 4.0     | 0.042  | 0.214  | -169.4| 0.046          | 0.232          |
| 6.0     | 0.018  | 0.090  | -207.7| 0.019          | 0.094          |
| 10.0    | 0.004  | 0.022  | -252  | 0.005          | 0.023          |
TABLE (4-16): Gain and phase Angle

For Case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

"Four Lumped Models"

| \( \Omega \) | \( |G| \) | \( |G|_N \) | \( \theta^\circ \) | \( |G|_{\text{Fitted}} \) | \( |G|_{N,\text{Fitted}} \) |
|-------|------|-------|-------|--------|----------------|
| 0.0   | 0.1922 | 0.968 | 0     | 0.1986 | 1.0            |
| 0.1   | 0.1920 | 0.967 | -5.9  | 0.1985 | 0.999          |
| 0.3   | 0.1900 | 0.957 | -17.7 | 0.196  | 0.988          |
| 0.7   | 0.181  | 0.910 | -40.9 | 0.187  | 0.941          |
| 1.0   | 0.170  | 0.857 | -57.8 | 0.176  | 0.885          |
| 2.0   | 0.123  | 0.619 | -109.5| 0.127  | 0.640          |
| 4.0   | 0.05   | 0.245 | -186.2| 0.051  | 0.253          |
| 10.0  | 0.004  | 0.020 | -364.2| 0.004  | 0.020          |
TABLE (4-17): Gain and Phase Angle (Parallel Flow)

For Case: $a_1 = 2$, $a_2 = 1$, $r = 2$

"The Exact Results"

| $\Omega$ | $|G|$  | $|G|_{\mu}$ | $\phi^\circ$ |
|----------|-------|-------------|-------------|
| 0        | .3167 | 1.0         | 0           |
| .1       | .3166 | .999        | -8          |
| .3       | .3155 | .996        | -24         |
| .7       | .3110 | .982        | -63         |
| 1.0      | .3055 | .965        | -89         |
| 2.0      | .2730 | .864        | -177        |
| 4.0      | .1720 | .542        | -324        |
| 6.0      | .0623 | .197        | -572        |
| 10.0     | .0525 | .166        | -688        |
**TABLE (4-18): Gain and Phase Angle**

For Case: \( a_f = 2 \), \( a_2 = 1 \), \( r = 2 \)

"Two Lumped Models"

| \( \Omega \) | \(| G |\) | \(| G |_M\) | \( \theta^\circ \) | \(| G |_{\text{fitted}}\) | \(| G |_{M,\text{fitted}}\) |
|-------------|-------|--------|----------|----------------|----------------|
| 0.0         | 0.280 | 0.884  | 0.0      | 0.3167         | 1.00           |
| 0.1         | 0.278 | 0.878  | -10.1    | 0.314          | 0.993          |
| 0.3         | 0.263 | 0.830  | -30.0    | 0.297          | 0.940          |
| 0.7         | 0.208 | 0.656  | -63.3    | 0.235          | 0.742          |
| 1.0         | 0.165 | 0.520  | -84.8    | 0.186          | 0.588          |
| 2.0         | 0.076 | 0.240  | -129     | 0.085          | 0.270          |
| 4.0         | 0.023 | 0.072  | -173     | 0.026          | 0.082          |
| 6.0         | 0.009 | 0.030  | -198     | 0.011          | 0.034          |
| 10.0        | 0.0025| 0.008  | -223     | 0.003          | 0.009          |
TABLE (4-19): Gain and Phase Angle

For Case: \( a_1 = 2 \), \( a_2 = 1 \), \( r = 2 \)

"Three Lumped Models"

| \( \Omega \) | \(|G|\) | \(|G|_N\) | \(\phi\) | \(|G|_{\text{Fitted}}\) | \(|G|_{\text{H Fitted}}\) |
|---|---|---|---|---|---|
| 0.0 | 0.292 | 0.920 | 0.0 | 0.3167 | 1.0 |
| 0.1 | 0.290 | 0.916 | -9.7 | 0.315 | 0.996 |
| 0.3 | 0.280 | 0.882 | -29 | 0.303 | 0.960 |
| 0.7 | 0.234 | 0.740 | -64.6 | 0.255 | 0.804 |
| 1.0 | 0.193 | 0.610 | -88 | 0.210 | 0.663 |
| 2.0 | 0.090 | 0.283 | -144.7 | 0.100 | 0.310 |
| 4.0 | 0.023 | 0.074 | -205 | 0.025 | 0.080 |
| 6.0 | 0.008 | 0.026 | -240 | 0.009 | 0.028 |
| 10.0 | 0.0016 | 0.0053 | -279 | 0.0018 | 0.0057 |
TABLE (4-20): Gain and Phase Angle

For Case: $a_1 = 2$, $a_2 = 1$, $r = 2$

"Four Lumped Models"

| $\Omega$ | $|G|$ | $|G|_W$ | $\theta^\circ$ | $|G|_{Fitted}$ | $|G|_{N,Fitted}$ |
|---------|-------|---------|---------------|----------------|-----------------|
| 0.0     | 0.298 | 0.94    | 0.0           | 0.3167         | 1.0             |
| 0.1     | 0.297 | 0.937   | -9.5          | 0.315          | 0.996           |
| 0.3     | 0.288 | 0.910   | -28.4         | 0.306          | 0.967           |
| 0.7     | 0.250 | 0.790   | -64.5         | 0.266          | 0.840           |
| 1.0     | 0.213 | 0.673   | -89.4         | 0.227          | 0.716           |
| 2.0     | 0.164 | 0.330   | -154.6        | 0.110          | 0.350           |
| 4.0     | 0.025 | 0.080   | -229.3        | 0.026          | 0.084           |
| 6.0     | 0.008 | 0.025   | -274.0        | 0.008          | 0.027           |
| 10.0    | 0.0013| 0.004   | -328.0        | 0.0014         | 0.0043          |
TABLE (5-1): Gain and phase Angle (Using Different Ψ')

For case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \), \( \Psi_1 = \frac{1}{3} \), \( \Psi_2 = \frac{2}{3} \)

"Two Lumped Models"

COUNTER-FLOW

| \( \Omega \) | \( |G| \) | \( |G|_N \) | \( \theta \) | \( |G|_{\text{fitted}} \) | \( |G|_{\Psi_3,\text{fitted}} \) |
|------------|--------|--------|--------|----------------|----------------|
| 0.0        | .198   | .823   | 0.0    | .2405          | 1.00           |
| 0.1        | .197   | .820   | -4.7   | .2400          | .997           |
| 0.3        | .193   | .800   | -13.9  | .2340          | .973           |
| 0.7        | .174   | .723   | -30.0  | .2110          | .878           |
| 1.0        | .157   | .652   | -40.0  | .1900          | .790           |
| 2.0        | .110   | .460   | -62.3  | .1350          | .560           |
| 4.0        | .065   | .270   | -87.0  | .0800          | .328           |
| 6.0        | .043   | .180   | -104.6 | .0520          | .217           |
| 10.0       | .022   | .091   | -125.0 | .0260          | .110           |
| 20.0       | .007   | .030   | -149.5 | .0090          | .036           |
| 50.0       | .001   | .005   | -167.5 | .0010          | .006           |
| 80.0       | .0004  | .002   | -171.5 | .0004          | .002           |
TABLE (5-2): Gain and Phase Angle for Improved Case:

\[ a_1 = 4, \quad a_2 = 1, \quad r = 1 \]

"Two Lumped-Models"  "Exact"

| \( \Omega \) | \( |G| \) | \( |G|_W \) | \( \theta^\circ \) | \( |G|_W \) | \( \theta^\circ \) |
|-------------|---------|---------|---------|---------|---------|
| 0.0         | 0.2405  | 1.000   | 0.0     | 1.00    | 0.0     |
| 0.1         | 0.2400  | 0.997   | -3.75   | 0.99    | -3.8    |
| 0.3         | 0.2350  | 0.977   | -11.0   | 0.97    | -11.2   |
| 0.7         | 0.2140  | 0.890   | -23.8   | 0.88    | -23.0   |
| 1.0         | 0.195   | 0.812   | -31.0   | 0.82    | -29.5   |
| 2.0         | 0.144   | 0.600   | -46.0   | 0.63    | -45.0   |
| 4.0         | 0.094   | 0.390   | -58.0   | 0.41    | -60.0   |
| 6.0         | 0.071   | 0.295   | -64.0   | 0.30    | -69.0   |
| 10.0        | 0.048   | 0.200   | -71.0   | 0.20    | -76.0   |
| 20.0        | 0.025   | 0.106   | -80.0   | 0.10    | -82.7   |
| 50.0        | 0.010   | 0.043   | -86.0   | 0.041   | -86.7   |
| 80.0        | 0.006   | 0.027   | -87.3   | 0.026   | -88.0   |
TABLE (5-3): Gain and Phase Angle for Improved Case:

\[ a_4 = 2, \quad a_2 = 1, \quad r = 2 \]

"Two Lumped Models"  "Exact"

| \( \Omega \) | \( |G| \) | \( |G|_W \) | \( \phi^\circ \) | \( |G|_N \) | \( \phi^\circ \) |
|--------|--------|--------|--------|--------|--------|
| 0.0    | 0.400  | 1.037  | 0.0    | 1.00   | 0.0    |
| 0.1    | 0.397  | 1.026  | -8.0   | 0.996  | -7.0   |
| 0.3    | 0.368  | 0.950  | -23.0  | 0.954  | -20.0  |
| 0.7    | 0.280  | 0.720  | -46.0  | 0.790  | -41.0  |
| 1.0    | 0.222  | 0.573  | -55.0  | 0.642  | -53.5  |
| 2.0    | 0.125  | 0.323  | -67.0  | 0.370  | -72.0  |
| 4.0    | 0.074  | 0.190  | -73.2  | 0.196  | -77.0  |
| 6.0    | 0.051  | 0.132  | -78.5  | 0.135  | -82.0  |
| 10.0   | 0.031  | 0.081  | -84.0  | 0.086  | -87.0  |
| 20.0   | 0.016  | 0.041  | -87.4  | 0.045  | -87.5  |
| 50.0   | 0.007  | 0.018  | -87.7  | 0.018  | -87.7  |
| 80.0   | 0.004  | 0.011  | -89.7  | 0.011  | -89.7  |
TABLE (5-4): Step Response for Counter-Flow Heat-Exchanger

For Case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \frac{T_{sc}}{T_{in}} ) Lumped</th>
<th>( \frac{T_{sc}}{T_{in}} ) Exact</th>
<th>( \frac{T_{sc}}{T_{in}} ) Bounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.25</td>
<td>0.373</td>
<td>0.375</td>
<td>0.400</td>
</tr>
<tr>
<td>0.5</td>
<td>0.580</td>
<td>0.580</td>
<td>0.644</td>
</tr>
<tr>
<td>1.0</td>
<td>0.780</td>
<td>0.785</td>
<td>0.873</td>
</tr>
<tr>
<td>1.5</td>
<td>0.884</td>
<td>0.896</td>
<td>0.955</td>
</tr>
<tr>
<td>2.0</td>
<td>0.925</td>
<td>0.94</td>
<td>0.984</td>
</tr>
<tr>
<td>2.5</td>
<td>0.968</td>
<td>0.97</td>
<td>0.989</td>
</tr>
<tr>
<td>3.0</td>
<td>0.983</td>
<td>0.983</td>
<td>0.992</td>
</tr>
<tr>
<td>3.5</td>
<td>0.991</td>
<td>0.991</td>
<td>0.993</td>
</tr>
<tr>
<td>4.0</td>
<td>0.995</td>
<td>0.995</td>
<td>0.995</td>
</tr>
</tbody>
</table>
TABLE (5-5): Impulse Response for Counter Flow

For case: $a_1 = 4$, $a_2 = 1$, $r = 1$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\frac{T_{ec}}{T_{th}}_{Lumped}$</th>
<th>$\frac{T_{ec}}{T_{th}}_{Exact}$</th>
<th>$\frac{T_{ec}}{T_{th}}_{Exact}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2.16</td>
<td>2.065</td>
<td>2.065</td>
</tr>
<tr>
<td>0.25</td>
<td>1.01</td>
<td>1.05</td>
<td>1.23</td>
</tr>
<tr>
<td>0.5</td>
<td>0.605</td>
<td>0.625</td>
<td>0.735</td>
</tr>
<tr>
<td>0.75</td>
<td>0.410</td>
<td>0.410</td>
<td>0.440</td>
</tr>
<tr>
<td>1.00</td>
<td>0.285</td>
<td>0.275</td>
<td>0.262</td>
</tr>
<tr>
<td>1.25</td>
<td>0.200</td>
<td>0.190</td>
<td>0.160</td>
</tr>
<tr>
<td>1.50</td>
<td>0.150</td>
<td>0.150</td>
<td>0.093</td>
</tr>
<tr>
<td>1.75</td>
<td>0.112</td>
<td>0.120</td>
<td>0.055</td>
</tr>
<tr>
<td>2.00</td>
<td>0.062</td>
<td>0.050</td>
<td>0.03</td>
</tr>
<tr>
<td>2.25</td>
<td>0.051</td>
<td>0.04</td>
<td>0.02</td>
</tr>
<tr>
<td>2.50</td>
<td>0.032</td>
<td>0.03</td>
<td>0.01</td>
</tr>
<tr>
<td>3.00</td>
<td>0.020</td>
<td>0.01</td>
<td>0.005</td>
</tr>
</tbody>
</table>
FIGURE(3-12a): Static Case of Parallel-Flow

For case: $a_1=4$, $a_2=1$
FIGURE (3-12b): Static Case of Counter-Flow

For case: $a_1 = 4$, $a_2 = 1$
FIGURE (3-13a): Static Case of Parallel-Flow

For case: $a_1 = 1$, $a_2 = 1$
FIGURE (3-13b): Static Case of Counter-Flow

For case: $a_1 = 1$, $a_2 = 1$
FIGURE (3-14): Effect of Fitting Counter-Flow by Exact Value. \( a_4 = 4 \), \( a_2 = 1 \)
FIGURE(3-15): Effect of Fitting Parallel-Flow by Exact Values. \( a_1 = 4 \), \( a_2 = 1 \)
FIGURE(3-16): Effect of Fitting Counter-Flow by Exact Value. \( a_1 = 1, \ a_2 = 1 \)

(PARALLEL)
FIGURE (3-17): Effect of Fitting Parallel-Flow by Exact Value. \( a_1 = 1 \), \( a_2 = 1 \)
FIGURE (4-13): Gain of Frequency Response of Lumped-Models
"Counter-Flow" $a_1 = 4$, $a_2 = 1$, $r = 1$
FIGURE (4-14): Phase Angle of Lumped Models "Counter-Flow"

For case: \( a_1 = 4 \), \( a_2 = 1 \), \( r = 1 \)
FIGURE 4-15: Phase Angle of Lumped-Models "Counter-Flow"
For case $a_1 = 1$, $a_2 = 1$, $r = 1$
FIGURE(4-16): Gain of Frequency Response of Lumped-Models "Counter-Flow" \( a_1 = 1 \), \( a_2 = 1 \), \( r = 1 \)
FIGURE (4-17): Gain of Frequency Response of Lumped-Models "Counter-Flow" $a_1=2$, $a_2=1$, $r=2$
FIGURE (4-18): Phase Angle of Lumped Models "Counter Flow"

For case: $a_1 = 2$, $a_2 = 1$, $r = 2$
FIGURE (4-19): Frequency Response of Lumped-Models

"Parallel-Flow" $a_1 = 4, a_2 = 1, r = 1$

Polar-Plot

- Two Lumps
- Three Lumps
- Four Lumps
FIGURE (4-20): Frequency Response of Lumped-Models

"Parallel Flow" $a_1 = 2$, $a_2 = 1$, $r = 2$

POLAR- PLOT

- TWO LUMPS
- THREE LUMPS
- FOUR LUMPS

EXACT
FIGURE (5-1): Frequency Responses of Fitted Lumped-Models
For case: \( a = 4 \), \( a = 1 \), \( r = 1 \), \( \varphi_1 = \frac{1}{3} \), \( \varphi_2 = \frac{2}{3} \)

COUNTER-FLOW
FIGURE(5-3): Gain and Phase Angle for Improved Case

For case: $a_1 = 4$, $a_2 = 1$, $r = 1$

COUNTER-FLOW

NORMALIZED GAIN

FRIEDLY

EXACT

- TWO LUMPS

PHASE

FREQUENCY
FIGURE (5-4): Gain and Phase Angle for Improved Case

For case: $a_1 = 2$, $a_2 = 1$, $r = 1$
FIGURE (5-5): Step Response of Improved Case

For case: $a_1 = 4$, $a_2 = 1$, $r = 1$
FIGURE(5-6): Impulse Response of Improved Case

For case: $a_1 = 4$, $a_2 = 1$, $r = 1$
FOR CASE: $a_1 = 4$, $a_2 = 1$, $f = 1$

FIGURE (5-7): Monotonic Parameters of Lumped-Models
FIGURE (5-8): Monotonic Parameters of Exact Solution

For case: \(a_1 = 4\), \(a_2 = 1\), \(r = 1\)
FIGURE (5-9): Monotonic Parameters of Friedly's Method

For case: $a_1 = 4$, $a_2 = 1$, $r = 1$