WAVE PROPAGATION IN ELASTIC BEAMS AND RODS

by

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B.S., Escola Politécnica da Universidade de São Paulo
(1963)

M.S., Escola Politécnica da Universidade de São Paulo
(1972)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August, 1978

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Submitted to the Department of Ocean Engineering on August 29, 1978 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

ABSTRACT

This work is divided into three different parts. In the first part, an extensive study is performed for the problem of sinusoidal flexural wave propagation in infinite and semi-infinite Timoshenko beams. The concept of a modal mobility matrix is introduced and the elements of the matrix actually obtained. The matrix allows for the solution of most sinusoidal forced problems of infinite and semi-infinite beams. A comparison is made between the Timoshenko theory results and the results of other approximate theories.

The second part treats the problem of sinusoidal wave propagation in infinite periodically loaded strings and beams. The main results are the dispersion relation and the modal mobility matrix for the beam treated with the Timoshenko beam model.

Finally, the problem of the interaction of a high intensity compressional pulse propagating in a long rod, with initial imperfections of the rod, is studied. It is shown experimentally that the interaction may lead to significant rates of transformation of the compressional energy contained in the initial pulse into bending energy, which then propagates as bending waves at substantially lower speeds.

Thesis Supervisor: Professor Stephen Harris Crandall
Title: Ford Professor of Engineering
ACKNOWLEDGEMENTS

The author wishes to express his deep gratitude to Professor Stephen H. Crandall for his friendship, guidance, and patience during the development of this thesis.

Thanks are also due to the other members of his doctoral committee for their comments, especially to Professor John Dugundji.

The advice and assistance given by Professor Chrysostomos Chrysostomidis were invaluable, as was the help with computer programming given by Michael B. Kennedy, a fellow student.

The author also wishes to extend his deep appreciation and gratitude to his employer, the "Instituto de Pesquisas Tecnológicas", as well as to the "Fundacão de Amparo à Pesquisa do Estado de São Paulo" for their continuous support throughout these last several years.

The excellent typing by Ms. Patricia Schaffer and Ms. Debbie Schmitt is also very much appreciated.

It is at this point that the author wishes to express, in a very special way, his love for his wife, Bernadete, for her endless understanding and sacrifice during all these years, and for his sons, Vinicius and Gonçalo, for accepting the little in time and attention he was able to give them.

This thesis is dedicated to the memory of his parents, João and Waldelice.
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GLOSSARY OF SYMBOLS

\[ A = \text{cross-section area of beam or rod} \quad \text{Units} \quad \text{m}^2 \]

\[ a_1 = \left(\frac{G^*}{\rho}\right)^{1/2} \quad \text{m/sec} \]

\[ a_2 = \left(\frac{E}{\rho}\right)^{1/2} - \text{propagation speed of compressional waves in a rod} \quad \text{m/sec} \]

\[ c = \text{phase velocity} \quad \text{m/sec} \]

\[ c_1 = \text{phase velocity for the first mode in a Timoshenko beam} \quad \text{m/sec} \]

\[ c_2 = \text{phase velocity for the second mode in a Timoshenko beam} \quad \text{m/sec} \]

\[ c_{g1} = \text{group velocity for the first mode in a Timoshenko beam} \quad \text{m/sec} \]

\[ c_{g2} = \text{group velocity for the second mode in a Timoshenko beam} \quad \text{m/sec} \]

\[ C = \text{linear dashpot constant} \quad \text{N/(m/sec)} \]

\[ E = \text{modules of elasticity} \quad \text{N/m}^2 \]

\[ F = \text{amplitude of sinusoidal force} \quad \text{N} \]

\[ G = \text{shear modulus} \quad \text{N/m}^2 \]

\[ G^* = \text{KG - effective shear modulus} \quad \text{N/m}^2 \]

\[ H = \text{sum of kinetic and potential energy} \quad \text{N} \cdot \text{m} \]

\[ <H_f> = \text{average flexural energy per unit length in sinusoidal waves} \quad \text{N} \cdot \text{m} \]

\[ h = \text{height the pendulum is released from in the experimental work of Chapter 4} \quad \text{m} \]
$i = \sqrt{-1}$ imaginary unit  

$I =$ area moment of inertia of cross-section  

$J =$ mass moment of inertia  

$K =$ linear spring constant  

$k_1 =$ propagation constant for first mode  

$k_2 =$ propagation constant for second mode  

$L =$ length of segment in a periodic system  

$M =$ amplitude of external sinusoidal moment  

$M =$ bending moment at a section of a beam  

$m =$ mass of load for a periodic system  

$M_{Fw} =$ force-transverse velocity mobility  

$M_{Mw} =$ moment-transverse velocity mobility  

$M_{F\psi} =$ force-angular velocity mobility  

$M_{M\psi} =$ moment-angular velocity mobility  

$T =$ tension in a string  

$T^* =$ kinetic coenergy  

$V =$ potential energy  

$\omega(x,t) =$ transverse velocity of neutral line of beam at point $x$
\( W_1 = \) first mode amplitude for sinusoidal \( \omega(x,t) \). 
\( W_2 = \) second mode amplitude for sinusoidal \( \omega(x,t) \). 
\( \alpha_{1,2}', \beta_{1,2}' \) = defined by equations (3.32)
\( \gamma_{1,2} \)
\( \eta(x,t) = \) transverse displacement of point \( x \) of the neutral line 
\( \kappa = \) shear correction coefficient 
\( \lambda = \) wave length 
\( \mu = \) propagation constant for a periodic system 
\( \mu_r = \) real part of \( \mu \) 
\( \mu_i = \) imaginary part of \( \mu \) 
\( \nu = \) Poisson's ratio 
\( \xi = \) longitudinal displacement of a point of the neutral line of a rod 
\( \rho = \) mass density 
\( \phi(x,t) = \) angular displacement of cross-section of a beam at \( x \) 
\( \phi_1 = \) first mode amplitude for sinusoidal \( (x,t) \) 
\( \phi_2 = \) second mode amplitude for sinusoidal \( (x,t) \) 
\( \psi(x,t) = \) angular velocity of cross-section of a beam at \( x \).
$\psi_1 = \text{first mode amplitude for sinusoidal } \psi(x,t)$ \hspace{1cm} \text{Units: } \text{rd/sec}

$\psi_2 = \text{second mode amplitude for sinusoidal } \psi(x,t)$ \hspace{1cm} \text{rd/sec}

$\omega = \text{angular frequency}$ \hspace{1cm} \text{rd/sec}

\textbf{Nondimensional groups:}

$\tilde{a} = \frac{a_r}{a_1}$

$\tilde{c}_1 = \frac{c_1}{a_1}$

$\tilde{c}_2 = \frac{c_2}{a_2}$

$\tilde{c}_{gL} = \frac{c_{g2}}{a_2}$

$\tilde{k} = \frac{rk}{L}$

$\tilde{L} = \frac{L}{r}$

$\tilde{M}_{FW} = \frac{M_{FW}}{a_1 \rho A}$

$\tilde{M}_{F\psi} = \frac{M_{F\psi}}{a_1 \rho A r}$

$\tilde{M}_{Mw} = \frac{M_{Mw}}{a_1 \rho A r}$

$\tilde{M}_{M\psi} = \frac{M_{M\psi}}{a_1 \rho A r^2}$

$\tilde{C} = \frac{C}{\rho A a_1}$

$\tilde{K} = \frac{rK}{G^*A}$

$\tilde{m} = \frac{m}{\rho AL}$

$\tilde{J} = \frac{J}{\rho IL}$

$\tilde{F} = \frac{F}{a_1^2 \rho A}$
\[ \bar{t} = \frac{t}{L/a_2} \]

\[ \bar{\mu} = \mu L \]

\[ \bar{\mu}_r = \mu_r L \]

\[ \bar{\mu}_i = \mu_i L \]

\[ \bar{\omega}_1 = \frac{\omega_r}{a_1} \]

\[ \bar{\omega}_2 = \frac{\omega_r}{a_2} \]

\( \bar{E}, \bar{\rho}, \bar{A}, \bar{I} \) are defined at the bottom of Table C.1 in Appendix C.
CHAPTER 1
INTRODUCTION

Pulverized-coal fired boilers in electric generating stations produce exhaust gases loaded with fly ash. Before such gases can be discharged into the atmosphere, it is necessary to clean them by removing the fly ash. A very efficient cleaning method which is being used almost exclusively today is based on the principle of electrostatic precipitation.

A Cottrell type electrostatic precipitator consists of rows of wire electrodes suspended between grounded parallel plates, which in the present days may be as long as 50 feet. High voltage applied to the wire electrodes charges the ash particles present in the gas flowing through the system. Charged particles precipitate into the grounded plates (collecting electrodes) where they form a dust layer of growing thickness. Periodically the plates have to be freed from the collected dust layer. One of the ways to do this is to rap the structure at specifically chosen points. The impact excites the plates and dislodges the dust layer which then falls under gravity into a hopper beneath. Figure 1.1 is a sketch of one kind of electrostatic precipitator.
Point of application of transient load

Figure 1.1 Sketch of an electrostatic precipitator
The magnitude of the impact must be such as to allow for an efficient cleaning of the plates, but at the same time it shall not cause the dust to reenter the flow (cleaning is done under operating conditions). According to Sproull (1972), the required efficiency of modern electrostatic precipitators is above 99.5% in collected ash by weight. However, proposed installations are now so large that it is difficult to clean the plates without overstressing the structure or without losing large amounts of ash to the gas flow. In order to improve the design, it is necessary to develop better insight into rapping dynamics.

The objective of this work is to develop analytical tools to aid in the theoretical analysis of the behavior of electrostatic precipitators under rapping. From Figure 1.1 we see that an electrostatic precipitator can be represented by a long horizontal beam periodically loaded with built-in long vertical plate strips. We also note that according to the existing literature, it appears to be the case that the dust layer disloging is accomplished in a very short time after rapping. This indicates that the first arrival of the disturbance is mostly responsible for the cleaning.

Therefore, we conclude that a wave propagation
approach rather than a vibration approach seems to be a convenient way to study the problem. We, therefore, engaged in the study of sinusoidal wave propagation in infinitely long periodically loaded beams.

However, in order to develop some feeling about the problem of wave propagation in beams, we started by studying sinusoidal wave propagation in unloaded beams modeled by the Timoshenko beam theory. In doing so, we found a gap in the literature in the problem of sinusoidal waves in Timoshenko beams. There certainly are works published in the area, but most of them do not go beyond the dispersion relation, phase and group velocities. Cremer and Heckl (1967) present a formula for the driving point mobility of an infinite Timoshenko beam, but we have shown that their formula does not appear to be correct.

We have, therefore, developed an extensive study of wave propagation in infinite Timoshenko beams, which forms Chapter 2 of this work. The results are presented in nondimensional form which makes them very general. Curves have been obtained for a wide range of parameters which include most materials and cross-sectional shapes used in practice. Our main contribution in this chapter was the
introduction of the concept of a modal mobility matrix and the actual derivation of the elements of the matrix. The numerical results for all elements are presented in graphical form. This matrix can be used to compute the response of an effectively infinite Timoshenko beam to a general (combination of force and moment) sinusoidal load. We have also in Chapter 2 performed a comparison between the mobility for a Timoshenko beam with the mobility for four other approximate beam theories.

We then studied in Chapter 3 the problem of the propagation of sinusoidal waves in periodically loaded one-dimensional structures. The resulting waves were named nodal waves. Again, in order to gain some feeling for the problem, the periodically loaded string was studied first.

The problem of a Timoshenko beam periodically loaded with arbitrary translational and rotational impedances was then studied in great detail. This work is completely original because the literature does not contain any studies of periodically loaded Timoshenko beams. The dispersion relation as well as the modal mobility matrix for the system driven at a loading point were obtained. The properties of the mobility matrix in this case are
basically the same as for the unloaded beam, although the process leading to the matrix elements in this case is much more involved.

We also obtained the dispersion relation for a periodically loaded Bernoulli-Euler beam. The idea here was to verify our results for the Timoshenko beam by showing that in the limit, the Timoshenko beam results converge to the Bernoulli-Euler beam results.

The dispersion relations for the periodically loaded string, Timoshenko beam and Bernoulli-Euler beam were obtained graphically for a significant set of cases of loading impedances. All the curves are in Appendix E. These graphs clearly show that structures loaded with reactive (mass-like and spring-like) impedances behave in any of the three cases as passband filters with an infinite number of passbands and stopbands. Structures loaded with resistive (dashpot like) impedances have the same pattern for the dispersion relations, but attenuation is always present.

The main difference between the string case and the beam case is that the string has only one mode of propagation of nodal waves and the beam has two modes of propagation. The first mode of nodal waves of the Bernoulli-Euler beam
is mostly nonpropagating.

Finally, in Chapter 4, we have studied a problem which may be very important for the rapping dynamics of electrostatic precipitators as far as the wave propagation in the vertical plates goes. This is the problem of interaction of a high intensity compressional pulse with the initial imperfections of a long slender rod (or plate). Our objective was to determine if this interaction leads to sizable transformation of compressional energy into flexural energy as the compressional pulse propagates down in the long plate strip, through a phenomenon similar to dynamic buckling. In other words, the region of the rod which is under compression at a given instant will tend to grow its imperfections introducing flexural energy which will then propagate as bending waves. We have developed the "exact" nonlinear equations of motion for such a phenomenon but did not solve it. A decision was made to perform experiments to study the importance of the phenomenon. A simplified linear theory was used to obtain some numerical data and the experiments were then performed. The results are very interesting and actually show that the energy conversion can happen at significant rates.
Therefore, it is quite possible that in long plates, transverse motion plays the most important role in dislodging the dust layer in electrostatic precipitators. The other possible mechanism would be the shear between the plate and the dust layer, introduced by compressional waves. However, as the plates are thin and, of course, contain many imperfections, the compressional waves shall become weaker, due to the mechanism here discussed, as they propagate through the plate.
CHAPTER 2

FLEXURAL WAVES IN INFINITELY LONG TIMOSHENKO BEAMS

2.1 Introduction

The problem of propagation of flexural waves in elastic beams is long standing, with the first studies dating from the mid eighteenth century. Love (1927)* summarizes in his historical introduction the work done in the subject until the beginning of the present century.

The classical theory (Bernoulli-Euler beam theory) of beams, which includes only transverse inertia and bending deformation, was modified by Rayleigh in 1877 with the introduction of the correction for rotary inertia (Rayleigh, 1894). This eliminated the contradiction of infinite phase and group velocities for infinitely small wavelengths that results from the classical theory. The Rayleigh beam theory was then modified by Timoshenko (1921), who introduced the shear deformation effects and gave origin to the so-called Timoshenko beam theory. It is, however, of historical interest to notice that, according to Mindlin (1954),

* Author names followed by the year refer to publications listed alphabetically under References.
both the rotary inertia and the shear deformation corrections were actually first introduced by M. Bresse in his "Cours de Mécanique Appliquée" published in Paris in 1859.

The Timoshenko beam theory has recently been extensively used in the study of vibrations and wave propagation problems. Much has been done in the area of response of infinite beams to impact and other transient loads. As examples, the reader can refer to the works of Dengler and Goland (1951), Goland, et.al. (1955), Fu (1967), Paul and Fu (1967), Kelly (1967), Garrellick (1969), and Forrestal, et.al. (1975).

For propagation of natural sinusoidal waves in Timoshenko beams, various authors have obtained and discussed the dispersion relation and the phase velocity for both modes of propagation. Abramson et.al. (1958) has compared first and second mode phase velocities for the Timoshenko beam with the phase velocities for the first two modes for the exact three-dimensional theory in the particular case of circular cylindrical beams. Kolsky (1963) compared also phase and group velocities for the first mode of the exact and the Timoshenko theories. He used Hudson's (1943) results for the exact theory. It is interesting to note that neither Hudson
in 1943 nor Kolsky in 1953 knew of the existence of other modes than the fundamental mode for the exact theory. See Abramson et. al. (1958) for the discussion of this point.

To the best of our knowledge, the problem of the response of an infinite Timoshenko beam to sinusoidal loads has been treated in the literature exclusively by Cremer and Heckl (1967). They give, without proof, a formula for the driving-point modal mobility of an infinite Timoshenko beam driven at the "middle" by a sinusoidal transverse force. We show in Section 2.7 that Cremer and Heckl's results do not have the correct limiting value for very high frequencies, while our results do behave correctly in that limit.

The present study develops a body of results that allows quick determination of the response of an infinite or semi-infinite beam of arbitrary uniform cross-section to a general sinusoidal point load. All results are obtained through the Timoshenko beam model. A general sinusoidal point load consists of an arbitrary combination of sinusoidal force and sinusoidal moment of same frequency. Once this response, which is a form of Green's function, has been determined, the response to a transient point load can be obtained with the use of Fourier transforms.
The treatment starts with the development of the equations of motion for the Timoshenko beam and an analysis of the range of variation of the shear correction coefficient $\kappa$, to be defined later. The next step is the study of natural sinusoidal waves in the beam. This leads to the expressions for the dispersion relation, the propagation constants (generalized wave numbers), and for the phase and group velocities. Next we study the shape of each of the two modes of propagation as functions of frequency. The ratios angular displacement/transverse displacement and bending moment/shear force are used for this study.

The second phase of the study is directed to the determination of the driving-point mobility matrix, which is defined later. This matrix allows for the determination of the modal response of the beam when any two of the four sinusoidal variables (transverse force, moment, transverse and angular velocities) are imposed at the driving point.

Finally, the components of the mobility matrix for the Timoshenko model are compared to the results of four other approximate theories:

(a) Shear beam,

(b) Bernoulli-Euler beam,

(c) Rayleigh beam,
(d) Shear-bending beam.

To simplify comprehension all the algebraic developments are made in terms of dimensional quantities, but the final equations are systematically non-dimensionalized. The numerical results are non-dimensional and in general presented in graphical form.
2.2 The Timoshenko Beam Model

2.2.1 The Equations of Motion

The Timoshenko beam model includes transverse inertia, rotary inertia, and two modes of elastic deformation. Figure 2.1a shows a deformed segment of the beam and depicts at a point x the transverse displacement η(x,t) of the neutral line, and the angular displacement ϕ(x,t) of the cross-section, which is assumed to remain plane after deformation. The total rotation of the neutral line at x is given by ∂η/∂x. It differs from the rotation ϕ(x,t) of the cross-section by an amount equal to the shear deformation γ at the section. Therefore:

\[ γ = \frac{∂η}{∂x} - ϕ \]  (2.1)

Figures 2.1b and c present the constitutive relations for the beam, where E is the modulus of elasticity and G the shear modulus of the material, A is the area, and I the moment of inertia of the cross-section. The bending moment-curvature relation is a basic result in mechanics of solids (Crandall, et.al., 1972). The shear force-shear angle relation (c) is also a basic result but it is complicated by the fact that the shear stress and consequently γ are non-uniform along the cross-section.
Figure 2.1  Beam model, constitutive equations, and dynamic element for a Timoshenko beam
This is due to the presence of shear-stress-free boundaries in the beam. In order to retain a one-dimensional model, an average of the shear angle is obtained by multiplying it by a shear correction coefficient $\kappa$. This coefficient is discussed in Section 2.2.2.

The transverse momentum and the angular momentum equations can now be written with the aid of the element of beam shown in Figure 2.1d.

\[
\frac{\partial Q}{\partial x} \, dx = \rho A dx \, \frac{\partial^2 \eta}{\partial t^2} \tag{2.2a}
\]

\[
\frac{\partial M}{\partial x} \, dx + Qdx = \rho I dx \, \frac{\partial^2 \phi}{\partial t^2} \tag{2.2b}
\]

Using the constitutive relations (Figures 2.1b and c) in Equations (2.2) and assuming the beam to be uniform with $x$, we obtain:

\[
a_1^2 \left( \frac{\partial^2 \eta}{\partial x^2} - \frac{\partial \phi}{\partial x} \right) - \frac{\partial^2 \eta}{\partial t^2} = 0 \tag{2.3a}
\]

\[
a_2^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{a_1^2}{r^2} \left( \frac{\partial \eta}{\partial x} - \phi \right) - \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2.3b}
\]

where the following definitions apply:

\[
a_1^2 = \frac{G^*}{\rho} \quad a_2^2 = \frac{E}{\rho} \quad r^2 = \frac{I}{A} \tag{2.4}
\]
In here, \( r \) is the radius of gyration of the section and \( a_1 \) and \( a_2 \) have dimensions of propagation velocities. Also, \( G^* = \kappa G \).

Equations (2.3) are coupled partial differential equations in \( \eta \) and \( \phi \) and are the equations of motion of a Timoshenko beam.

2.2.2 The Shear Correction Coefficient \( \kappa \)

As already seen above, \( \kappa \) is a coefficient that multiplies the shear angle \( \gamma \) at the neutral line of a beam to give an average value of \( \gamma \) for the entire cross-section. Since its introduction by Timoshenko (1921), \( \kappa \) has been defined in many different ways. According to Cowper (1966), although the definitions differ in nature, the numerical results are close to each other.

For dynamic problems, \( \kappa \) is in general a function of:

(a) cross-section shape

(b) frequency

(c) beam material.

However, the available definitions for \( \kappa \) include at the most two of these parameters. In the simplest cases it has been defined for static deformation as function of cross-section shape. Mindlin and Deresiewicz (1954) found \( \kappa \) by imposing that the cutoff frequency of the second mode
of propagation for infinite Timoshenko beams be equal to the frequency of the first thickness-shear mode as given by the three-dimensional theory of elasticity. Goodman and Sutherland (1951) determined \( \kappa \) by imposing that the Timoshenko frequency equation be correct in the limit of zero wavelength. Cowper (1966) obtained a general expression for \( \kappa \) during the process of specialization of the equations of three-dimensional elasticity to the Timoshenko beam equation. In this case, \( \kappa \) is a function of cross-section shape and Poisson's ratio \( \nu \).

From all the existent results, Cowper's seem to be the most convenient to use. The reason is that he has specialized his general formula for more than ten different cross-section shapes. Although he recognizes that his results are most satisfactory for static and long wavelength deformations, they are within a few percent of the Mindlin and Goodman results. Cowper's formulas for \( \kappa \) are reproduced in Table A.1 of Appendix A.

In order to obtain a range of variation for the coefficient \( \kappa \), we have computed Table 2.1 by using the formulas of Table A.1. The table includes a very broad range of sections and all possible materials \((0 \leq \nu \leq 0.5)\). Therefore, we can state that in practice:
TABLE 2.1 - Values of the Shear Correction Coefficient $K$ for a Wide Set of Cross-Section Shapes and for Poisson Ratio $0 < \nu < 0.5$ (computed with the formulas of Table A.1, Appendix A).

<table>
<thead>
<tr>
<th>Shapes</th>
<th>Ellipse $a/b=0.5$</th>
<th>Ellipse $a/b=1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>.877</td>
<td>.877</td>
</tr>
<tr>
<td>0.1</td>
<td>.887</td>
<td>.887</td>
</tr>
<tr>
<td>0.2</td>
<td>.895</td>
<td>.902</td>
</tr>
<tr>
<td>0.3</td>
<td>.907</td>
<td>.913</td>
</tr>
<tr>
<td>0.4</td>
<td>.907</td>
<td>.907</td>
</tr>
<tr>
<td>0.5</td>
<td>.913</td>
<td>.913</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1-beams ($tf/tw=1$)</th>
<th>Thin Walled Square Tube</th>
<th>Thin Walled Round Tube</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b/h=2$</td>
<td>.472</td>
<td>.472</td>
<td>.472</td>
</tr>
<tr>
<td>$b/h=2.5$</td>
<td>.637</td>
<td>.641</td>
<td>.645</td>
</tr>
<tr>
<td>$b/h=5$</td>
<td>.500</td>
<td>.512</td>
<td>.522</td>
</tr>
<tr>
<td>$b/h=10$</td>
<td>.599</td>
<td>.602</td>
<td>.606</td>
</tr>
</tbody>
</table>
0.150 < \kappa < 0.920 \quad \quad (2.5)

This is indeed a surprisingly broad range of variation for a factor which is considered as "weak" (close to unity) by many authors.

We have also computed the values of the shear correction coefficient \kappa for most steel I-beams listed in the Manual of Steel Construction (1976). Table A.2 of Appendix A contains these values.

2.3 Dimensional Analysis

In this section we will try to find a set of non-dimensional parameters that completely describe each phenomenon to be studied.

The following is a general functional equation for a Timoshenko beam:

\[ X = X(\rho, E, G^*, A, r, \omega) \quad \quad (2.6) \]

where:

- \( X \) - propagation constant, phase velocity, etc.

The independent variables are classified in three categories:

- (a) material properties: \( \rho, E, G^* \)
  
  \( \rho \) is the density of the material, \( E \) is the Young's modulus, and \( G^* \) is the shear modulus.
(b) cross-section shape: $G^*, A, r$

(c) frequency: $\omega$

Notice that $G^* = \kappa G$ involves material and cross-section shape properties.

Equation (2.6) contains seven variables. Therefore, the $\pi$-theorem predicts four non-dimensional* groups, which by inspection are:

\[
\begin{align*}
\bar{X} & \text{ - dependent variable} \\
\bar{\omega}_1 & = \omega r/a_1 \text{ - frequency} \\
\bar{\alpha} & = a_2/a_1 = \sqrt{E/G^*} \text{ - material and cross-section shape} \\
\bar{A} & = A/r^2 \text{ - cross-section shape}
\end{align*}
\]

However, in the results obtained in the following sections, $\bar{A}$ does not appear. Therefore, the two parameters $\bar{\omega}_1$ and $\bar{\alpha}$ completely describe the phenomena. This does not mean that the cross-section shape does not have its influence. It enters through the shear correction coefficient in $\bar{\alpha}$ and through the radius of gyration in $\bar{\omega}_1$.

---

* Non-dimensional groups will always be indicated by a bar above the symbol.
Finally, the dimensionless functional equation is:

\[ \overline{X} = \overline{X}(\overline{\omega}_1, \overline{a}) \]  \hspace{1cm} (2.8)

The parameter \( \overline{\omega}_1 \) ranges from zero to plus infinity. The range for parameter \( \overline{a} \) is obtained by considering its definition (2.7). From the theory of elasticity:

\[ \frac{E}{G} = 2(1 + \nu) \]

Therefore, (2.8) becomes:

\[ \overline{a} = \sqrt{\frac{2(1 + \nu)}{\kappa}} \]  \hspace{1cm} (2.9a)

Considering that

\[ 0 < \nu < 0.5 \]

and that \( \kappa \) ranges as given by Equation (2.5), we are lead to:

\[ 1.5 < \overline{a} < 4.5 \]  \hspace{1cm} (2.9)

2.4 Dispersion Relation, Phase and Group Velocities

Consider sinusoidal waves propagating along a
Timoshenko beam. The transverse displacements will be described by:

\[ \eta(x,t) = Ne^{i(kx - \omega t)} \]  \hspace{1cm} (2.10)

and the angular displacements by:

\[ \phi(x,t) = \phi e^{i(kx - \omega t)} \]  \hspace{1cm} (2.11)

The amplitudes N and \( \phi \) may be complex. A difference in their arguments indicates that N and \( \phi \) are out of phase.

2.4.1 Dispersion Relation

Substituting (2.10) and (2.11) into the equations of motion (2.3), and performing some simplifications, we find:

\[ (\omega^2 - a_{l2}^2k^2)N - ika_{l1}^2 \phi = 0 \]  \hspace{1cm} (2.12a)

\[ \frac{ika_{l1}^2}{r^2} N + (\omega^2 - a_{22}^2k^2 - \frac{a_{l1}^2}{r^2}) \phi = 0 \]  \hspace{1cm} (2.12b)

This is a set of two coupled homogeneous linear equations in N and \( \phi \). For a nontrivial solution to exist, the determinant of the coefficients must vanish. This leads to the dispersion relation:
\[
a_1^2 a_2^2 k^4 - (a_1^2 + a_2^2) \omega^2 k^2 + \omega^4 (1 - \frac{a_1^2}{\omega^2 r^2}) = 0
\] (2.13)

Equation (2.13) is a biquadratic equation in \( k \) which has the following roots:

\[
k = \pm \frac{\omega}{\sqrt{2} a_2} \sqrt{(1 + \frac{a_2^2}{a_1^2}) \pm \sqrt{(1 - \frac{a_2^2}{a_1^2})^2 + 4 \left(\frac{a_2}{a_1}\right)^2 \left(\frac{1}{\omega r}\right)^2}}
\] (2.14)

We have then four different values for \( k \), which represent two right going waves (plus sign outside) and two left going waves (minus sign outside). Therefore, there are two different natural modes that can propagate in a Timoshenko beam and each mode can be left going and right going. Considering only right going waves, we see that Equations (2.10) and (2.11) lead to:

\[
\begin{align*}
\eta(x,t) &= N_1 e^{i(k_1 x - \omega t)} + N_2 e^{i(k_2 x - \omega t)} \\
\phi(x,t) &= \phi_1 e^{i(k_1 x - \omega t)} + \phi_2 e^{i(k_2 x - \omega t)}
\end{align*}
\] (2.15)

Equation (2.14) can be non-dimensionalized as follows:

\[
\bar{k}_{1,2} = \frac{\bar{\omega}_1}{\sqrt{2} \bar{a}} \sqrt{(1 + \bar{a}^2) \pm \sqrt{(1 - \bar{a}^2)^2 + \left(\frac{2 \bar{a}}{\bar{\omega}_1}\right)^2}}
\] (2.14a)
where \( \bar{\omega}_1 \) and \( \bar{a} \) are defined in (2.7) and \( \bar{k} = r k \), where \( r \) is the radius of gyration of the section.

From (2.14a) we see that \( \bar{k}_1 \) (plus sign in 2.14a) is real for any positive value of \( \bar{\omega}_1 \), and that \( \bar{k}_2 \) (negative sign in 2.14a) is imaginary for \( \bar{\omega}_1 < 1 \) and real for \( \bar{\omega}_1 > 1 \). This means that the first mode is a propagating mode for any frequency. The second mode, however, is exponentially attenuated for \( \bar{\omega}_1 < 1 \) and propagates for \( \bar{\omega}_1 > 1 \). Therefore, \( \bar{\omega}_1 = 1 \) is the cutoff frequency for the second mode.

The two branches of the dispersion relation (2.13) have been plotted in Figures 2.2 and 2.3 respectively. The curves are parametrized in \( \bar{a} \) over the range indicated by Equation (2.9).

2.4.2 Phase and Group Velocities

For sinusoidal waves propagating in a system, phase and group velocities are defined (Crandall, et.al., 1968) as:

\[
. c = \frac{\bar{\omega}}{\bar{k}}
\]  

(2.16)

and
Figure 2.2 First branch of the dispersion relation for the Timoshenko beam model.

\[ \omega_l = \frac{\omega}{a_l} \]
Figure 2.3  Second branch for the dispersion relation for the Timoshenko model.
\[ c_g = \frac{d\omega}{dk} \] (2.17)

respectively.

In order to compute \( c \) and \( c_g \), we consider \( \omega \) as the dependent variable in the dispersion relation (2.13) and rewrite it as:

\[ \omega^4 - [(a_1^2 + a_2^2)k^2 + \frac{a_1^2}{r^2}]\omega^2 + a_1a_2k^4 = 0 \] (2.13a)

This is again a biquadratic equation in \( \omega \) which has the following roots:

\[
\omega = \frac{1}{\sqrt{2r}} \sqrt{\frac{a_2}{a_1} (rk)^2 + 1 \pm \sqrt{(1 - \frac{a_2^2}{a_1^2}) (rk)^4 + 2(1 + \frac{a_2^2}{a_1^2}) (rk)^2 + 1}}
\] (2.18)

where we have discarded negative values of \( \omega \).

From Equation (2.14a) we see that \( k_1 = 0 \) for \( \bar{w}_1 = 0 \) and \( k_2 = 0 \) for \( \bar{w}_1 = 1 \). Therefore, for consistency, the minus sign in Equation (2.18) corresponds to the first mode and the plus sign to the second mode.

Using Equation (2.18) in (2.16) and (2.17), we obtain after some algebraic manipulation:
\[ c_{1,2} = \frac{a_1}{\sqrt{2}rk} \sqrt{(1 + \frac{a_2^2}{a_1^2})(rk)^2 + 1 + \sqrt{(1 - \frac{a_2^2}{a_1^2})(rk)^4 + 2(1 + \frac{a_2^2}{a_1^2})(rk)^2 + 1}} \]

\[ c_{g_{1,2}} = \frac{a_1(rk)}{\sqrt{2}} \left( \frac{a_2}{a_1} \right) \sqrt{\left(1 - \frac{a_2^2}{a_1^2}\right)(rk)^4 + 2(1 + \frac{a_2^2}{a_1^2})(rk)^2 + 1} \left[ \frac{a_2}{a_1} \right] \left( \frac{a_2}{a_1} \right) \]

\[ \sqrt{(1 - \frac{a_2^2}{a_1^2})(rk)^4 + 2(1 + \frac{a_2^2}{a_1^2})(rk)^2 + 1} \left( \frac{a_2}{a_1} \right) \left( \frac{a_2}{a_1} \right) \]

The non-dimensional forms are:

\[ \overline{c}_1 = \frac{c_1}{a_1} = A_1 \] (2.19a)

\[ \overline{c}_2 = \frac{c_2}{a_2} = \frac{A_2}{a} \] (2.19b)

\[ \overline{c}_{g_1} = \frac{c_{g_1}}{a_1} = B_1 \] (2.20a)

\[ \overline{c}_{g_2} = \frac{c_{g_2}}{a_2} = \frac{B_1}{a} \] (2.20b)

where:

\[ A_{1,2} = \frac{1}{\sqrt{2k}} \sqrt{(1 + \frac{a_2^2}{k^2}) + 1 + \sqrt{(1 - \frac{a_2^2}{k^2})(k^4 + 2(1 + \frac{a_2^2}{k^2}))(k^2 + 1)}} \]

\[ ... (2.21) \]
Figure 2.4 Phase velocity for the first mode in a Timoshenko beam.
Figure 2.5 Phase velocity for the second mode in a Timoshenko beam.
Figure 2.6 Group velocity for the first mode in a Timoshenko beam.
\( \dfrac{c}{c_g} = \dfrac{c g_2}{a_2} \)

**Figure 2.7** Group velocity for the second mode in a Timoshenko beam.
\[
B_{1,2} = \frac{\bar{k}}{\sqrt{2}} \sqrt{\frac{(1-a^2)^2 \bar{k}^4 + 2(1+a^2)\bar{k}^2 + 1}{(1-a^2)^2 \bar{k}^4 + 2(1+a^2)\bar{k}^2 + 1}} \sqrt{(1-a^2)^2 \bar{k}^2 + 1} + \sqrt{(1-a^2)^2 \bar{k}^2 + 2(1+a^2)\bar{k}^2 + 1} 
\]

...(2.22)

It can be shown from Equations (2.19) and (2.20) that in the limit of very large \(\bar{\omega}_1\), \(c_1\) and \(c_{g_1}\) tend to \(a_1\), and \(c_2\) and \(c_{g_2}\) tend to \(a_2\). Therefore, the dimensionless velocities of (2.19) and (2.20) all have limit equal to one for large \(\bar{\omega}_1\).

Figures 2.4 through 2.7 present the curves for Equations (2.19a,b) and (2.20a,b) respectively.

2.5 Modal Configurations

In this section we will try to develop some insight into the configuration of each of the two modes of propagation for a Timoshenko beam, for all values of frequency.

In order to do so, one can look at the ratios \(\phi/\eta\) of angular displacement to transverse displacement and \(M/Q\) of bending moment to shear force for each mode at a section.

The ratio \(\phi/\eta\) is obtained by solving one of Equations (2.12). Solving, for example, (2.12a):
\[
\frac{\phi}{N} = i(k - \frac{\omega^2}{a_1 k})
\]  
(2.23)

In non-dimensional terms:

\[
\frac{r\phi}{N} = i(\bar{k} - \frac{\bar{\omega}^2}{\bar{k}})
\]
(2.23a)

This result applies for both modes where the values of \(\bar{k}\) for each mode are given by Equation (2.14a). Figures 2.8 and 2.9 contain the absolute value and argument for the ratio given by Equation (2.23a), for both modes of propagation.

From Figure 2.8 we notice that in general the two modes of propagation involve a combination of transverse (\(\eta\)) and angular (\(\phi\)) displacements. It is also clear that the first mode is dominated by transverse motion and the second mode by angular motion. Figure 2.10 depicts the shapes of Mode 1 and Mode 2 for the limiting case of \(k \to 0\) and \(k \to \infty\).

Figure 2.9 indicates that in the first mode \(\phi\) leads \(\eta\) by 90°. In the second mode, \(\phi\) and \(\eta\) are in phase opposition below cutoff (\(\bar{\omega}_1 < 1\)) and \(\phi\) lags \(\eta\) by 90° above cutoff (\(\bar{\omega}_1 > 1\)).

Let us now consider the ratio \(M/Q\). From Figure 2.1b and c, we rewrite the constitutive equations for the beam:
Figure 2.8 Absolute value of the ratio of angular displacement to transverse displacement ($\phi/\eta$) for sinusoidal waves in a Timoshenko beam.
Figure 2.9 Argument of the ratio ($\phi/\eta$) for sinusoidal waves in a Timoshenko beam.
Figure 2.10 Limiting wave configurations.

(a) First mode for $k_1 \to 0$, rigid translation.
(b) First mode for large $k_1$ ($\lambda \to 0$).
(c) Second mode for $k_2 \to 0$ ($\bar{\omega}_1 = 1$), thickness shear.
(d) Second mode for large $k_2$ ($\lambda \to 0$).

(From Crandall, 1968)
\[ Q = G^* A \left( \frac{\partial^3 \eta}{\partial x^3} - \phi \right) \]  \hspace{1cm} (2.24)

\[ M = EI \frac{\partial \phi}{\partial x} \]  \hspace{1cm} (2.25)

Using now Equations (2.15) for \( \eta(x,t) \) and \( \phi(x,t) \) in (2.24) and (2.25), and also considering Equation (2.23) to solve for \( \phi \) as a function of \( N \), we obtain:

\[ Q(x,t) = \frac{iG^* A \omega^2}{a_1^2} \left( \frac{N_1}{k_1} e^{ik_1x} + \frac{N_2}{k_2} e^{ik_2x} \right) e^{-i\omega t} \]  \hspace{1cm} (2.26)

\[ M(x,t) = -EI \left[ \frac{k_1^2 - \omega^2}{a_1^2} N_1 e^{ik_1x} + \frac{k_2^2 - \omega^2}{a_1^2} N_2 e^{ik_2x} \right] e^{-i\omega t} \]  \hspace{1cm} (2.27)

If we put the modal amplitudes \( N_1 \) and \( N_2 \) successively equal to zero and evaluate the ratio \( M/Q \), we obtain for both modes:

\[ \frac{M}{Q} = \frac{iEIa_1^2}{G^* \omega^2} \frac{k(k^2 - \omega^2)}{a_1^2} \]

Or, remembering that \( E = \rho a_2^2 \), \( G^* = \rho a_1^2 \), and \( I = Ar^2 \) (see Equations 2.4), we obtain:
\[
\frac{M}{Q} = \frac{ia_2^2 r^2}{\omega^2} k(k^2 - \frac{\omega^2}{a_1^2}) \quad (2.28)
\]

In non-dimensional terms, we have:

\[
\frac{M}{rQ} = \frac{ia_2^2 k(k^2 - \frac{\omega^2}{\omega_1^2})}{\frac{\omega^2}{\omega_1}} \quad (2.28a)
\]

Figures 2.11 and 2.12 show the absolute value and the argument for the ratio \(\frac{M}{rQ}\) given by Equation (2.28a).

Figure 2.11 shows that the first mode is dominated by shear above the cutoff frequency of the second mode \((\omega_1 > 1)\) and it is dominated by bending below that frequency \((\omega_1 < 1)\). The second mode is always dominated by bending, except around cutoff \((\omega_1 = 1)\) where the bending moment approaches zero and shear dominates. The thickness-shear motion of the beam for the second mode at cutoff explains this fact.

Finally, Figure 2.12 indicates that for the first mode the bending moment leads the shear force by 90° for all values of frequency. For the second mode, bending is in phase with shear below cutoff \((\omega_1 < 1)\) and bending lags shear by 90° above cutoff.
Figure 2.11 Absolute value of the ratio $M/rQ$ for sinusoidal waves in a Timoshenko beam.
Figure 2.12 Argument of the ratio $M/rQ$ for sinusoidal waves in a Timoshenko beam.
2.6 **The Mobility Matrix**

Up to this point we have been dealing with sinusoidal wave propagation in infinite Timoshenko beams, with no regard to the generation of such waves. It was concluded that in a Timoshenko beam, sinusoidal waves can propagate in two different natural modes, and that both modes can be right or left going. We also saw that each of the two modes has its own phase and group velocities and its own configuration.

We now turn to the problem of the generation of sinusoidal waves in infinite Timoshenko beams. The objective is to determine the general expressions for the elements of the modal mobility matrix, to be defined in the next item.

2.6.1 **The Concept of Mobility Matrix**

Consider a semi-infinite beam driven at the free end by a sinusoidal load made up of a moment $Me^{-i\omega t}$ and a transverse force $Fe^{-i\omega t}$ (see Figure 2.13).

![Figure 2.13](image)

**Figure 2.13** Semi-infinite Timoshenko beam driven at the end by a general sinusoidal load.
From now on, we will be dealing with transverse velocity \( w(x,t) \) and angular velocity \( \psi(x,t) \), rather than transverse displacement \( \eta(x,t) \) and angular displacement \( \phi(x,t) \). For sinusoidal waves (see Equation 2.15), there exist the following relations between velocities and displacements.

\[
\begin{align*}
w(x,t) &= \frac{\partial}{\partial t} \eta(x,t) = -i\omega \eta(x,t) \\
\psi(x,t) &= \frac{\partial}{\partial t} \phi(x,t) = -i\omega \phi(x,t)
\end{align*}
\]

(2.29)

The concept of driving-point mobility will be of central importance for this development. It applies only to sinusoidal waves and is in general defined as:

\[
\text{driving-point mobility} = \frac{\text{amplitude of linear or angular velocity at the driving point}}{\text{amplitude of driving force or moment}}
\]

\( \cdots (2.30) \)

From this definition we see that there are four individual definitions for driving-point mobilities. This is because a moment alone or a force alone will drive both transverse and angular velocities.

As the Timoshenko beam model is linear, superposition
applies. Therefore, the transverse and angular velocities responses are equal to the sum of the responses due to force and moment individually:

\[
W_0 e^{-i\omega t} = M_{FW} F e^{-i\omega t} + M_{MW} M e^{-i\omega t}
\]

(2.31)

\[
\psi_0 e^{-i\omega t} = M_{F\psi} F e^{-i\omega t} + M_{M\psi} M e^{-i\omega t}
\]

Or in matrix form, after concealing the time dependence:

\[
\begin{pmatrix}
W \\
\psi
\end{pmatrix} =
\begin{bmatrix}
M_{FW} & M_{MW} \\
M_{F\psi} & M_{M\psi}
\end{bmatrix}
\begin{pmatrix}
F \\
M
\end{pmatrix}
\]

(2.32)

The square matrix in (2.21) is the mobility matrix and its components are the driving-point mobilities. The first subscript on the symbol of the driving-point mobilities refers to the driving element (F or M) and the second subscript refers to the response being considered (W or \(\psi\)).

The individual driving-point mobilities in (2.32) yield the total transverse or angular velocity response at the driving point for force or moment driving. In this sense, they are total driving-point mobilities. However,
to obtain the response of the beam at any point other than
the driving point, we need the modal responses rather
than the total responses at the driving point. The reason
for this is that, as Equations (2.15) indicate, the two
modes propagate at different velocities. The response
at a point \( x \) is then given by the combination of the
two modes at that particular point. This being so, it
is clearly more convenient and more general to obtain the
modal responses at the driving-point.

There are two ways to do this:

(a) Split the excitation matrix into modal excitation
matrices and use each one with the total mobility matrix
of (2.32) to obtain the modal responses.

(b) Determine from the onset a modal mobility matrix
which used with the excitation matrix directly gives the
modal responses at the driving point.

We will adopt Case (b) procedure as it appears to be
considerably simpler. Therefore, in the next item we
will engage in determining the expressions for the modal
mobilities contained in the modal mobility matrix below:

\[
\begin{align*}
\begin{bmatrix}
W_1 + W_2 \\
\psi_1 + \psi_2
\end{bmatrix}
= & \begin{bmatrix}
M_{FW1} + M_{FW2} & M_{MW1} + M_{MW2} \\
M_{F\psi1} + M_{F\psi2} & M_{M\psi1} + M_{M\psi2}
\end{bmatrix}
\begin{bmatrix}
F \\
M
\end{bmatrix}
\end{align*}
\] (2.33)
The meaning of (2.33) is:

\[ W_1 = M_{FW1} \cdot F + M_{MW1} \cdot M \]

\[ W_2 = M_{FW2} \cdot F + M_{MW2} \cdot M \]  \hspace{1cm} (2.33a-d)

\[ \Psi_1 = M_{F\Psi1} \cdot F + M_{M\Psi1} \cdot M \]

\[ \Psi_2 = M_{F\Psi2} \cdot F + M_{M\Psi2} \cdot M \]

2.6.2 **The Components of the Modal Mobility Matrix**

Consider again the semi-infinite beam of Figure 2.13, driven at the free end by a general sinusoidal load, and recall that causality says that there will only be right going waves.

Adopting the sign conventions of Figure 2.1b and c, we can write the boundary conditions at x=0 as follows:

\[ Q(0,t) = -Fe^{-i\omega t} \]  \hspace{1cm} (2.34)

\[ M(0,t) = -Me^{-i\omega t} \]  \hspace{1cm} (2.35)

Equations (2.26) and (2.27) are the expressions for the
shear force \( Q(x,t) \) and the bending moment \( M(x,t) \) at a section \( x \) of a Timoshenko beam that contains only right going waves. Rewrite Equations (2.26) and (2.27) for velocity components by using Equations (2.29):

\[
Q(x,t) = -\frac{G\alpha \omega}{a_1^2} \left( \frac{W_1}{k_1} e^{ik_1x} + \frac{W_2}{k_2} e^{ik_2x} \right) e^{-i\omega t} \quad (2.36)
\]

\[
M(x,t) = -\frac{iEI}{\omega} \left[ (k_1^2 - \frac{\omega^2}{a_1^2})W_1 e^{ik_1x} + (k_2^2 - \frac{\omega^2}{a_1^2})W_2 e^{ik_2x} \right] e^{-i\omega t}
\]

\[
\ldots (2.37)
\]

Using now (2.36) and (2.37) into (2.34) and (2.35), we obtain:

\[
\frac{1}{k_1} W_1 + \frac{1}{k_2} W_2 = \frac{a_1^2F}{G\alpha \omega} \quad (2.38)
\]

\[
(k_1^2 - \frac{\omega^2}{a_1^2})W_1 + (k_2^2 - \frac{\omega^2}{a_1^2})W_2 = -\frac{i\omega M}{EI} \quad (2.39)
\]

If we now solve the set (2.38)-(2.39) for \( W_1 \) and \( W_2 \) and remember that:
\[ G^* = \rho a_1^2 \quad E = \rho a_2^2 \quad I = Ar^2 \]

we obtain:

\[ \bar{W}_1 = F \frac{k_1 k_2 (k_2^2 - \omega^2/a_1^2)}{\rho A \omega d} + M \frac{i \omega k_1}{\rho A r^2 a_2^2 d} \]

\[ \bar{W}_2 = -F \frac{k_1 k_2 (k_1^2 - \omega^2/a_1^2)}{\rho A \omega d} - M \frac{i \omega k_2}{\rho A r^2 a_2^2 d} \]

(2.40a,b)

where:

\[ d = k_2 (k_2^2 - \frac{\omega^2}{a_1^2}) - k_1 (k_1^2 - \frac{\omega^2}{a_2^2}) \]

(2.41)

Using now Equation (2.23) in conjunction with

Equations (2.40), we obtain the resulting modal amplitudes

for the angular velocities at the driving point.

\[ \psi_1 = F \frac{i k_2 (k_1^2 - \omega^2/a_1^2) (k_2^2 - \omega^2/a_1^2)}{\rho A \omega d} - M \frac{\omega (k_1^2 - \omega^2/a_1^2)}{\rho A r^2 a_2^2 d} \]

(2.42a,b)

\[ \psi_2 = -F \frac{i k_1 (k_1^2 - \omega^2/a_1^2) (k_2^2 - \omega^2/a_1^2)}{\rho A \omega d} + M \frac{\omega (k_2^2 - \omega^2/a_1^2)}{\rho A r^2 a_2^2 d} \]
If we impose now $M = 0$ in Equations (2.40) and (2.42), we obtain the following driving-point modal mobilities:

$$M_{FW1} = \frac{W_1}{F} = \frac{k_1 k_2 (k_2^2 - \omega^2/a_1^2)}{\rho A \omega d} \quad (2.43a)$$

$$M_{FW2} = \frac{W_2}{F} = -\frac{k_1 k_2 (k_1^2 - \omega^2/a_1^2)}{\rho A \omega d} \quad (2.43b)$$

$$M_{F\psi 1} = \frac{\psi_1}{F} = \frac{i k_2 (k_1^2 - \omega^2/a_1^2) (k_2^2 - \omega^2/a_1^2)}{\rho A \omega d} \quad (2.43c)$$

$$M_{F\psi 2} = \frac{\psi_2}{F} = -\frac{i k_1 (k_1^2 - \omega^2/a_1^2) (k_2^2 - \omega^2/a_1^2)}{\rho A \omega d} \quad (2.43d)$$

Now if we impose $F = 0$ in Equations (2.40) and (2.42), we obtain the other four driving-point modal mobilities,

$$M_{MW1} = \frac{W_1}{M} = \frac{i \omega k_1}{\rho A r^2 a_2^2 d} \quad (2.43e)$$

$$M_{MW2} = \frac{W_2}{M} = -\frac{i \omega k_2}{\rho A r^2 a_2^2 d} \quad (2.43f)$$
\[ M_{\psi 1} = \frac{\psi_1}{M} = - \frac{\omega (k_1^2 - \omega^2/a_1^2)}{\rho Ar^2 a_2^2 d} \]  

\[ M_{\psi 2} = \frac{\psi_2}{M} = \frac{\omega (k_2^2 - \omega^2/a_1^2)}{\rho Ar^2 a_2^2 d} \]  

(2.43g)  

(2.43h)

Table 2.2 contains the dimensional and non-dimensional forms for the propagation constants \( k_1 \) and \( k_2 \) as given by Equation (2.14) and for the modal mobilities as given by Equations (2.43a-h).

The non-dimensional equations for the modal mobilities presented in Table 2.2 have been plotted in Figures 2.14 through 2.23. The range used for the parameter \( \bar{a} \) was taken from Equation (2.9). The range for the frequency parameter \( \bar{\omega}_1 \) is:

\[ 0 < \bar{\omega}_1 < 2.6 \]

This range was chosen such that at the maximum value it is clear what values the curves tend to for very high values of \( \bar{\omega}_1 \). We have indicated in the figures the limiting values.
TABLE 2.2 Summary of dimensional and non-dimensional equations for the propagations constants and the modal mobilities for a semi-infinite Timoshenko beam.

<table>
<thead>
<tr>
<th>DIMENSIONAL</th>
<th>NON-DIMENSIONAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{1,2} = \frac{\omega}{\sqrt{2a_2}} \sqrt{\left(1 - \frac{a_2^2}{a_1^2}\right)^2 \pm \sqrt{\left(1 - \frac{a_2^2}{a_1^2}\right)^2 + \frac{4a_2^2}{\omega^2}}}$</td>
<td>$k_{1,2} = r k_{12} = \frac{\bar{\omega}_1}{\sqrt{2a_1}} \sqrt{\left(1 + \bar{a}^2\right)^2 \pm \sqrt{\left(1 - \bar{a}^2\right)^2 + \left(\frac{2\bar{a}}{\bar{\omega}_1}\right)^2}}$</td>
</tr>
<tr>
<td>$M_{FW1} = \frac{\psi}{F_1} = k_1k_2(k_2^2 - \omega^2/a_1^2)/\rho Aw_d$</td>
<td>$\bar{M}<em>{FW1} = (a_1 \rho A) M</em>{FW1} = \bar{k}_1\bar{k}_2(v_2^2 - \bar{\omega}_1^2)/\bar{\omega}_1 \bar{d}$</td>
</tr>
<tr>
<td>$M_{FW2} = \frac{\psi}{F_2} = -k_1k_2(k_1^2 - \omega_1^2/a_1^2)/\rho Aw_d$</td>
<td>$\bar{M}<em>{FW2} = (a_1 \rho A) M</em>{FW2} = -\bar{k}_1\bar{k}_2(k_1^2 - \bar{\omega}_1^2)/\bar{\omega}_1 \bar{d}$</td>
</tr>
<tr>
<td>$M_{F\psi_1} = \frac{\psi}{F_1} = i k_2(k_2^2 - \omega^2/a_1^2)(k_2^2 - \omega^2/a_1^2)/\rho Aw_d$</td>
<td>$\bar{M}<em>{\psi_1} = (a_1 \rho Ar) M</em>{F\psi_1} = i \bar{k}_2(k_1^2 - \bar{\omega}_1^2)(k_2^2 - \bar{\omega}_1^2)/\bar{\omega}_1 \bar{d}$</td>
</tr>
<tr>
<td>$M_{F\psi_2} = \frac{\psi}{F_2} = -i k_1(k_1^2 - \omega^2/a_1^2)(k_2^2 - \omega^2/a_1^2)/\rho Aw_d$</td>
<td>$\bar{M}<em>{\psi_2} = (a_1 \rho Ar) M</em>{F\psi_2} = -i \bar{k}_1(k_1^2 - \bar{\omega}_1^2)(k_2^2 - \bar{\omega}_1^2)/\bar{\omega}_1 \bar{d}$</td>
</tr>
<tr>
<td>$M_{MW1} = \frac{\omega}{M} = i \omega k_1/\rho Ar a_2 d$</td>
<td>$\bar{M}<em>{MW1} = (a_1 \rho Ar) M</em>{MW1} = i \bar{\omega}_1 \bar{k}_1/\bar{a}_2 \bar{d}$</td>
</tr>
<tr>
<td>$M_{MW2} = \frac{\omega}{M} = -i \omega k_2/\rho Ar a_2 d$</td>
<td>$\bar{M}<em>{MW2} = (a_1 \rho Ar) M</em>{MW2} = -i \bar{\omega}_1 \bar{k}_2/\bar{a}_2 \bar{d}$</td>
</tr>
<tr>
<td>$M_{M\psi_1} = \frac{\psi}{M} = -\omega(k_1^2 - \omega^2/a_1^2)/\rho Ar a_2 d$</td>
<td>$\bar{M}<em>{M\psi_1} = (a_1 \rho Ar^2) M</em>{M\psi_1} = -\bar{\omega}_1(k_1^2 - \bar{\omega}_1^2)/\bar{a}_2 \bar{d}$</td>
</tr>
<tr>
<td>$M_{M\psi_2} = \frac{\psi}{M} = \omega(k_2^2 - \omega_1^2/a_1^2)/\rho Ar a_2 d$</td>
<td>$\bar{M}<em>{M\psi_2} = (a_1 \rho Ar^2) M</em>{M\psi_2} = \bar{\omega}_1(k_2^2 - \bar{\omega}_1^2)/\bar{a}_2 \bar{d}$</td>
</tr>
<tr>
<td>$d = k_2(k_2^2 - \omega^2/a_1^2) - k_1(k_1^2 - \omega^2/a_1^2)$</td>
<td>$\bar{d} = \bar{k}_2(k_2^2 - \bar{\omega}_1^2) - \bar{k}_1(k_1^2 - \bar{\omega}_1^2)$</td>
</tr>
</tbody>
</table>
Figure 2.14 Absolute value of the modal mobility $|\overline{M}_{FW1}|$

$\overline{\omega_1} = \omega r/a_1$
Figure 2.5 Absolute value of the modal mobility $|\overline{M}_{FW2}|$

$\overline{\omega}_1 = \frac{\omega a_1}{r}$
Argument of \( \bar{M}_{FW1} \) and \( \bar{M}_{FW2} \) (degrees)

\[ \bar{\omega}_1 = \omega r/a_1 \]

Figure 2.16 Argument of the modal mobilities \( \bar{M}_{FW1} \) and \( \bar{M}_{FW2} \)
Figure 2.17 Absolute value of the modal mobilities $\bar{M}_{\psi 1}$ and $\bar{M}_{\psi 2}$
Argument of $\overline{M}_{F\psi_1}$ and $\overline{M}_{MW2}$ (degrees)

$\bar{\omega}_1 = \omega r/a_1$

Figure 2.18 Argument of the modal mobility $\overline{M}_{F\psi_1}$ and $\overline{M}_{MW2}$. 
Figure 2.19 Absolute value of the modal mobility $\overline{M}_F \psi_2$ and $\overline{M}_{MW1}$. 

$\overline{\omega}_1 = \omega r/a_1$
Figure 2.20 Argument of the modal mobilities $\bar{M}_{F,\psi}$ and $\bar{M}_{MWL}$.
Figure 2.21 Absolute value of the modal mobility $\bar{M}_{M\psi_1}$. 

$\bar{\omega}_1 = \omega r/a_1$
Figure 2.22 Absolute value of the modal mobility $\bar{M}_{M\psi_2}$.

$\bar{\omega}_1 = \omega r/a_1$
Arguments of $\bar{\bar{M}}_{M\psi_1}$ and $\bar{\bar{M}}_{M\psi_2}$ (degrees)

\[ \bar{\omega}_1 = \omega r/a_1 \]

Figure 2.23 Argument for the modal mobilities $\bar{\bar{M}}_{M\psi_1}$ and $\bar{\bar{M}}_{M\psi_2}$. 
2.6.3 Analyses of the Mobility Matrix Elements

We will now discuss some interesting aspects of the modal mobility matrix elements we have obtained in Item 2.6.2. This discussion is based on Figures 2.14 through 2.23.

(a) The first thing to notice is that there are only ten figures instead of the sixteen we would need for the eight modal mobilities with one figure for the amplitude and one for the argument of each mobility. The reason for this is that the arguments of $\overline{M}_{\text{FW1}}$ and $\overline{M}_{\text{FW2}}$ are equal to each other, and the arguments of $\overline{M}_{\text{F} \psi 1}$ and $\overline{M}_{\text{F} \psi 2}$ are also equal to each other. This means that the transverse velocities $W_1(0,t)$ and $W_2(0,t)$ generated by a sinusoidal force are in phase. The same is true for the angular velocities $\psi_1(0,t)$ and $\psi_2(0,t)$ generated by a sinusoidal moment. Furthermore, $\overline{M}_{\text{F} \psi 1} = \overline{M}_{\text{MW2}}$ and $\overline{M}_{\text{F} \psi 2} = \overline{M}_{\text{MW1}}$. In other words, the off-diagonal total mobilities $\overline{M}_{\text{F} \psi}$ ($= \overline{M}_{\text{F} \psi 1} + \overline{M}_{\text{F} \psi 2}$) and $\overline{M}_{\text{MW}}$ ($= \overline{M}_{\text{MW1}} + \overline{M}_{\text{MW2}}$) are equal, which makes the total mobility matrix in (2.32) symmetric for a Timoshenko beam. Yet in modal terms, only opposite modes match in the off-diagonal terms.

(b) For $\overline{\omega}_1 = 1$ (cutoff frequency for the second mode), we observe that $\overline{M}_{\text{FW1}} = \overline{M}_{\text{FW2}} = \overline{M}_{\text{F} \psi 1} = 0$ and that $\overline{M}_{\text{F} \psi 2} = 1$. This means that a semi-infinite Timoshenko beam driven
at the free end by a transverse sinusoidal force will show no transverse motion but only angular motion for the second mode. As the wave length for the second mode is infinite at \( \tilde{\omega}_1 = 0 \), the resulting motion is thickness shear (see Figure 2.10c). This might seem strange because we are driving a longitudinal motion with a transverse force. The explanation is that the transverse force induces shear deformation which, for the second mode, has infinite velocity of propagation. Therefore, the whole length of the beam suffers the same amount of shear simultaneously and the resultant motion is thickness shear. The first mode just happens not to be driven by the force at this frequency.

(c) Now we will study the low and high frequency limits for all the mobilities. The limits and discussions are in Table 2.3.

(d) Finally, we would like to notice the sharp change in slope for all curves of absolute value of mobilities, as well as the sudden change of phase in some cases, at the cutoff frequency of the second mode \( \tilde{\omega}_1 = 1 \). This fact has no quick and simple physical explanation. However, there is the significant fact that at this frequency the second mode changes from a nonpropagating condition (\( \tilde{\omega}_1 < 1 \)) to a propagating condition (\( \tilde{\omega}_1 > 1 \)). In other words (see Figure
### TABLE 2.3 Limiting Values for the Modal and Total Mobility for a Semi-Infinite Timoshenko beam driven at the free end.

<table>
<thead>
<tr>
<th>( \omega_1 \to 0 )</th>
<th>( \omega_1 \to \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{FW} = M_{FW}(a_1 \rho A) )</td>
<td>( -m_1 + i m_1^* )</td>
</tr>
</tbody>
</table>

| Discussion | \( M_{FW} \) is equal to \( M_{FW} \) for a shear beam \( (l/a_1 A) \). This agrees with conclusion from Fig. 2.11 that at high frequencies mode 1 is dominated by shear (see Fig. 2.10b) |
| Discussion | Very high frequency transverse force causes no angular motion. |

\(*m_1, m_2 \& m_3 \) are constant functions of \( a \).
TABLE 2.3 (Continued)

<table>
<thead>
<tr>
<th></th>
<th>( \lim _{\omega _1 \to 0} )</th>
<th>( \lim _{\omega _1 \to \infty} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mode 1</td>
<td>Mode 2</td>
</tr>
<tr>
<td>( M_{MW} )</td>
<td>-( m_1 + im_2 )</td>
<td>-( m_1 - im_2 )</td>
</tr>
<tr>
<td>( M_{MW}(a_1 \theta Ar) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

|                  | | | | | | |
| \( M_{M\psi} = \) | | | | | | |
| \( M_{M\psi}(a_0 \theta Ar^2) \) | 0 | 0 | 0 | Static moment causes no steady-state angular velocity at the driving-point. This means that the cross-section will have a constant angle of which is function of the intensity of the moment \( M \). |
|                  | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |

\( \frac{m_3}{m_3} \) Very high frequency moment causes no transverse motion.

\( \frac{m_3}{m_3} \) Very high frequency moment causes angular velocity only for the second mode. \( r^2 M_{M\psi} \) equals the driving point mobility for compressional waves in a rod. Fig. 2.10b shows that motion is equivalent to compressional waves of varying amplitude through the cross-section. Motion changes phase at the neutral line.
2.3 for $\bar{k}_2$) for $\bar{\omega}_1$ slightly below unity, the second mode is a very slowly decaying exponential curve and for $\bar{\omega}_1$ slightly greater than unity it is a propagating mode with large wavelength. In the former case no energy is propagated in the second mode and in the latter case energy is propagated. The fact that the effect is extended also to the first mode mobilities suggests that there is some kind of coupling between the two modes of propagation. This can be seen from Equations (2.40) and (2.42) where the amplitude of the responses for any of the two modes depend on both propagation constants $k_1$ and $k_2$.

Mindlin and Deresiewicz (1954) point out that at the frequency of the first thickness-shear motion, for a beam studied with three-dimensional elasticity, there is strong interaction between the flexural modes and the thickness-shear modes. This causes a change in character of the frequency spectrum, they state. They associated the first thickness-shear frequency from three-dimensional elasticity with the thickness-shear of the second mode at $\bar{\omega}_1 = 1$ for the Timoshenko beam.

2.7 Use of the Modal Mobility Matrix

The modal mobility matrix can be used to solve for
the modal responses for virtually any problem of semi-infinite Timoshenko beam driven at the end by a sinusoidal load. An infinite beam loaded at a single point can be treated as a pair of semi-infinite beams.

Table 2.4 lists the six different sinusoidal forced problems that can be encountered in practice.

Table 2.4 The six different sinusoidal forced problems for semi-infinite beams

<table>
<thead>
<tr>
<th>Imposed</th>
<th>Resulting</th>
</tr>
</thead>
<tbody>
<tr>
<td>F,M</td>
<td>W,ψ</td>
</tr>
<tr>
<td>W,ψ</td>
<td>F,M</td>
</tr>
<tr>
<td>F,W</td>
<td>M,ψ</td>
</tr>
<tr>
<td>F,ψ</td>
<td>M,W</td>
</tr>
<tr>
<td>M,W</td>
<td>F,ψ</td>
</tr>
<tr>
<td>M,ψ</td>
<td>F,W</td>
</tr>
</tbody>
</table>

It is important to notice that the modal responses can be computed only if F and M are known. In other words,
if we consider any case but the first in Table 2.4, we first have to solve for the other two variables using the total mobility matrix (modal matrix with modal mobilities added together). Then F and M are used to solve for the modal velocity responses. For example, if sinusoidal velocities W and \( \psi \) are imposed, we solve for F and M by inverting (2.32) as follows:

\[
\begin{bmatrix}
F \\
M
\end{bmatrix} = \begin{bmatrix}
M_{FW} & M_{MW} \\
M_{F\psi} & M_{M\psi}
\end{bmatrix}^{-1} \begin{bmatrix}
W \\
\psi
\end{bmatrix}
\] (2.44)

Then we use F and M in (2.33) to solve for the modal velocity responses.

The inverse of the total mobility matrix in (2.44) is known as total impedance matrix.

We will now give two examples to illustrate the use of the modal mobility matrix we have developed in Item 2.6.

**Example 1:** Consider a semi-infinite beam driven at the end by a sinusoidal force \( F e^{-i\omega t} \) (amplitude \( F = 2000 \text{N} = 449 \text{ lb} \)) and a sinusoidal moment \( M e^{-i\omega t} \). The condition imposed is that the total motion of the beam consists purely of the second mode. In other words, the first mode shall not be excited. This is an important problem
because for frequencies below the cutoff frequency of the second mode ($\bar{\omega}_1 < 1$) the second mode does not propagate and therefore no power is introduced into the beam. Therefore, if a beam is excited by a sinusoidal force, it is possible to add a sinusoidal moment whose amplitude and phase bear a certain relationship to the amplitude and phase of the force, such that the power input is minimized.

For this example we have chosen the I-beam Wl6x88 (web 16 in. high; total weight 88 lbs/ft) from the Manual of Steel Construction (1973). The characteristics of interest for this beam are:

- modulus of elasticity: $E = 30 \times 10^6$ psi
  
  $= 2.07 \times 10^{11}$ N/m$^2$

- mass density: $\rho = 7700$ kg/m$^3$

- Poisson's ratio: $\nu = 0.275$

- cross-section area: $A = 25.9$ in$^2$
  
  $= 1.67 \times 10^{-2}$ m$^2$

- radius of gyration: $r = 6.87$ in $= 0.174$ m

- shear correction coefficient (from Table A.2):
  
  $\kappa = 0.281$

- from Equation (2.9a): $\bar{a} = \sqrt{2(1+\nu)/\kappa} = 3.01$

- from Equation (2.4): $a_1 = \sqrt{G^*/\rho} = \sqrt{\kappa E/2(1+\nu)\rho}$
  
  $= 1721$ m/sec
- the product $\rho A a_1 = 2.21 \times 10^5 \text{ N/(m/sec)}$

Equation (2.28a) gives the relation between bending moment and shear force for both modes of propagation. At the driving-point, the bending moment and the shear force equal the driving moment and the driving force respectively. Therefore, in order to drive the second mode only we must impose for the driving moment, according to Equation (2.28a):

$$M/rF = i \frac{\bar{a}^2 \bar{k}_2 (\bar{k}_2^2 - \omega_1^2)}{\bar{\omega}_1^2}$$  \hspace{1cm} (2.45)

This ratio could be directly estimated from Figures 2.11 and 2.12, but the logarithmic scale used for the absolute values makes it difficult to read. Therefore, we prefer to estimate the propagation constant $\bar{k}_2$ from Figure 2.3 and then compute $M/rF$ with Equation (2.45). Notice that although we are considering the amplitude $F$ as a constant, the amplitude $M$ is a function of frequency.

The computation process is all shown in Table 2.5. The last four lines on the table are the amplitudes of the modal responses for transverse and angular velocities.

We notice that the zero values for $W_1$ and $\psi_1$ were in the actual computations different from zero, but at least
TABLE 2.5 Solution for the response of a semi-infinite W16X88 beam, driven at the end by a sinusoidal load (F = 2000N), such that only second mode is driven (symbols written within parentheses stand for the units).

<table>
<thead>
<tr>
<th>$\overline{ω}_1$</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ω = \overline{ω}_1 a_1 / r$ (rd/sec)</td>
<td>4946</td>
<td>9892</td>
<td>14838</td>
<td>19784</td>
<td>24729</td>
</tr>
<tr>
<td>$f = ω / 2π$ (Hz)</td>
<td>787</td>
<td>1574</td>
<td>2361</td>
<td>3149</td>
<td>3936</td>
</tr>
<tr>
<td>$k_1'$, $k_1 = \overline{k_1} / r$ (m$^{-1}$)</td>
<td>0.58, 3.33</td>
<td>1.05, 6.32</td>
<td>1.54, 8.85</td>
<td>2.02, 11.61</td>
<td>2.51, 14.37</td>
</tr>
<tr>
<td>$λ_1 = 2π/k_1$ (m)</td>
<td>1.88</td>
<td>0.99</td>
<td>0.71</td>
<td>0.54</td>
<td>0.44</td>
</tr>
<tr>
<td>$k_2'$, $k_2 = \overline{k_2} / r$ (m$^{-1}$)</td>
<td>0.25i, 1.44i</td>
<td>0,0</td>
<td>0.37, 2.13</td>
<td>0.57, 3.28</td>
<td>0.76, 4.37</td>
</tr>
<tr>
<td>$λ_2 = 2π/k_2$ (m)</td>
<td>2.95</td>
<td>1.92</td>
<td>1.44</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M/rF$ (eqn. (2.45))</td>
<td>2.83</td>
<td>0</td>
<td>-3.15i</td>
<td>-4.74i</td>
<td>-6.25i</td>
</tr>
<tr>
<td>M(N.m)</td>
<td>985.3</td>
<td>0</td>
<td>-1095.6i</td>
<td>-1651.2i</td>
<td>-2174.8i</td>
</tr>
<tr>
<td>$</td>
<td>\overline{M}_{FW1}</td>
<td>$, argument</td>
<td>0.97, 34.2°</td>
<td>0, 90°</td>
<td>0.83, 0°</td>
</tr>
<tr>
<td>$</td>
<td>\overline{M}_{FW2}</td>
<td>$, argument</td>
<td>0.28, 33.9°</td>
<td>0, 90°</td>
<td>0.05, 0°</td>
</tr>
<tr>
<td>$</td>
<td>\overline{M}_{ψ1}</td>
<td>$, argument</td>
<td>0.15, 124°</td>
<td>0, 180°</td>
<td>0.06, 90°</td>
</tr>
<tr>
<td>$</td>
<td>\overline{M}_{ψ2}</td>
<td>$, argument</td>
<td>0.35, -146°</td>
<td>1.0, -90°</td>
<td>0.27, -90°</td>
</tr>
<tr>
<td>$</td>
<td>\overline{M}_{MW1}</td>
<td>$, argument</td>
<td>0.35, -146°</td>
<td>1.0, -90°</td>
<td>0.27, -90°</td>
</tr>
<tr>
<td>(</td>
<td>\tilde{M}_{MW2}</td>
<td>), argument</td>
<td>0.15, 124°</td>
<td>0, 180°</td>
<td>0.06, 90°</td>
</tr>
<tr>
<td>(</td>
<td>\tilde{M}_{M\psi1}</td>
<td>), argument</td>
<td>0.054, -56°</td>
<td>0.105, 0°</td>
<td>0.02, 0°</td>
</tr>
<tr>
<td>(</td>
<td>\tilde{M}_{M\psi2}</td>
<td>), argument</td>
<td>0.34, -56°</td>
<td>0.949, 0°</td>
<td>0.371, 0°</td>
</tr>
</tbody>
</table>

\[
W_1 = \frac{F}{\rho A a_1} (M_{FW1} + \frac{M}{r F} M_{MW1}) (m) = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
W_2 = \frac{F}{\rho A a_1} (M_{FW2} + \frac{M}{r F} M_{MW2}) (m) = \begin{pmatrix}
4.60 \times 10^{-3} i \\
0 \\
2.31 \times 10^{-3} \\
2.63 \times 10^{-3} \\
2.72 \times 10^{-3}
\end{pmatrix}
\]

\[
\psi_1 = \frac{F}{r \rho A a_1} (M_{F\psi1} + \frac{M}{r F} M_{M\psi1}) (rd) = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\psi_2 = \frac{F}{r \rho A a_1} (M_{F\psi2} + \frac{M}{r F} M_{M\psi2}) (rd) = \begin{pmatrix}
-0.05i \\
-0.05i \\
-0.075i \\
-0.096i \\
-0.119i
\end{pmatrix}
\]
one order of magnitude smaller than the other values. This is caused by approximations that result from reading numbers out of graphs for $\bar{k}_2$ and for the modal mobilities.

**Example 2:** Let us now consider an infinite beam driven at the "middle" by a sinusoidal transverse force. Despite the fact that there is no external moment, there will exist a bending moment at the driving section because it no longer is a free end. In this case, however, we can state that the driving section will have no rotational motion due to the symmetry of the infinite beam around the driving-point. Therefore, we can split the infinite beam into two semi-infinite beams (see Figure 2.23a) and solve the problem for the right-hand side only with the following boundary conditions:

\[
Q(0,t) = -\frac{1}{2} Fe^{-i\omega t} \\
\psi(0,t) = 0
\]

(2.46a,b)

\[
Me^{-i\omega t}
\]

\[
\begin{array}{c}
\uparrow \\
\lambda = 0
\end{array}
\]

Figure 2.23a
The response of this system is the same as the response of the proposed problem for the infinite beam. From (2.32), we have:

\[ W = F \cdot M_{FW} + M \cdot M_{MW} \]  
\[ \Psi = F \cdot M_{F\psi} + M \cdot M_{M\psi} \]  
(2.47a,b)

Using the boundary conditions (2.46a,b) in Equations (2.47a,b), we obtain:

\[ W = -\frac{1}{2} F \cdot M_{FW} + M \cdot M_{MW} \]  
\[ 0 = -\frac{1}{2} F \cdot M_{F\psi} + M \cdot M_{M\psi} \]  
(2.48a,b)

Equation (2.48b) yields:

\[ M = \frac{1}{2} F \cdot \frac{M_{F\psi}}{M_{M\psi}} \]  
(2.49)

This is the amplitude of the external moment to be applied at the free end, for the boundary condition (2.46b) to be satisfied. It also is the amplitude of the bending moment that exists at the driving section of the infinite beam of the proposed problem. In order to solve
for the modal responses all we have to do now is to use
the given $F$ and the $M$ determined in Equation (2.49) in
conjunction with the modal mobility matrix of (2.33).
We will not continue because the solution procedure from
now on is identical to the one performed in detail in
Example 1 above.

We will, however, use this opportunity to determine
the driving-point mobility of the infinite beam driven
by a transverse force. Substituting Equation (2.49)
into (2.48a), we obtain, after dividing by $F$,

$$ M_{FW\infty} = \frac{W}{F} = \frac{1}{2} \left( -M_{FW} + M_{MW} \cdot \frac{M_{F\psi}}{M_{M\psi}} \right) \quad (2.50) $$

where $M_{FW\infty}$ is the driving-point mobility we are looking for.

Using now the expressions for the modal mobilities
in Table 2.1 and remembering that each total mobility
is the sum of the two modal mobilities, we obtain from
Equation (2.50), after some simplification,

$$ M_{FW\infty} = \frac{k_1 k_2 + \omega^2/a_1^2}{2 \rho A \omega (k_1 + k_2)} \quad (2.51) $$

Or in non-dimensional terms:

$$ \bar{M}_{FW\infty} = (a_1 \rho A) M_{FW\infty} = \frac{1}{2} \frac{\bar{k}_1 \bar{k}_2 + \bar{\omega}^2}{\bar{\omega}_1 (\bar{k}_1 + \bar{k}_2)} \quad (2.51a) $$
A different expression for this driving-point mobility is presented by Cremer and Heckl (1967) in their Chapter 4, Equation (78b). As stated in the introduction of our present chapter, their expression does not seem to be equivalent to our Equation (2.51). In order to check the validity of these expressions we can verify their asymptotic behavior in the limit of infinite frequency.

Using the expressions for $\bar{k}_1$ and $\bar{k}_2$ in Table 2.1 in Equation (2.51a), and performing the limit for $\bar{\omega}_1 \to \infty$ we obtain

$$\lim_{\bar{\omega}_1 \to \infty} \bar{M}_{FW\infty} = 1$$

or

$$\lim_{\bar{\omega}_1 \to \infty} M_{FW\infty} = \frac{1}{2a_1 \rho A} \quad (2.52)$$

Thus the high frequency limit of (2.51) is the same as the driving-point mobility of an infinite shear-beam, which is the expected result.

If we perform the same limit for the expression by Cremer and Heckl, we get:
\[ \lim_{\omega \to \infty} M_{FW} = \infty \]

which does not seem to be correct.

2.8 **Comparison of the Timoshenko Beam Theory with Other Approximate Theories**

In this section we will compare graphically the elements of the total mobility matrix for the Timoshenko theory with the ones for the following theories:

(a) Shear beam

(b) Bernoulli-Euler beam

(c) Rayleigh beam

(d) Shear-bending beam.

The equations of motion and the expressions for the total mobilities for all theories, including the Timoshenko theory, are presented in Tables B.1 and B.2 of Appendix B. Notice that the shear beam theory has only the mobility \( M_{FW} \). This is because no angular motion is allowed in a shear beam, and also because an external moment shall cause no transverse displacement on it.

Figures 2.24 through 2.31 contain the curves for the four elements of the total mobility matrix for all the theories. Note that the Bernoulli-Euler and the Rayleigh beams do not have \( \tilde{a} \) \( (= \frac{a_2}{a_1} = \sqrt{E/G^*}) \) as a parameter. This is because in both theories no shear deformation is allowed,
or the beam is infinitely rigid to shear \((G = \infty)\) and \(\bar{a} = 0\). The other three theories contain the parameter \(\bar{a}\), and for them we took the extreme values of \(\bar{a}\) given by Equation (2.9), namely \(\bar{a} = 1.5\) and \(\bar{a} = 4.5\).

We would like to call attention to the fact that in this section the nondimensional groups for angular frequency \((\bar{\omega}_2 = \omega r / a_2)\) and mobilities (see Table B.2) contain the parameter \(a_2 (= \sqrt{E / \rho})\) instead of the parameter \(a_1 (= \sqrt{G^* / \rho})\) used in the previous sections. Again, the reason is that \(a_1\) is not defined for two of the theories being compared. For this reason the numerical values of the nondimensional mobilities for the Timoshenko beam in this section differ from the results of Section 2.6. Also, the cutoff frequency for the second mode in the Timoshenko beam is no longer unity but it is the inverse of \(\bar{a}\).

As a general rule, we will consider the Timoshenko theory as the most accurate of the five theories being compared. Therefore, the results for the other theories will be judged in the light of the Timoshenko theory results. It is, however, important to note that the Timoshenko theory is also an approximate theory and has limitations of its own. The three-dimensional theory of elasticity (Pochhammer-Chree theory) indicates that a beam has an infinite number of modes of propagation. The
two modes of propagation for the Timoshenko beam correspond to the two first modes of the exact theory. Abramson has also shown graphically that, for a circular cylindrical bar, the curve for the phase velocity versus frequency for the first mode of the Timoshenko beam has a good agreement with the same curve for the first mode of the exact theory. For the second mode, the overall shape of the curves is the same but the agreement is poor. In addition, the exact theory predicts a high frequency limit for the phase velocity of all its modes equal to the phase-velocity of Rayleigh waves $c_R$. This is reasonable because for very short wavelengths, the beam looks to the waves as a semi-infinite solid. The Timoshenko theory predicts for the first mode a high frequency limit for the phase velocity equal to the modified shear velocity $a_1 = \sqrt{G^s/\rho}$. The phase velocity of Rayleigh waves is slightly lower than the velocity of shear waves in an unbounded medium $c_s = \sqrt{G/\rho}$. More specifically (Viktorov, 1967):

$$0.87c_s < c_R < 0.96c_s$$

where $c_R$ is the phase velocity of Rayleigh waves and it varies with Poisson ratio $\nu$ (the above range is for $0 < \nu < 0.5$). Therefore, for full beams (shear correction
coefficient $\kappa$ close to unity and $G^*$ slightly lower than $G$) the high frequency limit for the phase velocity of the first mode of the Timoshenko beam is in good agreement with the results of the exact theory. However, for less full sections (I-beams for example), $\kappa$ is much smaller than unity and the limit $a_1 = \sqrt{G^*/\rho}$ is lower than $c_R$, and the agreement with the exact theory is not so good in the high frequencies.

For the second mode the Timoshenko beam predicts a high frequency limit for the phase velocity equal to the bar velocity $a_2 = \sqrt{E/\rho}$ which for $\nu = 0.3$ is $1.67c_R$. A poor result as compared to the limit $c_R$ for the exact theory.

Despite these limitations, the Timoshenko beam theory bridges very well the gap between the excessively complicated Pochhammer-Chree or exact theory and the simpler theories.

2.8.1 Interpretation of Curves

We will now study the curves of Figures 2.24 to 2.29.

These curves allow for a quick visualization of the differences among the predictions of the response of a semi-infinite beam to sinusoidal excitation, as given by the various approximate theories. The physical interpretation
of the results may be easy in some instances, but in
general it is difficult, due to the complexity of the
phenomena involved.

Looking at Table B.2, we see that the mobilities and
the propagation constants for the Rayleigh beam are
directly obtained by making $\bar{a} \to 0$ (eliminating shear
deformation) for the Timoshenko beam. The same is true in
obtaining the Bernoulli-Euler beam results from the
shear-bending beam results. However, going from the
Timoshenko beam results to the shear-bending beam results
or from the Rayleigh beam results to the Bernoulli-Euler
beam results is more difficult. There is no easy
limiting process that leads to the transformations. The
reason for this is that we are trying to eliminate rotary
inertia and there is no clear indication on how it
modifies the equations.

Let us now study the individual mobilities.

(a) Total Mobility $\bar{M}_{FW}$

The striking characteristic of Figure 2.24 is the
existence of two different groups of curves. The upper
group contains the curves for the theories that include
shear deformation with $\bar{a} = 4.5$ (low shear coefficient
$\kappa$ — see Section 2.2.2). We see that both the Timoshenko
beam and the shear-bending beam have for the high
frequency limit the shear beam behavior, for any \( \bar{a} \). The
Bernoulli-Euler beam and the Rayleigh beam responses
approach reasonably well the low \( \bar{a} \) behavior for the other
beams. However, for high frequencies they predict values
which are too low and in the limit the Bernoulli-Euler
beam results approach zero. For very low frequencies,
all results are close together.

Above a minimum frequency \((\bar{\omega}_2 > 0.2)\), beams with
high \( \bar{a} \) (low \( \kappa \)) are poorly modeled by the Bernoulli-Euler
and the Rayleigh theories. In general, the shear-bending
beam predicts results too high as compared to the
Timoshenko results in the vicinity of the cutoff
frequency \((\bar{\omega}_2 = 1/\bar{a})\) of the second mode. The fact that
the beams with \( \bar{a} = 4.5 \) have so high values for \( |\bar{M}_{FW}| \) means
that beams with high \( \bar{a} \) (low \( \kappa \)) respond with higher
velocity amplitudes to a sinusoidal force, as compared to
beams with low \( \bar{a} \). The reason for this is that low \( \kappa 
\) beams take more shear deformation to equilibrate the same
shear force as compared to high \( \kappa \) beams.

(b) Total Mobilities \( \bar{M}_{F\psi} \) and \( \bar{M}_{MW} \)

The off-diagonal terms \( \bar{M}_{F\psi} \) and \( \bar{M}_{MW} \) of the mobility
matrix are equal for all models being studied. The matrix
is therefore symmetric, what is a consequence of the reciprocity theory for conservative dynamic systems.

From Figure 2.26 we see that the shear-bending and the Bernoulli-Euler theories predict the same behavior and the responses are independent of frequency. The Timoshenko theory results are more elaborate with peaks for the response at cutoff and zero high frequency limit. The Rayleigh beam results start off with $\bar{M}_F\psi = \bar{M}_{MW} = -1$ for zero frequency, as all the other models but it falls down to zero monotonically. Therefore, for high frequencies the Rayleigh results have the same tendency as the Timoshenko results but they fall off to zero slower than the Timoshenko results. Things get worse in general for high $\bar{a}$ beams.

(c) Total Mobility $\bar{M}_{M\psi}$

Figure 2.28 indicates that for very low frequencies all theories predict approximately the same results for $\bar{M}_{M\psi}$ (angular velocity response driven by a moment $Me^{-i\omega t}$). Also for a significant range of frequencies ($0 < \bar{\omega}_2 < 1$) all theories are close to Timoshenko's for high $\kappa$ beams (low $\bar{a}$). For high $\bar{a}$, the shear-bending beam results are excessively high for frequencies $\bar{\omega}_2 > 0.4$. The apparent
reason for this is that rotary inertia (higher for higher \( \bar{a} \)) which is not considered in the shear-bending beam plays a significant role in reducing the angular velocity response to a moment when shear deformation is taken into account. It is interesting that for frequencies \( \bar{\omega}_2 > 1 \) the Rayleigh beam has a very close agreement with the Timoshenko beam results for any \( \bar{a} \). This means that rotary inertia controls the angular motion at high frequencies.

The arguments for all mobilities differ a lot for the various theories. It has to do with the phase between force or moment and the responses.

In conclusion, we see that for low \( \bar{a} \) (full sections), all theories predict reasonable results, as compared to the Timoshenko results, within a significant frequency range (0 < \( \bar{\omega}_2 < 0.8 \)). However, for high \( \bar{a} \) the Bernoulli-Euler and Rayleigh results are bad for almost every frequency (good for very low frequencies) for all four mobilities. A surprising conclusion is that the shear-bending beam is worse than the Bernoulli-Euler and Rayleigh theories, for \( \bar{M}_{M\Psi} \), specially for high \( \bar{a} \). The Rayleigh results are good for \( \bar{M}_{M\Psi} \) in the high frequencies.

Therefore, for \( \bar{M}_{FW} \) the shear correction significantly improves the classical (Bernoulli-Euler) theory. The rotary inertia improves slightly the results for \( \bar{\omega}_2 < 1 \) for
low $\bar{a}$ beams and significantly worsens results for $\bar{\omega}_2 > 1$. For very high frequencies it improves slightly because $\bar{M}_{FW}$ does not go to zero while it does for the classical theory.

For $\bar{M}_{F\psi}$ and $\bar{M}_{MW}$ shear correction does not change the classical results. Rotary inertia improves results for mid to high frequencies ($\bar{\omega}_2 > 0.7$) and worsens results for $\bar{\omega}_2 < 0.7$.

For $\bar{M}_{MW}$ shear correction significantly deteriorates the classical results for most frequencies, specially for high $\bar{a}$ beams. The rotary inertia lowers the classical results, which is desirable for high frequencies ($\bar{\omega}_2 > 1$) but undesirable for lower frequencies.

2.9 Conclusions

We have in this chapter performed an extensive study of sinusoidal wave propagation in infinite and semi-infinite beams, according to the Timoshenko beam theory.

In our opinion, the most significant contributions given here were the introduction of the concept of modal mobility matrix as opposed to the total mobility matrix, and the development of the expressions for the elements of the matrix. We were able to generalize the study by determining the practical range of variation of the shear
correction coefficient $\kappa$ and consequently for the parameter $\bar{a}$, and by obtaining all the results in nondimensional form.

An interesting aspect of the study is the determination of the modal shapes for all frequencies. This provides us with a physical understanding of the behavior of the two modes of propagation in a Timoshenko beam.

The collection of equations and graphs we have developed have enough information for the solution of any sinusoidal single point driven infinite or semi-infinite beam, by means of the Timoshenko theory. Transient problems can then be solved by means of Fourier techniques. It would be interesting to see some work done in transient problems, making use of our sinusoidal results.

In comparing the various approximate theories, we introduced the idea of examining the driving-point mobilities. This way we could develop a better feeling for the importance of either the shear correction or the rotary inertia correction in each case. The rotary inertia correction contribution was shown to be less significant than the shear correction. However, the shear correction alone was shown to be desirable for $\bar{M}_P$ but undesirable for $\bar{M}_\Psi$. Figures 2.24 through 2.29 contain a wealth of information and can with more time be examined in a deeper
sense and eventually lead to other interesting conclusions.
\[ \bar{\omega}_2 = \frac{\omega r}{a^2} \]

Figure 2.24 Absolute value of the total mobility \( \bar{M}_{FW} \)
Figure 2.25 Argument of the total mobility $\bar{M}_{FW}$
Figure 2.26 Absolute value of the total mobility $M_{F\psi}$ and $M_{MW}$
1. Timoshenko beam $\bar{a} = 1.5$
2. Timoshenko beam $\bar{a} = 4.5$
3. Shear-bending beam $\bar{a} = 1.5$
4. Shear-bending beam $\bar{a} = 4.5$
5. Rayleigh beam
6. Bernoulli-Euler beam

$\bar{M}_{F\psi}$ and $\bar{M}_{MW}$

(degrees)

$\bar{\omega}_2 = \frac{\omega r}{a_2}$

Figure 2.27 Argument of the total mobilities $\bar{M}_{F\psi}$ and $\bar{M}_{MW}$
Figure 2.29 Argument of the total mobility $\bar{M}_{M\psi}$
CHAPTER 3

SINUSOIDAL WAVE PROPAGATION IN PERIODICALLY LOADED ELASTIC SYSTEMS

3.1 Introduction

According to Mead (1973), a periodic system consists of a number of identical elements, coupled together in identical ways to form the whole system. There is a number of natural and man-made structures which can be included in the category of periodic structures.

Brillouin (1946) has indicated that the problem of propagation of waves in periodic structures has been studied for nearly 300 years. It all began with the attempt by Newton to derive, in 1686, the formula for the velocity of sound in air. Other names connected to the problem are those of John and Daniel Bernoulli, Euler, Lagrange, Baden-Powell and Kelvin. These early studies were related specially to wave propagation in crystal lattices and the corpuscular theory of light. Other problems of interest, at the end of the nineteenth century and the beginning of the present century, were the transmission line theory and the theory of electrical and mechanical filters.
It is only in the last thirty years that investigations have been undertaken with periodic structures of the engineering kind (containing strings, beams and plates). In this case, the one-dimensional structures consist of one long element periodically loaded with some kind of impedance. For this reason, we will also refer to them as "periodically loaded structures." Cremer and Leilich (1953) studied sinusoidal flexural motions in periodically loaded beams and showed that waves can propagate only in certain frequency bands. Ungar (1966) and Bobrovinitskii and Maslov (1966) studied the same problem from a more general point of view. Mead (1970) has worked with a periodically supported beam. However, in all these works, the authors have consistently used the Bernoulli-Euler or classical beam theory. As far as we know, there have been no attempts, prior to ours, to use the Timoshenko beam theory to study periodically loaded beams.

We start our development with a study of sinusoidal wave propagation in periodically loaded infinite taut strings. The loads are general translational impedances containing reactive and/or resistive components. The study contains two parts: a) propagation of natural sinusoidal waves which yields the dispersion relation;
b) response of an infinite or semi-infinite periodically loaded string driven at one loading point (or node) by a sinusoidal force.

We then go on to study the problem of sinusoidal wave propagation in periodically loaded beams, where we use the Timoshenko beam theory (see Appendix B for definition). In this case, the loads are general translational and rotational impedances. This development also contains two parts as in the string case. The differences are that in this case there are two modes of propagation instead of one, and in the forced problem we determine the modal mobility matrix (see Chapter 2, section 2.6 for definition) instead of a single driving-point mobility.

We then determine the dispersion relations for the same beam problem but using now the Bernoulli-Euler beam theory. The reasons for this last study are:

a) To check the results with some existent results in the literature for the Bernoulli-Euler theory.

b) To show that, in the limit, when no rotary inertia and shear deformation are considered, the Timoshenko beam results converge to the Bernoulli-Euler beam results. This consistsutes a check for our Timoshenko beam results.
c) To compare the dispersion relations yielded by the two theories.

In Appendix D, we present a set of plots of dispersion relations for the string case and for the two beam theory cases. The curves are discussed and compared in the main text.
3.2 Periodically Loaded Infinite Taut String

In order to develop some physical understanding about wave propagation in infinitely long periodically loaded structures we have first developed the theory for a taut string. This is by no means an original work (see Wallace, 1972). However, we have never seen in the literature a general development like that given in this section.

Consider an infinitely long taut string periodically loaded with general translational impedances $Z_F$ (see figure 3.1). The impedances $Z_F$ may have reactive (mass-like or spring-like) and/or resistive (dashpot like) components.

In this section we will study the propagation of sinusoidal waves in such a system.

---

Figure 3.1 Infinitely long taut string periodically loaded with impedances $Z_F$.
3.2.1 Natural Wave Propagation

Let us first quickly establish the properties of waves propagating in an infinite unloaded taut string. The equation of motion is the classical wave equation as follows:

\[ \frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 w}{\partial t^2} \]  

(3.1)

where

\[ c = \sqrt{\frac{T}{\rho A}} \]  

(3.2)

is the phase velocity of the waves, \( T \) is the tension, \( \rho \) the density and \( A \) the cross-sectional area of the string. In equation (3.1) \( w(x,t) \) is the transverse velocity of the string.

Let us now investigate the existence of progressive sinusoidal waves propagating in a string. Assume a sinusoidal wave of the form:

\[ w(x,t) = \text{We}^{i(kx-\omega t)} \]  

(3.3)

traveling in the taut string. In (3.3) \( k \) is the wave number of the waves and \( \omega \) is the angular frequency.

Substitution of (3.3) into equation (3.1) leads to the dispersion relation

\[ k^2 = \frac{\rho A}{T} \omega^2 \]  

(3.4)

Therefore we obtain a positive and a negative value of \( k \) for every frequency. This indicates that, according
to equation (3.3) the system can support right and left
going sinusoidal waves.

If we not return to the periodic string of Figure 3.1 we see that the loading points are discontinuities in the
taut string. Therefore at these points the waves will be
partly reflected, partly transmitted and if the loading
impedances contain resistive components, part of the
wave energy will in general be absorbed. Thus in general
each segment of string between two loading points or
nodes will exhibit left and right going waves.

Note from Figure 3.1 that the segments are labeled
... j - 1, j, j + 1, ... and that the node on the left-hand
side of each segment bears the same label as the segment.
Furthermore, each segment has its own coordinate system
with origin at the left node.

We will now concentrate our attention at a node j
and write for it dynamic and kinematic compatibility
conditions. A string does not react to moments due
to its lack of bending rigidity. Also the only coupling
coordinate between two segments is transverse motion.
Therefore, we can write a force equilibrium equation
and a transverse velocity compatibility condition
at the node. The force condition is (see Figure 3.2):

\[ Q_j(0,t) - Q_{j-1}(L,t) - Z_F w_j(0,t) = 0 \]  \hspace{1cm} (3.5)
The velocity condition is, according to Figure 3.1:

\[ w_j(0,t) = w_{j-1}(L,t) \]  \hspace{1cm} (3.6)

where the total velocities \( w_j \) and \( w_{j-1} \) have two components each, as indicated by equation (3.7) below.

\[ Q_{j-1}(L,t) \quad \downarrow \quad Q_j(0,t) \]

\[ Z_F \quad w_j(0,t) \]

Figure 3.2 Free-body diagram of node \( j \).

Equations (3.1), (3.5) and (3.6) constitute the general equations of motion for the system illustrated in Figure 3.1. Therefore, any function that satisfies this set of equations constitutes a possible motion of the system.

In the present study we are interested in looking at the possibility of sinusoidal progressive waves to exist in a periodically loaded infinite string. We have seen that within a segment there will exist right-going and left-going waves. Therefore for sinusoidal motion of the structure, a point \( x_j \) of a segment \( j \) will have its transverse velocity given by:

\[ w_j(x_j,t) = (W_j^+ e^{ikx_j} + W_j^- e^{-ikx_j}) e^{-i\omega t} \]  \hspace{1cm} (3.7)

This means that looking at the structure, it will
be difficult to identify a progressive wave of the form one observes in infinite unloaded structures.

However it is known that, looking at the nodes only, one might be able to identify such a progressive wave pattern. Therefore we will try, as a solution, a modified version of the waveform given by equation (3.3), that is:

$$w_j(jL,t) = W_0 e^{i\mu(jL)} e^{-i\omega t}$$  \hspace{1cm} (3.8)

In this expression, $j$ is the number of the node being considered, $L$ is the length of a segment, and $W_0$ is the amplitude of the velocity of node zero. The symbol $\mu$ is called "propagation constant" and it is a generalized wave number, in the sense that it can be a general complex number, or

$$\mu = \mu_r + i \mu_i$$  \hspace{1cm} (3.9)

In this expression, $\mu_r$ is the real part of the propagation constant $\mu$. It dictates the propagation characteristics of the wave. The imaginary part $\mu_i$ governs the change in amplitude as the wave progresses. It is also called attenuation.

Equation (3.8) can also be written

$$w_j(jL,t) = W_0 e^{i(j\bar{\mu} - \omega t)}$$  \hspace{1cm} (3.8a)

where $\bar{\mu} = \mu L$ is the dimensionless propagation constant.
Before we use (3.8a) in conjunction with the compatibility conditions (3.5) and (3.6), we have to obtain (3.5) in terms of transverse velocities. To do this, we recall that the transverse force at a section of a taut string is given by

\[ Q = T \frac{\partial n}{\partial x} = \frac{T}{-i\omega} \frac{\partial w}{\partial x} \quad (3.10) \]

where the last form on the right is only valid for sinusoidal waves. Substituting (3.10) into (3.5) we obtain:

\[ \frac{T}{-i\omega} \frac{\partial}{\partial x} w_j(x_j, t) \bigg|_{x_j = 0} - \frac{T}{-i\omega} \frac{\partial}{\partial x} w_{j-1}(x_{j-1}, t) \bigg|_{x_{j-1} = L} - Z_F w_j(0, t) = 0 \quad (3.11) \]

Equations (3.6) and (3.11) contain the transverse velocity in segments \( j \) and \( j-1 \). With the help of equation (3.8a) we can transform (3.6) and (3.11) such that they contain only the transverse velocity in segment \( j \). If we then use equation (3.7), we obtain the two equations in terms of the amplitudes of the right and left-going waves within segment \( j \). The resulting equations are:

\[ \begin{bmatrix} 1 - e^{i(kL - \mu)} + \frac{Z_F}{\sqrt{\rho A T}} \end{bmatrix} W_j^+ - \begin{bmatrix} 1 - e^{-i(kL - \mu)} - \frac{Z_F}{\sqrt{\rho A T}} \end{bmatrix} W_j^- = 0 \]

\[ \begin{bmatrix} 1 - e^{i(kL + \mu)} \end{bmatrix} W_j^+ + \begin{bmatrix} 1 - e^{-i(kL + \mu)} \end{bmatrix} W_j^- = 0 \quad (3.12a,b) \]
From now on, we will refer to the waves that propagate within a segment as "internal waves." The progressive waves being here investigated, and given by equation (3.8a) will be called "nodal waves."

The equations (3.12a,b) form a set of homogeneous linear equations in $W^+_j$ and $W^-_j$. Therefore, in order for a nontrivial solution to exist, the determinant of the coefficients in (3.12a,b) must vanish. The expression for the determinant is:

$$\Delta = e^{-i\overline{\mu}} \left( 4 \cos \overline{\mu} - 4 \cos kL + 2i \frac{Z_F}{\sqrt{\rho AT}} \sin kL \right) \quad (3.13)$$

If we now impose the condition $\Delta = 0$, we obtain the dispersion relation which can be explicitly solved for the eigenvalue $\overline{\mu}$ to yield:

$$\overline{\mu} = \arccos \left( \cos kL - 2i \frac{Z_F}{\sqrt{\rho AT}} \sin kL \right) \quad (3.14)$$

With the help of equation (3.4), we can write:

$$\overline{k} = kL = \omega L \sqrt{\frac{\rho A}{T}} = \overline{\omega} \quad (3.15a)$$

and we choose $\overline{\omega}$ as the dimensionless frequency for the present problem. We also define the dimensionless loading impedance as:

$$\overline{Z_F} = \frac{Z_F}{\sqrt{\rho AT}} \quad (3.15b)$$
Substituting equations (3.15a) and (3.15b) into equation (3.14), we obtain:

\[ \bar{\mu} = \arccos (\cos \bar{\omega} - 2i \bar{\omega}_p \sin \bar{\omega}) \quad (3.14a) \]

Therefore, the waveform given by equation (3.8) can actually satisfy the equations of motion of the periodic string if the propagation constant \( \bar{\mu} \) is given by (3.14a). In other words, nodal waves are physically possible in periodically loaded strings.

In this point, we will try to establish what kind of relationship exists between the motion at an internal point \( x_{j-1} \) of a segment \( j-1 \) and the motion at the corresponding point \( x_j = x_{j-1} \) in the adjacent segment \( j \). With equation (3.8a), we can write an equation relating the motions of nodes \( j-1 \) and \( j \), and an equation relating the motions of nodes \( j \) and \( j+1 \). If we then use equation (3.7) in the two equations just described, we obtain:

\[
W_{j-1}^+ + W_{j-1}^- = (W_j^+ + W_j^-) e^{-i\bar{\mu}} \quad (3.16a,b)
\]

\[
W_{j-1}^+ e^{ikL} + W_{j-1}^- e^{-ikL} = (W_j^+ e^{ikL} + W_j^- e^{-ikL}) e^{-i\bar{\mu}}
\]

If we now solve (3.17a,b) for \( W_{j-1}^+ \) and \( W_{j-1}^- \) as functions of \( W_j^+ \) and \( W_j^- \) we obtain:

\[
W_{j-1}^+ = W_j^+ e^{-i\bar{\mu}} \quad (3.17a,b)
\]

\[
W_{j-1}^- = W_j^- e^{-i\bar{\mu}}
\]
This result indicates that the individual components of the internal waves propagating within the segments are also related according to equation (3.8a). The motion of a point $x_{j-1}$ of segment $j-1$ is given, for sinusoidal waves, by:

$$w_{j-1}(x_{j-1}, t) = w_{j-1}^+ e^{ikx_{j-1}} + w_{j-1}^- e^{-ikx_{j-1}} \quad (3.18)$$

Substituting now equations (3.17a,b) into equation (3.18) and making $x_j = x_{j-1}$, we obtain:

$$w_{j-1}(x_{j-1}, t) = (w_j^+ e^{ikx_j} + w_j^- e^{-ikx_j}) e^{-i\mu}$$

$$= w_j(x_j = x_{j-1}, t) e^{-i\mu} \quad (3.18a)$$

This means that if we look at corresponding points within the segments (points separated by a distance $L$ in adjacent segments), we shall see the same progressive wave pattern of the nodal waves.

Therefore, in a sense, the periodic loading of a structure discretizes it in space from the wave propagation point of view. In other words, the waveform given by equation (3.3) still applies for periodic structures, but the coordinate $x$ is no longer continuously varying. It is now a discrete variable of unit step $L$.

In Table C1 of Appendix C, we have listed a set of loading impedances in dimensional form, including translational impedances $Z_T$ and rotational impedances $Z_M$. 
The translational impedances are also given in dimensionless form for the string case and for the beam case to be studied in the following sections. The rotational impedances are given in dimensionless form only for the beam as they are not used for the string.

Equation (3.14a) indicates that the propagation constant $\mu$ is given by an inverse cosine, as will also be the case for periodic beams. Therefore an understanding of the behavior of the inverse cosine for general argument is an important part of this study, and we have for this reason included here Table 3.1.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$y = \arccos (Z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>real with $</td>
<td>Z</td>
</tr>
<tr>
<td>real with $Z &gt; 1$</td>
<td>purely imaginary</td>
</tr>
<tr>
<td>real with $Z &lt; -1$</td>
<td>complex with real $(y) = \pi$</td>
</tr>
<tr>
<td>complex</td>
<td>complex</td>
</tr>
</tbody>
</table>

It is important to notice that the inverse cosine function is a logarithm function (Hildebrand, 1962, page 519), and it has two infinite sets of values. However, we will in general use only the main branches (or the principal values) for each set. One case stands for right going waves and the other for left going
waves. The upper branches yield the same results as the main branches. The reason for this is that a propagation constant greater than \( \pi \) means a wavelength smaller than \( 2L \). Such short wavelengths cannot be distinguished from the corresponding longer wavelengths. This phenomenon is similar to the aliasing phenomenon that occurs when the Nyquist frequency, for a given sampling rate of a band limited analog signal, is lower than the highest frequency in the signal (Blackman and Tukey, 1959, page 31).

Equation (3.14a) indicates that for reactive loadings (purely imaginary impedances), the argument of the inverse cosine is real. This means that \( \bar{\mu} \) will behave according to the first three cases of Table 3.1. For loadings with resistive components, the argument of the inverse cosine will be complex and \( \bar{\mu} \) will be a general complex.

3.2.2 Examples of Dispersion Relations for Some Specific Cases

According to equation (3.14a), the propagation constant \( \bar{\mu} \) for a periodic string is a function of the dimensionless parameters \( \bar{w} \) and \( \bar{z}_f \). However, the loading impedances \( \bar{z}_f \) can have many different forms and in each case the dispersion relation will have its special characteristics. We have, therefore, chosen four
different cases of loadings, including masses, springs, mass-spring systems and dashpots. For each of these cases, we have plotted the dispersion relation for two different values of the loading parameters. The curves are contained in Figures dl.1 through dl.8 of Appendix D.

- mass loads

Figures dl.1 and dl.2 are the dispersion relations of a string periodically loaded with masses \( \overline{m} = 0.5 \) and \( \overline{m} = 2.0 \) respectively. The various parameters of interest are defined at the bottom of Table Cl in Appendix C.

Figures dl.1 and dl.2 indicate that strings periodically loaded with masses, show alternating passbands \( (\overline{\mu}_i = 0) \) and stopbands \( (\overline{\mu}_i \neq 0) \). In addition, as frequency grows, the passbands become narrower and the magnitude of the attenuation \( \overline{\mu}_i \) increases within the stopbands. This is physically sound because at high frequencies the lumped masses tend to remain still. Also comparing figures dl.1 and dl.2, we notice that increasing the magnitude of the loading masses causes a decrease of the width of the passbands and increases attenuation.

- spring-loads

Figures dl.3 and dl.4 are the dispersion relations for a periodic string loaded with springs with \( \overline{K} = 2.0 \)
and $\bar{K} = 4.0$. This case is similar to the case of the lumped masses, with alternating passbands and stopbands. However in this case, as frequency increases, the passbands become wider and the attenuation magnitude decreases. The reason for this is that at higher frequencies the inertia of the string overcomes the spring action. We also notice by comparing Figures dl.3 and dl.4 that increasing $\bar{K}$ increases the width of the stopbands and increases attenuation.

- **mass-spring loads**

  Figure dl.5 is for a string loaded with $\bar{m} = 0.5$ and $\bar{K} = 2.0$ in series and Figure dl.6 is for $\bar{m} = 2$ and $\bar{K} = 4.0$. Comparing with Figures dl.1 through dl.4, we notice that for low frequencies the system behaves as if it were loaded with masses only and at high frequencies it behaves as if it were loaded with springs only. The explanation is that for low frequencies each mass will move essentially the same as the corresponding node while for high frequencies the masses stand still and the string only "sees" the springs.

- **dashpot loads**

  Figures dl.7 and dl.8 stand for a periodic string loaded with dashpots, with $\bar{C} = 0.1$ and $\bar{C} = 1.0$ respectively.
As discussed earlier, in this case the propagation constant is always complex. Attenuation is always present when the nodes move because the dashpots absorb energy.

Comparing the two figures, we see that once more increasing the magnitude of the loading parameter causes attenuation to grow. In this case, however, the curve seems to repeat itself periodically (period $2\pi$) as the frequency increases, and while $\bar{\mu}_r$ is always positive $\bar{\mu}_i$ will alternate between positive and negative values.

There are two points here which deserve explanation. The first one is that $\bar{\mu}_i$ is zero at frequencies $\bar{\omega}$ which are multiples of $\pi$, and at these points $\bar{\mu}_r$ is equal to $\pi$ where $\bar{\omega}$ is an odd multiple of $\pi$, and equal to zero where $\bar{\omega}$ is an even multiple of $\pi$. As attenuation is not present in these points, we necessarily conclude that the nodes do not move. For $\bar{\mu}_r = 0$ corresponding points of the system (separated by multiples of $L$) are in phase and the resulting motion is, as shown in Figure 3.3a, a standing wave in each segment, with all segments moving in phase. In this

![Figure 3.3 Behavior of a periodically loaded string](image)

for (a) $\bar{\mu} = 0$ and (b) $\bar{\mu} = \pi$
case the nodes clearly do not move. For $\bar{\mu}_r = \mu$

corresponding points move in phase opposition and the
nodes stand still, as indicated in Figure 3.3b.

The second point of interest is the fact that
$\bar{\mu}_r$ assumes negative values for certain frequency ranges.
This means that the wave amplitude is growing instead
of decaying in the direction of propagation. We
conclude that although the phase velocity is
to the right, energy propagates to the left. In general,
it does not make sense to speak about phase and group
velocities when $\bar{u}$ is complex. In the case of
resistive loads, however, it makes sense because energy
would actually be injected into the system if sinusoidally
driven. The fact is that for the frequency ranges where
$\bar{\mu}_r$ is negative, the phase would move to the right and
energy to the left, or vice-versa for left going waves.
Unattenuated waves may also show such a behavior. In
Figure 1.1.1, for example, we note that there are frequency
ranges where $\bar{\mu}_i$ is zero and the slope of $\bar{\mu}_r$ is negative,
or the group velocity is negative.

3.2.3 Semi-infinite Periodic String Driven
at the End Node by a Sinusoidal Force

We will now study the problem of driving sinusoidal
waves in periodic strings. The study will be performed
for a semi-infinite string driven at the end node by a
sinusoidal force $Fe^{-i\omega t}$. (See Figure 3.4)

Figure 3.4 Semi-infinite periodic string driven at the end by a force $Fe^{-i\omega t}$.

We are not solving the general problem of driving at a point within a segment because we would obtain twice the number of simultaneous equations and in consequence the solution would be much more involved.

At the driving section we can state that the shear force is given by the resultant between the driving-force and the force introduced by loading impedance (see Figure 3.4). Therefore, at $x_0 = 0$, we have the boundary condition

$$Q(0,t) = Fe^{-i\omega t} - Z_F w_0(0,t) \quad (3.19)$$

The shear force in a taut string is given by equation (3.10) which with the use of equation (3.7) for segment zero and of equation (3.4) reduces to:

$$Q(x_0,t) = \sqrt{pA_T} \left( w_0^+ e^{ikx_0} - w_0^- e^{-ikx_0} \right) e^{-i\omega t} \quad (3.20)$$
Substituting now equations (3.5) and (3.20) into equation (3.19) we obtain after some simplifications

\[
(\sqrt{\rho A T} + Z_F) \, W_0^+ + (\sqrt{\rho A T} - Z_F) \, W_0^- = F
\]  

(3.21)

In our study of natural waves in periodic strings, we have obtained the set (3.12) of two homogeneous equations relating the amplitudes of the right and left-going waves in a general segment \( j \). Therefore, we can solve any one of the two equations (3.12) for one amplitude as a function of the other. Picking the second equation, we obtain

\[
W_j^- = - \frac{e^{i \mu}}{e^{i \mu} - e^{-ikL}} \, W_j^+
\]  

(3.22)

Adapting now (3.20) for segment zero and substituting it in equation (3.19), we obtain

\[
W_0^+ = \frac{1}{2} \, \frac{F(e^{i \mu} - e^{-ikL})}{\sqrt{\rho A T} \, (e^{i \mu} - \cos kL) + i \, Z_F \, \sin kL}
\]  

(3.23a)

Using again equation (3.20), we obtain

\[
W_0^- = \frac{1}{2} \, \frac{F(e^{ikL} - e^{i \mu})}{\sqrt{\rho A T} \, (e^{i \mu} - \cos kL) + i \, Z_F \, \sin kL}
\]  

(3.23b)

The total response at node zero is computed by adding (3.23a) and (3.23b).

We have in section 2.6 of Chapter 2 introduced the concept of driving-point mobility (equation (2.30)). Making use of this concept here and considering
equations (3.23a) and (3.23b), we can write the driving-point mobility of the semi-infinite periodic string as

$$M_{Fw} = \frac{1}{\sqrt{\rho A T} \left( e^{i \overline{\mu}} - \cos kL \right) + z_F} \frac{i \sin kL}{i \sin \overline{\omega}} \tag{3.24}$$

In dimensionless form, equation (3.23) can be written as

$$\overline{M}_{Fw} = \sqrt{\rho A T} \quad M_{Fw} = \frac{1}{(e^{i \overline{\mu}} - \cos \overline{\omega}) + \overline{z}_F} \tag{3.24a}$$

where $\overline{\omega}$ and $\overline{z}_F$ have been defined in equations (3.15a) and (3.15b).

With the response at node zero known, the response at any other node can be found with the help of equation (3.8a) and of course by using equation (3.14a) to compute the propagation constant $\overline{\mu}$.

As a quick check for the mobility expressed by equation (3.24), we observe that if we make $z_F \to 0$, $M_{Fw}$ should reduce to the driving-point mobility of an unloaded taut string. Performing such a limit and noticing from equation (3.14) that $\overline{\mu} \to kL$ as $z_F \to 0$, we obtain

$$M_{Fw} = \frac{1}{\sqrt{\rho A T}} \tag{3.24b}$$
which is exactly the driving-point mobility of a semi-
infinite unloaded taut string.

3.3 **Periodically Loaded Infinite Timoshenko Beam**

In section 3.2, we have studied the problem of
sinusoidal wave propagation in periodically loaded strings.
Most of the concepts to be used in the present section
were carefully introduced in section 3.2 in detail.
Therefore, we will from now on just refer to those
concepts.

In this case, we will consider rotational impedances
\( Z_M \) in addition to translational impedances \( Z_F \) attached
to the loading points. Just as a reminder, the im-
pedance \( Z_F \) of a load is the ratio between the amplitude
of the applied sinusoidal force and the resulting
amplitude of the linear velocity. The impedance \( Z_M \) of
a load is the ratio between the amplitude of the sinu-
soidal driving moment and the resulting amplitude of the
angular velocity.

Figure 3.5 depicts the system we will be studying
in this section.
Figure 3.5 Infinite beam periodically loaded with translational impedances $Z_F$ and rotational impedances $Z_M$.

3.3.1 Natural Wave Propagation

Consider now the problem of sinusoidal wave propagation through the system of Figure 3.5. In Chapter 2 we have studied the problem of sinusoidal wave propagation in unloaded Timoshenko beams. There we learned that in a Timoshenko beam, waves propagate in two modes and that the propagation constants $k_1$ and $k_2$ for the first and second modes are given by equation (2.14). The wave form was given by equations (2.10) and (2.11).

As in the case of the string, each loading impedance introduces a discontinuity in the beam. This causes an incoming wave to be partly reflected and
partly transmitted and if the loading impedances contain resistive components, part of the wave energy will in general be absorbed. Therefore, a segment of beam between two consecutive loading stations will contain right and left-going waves and as the Timoshenko beam possesses two modes of propagation, we shall in general, for sinusoidal waves, have four internal wave components traveling in each segment.

For transverse velocities, we have
\[ w_j(x_j,t) = \left( W_{j1}^+ e^{ik_1x_j} + W_{j2}^+ e^{ik_2x_j} + W_{j1}^- e^{-ik_1x_j} + W_{j2}^- e^{-ik_2x_j} \right) e^{-i\omega t} \] (3.25)

while for angular velocity we have
\[ \psi_j(x_j,t) = \left( \psi_{j1}^+ e^{ik_1x_j} + \psi_{j2}^+ e^{ik_2x_j} + \psi_{j1}^- e^{-ik_1x_j} + \psi_{j2}^- e^{-ik_2x_j} \right) e^{-i\omega t} \] (3.26)

As explained in subsection 3.2.1, after equation (3.18a), the periodic loading of a structure discretizes the structure such that, in general, a progressive wave pattern can be identified only at discrete equispaced points where two consecutive points are spaced by L. In the present case, we will assume the equivalent of equation (3.8a) for both transverse and angular velocities.
as follows

$$w_j(jL,t) = w_0 e^{ij\bar{\mu}} e^{-i\omega t} \quad (3.27a,b)$$

$$\psi_j(jL,t) = \psi_0 e^{ij\bar{\mu}} e^{-i\omega t}$$

where $\bar{\mu} = \mu L$ is the dimensionless propagation constant as defined in section 3.2.

We will now concentrate our attention in node $j$ and write for it two dynamic compatibility conditions, one for forces and one for moments, and two kinematic compatibility conditions, one for transverse motion and one for angular motion.

The kinematic conditions are

$$w_j(0,t) = w_{j-1}(L,t) \quad (3.28a,b)$$

$$\psi_j(0,t) = \psi_{j-1}(L,t)$$

The dynamic conditions are written with the aid of the free-body diagrams of Figure 3.6 as follows

![Free-body diagrams](image)

Figure 3.6 Free-body diagrams of node $j$. (a) for forces; (b) for moments.
\[ Q_j(0,t) - Q_{j-1}(L,t) - Z_F w_j(0,t) = 0 \]

\[ M_j(0,t) - M_{j-1}(L,t) - Z_M \psi_j(0,t) = 0 \]

(3.28c,d)

We know from equations (2.24) and (2.25) in Chapter 2 that at any section of a Timoshenko beam, undergoing sinusoidal wave transmission, the shear force and the moment are given by

\[ Q(x,t) = G^A \left( \frac{\partial \eta}{\partial x} - \phi \right) = \frac{G^A}{-i\omega} \left( \frac{\partial w}{\partial x} - \psi \right) \]

\[ M(x,t) = EI \frac{\partial \phi}{\partial x} = \frac{EI}{-i\omega} \frac{\partial w}{\partial x} \]

(3.29a,b)

Equation (2.23) provides us with the relation between angular and transverse motions at any section of a Timoshenko beam, undergoing sinusoidal wave transmission. In terms of velocities, equation (2.23) can be written as

\[ \frac{\psi}{W} = i \left( k - \frac{\omega^2}{Q_1^2k} \right) \]

(3.30)

where \( k = k_1 \) for the first mode of propagation and \( k = k_r \) for the second mode of propagation, and \( k_1 \) and \( k_2 \) are given by equation (2.14).

If we now substitute equations (3.29a,b), (3.27a,b), (3.25), (3.26) and (3.30) into the four compatibility conditions (3.28), we obtain after some algebraic
manipulations and simplifications

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
\gamma_1 & \gamma_2 & -\gamma_1 & -\gamma_2 \\
(\delta_1 + \frac{\beta_F}{\alpha_1^+}) & (\delta_2 + \frac{\beta_F}{\alpha_2^+}) & (-\delta_1 + \frac{\beta_F}{\alpha_1^-}) & (-\delta_2 + \frac{\beta_F}{\alpha_2^-}) \\
\gamma_1(k_1 + \frac{\beta_M}{\alpha_1^+}) & \gamma_2(k_2 + \frac{\beta_M}{\alpha_2^+}) & \gamma_1(k_1 - \frac{\beta_M}{\alpha_1^-}) & \gamma_2(k_2 - \frac{\beta_M}{\alpha_2^-})
\end{bmatrix}
\begin{bmatrix}
\alpha_1^{+}w_{jl}^+ \\
\alpha_1^{+}w_{j2}^+ \\
\alpha_1^{-}w_{jl}^- \\
\alpha_2^{-}w_{j2}^-
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

...(3.31)

Where the following definitions apply

\[
\begin{align*}
\alpha_1^+ &= e^{i\mu} - e^{ik_1L} \\
\alpha_1^- &= e^{i\mu} - e^{-ik_1L} \\
\alpha_2^+ &= e^{i\mu} - e^{ik_2L} \\
\alpha_2^- &= e^{i\mu} - e^{-ik_2L} \\
\delta_1 &= \frac{\omega^2}{a_1^2k_1} \\
\delta_2 &= \frac{\omega^2}{a_1^2k_2} \\
\gamma_1 &= k_1 - \delta_1 \\
\gamma_2 &= k_2 - \delta_2 \\
\beta_F &= \frac{\omega_{ZF}}{GA} e^{i\mu} \\
\beta_M &= \frac{\omega_{ZM}}{EI} e^{i\mu}
\end{align*}
\]

(3.32a-j)
System (3.31) is homogenous and in order for a nontrivial solution for the unknowns to exist, the determinant of the coefficients must vanish.

The expression we obtain for the determinant after a very long algebraic process is given by

\[
\Delta = 4(\gamma_2 k_2 - \gamma_1 k_1) (\gamma_2 \delta_2 - \gamma_1 \delta_1) + \beta M^2_F \left[ (\gamma_1 + \gamma_2)^2 \frac{a_2}{a_1} \frac{a_1^-}{a_1^+} + \frac{a_1^+}{a_1^-} \frac{a_2^+}{a_2^-} \right]
\]

\[
- (\gamma_1 - \gamma_2)^2 \frac{a_1^+}{a_2^+} \frac{a_1^-}{a_2^-} - \frac{a_1^+}{a_1^-} \frac{a_2^-}{a_2^+} - 4\gamma_1 \gamma_2 \frac{a_2^+}{a_2^-} \frac{a_1^-}{a_1^+} \frac{a_2}{a_1} \frac{a_1}{a_2}
\]

\[
+ 2\beta_F (\gamma_1 k_1 - \gamma_2 k_2) \left[ \gamma_2 \frac{a_1^+}{a_1^0} - \frac{a_1^-}{a_1^-} \right] - \gamma_1 \frac{a_2^+}{a_2^-} \frac{a_1^-}{a_1^+}
\]

If we not use equations (3.32) in equation (3.33), we obtain after another long series of algebraic manipulations
\[ \Delta = - \frac{4\omega^2 (k_1^2 - k_2^2)^2}{a_1^2 k_1 k_2} \]

\[ + \frac{\omega^2 Z_F Z_M}{(EI)(G*A)} \left[ \frac{2\gamma_1 \gamma_2 (\cos k_1 L \cos k_2 L - 1) + (\gamma_1^2 + \gamma_2^2) \sin k_1 L \sin k_2 L}{(\cos \mu - \cos k_1 L)(\cos \mu - \cos k_2 L)} \right] \]

\[ + \frac{2i\omega Z_F (k_1^2 - k_2^2)}{G*A} \left[ \frac{\gamma_2 \sin k_1 L}{\cos \mu - \cos k_1 L} - \frac{\gamma_1 \sin k_2 L}{\cos \mu - \cos k_2 L} \right] \]

\[ - \frac{2i\omega^3 Z_M (k_1^2 - k_2^2)}{EI a_1^2 k_1 k_2} \left[ \frac{\gamma_1 \sin k_1 L}{\cos \mu - \cos k_1 L} - \frac{\gamma_2 \sin k_2 L}{\cos \mu - \cos k_2 L} \right] \]

\[ \cdots (3.34) \]

As a matter of convenience, we have kept some of the symbols \( \gamma_1 \) and \( \gamma_2 \) and all \( k_1 \) and \( k_2 \) in equation (3.34), instead of substituting for their values. Equations (3.32 g, h) give the expressions for \( \gamma_1 \) and \( \gamma_2 \) and equations (2.14) give the expressions for \( k_1 \) and \( k_2 \).

We now impose the condition \( \Delta = 0 \) in equation (3.34) and transform it to obtain

\[ \cos^2 \mu + H_1 \cos \mu + H_2 = 0 \]

(3.35)
\[ H_1 = -iz_F \frac{k_1 k_2}{2\omega (k_1^2 - k_2^2) \rho A} \left[ \frac{1}{k_2} \left( k_2^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_1 - \frac{1}{k_1} \left( k_1^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_2 \right] \]

\[ + iZ_M \frac{\omega}{2(k_1^2 - k_2^2) EI} \left[ \frac{1}{k_1} \left( k_1^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_1 - \frac{1}{k_2} \left( k_2^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_2 \right] \]

\[- (\cos k_1 L + \cos k_2 L) \] \quad \ldots(3.36a)

and

\[ H_2 = - \frac{Z_F Z_M k_1 k_2}{4(k_1^2 - k_2^2)(EI)(\rho A)} \left( \frac{2}{k_1 k_2} \right) \left( k_1^2 - \frac{\omega^2}{a_1^2} \right) \left( k_2^2 - \frac{\omega^2}{a_1^2} \right) \left( \text{cosk}_1 L \text{cosk}_2 L - 1 \right) \]

\[ + \left[ \frac{1}{k_1^2} \left( k_1^2 - \frac{\omega^2}{a_1^2} \right)^2 + \frac{1}{k_2^2} \left( k_2^2 - \frac{\omega^2}{a_1^2} \right)^2 \right] \text{sink}_1 L \text{sink}_2 L \]

\[- \frac{iz_F k_1 k_2}{2\omega (k_1^2 - k_2^2) \rho A} \left[ \frac{1}{k_2} \left( k_2^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_1 L \text{cosk}_2 L \right] \]

\[- \frac{1}{k_1} \left( k_1^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_2 L \text{cosk}_1 L \]

\[- \frac{iZ_M \omega}{2(k_1^2 - k_2^2) EI} \left[ \frac{1}{k_1} \left( k_1^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_1 L \text{cosk}_2 L \right] \]

\[- \frac{1}{k_2} \left( k_2^2 - \frac{\omega^2}{a_1^2} \right) \text{sink}_2 L \text{cosk}_1 L \]

\[ + \text{cosk}_1 L \text{cosk}_2 L \] \quad (3.36b)
The expressions for $H_1$ and $H_2$ can also be written in terms of nondimensional parameters as follows

$$H_1 = -i\bar{Z}_F \frac{\kappa_1 \kappa_2}{2\omega_2 (\kappa_1^2 - \kappa_2^2)} \left[ \frac{1}{\kappa_2} (\kappa_2^2 - \omega_2^2 a^2) \sin \kappa_1 L - \frac{1}{\kappa_1} (\kappa_1^2 - \omega_2^2 a^2) \sin \kappa_2 L \right]$$

$$+ i\bar{Z}_M \frac{\omega_2}{2 (\kappa_1^2 - \kappa_2^2)} \left[ \frac{1}{\kappa_1} (\kappa_1^2 - \omega_2^2 a^2) \sin \kappa_1 L - \frac{1}{\kappa_2} (\kappa_2^2 - \omega_2^2 a^2) \sin \kappa_2 L \right]$$

$$- (\cos \kappa_1 L + \cos \kappa_2 L) \quad (3.37a)$$

and

$$H_2 = -\bar{Z}_F \bar{Z}_M \frac{\kappa_1 \kappa_2}{4 (\kappa_1^2 - \kappa_2^2)} \left\{ \frac{2}{\kappa_1^2} (\kappa_1^2 - \omega_2^2 a^2) (\kappa_2^2 - \omega_2^2 a^2) (\cos \kappa_1 L \cos \kappa_2 L - 1) \right\}$$

$$+ \left[ \frac{1}{\kappa_1^2} (\kappa_1^2 - \omega_2^2 a^2)^2 + \frac{1}{\kappa_2^2} (\kappa_2^2 - \omega_2^2 a^2)^2 \right] \sin \kappa_1 L \sin \kappa_2 L \right\}$$

$$+ i\bar{Z}_F \frac{\kappa_1 \kappa_2}{2\omega_2 (\kappa_1^2 - \kappa_2^2)} \left[ \frac{1}{\kappa_2} (\kappa_2^2 - \omega_2^2 a^2) \sin \kappa_1 L \cos \kappa_2 L \right]$$

$$- \frac{1}{\kappa_1} (\kappa_1^2 - \omega_2^2 a^2) \sin \kappa_2 L \cos \kappa_1 L \right]$$

$$+ \frac{i\bar{Z}_M \omega_2}{2 (\kappa_1^2 - \kappa_2^2)} \left[ \frac{1}{\kappa_1} (\kappa_1^2 - \omega_2^2 a^2) \sin \kappa_1 L \cos \kappa_2 L \right]$$

$$- \frac{1}{\kappa_2} (\kappa_2^2 - \omega_2^2 a^2) \sin \kappa_2 L \cos \kappa_1 L \right]$$

$$+ \cos \kappa_1 L \cos \kappa_2 L \quad (3.37b)$$
In equations (3.37a,b) \( \bar{k}_1 \) and \( \bar{k}_2 \) are defined by equation (2.14a). The other parameters are defined as

\[
\bar{\omega}_2 = \frac{\omega r}{a_2}
\]

\[
\bar{L} = \frac{L}{r}
\]

\[
\bar{Z}_F = \frac{Z_F}{a_2 \rho A}
\]

\[
\bar{Z}_M = \frac{Z_M}{a_2 \rho I}
\]  \hspace{1cm} (3.38a-d)

Finally, by solving equation (3.35) we obtain

\[
\bar{\mu}_{1,2} = \arccos \left[ -\frac{H_1}{2} \pm \sqrt{\left(\frac{H_1}{2}\right)^2 - H_2} \right] \]  \hspace{1cm} (3.39)

Where the plus sign corresponds to \( \bar{\mu}_1 \) and the minus sign to \( \bar{\mu}_2 \).

Therefore according to equation (3.39) there are two modes of propagation of nodal waves in periodically loaded Timoshenko beams.

By inspection of (3.37a,b) we see that \( H_1 \) and \( H_2 \) are real if the load impedances are purely reactive and will be complex if the load impedances have resistive components. Table Cl in Appendix C gives the expressions
for \( \overline{Z}_F \) and \( \overline{Z}_M \) for different loads.

Therefore, if the loads are reactive, the arguments of the inverse cosine function in (3.39) will be real or complex conjugates. In the case when the loading impedances have resistive components, the arguments will be general complex numbers. Table 3.1 helps in determining the behavior of \( \overline{\mu}_1 \) and \( \overline{\mu}_2 \) for the cases discussed above.

As discussed in the case of the string, after Table 3.1, we observe that the inverse cosine function has two infinite sets of values (Hildebrand, 1962, page 519). However, we only consider the principal value (or main branch) of each one of the sets. One case stands for right-going waves while the other stands for left-going waves. The upper branches yield the same results as the main branches.

3.3.2 Examples of Dispersion Relations for Periodic Timoshenko Beams

According to equations (3.37) and (3.39), the propagation constants \( \overline{\mu}_1 \) and \( \overline{\mu}_2 \) for a periodic Timoshenko beam are functions of \( \overline{k}_1, \overline{k}_2, \overline{a}, \overline{L}, \overline{\omega}_2, \overline{Z}_F \) and \( \overline{Z}_M \). However, \( \overline{k}_1 \) and \( \overline{k}_2 \) are according to equation (2.14a) functions of \( \overline{a} \) and \( \overline{\omega}_2 \). Therefore, we can state that

\[
\overline{\mu}_{1,2} = f(\overline{a}, \overline{L}, \overline{\omega}_2, \overline{Z}_F, \overline{Z}_M)
\] (3.40)
We have obtained graphically the dispersion relations for the cases of periodic Timoshenko beams loaded with masses, springs, dashpots and transverse built-in semi-infinite plate strips. All the examples were worked for $\bar{a} = 2.3$ and $\bar{L} = 4$. Figures d2.1 through d2.20 of Appendix D contain the curves, which we will now discuss.

- **mass loads**

In general, for the periodic Timoshenko beams, figure numbers ending with an odd number are for the first mode and the ones ending with an even number are for the second mode of the nodal waves.

Figures d2.1 through d2.4 are the dispersion relations for periodic Timoshenko beams loaded with masses with $\bar{m} = 0.5$ and $\bar{m} = 2.0$, considering their moment of inertia to be zero. Figures d2.5 through d2.8 are also for masses with $\bar{m} = 0.5$ and $\bar{m} = 2.0$, but with moment of inertia different from zero, with $\bar{J} = 0.1$ and $\bar{J} = 0.2$ respectively. Studying these figures, we draw the following conclusions:

a) Both the first and the second modes of propagation for all cases possess alternating passbands and stopbands. Attenuation $\bar{\mu}_i$ grows with frequency, as in the case of the string (Figures d1.1
and d1.2).

b) Comparing first and second modes for all cases, we see that the attenuation regions for the first mode alternate with the attenuation regions for the second mode. The result is that the two modes never propagate at the same time, except for eventual very narrowbands of frequency, where the modes change from propagating to nonpropagating and vice-versa.

c) Increasing the loading masses increases the attenuation \( \overline{\mu}_i \) for both modes, but does not change the character of the curves for all cases. The real part \( \overline{\mu}_r \) is almost unchanged by changing the magnitude of the loading masses.

d) The introduction of rotary inertia modifies the dispersion relation for the first mode by creating new attenuation bands in conjunction with regions of constant \( \overline{\mu}_r = \pi \) which did not exist in the case of mass loads without rotary inertia. The curves for the second-mode are almost unchanged by the introduction of rotary inertia.

- spring loads

Figures d2.9 through d2.12 show the dispersion curves for periodic Timoshenko beams loaded with springs with \( \overline{K} = 0.5 \) and \( \overline{K} = 2.0 \). Going through these
figures, we come up with the following conclusions:

a) Both modes show stopbands and passbands. The attenuation decreases and the attenuation bands get narrower as frequency increases.

b) Increasing the stiffness of the springs widens the attenuation bands and increases the attenuation. It does not change the general pattern of the curves though.

c) First mode does not show regions of constant $\bar{\nu}_r = \pi$ while the second mode does.

d) First and second mode alternate attenuation bands, but for the two cases we are studying, namely $\bar{K} = 0.5$ and $\bar{K} = 2.0$, there are frequency bands where both modes propagate. As $\bar{K}$ increases, these bands tend to become narrower.

- **dashpot loads**

Figures 2.13 through 2.16 show the dispersion relations for periodic Timoshenko beams loaded with translational dashpots with $\bar{C} = 0.1$ and $\bar{C} = 0.4$. The conclusions in these cases are:

a) Attenuation should be always present whenever the nodes have transverse motion because the dashpots absorb energy. There are frequency bands where
the attenuation is very close to zero. The conclusion is that in those bands the nodes move very little, transversally, although they might have larger angular motions.

b) Increasing the magnitude of the dashpots causes attenuation to grow but does not change the general pattern of the curves neither appreciably changes $\bar{\mu}_r$.

c) The presence of negative imaginary part has already been explained in the case of the string and it simply means that energy and phase propagate in opposite directions.

- **transverse built-in plate strips**

Transverse built-in plate strips represent translational and rotational impedances according to the formulas given in Table C1 of Appendix C. The translational impedance is purely real, therefore dashpot-like. The rotational impedance is complex with positive imaginary part, therefore it behaves like a rotational dashpot in parallel with a rotational spring.

Figures d2.17 through d2.20 show the dispersion relations for periodic Timoshenko beams loaded with built-in plate strips, with $\bar{E} = 1.0$, $\bar{\rho} = 1.0$, $\bar{A} = 0.1$, $\bar{I} = 0.01$ and $\bar{E} = 1.0$, $\bar{\rho} = 1.0$, $\bar{A} = 0.2$ and $\bar{I} = 0.1$. 
The symbols above are defined at the bottom of Table C1 in Appendix C.

Comparing Figures d2.17 through d2.20 with Figures d2.13 through d2.16, we notice a striking similarity among corresponding curves. This indicates that the vertical built-in plates considered in the examples might behave more like translational dashpots. In other words, it is possible that the plate strips take up mostly compressional energy from the main beam and less flexural energy. However, in order to be sure of this, we should have to actually compute the compressional and bending energy being introduced in each plate.

3.3.3 The Modal Mobility Matrix for Periodic Timoshenko Beams

We have studied above the problem of natural sinusoidal wave propagation in infinite periodic Timoshenko beams. It was learned there that in such structures waves propagate in two modes and that these waves are only visible at sets of equispaced points. Although the waves can be seen for any set of points of the structure with consecutive points spaced by L (the distance between consecutive loading points), we chose to call them nodal waves. This is because the waves can be observed at the nodes, and our ultimate interest is to find the nodal motions. However, it is possible
to compute the motions within the segments by using some of our intermediate equations, to be developed in this section.

We have, in section 2.6 of Chapter 2, introduced the concept of driving-point mobility and the concept of modal mobility matrix (see equations (2.30) and (2.33)). We will in this section use the same concepts and obtain for a semi-infinite periodic Timoshenko beam, the components of the modal mobility matrix. The driving-point is the end of the semi-infinite beam and it is chosen to be a node (point of attachment of loading impedances). The driving load consists of a sinusoidal force and a sinusoidal moment of the same frequency, as illustrated in Figure 3.7. Invoking the causality principle, we see that we shall have only right-going nodal waves.

Figure 3.7 Semi-infinite periodic Timoshenko beam driven at the end node by a general sinusoidal load
The boundary condition for forces is that at the driving section the shear force be equal to the combination of the driving force and the force introduced by the translational impedance $Z_F$ (see Figure 3.8a) or:

$$Q(0,t) = -Fe^{-i\omega t} + Z_F w_0(0,t)$$

(3.41a)

The boundary condition for moments is that at the driving section the bending moment be equal to the combination of the driving moment and the moment introduced by the rotational impedance $Z_M$ (see Figure 3.8b) or:

$$M(0,t) = -Me^{i\omega t} + Z_M \psi_0(0,t)$$

(3.41b)

Figure 3.8  a) forces at the boundary, b) moments at the boundary.

The sign conventions used for writing equations (3.41a,b) are the same shown in Figure 2.1b,c of Chapter 2.

The shear force and the bending moment at a section of a Timoshenko beam are given by equations (3.29a,b). We also know that the motion of a node $j$ or of any
internal point of a segment j is given by the combination of four components of internal waves as indicated by equations (3.25) and (3.26). Therefore, substituting equations (3.29a,b) into the boundary conditions (3.41a,b), and then substituting equations (3.25) and (3.26), adapted to segment zero, into the resulting equations, we obtain, after cancelling out the time dependence:

\[
\frac{G*\lambda}{i\omega} (ik_1 \psi_0^+ - ik_1 \psi_0^- + ik_2 \psi_0^+ - ik_2 \psi_0^- - \psi_0^+ - \psi_0^-)
- Z_F (W_0^+ + W_0^- + W_0^+ + W_0^-) = -F
\]

(3.42a)

\[
-\frac{E\lambda}{i\omega} (ik_1 \psi_0^+ - ik_1 \psi_0^- + ik_2 \psi_0^+ - ik_2 \psi_0^-)
- Z_M (\psi_0^+ + \psi_0^- + \psi_0^+ + \psi_0^-) = -M
\]

(3.42b)

Considering now equation (3.30) we can transform, in equations (3.42a,b), angular velocity amplitudes into transverse velocity amplitudes, to obtain

\[
\left(\frac{\omega^2}{a_1^2 k_2} + \frac{\omega Z_F}{G*\lambda}\right) W_0^+ + \left(-2k_1 + \frac{\omega^2}{a_1^2 k_1} + \frac{\omega Z_F}{G*\lambda}\right) W_0^-
\]

\[
+ \left(\frac{\omega^2}{a_1^2 k_2} + \frac{\omega Z_F}{G*\lambda}\right) W_0^+ + \left(-2k_2 + \frac{\omega^2}{a_1^2 k_2} + \frac{\omega Z_F}{G*\lambda}\right) W_0^-
\]

\[
= \frac{\omega F}{G*\lambda}
\]

(3.43a)
\[-(k_1^+ \frac{\omega Z_M}{EI}) (k_1^- \frac{2}{a_1^2k_1}) \bar{W}_{01}^+ + (k_1^- \frac{\omega Z_M}{EI}) (k_1^- \frac{2}{a_1^2k_1}) \bar{W}_{01}^- \]

\[-(k_2^+ \frac{\omega Z_M}{EI}) (k_2^- \frac{2}{a_1^2k_2}) \bar{W}_{02}^+ + (k_2^- \frac{\omega Z_M}{EI}) (k_2^- \frac{2}{a_1^2k_2}) \bar{W}_{02}^- \]

\[= \frac{i\omega M}{EI} \quad (3.43b)\]

This is a set of two equations with four unknowns, the amplitudes of the four internal wave components in segment zero.

At this point, we recall that in the study of natural waves in periodic Timoshenko beams (section 3.2), we obtained a set of four homogeneous simultaneous equations. This set relates the amplitudes of the four internal wave components in a segment \(j\) (equation (3.31)). It is then possible, for any segment, to solve for three of the components amplitudes as functions of the fourth one.

We have solved (3.31) for \(\bar{W}_{j1}^-\), \(\bar{W}_{j2}^+\) and \(\bar{W}_{j2}^-\) as functions of \(\bar{W}_{j1}^+\), by using Cramer's rule. The resulting solutions are written in Appendix E, equations (el), (e2) and (e3). As these solutions are functions of the propagation constant \(\overline{\mu}\), we recognize that we have one
set of solutions for each one of the two modes of propagation of nodal waves. In other words, the amplitudes of the internal waves in segment zero (as in all other segments) have to be split in two parts, such that at a given point \( x_0 \) of the segment the resulting motion will contain two parts as follows

\[
w_{01}(x_0, t) = (w^{+}_{011} e^{i k_1 x_0} + w^{-}_{011} e^{-i k_1 x_0} + w^{+}_{021} e^{i k_2 x_0} + w^{-}_{021} e^{-i k_2 x_0}) e^{-i \omega t}
\]

\[
.. (3.44a)
\]

\[
w_{02}(x_0, t) = (w^{+}_{012} e^{i k_1 x_0} + w^{-}_{012} e^{-i k_1 x_0} + w^{+}_{022} e^{i k_2 x_0} + w^{-}_{022} e^{-i k_2 x_0}) e^{-i \omega t}
\]

\[
.. (3.44b)
\]

Where we have defined:

\[
w^{+}_{01} = w^{+}_{011} + w^{+}_{012}
\]

\[
w^{-}_{01} = w^{-}_{011} + w^{-}_{012}
\]

\[
w^{+}_{02} = w^{+}_{021} + w^{+}_{022}
\]

\[
w^{-}_{02} = w^{-}_{021} + w^{-}_{022}
\]

\[
(3.45a-d)
\]

The third subscripts in the symbols on the right-hand side of equations (3.45) refer to the mode of the nodal waves.
Equations (e1), (e2) and (e3) then yield \( W_{011}^{-} \), \( W_{021}^{+} \) and \( W_{021}^{-} \) as functions of \( W_{011}^{+} \) for \( \bar{\mu} = \bar{\mu}_1 \), and also yield \( W_{021}^{-} \), \( W_{022}^{+} \) and \( W_{022}^{-} \) as functions of \( W_{021}^{+} \) for \( \bar{\mu} = \bar{\mu}_2 \), as follows:

\[
\begin{align*}
W_{011}^{-} &= A_{11}^{+} W_{011}^{+} \\
W_{021}^{+} &= A_{21}^{+} W_{011}^{+} \\
W_{021}^{-} &= A_{21}^{-} W_{011}^{+} \\
W_{012}^{-} &= A_{12}^{-} W_{012}^{+} \\
W_{022}^{+} &= A_{22}^{+} W_{012}^{+} \\
W_{022}^{-} &= A_{22}^{-} W_{012}^{+}
\end{align*}
\]

Note that the coefficients \( A \) were given a second subscript which did not exist in their definitions in Appendix E. This second subscript stands for the mode of the nodal waves.

With the aid of (3.45), (3.46) and (3.47), we can now transform equations (3.43a) and (3.43b) into a set of two simultaneous equations in the variables
and \( W_{011}^+ \) and \( W_{012}^+ \), as follows:

\[
C_1 \, W_{011}^+ + C_2 \, W_{012}^+ = \frac{\omega F}{G^*A} \tag{3.48a,b}
\]

\[
C_3 \, W_{011}^+ + C_4 \, W_{012}^+ = \frac{i \omega M}{E^*I}
\]

where the coefficients \( C \) are defined as:

\[
C_1 = D_1 + D_2 A_{11}^- + D_3 A_{21}^+ + D_4 A_{21}^-
\]

\[
C_2 = D_1 + D_2 A_{12}^- + D_3 A_{22}^+ + D_4 A_{22}^-
\]

\[
C_3 = -D_5 + D_6 A_{11}^- - D_7 A_{21}^+ + D_8 A_{21}^-
\]

\[
C_4 = -D_5 + D_6 A_{12}^- - D_7 A_{22}^+ + D_8 A_{22}^-
\]

and the coefficients \( D \) are defined as:

\[
D_1 = \frac{\omega^2}{a_1^2 k_1} + \frac{\omega Z_F}{G^*A}
\]

\[
D_2 = -2k_1 + \frac{\omega^2}{a_1^2 k_1} + \frac{\omega Z_F}{G^*A} \tag{3.50a-d}
\]

\[
D_3 = \frac{\omega^2}{a_1^2 k_2} + \frac{\omega Z_F}{G^*A}
\]

\[
D_4 = -2k_2 + \frac{\omega^2}{a_1^2 k_2} + \frac{\omega Z_F}{G^*A}
\]
\[ D_5 = \left( k_1 + \frac{\omega Z_M}{EI} \right) \left( k_1 - \frac{\omega^2}{a_1^2 k_1} \right) \]

\[ D_6 = \left( k_1 - \frac{\omega Z_M}{EI} \right) \left( k_1 - \frac{\omega^2}{a_1^2 k_1} \right) \]

\[ D_7 = \left( k_2 + \frac{\omega Z_M}{EI} \right) \left( k_2 - \frac{\omega^2}{a_1^2 k_2} \right) \]

\[ D_8 = \left( k_2 - \frac{\omega Z_M}{EI} \right) \left( k_2 - \frac{\omega^2}{a_1^2 k_2} \right) \]

(S.51a–d)

Solving now the system (3.48) for \( w_{011}^+ \) and \( w_{012}^+ \) we obtain:

\[ w_{011}^+ = \frac{\omega C_4}{G*A} F - \frac{i \omega C_2}{EI} M \]

\[ \frac{1}{C_1 C_4 - C_2 C_3} \]

\[ w_{012}^+ = -\frac{\omega C_3}{G*A} F - \frac{i \omega C_1}{EI} M \]

\[ \frac{1}{C_1 C_4 - C_2 C_3} \]

(3.52a,b)

Equations (3.46) and (3.47) yield the values of the other six transverse velocity components. The eight angular velocity components are obtained from the transverse velocity components by means of equation (3.30).
The resulting amplitudes for the two modes of nodal waves at node zero are:

\[ W_{01} = (1 + A_{11}^- + A_{12}^+ + A_{12}^-) \left( W_{011}^+ \right) \]
\[ W_{02} = (1 + A_{12}^- + A_{22}^+ + A_{22}^-) \left( W_{012}^+ \right) \]

(3.53a-d)

\[ \psi_{01} = i \left[ (1+A_{11}^-) \left( k_1 - \frac{\omega^2}{a_1^2 k_1} \right) + (A_{21}^- + A_{21}^+) \left( k_2 - \frac{\omega^2}{a_1^2 k_1} \right) \right] \left( W_{011}^+ \right) \]
\[ \psi_{02} = i \left[ (1+A_{12}^-) \left( k_1 - \frac{\omega^2}{a_1^2 k_1} \right) + (A_{22}^- + A_{22}^+) \left( k_2 - \frac{\omega^2}{a_1^2 k_2} \right) \right] \left( W_{012}^+ \right) \]

If we now substitute equations (3.52a,b) for \( W_{011}^+ \) and \( W_{012}^+ \) into equations (3.53a-d) and make \( F \) and \( M \) equal to zero one at a time, we obtain the modal mobilities as follows:

\[ M_{Fw1} = \frac{W_{01}}{F} = \frac{1 + A_{11}^- + A_{21}^+ + A_{21}^-}{E_1} \frac{\omega C_4}{G^*A} \]

\[ M_{Fw2} = \frac{W_{02}}{F} = \frac{1 + A_{12}^- + A_{22}^+ + A_{22}^-}{E_1} \frac{\omega C_3}{G^*A} \]

(3.54a-d)

\[ M_{Mw1} = \frac{W_{01}}{M} = -i \frac{1 + A_{11}^- + A_{21}^+ + A_{21}^-}{E_1} \frac{\omega C_2}{E^I} \]

\[ M_{Mw2} = \frac{W_{02}}{M} = i \frac{1 + A_{12}^- + A_{22}^+ + A_{22}^-}{E_1} \frac{\omega C_1}{E^I} \]
\[ M_F \psi_1 = \frac{\psi_{01}}{F} = i \frac{(1+A_{11}^-) E_2^+ (A_{21}^+ + A_{21}^-) E_3}{E_1^*} \frac{\omega C_4}{G^* A} \]

\[ M_F \psi_2 = \frac{\psi_{02}}{F} = -i \frac{(1+A_{12}^-) E_2^+ (A_{22}^+ + A_{21}^-) E_3}{E_1^*} \frac{\omega C_3}{G^* A} \]

\[ M_M \psi_1 = \frac{\psi_{01}}{M} = \frac{(1+A_{11}^-) E_2^+ (A_{21}^+ + A_{21}^-) E_3}{E_1} \frac{\omega C_2}{E I} \]

\[ M_M \psi_2 = \frac{\psi_{02}}{M} = -\frac{(1+A_{12}^-) E_2^+ (A_{22}^+ + A_{22}^-) E_3}{E_1} \frac{\omega C_1}{E I} \]

where

\[ E_1 = C_1 C_4 - C_2 C_3 \]

\[ E_2 = k_1 - \frac{\omega^2}{a_1^2 k_1} \]

\[ E_3 = k_2 - \frac{\omega^2}{a_1^2 k_2} \]

Therefore, equations (3.54a-h) are the expressions for the eight components of the modal mobility matrix.

It is needless to discuss the complexity of the results, but they can, with not much difficulty, be implemented in a computer. Such a program can be used
as one of the subroutines for a program intended to simulate the behavior of effectively infinite periodically loaded beams.

The use and the properties of the mobility matrix and its elements has been thoroughly studied in Chapter 2 for the case of the unloaded Timoshenko beam. The same concepts and procedures can be directly used here.

3.4 Periodically Loaded Infinite Bernoulli-Euler beam

We have performed a study of natural sinusoidal wave propagation in periodic beams by using the Bernoulli-Euler beam model. The reasons for this study are:

a) To compare our dispersion relation for periodic Bernoulli-Euler beams with the ones existent in the literature, by Ungar (1966) and Bobrovnistkii and Maslov (1966).

b) To show that our dispersion relation for a periodic Timoshenko beam has for limiting value the Bernoulli-Euler beam dispersion relation.

c) To compare graphically the dispersion relation of periodic beams as given by the Timoshenko beam model and by the Bernoulli-Euler beam model.
3.4.1 **Natural Waves**

The derivations in here are completely analogous to the ones performed in section 3.3.1 for the Timoshenko beam. We will, therefore, just transcribe the important equations and final results.

The set of simultaneous homogeneous equations equivalent to the set (3.31) is:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 \\
(1 + \frac{\beta_F}{\alpha_1^+}) & (-1 + \frac{\beta_F}{\alpha_2^+}) & (-1 + \frac{\beta_F}{\alpha_1^-}) & (i + \frac{\beta_F}{\alpha_2^-}) & (1 + \frac{\beta_F}{\alpha_2^-}) \\
-(1 + \frac{\beta_M}{\alpha_1^+}) & (1 - \frac{i\beta_M}{\alpha_2^+}) & -(1 - \frac{\beta_M}{\alpha_1^-}) & (i\beta_M/\alpha_2^-) & (1 + \frac{\beta_M}{\alpha_2^-})
\end{bmatrix}
\begin{bmatrix}
\alpha_{W_1}^+ \\
\alpha_{W_2}^+ \\
\alpha_{W_1}^- \\
\alpha_{W_2}^-
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

...(3.56)

where

\[
\beta_F = \frac{\omega}{EI k^3} Z_F e^{i\mu} \\
\beta_M = \frac{\omega}{EI} Z_M e^{i\mu}
\]

(3.57a,b)

\[
k = \left(\frac{pA}{EI}\omega^2\right)^{1/4}
\]
and \( \alpha_1^+, \alpha_2^+, \alpha_1^- \) and \( \alpha_2^- \) have been defined in equations (3.32a-d).

The expression for the determinant of the coefficients of (3.56) is:

\[
\Delta = 16 - \frac{2\omega^2 Z_F Z_M}{(EI)^2 k^4} \left[ \frac{1 - \cos kL \cosh kL}{(1 - \cos \bar{u} - \cos kL)(1 - \cos \bar{u} - \cosh kL)} \right] \\
+ \frac{4i\omega Z_F}{EI k^3} \left[ \frac{\sin kL}{1 - \cos \bar{u} - \cos kL} - \frac{\sinh kL}{1 - \cos \bar{u} - \cosh kL} \right] \\
+ \frac{4i\omega Z_M}{EI k} \left[ \frac{\sin kL}{1 - \cos \bar{u} - \cos kL} + \frac{\sinh kL}{1 - \cos \bar{u} - \cosh kL} \right] 
\]

\[\ldots (3.58)\]

Finally, the propagation constants are given by

\[
\bar{u}_{1,2} = -\frac{H_1}{2} \pm \sqrt{\left(\frac{H_1}{2}\right)^2 - H_2} 
\]

where \( H_1 \) and \( H_2 \) are given in terms of dimensionless parameters by

\[
H_1 = \frac{i\bar{Z}_F}{4\omega_2^{1/2}} (\sin kL - \sinh kL) \\
+ \frac{i}{4}\bar{Z}_M \omega_2^{1/2} (\sin kL + \sinh kL) \\
- (\cos kL + \cosh kL) 
\]

\[\ldots (3.60a)\]
\[ H_2 = \frac{1}{8} \frac{\bar{Z}_F \bar{Z}_M}{\bar{k}} \left( \cos \bar{kL} \cosh \bar{kL} - 1 \right) \]

\[ - \frac{i}{4} \frac{\bar{Z}_F}{\bar{k}} \left( \sin \bar{kL} \cosh \bar{kL} - \cos \bar{kL} \sinh \bar{kL} \right) \]

\[ - \frac{i}{4} \frac{\bar{Z}_M \bar{k}}{\bar{k}} \left( \sin \bar{kL} \cosh \bar{kL} + \cos \bar{kL} \sinh \bar{kL} \right) \]

\[ + \cos \bar{kL} \cosh \bar{kL}. \quad (3.60b) \]

where

\[ \bar{k} = rk \quad (3.61) \]

and the other dimensionless parameters have been defined by equations (3.38a-d).

The dispersion relation obtained by making \( \Delta = 0 \) in (3.58) agrees completely with Bobrovnitskii and Maslov (1966) results. It also agrees with Ungar (1966) partial results, by making \( Z_M = 0 \).

We have also shown that by making \( k_1 \to k \) and \( k_2 \to ik \), the results for the propagation constants for a periodic Timoshenko beam (equation (3.39)) converges perfectly to the results for the periodic Bernoulli-Euler beam given by equation (3.59).
3.4.2 Examples of Dispersion Relations for Periodic Bernoulli-Euler beams

We have plotted, in Figures d3.1 through d3.10 in Appendix D, the dispersion relations for periodic Bernoulli-Euler beams, for some of the sets of parameters we have used when plotting the same curves for periodic Timoshenko beams (Figures d2.1 through d2.20).

We observe that, for the sets of parameters used, the first mode of the nodal waves for the periodic Bernoulli-Euler beam is in general a nonpropagating mode. There is only one case in the set of examples where the first mode propagates in a very narrow frequency band around \( \bar{\omega}_2 \approx 0.6 \) (Figure d3.5). It is, however, possible that, for different sets of beam and loading parameters, the first mode presents wider propagation bands.

The second mode behavior shows the characteristics we are used to seeing for the string and the Timoshenko beam cases. In general, comparing the second mode for the two beam theories, they have the same general shape but the Bernoulli-Euler beam results change much slower with frequency. They can be seen as a stretched (in the \( \bar{\omega}_2 \) coordinate) version of the Timoshenko beam results.
Therefore, the two theories predict quite different results. This is to be expected because in the case of the unloaded beams, a much simpler case, the results are also quite different for the dispersion relations.

3.5 Conclusions

We have in this chapter studied the problem of sinusoidal wave propagation in infinite periodically loaded strings and beams. The loading elements were treated as impedances. This gives the study a general character because for any conceivable kind of loading, we can use our results by substituting the corresponding value of the impedance.

In the case of the string, we have obtained the dispersion relation as well as the driving-point mobility $M_{FW}$ for a semi-infinite string driven at the end node by a sinusoidal force $Fe^{-i\omega t}$. We have also explored the shape of the dispersion relation for various kinds of loadings. Figures dl.1 through dl.8 of Appendix D contain the curves.

In the case of the beam, we have primarily treated it with the Timoshenko beam model. The results include the dispersion relations for the two modes of propagation as well as the elements of the modal mobility matrix.
These results allow for the computation of the response of any infinite or semi-infinite periodic beam driven at a node by a general sinusoidal load. In this case also, the use of Fourier techniques will extend the use of our results to transient loads, applied at one node. We have obtained for the periodic Timoshenko beam case, plots of dispersion relations for a set of different loading parameters. Figures d2.1 through d2.20 contain the curves.

We have also, for comparison purposes, obtained the dispersion relation for sinusoidal waves in periodic beams by using the Bernoulli-Euler beam model. Figures d3.1 through d3.10 contain some plots of the dispersion relation for this case.

The general conclusions we have reached are:

a) Periodic structures loaded with purely reactive impedances behave as filters with alternating passbands and stopbands. This is true for strings and the two modes of Timoshenko beams, and the second mode of periodic Bernoulli-Euler beams. The first mode for the Bernoulli-Euler model, however, is mostly non-propagating.

b) The presence of resistive components in the loading impedances make every wave in all cases to be
attenuated, except at frequencies where the nodes do not move.

c) There is a marked difference between the dispersion relations for periodic Timoshenko beams and periodic Bernoulli-Euler beams, in the first mode for nodal waves. For the second mode, the curves have the same general shape but the Bernoulli-Euler beam results appear to be stretched versions of the Timoshenko results. Therefore, the results are also quite different.

Our main contributions in this chapter have been the complete study of sinusoidal waves in periodic Timoshenko beams. We have never seen in the literature any treatment of periodic beams using the Timoshenko beam model. We also consider the result for the driving-point mobility of the periodic string as original. The curves we have obtained for the dispersion relations of the three models constitute also a nice step towards the better understanding of wave propagation in periodic one-dimensional infinite structures.
CHAPTER 4

PROPAGATION OF COMPRESSIONAL PULSES
IN RODS WITH INITIAL IMPERFECTIONS

4.1 Introduction

In this chapter we develop a study of the interaction of a short high intensity compressional pulse, propagating in a long rod, with initial imperfections of the rod. The intensity of the pulse is assumed to be of such a magnitude as to make the region under compression unstable.

The phenomenon of dynamic buckling of rods and shells with initial imperfections has been extensively studied in the last fifteen years. These studies treat the problem of the behavior of a rod or a shell subject to dynamic loads. The loads can be, for example, a step function (Budiansky and Hutchinson, 1964) or a pulse of finite time duration (Hutchinson and Budiansky, 1966). Juricic and Herrmann (1977) studied the response of a plate strip with initial imperfections for an in plane hammer blow, to simulate the behavior under rapping of a certain kind of electrostatic precipitator plate.

In all these studies, the whole length of the elements being analyzed is assumed to be instantly subject to the load. However, it is well known that any kind of mechanical
disturbance will propagate through a medium at a finite speed. The consequence of this is that compressional pulses, with a length much smaller than the length of the element being studied, cannot be assumed to be instantly applied to the whole length of the element. In fact, as compressional waves are nondispersive, such a condition will never be reached for any length of time for such short pulses.

Therefore, we are proposing here the new idea of studying "dynamic buckling" from a wave propagation point of view. We observe that at a given instant, a length of the rod equal to the length of the pulse is subject to a high intensity compression. This will cause the irregularities to grow, and the pulse will leave behind it flexural energy which will then propagate as flexural waves in both directions. Our main concern here is to estimate the rate of transformation of compressional energy into bending energy, as a function of the rod dimensions initial irregularities and pulse intensity. In other words, we will try to assess the importance of the described phenomenon.
We start the study by obtaining the equations of motion of an infinite beam with initial irregularities, with coupled transverse and longitudinal motions. The result is a set of two nonlinear, nonhomogeneous, coupled partial differential equations in two dependent and two independent variables.

As we are trying to introduce a new idea and sense its importance, we felt it not to be appropriate to embark in the solution of such a complex set of equations. We, therefore, performed a simplified linear analysis by studying the behavior of a finite simply supported beam, with initial sinusoidal imperfections, subject to a short, high intensity compressional load. The duration of the load was taken to be equal to the time a compressional wave takes to travel the length of the beam. The numerical results of this approximate study were encouraging and we decided to undertake an experimental verification of the phenomenon.

The experiments were performed in a long slender aluminum beam suspended by strings from a supporting structure. The compressional pulse was introduced by the impact of a pendulum and the desired pulse intensity was obtained by concentrating energy by means of a
mechanical transformer. Strain gages were used as sensors and the results were obtained as traces in a storage oscilloscope.

The experimental results indicated that the energy transformation actually happens at higher rates than predicted by the approximate theory. It also indicates that the phenomenon may be important for structures subject to short dynamic loads such as a square pulse.

It is interesting to note that the experimental apparatus enabled us to also obtain other kinds of results. These results are mostly actual demonstrations of simple wave phenomena which have been theoretically studied for many years. We consider these results very interesting from the educational point of view.
4.2 Equations of Motion of a Rod with Coupled Transverse and Longitudinal Motions

Consider a long rod where \( \eta(x,t) \) indicates transverse displacements and \( \xi(x,t) \) longitudinal displacements of a point \( x \) of the beam (see figure 4.1).

![Figure 4.1](image)

We will in this section obtain the equations of motion of this beam with coupling between \( \xi(x,t) \) and \( \eta(x,t) \). The Bernoulli-Euler beam theory (see table B.1 in Appendix B for definition) and a variational approach will be used in the derivation.

The total kinetic coenergy and potential energy of the rod are respectively given by:

\[
T^* = \int_0^L \frac{1}{2} \rho A \left[ \left( \frac{\partial \eta}{\partial t} \right)^2 + \left( \frac{\partial \xi}{\partial t} \right)^2 \right] dx \tag{4.1}
\]

and

\[
V = \int_0^L \left[ \frac{1}{2} EI \left( \frac{\partial^2 \eta}{\partial x^2} \right)^2 + \frac{1}{2} EA \left( \epsilon_x \right)^2 \right] dx \tag{4.2}
\]
where in both equations the first term of the sum refers to the energy associated with transverse motion and the second term with the energy associated with longitudinal motion.

In order to couple $\eta$ and $\xi$, we will take the expression for the longitudinal strain $\varepsilon_x$ with two terms as given by the large deflections theory (see Fung, 1967). Therefore

$$\varepsilon_x = \frac{\partial \xi}{\partial x} + \frac{1}{2} \left( \frac{\partial \eta}{\partial x} \right)^2$$  \hspace{1cm} (4.3)

If we now substitute (4.3) into (4.2) we obtain:

$$v = \int_0^L \left\{ \frac{1}{2} EI \frac{\partial^2 \eta}{\partial x^2} + \frac{1}{2} EA \left[ \frac{\partial \xi}{\partial x} + \frac{1}{4} \left( \frac{\partial \eta}{\partial x} \right)^4 + \frac{\partial \xi}{\partial x} \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \right\} \, dx$$  \hspace{1cm} (4.4)

From Crandall et al (1968), we know that the variational indicator is given by:

$$v.I. = \int_{t_1}^{t_2} \left[ \delta T^* - \delta V + \sum_{j=1}^{n} \xi_j \delta \xi_j \right] \, dt$$  \hspace{1cm} (4.5)

where the symbol $\delta$ indicates the variation of the quantity and the last term (a summation) stands for the generalized forces which we will assume to be zero in this case.
From equations (4.1) and (4.4) we obtain:

\[ \delta T^* = \int_0^L \rho A \left[ \frac{\partial}{\partial t} \left( \frac{\partial \eta}{\partial t} \right) (\delta \eta) + \left( \frac{\partial \xi}{\partial t} \right) \frac{\partial}{\partial t} (\delta \xi) \right] \, dx \quad (4.6) \]

and

\[ \delta V = \int_0^L \left\{ EI \left( \frac{\partial^2 \eta}{\partial x^2} \right) \frac{\partial^2}{\partial x^2} (\delta \eta) + EA \left( \frac{\partial \xi}{\partial x} \right) \frac{\partial}{\partial x} (\delta \xi) \right. \]

\[ + \frac{1}{2} \, EA \left( \frac{\partial \eta}{\partial x} \right)^3 \frac{\partial}{\partial x} (\delta \eta) + \frac{1}{2} \, EA \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial}{\partial x} (\delta \xi) \]

\[ + EA \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial}{\partial x} (\delta \eta) \left\} \, dx \quad (4.7) \]

Substituting (4.6) and (4.7) into the variational indicator (4.5) and then integrating by parts the kinetic coenergy terms in time, and the potential energy terms in space \((x)\) and collecting terms, we obtain:

\[ V.I. = \int_{t_1}^{t_2} dt \left\{ \int_0^L \left[ -\rho A \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 \eta}{\partial x^2}) \right. \right. \]

\[ + \frac{\partial}{\partial x} \left( \frac{1}{2} \, EA \left( \frac{\partial \eta}{\partial x} \right)^3 \right) + \frac{\partial}{\partial x} \left( EA \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \eta}{\partial x} \right) \right) \left\} \delta \eta \, dx \]
\begin{align*}
&+ \int_0^L \left[ - \rho A \frac{\partial^2 \xi}{\partial t^2} + \frac{3}{\partial x} \left( EA \frac{\partial \xi}{\partial x} \right) + \frac{3}{\partial x} \left( \frac{1}{2} EA \left( \frac{\partial \eta}{\partial x} \right)^2 \right) \right] \delta \xi \, dx \\
&- \left[ EI \frac{\partial^2 \eta}{\partial x^2} \frac{\partial}{\partial x} (\delta \eta) \right]_0^L + \left[ \frac{3}{\partial x} \left( EI \frac{\partial^2 \eta}{\partial x^2} \right) \delta \eta \right]_0^L \\
&- \frac{1}{2} EA \left( \frac{\partial \eta}{\partial x} \right)^3 \delta \eta - EA \left( \frac{\partial \xi}{\partial x} \left( \frac{\partial \eta}{\partial x} \right) \delta \eta \right]_0^L \\
&- \frac{3A}{\partial x} \delta \xi + \frac{1}{2} EA \left( \frac{\partial \eta}{\partial x} \right)^2 \delta \xi \right]_0^L \\
\end{align*}
(4.8)

Making now use of the variational principle (see Crandall et al, 1968, page 27), we impose V.I. = 0 and obtain two equations of motion as follows:

\begin{align*}
&\text{EI} \frac{\partial^4 \eta}{\partial x^4} + \rho A \frac{\partial^2 \eta}{\partial t^2} - \frac{3}{2} EA \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 \eta}{\partial x^2} - EA \frac{\partial}{\partial x} \left[ \left( \frac{\partial \xi}{\partial x} \right) \frac{\partial \eta}{\partial x} \right] = 0 \\
&(4.9a,b) \\
&\text{EA} \frac{\partial^2 \xi}{\partial x^2} - \rho A \frac{\partial^2 \xi}{\partial t^2} + EA \frac{\partial \eta}{\partial x} \frac{\partial^2 \eta}{\partial x^2} = 0
\end{align*}

We do not consider the terms for the boundaries x=0 and x = L because we are interested in the behavior of very long rods with no concern for boundaries.
We notice that the third term in equation (4.9a,b) is one order of magnitude smaller than all other terms and it can be dropped.

Therefore, the final set of equations of motion is:

\[ EI \frac{\partial^4 \eta}{\partial x^4} + \rho A \frac{\partial^2 \eta}{\partial t^2} - EI \frac{\partial}{\partial x} \left[ \frac{\partial \xi}{\partial x} \right] \left( \frac{\partial \eta}{\partial x} \right) = 0 \]

\[ (4.10a,b) \]

\[ EA \frac{\partial^2 \xi}{\partial x^2} - \rho A \frac{\partial^2 \xi}{\partial t^2} + EA \frac{\partial \eta}{\partial x} \frac{\partial^2 \eta}{\partial x^2} = 0 \]

This is a set of coupled, nonlinear partial differential equations in the dependent variables \( \eta(x,t) \) and \( \xi(x,t) \).

In this particular case we are interested in the rod which contains initial transversal imperfections which we assume to be described by \( \eta_0(x) \). In this case, the transverse motion with respect to the \( x \) axis is given by

\[ \eta_1(x,t) = \eta(x,t) + \eta_0(x) \quad (4.11) \]

Using now equation (4.11) to substitute for \( \eta(x,t) \) in equations (4.10a,b), we obtain after some algebraic manipulations:
\[
\frac{a^4 \eta_1}{\partial x^4} + \frac{1}{a_2^2 r^2} \frac{\partial^2 \eta_1}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial x} \left[ \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \eta_1}{\partial x} \right) \right] + \frac{\partial}{\partial x} \left[ \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \eta_0}{\partial x} \right) \right] = \frac{a^4 \eta_0}{\partial x^4}
\]

(4.11a,b)

\[
\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{a_2^2} \frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \eta_0}{\partial x^2} \left( \frac{\partial \eta_1}{\partial x} \right) - \frac{\partial \eta_0}{\partial x} \left( \frac{\partial^2 \eta_1}{\partial x^2} \right) + \frac{\partial \eta_1}{\partial x} \left( \frac{\partial^2 \eta_1}{\partial x^2} \right)
\]

\[= \left( \frac{\partial \eta_0}{\partial x} \right) \left( \frac{\partial^2 \eta_0}{\partial x^2} \right) \]

where

\[r^2 = \frac{I}{A} \quad \text{and} \quad a_2^2 = \frac{E}{\rho} \quad \text{(4.12)}\]

Equations (4.11a,b) are the equations of motion of a rod with initial imperfections given by \(\eta_0(x)\) and with coupling of the independent variables \(\xi(x,t)\) and \(\eta(x,t)\).

We recall that the problem we are trying to study is the result of the interaction of a high intensity short compressional pulse with the initial imperfections \(\eta_0(x)\). Therefore, the path to follow now would be to solve system (4.11a,b) imposing as an initial condition a high intensity compressional pulse at a given region of the beam.
This pulse will, of course, split in two parts which will propagate in opposite directions, away from the region of its application. We, therefore, want to look to one of the sides only.

However, as discussed in the introduction to this chapter, we are trying to introduce a new idea and it would be undesirable to do it by solving such a complex mathematical problem. We, therefore, decided to leave this set of equations as a first contribution to further "exact" mathematical study and perform a simplified mathematical analysis associated with some experimental work.

4.3 Simplified Analysis

Before going to an experimental work, we have first performed a simplified analysis of the problem. The idea with this approximate study is to gain some feeling of the order of magnitude of the transformation of compression into bending energy. The experimental work will be undertaken only if the approximate analysis gives significant values of transformation of energy for certain sets of parameters.
Consider a simply supported slender rod of length $L$, with initial imperfections given by $\eta_0(x)$, as shown in Figure 4.2.

$$\eta(x,t) = \eta_1(x,t) - \eta_0(x)$$

Figure 4.2 Finite rod with initial imperfections $\eta_0(x)$ subject to high intensity compressional pulse of finite length. (Note that the transverse deflections are grossly exaggerated).

Assume now that the rod is subject to a compressional pulse of intensity $P$, which is assumed to act instantly in the whole length of the rod. The duration of the pulse will be assumed to be equal to the time a compressional wave would take to travel the length of the rod. The idea with this study is to simulate the behavior of a piece of the long rod described in the previous section with length equal to the length of the compressional pulse. The assumptions which correspond to approximations are the
simply supported condition of the finite rod and the time span it stays under the load. The other thing is that we are not taking into account the effects of large deformations.

The motion of the rod starts from an unstressed condition in the position $\eta_0(x)$. Figure (4.3) depicts an element $dx$ of the rod.

![Figure 4.3 Element dx of rod](image)

The transverse momentum equation is:

$$\frac{\partial Q}{\partial x} \ dx - \frac{\partial}{\partial x} \left( P \frac{\partial \eta_1}{\partial x} \right) dx = \rho A \frac{\partial^2 \eta}{\partial x^2} dx \quad (4.13)$$

The angular momentum equation is:

$$\frac{\partial M}{\partial x} \ dx + Q dx = 0 \quad (4.14)$$

From the elementary beam theory, we know that the bending moment $M$ is given by:

$$M(x, t) = EI \frac{\partial^2 \eta}{\partial x^2} (x, t) \quad (4.15)$$
Substituting (4.15) into (4.14), we obtain

\[ Q(x,t) = - \frac{\partial^3 \eta}{\partial x^3} (x,t) \]  

(4.16)

Notice that we made \( M \) and \( Q \) functions of \( \eta \), not of \( \eta_1 \) (see Figure 4.2). The reason for this is that the rod is unstressed when its shape is given by \( \eta_0(x) \).

If we now substitute (4.16) into (4.13), we obtain the equation of motion as follows:

\[ \frac{\partial^4 \eta}{\partial x^4} + \frac{P}{EI} \frac{\partial^2 \eta_1}{\partial x^2} + \frac{1}{a^2 r^2} \frac{\partial^2 \eta}{\partial t^2} = 0 \]  

(4.17)

where \( r^2 \) and \( a^2 \) are given by equation (4.12).

If we now make the following substitution (see Figure 4.2)

\[ \eta_1(x,t) = \eta(x,t) + \eta_0(x) \]  

(4.18)

into equation (4.17), we obtain:

\[ \frac{\partial^4 \eta}{\partial x^4} + \frac{P}{EI} \frac{\partial^2 \eta}{\partial x^2} + \frac{1}{a^2 r^2} \frac{\partial^2 \eta}{\partial t^2} = - \frac{P}{EI} \frac{\partial^2 \eta_0}{\partial x^2} \]  

(4.19)

which is the equation of motion we are looking for.
4.3.1 Growth of amplitude for sinusoidal initial imperfections

Assume now that the initial imperfections of the beam are sinusoidal and given by:

\[ \eta_0(x) = N_0 \sin \frac{n\pi x}{L}; \quad n = 1, 2, 3, \ldots \quad (4.20) \]

Notice that this shape for \( \eta_0(x) \) satisfies the simply supported boundary conditions for \( x = 0 \) and \( x = L \).

Substitute now (4.20) into equation (4.19) to obtain:

\[ \frac{\partial^4 \eta}{\partial x^4} + \frac{P}{EI} \frac{\partial^2 \eta}{\partial x^2} + \frac{1}{a_2^2 r_2^2} \frac{\partial^2 \eta}{\partial t^2} = N_0 \left( \frac{n\pi^2}{L} \right) \frac{P}{EI} \sin \frac{n\pi x}{L} \]

\[ \ldots (4.21) \]

Assume for (4.21) a solution of the form:

\[ \eta(x,t) = N(t) \sin \frac{n\pi x}{L} \quad (4.22) \]

If we substitute (4.22) into (4.21), we obtain:

\[ \frac{\partial^2 N(t)}{\partial t^2} - \Omega^2 N(t) = A_1 N_0 \quad (4.23) \]

where we have defined

\[ \Omega^2 = a_2^2 r_2^2 \left( \frac{n\pi}{L} \right)^2 \left[ \frac{P}{EI} - \left( \frac{n\pi}{L} \right)^2 \right] \quad (4.24a) \]
and

\[ A_1 = a_x^2 \pi^2 \left( \frac{n \pi}{L} \right)^2 \frac{P}{EI} \]  \hspace{1cm} (4.24b)

Equation (4.23) is a nonhomogeneous ordinary differential equation in \( N(t) \). Its solution is given by

\[ N(t) = C_1 e^{\Omega t} + C_2 e^{-\Omega t} - \frac{A_1 N_0}{\Omega^2} \]  \hspace{1cm} (4.25)

The initial conditions for (4.25) are:

\[ N(t=0) = 0 \]  \hspace{1cm} (4.26a, b)

\[ N(t=0) = 0 \]

If we now substitute (4.25) into (4.26a,b) and solve for \( C_1 \) and \( C_2 \), we obtain:

\[ C_1 = C_2 = \frac{A_1 N_0}{2\Omega^2} \]  \hspace{1cm} (4.27)

Therefore, the total solution (4.22) can be written as

\[ n(x,t) = \frac{A_1 N_0}{2\Omega^2} \left( \cosh \Omega t - 1 \right) \sin \frac{n \pi x}{L} \]  \hspace{1cm} (4.28)

Considering now equations (4.24a,b), we can rewrite equation (4.28) as
\[ \eta(x,t) = \frac{N_0}{1 - \frac{n^2 \pi^2 EI/L^2}{P}} (\cosh \Omega t - 1) \sin \frac{n \pi x}{L} \quad \ldots (4.29) \]

At this point, we recall that the critical load of any buckling mode \( n \) for a simply supported beam is given by

\[ P_n = \frac{n^2 \pi^2 EI}{L^2} \quad (4.30) \]

Therefore, we can represent the applied load \( P \) as a function of the critical load \( P_1 \) of the first buckling mode as follows

\[ P = mP_1 = m \frac{\pi^2 EI}{L^2} \quad (4.31) \]

If we now substitute (4.31) into (4.29) and into (4.24a) we obtain:

\[ \eta(x,t) = N(t) \sin \frac{n \pi x}{L} \quad (4.32) \]

and

\[ \Omega t = n^2 \pi^2 \left( \frac{m}{n^2} - 1 \right)^{1/2} \frac{E}{L} \quad (4.33) \]

where

\[ N(t) = N_0 \frac{m/n^2}{m/n^2 - 1} (\cosh \Omega t - 1) \quad (4.34) \]
We have introduced in (4.33) the following definitions

\[
\bar{t} = \frac{t}{L/a_2} \quad (4.35a)
\]

and

\[
\bar{L} = \frac{L}{r} \quad (4.35b)
\]

We notice that a dimensionless time \( \bar{t} = 1 \) implies a time \( t \) equal to the time a compressional wave (phase velocity \( a_2 \)) takes to travel the length \( L \).

From equations (4.33) and (4.34) we observe that if \( m > n^2 \) for a given node \( n \), then the amplitude \( N(t) \) will grow continuously with time until \( \bar{t} = 1 \). However, if \( m < n^2 \), it becomes an imaginary number and the hyperbolic cosine in (4.34) turns into a cosine and the beam will oscillate around the initial shape \( \eta_0(x) \).

Table 4.1 shows the growth \( N(t)/N_0 \) for the following ranges of the parameters involved

\[
200 \leq \bar{L} \leq 1000
\]

\[
.3 \leq m \leq 40 \quad (4.36a-c)
\]

\[
1 \leq n \leq 6
\]
TABLE 4.1 Growth of the amplitude for a rod with sinusoidal initial imperfections under a compressional load $P$ of duration $t = 1.0$

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We observe that for high values of m, the growth of the higher modes can be quite significant. It is important to recall that this is an approximate linear theory and for such large growth, nonlinearities may become very important leading to large errors on the approximate theory.

4.3.2 Transformation of Compressional into Bending Energy for Sinusoidal Initial Imperfections

We will, using the same approximate model of the last subsection, study the growth of bending energy in the same cases.

We first of all notice that the potential energy for axial motion of a rod of length L is given by

\[ V_a = \int_0^L \frac{EA}{2} \left( \frac{\partial \xi}{\partial x} \right)^2 \, dx \]  \hspace{1cm} (4.37)

However, we recall that for a compressional load the strain is given by

\[ \frac{\partial \xi}{\partial x} = \frac{P}{AE} \] \hspace{1cm} (4.38)

If we not substitute (4.38) into (4.37), we obtain

\[ V_a = \frac{P^2 L}{2EA} \] \hspace{1cm} (4.39)
Recalling now equation (4.31), we obtain from (4.39)

\[ V_a = m^2 \frac{\pi^4}{2L^3} r^2 EI \]  \hspace{1cm} (4.40)

The kinetic energy of the rod for axial motion is

\[ K_a = \int_0^L \frac{\rho A \dot{\xi}^2}{2} \, dx \]  \hspace{1cm} (4.41)

but

\[ \dot{\xi} = \frac{P}{\rho a^2 A} = \frac{P}{\sqrt{\rho E A}} \]  \hspace{1cm} (4.42)

where \( a_2 \) has been defined in equation (4.12). Therefore, equation (4.41) becomes

\[ K_a = \frac{P^2 L}{2EA} \]  \hspace{1cm} (4.43)

Note that \( V_a \) and \( K_a \) are equal.

The total energy for axial motion is given by

\[ H_a = V_a + K_a = \frac{P^2 L}{EA} = \frac{m^2 \pi^4}{L^3} r^2 EI \]  \hspace{1cm} (4.44)

where \( H_a \) is called the Hamiltonian.
Let us now compute the potential and kinetic energy for flexural motion of the rod of length $L$.

Potential energy is given by

$$V_f = \int_0^L \frac{1}{2} EI \left( \frac{\partial \eta}{\partial x} \right)^2 dx. \tag{4.45}$$

If we now recall equation (4.22), we obtain after some simplifications:

$$V_f = \frac{(n\pi)^4}{4L^3} EI N^2(t). \tag{4.46}$$

The kinetic energy for flexural motion is given by

$$K_f = \int_0^L \frac{\rho A}{2} (\dot{\eta})^2 \, dx. \tag{4.47}$$

Using again equation (4.22), we obtain

$$K_f = \frac{\rho AL}{4} \dot{N}^2(t)$$

Therefore, the total flexural energy is given by

$$H_f = V_f + K_f = \frac{(n\pi)^4}{4L^3} EI N^2(t) + \frac{\rho AL}{r} \dot{N}^2(t) \tag{4.48}$$

If we now recall equations (4.32) through (4.34), we obtain from (4.48)

$$H_f = \frac{\pi^2 EI}{4L^3} N_0^2 \frac{m^2}{m/n^2 - 1} \left[ \frac{1}{(m/n^2 - 1)} (\cosh \Omega t - 1)^2 + \sinh^2 \Omega t \right] \tag{4.49}$$
The ratio of equations (4.49) and (4.44) gives the growth of flexural as compared to the axial energy

\[
\frac{H_f}{H_a} = \frac{(N_0/r)^2}{4(m/n^2-1)} \left[ \frac{1}{(m/n^2-1)} (\cosh \Omega t-1)^2 + \sinh^2 \Omega t \right]
\]

(4.50)

where \( r = \sqrt{I/A} \) is the radius of gyration of the rod.

Once more we notice that for \( m > n^2 \), for a given mode \( n \), the ratio \( H_f/H_a \) grows continuously with time until \( \bar{\varepsilon} = 1 \). For \( m < n^2 \) the mode is not unstable and the long time behavior will be oscillation around the initial position \( \eta_0(x) \). However, if \( \bar{\varepsilon} = 1 \) corresponds to a very short time, the deflection may actually grow continuously until \( \bar{\varepsilon} = 1 \). In other words, the time for the deflection \( \eta(x,t) \) to reach a maximum is greater than \( \bar{\varepsilon} = 1 \). This appears to be the case, in general, for this study.

Table 4.2 shows the growth \( H_f/H_a \) of flexural energy for the following ranges of parameters

\[
\begin{align*}
200 & \leq I \leq 1000 \\
0.9 & \leq m \leq 40 \\
1 & \leq n \leq 6 \\
0.5 & \leq N_0/r \leq 1.5
\end{align*}
\]

(4.51a–d)
TABLE 4.2 Growth of flexural energy $H_f/H_e$ for a rod with sinusoidal initial imperfections under a compressional load of duration $\bar{t} = 1$ (approximate theory).

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4.4 Experimental Work

We have in the previous section developed an approximate theory for the problem of the interaction of a high amplitude short compressional pulse with initial imperfections of a long rod. The whole idea behind the study is to verify if the rate of transformation of compressional energy into bending energy is high enough such as to significantly weaken the compressional pulse as it propagates through the long rod.

The results of the approximate theory (see Table 4.2) indicate that, for high compressional loads, energy can be converted at significant rates.

We, therefore, decided to undertake an experimental investigation of the phenomenon.

4.4.1 The Experimental Apparatus

Figure 4.4 is a sketch of the experimental apparatus we have designed and built to perform the experiments. The sketch indicates that we have used a pendulum to introduce the desired pulse. It also shows that the section of impact has a larger cross-sectional area than that of the long rod and that between this section and the rod there is a long tapered piece. The material as indicated in
Figure 4.4 is 6061-T6 aluminum alloy. The reason to use such a strong aluminum alloy (yield stress 40,000 psi) is that we need high intensity pulses without reaching the plastic regime. Table 4.3 contains the properties of the material and of the two cross-sections of the apparatus.

The impact section and the pendulum have a cross-section of $1^{3/4}\" \times 1^{3/4}\"$. The long rod has a cross-section of $1\" \times 1/8\"$. The taper was milled out of a single piece of material and the rod is welded to its thin end.

The reason for using a pendulum of a given length which has the same cross-section as the impact section is that it yields a square compressional pulse of length twice the length of the pendulum. The amplitude of the pulse is proportional to the velocity of the pendulum and to the area of impact. We have found from our calculations that an impact area equal to the cross-section of the rod would not yield the needed pulse intensity. That is the reason why we used a larger impact area.

It is known from the theory of horns (the taper actually behaves as an inverted horn) that an exponential horn reflects the least energy and that the longer the horn, the less energy it reflects. We have for practical
### Table 4.3

**Sections and Material Properties for the Experimental Apparatus**

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*Note: 6061-T6 Aluminum: properties taken from Baumeister (1967), page 6-10, and Table 12 page 6-87.*
reasons built a linear taper and as long as the milling machine would take.

Ideally, the taper should concentrate energy such that the ratio stress at the end of the taper/stress at the beginning of the taper should be equal to the square root of the ratio area at the beginning of the taper/area at the end of the tape. In our particular case, this ratio is equal to five. In the actual experiments, we have achieved a lower ratio. The possible reasons for this are the energy reflection at the taper and at the welded section at the end of the taper.

The measurements were made with strain gages attached to the two wider faces of the rod, and two of the faces of the thicker section of the taper. Figure 4.4 indicates the sections where the strain gages are located. Such sections are numbered 1 to 6. In order to permit measuring bending or compression (tension), the strain gages are in sets of two, one in each face of the rod. Sections 3 and 4 contain two sets of two strain gages each.

Figures 4.5a - 4.5d are pictures taken from the apparatus. The legends explain them in detail.

We will now discuss the experimental results.
Figure 4.5a Long rod is shown on the right suspended by strings from a supporting beam. The tapered section of Figure 4.6b is at the extreme right (not shown) of the beam. Over the table, the four-trace oscilloscope and the box with the four Whitstone bridges.
Figure 4.5b  Tapered beam. The impact section is at the top of the picture. Note the welding between taper and rod.
Figure 4.5c  The three pendulums used in the experiments
Figure 4.5d  Section 3 of the beam (see Figure 4.4) instrumented with two strain gages. The other face also has two strain gages.
4.4.2 The Experimental Results

Figure 4.7 through 4.21 show the results of a set of experiments performed with the apparatus. The description of each experiment is in the legend of each figure. In order to understand the meaning of the results, one must refer to Figure 4.4.

Three pendulums were used for the experiments. Two of these, which we refer to as "thick pendulums," have a section of $1^{3}/4" \times 1^{3}/4"$. One of these is 30 cm long and the other 60 cm long. The third pendulum is a "thin pendulum" with a cross-section $1" \times 1/8"$ (same as the long rod) and it is 30 cm long. Figure 4.6d shows the three pendulums.

Figures 4.7 through 4.11 correspond to experiments that illustrate general properties of axial and flexural waves as well as some characteristics of this particular system. These figures, plus Figures 4.12 and 4.13, correspond to experiments ran before the permanent deformation has been introduced.

Figure 4.4 shows the region of the beam where the permanent deformation was introduced. The length was chosen to be 60 cm to match the length of the square pulse. This permits comparison with the results we have
obtained for the approximate theory of the previous section. We chose a sinusoidal deformation, which corresponds to the fourth buckling mode of a simply supported beam of length equal to 60 cm. Figure 4.6 shows the shape of the deformation.

![Image of a sinusoidal deformation with a scale of 1 mm and dimensions of 60 cm and 15 cm.]

**Figure 4.6** Shape of permanent deformation introduced into the beam

However, the method we used to introduce the deformation (force applied at the points of maximum amplitude of the deformation) lead to a triangular rather than sinusoidal deformation. The consequences of this fact will be explained later.

We will now discuss the result of each experiment:

**Figure 4.7:** It is known from basic considerations that if a short rod with initial velocity impacts a stationary long rod, the force generated at the interface
is given by

$$F = \frac{Z_1 Z_2}{Z_1 + Z_2} v_0$$  \hspace{1cm} (4.52)

where:

- $Z_1 = \rho_1 C_1 A_1$ - interface impedance of short rod
- $Z_2 = \rho_2 C_2 A_2$ - interface impedance of long rod
- $v_0$ = initial velocity of short rod

If the two rods have the same impedances, equation 4.52 reduces to

$$F = Z_1 \frac{v_0}{2}$$  \hspace{1cm} (4.53)

If, in addition, the rods have uniform cross-section along their length, this force will be held for the time it takes a compressional wave to travel twice the length of the short rod. After this time, the force falls to zero, the short rod stops unstressed and all its energy travels in the long rod as a square pulse.

The picture shown in Figure 4.7 is a striking demonstration of the phenomenon.

**Figure 4.8**

The phase velocity of flexural waves in beams is a
function of frequency. This means that the medium is dispersive for flexural waves. Therefore, a transient, which is made up of many frequencies, shall with time become a sinusoid of varying frequency and amplitude. This experiment demonstrates this quite clearly.

**Figure 4.9**

If $Z_1 \gg Z_2$, equation (4.52) becomes

$$F = Z_2 v_0$$  \hspace{1cm} (4.54)

This is a small force for the high impedance rod. The consequence is that after a time equal to twice the time compressional waves take to travel the length of the beam, there will be a small step decrease in the interface force. As a consequence, one should observe a decaying pulse with an abrupt front. This experiment demonstrates this, and as in this case $Z_1 = 25 Z_2$ the decay is so slow that the pulse resembles a step function. Therefore, this is a technique that yields a compressional step function.

**Figure 4.10**

This experiment shows that the presence of the taper changes quite a bit the nearly perfect square pulse
introduced at the impact section. We associate the slope on the initial pulse with reflections from the taper walls. This fact associated with the welded section (different material) at the end of the taper are probably the main reasons for the tail developed on the main square pulse at the long rod.

Figure 4.11

This is, again, a demonstration of dispersion of flexural waves in beams. Although in this case the waves go through the taper, the results are qualitatively very similar to the results of Figure 4.8.

Figures 4.12 and 4.13

These two experiments were designed to show that no bending waves appear in a strait beam with a high intensity compressional pulse propagating through it.

Figures 4.14 and 4.15

All the experiments from now on were performed after introducing the permanent deformation in the beam.

We consider Figure 4.15 as the most important result here. It actually shows the existence of the bending waves generated in the region of the beam with initial imperfections. The predicted waves are the long flexural waves. The short flexural waves which appear on top of the long waves are a consequence of the triangular rather than
sinusoidal shape of the initial irregularities. Therefore, the deformation grows as a triangular wave, and the Fourier components of the triangular waves will propagate at different speeds and, therefore, will show up separately after propagating for some length.

In Appendix F, we show the spectrum of a triangular wave. We see that it is a train of impulses and that the first Fourier component corresponds to the long waves, and the second to the short waves, in Figure 4.15.

As indicated in Figure 4.15, the rate of transformation of energy is about 7.4% if only the principal pulse is considered. If we consider also the first two pulses on the tail, this number falls to 5%. In any case, the rate is larger than 1.5% the rate predicted by the approximate theory.

Figure 4.16

This picture demonstrates clearly that the flexural waves have actually been generated in the middle of the beam where the initial imperfections are located.

Figures 4.17 through 4.19

These experiments are similar to the experiment of Figure 4.15, where the only difference is the initial velocity of the pendulum. In general, energy is transferred at higher rates than predicted by the
approximate theory.

Figure 4.20

This experiment has the same parameters as the one of Figure 4.15, but it was performed with the 15 cm thick pendulum. Although the compressional pulse has the same intensity of the compressional pulse of Figure 4.15, it is half as long. This leads to lower amplitude of the long flexural waves, but about the same amplitude for the short waves.

Figure 4.21

This experiment was performed by impacting the thin end of the rod with the thin pendulum. In this case, there is no tail in the pulse and the rate of transformation is the lowest of all cases, but still higher than the result predicted by the approximate theory.

We, therefore, conclude that energy conversion can actually happen at significant rates.

The phase velocity for compressional waves obtained from Figure 4.7 (measured time for wave to travel between 4A and 3A) is $a_2 = 5185 \text{ m/sec}$. The theoretical value as shown in Table 4.3 is $a_2 = 5203 \text{ m/sec}$.

The compressional energy of the compressional waves was computed from the pictures with the following formula
\[ H_a = \frac{P^2 L}{EA} \]  

(4.55)

where \( P \) is the intensity of the pulse and \( L \) its length.

The average flexural energy per unit length was computed with the following formula

\[ <H_f> = \frac{1}{2} \frac{EI}{k} k^4 N^2 \]  

(4.56)

where \( N \) is the amplitude of the sinusoidal wave.

The amplitude of the flexural waves was computed with the following formula

\[ N = \frac{M_{\text{max}}}{EI k^2} \]  

(4.57)

where \( M_{\text{max}} \) is the value of the bending moment at the point of maximum amplitude of the sinusoidal flexural wave.

Substituting (4.57) into (4.56), we obtain:

\[ <H_f> = \frac{1}{2} \frac{M_{\text{max}}^2}{EI} \]  

(4.58)

The calibration of constants of the strain gages in the rod for axial force and bending moment are:

\[ P = \frac{5.69 \times 10^6}{\Delta V} \quad \frac{N}{V} \]  

(4.59)
\[ M = 485 \cdot \Delta V \quad \text{N}\cdot\text{m/V} \quad (4.60) \]

where

\[ \Delta V \text{ - output voltage from the bridge} \]

\[ V \text{ - input voltage to the bridge} \]

For these measurements, we have used 6V batteries.

The energy of flexural waves was computed for a length of 60 cm corresponding to the length of the permanent deformation, for the long and the short waves.

In Figure 4.22, we have plotted the ratio \( <H_f>/H_a \) as a function of \( m = P/P_1 \) for the experiments of Figures 4.15, 4.17, 4.18 and 4.19. We can approximately say that \( <H_f>/H_a \) is proportional to \( 1/m \). It is interesting to notice that according to Table 4.2, the approximate theory predicts no variation of \( <H_f>/H_a \) with \( m \). It also shows (see dashed line in Figure 4.22) that the theoretical prediction is probably the asymptotic value of \( <H_f>/H_a \) for large values of \( m \). This somehow indicates that the theoretical results obtained with the approximate theory can be taken as a lower-bound for the values of \( H_f/H_a \). The values of \( <H_f>/H_a \) used to plot figure 4.22 correspond
to the compressional energy associated with the main square pulse only.

We finally observe that the flexural waves in Figures 4.15 through 4.21 have the following characteristics:

**long waves:**
- period: $T = 2.824 \text{ m/sec}$
- frequency: $f = 354.2 \text{ Hz}$
- angular frequency: $\omega = 2225.3 \text{ rad/sec}$
- wave number: $k = 21.65 \text{ m}^{-1}$
- wave length: $\lambda = \frac{2\pi}{k} = 0.0293 \text{ m}$
- phase velocity: $c = \omega/k = 103 \text{ m/sec}$

**short waves:**
- period: $T = 0.306 \text{ m/sec}$
- frequency: $f = 3269.2 \text{ Hz}$
- angular frequency: $\omega = 20541 \text{ rad/sec}$
- wave number: $k = 65.76 \text{ m}^{-1}$
- wave length: $\lambda = k/2\pi = 0.096 \text{ m}$
- phase velocity: $c = \omega/k = 312 \text{ m/sec}$

Recalling that compressional waves travel at 5185 m/sec we see that before the flexural waves can reach sections 3A and 4A (see Figure 4.4), the compressional pulse has traveled through the beam and reflected at the ends many times.
This adds to the uncertainties of the experimental results, because the axial pulse will act over the flexural waves many times before they go through sections 3A and 4A. In some passes, the pulse is a compressional pulse and in other passes it is a tension pulse.
Figure 4.7  Impact at the thin end with 30 cm long thin pendulum. It shows that a pendulum that matches the impedance of the rod generates a perfect square pulse. It also shows the nondispersive character of axial waves.
Figure 4.3  Impact at thin end with 30 cm thin pendulum. This picture demonstrates dispersion of flexural waves. The flexural wave starts off as a short pulse at the impact section (6) and appears as a frequency varying sinusoid in the other two sections.
section 6 (axial)
5 mV/div

section 4A (axial)
5 mV/div

section 3A (axial)
5 mV/div

0.1 msec/div

maximum pulse amplitude: 2111 N

h = 20 cm

Figure 4.9 Impact of the thin end with the 30 cm thick pendulum. It shows that the impact of a high impedance short rod with a low impedance long rod induces a nearly step wave in the long rod. The energy of the thick pendulum is slowly transferred to the thin long rod. The wave in the long rod actually decays by steps, where each step has twice the length of the pendulum. This experiment also demonstrates the nondispersive character of axial waves.
0.2 msec/div

pulse intensity in section 1: 42954 N
pulse intensity in sections 3A and 4A: 5690N
h = 40 cm

Figure 4.10 Impact at thick end with 30 cm thick pendulum. It shows an almost square pulse at section 1. In sections 3A and 4A the square pulses have the plateau with an increased slope and have developed a tail. As the pulses of Figure 4.6 do not show such a tail, we are lead to associate the tail with the presence of the taper.

$H_a$ (section 1) = 7.661 N·m
$H_a$ (section 3A, main pulse) = 3.297 N·m
$H_a$ (section 3A, three first pulses) = 5.428 N·m
figure 4.11 impact at thick end with 30 cm thick pendulum off center. this figure shows the dispersion of a short flexural pulse shown in section 1. notice the similarity with figure 4.7.
section 3A (axial)  
10 mV/div

section 3B (flexural)  
2 mV/div

section 4A (axial)  
10 mV/div

section 4B (flexural)  
5 mV/div

0.2 msec/div

pulse intensity at 3A and 4A ≥ 5215 N

h = 40 cm

Figure 4.12 Impact at thick end with 30 cm thick pendulum. It shows axial waves and practically no flexural waves. (Notice the high sensitivity of the flexural channels)
0.2 msec/div

pulse intensity at 3A and 4A: 3764 N

h = 20 cm

Figure 4.13 Impact at thick end with 15 cm thick pendulum. Note the similarity with Figure 4.12. The main square pulse is half the length.
section 3A (axial)
10 mV/div

section 3B (flexural)
2 mV/div

section 4A (axial)
10 mV/div

section 4B (flexural)
2 mV/div

0.2 msec/div

pulse intensity at 3A and 4A = 5200 N

h = 40 cm

Figure 4.14 Impact at thick end with 30 cm thick pendulum. This experiment has been performed after the introduction of the permanent deformation in the middle of the rod. Comparing with Figure 4.11, we notice that after deformation was introduced bending waves appeared in sections 3B and 4B, starting at the same point in time.
Figure 4.15

Same case as previous (figure 4.14) with longer time scale and less amplification for flexural waves. The ripples over the low frequency flexural waves are the flexural waves seen in Figure 4.14. The low frequency flexural waves have wave number $k_1 = 2.17 \text{m}^{-1}$, and wave length $\lambda_1 = 0.293 \text{m}$. The short flexural waves have $k_2 = 65.8 \text{m}^{-1}$, and $\lambda_2 = 0.096 \text{m}$. The short flexural waves are the second Fourier component of triangular waves (actual shape of irregularity).

$m = 41.3$

- Energy of main compressional pulse: $H_a = 3214 \text{ N} \cdot \text{m}$
- Total energy of first three pulses: $H_a \text{ total} = 4.6853 \text{ N} \cdot \text{m}$
- Total average energy of flexural waves (60 cm): $<H_f> = 0.2341 \text{ N} \cdot \text{m}$
- $<H_f>/H_a \text{ total} = 0.05$; $<H_f>/H_a = 0.074$; theory $<H_f>/H_a = 0.015$
- Average amplitude of long waves: 0.53mm
- Average amplitude of short waves: 0.024mm
- Total energy of short waves/total energy of long waves = 0.158
Figure 4.16 Same experiment as in Figure 4.15. It demonstrates that the flexural waves were actually generated in the middle of the beam and propagate to the left (sections 3B and 2) and to the right (sections 4A and 5) away from the region of initial irregularities. (See Figure 4.4).
Figure 4.17  Impact at thick end with 30 cm thick pendulum. Results are similar to the ones in Figure 4.15, but amplitudes are smaller.

\[ m = 24.1 \]
\[ H_a = 1.0935 \text{ N} \cdot \text{m} \]
\[ H_{a\text{total}} = 1.8635 \text{ N} \cdot \text{m} \]
\[ <H_f> = 0.092 \text{ N} \cdot \text{m} \]
\[ <H_f>/H_{a\text{total}} = 0.049 \]
\[ <H_f>/H_a = 0.084 \]

theory \[ <H_f>/H_a = 0.015 \]

- average amplitude of long waves: 0.33 mm
- average amplitude of short waves: 0.013 mm
- total energy of short waves/total energy of long waves: 0.119
Figure 4.18  Impact at thick end with 30 cm thick pendulum

\[ m = 13 \]
\[ H_a = 0.3160 \text{ N} \cdot \text{m} \]
\[ H_{atotal} = 0.4756 \text{ N} \cdot \text{m} \]
\[ \langle H_f \rangle = 0.043 \text{ N} \cdot \text{m} \]
\[ \langle H_f \rangle / H_{atotal} = 0.0904 \]
\[ \langle H_f \rangle / H_a = 0.1361 \]
\[ \text{theory} \langle H_f \rangle / H_a = 0.0146 \]

- average amplitude of long waves: 0.225 mm
- average amplitude of short waves: 0.009 mm
- total energy of short waves/total energy of long waves: 0.108
1 msec/div

h = 0.5 cm

Figure 4.19  Impact at thick end with 30cm thick pendulum

\[ m = 4.2 \]
\[ H_a = 0.0331 \text{ N} \cdot \text{m} \]
\[ H_{a\text{total}} = 0.0596 \text{ N} \cdot \text{m} \]
\[ <H_f> = 0.0062 \text{ N} \cdot \text{m} \]
\[ <H_f>/H_{a\text{total}} = 0.0904 \]
\[ <H_f>/H_a = 0.1361 \]

theory \[ <H_f>/H_a = 0.0145 \]

- average amplitude of long waves: 0.1mm
- average amplitude of short waves: 0.004mm
- total energy of long waves/total energy of short waves = 0.148
Figure 4.20  Impact of thick end with 15 cm thick pendulum

\[ m = 43.2 \]
\[ H_a = 1.7557 \text{ N} \cdot \text{m} \]
\[ H_{\text{total}} = 2.9295 \text{ N} \cdot \text{m} \]
\[ \langle H_f \rangle = 0.0900 \text{ N} \cdot \text{m} \]
\[ \langle H_f \rangle / H_{\text{total}} = 0.044 \]
\[ \langle H_f \rangle / H_a = 0.1046 \]

theory \[ \langle H_f \rangle / H_a = 0.038 \]
Figure 4.21  Impact of thin end with 30 cm thin pendulum

\[ m = 12.8 \]
\[ H_a = 0.3084 \text{ N}\cdot\text{m} \]
\[ <H_f> = 0.0185 \text{ N}\cdot\text{m} \]
\[ \frac{<H_f>}{H_a} = 0.06 \]

theory \[ \frac{<H_f>}{H_a} = 0.0181 \]
Figure 4.22 Variation of the energy ratio of $\langle H_F^r \rangle / H_a$ with the load ratio $m$. 

$m = \frac{P}{P_1}$

+ experimental points

--- approximate theory

$0.015$ $0.1$ $0.2$ $\langle H_F^r \rangle / H_a$
4.5 Conclusions and Recommendations

We have in this chapter studied the problem of the interaction of a short high intensity compressional pulse with the initial imperfections of a long slender rod.

We have obtained the equations of motion for a rod with initial imperfections, and with coupled transverse and longitudinal motion. Instead of solving such a system of equations, we have computed numbers by using an approximate theory and then performed experiments to verify the results. The experimental results show that the phenomenon of conversion of compressional into bending energy is significant. It is interesting to notice that in this transformation the energy is slowed down, because flexural waves are much slower than compressional waves.

We consider as our most important contribution in this chapter the idea of studying a phenomenon which has not yet been treated in the literature.

As discussed in the Introduction, the results shown in Figures 4.7 up to 4.11 have a fine educational value.

The results of the approximate theory were, in general, low as compared to the experimental results. Therefore, such a theory cannot be used to predict results, but it might be useful to obtain lower-bound results.
We would like now to make some recommendations for future experimental work. The first item is that it would be convenient to avoid the use of the taper. To obtain high intensity pulses with the thin rod, one can use a track and shut the rod with an air gun. The other item is that one should try to use shorter wavelengths for the initial imperfections in order to obtain faster flexural waves. Finally, it would be convenient to have the strain gages very close to the ends of the deformed region to read the waves as soon as they start.
REFERENCES


Cremer, L., and M. Heckl, 1967, Körperschall, Springer-Verlag, Berlin, also published in English under the title "Structure Borne Sound," 1973. As noticed in our Chapter 2, after equation (2.51a), their equation (78b) in their Chapter 4 does not seem to be correct. It is interesting to notice that the same equation appears in the English edition and in the same incorrect form.


APPENDIX A

SHEAR CORRECTION COEFFICIENT
Table A.1 has been directly reproduced from Cowper (1966).

Table A.2 contains the numerical values of the shear correction coefficient $\kappa$ for the I-beams listed in the Manual of Steel Construction (1973). The list is complete in the sense that it includes all groups of beams. However, not all beams were considered within a group because of the slow variation of $\kappa$. The symbols used in the second and third columns of Table A.2 are defined in Table A.1. Note that for the I-beams of Table A.2, we have:

\[
0.195 \leq \kappa \leq 0.589 \quad \text{(A.1)}
\]

Using Equation (A.1) into Equation (2.9a) and assuming the Poisson ratio for steel to be $\nu = 0.275$, we obtain

\[
2.08 \leq \bar{a} \leq 3.62 \quad \text{(A.2)}
\]

Table A.1  Formulas for the shear correction coefficient $k$; $
u = \text{Poisson's ratio}; \text{neutral axis is shown as a chain-dotted line (from Cowper, 1966).}$

<table>
<thead>
<tr>
<th>Shape</th>
<th>Formula</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIRCLE</td>
<td>$k = \frac{6(1+\nu)}{7+6\nu}$</td>
<td>0.886</td>
</tr>
<tr>
<td>HOLLOW CIRCLE</td>
<td>$k = \frac{6(1+\nu)(1+m^2)}{(7+8\nu)(1+m^2)^2 + (20+12\nu)m^2}$</td>
<td>$\frac{7.8}{8.8}(1+m^2)^2 + 23.6m^2$</td>
</tr>
<tr>
<td>RECTANGLE</td>
<td>$k = \frac{10(1+\nu)}{12+11\nu}$</td>
<td>0.850</td>
</tr>
<tr>
<td>ELLIPSE</td>
<td>$k = \frac{12(1+\nu)a^2 (3a^2 + 5^2)}{(40+37\nu)a^2 + (16+10\nu)a^2 + \nu b^4}$</td>
<td>$\frac{15.6a^2 (3a^2 + 5^2)}{51.1a^2 + 19a^2 b^2 + 0.3b^4}$</td>
</tr>
<tr>
<td>SEMICIRCLE</td>
<td>$k = \frac{1+\nu}{1.305 + 1.273\nu}$</td>
<td>0.771</td>
</tr>
<tr>
<td>THIN-WALLED ROUND TUBE</td>
<td>$k = \frac{2(1+\nu)}{4+3\nu}$</td>
<td>0.531</td>
</tr>
</tbody>
</table>
Table A.1 (continued)

<table>
<thead>
<tr>
<th>Thin-Walled Square Tube</th>
<th>for $\nu = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = \frac{20(1+\nu)}{48 + 39\nu}$</td>
<td>0.436</td>
</tr>
</tbody>
</table>

Thin-Walled I-Section

$K = \frac{10(1+\nu)(1+3\nu)^2}{(12 + 72m + 150m^2 + 90m^3) + \nu(11 + 66m + 135m^2 + 90m^3) + 30m^2(3m + m^2) + 5m^2(8m + 9m^3)}$

WHERE $m = 2b_1/ht_w$, $n = b/h$

Thin-Walled Box Section

$K = \frac{10(1+\nu)(1+3\nu)^2}{(12 + 72m + 150m^2 + 90m^3) + \nu(11 + 66m + 135m^2 + 90m^3) + 10m^2(3m + m^0)}$

WHERE $m = 2b_h/ht$, $n = b/h$

Spar-And-Web Section

$K = \frac{10(1+\nu)(1+3\nu)^2}{(12 + 72m + 150m^2 + 90m^3) + \nu(11 + 66m + 135m^2 + 90m^3)}$

WHERE $m = 2\Lambda_0/ht$, $\Lambda_0$ - area of one spar

Thin-Walled T-Section

$K = \frac{10(1+\nu)(1+4m)^2}{(12 + 96m + 276m^2 + 192m^3) + \nu(11 + 88m + 264m^2 + 216m^3) + 30m^2(3m + m^2) + 10m^2(4m + 5m^1 + m^3)}$

WHERE $m = 3b_1/nt_1$, $n = b/h$
TABLE A.2 Numerical Values of the Shear Correction Coefficient $\kappa$ for the I-Beams Listed in the Manual of Steel Construction (1973) 7th Edition. The Values of $\kappa$ Were Computed with the Formulas of Table A.2 with Poisson's Ratio $\nu = 0.275$.

<table>
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<th>Nominal Size</th>
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<th>$tf/tw$</th>
<th>$\kappa$</th>
</tr>
</thead>
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<td>W36x300</td>
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<td>1.78</td>
<td>0.362</td>
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<td>x260</td>
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<td>1.71</td>
<td>0.370</td>
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<td>x230</td>
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<td>0.377</td>
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<td>W36x194</td>
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<td>0.462</td>
</tr>
<tr>
<td>x170</td>
<td>0.34</td>
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<td>0.465</td>
</tr>
<tr>
<td>x160</td>
<td>0.34</td>
<td>1.56</td>
<td>0.473</td>
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<tr>
<td>x135</td>
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<td>1.33</td>
<td>0.510</td>
</tr>
<tr>
<td>W33x240</td>
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<td>0.364</td>
</tr>
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<td>x200</td>
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<td>1.61</td>
<td>0.374</td>
</tr>
<tr>
<td>W33x152</td>
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<td>1.66</td>
<td>0.449</td>
</tr>
<tr>
<td>x118</td>
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<tr>
<td>W30x210</td>
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<td>W27x177</td>
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<td>Nominal Size</td>
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<td>tf/tw</td>
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<td>------------</td>
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<td>------</td>
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<td>$tf/tw$</td>
<td>$\kappa$</td>
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**M SHAPES**

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| M12x11.8     | .26 | 1.27  | .589        |
| M10x29.1 x22.9 | .63 | .91  | .436        |
|              | .61 | 1.61  | .325        |
| M10x9        | .27 | 1.31  | .569        |
| M8x34.3 x32.6 | 1.06| 1.21  | .250        |
|              | 1.05| 1.46  | .223        |
| M8x22.5 x18.5 | .71 | .94  | .397        |
|              | .69 | 1.53  | .305        |
| M8x6.5       | .29 | 1.40  | .539        |</p>
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<td>x13</td>
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<td>1.46</td>
<td>.217</td>
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**S SHAPES**

<p>| S24x120     | .35   | 1.38   | .496   |
| x105.9      | .34   | 1.76   | .444   |
| S24x100     | .31   | 1.17   | .563   |
| x79.9       | .30   | 1.74   | .480   |
| S20x95      | .38   | 1.15   | .520   |
| x85         | .37   | 1.40   | .479   |
| S20x75      | .33   | 1.23   | .536   |
| x65.4       | .33   | 1.58   | .484   |
| S18x70      | .36   | .97    | .568   |
| x54.7       | .35   | 1.50   | .480   |
| S15x50      | .39   | 1.13   | .513   |
| x42.9       | .38   | 1.51   | .453   |
| S12x50      | .48   | .96    | .495   |
| x40.8       | .46   | 1.43   | .417   |
| S12x35      | .44   | 1.27   | .455   |
| x31.8       | .44   | 1.55   | .413   |
| S10x35      | .52   | .83    | .508   |
| x25.4       | .49   | 1.58   | .380   |</p>
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**HP SHAPES**

| HP14x117 x73 | 1.11 | 1.00 | .271 |
|              | 1.11 | 1.00 | .271 |
| HP12x74 x53  | 1.06 | 1.00 | .281 |
|              | 1.06 | 1.00 | .281 |
| HP10x57 x42  | 1.08 | 1.00 | .277 |
|              | 1.08 | 1.00 | .276 |
| HP8x36       | 1.08 | 1.00 | .278 |
APPENDIX B

EQUATIONS OF MOTION AND TOTAL MOBILITIES FOR APPROXIMATE BEAM THEORIES
# APPENDIX B

## EQUATIONS OF MOTION AND TOTAL MOBILITIES FOR APPROXIMATE BEAM THEORIES

### TABLE B.1: Description and Equations of Motion for Five Approximate Beam Theories

<table>
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<tr>
<th>Model</th>
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<th>Equations of Motion</th>
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<td>[ \frac{\partial^4 \eta}{\partial x^4} + \frac{1}{a_2^2} \frac{\partial^2 \eta}{\partial x^2 \partial t^2} + \frac{1}{a_2^2} \frac{\partial^2 \eta}{\partial t^4} = 0 ]</td>
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<tr>
<td></td>
<td>Transverse Inertia</td>
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<td>Shear Deformation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bending Deformation</td>
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<tr>
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<td>Rotatory Inertia</td>
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<tr>
<td>Bernoulli-Euler</td>
<td>Transverse Inertia</td>
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<td>Beam</td>
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<td>Bending Deformation</td>
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<tr>
<td></td>
<td>Rotatory Inertia</td>
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<tr>
<td>Rayleigh beam</td>
<td>Transverse Inertia</td>
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<td>Shear Bending</td>
<td>Transverse Inertia</td>
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### TABLE B.2: Nondimensional Propagation Constants and Total Mobility Matrix Elements for Five Approximate Beam Theories

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<thead>
<tr>
<th></th>
<th>Shear Beam</th>
<th>Bernoulli- Euler Beam</th>
<th>Rayleigh Beam</th>
<th>Shear-Bending Beam</th>
<th>Timoshenko Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{k}<em>{1,2} = \frac{\gamma k</em>{1,2}}{a}$</td>
<td>$\frac{\overline{\omega}}{\omega}^{1/2}$</td>
<td>$(\frac{\overline{\omega}}{\omega})^{1/2}$</td>
<td>$\frac{\omega}{\sqrt{2}} \sqrt{1 + \frac{1}{\overline{\omega}} \left(\frac{2}{\omega}\right)^2}$</td>
<td>$\frac{\overline{\omega}}{\sqrt{2}} \sqrt{\frac{a^2}{\overline{\omega}} + \frac{(a^2 - 1)^2}{\omega}}$</td>
<td>$\frac{\omega}{\sqrt{2}} \sqrt{(1+a^2) + \frac{(1-a^2)^2}{\omega^2} + \frac{2}{\omega^2}}$</td>
</tr>
</tbody>
</table>

$\overline{M}_{FW} = (a_2 \rho a) \overline{M}_{FW}$

$= \frac{\overline{k}_1 + \overline{k}_2}{\omega_2} (1 + \frac{1}{\omega_2})$

$= \frac{\overline{k}_1 + \overline{k}_2}{\omega_2 (1 - i \omega_2)}$

$= \frac{\overline{k}_1 + \overline{k}_2}{\omega_2}$

$= \sqrt{\frac{\omega_2^2 - 1}{\omega_2 (\omega_2 + \frac{\omega_2^2 - 2}{\omega_2 a} - 1)}}$

$\overline{M}_{FW}$

$= (a_2 \rho a r) \overline{M}_{FW}$

$= - \frac{1}{\overline{\omega}_2}$

$= - \frac{1}{(1 - i \omega_2)}$

$= -1$

$= \frac{1}{(1 - i \omega_2)}$

$= \frac{i}{\omega_2}$

$= \frac{i}{\omega_2 + \sqrt{\frac{\omega_2^2 a}{\omega_2^2 a^2 - 1}}}$

$\overline{M}_{FW}$

$= (a_2 \rho a r) \overline{M}_{FW}$

$= - \frac{1/2}{\omega_2 (1 - i)}$

$= - \frac{i(\overline{k}_1 + \overline{k}_2)}{(1 - i \omega_2)}$

$= - \frac{i(\overline{k}_1 + \overline{k}_2)}{(1 - i \omega_2)}$

$= \frac{\overline{k}_1 + \overline{k}_2}{\omega_2 + \sqrt{\frac{\omega_2^2 a^2}{\omega_2^2 a^2 - 1}}}$

$\overline{M}_{FW}$

$= (a_2 \rho a r^2) \overline{M}_{FW}$

$= - \frac{1/2}{\omega_2 (1 - i)}$

$= - \frac{i(\overline{k}_1 + \overline{k}_2)}{(1 - i \omega_2)}$

$= - \frac{i(\overline{k}_1 + \overline{k}_2)}{(1 - i \omega_2)}$

$= \frac{\overline{k}_1 + \overline{k}_2}{\omega_2 + \sqrt{\frac{\omega_2^2 a^2}{\omega_2^2 a^2 - 1}}}$

$\overline{M}_{FW}$
APPENDIX C

LOADING IMPEDANCES
TABLE C.1: Impedances of Loading Elements for Periodic Structures

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<td>$Z_m$</td>
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<td>$m$</td>
<td>$-i\omega m$</td>
<td>$-i\omega J$</td>
</tr>
<tr>
<td>$K$</td>
<td>$\frac{iK}{\omega}$</td>
<td>$\frac{iK_F}{\omega}$</td>
</tr>
<tr>
<td>$C$</td>
<td>$C_r$</td>
<td>$C$</td>
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<tr>
<td>$\frac{1}{ik} \frac{1}{i\omega m}$</td>
<td>$\frac{1}{\omega - \frac{1}{iK}} \frac{1}{i\omega m}$</td>
<td>$\frac{1}{\omega_2 - \frac{1}{iK}} \frac{1}{i\omega_2 m L}$</td>
</tr>
</tbody>
</table>

The following definitions apply:

- for strings:
  \[
  \omega = \omega_L \sqrt{\frac{\rho A}{T}} \quad \bar{K} = \frac{K}{T/L} \quad \bar{m} = \frac{m}{\rho A L} \quad \bar{C} = \frac{C}{\sqrt{\rho A T}} \quad \bar{L} = L/r \quad \bar{J} = \frac{J}{\rho IL} \quad \bar{K}_r = \frac{K_r}{a_2^2 \rho I/r} \quad \bar{E} = \frac{E}{E_L/E} \quad \bar{I} = \frac{I}{I_L/I}
  \]

- for beams:
  \[
  \omega_2 = \frac{\omega r}{a_2} \quad \bar{K} = \frac{K}{a_2 \rho A / r} \quad \bar{C} = \frac{C}{a_2 \rho A} \quad \bar{A} = \frac{A_L}{A} \quad \bar{I} = \frac{I_L}{I}
  \]

the subscript L stands for load
APPENDIX D

DISPERSION RELATIONS FOR PERIODIC STRUCTURES
APPENDIX D

DISPERSION RELATIONS FOR PERIODIC STRUCTURES

This appendix contains, in graphical form, the dispersion relations for the nodal waves in periodically loaded strings, Timoshenko beams, and Bernoulli-Euler beams, for some specific sets of parameters.

For the string and the Timoshenko beam cases, we have plotted the curves for a similar case with two different values of the loading parameters, in order to study the effect of such variations on the behavior of the system.

The curves for the dispersion relations in periodic Bernoulli-Euler beams were obtained with the purpose of comparison with the periodic Timoshenko beam results.

In general, in the graphs, full lines correspond to the real part $\mu_r$, and the lines made up of crosses correspond to the imaginary part $\mu_i$ of the propagation constant $\mu$. 
1. DISPERSION RELATIONS FOR THE NODAL WAVES
   IN PERIODIC STRINGS
\[ \bar{\omega} = \omega L \sqrt{\frac{\rho A}{T}} \]

Figure d1.1 Dispersion relation for the nodal waves in a periodic string loaded with masses, \( \bar{m} = 0.5 \)
Figure d1.2 Dispersion relation for the nodal waves in a periodic string loaded with masses, $\bar{m} = 2.0$
\[ \tilde{\omega} = \omega L \sqrt{\frac{\rho A}{T}} \]

\[ \tilde{\mu} = \tilde{\mu}_L \]

**Figure dl.3** Dispersion relation for the nodal waves in a periodic string loaded with springs, \( \overline{K} = 2.0 \)
\[ \bar{\omega} = \omega L \sqrt{\frac{\rho A}{T}} \]

Figure d1.4 Disperions relation for the nodal waves in a periodic string loaded with springs, \( \bar{K} = 4.0 \)
\( \bar{\omega} = \omega L \sqrt{\frac{\rho A}{T}} \)

\( \bar{\mu} = \mu L \)

Figure d1.5 Dispersion relation for the nodal waves in a periodic string loaded with mass-spring systems, \( \bar{m} = 0.5 \) and \( \bar{K} = 2.0 \)
\[ \bar{\omega} = \omega L \sqrt{\frac{\rho A}{T}} \]

Figure d1.6 Dispersion relation for the nodal waves in a periodic string loaded with mass-spring systems, \( \bar{m} = 2.0 \) and \( \bar{K} = 4.0 \)
Figure 11.7 Dispersion relation for the nodal waves in a periodic string loaded with dashpots, $C = 0.1$.

$$\omega = \sqrt{\frac{\rho A}{T_p}}$$
\[ \bar{\omega} = \omega L \sqrt{\frac{\rho A}{T}} \]

Figure d1.8 Dispersion relation for the nodal waves in a periodic string loaded with dashpots, \( C = 1.0 \)
2. DISPERSION RELATIONS FOR THE NODAL WAVES
IN PERIODIC TIMOSHENKO BEAMS
\[ \overline{\omega}_2 = \frac{\omega r}{a_2} \]

Figure d2.1 Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with masses, \( \overline{m} = 0.5 \), \( \overline{J} = 0 \).
\[ \omega^2 = \frac{\omega_r}{a^2} \]

\[ \bar{\mu}_2 = \mu_2 L \]

Figure d2.2 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with masses, \( \bar{m} = 0.5, \bar{f} = 0 \)
\[ \bar{\omega}_2 = \frac{\omega r}{a_2} \]

\[ \bar{\mu}_1 = \nu_2 L \]

Figure d2.3 Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with masses, \( \bar{m} = 2 \), \( \bar{J} = 0 \).
\[ \bar{\omega}_2 = \frac{\omega r}{a^2} \]

Figure d2.4 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with masses, \( \bar{m} = 2, \bar{J} = 0 \)
Figure d2.5  Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with masses, $\bar{m} = 0.5$, $\bar{J} = 0.1$
\[ \tilde{\omega}_2 = \frac{\omega_r}{a_2} \]

\[ \tilde{\mu}_2 = \mu_2 L \]

Figure d2.6 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with masses, \( \bar{m} = 0.5 \), \( \bar{J} = 0.1 \)
Figure d2.7 Disperions relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with masses, $\bar{m} = 2, \bar{J} = 0.2$
Figure d2.8 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with masses, \( \bar{m} = 2 \), \( \bar{J} = 0.2 \)

\[ \omega_2 = \frac{\omega r}{a_2} \]

\( \bar{L} = 4.0 \)

\( \bar{a} = 2.3 \)

\( \bar{\mu}_2 = \mu_2 L \)
Figure d2.9 Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with translational springs, $\bar{K} = 0.5$
Figure d2.10 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with translational springs, $K = 0.5$.
Figure d2.11 Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with translational springs, $\bar{K} = 2.0$.
\[ \omega_2 = \frac{\omega r}{a_2} \]

\[ \bar{\nu}_2 = \nu_2L \]

\[ \bar{L} = 4.0 \]

\[ \bar{a} = 2.3 \]

Figure d2.12 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with translational springs, \( \bar{K} = 2.0 \)
\bar{\omega}_2 = \frac{\omega r}{a_2}

\bar{\mu}_{1r}

+++ \bar{\mu}_{1i}

\bar{L} = 4.0
\bar{a} = 2.3

\bar{\mu}_1 = \mu_1^L

Figure d2.13 Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with translational dashpots, \( \bar{C} = 0.1 \)
Figure d2.14 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with translational dashpots, \( \bar{C} = 0.1 \)
Figure d2.15 Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with translational dashpots, \( \bar{C} = 0.4 \)
Figure d2.16  Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with translational dashpots, $\overline{C} = 0.4$
Figure d2.17 Dispersion relation for the first mode of nodal waves in a periodic Timoshenko beam loaded with transverse built-in plates, $\bar{E} = 1.0$, $\bar{\rho} = 1.0$, $\bar{A} = 0.1$, $\bar{I} = 0.01$
\[ \tilde{\omega}_2 = \frac{\omega_r}{a_2} \]

---

\[ \tilde{\mu}_2 = \mu_2 L \]

---

Figure 2.18 Dispersion relation for the second mode of nodal waves in a periodic Timoshenko beam loaded with transverse built-in semi-infinite plates, \( \bar{E} = 1.0, \bar{\rho} = 1.0, \bar{A} = 0.1, \bar{I} = 0.01 \)
Figure d2.19 Dispersion relation for the first mode of the nodal waves in a periodic Timoshenko beam loaded with transverse built-in semi-infinite plates, $\overline{E} = 1.0$, $\overline{\rho} = 1.0$, $\overline{A} = 0.2$, $\overline{I} = 0.1$
\[ \bar{\omega}_2 = \frac{\omega r}{\bar{a}_2} \]

\[ \mu_2 = \mu_2 L \]

\[ \bar{L} = 4.0 \]

\[ \bar{a} = 2.3 \]

Figure d2.20 Dispersion relation for the second mode of the nodal waves in a periodic Timoshenko beam loaded with transverse built-in semi-infinite plates, \( \bar{E} = 1.0, \bar{\rho} = 1.0, \bar{A} = 0.2, \bar{I} = 0.1 \)
3. DISPERSION RELATIONS FOR THE NODAL WAVES
   IN PERIODIC BERNOULLI-EULER BEAMS
\[ \overline{\omega}_2 = \frac{\omega r}{\overline{a}_2} \]

\[ \overline{\mu}_1 = \mu_1 L \]

Figure d3.1 Dispersion relation for the first mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with masses, \( \overline{m} = 2 \), \( \overline{J} = 0 \).
\[ \bar{\omega}_2 = \frac{\omega_1}{a_2} \]

\[ \bar{\mu} = \mu_2 L \]

\[ \bar{L} = 4.0 \]

Figure d3.2 Dispersion relation for the second mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with masses, \( \bar{m} = 2, \bar{J} = 0 \)
\[ \omega_2 = \frac{\omega_r}{a_2} \]

\( L = 4.0 \)

Figure d3.3 Dispersion relation for the first mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with masses, \( \bar{m} = 2, \bar{J} = 0.2 \)
Figure 3.4 Dispersion relation for the second mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with masses, $\bar{m} = 2$, $\bar{j} = 0.2$.

$\bar{\omega}_2 = \frac{\omega_2}{a_2}$

$\bar{\mu}_2 = \mu_2 L$

$L = 4.0$
\[ \bar{\omega}_2 = \frac{\omega r}{a_2} \]

\[ \bar{\mu}_1 = \mu_1 L \]

\[ L = 4.0 \]

Figure d3.5 Dispersion relation for the first mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with springs, \( \bar{K} = 2.0 \)
\bar{\omega}_2 = \frac{\omega_r}{a_2}

Figure d3.6 Dispersion relation for the second mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with springs, \bar{K} = 2.0
\[ \tilde{\omega}_2 = \frac{\omega r}{a_2} \]

--- \( \mu_{1r} \)

+++ \( \mu_{2i} \)

\[ \bar{L} = 4.0 \]

\[ \bar{\mu}_1 = \mu_1 L \]

**Figure d3.7** Dispersion relation for the first mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with dashpots, \( \bar{C} = 0.4 \).
Figure d3.8 Dispersion relation for the second mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with dashpots, $\bar{C} = 0.4$
Figure d3.9 Dispersion relation for the first mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with built-in transverse semi-infinite plate strips, $\bar{E} = 1.0$, $\bar{\rho} = 1.0$, $\bar{I} = 0.1$, $\bar{A} = 0.2$
Figure d3.10 Dispersion relation for the second mode of the nodal waves in a periodic Bernoulli-Euler beam loaded with built-in transverse semi-infinite plate strips, $\bar{E} = 1.0$, $\bar{\rho} = 1.0$, $\bar{I} = 0.1$, $\bar{A} = 0.2$
APPENDIX E

SOLUTION OF HOMOGENEOUS SYSTEM
OF EQUATIONS
APPENDIX E

SOLUTION OF HOMOGENEOUS SYSTEM OF EQUATIONS

The solution of the homogeneous system of equations (3.31) for \( W_{j1}^- \), \( W_{j2}^+ \) and \( W_{j2}^- \) as functions of \( W_{j1}^+ \) is given below by equations (e1), (e2) and (e3).

\[
A_1^- = \frac{W_{j1}^-}{W_{j1}^+} = \frac{\alpha_1^+}{\alpha_1^-} \frac{1 + B_1[B_2B_5 + B_3(B_7 - B_8)]}{B_{12}} \quad (e1)
\]

\[
A_2^+ = \frac{W_{j2}^+}{W_{j1}^+} = \frac{\alpha_1^+}{\alpha_2^+} \frac{1 + B_1[B_2(B_6 - B_{11}) - B_3B_4]}{B_{12}} \quad (e2)
\]

\[
A_2^- = \frac{W_{j2}^-}{W_{j1}^+} = \frac{\alpha_1^+}{\alpha_2^-} \frac{1 - B_1[B_2(B_6 - B_{10}) + B_3B_4]}{B_{12}} \quad (e3)
\]

The subscript to the symbol \( A \) indicates the mode of propagation of the particular internal wave and the superscript sign indicates its propagation direction.

The symbols \( \alpha_1^+, \alpha_1^-, \alpha_2^+ \) and \( \alpha_2^- \) have been defined in equations (3.32a–d) and the other symbols are defined below.

\[
B_1 = \frac{Z_F}{2 \omega (k_1^2 - k_2^2) G*A} \quad (e4)
\]

\[
B_2 = k_2 (a_2^2 k_1^2 - \omega^2) \quad (e5)
\]

\[
B_3 = k_1 (a_1^2 k_2^2 - \omega^2) \quad (e6)
\]
\[ B_4 = \frac{i \sin k_1 L}{\cos \bar{\mu} - \cos k_1 L} \]  
(e7)

\[ B_5 = \frac{i \sin k_2 L}{\cos \bar{\mu} - \cos k_2 L} \]  
(e8)

\[ B_6 = \frac{e^{i\bar{\mu}} - \cos k_1 L}{\cos \bar{\mu} - \cos k_1 L} \]  
(e9)

\[ B_7 = \frac{e^{i\bar{\mu}} - \cos k_2 L}{\cos \bar{\mu} - \cos k_2 L} \]  
(e10)

\[ B_8 = \frac{2}{1 - e^{-i\bar{\mu}} e^{ik_1 L}} \]  
(e11)

\[ B_9 = \frac{2}{1 - e^{-i\bar{\mu}} e^{-ik_1 L}} \]  
(e12)

\[ B_{10} = \frac{2}{1 - e^{-i\bar{\mu}} e^{ik_2 L}} \]  
(e13)

\[ B_{11} = \frac{2}{1 - e^{-i\bar{\mu}} e^{-ik_2 L}} \]  
(e14)

\[ B_{12} = 1 + B_1 [B_2 B_5 - B_3 (B_7 - B_9)] \]  
(e15)
APPENDIX F

THE FOURIER TRANSFORM OF A TRIANGULAR WAVE
APPENDIX F

THE FOURIER TRANSFORM OF A TRIANGULAR WAVE

Consider a triangular wave as shown in Figure f1.

\[ \eta(x) \]

\[ -\lambda/2 \quad \lambda/2 \]

\[ x \]

Figure f1. Triangular wave with wave length \( \lambda \) and amplitude equal to unity.

The Fourier transform of such a function is a train of impulses which has for envelope a \( \text{sinc}^2 a = (\sin a/a)^2 \) function. Figure f2 shows the transform.

\[ F \{ \eta(x) \} \]

\[ \pi \text{sinc}^2 (\pi/2) = 1.27 \]

\[ \pi \text{sinc}^2 (3\pi/2) = 0.141 \]

\[ \pi \text{sinc}^2 (5\pi/2) = 0.051 \]

Figure f2. Fourier transform of the triangular wave of figure f1.