ELECTROOSMOTIC DEWATERING OF WASTEWATER SLUDGES

by

LI-SHIANG LIANG

B.S., Massachusetts Institute of Technology (1970)
S.M., Massachusetts Institute of Technology (1971)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

OCTOBER 1976

Signature of Author ...........................................
Department of Mechanical Engineering, October 29, 1976

Certified by ..................................................
Thesis Supervisor

Accepted by ..................................................
Chairman, Department Committee on Graduate Students

MAR 8 1977
ELECTROOSMOTIC DEWATERING OF WASTEWATER SLUDGES

by

LI-SHIANG LIANG

Submitted to the Department of Mechanical Engineering on October 29, 1976 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

ABSTRACT

The theory of electroosmosis is reviewed and applied to models of porous media. Experimental data on Kaolin clay and digested sewage sludge support the "cluster model," in which a fraction of the fluid is bound within clusters of solid particles and unavailable for flow.

Models are presented for electroosmotic dewatering of wastewater sludges by filtration and by consolidation.

A major criterion for the utility of any sludge handling process is the cost of dewatering and disposal. We have therefore reviewed the theories for two conventional methods, sand-bed drying and vacuum filtration, and derived cost equations for comparison with equivalent equations for electroosmotic dewatering.

It is unlikely that electroosmosis will ever be an economically attractive method for dewatering ordinary wastewater sludges. A comparison between electroosmotic and pressure filtration of an anaerobically digested sewage sludge shows that the ratio of the electrical to the hydraulic energy required is about 10³.

However, electroosmosis may find applications in the dewatering of other sludges which are more resistant to the conventional dewatering methods. The conditions which favor electroosmosis are discussed in the thesis and listed in the Summary.

Thesis Supervisor: Ain A. Sonin
Title: Professor of Mechanical Engineering
ACKNOWLEDGEMENT

This research was motivated and guided by Professor Ain A. Sonin. I wish to express my gratitude for his patience and confidence, and his perceptiveness regarding complex situations, both scientific and human.

Professors Ronald Probstein and Richard Donnelly, the other two members of my thesis committee, have also given valuable advice and encouragement.

Many thanks to Cindy Polansky, who typed the thesis through long evenings and weekends; to Richard Fenner, who built most of the experimental apparatus; and to Charles Nakamura, who helped with the graphics.

This research was sponsored by the National Science Foundation under Grant No. ENG 74-20951 A01 of the Division of Engineering, Fluid Mechanics Section.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Acknowledgements</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>7</td>
</tr>
<tr>
<td>CHAPTER 1: INTRODUCTION</td>
<td>13</td>
</tr>
<tr>
<td>CHAPTER 2: THEORY OF ELECTROOSMOSIS</td>
<td></td>
</tr>
<tr>
<td>2.1 Introductory Remarks</td>
<td>18</td>
</tr>
<tr>
<td>2.2 Origin of Charge on Solids</td>
<td>21</td>
</tr>
<tr>
<td>2.3 Capillary Model</td>
<td>23</td>
</tr>
<tr>
<td>2.4 Application of Capillary Model to a Porous Medium</td>
<td>42</td>
</tr>
<tr>
<td>2.5 &quot;Efficiency&quot; of Electroosmosis</td>
<td>48</td>
</tr>
<tr>
<td>2.6 A Modification of the Simple Capillary Model: the Cluster Model</td>
<td>52</td>
</tr>
<tr>
<td>2.7 Summary</td>
<td>66</td>
</tr>
<tr>
<td>Table</td>
<td>69</td>
</tr>
<tr>
<td>Figures</td>
<td>70</td>
</tr>
<tr>
<td>CHAPTER 3: EXPERIMENTS ON ELECTROOSMOTIC FLOW</td>
<td>77</td>
</tr>
<tr>
<td>3.1 The Clay and Sludge Materials</td>
<td>78</td>
</tr>
<tr>
<td>3.2 Experimental Apparatus and Sample Preparation</td>
<td>80</td>
</tr>
<tr>
<td>3.3 Experimental Procedure</td>
<td>86</td>
</tr>
<tr>
<td>3.4 Experimental Results and Comparison with Cluster Model</td>
<td>88</td>
</tr>
<tr>
<td>3.5 Comparison of Cluster Model with Data from Previous Researchers</td>
<td>94</td>
</tr>
<tr>
<td>3.6 Summary and Conclusions</td>
<td>97</td>
</tr>
<tr>
<td>Tables</td>
<td>100</td>
</tr>
<tr>
<td>Figures</td>
<td>106</td>
</tr>
<tr>
<td>CHAPTER 6: DEWATERING BY DRYING BEDS</td>
<td>216</td>
</tr>
<tr>
<td>-----------------------------------</td>
<td>-----</td>
</tr>
<tr>
<td>6.1 Dewatering by Drainage</td>
<td>220</td>
</tr>
<tr>
<td>6.2 Dewatering by Evaporation; General Remarks</td>
<td>225</td>
</tr>
<tr>
<td>6.3 Evaporation; Theory</td>
<td>230</td>
</tr>
<tr>
<td>6.4 Total Dewatering Time</td>
<td>246</td>
</tr>
<tr>
<td>6.5 Cost Analysis</td>
<td>249</td>
</tr>
<tr>
<td>6.6 Design Application of Cost Equation</td>
<td>255</td>
</tr>
<tr>
<td>Tables</td>
<td>262</td>
</tr>
<tr>
<td>Figures</td>
<td>266</td>
</tr>
<tr>
<td>CHAPTER 7: VACUUM FILTRATION</td>
<td>280</td>
</tr>
<tr>
<td>7.1 Introductory Remarks</td>
<td>283</td>
</tr>
<tr>
<td>7.2 Filtration</td>
<td>288</td>
</tr>
<tr>
<td>7.3 Drying Period</td>
<td>299</td>
</tr>
<tr>
<td>7.4 Energy Requirement</td>
<td>303</td>
</tr>
<tr>
<td>7.5 Cost Analysis</td>
<td>309</td>
</tr>
<tr>
<td>7.6 Applications of the Cost Equation</td>
<td>322</td>
</tr>
<tr>
<td>7.7 Summary</td>
<td>325</td>
</tr>
<tr>
<td>Tables</td>
<td>328</td>
</tr>
<tr>
<td>Figures</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 8: SUMMARY AND CONCLUSION</td>
<td>336</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>341</td>
</tr>
<tr>
<td>APPENDIX A. CONCENTRATION POLARIZATION</td>
<td>346</td>
</tr>
<tr>
<td>APPENDIX B. DERIVATIONS OF EQUATIONS FOR CLUSTER MODEL</td>
<td>351</td>
</tr>
<tr>
<td>APPENDIX C. DATA AND CALCULATIONS FOR EXPERIMENT SERIES III</td>
<td>357</td>
</tr>
<tr>
<td>APPENDIX D. PATENTS BIBLIOGRAPHY</td>
<td>369</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>370</td>
</tr>
<tr>
<td>Biographical Note</td>
<td>380</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

FIGURE 1-1 Sources of sludge in a typical wastewater treatment plant

FIGURE 2-1 Schematic representation of the structure of the double layer according to Stern's theory (after Shaw21)

FIGURE 2-2 Pore model

FIGURE 2-3 Dimensionless potential distribution across a cylindrical pore for different values of the Debye ratio ($\lambda = \lambda_D/a$) and a constant surface potential of $\bar{\Psi}_k = 2.79$ (after Gross and Osterle30)

FIGURE 2-4 $\lambda << 1$

FIGURE 2-5 $\lambda >> 1$

FIGURE 2-6 $\eta$ vs. $(a/\lambda_D)$ for $\psi_k = 70$ mV (from Gross and Osterle30)

FIGURE 2-7 Cluster model

FIGURE 2-8 Relationship between $e_c$, $e_p$, and $e_T$

FIGURE 2-9 Current across a cross-section of the porous medium
FIGURE 3-1 Variation of $q_e/E$ with time, for a sample of raw Kaolin

FIGURE 3-2 Apparatus used by two previous researchers for measuring electroosmotic flow

FIGURE 3-3 Test cell

FIGURE 3-4 Compression cell for preparing samples
(a) Sewage Sludge
(b) Clay

FIGURE 3-5 Conductivity cell and bridge circuit

FIGURE 3-6 Experimental apparatus

FIGURE 3-7 Nonlinear relationship between the electroosmotic flow rate and the electric field

FIGURE 3-8 Electroosmotic permeability $k_e$ vs. porosity $n$, for sewage sludge

FIGURE 3-9 Dimensionless conductivity $\kappa/\sigma_0$ vs. porosity: experimental data for sewage sludge

FIGURE 3-10 $\eta$ vs. $(a_p/\lambda_D)$ for sewage sludge

FIGURE 3-11 $(q_e\sigma_0/i)$ vs. $(a_p/\lambda_D)$ for sewage sludge

FIGURE 3-12 $\eta$ vs. $(a/\lambda_D)$ for Kaolin clay

FIGURE 3-13 $(q_e\sigma_0/i)$ vs. $(a/\lambda_D)$ for Kaolin clay

FIGURE 3-14 $k_e$ vs. $n$; data from Olsen$^{33}$

FIGURE 3-15 $\kappa/\sigma_0$ vs. $n$; data from Olsen$^{33}$

FIGURE 3-16 $k_e$ and $\kappa/\sigma_0$ vs. $n$; data from Gray$^{34}$

FIGURE 3-17 $\eta$ vs. $(a_p/\lambda_D)$; data from Olsen$^{33}$ and Gray$^{34}$
FIGURE 4-1  An isolated charged particle in an electric field

FIGURE 4-2  A charged particle closely packed with others in a cake

FIGURE 4-3  Consolidation by pressure

FIGURE 4-4  Consolidation by electroosmosis with a porous anode

FIGURE 4-5  Details of the cake surface adjacent to a porous anode

FIGURE 4-6  Consolidation by electroosmosis, with an impermeable anode

FIGURE 4-7  Electroosmotic filtration with fixed anode

FIGURE 4-8  Electroosmotic filtration with movable anode

FIGURE 4-9  Schematic of a continuous electroosmotic filter with movable anode

FIGURE 4-10  Schematic of a continuous electroosmotic filter with fixed anode

FIGURE 5-1  Test cell

FIGURE 5-2  Experimental apparatus

FIGURE 5-3  Experiment III-1. Filtrate flow velocity, with and without electroosmosis, as function of time

FIGURE 5-4  Experiment III-1. Electroosmotic flow velocity vs. time

FIGURE 5-5  Experiment III-1. Electrical energy required per unit area vs. time

FIGURE 5-6  Experiment III-1. η vs. time

FIGURE 5-7  Experiment III-2. Filtrate volume per unit area vs. time: the numbers above the data points refer to sludge temperature in °C
FIGURE 5-8  Experiment III-2. Electrical energy required per unit area vs. time

FIGURE 5-9  Experiment III-2 \((\mu q_e \sigma_0 / i)\) vs. time

FIGURE 5-10 Experiment III-3. \(q_h\) and \(q_e\) vs. time

FIGURE 5-11 Experiment III-3. \(P_e\) vs. time

FIGURE 5-12 Experiment III-3. \((\eta)^{-1}\) vs. time

FIGURE 5-13 Experiment III-4. Filtrate volume per unit area vs. time

FIGURE 5-14 Experiment III-4. Electrical energy required per unit area vs. time

FIGURE 5-15 Determination of the polarization voltage \(\Delta \phi_e\) at the anode by extrapolation from voltages measured at two probes in the slurry

FIGURE 6-1  (a) Sand drying bed
(b) Paved drying bed

FIGURE 6-2  Typical dewatering curve

FIGURE 6-3  Relationship between two expressions of the moisture content

FIGURE 6-4  Profile of fluid pressure during drainage

FIGURE 6-5  Dewatering by evaporation

FIGURE 6-6  Idealized drawing of the movement of water by capillarity in a porous medium (after Sherwood

FIGURE 6-7  Dewatering by drainage; filtrate volume and sludge depth vs. time

FIGURE 6-8  Hysteresis in the relationship of mass transfer potential \(\Theta\) to the moisture content \(u\)
FIGURE 6-9 Typical data for coefficients in evaporation theory (after Luikov\textsuperscript{56})

FIGURE 6-10 Data on critical moisture content $U_{cr}$ vs. $(q_{m,w})$ for sewage sludge (from Nebiker\textsuperscript{60})

FIGURE 6-11 A possible explanation of the critical surface moisture content $(U_s)_{cr}$

FIGURE 6-12 Non-dimensional drying rate vs. $U^*$ (after Luikov\textsuperscript{56})

FIGURE 6-13 Model of evaporation during second falling rate period

FIGURE 6-14 Non-dimensional heat flux $\overline{q}(\Gamma)$ as a function of the non-dimensional variable $\Gamma$

FIGURE 6-15 Model of cracking in the sludge cake during drying

FIGURE 6-16 Dependence of the drying rate on $q_{me}$ and $H$

FIGURE 6-17 Sample calculation of cost effectiveness of chemical conditioning

FIGURE 6-18 Sample calculation of the effect of evaporation rate $q_{me}$ on the dewatering cost

FIGURE 7-1 Schematic of a continuous belt vacuum filter

FIGURE 7-2 Schematic diagram of cake illustrating internal variables and compaction of cake near the filter medium (after Tiller\textsuperscript{68})

FIGURE 7-3 Frictional drag and compressive forces on the solid particles in the cake (after Tiller\textsuperscript{68})

FIGURE 7-4 (a) Fluid saturated pores at the cake-air interface at the beginning of the dewatering period
   (b) Dewatering by consolidation
   (c) Dewatering by vacuum drying
FIGURE 7-5  Porosity variation in highly compressible filter cakes
   (a)  Data from Tiller and Cooper 75
   (b)  Schematic illustrating effect of increasing filtration pressure

FIGURE 7-6  $\Omega_b$ vs. $P$ [see Eqs. (7-81) and (7-82)]

FIGURE 7-7  Effect of $U_d$, the moisture content at disposal, on filtration cost ($C_v$), disposal cost ($C_d$), and total dewatering cost ($C_T$)

FIGURE 7-8  Constraints on $P$, the level of vacuum

FIGURE 7-9  Total dewatering cost $C_T$ as a function of moisture content at removal $U_d$, for three different solid input rates
CHAPTER 1:
INTRODUCTION

Electroosmosis refers to the movement of a fluid through a porous medium under an electric field. Since its discovery by Reuss in 1808, this electrokinetic phenomenon has been studied in many fields of science and has been applied to the dewatering of various types of suspensions and porous media. In this thesis we will investigate the use of electroosmosis in dewatering wastewater sludges.

Wastewater sludges are suspensions of fine solid particles in water; they are produced in many of the processes in wastewater treatment, as shown in Fig. 1-1. There are many methods of ultimate disposal of sludges, such as:

1. discharge into rivers, lakes, and oceans - to be prohibited in the near future by pollution regulations;
2. discharge into lagoons and stabilization ponds;
3. direct application onto land, with possible uses of the sludges as crop fertilizers;
4. landfill;
5. incineration.

The cost of disposal by the last two methods decreases with the water content of the sludge; the volume to be handled and transported is reduced and the intrinsic fuel value of the sludge is improved. Most sludges, however, cannot be dewatered readily, and a large fraction of the capital and operating cost of a wastewater treatment plant is committed to this
Two widely used dewatering methods are drying on sand beds and vacuum filtration. The initial steps in these two methods are similar: the sludge fluid is forced to follow through a porous filter medium under a pressure gradient (due to gravity or mechanically applied), and the sludge solid is retained on the filter medium in the form of a porous cake. The thickness of the cake increases with time, and its hydraulic resistance can dominate that of the filter medium. A major reason for the difficulty in filtration is the compressibility of the cake. Under the pressure gradient, the cake compacts and the sizes of the pores within decrease. Since the hydraulic flow rate through a pore is roughly proportional to the fourth power of the hydraulic radius of the pore, the overall flow rate through the cake can become economically small.

Electroosmotic flow through\(^{+}\) a porous medium, on the other hand, does not result in a net stress on the solid matrix. Furthermore, for typical sludges the electroosmotic flow rate through a pore in the cake is approximately proportional to the square of the hydraulic radius of the pore.\(^{++}\) For these reasons, electroosmosis would seem to be a promising method of sludge dewatering.

We will first briefly describe some applications of electroosmosis;

\(^{+}\)In filtration, the fluid flows through the cake; in consolidation, out of the cake.

\(^{++}\)As explained in Chapter 2, this approximation holds when the Debye length ratio \(\lambda\) [defined in Eq. (2-20)] is much less than one.
the reader is referred to Curtis, Casagrande, and Rampacek for detailed reviews. The first patent for an industrial application of electroosmosis, for the dewatering of peat, was awarded to von Schwerin in 1900. Subsequently, in 1915 he developed a revolving anode electrophoresis machine and an electroosmotic filter press, which were used for dewatering clay at two industrial plants in Germany and Czechoslovakia. The commercial success of these operations in 1920-1930's stimulated extensive research on the effect of variables such as voltage and current applied, electrolyte additions, and electrode spacings on the electroosmotic flow rate and energy requirement.

In 1937, Casagrande patented the method of dewatering and stabilizing soils and silt in situ by electroosmosis. This method has been used since then in a number of excavation and construction projects where necessity has justified the cost of the electrical energy required; it is still the subject of active research.

Electroosmosis can dewater and harden the soil irreversibly; this effect has been attributed to the creation of negative pore fluid pressures during electroosmosis and the consequent soil consolidation, to chemical reactions between the products of anodic decomposition and the soil minerals, and to ion-exchange. Electroosmosis has also been used to inject electrolytes into soils, again for electrochemical hardening and stabilization.

Another proposed application of electroosmosis is to increase the flow rate of reservoir fluid during petroleum recovery. To my knowledge, this application has not been tested outside of the laboratory.
The U.S. Bureau of Mines has conducted over the years a series of experiments on electroosmotic dewatering of clays and mining products.\textsuperscript{5} The most recent work is by Sprute and Kelsh,\textsuperscript{14-17} who have successfully dewatered mine tailings in hydraulic backfills.

There have been few previous publications on electroosmotic dewatering of sewage sludges. Beaudoin\textsuperscript{18} in 1943 described experiments on the use of electroosmosis to further dewater filter cakes formed by vacuum filtration. Cooling et al.\textsuperscript{19} in 1952 reported success in accelerating the rate of drainage of water from sludge on drying beds. These authors, however, conclude that electroosmosis is uneconomical due to the high energy consumption and the rapid electrochemical corrosion of the metal anodes.

The most extensive investigation to date is by Greyson,\textsuperscript{20} who conducted experiments with various types of wastewater sludges and proposed a device for practical application. He also found that anodes of stannic oxide ceramics are oxidation-resistant and relatively inexpensive.

Several patents have been awarded for devices to electroosmotically dewater sewage sludges; they are included in the bibliography.

It is the intention of this thesis to first review the theory of electroosmosis, and then to apply the theory to models of filtration and consolidation by electroosmosis. A cost analysis is included for dewatering by electroosmosis, as well as by two conventional methods of vacuum filtration and drying beds.
FIGURE 1-1. Sources of sludges in a typical wastewater treatment plant.
CHAPTER 2:
THEORY OF ELECTROOSMOSIS

Electrokinetic phenomena (including electroosmosis, electrophoresis, streaming potential, and sedimentation potential) can occur when a solid with a net electrical charge of one polarity is in contact with a polar (e.g., aqueous) liquid containing ions. Ions of the opposite charge polarity (counterions) are attracted to the solid, while ions of the same polarity (co-ions) are repelled. In equilibrium the electrostatic forces on the ions are balanced by diffusional forces, so that a distribution of ions occurs in the "double layer" adjacent to the solid with an excess of counterions. When an electric field is applied tangentially to the solid surface, the net charge in the double layer will cause a net migration of ions towards one of the electrodes, with momentum transfer to the liquid through viscous drag. If the solid is constrained from moving, then the resulting flow of liquid relative to the solid is called electroosmosis.

The study of electrokinetic phenomena has been closely associated with the field of surface and colloid chemistry, particularly with regard to the origin and distributions of charges at the solid-liquid interface. The reader is referred to the bibliography for discussions of double layer theory.

The model of the double layer which we will adopt is shown in Fig. 2-1. The double layer consists of an immobile Stern layer and a diffuse layer. There are three electrical potentials to note: the solid potential $\psi_0$, 


the Stern potential $\psi_\sigma$ at the outer edge of the Stern layer, and the
zeta potential $\zeta$ at the plane of shear between the bulk liquid and the
sheath of liquid molecules bound to the solid. The shear plane can also
be regarded as a region of rapidly changing liquid viscosity.\textsuperscript{21} Since
its exact location is unknown, the usefulness of the zeta-potential has
been questioned.\textsuperscript{22} For simplicity, however, we will assume that $\zeta$ can
be defined in terms of its effect on electroosmotic flow, and refer to
it also as an effective "surface potential" $\psi_S$. Similarly, the net
charge exposed to the bulk liquid at the shear plane (solid charge minus
the charge in Stern layer) will be referred to as the effective "surface
charge" $\sigma_S$. The Debye length is defined by Eq. (2-21), and its signi-
ficance will be discussed later. It is a measure of the distance from
the outer edge of the Stern layer into the bulk solution over which the
surface potential or charge exerts an influence on the ionic distribution.

Another field which has contributed to the understanding of electro-
kinetic phenomena is irreversible thermodynamics.\textsuperscript{23-24} The various fluxes
through the porous medium of fixed solid are related to the driving forces
by phenomenological coefficients cross-coupled through Onsager relations.
The coefficients, however, can be obtained only through experiments or
through separate models of the physico-chemical structure of the porous
medium.

Consequently, macroscopic, mechanistic models for a porous medium
have been developed; the fluxes of ions and liquid are assumed to be
governed by continuum mechanics. Manning, for example, has studied a
model in which the solid is in the form of parallel plates, with fixed charges forming a periodic lattice. \(^{25,26}\) The most widely used model, however, is the capillary tube model, in which the pore spaces are assumed to be circular tubes of uniform radii. \(^{27-31}\) The solid surface is usually assumed to have either a uniform, constant density of charges or a constant potential with respect to the bulk solution. The basic capillary tube model has been modified and applied to tubes of elliptical cross-section (Koh and Anderson\(^{31}\)) and of rectangular cross-section (Yung\(^{31}\)), as well as to circular tubes of different radii in series (Yung\(^{31}\)).

In Section 2.2 of this chapter, we will discuss briefly the origins of charges on clays and on sewage sludges. In Section 2.3 we will review the theory for the capillary tube model; in Section 2.4 the model for a single pore is related to the porous medium through bulk macroscopic quantities, such as porosity, specific surface area, and permeability. In Section 2.5 we will discuss the "efficiency" of electroosmosis, and in Section 2.6 we will consider a cluster model for the porous medium, which has been applied successfully by Olsen\(^{33}\) to hydraulic flow through saturated clays. We will derive the electroosmotic and current flow through such a model, and show in Chapter 3 that the model can be used to explain electroosmotic flow data by previous workers. \(^{33-35}\)
SECTION 2.2. ORIGINS OF CHARGE ON SOLIDS

The sources of inputs to a wastewater treatment plant are usually numerous and varied, so that it is practically impossible to predict the constituents and properties of the sludges produced in the treatment processes. For example, municipal wastewater is a combination of industrial and domestic wastewater, as well as surface and street runoffs. Therefore we can at best mention the possible origins of charge at the solid-liquid interfaces in wastewater sludges without being quantitatively precise.

(1) Acid-base equilibria in inorganic and organic compounds. The charge on amino acids, for example, depend on the pH in the following manner:

\[
\begin{align*}
\text{low pH:} & \quad R - CH_2 - CO_2H & \text{charge} & +1 \\
& \quad NH_3^+ \\
\text{high pH:} & \quad R - CH_2 - CO_2^- & \text{charge} & -1 \\
& \quad NH_2 \\
\text{intermediate pH:} & \quad R - CH_2 - CO_2^- & \text{charge} & 0 \\
& \quad (\text{isoelectric}) & \quad NH_3^+ 
\end{align*}
\]

The surface charge on microorganisms can sometimes be explained in terms of charged groups similar to the amino and carboxyl groups above.\textsuperscript{50}
(2) **Lattice imperfections in crystalline inorganic materials such as clays.** The isomorphous substitution of $\text{Al}^{+3}$ for $\text{Si}^{+4}$ and $\text{Mg}^{+2}$, $\text{Li}^+$, etc. for $\text{Al}^{+3}$ in the clay lattice results in net negative charges, which are relatively constant and independent of the ion concentration and pH of the fluid.\textsuperscript{22}

(3) **Physical adsorption due to Van der Waals attraction and hydrogen bonding.** The occurrence of negative charges on hydrocarbons, for example, is due to the preferential adsorption of simple anions (e.g., $\text{Cl}^-$) over simple cations, enhanced by the hydrophobic nature of the interface with water.\textsuperscript{51} The cations are more polarizable and have smaller hydrated radii than cations.

(4) **Specific chemical adsorption due to chemical bonds between charged groups and the surface of the solid.**

(5) **Adsorption of ions which are identical to those in the solid lattice.** For example, the charge on aluminum hydroxide sols depends on the relative concentrations of $\text{Al}^{+3}$ and $\text{OH}^-$ ions (potential-determining ions) which in turn depend on the pH. The charges on the broken edges of clay particles, where $\text{AlO}_3^{-3}$ complex ions are exposed to the fluid, are believed to be determined by a similar mechanism.\textsuperscript{22}
SECTION 2.3. CAPILLARY MODEL

We will model the pores in a porous medium as a collection of parallel cylindrical microcapillaries, each of radius \( a \) and length \( l \). In this section we will derive equations for the flow of liquid and ions through one of the capillaries, as shown in Fig. 2-2. The length \( l \) is assumed to be the product of the medium thickness \( h \) and a tortuosity \( \tau \); in practice \( \tau \) is an experimentally determined constant which not only specifies the effective pore length but in part also corrects for the deviation of the actual pore geometry from the idealized capillary tube model.

We will make the following assumptions:

(1) The fluid flowing through the pores is assumed to an ideal solution of fully dissociated, non-chemically-reacting salt, so that the activities of the ions can be approximated by the concentrations.

(2) Short range forces, such as van der Waals' and specific adsorption, are important only in the Stern layer. The ions in the diffuse layer are considered to be "point charges" subject only to electrostatic forces.

(3) The electrical nature of the solid affects electroosmosis only through an effective surface potential \( \psi_s \) (or zeta potential \( \zeta \)) and an effective surface charge density \( \sigma_s \) at the shear plane. In our analysis we will consider two special cases, and assume that at the pore walls, either \( \psi_s \) or \( \sigma_s \) is constant, regardless of the ion
concentration in the fluid.

(4) The fluid and ion fluxes are steady and isothermal.

(5) The Reynolds number based on radius $a$ and on average fluid velocity $\bar{v}$ in the pore is small:

$$\text{Re}_a = \frac{\bar{v} a}{\nu} \ll 1 ; \quad (2-1)$$

where $\nu$ is the kinematic viscosity of the fluid.

(6) For clarity, the solution is assumed to be of a single symmetrical binary salt, with ions of equal mobilities:$^+$

$$Z_1 = -Z_2 = Z$$
$$D_1 = D_2 = D , \quad (2-2)$$

where the subscript "1" refers to cations and "2" to anions.

(7) $\lambda/a >> 1$.

(8) The external solution on both sides of the porous medium is of uniform concentration $C_0$. $^{++}$

(9) The Péclet number, based on the Debye length $\lambda_D$ (defined later), is small:

$$\text{Pe}_{\lambda_D} = \frac{\bar{v} \lambda_D}{D} \ll 1 . \quad (2-3)$$

$^+$The general case of non-symmetrical salts of different mobilities has been studied by Sonin.$^{3}$

$^{++}$The problem of concentration polarization is discussed in Appendix A.
(10) The viscosity $\mu$ and the permittivity $\varepsilon$ of the solution is everywhere constant and equal to the bulk values.\textsuperscript{†}

The ion flux densities are given by the Nernst-Planck equation, which can be written in the following form under assumption (1):

$$\mathbf{j}_i = C_i \mathbf{v} - D_i \nabla C_i - \frac{Z_i F D_i}{RT} C_i \nabla \phi \ . \quad (2-4)$$

The terms on the right represent the contributions due to convection, diffusion, and electrostatic migration, respectively. $C_i$ is the molar concentration of ion species $i$; $D_i$ is the diffusion coefficient; and $Z_i$ is the valence. $\mathbf{v}$ is the fluid velocity vector; $F$ is Faraday's constant, $R$ the gas constant, $T$ the absolute temperature, and $\phi$ the electrical potential.

The fluid flux is described by the Navier-Stokes equation, with the inertial terms set to zero under assumption (5):

$$0 = -\nabla p + \mu \nabla^2 \mathbf{v} - F(Z_1 C_1 + Z_2 C_2) \nabla \phi \quad \Rightarrow \quad 0 = -\nabla p + \mu \nabla^2 \mathbf{v} - FZ(C_1 - C_2) \nabla \phi \quad ; \quad (2-5)$$

where $p$ is the fluid pressure. In an axially symmetric uniform capillary tube, the only non-zero component of $\mathbf{v}$ is the axial velocity $v$, for which the Navier-Stokes equation is:

\textsuperscript{†} There is disagreement as to the importance and magnitude of changes in $\mu$ and $\varepsilon$ within the double layer; for example, see Shaw.\textsuperscript{21}
\[ \frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) = \frac{\partial p}{\partial z} + FZ(C_1 - C_2) \frac{\partial \phi}{\partial z} \]  \quad (2-6)

The current density is given by:

\[ \mathbf{\dot{J}} = F(Z_1 \mathbf{j}_1 + Z_2 \mathbf{j}_2) \]

\[ = FZ(C_1 - C_2) \mathbf{v} - D \frac{\partial}{\partial z} \nabla (C_1 - C_2) - \frac{(ZF)^2D}{RT} (C_1 + C_2) \nabla \phi \]  \quad (2-7)

Let \( i_a \) be the axial component of \( \mathbf{\dot{J}} \) (the only non-zero component); then,

\[ i_a = FZ(C_1 - C_2) \mathbf{v} - D \frac{\partial}{\partial z} (C_1 - C_2) - \frac{(ZF)^2D}{RT} (C_1 + C_2) \frac{\partial \phi}{\partial z} \]  \quad (2-8)

The continuity equations for steady flow are:

\[ \nabla \cdot \mathbf{\dot{J}} = 0 \]  \quad (2-9)

\[ \nabla \cdot \mathbf{j}_1 = 0 \]  \quad (2-10)

\[ \nabla \cdot \mathbf{\dot{J}} = 0 \]  \quad (2-11)

Finally, the interaction between the electric field and the ion concentration is given by Poisson's equation,

\[ \nabla^2 \phi = -\frac{F}{e} \sum_i Z_i C_i = -\frac{F Z}{e} (C_1 - C_2) \]  \quad (2-12)

In cylindrical coordinates, with symmetry about the axis, the left hand side of Eq. (2-12) can be approximated as (to order of magnitude):

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \sim \frac{\phi}{\alpha^2} + \frac{\phi}{\kappa^2} \]  \quad (2-13)
For $\lambda \gg a$, the axial derivative may be neglected, so that Eq. (2-12) can be written as

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = -\frac{FZ}{\varepsilon} (C_1 - C_2) .
$$

(2-14)

In other words, at any small segment of the pore, the ion concentrations are in local quasi-equilibrium, determined solely by radial variation in $\phi$.

It is convenient then, to divide the potential $\phi$ into two parts:

$$
\phi(z,r) = \phi(z) + \psi(z,r) ,
$$

(2-15)

and write Poisson's equation as

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = -\frac{FZ}{\varepsilon} (C_1 - C_2) .
$$

(2-16)

Since there is no radial flux of ions and of fluid, we can integrate the radial component of the Nernst-Planck equation to obtain Boltzmann's equation:

$$
C_i(z,r) = C_i^0(z) \exp \left( -\frac{Z_i F \psi}{RT} \right) .
$$

(2-17)

As explained later, we can make the approximation:

$$
C_i^0(Z) = C_2^0(Z) = C(Z) .
$$

(2-18)

Using Eqs. (2-17) and (2-18), we can write Eq. (2-16) in the following non-dimensional form:

$$
\chi^2 \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial \bar{\psi}}{\partial \bar{r}} \right) = \sinh \bar{\psi} ,
$$

(2-19)
where
\[
\lambda = \frac{\lambda_D}{\alpha} \quad ; \quad \overline{r} = \frac{r}{\alpha} \quad ; \quad \overline{\psi} = \frac{ZF\psi}{RT}
\]  
(2-20)

and
\[
\lambda_D = \left[ \frac{eRT}{2(ZF)^2c} \right]^{1/2}.
\]  
(2-21)

\(\lambda_D\) is referred to as the Debye length, and \(\lambda\) as the Debye length ratio. The boundary conditions are:
\[
\overline{r} = 0 \quad \frac{\partial \overline{\psi}}{\partial \overline{r}} = 0 \quad \text{from axial symmetry} \quad (2-22)
\]
\[
\begin{cases}
\overline{\psi} = \overline{\psi}_s \\
\overline{r} = 1
\end{cases}
\]
and
\[
\frac{\partial \overline{\psi}}{\partial \overline{r}} = \frac{ZF\alpha\sigma_s}{RT\epsilon} \quad (2-24)
\]

Equation (2-24) relates \(\sigma_s\) to the potential distribution. We will consider two special cases:
\[
\psi_s = \text{constant} = \psi_k \quad \text{and} \quad \sigma_s \quad \text{is the dependent variable} \quad (2-25)
\]

or
\[
\sigma_s = \text{constant} = \sigma_k \quad \text{and} \quad \psi_s \quad \text{is the dependent variable.} \quad (2-26)
\]

Equation (2-19) can be solved analytically for a few special cases, as discussed later. In general, it can be solved only by numerical means, as done by Gross and Osterle\textsuperscript{30} for a constant surface potential of \(\overline{\psi}_k = 2.79\) and for various values of \(\lambda\). Their results are shown in Fig. 2-3.
We see that for $\lambda < .01$, the potential is zero over most of the pore cross-section, and for $\lambda > 10$, the potential is essentially uniform over the entire pore cross-section.

To derive the fluid velocity, we integrate the Navier-Stokes equation [Eq. (2-6)] with the following boundary conditions:

$$ r = 0 ; \quad \frac{\partial v}{\partial r} = 0$$

$$ r = a ; \quad v = 0 .$$

The result is:

$$ v(r) = \frac{FZ}{\mu} \int_0^r \left[ \int_0^{r_2} \int_0^{r_1} (C_1 - C_2) \frac{\partial \phi}{\partial z} dr_1 dr_2 + \frac{1}{4\mu} \frac{\partial \rho}{\partial z} (r^2 - a^2) \right] r \ dr \quad (2-28)$$

where $r_1$ and $r_2$ are dummy variables. The first term on the right is the electroosmotic flow velocity; the second is the familiar Poiseuille flow velocity.

Integration of Eqs. (2-32) and (2-8) over the pore cross-section will give the volume flow rate $V$ and the current $I$ through the pore, respectively.

$$ V = \frac{2\pi FZ}{\mu} \int_0^a \left[ \int_0^r \left[ \int_0^{r_2} \int_0^{r_1} (C_1 - C_2) \frac{\partial \phi}{\partial z} dr_1 dr_2 \right] r \ dr \right] \ dr \quad (2-29)$$

$$ I = FZ \int_0^a (C_1 - C_2) v 2\pi r \ dr - DFZ \int_0^a (C_1 - C_2) 2\pi r \ dr \quad (2-30)$$

$$ \quad - \frac{(ZF)^2D}{RT} \int_0^a (C_1 + C_2) \frac{\partial \phi}{\partial z} 2\pi r \ dr .$$
At this point it would be helpful to refer to Figs. 2-3 and 2-4.
For \( \lambda \ll 1 \), at the center of the pore \( \psi = 0 \) and the solution is electrically neutral, so that Eq. (2-18) holds exactly. From the continuity equation, we can show that:

\[
C(Z) = C_0 \tag{2-31}
\]

and

\[
\frac{\partial \phi}{\partial z} = \text{constant}. \tag{2-32}
\]

For \( \lambda_D/a >> 1 \), the entire pore is within the double layer (Fig. 2-5). Let "a" and "d" designate locations in the external solution just outside of the double layer, and "b" and "c" designate locations inside the pore entrance, as shown in Fig. 2-5. At equilibrium, with \( v = i = j_i = 0 \), the electrochemical potential of each ion specie is constant, so that with \( \psi_a = \psi_d = 0 \) and \( \psi_b = \psi_c = \psi_s \),

\[
\frac{C_{1b}}{C_0} = \frac{C_{1c}}{C_0} = \exp \left( -\frac{Z\psi_s}{RT} \right)
\]

and

\[
\frac{C_{2b}}{C_0} = \frac{C_{2c}}{C_0} = \exp \left( \frac{Z\psi_s}{RT} \right).
\tag{2-33}
\]

Equations (2-33) describe the well-known Donnan exclusion phenomenon.
For non-equilibrium, if assumption (9) holds, i.e.,

\[
Pe^\lambda_D = \frac{\bar{v} \lambda_D}{D} \ll 1, \tag{2-3}
\]

then the diffusion and migration terms dominate the ion fluxes at the
entrance regions, and Eq. (2-33) is still approximately true.\(^\dagger\) Equation (2-31) is therefore a good approximation regardless of \(\lambda\).

Equations (2-28), (2-29), and (2-30) can also be expressed in terms of the potential \(\psi\). Using Eqs. (2-6) and (2-16), the axial component of the Navier-Stokes equation can be written as follows:

\[
\frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) = -\frac{\varepsilon}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) \frac{\partial \phi}{\partial z} + \frac{\partial p}{\partial z}, \quad (2-34)
\]

which can be integrated with the boundary conditions:

\[
\begin{align*}
    r &= 0 ; \quad \frac{\partial v}{\partial r} = 0 ; \quad \frac{\partial \psi}{\partial r} = 0 \\
    r &= \alpha ; \quad v = 0 ; \quad \psi = \psi_s \quad \text{(not necessarily constant)}
\end{align*} \quad (2-35)
\]

\[
v(r) = -\frac{\varepsilon(\psi - \psi_s)}{\mu} \frac{\partial \phi}{\partial z} + \left( \frac{r^2 - \alpha^2}{4\mu} \right) \frac{\partial p}{\partial z}. \quad (2-37)
\]

Equation (2-38) can then be integrated over the pore cross-section to give \(V\) :

\[
V = -\int_0^\alpha \frac{\varepsilon(\psi - \psi_s)}{\mu} \frac{\partial \phi}{\partial z} 2\pi r \, dr - \frac{\pi \alpha^4}{8\mu} \frac{\partial p}{\partial z}. \quad (2-38)
\]

Finally we use Eqs. (2-17), (2-18) and (2-31) to rewrite Eq. (2-30) in the form:

\[
I = -2ZFC_0 \int_0^\alpha (\sinh \bar{\psi})v2\pi r \, dr - \sigma_0 \frac{\partial \phi}{\partial z} \int_0^\alpha (\cosh \bar{\psi})2\pi r \, dr, \quad (2-39)
\]

\(^\dagger\) For a more rigorous discussion, see Carlin.\(^3\)
where
\[ \sigma_0 = \frac{2(ZF)^2DC_0}{RT} = \text{conductivity of the external solution}. \quad (2-40) \]

Equations (2-19), (2-37), (2-38), and (2-39) can be solved sequentially in that order. These equations are the governing equations for electroosmosis, and in general can be solved only by numerical means.

We will next present the analytical solutions for three special cases:

1. \( \lambda = \frac{\lambda_D}{a} \gg 1 \)

2. \( \lambda = \frac{\lambda_D}{a} \ll 1 \)

3. \( \bar{\psi} = \frac{ZF\psi}{RT} \ll 1 \).

\( \lambda \gg 1 \), LARGE DEBYE LENGTH RATIO

The solution of Eq. (2-19) plotted in Fig. 2-3 shows that

as \( \lambda \to \infty \), \( \psi(z,r) \to \psi_S(z) \) and

\[ C_1(z,r) \to C_0 \exp \left( -\frac{Z_iF\psi_s}{RT} \right) ; \quad (2-41) \]

i.e., the ion concentrations are uniform across the pore. We can then replace Poisson's equation by a statement of overall electroneutrality for the pore solution and the wall, i.e.,
\[ FZ(C_1 - C_2) = -\frac{2\sigma_s}{a} \]  \hspace{1cm} (2-42)

For the assumption of constant surface charge density, \( \sigma_s = \sigma_k \), and

\[ C_1 - C_2 = -\frac{2\sigma_k}{dFZ} \equiv C_m \]  \hspace{1cm} (2-43)

For a fully dissociated salt,

\[ C_1 C_2 = C_0^2 \]  \hspace{1cm} (2-44)

Equations (2-43) and (2-44) can be combined to solve for \( C_1 \) and \( C_2 \):

\[ C_1 = \frac{C_m}{2} + \frac{C_m}{2} \left(1 + \frac{4C_0^2}{C_m^2}\right)^{\frac{1}{2}} \]  \hspace{1cm} (2-45)

\[ C_2 = \frac{C_m}{2} + \frac{C_m}{2} \left(1 + \frac{4C_0^2}{C_m^2}\right)^{\frac{1}{2}} \]  \hspace{1cm} .

\( C_1 \) and \( C_2 \) are constant throughout the pore, and the potential \( \psi_s \) can be determined through Eq. (2-33).

If the wall potential is constant, then the ion concentrations are given by Eq. (2-33) with \( \psi_s = \psi_k \).

\[ \frac{C_1}{C_0} = \exp \left(-\frac{ZF\psi_k}{RT}\right) ; \quad \frac{C_2}{C_0} = \exp \left(\frac{ZF\psi_k}{RT}\right) \]  \hspace{1cm} (2-46)

We can derive \( v, V, \) and \( I \) for the limit \( \lambda \gg 1 \) by substituting for \( C_1 \) and \( C_2 \) from Eqs. (2-45) or (2-46) into Eqs. (2-28), (2-29), and (2-30).
For constant surface potential \( \psi_s = \psi_k \),

\[
V = -\frac{(a^2 - r^2)}{4\mu} \frac{\partial}{\partial z} \left[ p - 2ZFC_0 \phi \sinh \left( \frac{ZF\psi_k}{RT} \right) \right] \tag{2-47}
\]

\[
V = -\frac{\pi a^4}{8\mu} \frac{\partial}{\partial z} \left[ p - 2ZFC_0 \phi \sinh \left( \frac{ZF\psi_k}{RT} \right) \right] \tag{2-48}
\]

\[
I = -2ZFC_0 \sinh \left( \frac{ZF\psi_k}{RT} \right) V - \sigma_0 \pi a^2 \frac{\partial \phi}{\partial z} \cosh \left( \frac{ZF\psi_k}{RT} \right)
\]

For constant surface charge density \( \sigma_s = \sigma_k \),

\[
V = -\frac{(a^2 - r^2)}{4\mu} \frac{\partial}{\partial z} \left( p - \frac{2\sigma_k \phi}{a} \right) \tag{2-50}
\]

\[
V = -\frac{\pi a}{8\mu} \frac{\partial}{\partial z} \left( p - \frac{2\sigma_k \phi}{a} \right) \tag{2-51}
\]

\[
I = -\frac{2\sigma_k}{a} V + \pi a^2 \sigma_0 \left[ 1 + \left( \frac{\sigma_k}{C_0 aFZ} \right)^2 \right]^{1/2} \frac{\partial \phi}{\partial z} \tag{2-52}
\]

\( \lambda \ll 1 \), SMALL DEBYE LENGTH RATIO

For \( \lambda \ll 1 \), the exact solutions of Poisson's equation [Eq. (2-19)] show that \( \psi = 0 \) over most of the pore cross-sections (for example, see Fig. 2-3); therefore, we need to solve for \( \psi \) only near the pore wall, where

\[
r \sim a >> \lambda_D
\]
It will be more useful to non-dimensionalize the distance coordinate in Poisson's equation (2-19) with respect to $\lambda_D$, so that

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{\psi}}{\partial \tilde{r}} \right) = \frac{\partial^2 \tilde{\psi}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{\psi}}{\partial \tilde{r}} = \sinh \tilde{\psi} \quad (2-53)$$

where

$$\tilde{\psi} = \frac{ZF \psi}{RT} \quad \tilde{r} = \frac{r}{\lambda_D} . \quad (2-54)$$

Since $r/\lambda_D \sim a/\lambda_D >> 1$, we can neglect the second term on the left hand side; i.e., we neglect the curvature of the pore wall. If we also make a coordinate transformation $y = a - r$, then

$$\frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2} = \sinh \tilde{\psi} \quad \left( \tilde{y} = \frac{y}{\lambda_D} \right) , \quad (2-55)$$

with the boundary conditions

$$\tilde{y} = \infty \quad \frac{\partial \tilde{\psi}}{\partial \tilde{y}} = 0$$

$$\tilde{y} = 0 \quad \tilde{\psi} = \psi_s . \quad (2-56)$$

We have taken the upper limit of $\tilde{y}$ to be effectively $\infty$, since $\tilde{\psi}$ decreases rapidly with $\tilde{y}$, as shown below.

The solution for $\tilde{\psi}$ is:

$$\tilde{\psi} = \frac{ZF}{2RT} = \ln \left[ \frac{\psi_s/2}{\ln \left( \frac{\psi_s/2}{e^{\psi_s/2} - 1} e^{-\tilde{y}} \right)} \right] . \quad (2-57)$$
Equation (2-57) is not particularly useful. Hypothetically, it can be substituted into Eqs. (2-37), (2-38), and (2-39), and the resulting equations solved numerically to obtain \( v, V, \) and \( I \), respectively. However, if numerical integrations are required, then we may as well start with the basic equation (2-19) and obtain the exact solutions.

Analytical solutions are possible, however, for \( \bar{\psi}_s << 1 \) (\( \lambda << 1 \)). We can then make the approximation known as the Debye-Hückel linearization:

\[
\sinh \bar{\psi} \approx \bar{\psi}, \quad \cosh \bar{\psi} \approx 1 + \frac{\bar{\psi}^2}{2}, \quad (2-58)
\]

and replace Eq. (2-55) with

\[
\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = \bar{\psi} \quad (2-59)
\]

and Eq. (2-57) with

\[
\bar{\psi} = \bar{\psi}_s e^{-\bar{y}}
\]

or

\[
\psi = \psi_s e^{-y/\lambda_d} = \psi_s e^{-(a-r)/\lambda_d} \quad . \quad (2-60)
\]

We see that \( \psi \rightarrow 0 \) within a few multiples of \( \lambda_d \) from the pore wall.

Substituting Eq. (2-60) into Eqs. (2-37), (2-38), and (2-39), we have

\[
v = \frac{\varepsilon \psi_s}{\mu} \left[ 1 - e^{-\frac{(a-r)}{\lambda_d}} \right] \frac{\partial \phi}{\partial z} + \frac{(r^2 - a^2)}{4 \mu} \frac{\partial p}{\partial z} \quad (2-61)
\]

\[
V = \frac{\varepsilon \psi_s}{\mu} \frac{\partial \phi}{\partial z} \pi a^2 \left( 1 - \frac{2 \lambda_D}{a} \right) - \frac{\pi a^4}{8 \mu} \frac{\partial p}{\partial z} \quad (2-62)
\]
I = - \frac{\pi a}{\mu \lambda_D} (\epsilon \psi_S)^2 \frac{\partial \phi}{\partial z} + \frac{\pi a^2 \epsilon \psi_S}{\mu} \frac{\partial \rho}{\partial z} - \pi a^2 \frac{\partial \phi}{\partial z} \left[ \sigma_0 + \frac{2}{\alpha} \frac{D \epsilon}{\lambda_D} \left( \frac{ZF \psi_S}{RT} \right)^2 \right]

or

I = - \pi a^2 \frac{\partial \phi}{\partial z} \left\{ \sigma_0 + \frac{2}{\alpha} \left[ \frac{\epsilon \psi_S}{2 \mu \lambda_D} + \frac{D \epsilon}{\lambda_D} \left( \frac{ZF \psi_S}{RT} \right)^2 \right] \right\} + \pi a^2 \epsilon \psi_S \frac{\partial \rho}{\partial z}

(\psi_S = \psi_k \text{ for constant surface potential}).

The surface charge density is related to the surface potential by Eq. (2-24), so that

$$\sigma_S = \epsilon \left( \frac{\partial \psi}{\partial r} \right)_S = \frac{\epsilon \psi_S}{\lambda_D} \quad ,$$

and Eqs. (2-61) - (2-63) can be written in terms of the surface charge density.

$$v = \frac{\sigma_S \lambda_D}{\mu} \frac{\partial \phi}{\partial z} \left[ 1 - e^{-(a-r)/\lambda_D} \right] + \frac{(r^2 - a^2)}{4 \mu} \frac{\partial \rho}{\partial z} \quad (2-65)$$

$$v = \frac{\sigma_S \lambda_D}{\mu} \frac{\partial \phi}{\partial z} \pi a^2 \left( 1 - \frac{2 \lambda_D}{a} \right) - \frac{\pi a^2}{\mu} \frac{\partial \rho}{\partial z} \quad (2-66)$$

$$I = - \pi a \frac{\lambda_D}{\mu} \sigma_S^2 \frac{\partial \phi}{\partial z} + \frac{\pi a^2}{\mu} \lambda_D \sigma_S \frac{\partial \rho}{\partial z} - \pi a^2 \frac{\partial \phi}{\partial z} \left[ \sigma_0 + \frac{2}{\alpha} \frac{\lambda_D}{\epsilon} \left( \frac{ZF \sigma_S}{RT} \right)^2 \right]

\text{or}

I = - \pi a^2 \frac{\partial \phi}{\partial z} \left\{ \sigma_0 + \frac{2}{\alpha} \left[ \frac{\lambda_D}{\epsilon} \left( \frac{ZF \sigma_S}{RT} \right)^2 + \left( \frac{\lambda_D \sigma_S}{2 \mu} \right)^2 \right] \right\} + \pi a^2 \frac{\lambda_D}{\mu} \sigma_S \frac{\partial \rho}{\partial z}

(\sigma_S = \sigma_k \text{ for constant surface charge density}).
For zero pressure gradient, we can write Eqs. (2-63) and (2-67) in the following form:

\[ I = -\pi a^2 \frac{\partial \phi}{\partial z} \left( \sigma_0 + \frac{2\sigma_w}{a} \right) \]  

(2-68)

\( \sigma_w \) is called the "surface conductance." In an actual porous medium, \( \sigma_w \) depends on the pore geometry and is difficult to measure experimentally. Its origins are also not well understood.\(^3\) Our expressions for \( \sigma_w \) in Eqs. (2-63) and (2-67) apply only to our particular model of the double layer and pore geometry, with \( \lambda \ll 1 \); our \( \sigma_w \) accounts for the current density in the double layer due to convection and electrostatic migration of the excess ions.

\[ \overline{\psi} \ll 1 \text{, WITH DEBYE-HÜCKEL LINEARIZATION} \]

For \( \overline{\psi} \ll 1 \),\( ^+ \) we can make use of the Debye-Hückel linearization [Eq. (2-58)], so that the Poisson-Boltzmann equation (2-19) can be approximated as:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = \frac{2(FZ)^2C}{RTe} \psi = \frac{\psi}{\lambda_D^2} \]  

(2-69)

The boundary conditions are:

\[ \]

\(^+\)Actually, the approximation is good for \( \overline{\psi} \) as large as .25 (about 1% error).
\[ r = 0 \quad \frac{\partial \psi}{\partial r} = 0 \]

\[
\begin{align*}
\frac{\partial \psi}{\partial r} &= \frac{\sigma_k}{\varepsilon} & \text{for constant surface charge density} \\
\text{or} \quad \psi &= \psi_i & \text{for constant surface potential.}
\end{align*}
\]

The analytical solution of Eq. (2-69) can be substituted into Eqs. (2-37) - (2-39), so that:

For constant surface potential,

\[ \psi = \psi_k \frac{I_0(r/\lambda_D)}{I_0(a/\lambda_D)} \quad (2-70) \]

\[ \begin{align*}
V &= \frac{\varepsilon \psi_k}{\mu I_0 \left( \frac{a}{\lambda_D} \right)} \left[ I_0 \left( \frac{a}{\lambda_D} \right) - I_0 \left( \frac{r}{\lambda_D} \right) \right] \frac{\partial \phi}{\partial z} + \left( \frac{r^2 - a^2}{4\mu} \right) \frac{\partial \rho}{\partial z} \\
V &= \frac{\varepsilon \psi_k}{\mu I_0 \left( \frac{a}{\lambda_D} \right)} \frac{\partial \phi}{\partial z} \int_0^a \left[ I_0 \left( \frac{a}{\lambda_D} \right) - I_0 \left( \frac{r}{\lambda_D} \right) \right] 2\pi r \, dr - \frac{\pi \alpha^4}{\mu} \frac{\partial \rho}{\partial z} \quad (2-71)
\end{align*} \]

\[ \begin{align*}
I &= -\frac{\varepsilon \psi_k}{\lambda_D^2 I_0 \left( \frac{a}{\lambda_D} \right)} \int_0^a I_0 \left( \frac{r}{\lambda_D} \right) \nu 2\pi r \, dr - \sigma_0 \frac{\partial \phi}{\partial z} \int_0^a \cosh \psi \, 2\pi r \, dr \\
I &= \frac{\varepsilon \psi_k}{\lambda_D^2 I_0 \left( \frac{a}{\lambda_D} \right)} \int_0^a I_0 \left( \frac{r}{\lambda_D} \right) \nu 2\pi r \, dr - \sigma_0 \frac{\partial \phi}{\partial z} \int_0^a \cosh \psi \, 2\pi r \, dr \quad (2-73)
\end{align*} \]

For constant surface charge density:

\[ \psi = \frac{\sigma_k \lambda_D}{\varepsilon} \frac{I_0(r/\lambda_D)}{I_0(a/\lambda_D)} \quad (2-74) \]
\[ v = \frac{\sigma_k \lambda_D}{\mu I_1 \left( \frac{a}{\lambda_D} \right)} \frac{\partial \phi}{\partial z} \left[ I_0 \left( \frac{a}{\lambda_D} \right) - I_0 \left( \frac{r}{\lambda_D} \right) \right] + \left( \frac{r^2 - a^2}{4\mu} \right) \frac{\partial p}{\partial z} \quad (2-75) \]

\[ v = \frac{\sigma_k \lambda_D}{\mu I_1 \left( \frac{a}{\lambda_D} \right)} \frac{\partial \phi}{\partial z} \int_0^a \left[ I_0 \left( \frac{a}{\lambda_D} \right) - I_0 \left( \frac{r}{\lambda_D} \right) \right] 2\pi r \, dr - \frac{\pi a^4}{8\mu} \frac{\partial p}{\partial z} \quad (2-76) \]

\[ I = -\frac{\sigma_k}{\lambda_D I_1 \left( \frac{a}{\lambda_D} \right)} \int_0^a I_0 \left( \frac{r}{\lambda_D} \right) v 2\pi r \, dr - \sigma_0 \frac{\partial \phi}{\partial z} \int_0^a \cosh \psi \, 2\pi r \, dr \quad (2-77) \]

In the limit \( \lambda >> 1 \), \( r/\lambda_D < a/\lambda_D << 1 \), and we can make the approximations:

\[ I_0(\xi) \approx 1 + \frac{\xi^2}{4} \quad I_1(\xi) \approx \frac{\xi}{2} \quad \xi = \frac{r}{\lambda_D}, \frac{a}{\lambda} \quad . \quad (2-78) \]

Using Eq. (2-78), Eqs. (2-70)-(2-77) become identical to Eqs. (2-47) - (2-52), with

\[ \bar{\psi}_k << 1 \quad \text{and} \quad C_m << C_0 \quad . \]

For \( \lambda << 1 \), the potential \( \psi \) is non-zero only near the pore wall, where

\[ \frac{r}{\lambda_D} \sim \frac{a}{\lambda_D} >> 1 \quad . \]

We can therefore make the approximations that:
\[ I_0 \left( \frac{a}{\lambda_D} \right) = I_1 \left( \frac{a}{\lambda_D} \right) \approx \left( \frac{\lambda_D}{2\pi a} \right)^{1/2} \left( e^{a/\lambda_D} \right) \]

\[ I_0 \left( \frac{r}{\lambda_D} \right) \approx \left( \frac{\lambda_D}{2\pi r} \right)^{1/2} \left( e^{r/\lambda_D} \right). \]

(2-79)

Substituting of Eq. (2-79) into Eqs. (2-70)-(2-77) will result in equations identical to Eqs. (2-61) - (2-67).
SECTION 2.4. APPLICATION OF CAPILLARY MODEL
TO A POROUS MEDIUM

In this section we apply the results of the capillary tube model to the overall porous media. We will use the coordinate system defined in Fig. 2-2. The coordinate \( x \) is parallel to the mean direction of the externally-applied electric field and pressure gradient. We wish to determine \( q \), the average volume flow rate per unit area,\(^{\dagger} \) and \( i \), the average current per unit area. By "unit area" we mean a unit cross-sectional area of the overall porous medium, perpendicular to \( x \).

We have assumed that the solid in the porous medium is impermeable and non-conducting, and that the capillary tubes are circular. The actual directions of \( V \), \( I \), \( \nabla \phi \), and \( \nabla p \) in the pore would be parallel to the pore axis. If there are \( m \) tubes per unit area, then:

\[
q = mV \quad \text{(2-80)}
\]

and

\[
i = mI \quad \text{(2-81)}
\]

\( V \) and \( I \) are given by the equations tabulated below:

<table>
<thead>
<tr>
<th>V</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. (2-29) or Eq. (2-38)</td>
<td>Eq. (2-30) or Eq. (2-39)</td>
</tr>
</tbody>
</table>

\(^{\dagger}q \) is also referred to as the "superficial velocity."
(2) For $\sigma_s = \sigma_k$
(constant surface charge density)

(a) $\lambda \ll 1$  Eq. (2-66)  Eq. (2-67)

(b) $\lambda \gg 1$  Eq. (2-51)  Eq. (2-52)

(3) For $\psi_s = \psi_k$
(constant surface potential)

(a) $\lambda \ll 1$  Eq. (2-62)  Eq. (2-63)

(b) $\lambda \gg 1$  Eq. (2-48)  Eq. (2-49)

All of the equations are of the form

$$V = -\pi a^2 \left( \frac{k_1}{\mu} \frac{\partial \phi}{\partial z} + \frac{k_2}{\mu} \frac{\partial p}{\partial z} \right)$$

(2-82)

$$= -\pi a^2 \left( \frac{k_1}{\mu} \frac{\partial \phi}{\partial x} + \frac{k_2}{\mu} \frac{\partial p}{\partial x} \right) \left( \frac{dx}{dz} \right)$$

and

$$I = -\pi a^2 \left( k_3 \frac{\partial \phi}{\partial z} + k_4 \frac{\partial p}{\partial z} \right)$$

(2-83)

$$= -\pi a^2 \left( k_3 \frac{\partial \phi}{\partial x} + k_4 \frac{\partial p}{\partial x} \right) \left( \frac{dx}{dz} \right)$$

where $k_1$, $k_2$, $k_3$, and $k_4$ are functions of ($\sigma_k$ or $\psi_k$, $\mu$, $\alpha$, $\lambda_D$, $C_0$, $Z$, $D$, $T$).

The porosity $n$ of the porous medium is

$$n = \frac{m \pi a^2 \rho \hbar}{h^3} = m \pi a^2 \tau$$

or

$$m \pi a^2 = \frac{n}{\tau}.$$
Equations (2-82) and (2-83) apply at any point in the medium. \( \frac{dx}{dz} \) however, varies throughout the medium. If we make the approximation that

\[
\left( \frac{dx}{dz} \right)_{\text{average}} \sim \frac{h}{ar{r}} = \frac{1}{\tau}, \tag{2-85}
\]

then, using Eqs. (2-82) - (2-85), we have

\[
q = -\frac{n}{\tau^2} \left( \frac{k_1}{\mu} \frac{\partial \phi}{\partial x} + \frac{k_2}{\mu} \frac{\partial p}{\partial x} \right) = q_e + q_h \tag{2-86}
\]

\[
q_e = -\frac{k_e}{\mu} \frac{\partial \phi}{\partial x} \tag{2-87}
\]

\[
q_h = -\frac{k}{\mu} \frac{\partial p}{\partial x} \tag{2-88}
\]

and

\[
i = -\frac{n}{\tau^2} \left( k_3 \frac{\partial \phi}{\partial x} + k_h \frac{\partial p}{\partial x} \right) = -\kappa \frac{\partial \phi}{\partial x} - \kappa_h \frac{\partial p}{\partial x} \tag{2-89}
\]

where

\[
k_e \equiv \frac{nk_1}{\tau^2} \equiv \text{electroosmotic permeability}; \tag{2-90}
\]

\[
k \equiv \frac{nk_2}{\tau^2} \equiv \text{hydraulic permeability}; \tag{2-91}
\]

\[
\kappa = \frac{nk_3}{\tau^2} = \text{electrical conductivity of the porous medium as a whole}; \tag{2-92}
\]

\[
\kappa_h = \frac{nk_h}{\tau^2} = \text{current density per unit pressure gradient due to convection by the pressure-driven flow} \tag{2-93}
\]

From the general solution,
\[ k_e = \frac{2\varepsilon n}{\tau^2} \int_0^1 (\psi - \psi_S) \left( \frac{r}{a} \right) \, d\left( \frac{r}{a} \right) \]  

(2-94)

\[ k = \frac{na^2}{8\tau^2} \]  

(2-95)

\[ \kappa = \frac{2n\sigma_0}{\tau^2} \int_0^1 \left( \cosh \bar{\psi} \right) \left( \frac{r}{a} \right) \, d\left( \frac{r}{a} \right) \]  

(2-96)

\[ \kappa_h = \frac{nFZC_0}{a^2\tau^2\mu} \int_0^a \left( \sinh \bar{\psi} \right) \left( r^2 - a^2 \right) r \, dr . \]  

(2-97)

For a uniform porous medium,

\[ -\frac{\partial \phi}{\partial x} = \frac{\Delta \phi}{h} \]  

(2-98)

\[ -\frac{\partial \rho}{\partial x} = \frac{\Delta \rho}{h} \]  

(2-99)

where \( \Delta \rho \) and \( \Delta \phi \) are the magnitudes of the applied pressure and electrical potential difference across the medium, respectively.

\( k_e \), \( \kappa \), and \( \kappa_p \) are listed below for four special cases:

1. \( \lambda \ll 1 \), \( \bar{\psi} \ll 1 \), \( \psi_S = \psi_k \)

\[ k_e = -\frac{n}{\tau^2} \varepsilon \psi_k \left( 1 - \frac{2\lambda D}{a} \right) \]  

(2-100)
\[ \kappa = \frac{n}{\tau^2} \left\{ \sigma_0 + \frac{2}{a} \left[ \frac{(e\psi_k)^2}{2\mu \lambda_D^2} + \frac{D_{\psi_k}}{\lambda_D} \left( \frac{ZF\psi_k}{RT} \right)^2 \right] \right\} \]  \hspace{1cm} (2-101)

\[ \kappa_h = -\frac{n}{\tau^2} \frac{e\psi_k}{\mu} \]  \hspace{1cm} (2-102)

(2) \( \lambda \ll 1 \), \( \bar{\psi} \ll 1 \), \( \sigma_s = \sigma_k \)

\[ k_e = -\frac{n}{\tau^2} \sigma_k \lambda_D \left( 1 - \frac{2\lambda_D}{a} \right) \]  \hspace{1cm} (2-103)

\[ \kappa = \frac{n}{\tau^2} \left\{ \sigma_0 + \frac{2}{a} \left[ \frac{\lambda_D}{e} \left( \frac{ZF\sigma_k}{RT} \right)^2 + \frac{\lambda_D \sigma_k^2}{2\mu} \right] \right\} \]  \hspace{1cm} (2-104)

\[ \kappa_h = -\frac{n}{\tau^2} \frac{\lambda_D \sigma_k}{\mu} \]  \hspace{1cm} (2-105)

(3) \( \lambda \gg 1 \), \( \sigma_s = \sigma_k \)

\[ k_e = -\frac{a\sigma_k}{4} \frac{n}{\tau^2} \]  \hspace{1cm} (2-106)

\[ \kappa = \frac{n}{\tau^2} \left\{ \frac{\sigma_k^2}{2\mu} + \sigma_0 \left[ 1 + \left( \frac{\sigma_k}{C_0 a FZ} \right)^2 \right] \right\} \]  \hspace{1cm} (2-107)

\[ \kappa_h = -\frac{na\sigma_k}{4\mu t^2} \]  \hspace{1cm} (2-108)
\[ (4) \; \lambda \gg 1 \; , \; \psi_s = \psi_k \]

\[ k_e = - \frac{na^2}{\tau^2} F_{Z}C_0 \sinh \bar{\psi}_k \quad (2-109) \]

\[ \kappa = \frac{n}{\tau^2} \left[ \sigma_0 \cosh \bar{\psi}_k + \frac{(aZ_{F}C_0 \sinh \bar{\psi}_k)^2}{2\mu} \right] \quad (2-110) \]

\[ \kappa_h = - \frac{n}{\tau^2} \frac{Z_{F}C_0 a^2 \sinh \bar{\psi}_k}{\mu} \quad (2-111) \]

\[ \bar{\psi}_k = \frac{Z_F \psi_k}{RT} . \]
SECTION 2.5. "EFFICIENCY" OF ELECTROOSMOSIS

Several variables have been used as measures of the effectiveness of electroosmosis, such as:

(1) \( \frac{k_e}{k} \) (C/m³) \(^{\dagger}\)

(2) \( \frac{q_e}{E} \) (C-s/m); \( E = -\frac{\partial \phi}{\partial x} \)

(3) \( \frac{q_e}{I} \) (m³/C)

(4) \( \eta \equiv \frac{q^* \Delta p}{I \Delta \phi} \) (dimensionless)

(q* to be defined below).

The first variable has been the most widely used. Suppose that we wish to obtain a given superficial velocity \( q^* \) by either applying an electric potential gradient or a pressure gradient. The ratio of the required gradients would be [Eqs. (2-87) and (2-88)]:

\[
\frac{(\Delta p/\Delta x)}{(\Delta \phi/\Delta x)} = \frac{k_e}{k} \quad . \tag{2-112}
\]

From Eqs. (2-95) and (2-100) we see that for \( \lambda \ll 1 \),

\[
\frac{k_e}{k} \approx \frac{1 - 2\lambda}{\alpha^2} \approx \frac{1}{\alpha^2} \quad , \tag{2-113}
\]

\(^{\dagger}\)C = coulomb, m = meter, s = second.
so that the ratio increases rapidly as the pore radius decreases. This is the usual argument given in favor of electroosmosis (Ref. 9, 20, and 38, for example). Most of the previous researchers, however, did not recognize that as $a$ decreases, $\lambda$ increases and would no longer be much less than 1, so that Eq. (2-113) would no longer be valid. In fact, when $\lambda >> 1$,

$$\frac{k_e}{k} \propto \frac{1}{a} \quad \text{for} \quad \sigma_S = \sigma_k \quad (2-114)$$

and

$$\frac{k_e}{k} = \text{constant for } \psi_S = \psi_k \quad . \quad (2-115)$$

For $\lambda >> 1$ and $\psi_S = \psi_k$, there is no particular advantage in using an electric field vs. a pressure gradient, in terms of the flow rates achievable.

The second variable has been used by Koh and Anderson, they have solved numerically for $(q_e/E)$ as a function of $(a/\lambda_0)$ for different values of $\sigma_k$, and compared the results with experimental data for electroosmotic flow through track-etched mica. The third variable, $(q_e/i)$, has been used by Gray to correlate experimental data as a function of the porosity $n$.

The final variable $n$ is perhaps the most appropriate measure of the efficiency of electroosmosis. It was first proposed by Osterle as a "conversion efficiency." It is the ratio of the hydraulic to the electrical energy required to achieve a given superficial flow velocity $q^*$ through the porous medium. From Eqs. (2-87), (2-88), (2-98), and (2-99), we have:
For \( q_p = q_e = q^* \):

\[
\Delta p = \frac{\mu q^* h}{k} \quad \text{and} \quad \Delta \phi = \frac{\mu q^* h}{k_e}.
\] (2-116)

From Eqs. (2-89) and (2-98)\(^\dagger\)

\[
i = \kappa \frac{\Delta \phi}{\hbar} = \frac{k \mu q^*}{k_e}.
\] (2-117)

\( \eta \) can then be written as:

\[
\eta = \frac{q^* \Delta p}{i \Delta \phi} = \frac{k_e^2}{\kappa \kappa k}.
\] (2-118)

Analytical expressions for \( \eta \) are available for four special cases:

(1) \( \lambda << 1 \); \( \bar{\psi} << 1 \); \( \psi_S = \psi_k \)

\[
\eta = \frac{8 \left( 1 - \frac{2 \lambda_D}{a} \right)^2}{\frac{\mu D}{\varepsilon \psi_k^2} \left( \frac{a}{\lambda_D} \right)^2 + \frac{a}{\lambda_D} \left[ 1 + \frac{2 \mu D}{\varepsilon} \left( \frac{Z_F}{RT} \right)^2 \right]}.
\] (2-119)

(2) \( \lambda << 1 \); \( \bar{\psi} << 1 \); \( \sigma_S = \sigma_k \)

\[
\eta = \frac{8 \left( 1 - \frac{2 \lambda_D}{a} \right)^2}{\left( \frac{D \mu}{\sigma_k^2} \frac{a^2}{\lambda_D^4} \right) + \left( \frac{a}{\lambda_D} \right) \left[ 1 + \frac{2 \mu D}{\varepsilon} \left( \frac{Z_F}{RT} \right)^2 \right]}.
\] (2-120)

\(^\dagger\)\( \kappa_p \) is not included since the pressure gradient and the electric field are applied separately.
(3) \(\lambda \gg 1\); \(\psi_s = \psi_k\)

\[\eta = \frac{16 \left(\frac{a}{\lambda_D}\right)^2}{8 \mu D \varepsilon \left(\frac{FZ}{RT \sinh \overline{\psi}_k}\right)^2 \cosh \overline{\psi}_k + \left(\frac{a}{\lambda_D}\right)^2}\]  \hspace{1cm} (2-121)

(4) \(\lambda \gg 1\); \(\sigma_s = \sigma_k\)

\[\eta = \frac{1}{1 + \left(\frac{4ZF D}{\sigma_k aRT}\right)\left[1 + \left(\frac{C_0 aFZ}{\sigma_k}\right)^2\right]^{\frac{1}{2}}}\]  \hspace{1cm} (2-122)

In Eqs. (2-119) - (2-121), the first term in the denominator is much larger than the second; the relative magnitudes of the two terms in Eq. (2-122) are less definite.

For constant wall potential \((\psi_s = \psi_k)\), the behavior of \(\eta\) is particularly simple. If we assume that the ion diffusivities \(D\) are approximately constant, then for \(\lambda \ll 1\),

\[\eta \approx \frac{8}{\mu D} \left(\frac{a}{\lambda_D}\right)^2 \propto \left(\frac{a}{\lambda_D}\right)^{-2}\]  \hspace{1cm} (2-123)

and for \(\lambda \gg 1\),

\[\eta \approx \frac{2 \left(\frac{a}{\lambda_D}\right)^2}{\mu D \varepsilon \left(\frac{FZ}{RT \sinh \overline{\psi}_k}\right)^2 \cosh \overline{\psi}_k} \propto \left(\frac{a}{\lambda_D}\right)^2\]  \hspace{1cm} (2-124)

Clearly \(\eta\) has a maximum at a \(\lambda\) between the two extremes, as shown in Figs. 2-6 and 3-12. We will discuss this maximum further in Chapter 3.
SECTION 2.6. A MODIFICATION OF THE SIMPLE
CAPILLARY MODEL: CLUSTER MODEL

The cluster model which we will use was initially proposed by Olsen\textsuperscript{33} to account for the inability of the Kozeny-Cormen equation\textsuperscript{42} to predict the hydraulic permeability of saturated clays. We will derive equations for electroosmotic flow through the cluster model for a similar reason: the results of the previous simple capillary tube model do not agree with experimental data for Kaolinite clays by Gray\textsuperscript{34} and Olsen.\textsuperscript{33}

Previously we had assumed that the fluid volume in a porous medium is uniformly distributed among mA capillaries of radius \(a\), where \(A\) is the cross-sectional area of the medium perpendicular to the mean direction of current and fluid flow, and \(m\) is the number of pores per unit area. In the cluster model, we assume that the solid in the porous medium consists of uniform particles arranged in clusters, with space between the particles filled with fluid (Fig. 2-7). The particles and the clusters are spherical in shape. The total fluid volume then is the sum of the fluid enclosed in the clusters (referred to as "cluster fluid") and the fluid between the clusters ("pore fluid"). We further assume that the interparticle distance between the particles inside a cluster is extremely small and the cluster fluid is bound to the particles by adsorptive forces, so that only the pore fluid is free to flow under applied pressure and/or electrical gradient.

Similar models have been suggested by workers in soil mechanics\textsuperscript{40}
and wastewater sludge treatment, who divide the total fluid between "bound" and "unbound" fluid.

The cluster model is probably a good approximation for wastewater sludges, especially for sludges which have been chemically conditioned; the organic matter and the microorganisms are then associated in "flocs" with tightly bound water.

In our following discussion, the fluid volumes can be described more conveniently by the void ratios:

\[
e_T = \text{total void ratio} = \frac{\text{total fluid volume}}{\text{total solid volume}}
\]

\[
e_c = \text{cluster void ratio} = \frac{\text{volume of fluid enclosed in the clusters}}{\text{total solid volume}}
\]

\[
e_p = \text{pore void ratio} = \frac{\text{volume of fluid between clusters}}{\text{total solid volume}}.
\]

As the porous medium as a whole is compressed (by an applied load, for example), the behavior of \(e_T, e_c,\) and \(e_p\) is assumed to be described by Fig. 2-8. Initially only \(e_p\) decreases, until \(e_T = e_{cr}\), when the clusters come into contact. Further compression of the medium results in compression of the clusters as well, so that decrease in \(e_T\) occurs by simultaneous decreases in \(e_p\) and \(e_c\). The dependence of \(e_p\) and \(e_c\) on \(e_T\) in Fig. 2-8 has been experimentally confirmed by Olsen.\(^{33}\)

Olsen has derived an equation for the hydraulic permeability \(k\) which agrees with experimental data:
\[
\frac{k}{k_0} = \frac{N^{2/3}(e_T - e_c)^3}{e_T^3(1 + e_c)^{4/3}}
\]

(2-125)

\[
k_0 = \frac{e_T^3}{C_k \tau^2 S^2(1 + e_T)} \quad \text{Kozeny-Cormany equation} \quad (2-126)
\]

where \( N \) is the number of particles per cluster, \( k_0 \) is the permeability from the Kozeny-Cormany equation, \( S \) is the specific area (surface area per unit volume of particle), \( C_k \) is a "shape factor," and \( \tau \) is the tortuosity. The value of \( C_k \tau^2 \) is usually taken to be 5.42.

Olsen has experimentally determined values of \( N \), \( e_{co} \), and \( e_{cr} \) for various clays (Table 2-1). He has also derived an equation for the electrical conductance through the cluster model, for the special case of negligible surface conduction. He assumes that there are three current paths: (1) through the pores between the clusters; (2) alternately through the pores and the clusters; and (3) through the clusters at their point of contact. A serious weakness in his derivation is the assumption that the electrical conductances of the pore and the cluster fluid are equal to each other and to that of the neutral external fluid. The effect of the surface potential or charge on the conductance through the double layer is ignored. We will derive later a different equation for the conductance.

For simplicity, we will only consider electroosmotic flow through the cluster model \((\partial p/\partial x = 0)\). We make the following assumptions:

1. The particles and the clusters are spherical in shape and uniform in size, with an average of \( N \) particles per cluster. The clusters are uniformly dispersed throughout the porous medium.
(2) For \( e_T, e_c, e = e_{co} = \text{constant} \) (Fig. 2-8) so that the pore void ratio \( e_p \) is:

\[
e_p = e_T - e_{co},
\]

(2-127)

and the pore porosity \( n_p \) is

\[
n_p = n - n_c = (1 + e_{co}) n - e_{co},
\]

(2-128)

where \( n = \) overall porosity = \( \frac{\text{total volume of fluid}}{\text{volume of porous medium}} \)

\( n_c = \) cluster porosity = \( \frac{\text{volume of fluid in clusters}}{\text{volume of porous medium}} \).

For \( e_T < e_{cr} \) (except for the transition region near \( e_{cr} \)),

\[
e_p = e_T - e_c = m_p e_T \quad m_p = \text{constant}
\]

(2-129)

and

\[
n_p = m_p n.
\]

(2-130)

(3) The pores can be modeled as parallel uniform capillary tubes of circular cross-section, with radii \( a_p \) equal to a "hydraulic radius" defined as

\[
a_p = \frac{2V_p}{S_p};
\]

(2-131)

where \( V_p = \) volume of pores between clusters;

\( S_p = \) area of pore walls = external area of clusters exposed to the pore fluid.

From Appendix B,

\[
a_p = \frac{2(e_T - e_c) N^{2/3}}{S(1 + e_c)^{2/3}}.
\]

(2-132)

If experimental data for the hydraulic permeability is available, then \( a_p \) can also be given by the equation:
\[ a_p = \left( \frac{4C_k \tau^2 k}{n_p} \right)^{\frac{1}{2}} \]  \hspace{1cm} (2-133)

We define two Debye length ratios:

\[ \lambda_p \equiv \frac{\lambda_D}{a_p} \quad \lambda_c \equiv \frac{\lambda_D}{d_c} \]  \hspace{1cm} (2-134)

where \( d_c \) is the distance between the particles within the cluster (Fig. 2-7):

\[ d_c \sim 2 \left[ 1 - (1 + e_c)^{1/3} \right] \]  \hspace{1cm} (2-135)

(4) \( \lambda_c \gg 1 \), so that the potential \( \psi \) and the ion concentrations \( C_1 \) and \( C_2 \) are uniform throughout the cluster fluids. We can apply electroneutrality to the cluster as a whole, so that (Appendix B):

\[ C_1 - C_2 = -\frac{S \sigma S}{Z F e_c} = C_n \]  \hspace{1cm} (2-136)

The binary salt is fully dissociated, so that

\[ C_1 C_2 = C_0^2 \]  \hspace{1cm} (2-44)

From Eqs. (2-44) and (2-136),

\[ C_1 = \frac{C_n}{2} + \frac{C_n}{2} \left[ 1 + \left( \frac{2C_0}{C_n} \right)^2 \right]^{\frac{1}{2}} \]  \hspace{1cm} (2-137)

\[ C_2 = -\frac{C_n}{2} + \frac{C_n}{2} \left[ 1 + \left( \frac{2C_0}{C_n} \right)^2 \right]^{\frac{1}{2}} \]  \hspace{1cm} .

\( C_1 \) and \( C_2 \) can also be expressed in terms of the surface potential:

\[ C_1 = C_0 \exp \left( -\frac{Z F \psi S}{R T} \right) \]  \hspace{1cm} (2-138)

\[ C_2 = C_0 \exp \left( \frac{Z F \psi S}{R T} \right) \]  \hspace{1cm} .
(5) \( d_c \ll a_p \). We will assume then that the flow rate through the clusters is negligible compared to that through the pores. For example, for constant surface potential, in the pore:

\[
\bar{v}_p = \begin{cases} 
\frac{\varepsilon \psi_k}{\mu} \frac{\partial \phi}{\partial x} & \lambda_p \ll 1 \\
\text{to} & \\
\frac{\varepsilon \psi_k}{\delta \mu \lambda_D^2} a_p^2 \frac{\partial \phi}{\partial x} & \lambda_p \gg 1 
\end{cases}
\]  

(2-139)

\[
q_p = \eta_p \bar{v}_p ;
\]  

(2-140)

and in the clusters:

\[
\bar{v}_c \approx \frac{\varepsilon \psi_k}{\delta \mu \lambda_D^2} d_c^2 \frac{\partial \phi}{\partial x} & \lambda_c \gg 1 
\]  

(2-141)

\[
q_c = \eta_c \bar{v}_c ,
\]  

(2-142)

where \( \bar{v}_p, \bar{v}_c \) are average fluid velocities, and \( q_p, q_c \) are volume flow rates per unit cross-sectional area of porous medium.

Therefore:

\[
\frac{q_c}{q_p} = \begin{cases} 
\eta_c \left( \frac{d_c}{a_p} \right)^2 & \eta_p \gg \eta_c \\
\frac{\eta_c}{\eta_p} \frac{d_c^2}{8 \lambda_D^2} & \lambda_p \ll 1 \\
\text{to} & \\
\frac{\eta_c}{\eta_p} \left( \frac{d_c}{a_p} \right)^2 & \lambda_p \gg 1 \\
\end{cases}
\]  

\ll 1 \]  

(2-143)
(6) The current paths are (1) through the pores and (2) through the clusters. The current through the second path may be negligible in certain cases.

The governing equations for electroosmotic flow through the cluster model are Eqs. (2-6), (2-19), (2-38), (2-87), and (2-94), with $a$ replaced by $a_p$ and $n$ replaced by $n_p$ ($\partial p/\partial x = 0$).

Potential distribution:

$$\lambda_p^2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \bar{\psi}}{\partial r} \right) = \sinh \bar{\psi} \quad (2-144)$$

$$\lambda_p = \frac{\lambda D}{a_p} \quad \bar{r} = \frac{r}{a_p} \quad \bar{\psi} = \frac{ZFe \psi}{RT} \quad (2-145)$$

Boundary conditions:

$$\bar{r} = 0 \quad \frac{\partial \bar{\psi}}{\partial \bar{r}} = 0$$

$$\begin{cases} 
\bar{\psi} = \bar{\psi}_k \quad \text{constant surface potential} \\
\bar{r} = 1 \\
\frac{\partial \bar{\psi}}{\partial \bar{r}} = \frac{ZFa_p \sigma_k}{\varepsilon RT} \quad \text{constant surface charge density}
\end{cases}$$

Fluid velocity:

$$v_p = -\frac{\varepsilon (\psi - \psi_s)}{\mu} \frac{\partial \phi}{\partial z} \quad (2-146)$$
Volume flow rate per pore of radius $a_p$:

$$V_p = - \frac{e}{\mu} \frac{\partial \phi}{\partial z} \int_0^{a_p} (\psi - \psi_s) 2\pi r \, dr \quad . \quad (2-147)$$

Superficial flow velocity (volume flow rate per unit cross-sectional area of porous medium) $q_e$:

$$q_e = q_p = n_p \bar{v}_p = - \frac{2n_p e}{\mu \tau^2} \frac{\partial \phi}{\partial x} \int_0^1 (\psi - \psi_s) \left( \frac{r}{a_p} \right) \, d \left( \frac{r}{a_p} \right) \quad . \quad (2-148)$$

Let $\sigma_p$ and $\sigma_c$ be the average electrical conductances of the fluid in the pores and in the clusters, respectively. $\sigma_p$ includes the surface conductance in the pores and is a function of $a_p$ and $\lambda_p$.† With uniformly dispersed clusters:

$$i = n_p \bar{\tau}_p \cdot \hat{x} + n_c \bar{\tau}_c \cdot \hat{x} \quad , \quad (2-149)$$

where $\hat{x}$ = unit vector in the mean direction of the externally applied electric field;

$\bar{\tau}_p, \bar{\tau}_c$ = current densities in the pores and clusters, respectively.

In general, $\bar{\tau}_p$ and $\bar{\tau}_c$ are not parallel to $\hat{x}$ (Fig. 2-9); the angular deviations are accounted for by the tortuosities with reasoning

†For example, for $\lambda_p \ll 1$, $\sigma_p = \sigma_0 + 2\sigma_w/a_p$, where $\sigma_w$ = surface conductance, given by Eq. (2-63) or (2-67).
similar to that in Section 2.4:

\[ \dot{\mathbf{t}}_p \cdot \dot{x} = -\frac{\sigma_p}{\tau^2} \frac{\partial \phi}{\partial x} \quad (2-150) \]

\[ \dot{\mathbf{t}}_c \cdot \dot{x} = -\frac{\sigma_c}{\tau_c^2} \frac{\partial \phi}{\partial x} \quad (2-151) \]

\[ i = -\left( \frac{n_p \sigma_p}{\tau^2} + \frac{n_c \sigma_c}{\tau_c^2} \right) \frac{\partial \phi}{\partial x} . \quad (2-152) \]

\( \sigma_p \) is derived from Eq. (2-30),

\[ \sigma_p = -\frac{2FZ}{(\partial \phi / \partial x)} \int_0^1 (C_1 - C_2) v_p \bar{r} \, d\bar{r} \quad (2-153) \]

\[ + \frac{2(ZF)^2D}{RT} \int_0^1 (C_1 + C_2) \bar{r} \, d\bar{r} \]

which can be rewritten in terms of \( \psi \):

\[ \sigma_p = -\frac{4RTC\alpha\varepsilon}{\mu} \int_0^1 \sinh \bar{\psi} (\bar{\psi} - \bar{\psi}_s) \bar{r} \, d\bar{r} + 2\sigma_0 \int_0^1 \cosh \bar{\psi} \bar{r} \, d\bar{r} \quad (2-154) \]

where \( \bar{r} = r/\alpha_p \).

Within the cluster, \( v = 0 \) and \( (C_1 - C_2) = \) constant, so that

\[ \dot{\mathbf{t}}_c = -\frac{(ZF)^2D}{RT} (C_1 + C_2) \nabla \phi = -\sigma_c \nabla \phi . \quad (2-155) \]

In terms of the surface charge density \( \sigma_s \):†

†See Appendix B.
\[ \sigma_c = \sigma_0 \left[ 1 + \left( \frac{\sigma_s S}{2zFe_c C_0} \right)^2 \right]^{\frac{1}{2}} \], \quad (2-156) \\

and in terms of the surface potential \( \psi_s \):

\[ \sigma_c = \sigma_0 \cosh \psi_s \]. \quad (2-157)

In general, numerical methods are required to solve for the electro-osmotic flow rate and the current. The procedure is as follows:

1. The dependence of \( e_c \) on \( e_T \) must be specified (as in Fig. 2-8); \( e_{co}, e_{cr}, \tau, \) and \( N \) can be obtained from experimental data on the hydraulic permeability (Appendix B) or must be guessed.

2. The pore radius \( \sigma_p \) is calculated for each value of \( e_T \) of interest.

3. Equation (2-144) is solved for the potential distribution in the pore.

4. Equations (2-146) - (2-148) are solved for the fluid velocity, flow rate, and superficial velocity.

5. Equation (2-152) is solved for the current density, with \( \sigma_p \) given by Eq. (2-154) and \( \sigma_c \) given by Eq. (2-156) or (2-157). \( \tau_c \) may be determined from experimental data or guessed.

Analytical solutions are possible for some limiting cases. We first assume that \( N, e_{co}, e_{cr}, \tau, \) and \( \tau_c \) are given, and that \( \sigma_p \) can be determined from Eq. (2-132) or (2-133). As in Section 2.4, we can write \( q_e \) and \( i \) in the following form:
\[ q_e = - \frac{k_e}{\mu} \frac{\partial \phi}{\partial x}; \quad i = -\kappa \frac{\partial \phi}{\partial x}. \]

Then, for:

1. \( e_T > e_{cr}, \lambda_p \ll 1, \bar{\psi}_s \ll 1 \), and \( \psi_s = \psi_k \) (constant surface potential)

\[ k_e = -n_p \frac{e \psi_k}{\tau^2} \left( 1 - \frac{2\lambda_D}{\alpha_p} \right) \]

\[ \kappa = \sigma_0 \frac{n_p}{\tau^2} \left[ 1 + \cosh \bar{\psi}_k \frac{n_c}{n_p} \left( \frac{\tau}{\tau_c} \right)^2 + \frac{2\sigma_w}{\sigma_0 \alpha_p} \right] \]

where

\[ \sigma_w = \frac{(e \psi_k)^2}{2\mu \lambda_D} + \frac{D_e}{\lambda_D} \left( \frac{ZF \psi_k}{RT} \right)^2 \]

\[ n_p = (1 + e_{co})n - e_{co}. \]

2. \( e_T > e_{cr}, \lambda_p \ll 1, \bar{\psi}_s \ll 1 \), and \( \sigma_s = \sigma_k \) (constant surface charge density)

\[ k_e = -\frac{n_p \sigma_k \lambda_D}{\tau^2} \left( 1 - \frac{2\lambda_D}{\alpha} \right) \]

\[ \kappa = \frac{n_p}{\tau^2} \sigma_0 \left\{ 1 + \left[ 1 + \left( \frac{\sigma_k S}{2ZF e_{co} C_0} \right)^2 \right] \frac{n_c}{n_p} \left( \frac{\tau}{\tau_c} \right)^2 + \frac{2\sigma_w}{\sigma_0 \alpha_p} \right\} \]

\[ \sigma_w = \frac{\lambda_D}{\epsilon} \left( \frac{ZF \sigma_k}{RT} \right)^2 + \frac{\lambda_D \sigma_k^2}{2\mu} \]

and \( n_p \) is again given by Eq. (2-128).
(3) \( e_T < e_{cr} , \lambda_p >> 1 , \psi_s = \psi_k \)

\[
k_e = - \frac{n_p}{\tau^2} \frac{a_p^2}{4} ZFC_0 \sinh \bar{\psi}_k
\]  \hspace{1cm} (2-164)

\[
\kappa = \frac{n_p}{\tau^2} \frac{\varepsilon RTC_0 \sinh^2 \bar{\psi}_k}{4} \left( \frac{a_p}{\lambda_D} \right)^2 + \frac{n_p \sigma_0}{\tau^2} \cosh \bar{\psi}_k
\]
\[
\times \left[ 1 + \frac{n_c}{n_p} \left( \frac{\tau}{\tau_c} \right)^2 \right]
\]  \hspace{1cm} (2-165)

\[
n_p = n_p n
\]

(4) \( e_T < e_{cr} , \lambda_p >> 1 , \sigma_s = \sigma_k \)

\[
k_e = - \frac{n_p}{\tau^2} \frac{a_p \sigma_k}{4}
\]  \hspace{1cm} (2-166)

\[
\kappa = \frac{n_p}{\tau^2} \frac{a_p \sigma_k^2}{2 \mu} + \frac{n_p}{\tau^2} \sigma_0 \left[ 1 + \left( \frac{\sigma_k}{C_0 a_p FZ} \right)^2 \right]^{\frac{1}{2}}
\]
\[
+ \frac{n_c}{\tau_c^2} \sigma_0 \left[ 1 + \left( \frac{\sigma_k S}{2 ZFC_0 e_c} \right)^2 \right]^{\frac{1}{2}}
\]  \hspace{1cm} (2-167)

\[
n_p \quad \text{given by Eq. (2-130)}
\]

We will also derive \( \eta \) and the group \( (q_e \sigma_0 / \kappa) \) for comparison with experimental data in Chapter 3.
(1) $e_T > e_{cr}$, $\lambda_p << 1$, $\psi_s = \psi_k$

\[
n = \frac{4C_k (1 - 2\lambda_p)^2}{\mu D (1 + \sigma_2) + \frac{1}{\lambda_p} \left[ 1 + \frac{2\mu D}{\varepsilon} \left( \frac{ZF}{RT} \right)^2 \right]} \quad (2-168)
\]

\[
q_{0}\sigma_0 = -\frac{\varepsilon \psi_k}{\mu} \frac{(1 - 2\lambda_p)}{1 + \sigma_1 + \psi_k^2 \lambda_p \left[ \frac{\varepsilon}{\mu D} + 2 \left( \frac{ZF}{RT} \right)^2 \right]} \quad (2-169)
\]

\[
\sigma_1 \equiv \cosh \psi_k \frac{n_c}{n_p} \left( \frac{\tau}{\tau_c} \right)^2 \quad (2-170)
\]

If we neglect the current through the clusters, then for $\lambda_p \to 0$,

\[
n \to \frac{4C_k \varepsilon (\psi_k \lambda_p)^2}{\mu D} \quad (2-171)
\]

and

\[
q_{0}\sigma_0 \to -\frac{\varepsilon \psi_k}{\mu} \quad . \quad (2-172)
\]

(2) $e_T > e_{cr}$, $\lambda_p << 1$, $\sigma_s = \sigma_k$

\[
n = \frac{4C_k (1 - 2\lambda_p)^2}{\mu \varepsilon D \sigma_p^2 \left( 1 + \sigma_2 \right) \lambda_D^4 + \frac{1}{\lambda_p} \left[ 1 + \frac{2\mu D}{\varepsilon} \left( \frac{ZF}{RT} \right)^2 \right]} \quad (2-173)
\]

\[
q_{e}\sigma_0 = -\frac{\sigma_k \lambda_D}{\mu} \frac{(1 - 2\lambda_p)}{1 + \sigma_2 + \frac{2\lambda_D^3 \sigma_k^2}{\varepsilon D \sigma_p} \left[ \frac{D}{\varepsilon} \left( \frac{ZF}{RT} \right)^2 + \frac{1}{2\mu} \right]} \quad (2-174)
\]
\[ \sigma_2 \equiv \frac{n_c}{n_p} \left( \frac{\tau}{\tau_c} \right)^2 \left[ 1 + \left( \frac{\sigma_k S}{2ZFce_0e_c} \right)^2 \right]^{\nu_q} \]  

(2-175)

For \( \lambda_p \to 0 \), again neglecting current through clusters \( (\sigma_2 = 0) \),

\[ \eta = \frac{4C_k \sigma_k^2 \lambda_D^4}{\mu \varepsilon D \sigma_p^2} \]  

(2-176)

\[ \frac{q_e \sigma_0}{I} = -\frac{\sigma_k}{\mu} \lambda_D \]  

(2-177)

(3) \( e_T < e_{cr}, \lambda_p \gg 1 \), \( \psi_s = \psi_k \):

\[ n = \frac{C_k}{\frac{16 \mu D \varepsilon}{\mu} \left( \frac{ZF \lambda_p}{RT \sinh \psi_k} \right)^2 \left( \cosh \psi_k + \sigma_1 \right) + 2} \]  

(2-178)

\[ \frac{q_e \sigma_0}{I} = -\frac{\varepsilon}{8} \frac{RT \sinh \psi_k}{ZF \sinh \psi_k} \left( \frac{RT}{ZF} \sinh \psi_k \right)^2 + \mu \lambda_p^2 \left( \cosh \psi_k + \sigma_1 \right) \]  

(2-179)

(4) \( e_T < e_{cr}, \lambda_p \gg 1 \), \( \sigma_s = \sigma_k \):

\[ n = \frac{(C_k/2)}{1 + \frac{2 \mu \sigma_0}{\sigma_k \sigma_2} \left[ 1 + \left( \frac{\sigma_k S}{C_0 \sigma_p FZ} \right)^2 \right]^{\nu_q} + \sigma_2} \]  

(2-180)

\[ \frac{q_e \sigma_0}{I} = -\frac{(\sigma_p \sigma_k/4 \mu)}{\sigma_2 + \frac{\sigma_k}{2 \mu \sigma_0} \left[ 1 + \left( \frac{\sigma_k S}{C_0 \sigma_p FZ} \right)^2 \right]^{\nu_q}} \]  

(2-181)

In Eqs. (2-168), (2-173), and (2-178) the first term in the denominator usually dominates the second. In Eq. (2-180) the relative magnitudes cannot be predicted as readily.
SECTION 2.7. SUMMARY

In this chapter, we have first reviewed the theory for electroosmosis through a single capillary of circular cross-section. Analytical expressions for the volume flow rate and the current are derived for the special cases of $\lambda << 1$ and $\lambda >> 1$, where $\lambda$ is the ratio of the Debye length to the capillary radius.

We then relate the capillary model to two different models of a porous medium, through bulk macroscopic properties such as the porosity, hydraulic permeability, and specific surface area. We derive the equations for the superficial flow velocity and current density, which must be solved by numerical means, except again for the special cases of $\lambda << 1$ and $\lambda >> 1$.

The first model assumes that the fluid volume in the medium is uniformly distributed among identical, parallel capillaries. This model has been widely used.\textsuperscript{27,30-32} For electroosmosis alone ($\partial \psi / \partial x = 0$), the superficial flow velocity and the current density through the medium can be non-dimensionalized as follows:

$$\frac{q_e}{\left(\frac{n e R T}{\mu Z F} \frac{\partial \phi}{\partial x}\right)} = -2 \int_{0}^{1} (\bar{\psi} - \bar{\psi}_S) \bar{r} \, d\bar{r}$$  \hspace{1cm} (2-182)

$\dagger$Correct to the first order in $\lambda$. 

\[ \frac{i}{(\sigma_{an} \frac{\partial}{\partial x} \frac{\partial}{\partial x})} = -2 \int_{0}^{1} (\cosh \psi) \overline{r} \, d\overline{r} \]

\[ + \frac{2e}{\mu D} \left( \frac{RT}{ZF} \right)^{2} \int_{0}^{1} (\sinh \psi) (\overline{\psi} - \overline{\psi}_{s}) \overline{r} \, d\overline{r} \]  

(2-183)

The right hand side of Eq. (2-182) is a function of the dimensionless parameters \( \lambda \) and \( \overline{\psi}_{k} \) or \( \overline{\sigma}_{k} \), where

\[ \lambda \equiv \text{Debye length ratio} = \frac{\lambda_{D}}{a} ; \]

\[ \overline{\psi}_{k} = \frac{ZF\psi_{k}}{RT} = \text{dimensionless surface potential for the special case of constant surface potential}; \]

\[ \overline{\sigma}_{k} = \frac{ZF \sigma_{k}}{\varepsilon RT} = \text{dimensionless surface charge density for the special case of constant surface charge density}. \]

The left hand side of Eq. (2-183) is the ratio of the actual current density to the current density through an uncharged porous medium of the same geometry. The right hand side is a function of \( \lambda \), \( \overline{\psi}_{k} \) or \( \overline{\sigma}_{k} \), and the dimensionless group \( \frac{e}{\mu D} (RT/ZF)^{2} \).

The second model is based on the "cluster model" proposed by Olsen. Only a fraction of the fluid volume in the medium is available for flow; the remainder is bound within clusters of solid particles. In other words, the effective porosity available to electroosmotic flow is less than the total porosity. The current, however, is assumed to flow through both the clusters and the pore spaces between the clusters; the tortuosities of the two paths are different.

The second model is expected to be more accurate than the first for
porous media such as soils, clays, and wastewater sludges, where the solid particles are in fact associated in "aggregates" or "flocs."

Several possible measures of the "efficiency" of electroosmosis are mentioned in Section 2.5, including the widely used ratio $k_e/k$. We propose that the most appropriate measure is $\eta$, the ratio of the hydraulic to the electrical energy required to produce a given flow rate through a porous medium. For the special case of constant surface potential ($\psi_s = \psi_k$), $\eta$ is approximately proportional to $(\psi_k \lambda)^2$ for $\lambda \ll 1$ and to $(\psi_k / \lambda)^2$ for $\lambda \gg 1$. There is therefore a maximum in $\eta$ at some intermediate value of $\lambda$. For the sewage sludge used in our experiments ($\psi_s \sim -7 \text{ mV}$), the maximum is $\sim 10^{-2}$ (see Fig. 3-10).

A discussion of the application of electroosmosis for sludge dewatering must include the cost of the energy required, as well as the costs for capital amortization and interest, labor, maintenance, and ultimate disposal of the sludge solids (see Section 4.11).
<table>
<thead>
<tr>
<th>Clay</th>
<th>Permeant</th>
<th>Initial Cluster Void Ratio $e_{co}$</th>
<th>Particles per Cluster N</th>
<th>Initial Cluster Diameter $2r_{cl}$ (μm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural kaolinite</td>
<td>Distilled H$_2$O</td>
<td>0.53</td>
<td>83</td>
<td>1.06</td>
</tr>
<tr>
<td>Calcium kaolinite</td>
<td>$10^{-1}$ N CaCl$_2$</td>
<td>0.50</td>
<td>34</td>
<td>0.78</td>
</tr>
<tr>
<td>Sodium kaolinite</td>
<td>$10^{-1}$ N NaCl</td>
<td>0.45</td>
<td>17</td>
<td>0.61</td>
</tr>
<tr>
<td>Sodium kaolinite</td>
<td>$10^{-3}$ N NaCl</td>
<td>0.40</td>
<td>9</td>
<td>0.49</td>
</tr>
<tr>
<td>Sodium kaolinite</td>
<td>$10^{-4}$ N NaCl</td>
<td>0.45</td>
<td>14</td>
<td>0.57</td>
</tr>
<tr>
<td>Sodium kaolinite</td>
<td>0.02% sodium tetraphosphate (by weight)</td>
<td>0.40</td>
<td>7</td>
<td>0.48</td>
</tr>
<tr>
<td>Sodium kaolinite</td>
<td>1.0% sodium tetraphosphate (by weight)</td>
<td>0.30</td>
<td>3</td>
<td>0.33</td>
</tr>
<tr>
<td>Sodium illite</td>
<td>$10^{-1}$ N NaCl</td>
<td>0.74</td>
<td>2150</td>
<td>0.34</td>
</tr>
<tr>
<td>Sodium illite</td>
<td>$10^{-4}$ N NaCl</td>
<td>0.60</td>
<td>110</td>
<td>0.12</td>
</tr>
<tr>
<td>Calcium Boston blue clay</td>
<td>$10^{-1}$ N CaCl$_2$</td>
<td>1.00</td>
<td>26</td>
<td>0.15</td>
</tr>
<tr>
<td>Calcium Boston blue clay</td>
<td>$10^{-3}$ N CaCl$_2$</td>
<td>0.90</td>
<td>19</td>
<td>0.13</td>
</tr>
<tr>
<td>Sodium Boston blue clay</td>
<td>$10^{-1}$ N NaCl</td>
<td>1.10</td>
<td>--</td>
<td>0.26</td>
</tr>
</tbody>
</table>
FIGURE 2-1. Schematic representation of the structure of the double layer according to Stern's theory (after Shaw²¹).
FIGURE 2-3. Dimensionless potential distribution across a cylindrical pore for different values of the Debye ratio ($\lambda = \frac{\lambda_D}{a}$) and a constant surface potential of $\bar{\Psi}_k = 2.79$ (from Gross and Osterle$^3$0)
FIGURE 2-4. \( \lambda \ll 1 \)

FIGURE 2-5. \( \lambda \gg 1 \)
FIGURE 2-6. $\eta$ vs. $(a/\lambda_D)$ for $\psi_K = 70$ mV
(from Gross and Osterle$^{30}$).
FIGURE 2-7. Cluster model.
FIGURE 2-8. Relationship between $e_c$, $e_p$, and $e_T$.

FIGURE 2-9. Current across a cross-section of the porous medium.
CHAPTER 3:
EXPERIMENTS ON ELECTROOSMOTIC FLOW

This chapter describes a series of experiments on electroosmotic flow through plugs of Kaolin clay and anaerobically digested sewage sludge from the Deer Island treatment plant serving Boston, Massachusetts. The principal experimental variables are the porosity of the plugs and the ionic concentration of the external solution. Hydraulic permeability is also measured.

The properties of the clay and the sewage sludge are described in Section 3.1. Section 3.2 describes the methods of sample preparation and the experimental apparatus, and Section 3.3 outlines the experimental procedures. The results are compared in Section 3.4 with the theory. Section 3.5 compares data on electroosmotic flow through Kaolin clay by previous workers\textsuperscript{33,34} with the theory for the cluster model. A summary is given in Section 3.6, along with recommendations for future research.

The comparison of the experimental results with theory is not absolute, since the surface charge density, surface potential, and the tortuosity are not measured independently.
SECTION 3.1. CLAY AND SLUDGE

The Kaolin used is mined and processed at Bath, South Carolina, and marketed as Peerless Kaolin #2 by R.T. Vanderbilt Company, Inc. Some properties of the original raw clay are listed in Table 3-1.

Initially the raw clay was mixed directly with solutions of NaCl in distilled water, prepared into plugs as described below, and tested. The electroosmotic flow was found to vary rapidly with time (Fig. 3-1), possibly due to changes in surface properties of the clay particles from ion exchange.

Subsequent experiments were then conducted on Kaolin made as homo-ionic in NaCl as possible, following the procedure described by Olsen. Batches of 200 grams of raw Kaolin are mixed with 1 liter of 1 N NaCl solution, agitated every 10 minutes for an hour, and allowed to settle over night. The supernatant is decanted, and replaced with fresh 1 N NaCl solution; the clay suspension is again agitated and settled as before. The excess NaCl is then removed by repeatedly replacing the supernatant with distilled water, agitating, and settling until the supernatant is of the desired NaCl concentration. The final suspension is stored at room temperature until used.

The electroosmotic flow through the resulting homoionic clay is found to be time-independent and reproducible, provided certain precautions are observed (see Section 3.3).
SEWAGE SLUDGE

The anaerobically digested sludge is obtained from well-mixed digestors at the Deer Island treatment plant. The mean residence time in the digestors is 28 days, at a temperature of 36°C. Attempts were made to test the sludge immediately after collection; otherwise, the sludge is stored at 4°C until used. The storage time never exceeded 1 week.

The properties of the sludge vary from day to day. Some typical values are given in Table 3-2. We decided not to condition the sludge before testing, based on the findings of Greyson\textsuperscript{20} that additions of poly-electrolytes did not significantly alter the zeta-potential of the sludge solid, and therefore the electroosmotic flow rate. Conditioning would have introduced further variables into the experiments. An unavoidable result of this decision was that the sludge could not be dewatered rapidly to the desired moisture content for testing, so that sludge samples had to be prepared in a different manner from Kaolin samples.
SECTION 3.2. EXPERIMENTAL APPARATUS AND SAMPLE PREPARATION

The apparatus used by some previous workers for measuring electro-osmotic flow is shown in Fig. 3-2. The samples are supported by porous discs of stone or alundum so that a pressure difference can be applied to determine the hydraulic permeability. In Gray's apparatus, the porosity of the sample can be varied by compression with the piston. We have tried a similar apparatus and found that concentration polarization was a severe problem since the circulating fluid on both sides of the sample could not penetrate the porous discs.

We have therefore developed a different method of supporting the sample (Fig. 3-3). The sample is sandwiched between membrane filters (Millipore-MF) and supported by stainless steel screens and backup plates of plexiglas with circulation grooves. Holes are drilled in the backup plates to permit electroosmotic and hydraulic flow through the sample. Fluid is circulated traversely across the face of the sample, with only the membrane filter (thickness 125-150 μm) between the circulating fluid and the sample. The entire assembly will be referred to as the test cell.

The samples are prepared as follows. The dilute suspension (clay or sewage sludge) is allowed to settle, and the supernatant is decanted and reserved. The settled slurry is poured into a compression cell, similar to the oedometer in soil mechanics and to the compressibility-permeability cells used in filtration studies. Figure 3-4 details the compression cell in various configurations. A load is applied to the upper piston and
the slurry is dewatered. Initially, the solid is deposited at the porous ends of the cell in the form of a cake whose thickness increases with time. This dewatering process is referred to as filtration. When the top and the bottom filter cakes meet, further dewatering then occurs by compaction, or consolidation, of the cake. At equilibrium, all internal flow of fluid in the porous cake has ceased, and the applied load is supported entirely by the solid matrix.

During filtration of the sewage sludge the flow resistance of the filter cake was found to be extremely high (specific resistance > $10^{15}$ m/kg), so that equilibrium could not be reached in a matter of hours. We therefore decided to stop the compression once a filter cake of the desired thickness was formed and to use the cake as the sample. Two major problems with this method are non-uniformity of porosity in the plug and possible leakages between the plug and the wall of the test cells. The problems are discussed in more detail in Appendix A.

The filter cakes from the sewage sludge are fragile and rip easily, so to minimize handling the filter cakes are formed directly in the test cell. The cylindrical cell body and one of the end plates are bolted together to form an integral unit. The unit is then bolted to the bottom of the compression cell (Fig. 3-4a). When a filter cake of a thickness greater than that of the cell body is formed, the load on the upper piston is released, and the test cell unbolted from the compression cell. The upper piston is used to gently push the filter cake out of the compression cell. Excess cake is cut off with a .020" stainless steel wire, and the remaining end plate, with its membrane filter and support screen, is bolted
onto the lower part to form a complete test cell.

For Kaolin clay, much higher pressures on the upper piston (up to 300 psi) are used to dewater the clay to the desired porosity. The higher pressure could not be supported by the backup plate of the test cell, so that a different method of sample preparation was used.

The test cell body and a filter assembly are bolted onto the compression cell (Fig. 3-4b). The filter assembly consists of a membrane filter supported by a porous disc of sintered brass which is in turn recessed inside and supported by a plexiglas plate with drainage holes. A load is applied to the upper piston until equilibrium is reached. The filter assembly is then removed, and a solid plexiglas plate is bolted onto the body to the test cell to prevent relative movement between the clay plug and the body. The upper piston is used to push the clay plug out of the compression cell and excess clay is removed again with the stainless steel wire. The body of the test cell, containing the test sample, is then bolted onto the two end plates of the test cell.

The conductivity of the reserved supernatant from the sewage sludge is measured with a conductivity cell (Fig. 3-5) against a reference solution of a known concentration of KCl in distilled water. The normality of a hypothetical KCl solution which would have the same conductivity is used to calculate the Debye length.

The NaCl concentration in the reserved supernatant from the Kaolin clay slurry is measured with a chloride specific ion electrode (Orion Model 94-17) and electrometer (Keithley Model 602) to ±2% accuracy.
The pH of both types of supernatant is measured with a combination electrode (Broadley-James No. 9005) and an electrometer (IL Model 165).

A sketch of the experimental apparatus is shown in Fig. 3-6. The sample is supported in the test cell as described before. For samples of Kaolin clay, the sample and electrode compartments are filled with NaCl solution with the same concentration as that of the supernatant above the original slurry. For sewage sludges, the solution is the sludge supernatant from the settling tanks. All of the compartments are referenced with respect to a common free surface for hydrostatic equilibrium.

To minimize possible problems with electrochemical corrosion, all of the surfaces in contact with the sample and the solution are of non-metals (glass, acrylic, polypropylene, vinyl and polyethylene) with the exception of the stainless steel support screen in the test cells, the epoxy-encapsulated circulation pump, and the platinum power electrodes.

Direct current is normally supplied to the platinum electrodes by a HP Model 6439B D.C. power supply, which can operate in a constant current (0-15 amp) or in a constant voltage mode (0-60 Vdc). For voltages greater than 60 V, the HP is connected in series with a NJE Model TC80-8 D.C. power supply. Passage of current through the inert platinum electrodes is accompanied by electrolysis of water, according to the following reactions:

at the anode: \[ 2H_2O \rightarrow 4H^+ + 4e + O_2 \text{ (gas)} \]

at the cathode: \[ 4e^- + 4H_2O \rightarrow 4OH^- + 2H_2 \text{ (gas)} \]
The electrode compartments are therefore separated from the sample by salt bridges made of ion exchange membranes (Ionics Type 61CZL 183 and 103 QZL 219) are are flushed continuously with the circulation pump. Each salt bridge consists of an anion and a cation exchange membrane placed side by side to allow passage of both Na\(^+\) and Cl\(^-\) ions.

The solution in compartment B is circulated also with the pump through grooves in the backup plates of the test cell. Change in volume in compartment C is used as a measure of the electroosmotic flow rate through the plug. The direction of electroosmotic flow through the plug can be reversed by reversing the polarity of the power electrodes.

Volume change in compartment C is measured by change in the height of the fluid in a pipette, and is recorded on a strip chart recorder via a pressure transducer (PACF Wiancko Model KP 15). The pipette-transducer-chart recorder system is calibrated by adding fluid to the compartment through a syringe. To avoid pressure fluctuations, the solution in compartment C is circulated by a falling piston pump. The circulation rate is regulated by valve 6. The volume swept by the piston in one pass downward (1200 cc) is large enough to maintain circulation for one electroosmotic run. The piston is subsequently raised to its starting position by the electrical circulation pump.

After closing and opening the appropriate valves, the compartments A and B can be pressurized either with a column of fluid or with compressed N\(_2\) gas. The hydraulic flow rate through the sample is again measured with the pipette.
Voltage drop across the sample is measured with a Keithly Model 602 electrometer ($10^{14}$ $\Omega$ internal resistance) through a pair of Ag/AgCl reversible electrodes located between the membrane filter and the sample (Fig. 3-3). The electrodes are formed by electrodeposition (see Ref. 43). Current is measured with a digital multimeter (Fluke Model 8000A).

The temperature of the solution is maintained at 25°C ±1°C by cooling and heating coils in the reservoir, unless otherwise noted.

The mass fraction of the fluid in the plug is determined after the experiment. The moist plug is weighed to .01 gram accuracy, dried in oven at 103°C for 12 hours, and then re-weighed. The porosity can then be calculated once the solid density is known (see Section 3.3).
SECTION 3.3. EXPERIMENTAL PROCEDURE

1. The sample is prepared in the form of a plug and loaded into the test cell, as described in the previous section.

2. The test cell and the electrode chambers are assembled, along with the fluid and electrical connections.

3. The electrical circulating pump is used to fill the compartments with the solution and to flush out all the air bubbles.

4. By closing valves 5 and 7, the fluid in compartment C is isolated from the rest of the system. The pipette-transducer-recorder system is calibrated by adding fluid to the compartment with a syringe.

5. Valve 6 is closed, and valves 5 and 7 opened, so that the piston in the gravity pump is raised to its starting position by the flow from the electrical pump.

6. Valves 5 and 7 are closed, and the fluid in compartment C is then circulated by the falling piston.

7. Each electroosmotic run consists of applying a constant voltage across the plug (as measured by the Ag/AgCl electrodes) and recording the current and the flow rate on the strip chart recorder.

8. The current is periodically reversed, and the duration of each run is typically less than a minute, so as to minimize possible effects of a direct current on the physico-chemical properties of the clay-
water system.

9. After each run, the appropriate valves are opened and the fluid in compartment C is flushed with the electrical pump. The piston in the gravity pump is returned to its starting position.

10. Each experiment consists of a series of runs with progressively increasing and then decreasing voltages to detect any possible hysteresis.

11. At the end of the experiment, the hydraulic permeability of the sample is measured by applying a pressure differential with the fluid column or with the compressed N₂ gas.

12. The apparatus is drained and disassembled. The mass fraction of fluid in the sample is determined.

13. The porosity is calculated with the following equation:

\[
\eta = \frac{1}{1 + \left( \frac{1 - m_c}{m_c} \right) \frac{\rho}{\rho_s}} ,
\]

where \(m_c\) is the mass fraction of fluid and \(\rho\) and \(\rho_s\) are densities of fluid and solid, respectively.
SECTION 3.4. EXPERIMENTAL RESULTS AND COMPARISON WITH CLUSTER MODEL

Two series of experiments were conducted: Series I, on anaerobically digested sewage sludge, and Series II, on Kaolin clay made homoionic in NaCl. The results are summarized in Tables 3-3 through 3-6.

In all of the experiments, the current is linearly proportional to the voltage across the sample. The electroosmotic flow rate, however, is non-linear with respect to the voltage in some of the experiments with \( \lambda < 0.1 \) (see Fig. 3-7, for example). These experiments are marked with an asterisk in Tables 3-3 and 3-5. At this moment we have no explanations for this behavior which would still be consistent with a constant electrical resistance.

We have calculated \( k_e \) and \( \kappa \) using the least-squared method, except for the non-linear cases for which we have used the asymptotic slope of \( q_e \) vs. \( E \) as \( E \to 0 \) (Fig. 3-7).

In Fig. 3-8 we have plotted \( k_e \) as a function of the overall porosity \( n \) for the sewage sludge. Despite the scatter in the data, it is clear that \( k_e \) is not linearly proportional to \( n \), as would be expected if the pore volume in the sample were uniformly distributed among identical pores. In fact the behavior of \( k_e \) suggests that the cluster model is more appropriate.

In Section 2.6, we have shown that for \( e_T > e_{cr} \), and \( \lambda_p \ll 1 \), \( k_e \) and \( (\kappa/\sigma_0) \) are proportional to \( n_p \) [Eqs. (2-158) and (2-159), for example].
Since
\[ n_p = (1 + e_{co})n - e_{co} \quad , \hspace{1cm} (2-128) \]
the tangents to the data points for \( k_e \) and \( (\kappa/\sigma_0) \) should intersect the \( n \) axis at
\[ n = \frac{e_{co}}{1 + e_{co}} \quad . \]

Figure 3-9 shows \( (\kappa/\sigma_0) \) as a function of \( n \) for the sludge. The data is too scattered to be useful. For lack of more data, we will assume that the intercept with the abscissa is 0.55, corresponding to \( e_{co} = 1.22 \).

We cannot determine \( e_{cr} \) without more data. However, if we assume that \( e_{cr} = e_{co} \), then all of the data points in both series satisfy the inequality \( e_T > e_{cr} \).

We should admit that the value of \( e_{co} \) does depend on the state of aggregation of the solid particles, which in turn depends partly on the ionic concentration of the fluid. For the sewage sludge, the equivalent ionic concentration of the supernatant (in molarity of KC\&l) varies within a narrow range (0.045 M to .097 M), so that a fixed \( e_{co} \) is a reasonable approximation.

Knowing \( e_{co} \), we can then calculate \( n_p \) [Eq. (2-128)], \( a_p \) [Eq. (2-133) with \( C_k / \tau^2 \) assumed equal to 5.0], and \( (a_p / \lambda_D) \). \( n \) and \( (q_e \sigma_0 / 1) \) are calculated from the data using the definitions:
\[ \eta = \frac{k_e}{\mu k} \quad ; \hspace{1cm} (2-118) \]
\[ \frac{q_e \sigma_0}{i} = \frac{k_e \sigma_0}{\mu \kappa} \]  

(3-2)\textsuperscript{†}

The two groups of variables are not redundant; only \( \eta \) contains the independently measured hydraulic permeability \( k \).

We next calculate \( \psi_k \) and \( \sigma_k \) using Eqs. (2-169) and (2-1974) and the data for \( (q_e \sigma_0/i) \) from the experiments for which \( \lambda \ll 1 \).\textsuperscript{††} Knowing \( \psi_k \) or \( \sigma_k \), we then calculate the tortuosity \( \tau \) using Eq. (2-158) or (2-161). The results are presented in Table 3-4; we cannot draw a definite conclusion as to whether the sewage sludge has a constant surface potential or a constant surface charge density.

There is a large scatter in the calculated values of \( \tau \). At best we can take the average \( (\tau_{\text{average}} = 1.46) \) which then gives a value of 2.34 for \( C_k \).

Figure 3-10 shows \( \eta \) plotted against \( (a_p/\lambda_D) \) for sewage sludge, along with the two theoretical asymptotes with an assumed constant surface potential of \( \psi_k = -6.9 \text{ mV} \) and \( C_k = 2.34 \). The agreement is reasonable for \( \lambda_p \ll 1 \).

In Fig. 3-11 we show \( (q_e \sigma_0/i) \) vs. \( (a_p/\lambda_D) \) for sewage sludge, as well as the theoretical asymptote for \( \lambda_p \ll 1 \) with \( \psi_w \) again equal to -6.9 mV.

\textsuperscript{†}Equation (3-2) is derived from Eqs. (2-87) and (2-89). \( \kappa_p \) is not included since we are applying the electric field and the pressure gradients separately.

\textsuperscript{††}We have neglected the current through the clusters: essentially we are assuming that \( \tau_C \gg \tau \).
It could be argued that $C_k$ is a "fudge factor" so that a single value of $\psi_k$ can fit the data for both $n$ and $(q_e\sigma_0/i)$. Our calculated value of $C_k$, however, is typical for a porous medium.\(^4\)\(^2\) \(^1\)

The data for Kaolin clay is presented in Table 3-5. There are three experiments for which we expect $\lambda_p << 1$ (to be verified later): Experiments II-4, II-5, and II-6. The variation in $n$ and $k_e$ among these experiments is not sufficient for us to decide between a uniform capillary tube model or a cluster model. The data from Olsen\(^3\)\(^3\) and Gray\(^3\)\(^4\) (Figs. 3-14 and 3-16) suggest that for Kaolin a cluster model is appropriate. The Kaolin they used, however, is from a different source, and we cannot use the value of $e_{co}$ inferred from their data. We will therefore assume, as a first approximation, that the porosity in the clay is uniformly distributed among the capillary tubes, i.e., $e_{co} = 0$.

From Eqs. (3-2), (2-100), (2-101), (2-103), and (2-104), we have, for $\lambda << 1$,

$$\frac{q_e\sigma_0}{i} = \begin{cases} \frac{\varepsilon\psi_k(1 - 2\lambda)}{\mu(1 + \frac{2\psi_k^2}{\alpha\sigma_0\lambda D} + \frac{\varepsilon ZF}{2\mu} + D\varepsilon \frac{ZF}{RT})^2} & \text{for } \psi_s = \psi_k \quad (3-3) \\ \frac{\sigma_k\lambda_D(1 - 2\lambda)}{\mu(1 + \frac{2\lambda D\sigma_k^2}{\alpha\sigma_0} + \frac{1}{2\mu} + \frac{D\varepsilon ZF}{RT})^2} & \text{for } \sigma_s = \sigma_k \quad (3-4) \end{cases}$$

\(^1\)For capillary tubes of circular cross-section, $C_k = 2$. 


As $\lambda \to 0$,

$$\frac{q_e \sigma_0}{i} \to \begin{cases} \frac{-e\psi_k}{\mu} & \text{for } \psi_s = \psi_k \\ \frac{-\sigma_k \lambda_D}{\mu} & \text{for } \sigma_s = \sigma_k. \end{cases} \quad (3-5)$$

Using the data for Expts. II-4 to II-6, and Eqs. (3-5) and (3-6), we first calculate $\psi_k$ and $\sigma_k$ approximately (see Table 3-6). It seems clear that the Kaolin can be modeled as a constant potential surface. We then use Eq. (2-100) to calculate the tortuosity $\tau$. The average value for the three experiments ($\tau = 1.42$) is used in Eq. (2-95) to solve for the pore radius $\alpha$ in all of the experiments. The results are shown in Table 3-6. The values for $\psi_k$ given in the table are obtained from a second calculation using the more exact Eq. (3-3).

In Fig. 3-12 we have plotted $\eta$ as a function of $(a/\lambda_D)$ for Kaolin, along with the theoretical asymptote for $\lambda \ll 1$ from Eq. (2-119), with an assumed value of $\psi_k = -6.9$ mV. We have also shown the asymptote for $\lambda \gg 1$ from Eq. (2-121), with the same $\psi_k$. The theoretical curve for $\lambda = 1 \to 10$ must be obtained by numerical methods, as done by Gross and Osterle$^{30}$ (Fig. 2-6). Despite the scatter, we see that the data points within that range of $\lambda$ do indicate a transition between the two asymptotes.

Figure 3-13 shows $(q_e \sigma_0/i)$ as a function of $(\lambda)^{-1}$ for Kaolin. Equation (3-3) predicts that as $\lambda \to 0$, $(q_e \sigma_0/i) \to$ constant given by Eq. (3-5). The data points do show this behavior. We have again used $\psi_k = -6.9$ mV in plotting the theoretical asymptotes for $\lambda \ll 1$ and $\lambda \gg 1$. We do
not have data for $\lambda \gg 1$; however, the points for $(\lambda)^{-1} < 10$ again do seem to make a smooth transition between the two asymptotes.
SECTION 3.5. COMPARISON OF CLUSTER MODEL WITH DATA FROM PREVIOUS RESEARCHERS

Previous work on sewage sludge has involved the dewatering of the liquid sludge by electroosmotic filtration\[19\] and the dewatering of a filter cake of sewage sludge by electroosmotic consolidation\[18\]. To my knowledge there has not been any previous work on electroosmotic flow through samples of sewage sludge of fixed volume and porosity.

Experimental results on electroosmotic flow through plugs of sodium kaolinite,\[35\] however, are available. Ballou\[33\] and Gray\[34\] measured the electroosmotic flow directly, whereas Olsen\[33\] measured the streaming potential. The electroosmotic permeability $k_e$ can be derived from the data on streaming potential through Saxen's relation:\[23\]

$$
\left( \frac{\Delta \phi}{\Delta \rho} \right)_{i=0} = -\left( \frac{q_{e}}{\Psi} \right)_{\Delta \rho=0} = - \frac{k_e}{\mu \kappa} .
$$

We will compare the data from Olsen and Gray with the theory for electroosmosis. As an approximation we will use the equations for $\Psi << 1$.

Figures 3-14 and 3-16 show the data for $k_e$ as a function of the

\[1\] The theory for these two processes are discussed in Chapter 4.

\[2\] Kaolinite clay made homoionic in NaCl by a procedure similar to that described in Section 3.1.
porosity \( n \). We note that at higher values of \( n \), for which we expect \( \lambda \ll 1 \), the points fall on a straight line which does not pass through the origin. This suggests that the cluster model is appropriate.

For \( \lambda \ll 1 \) and \( \bar{\psi} \ll 1 \), the surface charge density \( \sigma_s \) and the surface potential are related:

\[
\sigma_s \lambda_D = \varepsilon \psi_s. \tag{2-64}
\]

Both sets of data are from experiments with the ionic concentration of the external solution constant, so that \( \lambda_D \) is constant. \( \sigma_s \) is then directly proportional to \( \psi_s \), and we can for convenience assume that the surface has a constant potential of \( \psi_s = \psi_k \). The appropriate equations for \( \lambda \ll 1 \) are then Eqs. (2-128), (2-158) - (2-160), and (2-168) - (2-170).

Equation (2-159) can be rewritten in the following form:

\[
\frac{\kappa}{\sigma_0} = \frac{n_p}{\tau^2} \left\{ 1 + \cosh \bar{\psi}_k \frac{n_c}{n_p} \left( \frac{\tau}{\tau_c} \right)^2 + \frac{\lambda_D}{\sigma_p} \psi_k^2 \left[ \frac{e}{\mu D} + 2 \left( \frac{Z F}{R T} \right)^2 \right] \right\}.
\]

In the bracket on the right hand side, the first term is the conduction through the bulk pore fluid, the second the conduction through the clusters, and the third the surface conductance at the pore walls.

Figures 3-15 and 3-16 show \( \kappa/\sigma_0 \) as functions of \( n \). Olsen's data in Figs. 3-14 and 3-15 indicate that in his experiments the points at higher values of \( n \) correspond to \( \lambda_p \sim 0 \) and negligible cluster conductance; both \( k_e \) and \( \kappa/\sigma_0 \) are directly proportional to \( n_p \) and fall on lines which intersect the abscissa at \( n_p = 0 \), i.e., at
\[ n = \frac{e_{co}}{1 + e_{co}} \approx 0.3 \]

Gray's data in Fig. 3-16 suggest that the surface and/or the cluster conductance is not negligible, since values of \( \lambda_D \) and \( \psi_k \) are larger in Gray's experiment than in Olsen's (see Tables 3-7 and 3-8).

Figure 3-17 shows \( n \) vs. \( (\alpha_p/\lambda_D) \). In Figs. 3-14 through 3-17, we have also plotted the theoretical curves using the following parameters:

<table>
<thead>
<tr>
<th>Source of data</th>
<th>( \psi_k ) (mV)</th>
<th>( \tau^2 )</th>
<th>( (\tau/\tau_C)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Olsen(^{33} )</td>
<td>-15.0</td>
<td>1.76</td>
<td>0.11</td>
</tr>
<tr>
<td>Gray(^{34} )</td>
<td>-25.0</td>
<td>1.60</td>
<td>0.20</td>
</tr>
</tbody>
</table>

It seems from these two examples that the cluster model may be accurate in predicting electroosmotic flow through Kaolin clay. To be certain we would need more data and direct measurements of \( \psi_k \) (or \( \sigma_k \)^\(t\)) to compare with the best fit values.

\[ ^{t} \tau^2 \] and \( (\tau/\tau_C)^2 \) are to my knowledge impossible to measure directly.
SECTION 3.6. SUMMARY AND CONCLUSIONS

The electroosmotic flow through cakes of anaerobically digested sewage sludge can be described by the cluster model. There is considerable scatter in our data for $k_e$ and $\kappa/\sigma_0$ vs. porosity $\eta$, so that only an estimate of $e_{co}$ is possible ($e_{co} \approx 1.2$).

The data from Olsen$^{33}$ and Gray$^{34}$ (Figs. 3-14 through 3-16) show that the cluster model is also appropriate for Kaolin clay. We do not have sufficient data to determine $e_{co}$ for our particular Kaolin, so that as a first approximation, $e_{co}$ is assumed to be zero (the value of $e_{co}$ only affects the calculated pore radius $a_p$).

Figures 3-10 through 3-13 show that for $\lambda \ll 1$, $\eta \propto \lambda^2$ and $(q_e\sigma_0/i) \propto $ constant, as predicted by theory. The value of $\psi_k$ which best fits the data is -6.9 mV for both the sewage sludge and the Kaolin. This value must be confirmed by independent measurements (by electrophoresis, for example) before an absolute comparison can be made between the data and the theory.

The data in Fig. 3-12 seem to verify the theoretical expectation of a maximum in $\eta$. For both the sludge and the Kaolin, the maximum would be about $10^{-2}$. This means that even in the most favorable case, to achieve a given flow rate, the energy required by electroosmosis would have to be over a hundred times greater than that required by applying a pressure difference. Furthermore, for our particular sewage sludge, the maximum possible $\eta$ at $(a_p/\lambda_p) = 3$ would not be achievable in practice.
The ionic concentrations in the sludge fluid are relatively invariable \((\sim 0.05 \text{ N to } 0.1 \text{ N})\), so that \(\lambda_D\) is approximately constant and \(\lambda_p\) varies only with \(\alpha_p\). \(\alpha_p\) is a function of the porosity \(n\) through Eqs. (2-128) and (2-133). At \(n\) of \(\sim 0.75\), the sludge is already in a form of a hard porous mass which can be handled with ease and can be incinerated; this \(n\) corresponds to \((\alpha_p/\lambda_D) = 15\) with \(n \sim 10^{-3}\). There is no reason to dewater the sludge to a porosity \(n < 0.75\), so that the maximum \(n\) which we would encounter in practice is \(\sim 10^{-3}\).

It seems clear then that from an energy point of view, electroosmosis would be a far more costly method of forcing fluid through the sewage sludge than applying a pressure gradient. The overall economic comparison, including capital costs, will be discussed in Section 5.4.

We should note that for \(\psi_S = \psi_k\), \(\eta\) is approximately proportional to \(\psi_k^2\) [see Eqs. (2-119) or (2-168)].\(^\dagger\) There may be other wastewater sludges for which \(\psi_k\) is higher than the \(-6.9 \text{ mV}\) for our particular sludge, with a correspondingly higher \(\eta\).

We do not have data for \(\lambda >> 1\). For sewage sludge, with \(\lambda_D \sim 10 \text{ Å}\), such data would be virtually impossible to obtain. The sludge could be diluted with distilled water to decrease its ionic concentrations and increase \(\lambda_D\); however, other properties may then be affected, such as the surface potential/charge density and the state of aggregation (as

\(^\dagger\)The second term in the denominators of these equations is much less than the first.
expressed in $e_{\text{co}}$).

For Kaolin, $\lambda >> 1$ is possible with very dilute solutions ($C_0 < 10^{-4} \text{ N}$) and very high loads on the compression cell to produce a plug of low porosity (for example, in Test 121 in Olsen, a load of 21.6 tonne/m$^2$ is required to compress the Kaolin to a porosity of 0.23; with $C_0 = 10^{-4} \text{ N}$, the corresponding $\lambda$ would be approximately 2).

Any future research in electroosmotic phenomena should involve porous media with well-defined pore geometry and without the complications of particle aggregation. Koh and Anderson, for example, have used a thin membrane of mica with straight pores made by bombardment with heavy fission fragments and enlarged by etching in aqueous hydrofluoric acid. The number of pores is controlled by the irradiation time, and the pore sizes by the etching time.

The information below is from the manufacturer

1. Chemical Analysis: Air Floated

<table>
<thead>
<tr>
<th>Component</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>SiO₂</td>
<td>44.8%</td>
</tr>
<tr>
<td>Al₂O₃</td>
<td>39.1</td>
</tr>
<tr>
<td>Fe₂O₃</td>
<td>0.5</td>
</tr>
<tr>
<td>TiO₂</td>
<td>1.1</td>
</tr>
<tr>
<td>MgO</td>
<td>trace</td>
</tr>
<tr>
<td>CaO</td>
<td>0.4</td>
</tr>
<tr>
<td>Na₂O</td>
<td>0.0</td>
</tr>
<tr>
<td>K₂O</td>
<td>0.4</td>
</tr>
<tr>
<td>Ignition</td>
<td>13.7</td>
</tr>
</tbody>
</table>

100.0%

Average Sulphate Content: 25 ppm

2. X-ray Analysis:

Peerless Kaolin is well crystallized with Kaolinite being the predominate mineral. A trace of muscovite was noted and no free silica was apparent.

3. pH : 5.0

4. Base Exchange: 3.5 milliequivalents per 100 gm clay

5. Water of Plasticity: 35.2%

6. Dry Modulus of Rupture (50-50 with flint, deaired bars): 20 lbs/in²

7. Screen Analysis (air-floated grade):

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Residue</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0.10%</td>
</tr>
<tr>
<td>325</td>
<td>0.32%</td>
</tr>
</tbody>
</table>

8. Specific Gravity: 2.62 - 2.83

9. Average Particle Size: 1.2 μm
TABLE 3-2. Properties\footnote{Unless otherwise noted, all data is from the laboratory at the plant.} of Anaerobically Digested Sewage Sludge from Deer Island Treatment Plant, Winthrop, Massachusetts.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>pH</td>
<td>7.0 - 7.3</td>
</tr>
<tr>
<td>Solids</td>
<td>2.0 - 4.0% by weight</td>
</tr>
<tr>
<td>Volatile acids</td>
<td>300 - 800 ppm</td>
</tr>
<tr>
<td>Alkalinity</td>
<td>2400 - 2800 ppm</td>
</tr>
<tr>
<td>Chloride</td>
<td>2000 - 2600 ppm</td>
</tr>
<tr>
<td>Loss on ignition (1100°F)</td>
<td>50%</td>
</tr>
<tr>
<td>Digested for 14 days, 98°F</td>
<td></td>
</tr>
<tr>
<td>Specific gravity</td>
<td>1.9 g/cm$^3$ (by author)</td>
</tr>
</tbody>
</table>
TABLE 3-3. Experiments, Series I

Anaerobically Digested Sewage Sludge

<table>
<thead>
<tr>
<th>Experiment</th>
<th>I-1*</th>
<th>I-2</th>
<th>I-3</th>
<th>I-4</th>
<th>I-5</th>
<th>I-6*</th>
<th>I-7*</th>
</tr>
</thead>
<tbody>
<tr>
<td>h (m) × 10^2</td>
<td>1.19</td>
<td>.93</td>
<td>.90</td>
<td>.51</td>
<td>.75</td>
<td>.97</td>
<td>.54</td>
</tr>
<tr>
<td>A (m^2) × 10^4</td>
<td>33.5</td>
<td>21.7</td>
<td>21.7</td>
<td>32.9</td>
<td>33.3</td>
<td>33.6</td>
<td>32.8</td>
</tr>
<tr>
<td>(\sigma_0 (\Omega_m)^{-1})</td>
<td>1.23</td>
<td>.80</td>
<td>.84</td>
<td>.59</td>
<td>.59</td>
<td>.60</td>
<td>.63</td>
</tr>
<tr>
<td>(k (m^2) \times 10^{17})</td>
<td>1.88</td>
<td>.058</td>
<td>2.67</td>
<td>1.61</td>
<td>.71</td>
<td>.48</td>
<td>1.14</td>
</tr>
<tr>
<td>n</td>
<td>.75</td>
<td>.66</td>
<td>.70</td>
<td>.81</td>
<td>.75</td>
<td>.82</td>
<td>.82</td>
</tr>
<tr>
<td>(k_e (kg-m/V-s^2) \times 10^{13})</td>
<td>6.06</td>
<td>4.68</td>
<td>6.88</td>
<td>11.2</td>
<td>9.58</td>
<td>18.9</td>
<td>12.5</td>
</tr>
<tr>
<td>R = (\Delta \phi/I (\Omega))</td>
<td>22.7</td>
<td>50.0</td>
<td>38.3</td>
<td>9.48</td>
<td>10.6</td>
<td>9.89</td>
<td>9.77</td>
</tr>
</tbody>
</table>

**Calculated Data**

| Equivalent KCl concentration (M) | .097 | .064 | .062 | .045 | .045 | .046 | .048 |
| \(\lambda_D (A^0)\) | 8.8 | 10.8 | 11.0 | 12.9 | 12.9 | 12.7 | 12.5 |
| \(k (\Omega_m)^{-1}\) | .156 | .085 | .108 | .165 | .213 | .293 | .170 |
| \(\kappa/\sigma_0\) | .127 | .106 | .129 | .281 | .363 | .485 | .271 |
| \(\eta \times 10^4\) | 1.40 | 54.8 | 1.84 | 5.30 | 6.77 | 28.6 | 9.03 |
| \(q_e \sigma_0/i (m^2/V-s) \times 10^9\) | 5.38 | 4.94 | 6.00 | 4.47 | 2.96 | 4.37 | 5.20 |

*Experiments for which \(q_e\) is non-linear with \((\Delta \phi/\Delta x)\).*
TABLE 3-4. Calculated Data for Sewage Sludge

<table>
<thead>
<tr>
<th>Experiment</th>
<th>I-1*</th>
<th>I-2</th>
<th>I-3</th>
<th>I-4</th>
<th>I-5</th>
<th>I-6*</th>
<th>I-7*</th>
</tr>
</thead>
<tbody>
<tr>
<td>η_p</td>
<td>0.44</td>
<td>0.24</td>
<td>0.33</td>
<td>0.57</td>
<td>0.44</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>α_p (Å°)</td>
<td>308</td>
<td>73</td>
<td>423</td>
<td>250</td>
<td>190</td>
<td>133</td>
<td>205</td>
</tr>
<tr>
<td>α_p/λ_D</td>
<td>35</td>
<td>6.7</td>
<td>38</td>
<td>19</td>
<td>15</td>
<td>10</td>
<td>16</td>
</tr>
<tr>
<td>ψ_k (mV)</td>
<td>-7.36</td>
<td>-8.16</td>
<td>-6.45</td>
<td>-4.43</td>
<td>-7.04</td>
<td>-7.69</td>
<td></td>
</tr>
</tbody>
</table>

Average ψ_k = -6.9 mV

<table>
<thead>
<tr>
<th>σ_k × 10^3 (C^2/m^2)</th>
<th>5.80</th>
<th>5.14</th>
<th>3.46</th>
<th>2.38</th>
<th>3.84</th>
<th>4.26</th>
</tr>
</thead>
<tbody>
<tr>
<td>τ</td>
<td>1.87</td>
<td>1.60</td>
<td>1.43</td>
<td>1.10</td>
<td>1.12</td>
<td>1.50</td>
</tr>
</tbody>
</table>

Average τ = 1.46

*Experiments for which q_e is nonlinear, with (∂φ/∂x).*
<table>
<thead>
<tr>
<th>Experiment</th>
<th>II-1</th>
<th>II-2</th>
<th>II-3</th>
<th>II-4*</th>
<th>II-5*</th>
<th>II-6*</th>
<th>II-7</th>
<th>II-8</th>
<th>II-9</th>
<th>II-10</th>
<th>II-11</th>
</tr>
</thead>
<tbody>
<tr>
<td>h x 10^2 (m)</td>
<td>1.34</td>
<td>1.25</td>
<td>1.36</td>
<td>1.35</td>
<td>1.35</td>
<td>1.38</td>
<td>1.23</td>
<td>1.27</td>
<td>1.25</td>
<td>1.26</td>
<td>1.29</td>
</tr>
<tr>
<td>A x 10^4 (m^2)</td>
<td>33.5</td>
<td>33.5</td>
<td>33.4</td>
<td>33.6</td>
<td>33.4</td>
<td>33.5</td>
<td>33.5</td>
<td>33.6</td>
<td>33.5</td>
<td>33.5</td>
<td>33.5</td>
</tr>
<tr>
<td>C_0 (N)</td>
<td>7.5</td>
<td>7.5</td>
<td>7.0</td>
<td>0.15</td>
<td>0.15</td>
<td>0.375</td>
<td>2.8</td>
<td>8.5</td>
<td>8.5</td>
<td>5.6</td>
<td>5.6</td>
</tr>
<tr>
<td></td>
<td>x10^{-1}</td>
<td>x10^{-4}</td>
<td>x10^{-3}</td>
<td>x10^{-3}</td>
<td>x10^{-4}</td>
<td>x10^{-4}</td>
<td>x10^{-4}</td>
<td>x10^{-4}</td>
<td>x10^{-4}</td>
<td>x10^{-4}</td>
<td></td>
</tr>
<tr>
<td>n_T</td>
<td>0.52</td>
<td>0.64</td>
<td>0.55</td>
<td>0.52</td>
<td>0.50</td>
<td>0.51</td>
<td>0.57</td>
<td>0.56</td>
<td>0.55</td>
<td>0.56</td>
<td>0.56</td>
</tr>
<tr>
<td>k x 10^{17} (m^2)</td>
<td>3.15</td>
<td>10.0</td>
<td>2.89</td>
<td>12.8</td>
<td>1.55</td>
<td>1.49</td>
<td>5.35</td>
<td>3.29</td>
<td>2.56</td>
<td>3.58</td>
<td>2.96</td>
</tr>
<tr>
<td>k_e x 10^{12} (kg-m/V-s^2)</td>
<td>1.08</td>
<td>2.75</td>
<td>1.50</td>
<td>1.15</td>
<td>1.10</td>
<td>1.21</td>
<td>1.10</td>
<td>1.14</td>
<td>1.08</td>
<td>1.00</td>
<td>0.89</td>
</tr>
<tr>
<td>\sigma_0 x 10^{12} (\Omega_m)^{-1}</td>
<td>0.93</td>
<td>0.93</td>
<td>8.33</td>
<td>156</td>
<td>156</td>
<td>367</td>
<td>0.51</td>
<td>0.35</td>
<td>1.05</td>
<td>1.05</td>
<td>0.70</td>
</tr>
<tr>
<td>\lambda_D (\AA^-)</td>
<td>100</td>
<td>100</td>
<td>32.7</td>
<td>7.1</td>
<td>7.1</td>
<td>4.5</td>
<td>135</td>
<td>94</td>
<td>94</td>
<td>115</td>
<td></td>
</tr>
<tr>
<td>k x 10^{2} (\Omega_m)^{-1}</td>
<td>0.97</td>
<td>0.95</td>
<td>2.99</td>
<td>37.9</td>
<td>38.4</td>
<td>102</td>
<td>0.66</td>
<td>0.66</td>
<td>1.40</td>
<td>1.30</td>
<td>0.64</td>
</tr>
<tr>
<td>\kappa/\sigma_0</td>
<td>1.04</td>
<td>1.02</td>
<td>0.36</td>
<td>0.24</td>
<td>0.24</td>
<td>0.28</td>
<td>1.28</td>
<td>1.89</td>
<td>1.33</td>
<td>1.24</td>
<td>0.91</td>
</tr>
<tr>
<td>\eta x 10^{4}</td>
<td>43.2</td>
<td>89</td>
<td>29.1</td>
<td>0.31</td>
<td>2.30</td>
<td>1.09</td>
<td>38.7</td>
<td>67.2</td>
<td>37.1</td>
<td>24.4</td>
<td>47.7</td>
</tr>
<tr>
<td>\frac{q_e\sigma_0}{I} x 10^3 (m^2/V-s)</td>
<td>1.17</td>
<td>3.03</td>
<td>4.69</td>
<td>5.34</td>
<td>5.06</td>
<td>4.91</td>
<td>0.96</td>
<td>0.68</td>
<td>0.92</td>
<td>0.91</td>
<td>1.10</td>
</tr>
</tbody>
</table>

* experiments for which q_e is non-linear with (\partial \phi/\partial x).
TABLE 3-6. Calculated Data for Kaolin Clay

<table>
<thead>
<tr>
<th>First approximation Expt.</th>
<th>II-4*</th>
<th>II-5*</th>
<th>II-6*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_k$ (mV) using Eq. (2-5)</td>
<td>-6.86</td>
<td>-6.50</td>
<td>-6.31</td>
</tr>
<tr>
<td>$\sigma_k \times 10^3$ (C/m²) Eq. (2-6)</td>
<td>-6.94</td>
<td>-6.34</td>
<td>-9.72</td>
</tr>
<tr>
<td>$\tau$ with $\lambda \to 0$ (Eq. 2-123)</td>
<td>1.47</td>
<td>1.43</td>
<td>1.36</td>
</tr>
<tr>
<td>Average $\tau = 1.42$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The following calculations are based on $\tau = 1.42$:

<table>
<thead>
<tr>
<th>Experiment</th>
<th>II-1</th>
<th>II-2</th>
<th>II-3</th>
<th>II-4*</th>
<th>II-5*</th>
<th>II-6*</th>
<th>II-7</th>
<th>II-8</th>
<th>II-9</th>
<th>II-10</th>
<th>II-11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ (Å)</td>
<td>312</td>
<td>502</td>
<td>291</td>
<td>630</td>
<td>224</td>
<td>217</td>
<td>389</td>
<td>308</td>
<td>274</td>
<td>321</td>
<td>292</td>
</tr>
<tr>
<td>$a/\lambda_D$</td>
<td>3.12</td>
<td>5.02</td>
<td>8.9</td>
<td>88.7</td>
<td>31.5</td>
<td>48.2</td>
<td>2.88</td>
<td>1.89</td>
<td>2.91</td>
<td>3.41</td>
<td>2.54</td>
</tr>
<tr>
<td>$\psi_k$ (mV)</td>
<td>-7.03</td>
<td>-6.98</td>
<td>-6.60</td>
<td>Average = -6.87</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_k \times 10^3$ (C/m²)</td>
<td>-6.95</td>
<td>-6.90</td>
<td>-10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Experiments for which $q_e$ is non-linear with $(\partial \psi/\partial x)$. 
FIGURE 3-1. Variation of $q_e/E$ with time, for a sample of raw Kaolin.
FIGURE 3-2. Apparatus used by two previous researchers for measuring electroosmotic flow.
Stainless Steel Screen
Millipore Membrane

Backup Plate
Circulation Gasket
Test Cell Body
Gasket Circulation
Backup Plate

Circulation Grooves

Ag/AgCl Electrodes

Sample

Circulation

FIGURE 3-3. Test cell.
FIGURE 3-4. Compression cell for preparing samples.

(a) Sewage Sludge

(b) Clay

LOAD

SLURRY

CAKE

SINTERED METAL FILTERS

BODY OF TEST CELL

FILTER ASSEMBLY

BACKUP PLATE

LOAD
FIGURE 3-5. Conductivity cell and bridge circuit.
FIGURE 3.6. Experimental apparatus.
FIGURE 3-7. Nonlinear relationship between the electroosmotic flow rate and the electric field.
FIGURE 3-8. Electroosmotic permeability $k_e$ vs. porosity $n$, for sewage sludge.
FIGURE 3-9. $\frac{\pi}{\sigma_0}$ vs. porosity for sewage sludge.
FIGURE 3-10.

POINTS: EXPERIMENTS WITH SEWAGE SLUDGE

LINES: THEORETICAL ASYMPTOTES, ASSUMING $\psi_k = -6.9 \text{ mV}$
POINTS: EXPERIMENTS WITH SEWAGE SLUDGE
LINE: THEORETICAL ASYMPTOTE, ASSUMING \( \psi_k = -6.9 \text{ mV} \)

FIGURE 3-11.
POINTS: EXPERIMENTS WITH SODIUM KAOLINITE

LINES: THEORETICAL ASYMPTOTES, ASSUMING $\psi_k = -6.9 \text{ mV}$

FIGURE 3-12.
FIGURE 3-13.

POINTS: EXPERIMENTS WITH SODIUM KAOLINITE

LINES: THEORETICAL ASYMMPTOTES, ASSUMING $\psi_k = -6.9 \text{ mV}$
Figure 3-14. Electroosmotic permeability $k_e$ as a function of porosity; data from Olsen.33

Points: Data from Olsen
Sodium Kaolinite
$C_0 = 10^{-1} \text{ N NaCl}$

Line: Theory, assuming
$\psi_k = -15 \text{ mV}$
$\tau = 1.33$
$e_{CO} = 0.43$
POINTS: DATA FROM OLSEN
SODIUM KAOLINITE
$C_0 = 10^{-1}$ N NaCl

$\frac{k}{\sigma_0}$

LINE: THEORY, ASSUMING

$\psi_k = -15$ mV
$\tau = 1.33$
$e_{co} = 0.43$
$(\tau/\tau_c)^2 = 0.11$

FIGURE 3-15. Data from Olsen.\textsuperscript{33}
POINTS: DATA FROM GRAY
SODIUM KAOLINITE
C₀ = 10⁻² N NaCl

LINES: THEORY, ASSUMING
Ψₖ = -25 mV
τ = 1.26
eₗ₀ = .69
(τ/τ₀)² = 0.2

FIGURE 3-16. Data from Gray."
POINTS: DATA
- GRAY, $C_0 = 10^{-2}$ N
- OLSEN, $C_0 = 10^{-1}$ N

LINES: THEORY, ASSUMING

A $\psi_k = -15$ mV
B $\psi_k = -25$ mV

FIGURE 3-17. $\eta$ vs. $a_p / \lambda_D$; data from Olsen$^{33}$ and Gray.$^{34}$
CHAPTER 4:
MODELS FOR FILTRATION AND CONSOLIDATION
BY ELECTROOSMOSIS

In this chapter we will present models for the dewatering of suspensions of solid particles in a fluid (such as clay slurries or wastewater sludges) by electroosmosis, with or without the aid of applied pressure gradient. The influence of gravity will be included when it is significant. There are two physical mechanisms of dewatering: filtration and consolidation.

First we will distinguish between two possible forms of suspensions: slurry and gel. In a slurry the particles are freely dispersed, whereas in a gel the particles are held by attractive forces in a loose porous structure with the voids within filled with fluid. The form of a suspension depends on many variables, such as the shape, size distribution, and concentration of the particles, the electrochemical nature of the solid-fluid interface, and the type and the concentration of electrolytes in the solution (a detailed discussion of the gelation of clay particles is given in Van Olphen)
SECTION 4.1. FILTRATION: GENERAL REMARKS

In filtration the suspension is in the form of a slurry. A detailed discussion of filtration under an applied pressure differential is given in Chapter 7.

The important features of pressure filtration are:

(1) As the fluid is forced through the porous filter medium, the solid particles are deposited in the form of a cake on the medium (Fig. 7-2).

(2) The particles in the cake are in close contact, in the form of a porous matrix. The drag force due to the fluid flow through the pores results in a compressive stress on the matrix. The deformation of the matrix decreases the volume of the pores and increases the flow resistance.

(3) As filtration proceeds, the cake thickness increases, so that its flow resistance dominates that of the filter medium.

(4) The fluid in the slurry is virtually in hydrostatic equilibrium, since its velocity is small and the particles are not in contact and cannot transmit any force. All of the applied pressure drop therefore occurs across the cake and the filter medium.

(5) Filtration ends when there is no longer any slurry above the cake; i.e., when all the solid particles have been deposited.

Filtration by electroosmosis is a far more complex process. Consider first a single negatively-charged particle in an electrolyte solution
(Fig. 4-1).\textsuperscript{+} Under an applied electric field, the particle is attracted to the anode and the counterions in the double layer to the cathode. The particle will then accelerate until it reaches a steady state velocity $v_{se}$ (electrophoretic velocity) relative to the fluid velocity at which the electrostatic and the gravitational forces are balanced by three retarding forces:

1. the fluid drag force due to the particle velocity;
2. the drag force due to the migration of the counterions in the double layer and the consequent fluid flow towards the cathode. The counterions which migrate out of the double layer are continuously replaced by counterions in the fluid ahead of the particle;
3. the local electrostatic force due to the displacement of the electrostatic center of the counterions relative to that of the particle ("relaxation effect"). In other words, the externally applied electric field causes a redistribution of the charge density in the double layer, which in turn affects the effective electric field experienced by the particle.

The last two retarding forces are dependent on the ratio of the Debye length to the particle radius.

Let us consider next the other extreme of a particle closely packed with others in a cake and immobile (Fig. 4-2). Under an applied electric

\textsuperscript{+}The discussion also applies to a positively-charged particle by merely reversing the sign of the counterions and the direction of the electrophoretic and electroosmotic velocity.
field, only the counterions are free to migrate (towards the cathode) and to transfer their momentum to the fluid through viscous drag. The resulting fluid flow is called electroosmosis. In steady state, the electrostatic force on the counterions in the double layer, as a body force on the fluid, is transferred to the solid as a shear stress on the surface of the particle. The surface of the particle carries a fixed net charge equal in magnitude and opposite in polarity to the net charge in the double layer. The electrostatic force on the particle due to the fixed charge is therefore equal and opposite to the force due to the shear stress, and there is no net force on the particle due to electroosmosis. The particle can experience a net force only if it is applied externally and transmitted into the solid matrix through the points of contact between the particles. For electroosmotic flow through a cake, there is no such force.

The first discussion of the single particle describes electrophoresis in an infinitely dilute suspension; the second describes electroosmosis in a highly concentrated suspension, if the matrix of particles in the cake can be regarded as such. The transition between electrophoresis and electroosmosis occurs in the concentration range between the two extremes.

As the particle concentration increases, the average distance between the particles decreases. The average velocity of the particles $v_{se}$ then decreases due to mutual fluid flow interference.† Other particle-to-particle

†A similar phenomena is the retarded sedimentation of a suspension (see Refs. 45 and 46).
interactions are also possible, such as electrostatic repulsion if the
double layers overlap. At a high enough concentration, $v_{se}$ approaches
zero, so that the particles are convected by the fluid unless constrained,
by a filter for example.

From the above model we would expect that electroosmotic dewatering
of a suspension cannot begin unless the particle concentration in the
slurry is high enough or unless a cake is already formed on the filter
medium by some other means, such as pressure filtration or gravity drain-
age. This conclusion in fact has been confirmed experimentally by Greyson.\textsuperscript{20}
SECTION 4.2. CONSOLIDATION: GENERAL REMARKS

The cake formed during filtration can be further dewatered by consolidation. Our following discussion applies to the consolidation of other compressible porous mediums as well, such as soils. We begin with consolidation under pressure, which has been extensively studied in soil mechanics and is discussed in Chapter 7. We first make the simplifying assumptions that the particles in the solid matrix are in point contact and that the matrix is perfectly elastic. The important features of pressure consolidation are:

1. The solid matrix deforms as the fluid is forced out of the pores.
2. A force balance gives the following relationship between the compressive stress in the solid and the fluid pressure in the pores (Fig. 4-3):
   \[ P_0 = p_s + p \]  
   \( P_0 \) = externally applied pressure (relative to atmospheric);  
   \( p_s \) = solid stress = \( \frac{dF_s}{dA} \) = force on the solid matrix per unit cross-sectional area of the porous medium;  
   \( p \) = fluid pressure (relative to atmospheric).
3. At the end of consolidation, the fluid flow in the cake ceases and the pressure \( p \) is zero throughout. The applied pressure \( P_0 \) is then entirely supported by the solid matrix.

The distributions of fluid pressure, solid stress, and porosity in the cake are shown qualitatively in Fig. 4-3 as a function of \((x/L)\),
where $L$ is the instantaneous thickness of the cake.

Consolidation by electroosmosis is again far more complex than consolidation by pressure. We next consider an electrode arrangement as shown in Fig. 4-4, and assume again that the fixed charge on the solid is negative. The cake is in contact with an electrolyte solution only on the bottom surface near the cathode, and the anode is in contact with the upper surface under its own weight. When a field is applied, the electroosmotic flow would be away from the anode and towards the cathode. It is clear that electrical contact between the anode and the fluid would be broken and the electroosmotic flow stopped† unless the solid matrix deforms in such a way that there is no relative motion between the fluid and the matrix at the anode surface.

There are some situations in which this may be possible. We must first make some assumptions:‡‡

1. the cake, or porous medium, is fully saturated with fluid;
2. the physical-chemical properties of the solid-fluid interface are uniform throughout and constant with time;
3. no reactions occur at the electrodes which would affect the electroosmotic flow; in particular, there is no electrolysis

†The progressive decrease in the area of contact between the anode and the fluid, with consequent increase in electrical resistance, is a well-known problem in electroosmotic consolidation of soils (see Ref. 47).

‡‡The validity of the assumptions is discussed in Section 4.12.
and gas evolution at the anode;

(4) the particles in the solid matrix are in point contact.

The superficial velocity through the porous medium, \( q \), can be expressed as follows [Eqs. (2-86) - (2-88)]:

\[
q = - \frac{k_e}{u} \frac{\partial \phi}{\partial x} - \frac{k}{u} \frac{\partial p}{\partial x} = n(v_f - v_s) .
\] (4-2)

\( v_f \) and \( v_s \) are the average absolute velocities of the fluid and the solid, respectively.

At the anode \((x = 0)\), we want \( v_f = v_s \), so that

\[
\left( \frac{\partial p}{\partial x} \right)_o = - \frac{k_e}{k} \left( \frac{\partial \phi}{\partial x} \right)_o .
\] (4-3)

One of the ways in which Eq. (4-3) can be satisfied is to use an impermeable anode (e.g., a sheet of metal) as shown in Fig. 4-6. As the electric field is applied, the electroosmotic flow away from the anode will result in a negative fluid pressure (relative to atmospheric) near the anode such that the hydraulic backflow would balance the electroosmotic flow, and the anode would remain in electrical contact with the fluid. A force balance over an element of the medium as shown in Fig. 4-6 gives the relationship between the stress on the solid matrix and the fluid pressure:

\[
p_s + p = 0 ,
\] (4-4)

where \( p_s \) is defined as in Eq. (4-1).
A negative pressure $p$ will cause a compressive (positive) stress $p_s$ on the solid matrix; therefore, as the fluid is forced out of the porous medium by electroosmosis, the solid matrix is compressed and the thickness of the medium decreases with a decrease in the volume fraction of fluid.

We can also apply an additional pressure externally to the impermeable anode, so that the porous medium consolidates under both pressure and electroosmosis.

If the anode is porous (e.g., a wire screen), then Eq. (4-3) can still be satisfied if the surface tension forces are large enough so that the fluid-air interface is not drawn into the porous medium and away from the anode. Figure 4-5 shows a typical pore at the surface adjacent to the anode. When an electric field is applied, the electroosmotic flow away from the anode would draw the fluid-air interface into the shape of a concave meniscus in the pore (point 1 in Fig. 4-5). The pressure difference across the meniscus at any point is:

$$\Delta p = 0 - p = \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right), \quad (4-5)$$

where $p$ is again the gage pressure, $R_1$ and $R_2$ are two radii of curvature which define the meniscus, and $\gamma$ is the surface tension (joules per $m^2$). Note that $p$ goes more negative as the pore radius decreases.

In the coordinate system shown in Fig. 4-4, at $x = 0$,

$$\frac{\partial \phi}{\partial x} < 0 \quad q_e = -\frac{k}{\mu} \frac{\partial \phi}{\partial x} > 0$$

$$\frac{\partial p}{\partial x} > 0 \quad q_h = -\frac{k}{\mu} \frac{\partial p}{\partial x} < 0$$

$$\quad (4-6)$$
and 

\[ q = q_e + q_h \] 

Clearly, if \( q_e = -q_h \), then the meniscus is stationary; if \( |q_e| > |q_h| \), then \( q > 0 \) and the meniscus is drawn further into the pore. At the narrowest point in the pore (point 2) the fluid pressure reaches a minimum. If \( |q_e| \) is still greater than \( |q_h| \), then the meniscus will be pulled into the medium until it reaches the next narrow constriction in the pore (point 3).

We are assuming that the electrical resistance between the anode and the fluid is negligible if the meniscus is within a few particle diameters of the surface (~1 \( \mu \)m); the current can flow, for example, through surface films of fluid on the particles. Therefore, if \( q = 0 \) at \( x = 0 \), then electroosmosis continues.

A force balance as shown in Fig. 4-5 again results in Eq. (4-4). Since \( p < 0 \), the solid matrix is again under a compressive stress, and the cake thickness decreases as the fluid flows out of the medium under electroosmosis.

In actuality, the pores are of a distribution of sizes, so that \( q < 0 \) at \( x = 0 \) does not imply that the menisci in all of the pores would be uniformly drawn into the porous medium. The larger pores will empty first, then progressively smaller ones, so that the area available for current conduction from the anode to the fluid progressively decreases and the electrical resistance between the electrodes increases. We can keep the anode wetted by pressurizing the space above the cake and combining consolidation by pressure and electroosmosis.
SECTION 4.3. A MODEL FOR ELECTROOSMOTIC FILTRATION

This section describes models for filtration by electroosmosis with the electrodes arranged as shown in Fig. 4-7 or 4-8. For Fig. 4-8 the anode is movable so as to maintain electrical contact with the surface of the slurry, whereas in Fig. 4-7 the anode is in a fixed position immersed in the slurry. In both electrode arrangements the cathode is porous and supports a filter medium. The coordinate system is defined in the figures.

We first make the following assumptions:

(1) The suspension and the filter cake are fully saturated with fluid and can be described by the cluster model (Section 2.6).

(2) The surface charges on the solid particles are negative and uniformly distributed over the surfaces.

(3) The surface charge density and surface potential are constant; in particular, they are not affected by the products of electrode reactions.

(4) The temperature, the viscosity, and the electrical conductivity of the fluid are constant.

We must first form a cake on the filter medium by gravity drainage, and then apply the electric field. The subsequent fluid flow downward through the cake is due to the combined effects of electroosmosis and gravity drainage. The net stress on the matrix of particles in the cake due to the electric field is zero, as explained on p. 126 and the compressive stress due to the weight of the overlying slurry is negligible.
for small slurry depth, so that we can assume that the porosity in the cake is approximately uniform.

We first consider the motion of the clusters in the slurry. If a cluster is regarded as a single large particle, then its velocity $v_{sc}$ relative to the fluid velocity $v_{fs}$ is:

$$v_{sc} = \theta_1 \left[ -\frac{\theta_2 \psi}{\mu} \frac{\partial \phi}{\partial x} + \frac{2(\rho_s - \rho)g r_c^2}{9\mu} \right]. \quad (4-7)$$

$\theta_2$ is a non-dimensional coefficient which accounts for the effect of "relaxation" and "electrophoretic retarding" on the electrophoretic velocity. Numerical computations for $\theta_2$ are given in Wierserna et al. for spherical particles. For $\lambda D/r_c \ll 1$, $\theta_2 \approx 1$. $\theta_1$ is also non-dimensional and accounts for the mutual fluid flow interactions between the clusters. The dependence of $\theta_1$ on the volume fraction of the clusters, $f_c$, has been extensively studied with regard to retarded sedimentation (see Refs. 45 and 46). Batchelor, for example, has derived the

---

The maximum compressive stress is at the filter medium, where $p_s$ [as defined in Eq. (4-1)] is given by

$$p_s = \frac{(\rho_s - \rho)gh}{1 + e_{av}} ,$$

where $e_{av}$ is the average void ratio in the cake. The decrease in void ratio would be:

$$\Delta e \sim \frac{\partial e}{\partial p_s} p_s$$

for typical values for a cake of sewage sludge, with $h = 2$ cm, $\Delta e \sim 10^{-3}$. 

---
equation
\[ \theta_1 = 1 - 6.55 \, f_c \quad , \quad (4-8) \]

which is correct to the order \( f_c \) for \( f_c \ll 1 \).

\( f_c \) in the slurry can be expressed in terms of \( n_s \), the volume fraction of fluid in the slurry:
\[ f_c = (1 + e_{co})(1 - n_s) \quad . \quad (4-9) \]
\( e_{co} \) is defined in Fig. 2-8.

If the upward electrostatic force on the cluster is larger than the downward gravitational force, then \( v_{sc} < 0 \) and the slurry will become more concentrated with time; and vice versa. Furthermore, if \( v_{sc} \neq 0 \), then there would be a vertical gradient in the volume fraction of solid particles.

An analytical solution is possible only if \( v_{sc} = 0 \). \( n_s \) is then constant and equal to \( n_0 \), the original volume fraction of fluid in the slurry. From volume conservation:
\[ h = \left( \frac{1 - n_0}{n_0 - n_k} \right) v_1 \quad (4-10) \]
\[ H_s = H_0 - \frac{(1 - n_k)}{(n_0 - n_k)} v_1 \quad (4-11) \]

where \( n_k \) = volume fraction of fluid in the cake (or the porosity);
\( H_s \) = height of slurry above the cake;
\( h \) = thickness of the cake;
\( H_0 = \) initial height of slurry above the filter;
\( v_1 = \) volume of filtrate per unit area.

We will assume that:

1. The electric power supply can be regulated to provide either constant current density \( i_0 \) or a constant voltage difference \( \Delta \phi_0 \).
2. The surface of the solid can be characterized as either with a constant surface potential \( \psi_k \) or a constant surface charge density \( \sigma_k \).
3. \( \lambda_D/a_p \ll 1 \) , \( \lambda_D/a_c \ll 1 \).
4. \( q_e \) is given by Eq. (2-158) or (2-161).
5. \( \kappa_k \), the overall electrical conductance of the cake, is given by Eq. (2-159) or (2-162), with \( n_p \) and \( \tau \) replaced by \( n_{pk} \) and \( \tau_k \), respectively.
6. \( \kappa_s = n_{ps} \sigma_0/\tau_s^2 \). \( n_{ps} \) and \( \tau_s \) are the porosity and the tortuosity of the fluid passages between the clusters in the slurry. We are neglecting surface conductance and conduction through the clusters.

The voltage difference between the anode and the cathode is the sum:

\[
\Delta \phi = \Delta \phi_s + \Delta \phi_k + \Delta \phi_e, \quad (4-13)
\]

where \( \Delta \phi_s = \) voltage drop across the slurry above the cake;
\( \Delta \phi_k = \) voltage drop across the cake;
\( \Delta \phi_e = \) voltage drops at the electrodes due to polarization.
\( \Delta \phi_e \) depends on the electrode material and type and concentration of the electrolyte, as well as on the current density. We will assume for simplicity that \( \Delta \phi_e \approx \text{constant} \). The reader is referred to Vetter\(^6\) for a detailed discussion.

From continuity of current we can derive the following equations for \( \Delta \phi \):

For a movable anode,

\[
\Delta \phi = i \left\{ \frac{H_0}{\kappa_S} + \frac{v_1}{(n_0 - n_k)} \left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_S} \right] \right\} + \Delta \phi_e \tag{4-14}
\]

and for a fixed anode, at distance \( L_a \) from the cathode,

\[
\Delta \phi = \Delta \phi_e + i \left\{ \frac{L_a}{\kappa_S} + \left( \frac{\kappa_S - \kappa_k}{\kappa_S \kappa_k} \right) \left( \frac{1 - n_0}{n - n_k} \right) v_1 \right\} \tag{4-15}
\]

\( q_e \) is related to \( i \) by the following:

\[
q_e = -\frac{k_e}{\mu} \left( \frac{\partial \phi}{\partial x} \right)_k = \frac{k_e}{\mu} \frac{\Delta \phi_k}{h} = \frac{k_e i}{\mu \kappa_k} \tag{4-16}
\]

We may include a pressure-drive component in the superficial flow velocity. The pressure gradient \((\partial p/\partial x)\) in the cake may be due to the hydrostatic pressure of the fluid in the slurry above the cake or due to an externally applied pressure difference.

For example, if for \( 0 \leq t \leq t_0 \) we form a cake on the filter medium by gravity drainage alone, then

\[
q = q_r = \frac{k}{\mu} \frac{\rho g H_s}{h} \tag{4-17}
\]
Substituting for $H_s$ and $h$ from Eqs. (4-10) and (4-11), we have

$$q = \frac{dv_1}{dt} = \frac{a_1}{v_1} + a_2,$$  \quad (4-18)

where

$$a_1 = \frac{k p g}{\mu} \left( \frac{(n_0 - n_k) H_0}{(1 - n_0)} \right)$$

$$a_2 = -\frac{k}{\mu} \rho g \left( \frac{1 - n_k}{1 - n_0} \right)$$  \quad (4-19)

Equation (4-18) can be integrated to give:

$$\frac{a_2}{a_1} v_1 - \ln \left( 1 + \frac{a_2 v_1}{a_1} \right) = \frac{a_2^2}{a_1} t,$$  \quad (4-20)

The remainder of this chapter is divided into short sections. Each section describes a model for a particular application of electroosmotic dewatering, with or without the aid of pressure and of gravity.

In the final section, we will discuss the practical problems in applying electroosmosis.
SECTION 4.4. ELECTROOSMOTIC FILTRATION WITH MOVABLE ANODE, CONSTANT CURRENT DENSITY, AND GRAVITY

We assume that:

(1) the electrodes are arranged as in Fig. 4-8;

(2) the current density is constant \( i_0 \);

(3) the effect of gravity is included, i.e.,

\[
q = q_e + q_r \quad \text{for} \quad t \geq t_0.
\]

From Eqs. (4-16), (2-158), and (2-161), we have

\[
q_e = \text{constant}
\]

\[
\begin{align*}
&= - \frac{n_{pk} \sigma_k \lambda_0}{\mu \tau_k^2 \kappa_k} (1 - 2 \lambda_d) i_0 \quad (\sigma_s = \sigma_k) \\
&= \frac{n_{pk} \epsilon \psi_k}{\mu \tau_k^2 \kappa_k} (1 - 2 \lambda_d) i_0 \quad (\psi_s = \psi_k).
\end{align*}
\]  

(4-21)  

(4-22)

For \( 0 \leq t \leq t_0 \), \( q \) is due entirely to gravity drainage, and is given by Eq. (4-18). \( v_1 \) is related to \( t \) by Eq. (4-20). For \( t \geq t_0 \),

\[
q = \frac{dv_1}{dt} = q_e + q_r = q_e + \frac{a_1}{v_1} + a_2
\]  

(4-23)

\[
\frac{a_3}{a_1} (v_1 - v_{10}) - \ln \left( \frac{a_3 v_1 + a_1}{a_3 v_{10} + a_1} \right) = \frac{a_3^2}{a_1} (t - t_0).
\]  

(4-24)

\( a_1 \) and \( a_2 \) are defined in Eq. (4-19) and
\[ a_3 = q_e + a_2 \]  
\[ v_{10} = v_1(t_0) \]

\[ \Delta \phi \] is given by Eq. (4-14) with \( i = i_0 \).

\[ \Delta \phi = i_0 \left\{ \frac{H_0}{\kappa_S} + \frac{v_1}{(n_0 - n_k)} \left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_S} \right] \right\} + \Delta \phi_e \]

The electrical power required per unit filter area is then:

\[ P_e = \Delta \phi i_0 = i_0^2 \left\{ \frac{H_0}{\kappa_S} + \frac{v_1}{(n_0 - n_k)} \left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_S} \right] \right\} + i_0 \Delta \phi_e \]

and the energy required per unit filter area for \( t \geq t_0 \) is

\[ W_e = \int_{t_0}^{t} P_e \, dt \]

\[ = (t - t_0) \left( i_0 \Delta \phi_e + \frac{H_0 i_0^2}{\kappa_S} \right) + \frac{i_0^2}{(n_0 - n_k)} \]

\[ \times \left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_S} \right] \int_{t_0}^{t} v_1 \, dt \]

To evaluate the integral in the second term on the right hand side, we need Eq. (4-23), which can be written in the form:

\[ \frac{v_1 \, dv_1}{a_1 + a_3 v_1} = dt \]

Then,
\[
\int_{t_0}^{t} v_1 \, dt = \int_{t_0}^{t} \frac{v_1^2}{a_1 + a_3 v_1} \, dv_1
\]

\[
= \frac{a_1^2}{a_3^3} \ln \left( \frac{a_1 + a_3 v_1}{a_1 + a_3 v_{10}} \right) - \frac{a_1}{a_3^2} \left( v_1 - v_{10} \right)
+ \frac{1}{2a_3} \left( v_1^2 - v_{10}^2 \right). \quad (4-29)
\]

Suppose that filtration ends at \( t = t_f \), when all the solid in the initial suspension has been deposited as a cake. From mass conservation, the volume of filtrate per unit filter area at \( t = t_f \) is

\[
v_{1f} = \left( \frac{n_0 - n_k}{1 - n_k} \right) H_0. \quad (4-30)
\]

\( t_f \) is the filtration time required and is given by Eq. (4-24).

\[
t_f = t_0 + \frac{(v_{1f} - v_{10})}{a_3} - \frac{a_1}{a_3^2} \ln \left( \frac{a_1 + a_3 v_{1f}}{a_1 + a_3 v_{10}} \right). \quad (4-31)
\]

\( W_f \) is the energy required per unit filter area per filtration cycle, and is given by Eqs. (4-28) and (4-29), with \( t \) replaced by \( t_f \) and \( v_1 \) by \( v_{1f} \).

Finally, the mass of solid filtered per unit area per cycle is:

\[
w = \rho_s (1 - n_0) H_0. \quad (4-32)
\]

\( t_f \), \( W_f \), and \( w \) are used to derive the filtration cost per unit mass of solid, as described in Section 4.11.
SECTION 4.5. ELECTROOSMOTIC FILTRATION WITH MOVABLE ANODE, CONSTANT CURRENT DENSITY, AND NEGLCITING GRAVITY

We assume that:

1. the electrodes are arranged as in Fig. 4-8;
2. the current density \( i \) is constant = \( i_0 \);
3. for \( t \geq t_0 \), the effect of gravity can be neglected; i.e., \( q_e \gg q_r \) and \( q = q_e \);
4. the solid surface has a constant surface potential \( \psi_s = \psi_k \).\(^\dagger\)

We must still form an initial cake by gravity drainage, so that, for \( 0 \leq t \leq t_0 \), Eqs. (4-18) and (4-20) are applicable. For \( t \geq t_0 \),

\[
q = q_e = \text{constant, given by Eq. (4-22)}
\]

\[
v_1 - v_{10} = q_e(t - t_0) \quad . \tag{4-33}
\]

\( \Delta \phi \), \( P_e \), and \( \psi_e \) are given by Eqs. (4-26), (4-27), and (4-28), respectively, with \( v_1 \) given by Eq. (4-33) for \( t \geq t_0 \).

\[
\Delta \phi = \Delta \phi_e + i_0 \left\{ \frac{H_0}{\kappa_S} + \left[ \frac{v_{10} + q_e(t-t_0)}{n_0 - n_k} \right] \left[ \frac{(1-n_0)}{\kappa_k} - \frac{(1-n_k)}{\kappa_S} \right] \right\} \tag{4-34}
\]

\[
P_e = \Delta \phi i_0 \quad \tag{4-35}
\]

\(^\dagger\)For constant surface charge density \( \sigma_s = \sigma_k \), we merely replace \( (\varepsilon \psi_k) \) by \( (\sigma_k \lambda_D) \). See Eq.
\[ W_e = (t - t_0) \left( i_0 \Delta \phi_e + \frac{H_0 i_0^2}{\kappa_S} \right) + \frac{i_0^2}{(n_0 - n_k)} \left[ \frac{(1-n_0)}{\kappa_k} - \frac{(1-n_k)}{\kappa_S} \right] \]

\[ \times \left[ v_{10}(t - t_0) + \frac{q_e}{2} (t-t_0)^2 \right] \] (4-36)

At the end of filtration, \( t = t_f \), and the total volume of filtrate per unit area is given by Eq. (4-30). From Eq. (4-33), the filtration time \( t_f \) is:

\[ t_f = \frac{(v_{1f} - v_{10})}{q_e} + t_0 \] (4-37)

and the energy required per unit area per filtration cycle is:

\[ W_f = W_e(t_f) = \left( i_0 \Delta \phi_e + \frac{H_0 i_0^2}{\kappa_S} \right) \frac{(v_{1f} - v_{10})}{q_e} \] (4-38)

\[ + \frac{i_0^2}{(n_0 - n_k)} \left[ \frac{(1-n_0)}{\kappa_k} - \frac{(1-n_k)}{\kappa_S} \right] \left[ \frac{v_{10}(v_{1f} - v_{10})}{q_e} + \frac{(v_{1f} - v_{10})^2}{2q_e} \right] \]

Finally, the solid mass filtered per unit area is given by Eq. (4-32).
SECTION 4.6. ELECTROOSMOTIC FILTRATION WITH MOVABLE ANODE, CONSTANT VOLTAGE DIFFERENCE, AND NEGLECTING GRAVITY

We assume that:

1. the electrodes are arranged as in Fig. 4-8;
2. the voltage difference $\Delta \phi$ is constant $= \Delta \phi_0$;
3. for $t > t_0$, the effect of gravity is negligible;
4. the solid particles have constant surface potential ($\psi_s = \psi_k$).

The current density is given by Eq. (4-14), with $\Delta \phi = \Delta \phi_0$.

\[
i = \frac{\Delta \phi_0 - \Delta \phi_e}{\frac{H_0}{\kappa_s} + \frac{v_1}{(n_0 - n_k)} \left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_s} \right]} \quad , \quad (4-39)
\]

and from Eqs. (4-15), (4-14), and (2-158),

\[
q_e = - \frac{n_{pk} \varepsilon \psi_k}{\tau_k^2 \mu} \frac{(1 - 2\lambda_p)}{\kappa_k} \left( \Delta \phi_0 - \Delta \phi_e \right)
\]

\[
\frac{H_0}{\kappa_s} + \frac{v_1}{(n_0 - n_k)} \left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_s} \right] \quad . \quad (4-40)
\]

For $0 \leq t \leq t_0$, Eqs. (4-18) and (4-20) are applicable. For $t > t_0$

\[
q = \frac{dv_1}{dt} \approx q_e = \frac{b_1}{b_2 + b_3 v_1} \quad (4-41)
\]

\[
b_2(v_1 - v_{10}) + \frac{b_3}{2} (v_1^2 - v_{10}^2) = b_1(t - t_0) \quad (4-42)
\]
or
\[ \nu_1 = \frac{b_2}{b_3} \left\{ -1 + \left[ b_4 + b_5(t - t_0) \right]^{\frac{1}{2}} \right\}, \quad (4-43) \]

where
\[ b_1 \equiv -\frac{n_{pk}}{\tau_k^2} \frac{e\psi_k}{\mu} \frac{(1 - 2\lambda_p)}{\kappa_k} (\Delta \phi_0 - \Delta \phi_e) \quad (4-44) \]

\[ b_2 \equiv \frac{H_0}{\kappa_s} \]

\[ b_3 \equiv \frac{1}{(n_0 - n_k)} \left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_s} \right] \]

\[ b_4 \equiv 1 + \frac{2b_3}{b_2} \left( \frac{b_3}{2} \nu_{10}^2 + b_2 \nu_{10} \right) \]

\[ b_5 \equiv \frac{2b_3b_1}{b_2^2} \]

The electrical power required per unit area is:
\[ P_e = \Delta \phi_0 i = \frac{\Delta \phi_0(\Delta \phi_0 - \Delta \phi_e)}{b_2 + b_3 \nu_1} = \frac{\Delta \phi_0(\Delta \phi_0 - \Delta \phi_e)}{b_2 \left[ b_4 + b_5(t - t_0) \right]^{\frac{1}{2}}} \quad (4-46) \]

and the energy required per unit area is
\[ W_e = \int_{t_0}^{t} P_e dt = \frac{\Delta \phi_0(\Delta \phi_0 - \Delta \phi_e)b_2^{3/2}}{b_1b_3} \left\{ \left[ 1 + \frac{b_5}{b_2} (t - t_0) \right]^{\frac{1}{2}} - 1 \right\} \quad (4-47) \]

If filtration ends when there is no longer any slurry above the cake, then \( \nu_{1f} \) is given by Eq. (4-30), and
\[ t_f = t_0 + \frac{b_2}{b_1} (v_{1f} - v_{10}) + \frac{b_3}{2b_1} (v_{1f} - v_{10}^2) \]  \hfill (4-48) \\

\[ W_f = W_e(t_f) = \frac{\Delta \phi_0 \Delta \phi_0 - \Delta \phi_e}{b_1 b_3} \left\{ \left[ 1 + \frac{b_5}{b_1} (v_{1f} - v_{10}) \right] + \frac{b_5 b_3}{2b_1 b_2} (v_{1f} - v_{10}^2) \right\}^{\frac{1}{2}} - 1 \]  \hfill (4-49) \\

The mass of solid filtered per unit area is given by Eq. (4-42).
SECTION 4.7. ELECTROOSMOTIC FILTRATION WITH FIXED ANODE, CONSTANT VOLTAGE DIFFERENCE, AND GRAVITY

We assume that:

1. the electrodes are arranged as in Fig. 4-7;
2. the voltage difference $\Delta \phi$ is constant $= \Delta \phi_0$;
3. the effect of gravity is included, i.e.,
   \[ q = q_e + q_r \]
4. the solid particles have constant surface potential ($\psi_s = \psi_k$).

We first form a cake on the filter medium by gravity drainage, so that, for $0 \leq t \leq t_0$, Eqs. (4-18) and (4-20) apply.

For $t \geq t_0$, the current density is then given by Eq. (4-15) with $\Delta \phi = \Delta \phi_0$:

\[ i = \frac{\Delta \phi_0 - \Delta \phi_e}{b_6 + b_7 v_1} \quad (4-50) \]

where

\[ b_6 \equiv \frac{L_a}{\kappa_s} \quad ; \quad b_7 \equiv \left( \frac{\kappa_s - \kappa_k}{\kappa_s \kappa_k} \right) \left( \frac{1 - n_0}{n_0 - n_k} \right) \quad (4-51) \]

$q_e$ is given by Eq. (4-15) and (4-16):

\[ q_e = \frac{b_1}{b_6 + b_7 v_1} \quad (4-52) \]

$b_1$ is defined in Eq. (4-44).

The total superficial flow rate $q$ is then:
\[ q = q_1 = \frac{dv_1}{dt} = q_e + q_r, \]

or

\[ \frac{dv_1}{dt} = \left( \frac{b_1}{b_6 + b_7v_1} \right) + \left( \frac{a_1 + a_2v_1}{v_1} \right), \quad (4-53) \]

which can be integrated to give

\[ v_1 - v_{10} + \left[ \frac{a_1b_6 + s_2(b_1 + a_1b_7)}{a_2b_7(s_1 - s_2)} \right] \ln \left( \frac{v_1 - s_2}{v_{10} - s_2} \right) \]

\[ + \left[ \frac{a_1b_6 + s_1(b_1 + a_1b_7)}{a_2b_7(s_2 - s_1)} \right] \ln \left( \frac{v_1 - s_1}{v_{10} - s_1} \right) = a_2(t - t_0). \quad (4-54) \]

\( s_1 \) and \( s_2 \) are the roots of the quadratic

\[ v_1^2 + \frac{(b_1 + a_2b_6 + a_1b_7)v_1 + a_1b_6}{a_2b_7}. \]

The electric power required per unit area is:

\[ P_e = \Delta \phi_0i = \frac{\Delta \phi_0(\Delta \phi_0 - \Delta \phi_e)}{(b_6 + b_7v_1)} \quad (4-55) \]

and the electrical energy required per unit area is

\[ W_e = \int_{t_0}^{t} P_e \, dt = \int_{t_0}^{t} \frac{(\Delta \phi_0 - \Delta \phi_e)}{(b_6 + b_7v_1)} \, dt. \quad (4-56) \]

If \( \Delta \phi_e \approx \text{constant} \), then:
\[ W_e = \frac{\Delta \phi_0 (\Delta \phi_0 - \Delta \phi_e)}{a_2 b_7} \left[ \frac{s_1}{(s_1 - s_2)} \ln \left( \frac{v_1 - s_1}{v_{10} - s_1} \right) + \left( \frac{s_2}{s_2 - s_1} \right) \ln \left( \frac{v_1 - s_2}{v_{10} - s_2} \right) \right] \] (4-57)
SECTION 4.8. ELECTROOSMOTIC FILTRATION WITH FIXED ANODE, CONSTANT CURRENT DENSITY, AND GRAVITY

We assume that:

(1) the electrodes are arranged as in Fig. 4-7;

(2) the current density \( i \) is constant = \( i_0 \); 

(3) the effect of gravity is included, i.e.,

\[
q = q_e + q_r 
\]

(4) the solid particles have constant surface potential \( \psi_s = \psi_k \).

During \( 0 \leq t \leq t_0 \), we form a cake on the filter by gravity drainage, and Eqs. (4-18) and (4-20) are applicable. For \( t \geq t_0 \), \( q_e \) is given by Eq. (4-22), and the total filtrate flow rate per unit area is:

\[
q_1 = \frac{dv_1}{dt} = q_e + q_r = q_e + \frac{a_1}{v_1} + a_2. \tag{4-58}
\]

Equation (4-58) can be integrated easily:

\[
\frac{a_3}{a_1} (v_1 - v_{10}) - \ln \left( \frac{a_1 + a_3 v_1}{a_1 + a_3 v_{10}} \right) = \frac{a_3^2}{a_1} (t - t_0) \tag{4-59}
\]

\[
a_3 = a_2 + q_e.
\]

Using an approach similar to that in Section 4.6, we have the voltage difference between the electrodes:

\[
\Delta \phi = i_0 (b_6 + b_7 v_1) + \Delta \phi_e, \tag{4-60}
\]

where \( b_6 \) and \( b_7 \) are defined in Eq. (4-51); the power required per
unit area:

\[ P_e = \Delta \phi i_0 = i_0 \Delta \phi_e + i_0^2(b_6 + b_1 v_1) \]  \hspace{1cm} (4-61)

and the energy required per unit area:

\[ W_e = \int_{t_0}^{t} P_e \, dt = (\Delta \phi_e i_0 + b_6 i_0^2)(t - t_0) + i_0^2 b_7 \left[ - \frac{a_1}{a_3^2} (v_1 - v_{10}) + \frac{(v_1^2 - v_{10}^2)}{2a_3} \right. \\
+ \left. \frac{a_1^2}{a_3^3} \ln \left( \frac{a_1 + a_3 v_1}{a_1 + a_3 v_{10}} \right) \right] . \]  \hspace{1cm} (4-62)
SECTION 4.9. FILTRATION BY ELECTROOSMOSIS AND PRESSURE, WITH MOVABLE ANODE AND CONSTANT CURRENT DENSITY

In this section we will present an extremely simplified model for filtration under simultaneously applied pressure and electric potential gradient. The most important, and questionable, assumption is that the porosity in the cake is uniform; i.e., that the solid matrix in the cake is incompressible. Without this assumption we could not derive analytical expressions to describe the process. The electrodes are arranged as in Fig. 4-8. We assume that:

(1) conditions (1) to (6) on p. 136 hold;

(2) the fraction of the current due to convection by the pressure-driven flow is negligibly small (for \( \lambda \ll 1 \)). This can be verified by comparing the terms in Eqs. (2-101) and (2-102);

(3) the power supply is regulated so that the current density is constant.

The superficial flow rate \( q \) is then uniform in the cake, and equal to the filtrate velocity \( q_1 \). We must first deposit a cake on the filter by pressure filtration to initiate electroosmotic flow. Therefore:

\[
\text{for } 0 < t < t_0 \quad q_1 = \frac{dv_1}{dt} = -\frac{k}{\mu} \frac{\partial p}{\partial x} = \frac{k}{\mu h} \Delta p \quad (4-63)
\]

and

\[
\text{for } t > t_0 \quad q_1 = \frac{dv_1}{dt} = q_e + \frac{k}{\mu h} \Delta p \quad (4-64)
\]

\( q_e \) is given by Eq. (4-21) or (4-22) and is constant, and \( \Delta p \) is the applied pressure difference. By a material balance we can relate the
cake thickness \( h \) to the volume of filtrate per unit area \( v_1 \) [Eq. (4-10)]. We substitute for \( h \) in Eqs. (4-63) and (4-64), and integrate, with the results:

\[
0 \leq t \leq t_0 : \quad v_1 = \left[ \frac{2k\Delta p(n_0 - n_k) t}{\mu(1 - n_0)} \right]^{\frac{1}{2}}
\]

\[
t \geq t_0 : \quad \frac{q_e}{a_4} (v_1 - v_{10}) - \ln \left( \frac{a_4 + q_e v_1}{a_4 + q_e v_{10}} \right) = \frac{q_e}{a_4} (t - t_0)
\]

where

\[
a_4 = \frac{k\Delta p(n_0 - n_k)}{\mu(1 - n_0)}
\]

\[
v_{10} = v_1(t_0).
\]

We have assumed that the current is due entirely to the electric field, so that the voltage difference between the electrodes is again given by Eq. (4-26), and the electric power required per unit filter area by Eq. (4-27). The electrical energy required per unit area is then:

\[
W_e = (t - t_0) \left( i_0 \Delta \phi + \frac{H_0 i_0^2}{\kappa_S} \right)
\]

\[
+ \frac{i_0^2}{(n_0 - n_k)} \left[ \frac{1 - n_0}{\kappa_k} - \frac{1 - n_k}{\kappa_S} \right] \int_{v_1}^{v_1(t_0)} dt
\]

In this case,

\[
\int_{v_0}^{v_1(t_0)} dt = \int_{v_{10}}^{v_1} \frac{v_1^2}{(a_4 + q_e v_1)} dv
\]

\[
= \left( \frac{v_1^2 - v_{10}^2}{2q_e} \right) - \frac{a_4}{2q_e} (v_1 - v_{10}) + \frac{a_4^2}{q_e^2} \ln \left( \frac{a_4 + q_e v_1}{a_4 + q_e v_{10}} \right)
\]
The hydraulic power required per unit area is:

for \(0 \leq t \leq t_0\): \(P_p = \Delta p \cdot q_1 = \frac{k \Delta p^2}{\mu h}\), \hspace{1cm} (4-70)

and

\[P_p = \Delta p \cdot q_1 = pq_e + \frac{k \Delta p^2}{\mu h}.\] \hspace{1cm} (4-71)

The hydraulic energy required is the integral of \(P_p\) over time. We use Eqs. (4-10) and (4-65) to obtain:

\[0 \leq t \leq t_0:\quad W_p = \left[2 \frac{n_0 - n_k}{1 - n_0} \frac{k \Delta p^3 t}{\mu}\right]^{\frac{1}{2}}, \hspace{1cm} (4-72)\]

\[t \geq t_0:\quad W_p = q_e \Delta p(t - t_0) + \frac{k \Delta p^2}{\mu} \int_{t_0}^{t} \frac{dt}{h}\]

\[= q_e \Delta p(t - t_0) + \frac{k \Delta p^2}{\mu q_e} \left(\frac{n_0 - n_k}{1 - n_0}\right)\]

\[\times \ln\left(\frac{a_v + q_e v_1}{a_v + q_e v_{10}}\right)\]. \hspace{1cm} (4-73)

Suppose that filtration ends at \(t = t_f\), when all the solid has been deposited in the cake. \(v_{1f}\) is given by Eq. (4-30), and from Eq. (4-66):

\[t_f = t_0 + \frac{(v_1 - v_{10})}{q_e} - \frac{a_v}{q_e^2} \ln\left(\frac{a_v + q_e v_1}{a_v + q_e v_{10}}\right). \hspace{1cm} (4-74)\]

The total energy required per filtration cycle is the sum of the electrical and the hydraulic energy. We substitute \(t_0\) for \(t\) in Eq. (4-72) and \(t_f\) for \(t\) and \(v_{1f}\) for \(v_1\) in Eqs. (4-68) and (4-73),
and then sum the three equations to obtain \( W_f \), the total energy required per unit area per cycle.

The mass of solid filtered per unit area is given by Eq. (4-32). \( W_f \), \( t_f \), and \( w \) are used to derive the cost equation as described in Section 4.11. The cost equation is a function of the following parameters:

1. Four controllable parameters \( i_0 \), \( \Delta \rho \), \( H_0 \), and \( n_0 \).

2. Parameters which characterize the particular suspension, such as \( k \), \( \mu \), \( n_k \), \( \rho_s \), \( \kappa_k \), \( \kappa_s \), \( \sigma_k \), \( \lambda_D \), and \( \sigma_p \).
SECTION 4.10. ELECTROOSMOTIC CONSOLIDATION

In this section we will present the equations for the dewatering of a compressible porous medium by electroosmotic consolidation, with or without the aid of an externally-applied pressure.

A qualitative discussion of the physics involved has been given in the introduction to this chapter. We summarize the important points.

(1) The drag force on the solid matrix due to the electroosmotic flow is balanced by the electrostatic force on the fixed charges in the solid, so that the net force due to the electric field is zero.

(2) Electroosmosis can cause a net compressive stress on the solid matrix only by inducing a negative pressure\(^\dagger\) in the pore fluid.

(3) Electrical contact between the anode and the pore fluid can be maintained only if the interface between the medium and the anode remains saturated with fluid. This can occur in two ways:

(a) The anode is impermeable. There is no gas evolved with the passage of current and the fluid does not cavitate;

(b) The anode is porous, but the surface tension forces are large enough so that the fluid-air meniscus in the pores is not drawn into the medium and away from the anode.

Figure 4-4 shows the porous medium with the cathode porous and open

\(^\dagger\)Relative to atmospheric.
to the atmosphere. We will consider one-dimensional flow, with the coordinate system \( s \), where

\[
s = \int_0^x (1 - n) dx \quad (4\text{-}75)
\]

\[
s_0 = \int_0^L (1 - n) dx \quad (4\text{-}76)
\]

and

\[
ds = (1 - n) dx = \frac{dx}{(1 + e)} \quad (4\text{-}77)
\]

\( e \) is the local void ratio, \( x \) is the spatial coordinate, \( L \) is the instantaneous thickness of the medium, \( ds \) is the volume of solid in a volume element per unit area, and \( s_0 \) is the total volume of solid per unit area.\(^+\)

The superficial velocity \( q \) is given by Eq. (4-2),\(^++\) rewritten in terms of coordinate \( s \):

\[
q = \frac{k_e}{\mu(1 + e)} \frac{\partial \phi}{\partial s} - \frac{k}{\mu(1 + e)} \frac{\partial p}{\partial s} \quad (4\text{-}78)
\]

Volume conservation for the fluid in the element gives:

\(^+\)By unit area we again mean a unit cross-sectional area of the medium perpendicular to the mean flow direction.

\(^++\)For simplicity we will assume that the porous medium can be described by the simple capillary model (Section 2.3).
\[
\frac{\partial e}{\partial t} = - \frac{\partial q}{\partial s} = \frac{\partial}{\partial s} \left[ \frac{k_e}{\mu(1 + e)} \frac{\partial \phi}{\partial s} + \frac{k}{\mu(1 + e)} \frac{\partial p}{\partial s} \right]. \tag{4-79}
\]

For a perfectly elastic solid matrix,

\[
e = e(p_s). \tag{4-80}
\]

Using Eqs. (4-4) and (4-80), we have:

\[
\frac{\partial p}{\partial s} = - \frac{\partial p_s}{\partial s} = - \left( \frac{\partial p_s}{\partial e} \right) \left( \frac{\partial e}{\partial s} \right), \tag{4-81}
\]

so that Eq. (4-79) can be rewritten as

\[
\frac{\partial e}{\partial t} = \frac{\partial}{\partial s} \left[ \frac{k_e}{\mu(1 + e)} \left( \frac{\partial \phi}{\partial s} \right) - \frac{k}{\mu(1 + e)} \left( \frac{\partial p_s}{\partial e} \right) \left( \frac{\partial e}{\partial s} \right) \right]. \tag{4-82}
\]

\(\partial p_s/\partial e\) is a modulus of elasticity of the solid matrix; it may be a non-linear function of \(e\).

At the anode \((s = 0)\), \((v_f - v_s)\) must be zero, so that

\[
\left( \frac{\partial p}{\partial s} \right)_0 = - \frac{k_e}{k} \left( \frac{\partial \phi}{\partial s} \right)_0, \tag{4-83}
\]

or, using Eq. (4-81),

\[
\left( \frac{\partial e}{\partial s} \right)_0 = \frac{k_e}{k} \left( \frac{\partial e}{\partial p_s} \right)_0 \left( \frac{\partial e}{\partial s} \right)_0. \tag{4-84}
\]

At the cathode \((s = s_0)\), \(p = 0\), so that

\[
p_s = 0 \quad \text{and} \quad (e)_{s_0} = e_0 = e(0). \tag{4-85}
\]
Finally, we need the relationship between the current density and the voltage gradient. We assume that the current due to convection by the pressure-driven flow is negligible, and write Eq. (2-89) in terms of the coordinate $s$:

$$i = -\kappa \frac{\partial \phi}{\partial x} = -\frac{\kappa}{1 + e} \frac{\partial \phi}{\partial s}.$$  \hfill (4-86)

Integrating over $s$, we have

$$\phi(s) - \phi(s_0) = -\int_{s_0}^{s} \frac{i(1 + e)ds}{\kappa} = i \int_{s_0}^{s} \frac{(1 + e)ds}{\kappa}$$  \hfill (4-87)

and

$$\Delta \phi = \phi(0) - \phi(s_0) = i \int_{0}^{s_0} \frac{(1 + e)ds}{\kappa}.$$  \hfill (4-88)

$\Delta \phi$, the applied voltage difference, is related to the current density by the voltage-current characteristics of the electrical power supply, which we express as:

$$i = F(\Delta \phi).$$  \hfill (4-89)

Equations (4-82), (4-87), (4-88) and (4-89), with the boundary conditions Eqs. (4-84) and (4-85) are four non-linear coupled equations for.

---

$^\dagger$This must be verified for each case by comparing Eq. (2-101) with (2-102).
four unknowns $\phi, i, \Delta\phi$, and $e$. They must be solved by numerical means. Knowing the distribution of $e$ in the medium, we can then calculate for $p_s$ using the constitutive relationship for $(\partial p_s/\partial e)$, and then for $p$ using Eq. (4-4).

An analytical solution is possible if further assumptions are made. First, the quantity

$$\frac{k}{\mu(1+e)} \left( \frac{\partial p_s}{\partial e} \right)$$

is usually approximated as constant (see Refs. 48 and 49, for example), and is referred to as the consolidation constant $c_F$. Second, if the power source is regulated so that the current density is maintained constant, at $i_0$, then for $\lambda \ll 1$,

$$q_e \approx i_0 = \text{constant},$$

if we neglect the surface conductance and terms of order $\lambda$ [see Eqs. (4-21) and (4-22)]. Equation (4-82) then reduces to the form:

$$\frac{\partial e}{\partial t} = c_F \frac{\partial^2 e}{\partial s^2}. \quad (4-90)$$

**Boundary conditions:**

\begin{align*}
 s &= s_0; \quad e(s_0, t) = e_0 = e(0) \quad (4-91) \\
 s &= 0; \quad \frac{\partial e}{\partial s} = d = \text{constant},
\end{align*}

where:
\begin{equation}
\begin{array}{ll}
d \equiv \frac{q_e}{c_F} = \\
\left\{ \begin{array}{ll}
- \frac{\varepsilon \psi_k i_0}{\mu \sigma_0 c_F} & \text{for } \psi_s = \psi_k \\
- \frac{\sigma_k \lambda_D i_0}{\mu \sigma_0 c_F} & \text{for } \sigma_s = \sigma_k
\end{array} \right.
\end{array}
\end{equation}

\textbf{Initial conditions:}
\begin{equation}
t = 0 \quad e(s,0) = f(s)
\end{equation}

We will solve Eq. (4-90) for an initially uniform void ratio:
\begin{equation}
e(s,0) = e_0
\end{equation}

The solution is then:
\begin{equation}
\frac{e_0 - e}{q_e s_0} = \frac{1 - \frac{s}{s_0} - \frac{2}{\pi^2} \sum_{0}^{\infty} \left( \frac{2}{2n+1} \right)^2 \exp \left[ - \left( \frac{2n+1}{2} \right)^2 \pi^2 \bar{t} \right]}{c_F}
\end{equation}

where \( \bar{t} = \frac{c_F t}{s_0^2} \).

\begin{equation}
\text{For } \lambda \ll 1, \text{ and neglecting surface conductance,}
\end{equation}

\begin{equation}
\kappa = \frac{n \sigma_0}{\tau^2} = \frac{e \sigma_0}{(1 + e) \tau^2}
\end{equation}

We approximate \( \tau \) as a constant throughout the medium, and substitute for \( \kappa \) into Eq. (4-88), so that:
\[
\Delta \phi = \frac{i_0 \tau^2}{\sigma_0} \int_0^{S_0} \frac{(1 + e)^2}{e} \, ds ,
\]
\[
= \frac{i_0 \tau^2}{\sigma_0} \int_0^{S_0} (2 + e) \, ds ,^+ 
\]
\[
= \frac{i_0 \tau^2}{\sigma_0} \, s_0(2 + e_{av}) ,
\]
where \( e_{av} \) is the average void ratio, given by the following equation:
\[
\frac{(e_0 - e_{av})}{q_e s_0/c_F} = \frac{1}{2} - \frac{2}{\pi^2} \sum_0^\infty \left(\frac{2}{2n+1}\right)^2 \exp\left[-\left(\frac{2n+1}{2}\right)^2 \pi^2 \bar{t}\right]. 
\]
(4-99)

For \( \bar{t} > .067 \), the first term in the summation \( n = 0 \) is 99\% of the sum, so that
\[
e_0 - e_{av} = \left(\frac{q_e s_0}{c_F}\right) \left[\frac{1}{2} - \frac{8}{\pi^2} \exp\left(-\frac{\pi^2 \bar{t}}{4}\right)\right]. 
\]
(4-100)

Let \( e_d \) be the average void ratio of the end of the dewatering cycle \( (t = t_d) \) ; then Eq. (4-100) can be rearranged into the form:
\[
t_d = \frac{4s_0^2}{\pi^2 c_F} \ln \left\{ \frac{16 s_d}{\pi^2 \left[2(e_d - e_0) + s_d\right]} \right\} . 
\]
(4-101)

^+This approximation is actually quite good for wastewater sludges. For a typical moisture content of 85\% by weight in a cake and \( (\rho_s/\rho) = 1.9 \), \( e = 10.8 \) and \( (1/e) = .093 \).
The electrical power required per unit area is $P_e$:

$$P_e = \Delta \phi i_0 = \frac{i_0^2 \tau^2}{\sigma_0} s_0 (2 + e_{avg})$$  \hspace{1cm} (4-102)

and the electrical energy required per unit area per dewatering cycle is:

$$W_e = \int_0^{t_d} P_e \, dt = \frac{i_0^2 \tau^2 s_0}{\sigma_0} \left( 2 + e_0 - \frac{s_0 d}{2} \right) t_d$$  \hspace{1cm} (4-103)

\[ - \frac{32 d s_0^4 i_0^2 \tau^2}{\pi^4 \sigma_0 c_F} \left[ \exp \left( -\frac{\pi^2}{4} \frac{t_d}{t_d} \right) - 1 \right]. \]

We can substitute for the exponential in the second term on the right, using Eq. (4-100), and rewrite $W_e$ as:

$$W_e = \frac{i_0^2 \tau^2 s_0}{\sigma_0} \left( 2 + e_0 - \frac{s_0 d}{2} \right) t_d$$  \hspace{1cm} (4-104)

\[ + \frac{32 d s_0^4 i_0^2 \tau^2}{\pi^4 \sigma_0 c_F} \left[ 1 - \frac{\pi^2}{8} \left( \frac{1}{2} - \frac{e_0 - e_d}{s_0 d} \right) \right]. \]

$w$, the mass of solid dewatered per unit area, is:

$$w = \rho_s s_0$$  \hspace{1cm} (4-105)

$w$, $w (= W_e)$, and $t_d$ are used in the cost equation, as described in Section 4.11.

We can apply an external pressure $P_0$ to the porous medium by either loading an impermeable anode (Fig. 4-6) or by pressurizing the air space above a porous anode (Fig. 4-4). Consolidation under both $P_0$ and $\Delta \phi$ is described also by Eq. (4-90), with $e(0,t) = e_0 = e(P_0)$. We will not derive the solutions for this case.
SECTION 4.11. COST ANALYSIS

In this section we will present a general cost equation for sludge dewatering by electroosmosis and/or pressure, and then give an example of its application.

Following standard practice, we will express the cost in terms of $/unit mass of sludge solid. The total dewatering cost is assumed to be the sum of:

(1) Capital amortization and interest cost per unit mass of solid \( C_C \) ($/unit mass):

\[
C_C = \frac{K_C t_f}{w} ,
\]

(4-106)

where

- \( t_f \) = time required per filtration cycle;
- \( w \) = mass of solid dewatered per unit area per cycle;\(^\dagger\)
- \( K_C \) = amortization and interest cost coefficient ($ per unit area per unit time):

\[
K_C = \frac{I_C K_{cap}(A)}{A} ,
\]

(4-107)

where

- \( I_C \) = amortization and interest rate;
- \( K_{cap}(A) \) = capital cost, a function of \( A \);
- \( A \) = total area of filter medium.

\(^\dagger\)By "area" we mean the area of the filter medium.
(2) Labor cost per unit mass of solid = $C_L$ ($/\text{unit mass}$). We will assume that $C_L$ can be written in the form:

$$C_L = \frac{K_L t_f}{w};$$  \hspace{1cm} (4-108)

$$K_L = \frac{K_{\text{labor}}(A)}{A};$$  \hspace{1cm} (4-109)

where $K_{\text{labor}}(A)$ = total labor cost per unit time, a function of $A$.

(3) Energy cost per unit mass of solid = $C_e$ ($/\text{unit mass}$):

$$C_e = \frac{K_e W_f}{w};$$  \hspace{1cm} (4-110)

$K_e$ is the cost per unit energy and $W_f$ is the energy required per unit area per filtration cycle.

(4) Disposal cost per unit mass of solid = $C_d$ ($/\text{unit mass}$). For simplicity, we will assume that ultimate disposal of the sludge cake is by transportation to a landfill site. We would then expect the cost to be directly proportional to the mass of cake transported (consisting of the mass of fluid and of the solid). $C_d$ is then given by the equation:

$$C_d = K_d (1 + U_d),$$  \hspace{1cm} (4-111)

where $K_d$ = disposal cost per unit mass of cake;

$U_d$ = moisture content of cake at disposal (mass of fluid per mass of solid).
(5) Maintenance cost per unit mass of solid = \( C_r \). We will assume that

\[
C_r = C_{r1} + C_{r2}
\]

where \( C_{r1} = \) cost of maintenance and replacement of electrodes per unit mass of solid dewatered;

\( C_{r2} = \) cost of maintenance for the remainder of the equipment per unit mass of solid.

We would expect \( C_{r1} \) to depend on the amount of charge passed through the electrodes:

\[
C_{r1} = \frac{K_r \int_0^{t_f} i \, dt}{w} \quad (4-113)
\]

The units of \( K_r \) are \($/\text{unit charge}\). If the current density is maintained constant, then, for \( i = i_0 \),

\[
C_{r1} = \frac{K_r i_0 t_f}{w} \quad (4-114)
\]

If the voltage difference between the electrodes is held fixed, then for \( \Delta \phi = \Delta \phi_0 \),

\[
C_{r1} = \frac{K_r}{\Delta \phi_0} \int_0^{t_f} (\Delta \phi_0) i \, dt = \frac{K_r W_e}{(\Delta \phi)_0 w} \quad (4-115)
\]

\( C_{r2} \) will be assumed to be a fraction of \( C_c \):

\[
C_{r2} = f_r C_c \quad (4-116)
\]
The total dewatering cost per unit mass of solid, \( C_T \), can be written in the following form:

\[
C_T = \frac{K_c(1 + f_r) + K_L}{w} t_f + \frac{(K_e + K_r)w_f}{w} + K_d(1 + U_d). \tag{4-117}
\]

In each of the previous sections in this chapter, we have derived equations for \( W_f \), \( t_f \), and \( w \) for a specific filtration model. These equations can be substituted into Eq. (4-117) and \( C_T \) can then be optimized with respect to the control-label parameters.

For example, for electroosmotic filtration as described in section 4.5 with \( i = i_0 \), \( t_f \) is given by Eq. (4-37), \( W_f \) by Eq. (4-38), and \( w \) by Eq. (4-32). The resulting equation for \( C_T \) is algebraically complex; we can reduce it to an approximate but useful form by assuming that \( t_f \gg t_0 \) and \( v_f \gg v_{10} \); i.e., that we need to deposit only a negligibly thin cake by gravity drainage initially for electroosmosis to be effective.

The cost per unit mass of solid filtered is then

\[
C_T = \frac{K_{ce}t_f}{w} + \frac{K_eW_f}{w} + \frac{K_{r10}t_f}{w} + K_d(1 + U_d), \tag{4-118}
\]

where

\[
K_{ce} = K_c(1 + f_r) + K_L \quad \text{(for electroosmosis)}; \tag{4-119}
\]

\[
U_d = \frac{\rho}{\rho_s} \left( \frac{n_k}{1 - n_k} \right) = \text{constant}; \tag{4-120}
\]
\[ t_f = \frac{v_1 f}{q_e} = -\frac{(n_0 - n_k)H_0 \tau_k^2 \mu k_k}{(1 - n_k)n_{pk} \varepsilon \psi_k i_0(1 - 2\lambda_p)} \quad ; \quad (4-121) \]

\[ W_f = -\frac{\Delta \phi_e (n_0 - n_k)H_0 \tau_k^2 \mu k_k}{(1 - n_k)n_{pk} \varepsilon \psi_k (1 - 2\lambda_p)} \quad ; \quad (4-122) \]

\[ -\frac{i_0 H_0^2 (n_0 - n_k) \left[ k_k (1 - n_k) + k_s (1 - n_0) \right] \tau_k^2 \mu}{2(1 - n_k)^2 n_{pk} k_s \varepsilon \psi_k (1 - 2\lambda_p)} \quad . \]

We substitute for \( t_f \) and \( W_f \) into Eq. (4-140) and note that the controllable parameters are \( i_0, H_0, \) and \( n_0 \). The other parameters depend on the properties of the solid and the fluid, and are assumed constant for a given material.

\( C_T \) depends on \( i_0 \) as follows:

\[ C_T = \frac{A}{T_0} + B i_0 + G \quad , \quad (4-123) \]

where \( A, B, \) and \( G \) are functions of parameters other than \( i_0 \). We can then minimize \( C_T \) with respect to \( i_0 \), and obtain:

\[ (i_0)_{\text{optimum}} = \left( \frac{A}{B} \right)^{\frac{1}{2}} \quad ; \quad (4-124) \]

\[ (C_T)_{\text{optimum}} = 2(AB)^{\frac{1}{2}} + G \quad . \quad (4-125) \]

After some algebraic manipulation, we have:

\[ \text{†} \text{Remember that we have assumed } \psi_k < 0 \text{.} \]
\[
(C_T)_{opt} = C_d + \frac{(n_0 - n_k) \tau_k^2 \mu \kappa_k}{\varepsilon \psi_k \rho_s (1 - n_0)(1 - n_k)(1 - 2\lambda_p)n_{pk}} \\
\times \left[ \left( \frac{2K_{ce}H_0 \kappa_k (1-n_k) + \kappa_s (1-n_0)}{(1 - n_k) \kappa_k \kappa_s} \right)^{\frac{1}{2}} + K_r + K_e \Delta \phi_e \right] .
\]

\[
C_d = \frac{K_d \left[ \rho_s + n_k (n - \rho_s) \right]}{\rho_s (1 - n_k)} 
\] (4-127)

and

\[
(i_0)_{opt} = \left( \frac{2K_{ce} \kappa_s \kappa_k (1 - n_k)}{K_e H_0 \left[ (1 - n_k) \kappa_k + (1 - n_0) \kappa_s \right]} \right)^{\frac{1}{2}} .
\] (4-128)

Equation (4-117) can also be used for pressure filtration for cost comparison with electroosmosis. Suppose that we initially have a slurry of depth \( H_0 \) above the filter medium which we then dewater completely into a cake of thickness \( h \). The mass of solid filtered per unit area would be given by Eq. (4-32). The equations for constant pressure filtration are given in Chapter 7. If we assume again for simplicity that the cake is incompressible and has a uniform porosity \( n_k \), then:

\[
w = \left[ \frac{2k}{\mu} \frac{(1 - n_0)(1 - n_k)^2 \Delta p \ t_f}{(n_0 - n_k)} \right]^{\frac{1}{2}} \rho_s ,
\] (4-129)
or:

\[
t_f = \left( \frac{w}{\rho_s} \right)^2 \frac{\mu (n_0 - n_k)}{2k(1 - n_0)(1 - n_k)^2 \Delta p} .
\] (4-130)
The energy required per unit area would be:

\[
W_f = \int_0^{t_f} (\Delta p \ q_p) \ dt = (\Delta p) v_{1f}.
\]  

(4-131)

\(v_{1f}\) is given by Eq. (4-30). Substituting for \(t_f\), \(W_f\), and \(w\) into Eq. (4-117), we have \(C_T\) for constant pressure filtration:

\[
C_T = \frac{K_{cp} t_f}{w} + \frac{K_e W_f}{w} + K_d (1 + U_d)
\]

\[
= \frac{K_{cp} \ u(n_0 - n_k) H_0}{2\rho_s k (1 - n_k)^2 \Delta p} + \frac{K_e \Delta p (n_0 - n_k)}{\rho_s (1 - n_0)(1 - n_k)}
\]

\[
+ K_d \ \frac{\rho_s + n_k (\rho - \rho_s)}{\rho_s (1 - n_k)}\]

(4-132)

\[K_{cp} = K_c (1 + f) + K_L \quad \text{(for pressure filtration)}\]  

(4-133)

Equation (4-132) can be easily optimized with respect to \(\Delta p\), so that for pressure filtration,

\[
(C_T)_{\text{optimum}} = \frac{(n_0 - n_k)}{\rho_s (1 - n_k)} \left[ \frac{2\mu K_{cp} K_e H_0}{k (1-n_0)(1-n_k)} \right]^{\frac{1}{2}}
\]

\[
+ K_d \left[ \frac{\rho_s + n_k (\rho - \rho_s)}{\rho_s (1 - n_k)} \right]
\]

(4-134)
\[(\Delta p)_{\text{optimum}} = \left[ \frac{k_{cp} \mu H_0 (1 - n_0)}{2K_e k (1 - n_k)} \right]^\frac{1}{2} \]  
\[ (4-135) \]

The disposal cost is the same for both pressure and electroosmotic filtration, with our assumption of an incompressible cake, so that we should compare the other costs. Let:

\[ R_c = \frac{[(C_T)_{\text{opt}} - C_d]_{\text{electroosmosis}}}{[(C_T)_{\text{opt}} - C_d]_{\text{pressure}}} \]  
\[ (4-136) \]

For simplicity we assume that surface conductance is negligible and \( \lambda_p \simeq 0 \). Using Eqs. (4-126) and (4-134), we have:

\[ R_c = \frac{\sigma_0}{\varepsilon |\psi_K|} \left[ \frac{\mu k (1 - n_k)}{2k_{cp} K_e H_0 (1 - n_0)} \right]^\frac{1}{2} \]
\[ (4-137) \]

\[ \times \left\{ K_r + K_e \Delta \phi_e + \left[ \frac{2k_c e H_0 K_e}{\sigma_0 (1 - n_0)} \left( \frac{(1 - n_k) \tau_s^2}{n_{ps}} + \frac{(1-n_0) \tau_k^2}{n_{pk}} \right) \right]^{\frac{1}{2}} \right\} \]

As we would expect, electroosmotic filtration is favored over pressure filtration (\( R_c < 1 \)) only if \( k_{ce} \) is much less than \( k_{cp} \) and if the hydraulic permeability \( k \) of the cake and the electrical conductivity of the fluid \( \sigma_0 \) are low. What is unexpected is that \( R_c \) is more sensitive to \( \psi_k \) (or to \( \sigma_k \)) than to the other parameters.
SECTION 4.12. PRACTICAL PROBLEMS IN ELECTROOSMOTIC DEWATERING

The models in Sections 4.4 through 4.6 can be used to simulate electroosmotic dewatering in devices such as that shown in Fig. 4-9. With the movable anode positioned at the surface of the slurry, a large fraction of the electrical energy is initially wasted in ohmic losses in the slurry.

One could improve the energy efficiency by using a movable anode which is in contact with the surface of the cake; sensing and servo mechanisms would be required to move the anode as the cake thickness increases. A simpler arrangement would be to have the anode immersed in the slurry and fixed at a small distance away from the cathode (Sections 4.7 and 4.8). Continuous operation would be possible using a moving belt (Fig. 4-10).

The model for electroosmotic consolidation (Section 4.10) can be used directly for the dewatering of filter cakes, or can be modified and applied to the dewatering of soil in situ with an additional load on the soil matrix due to the weight of the overlying soil.

The assumptions which we have made in deriving the models will be violated in practice to varying extent. For example:

(1) The properties of a wastewater sludge will most likely not be uniform nor constant through the dewatering period.

(2) In particular, the surface charge density and potential will be affected by:
(a) products of electrode reactions;
(b) ion exchange and other reactions with the current;
(c) temperature rise due to ohmic losses.

(3) The conductivity of the fluid will be affected also by the products of electrode reactions. For example, the impervious anode which we have proposed in the model for electroosmotic consolidation will rapidly lose contact with the pore fluid if gas evolution occurs. A porous anode would allow the gas to escape; however, in that case the surface tension forces must be large enough so that the fluid is not drawn into the cake and away from the anode.

A major problem will be the corrosion of electrodes, especially the anode. Researchers usually use inert metals, which are too expensive for practical applications. Greyson\textsuperscript{20} has discovered, however, that relatively inexpensive electrodes of stannic oxide are oxidation-resistant and have high electrical conductivity and mechanical strength.

The difficulties with electrode reaction products can be partially solved by periodic current reversal, slurry agitation, and mechanical vibrations. The reader is referred to Sprute and Kelsh\textsuperscript{14-17} for details.
FIGURE 4-1. An isolated charged particle in an electric field.

FIGURE 4-2. A charged particle closely packed with others in a cake.
FIGURE 4-3. Consolidation by pressure.
FIGURE 4-4. Consolidation by electroosmosis with a porous anode.

FIGURE 4-5. Detail of the cake surface adjacent to a porous anode.
FIGURE 4-6. Consolidation by electroosmosis, with an impermeable anode.
Figure 4-7. Fixed anode.

Figure 4-8. Movable anode.

Electroosmotic Filtration
FIGURE 4-9. Schematic of a continuous electroosmotic filter with movable anode.
FIGURE 4-10. Schematic of a continuous electroosmotic filter with fixed anode.
CHAPTER 5:
EXPERIMENTS IN ELECTROOSMOTIC FILTRATION

This chapter describes a series of experiments on electroosmotic filtration of Kaolin clay slurries and anaerobically digested sewage sludge. In some of the experiments a pressure difference is also applied in addition to the electric field.

The experimental apparatus is set up so as to simulate as closely as possible the conditions assumed in the derivations of the theoretical filtration models in Chapter 4.

The variables measured in the experiments are:

1. mass fraction of solid in the slurry before and after the experiment;
2. current;
3. voltage at the anode and at various points in the slurry relative to the voltage at the filter medium;
4. applied pressure difference, if any;
5. filtrate volume and flow rate;
6. cake thickness;
7. conductivity and ionic concentration of the fluid;
8. mass fraction of solid in the cake.

In section 5.1 we describe the samples used and the experimental apparatus. In Section 5.2 we outline the experimental procedures, and in Section 5.3 we discuss the results. A summary is given in Section 5.4, along with recommendations for future research.
SECTION 5.1. MATERIALS USED AND EXPERIMENTAL APPARATUS

The anaerobically digested sewage sludge is collected from the Deer Island Treatment Plant, serving Boston, Massachusetts; it is briefly described in Section 3.1. The sludge is first allowed to settle and the supernatant is decanted and its conductivity measured with a conducting cell (Fig. 3-5). The settled sludge is then tested directly. The sludge is not chemically conditioned before testing, for the reasons mentioned on p.

The Kaolin clay slurries are made monoionic in NaCl and then repeatedly washed with distilled water until the supernatant is of the desired NaCl concentration (see Section 3.1). The Cl concentration is measured with a chloride specific ion electrode (Orion Model 94-17) and electrometer (Keithley Model 602). Before testing, the slurry is allowed to settle and the supernatant removed.

Sketches of the test cell and of the entire experimental apparatus are shown in Figs. 5-1 and 5-2, respectively. The apparatus has many parts in common with the one described in Chapter 3. To minimize problems with electrochemical corrosion, non-metallic materials are used whenever possible (glass, acrylic, polyethylene, etc.).

The anode is a grid made from 0.16 cm diameter stainless steel wire. It can either be held in a fixed position submerged in the slurry or manually moved so as to maintain contact with the surface of the slurry. The surface area of the anode is approximately 66 cm².

The test cell is separated from compartment B by a membrane filter
(Millipore-MF) backed up by a stainless steel screen and a perforated plexiglas plate with circulation grooves. The volume flow rate of filtrate into compartment B is measured either with a pipette or with a graduate. The change in the height of fluid in the pipette is measured by a pressure transducer and recorded on a strip chart recorder. The pipette-transducer-recorder combination is calibrated by injecting fluid into compartment B with a syringe.

To prevent concentration polarization, fluid is circulated transversely across the filter through the circulation grooves in the backup plate by a falling piston pump.

The cathode is platinum. Passage of current through the cathode is accompanied by electrolysis of water. The H₂ gas evolved is separated from compartment B by a salt bridge made of cation and anion ion-exchange membranes placed side by side (Ionics type 61CZL183 and 103QZL219). The fluid in the cathode compartment is circulated with an electric pump, and the temperature maintained constant with heating and cooling coils in the reservoir.

An HP Model 6439B D.C. power supply is used to supply either a constant voltage (0-60 V) or a constant current (0 - 15 amps) to the power electrodes. Current is measured with a digital multimeter (Fluke Model 8000A). The voltages at the anode and at two stainless steel probes immersed in the slurry are measured relative to the voltage at the stainless steel screen supporting the membrane filter with a Keithley Model 602 electrometer (10¹⁴ Ω internal resistance).
The temperature of the slurry is measured with a thermometer.

The space above the slurry in the test cell can be pressurized with compressed N₂ gas.
SECTION 5.2. EXPERIMENTAL PROCEDURE

Please refer to Figure 5-2 if necessary.

1. The suspension is allowed to settle, and the supernatant removed.

2. The reservoir is filled with a NaCl solution with the same electrical conductivity as the supernatant.

3. The apparatus is assembled, with a fresh membrane filter in the test cell.

4. The test cell is filled with solution and pressurized with N₂ gas, and the electrical circulating pump is started. The resulting combined flow of solution is used to fill the apparatus and to flush out all air bubbles, especially those in the circulating grooves in the backup plate for the filter (see Fig. 5-1).

5. When the test cell is emptied of solution, the pressure is released and all the valves are closed to prevent flow back into the cell.

6. The settled slurry is poured into the test cell, and the electrodes and the thermometer are inserted. A small sample of the slurry is reserved for moisture content determination.

7. The electrical pump is used to raise the piston in the gravity pump to its starting position. Valves 3 and 4 are then closed, isolating compartment B from the rest of the apparatus.
8. The fluid in compartment B is circulated as the piston falls; the circulation rate is controlled with valve 12.

For filtration by electroosmosis alone, the remaining steps are:

9a. Valve 11 is set so that the filtrate flow into compartment B is measured with the graduate. We first allow a cake to form on the filter by gravity drainage alone, and then apply the electric field.

10a. The pipette could not be used to measure the electroosmotic flow. To eliminate the effect of gravity, the level of the fluid in the pipette would have to be initially set equal to that of the slurry. The electroosmotic flow into the pipette would then create a hydrostatic back pressure across the cake. Since the electric field does not produce a net stress on the solid matrix in the cake, the back pressure would lift the cake off the filter and invalidate the experiment.

11a. During the course of an experiment the temperature in the slurry can rise 10°C or more from the ohmic losses. Attempts to cool the apparatus with an external fan were not successful. The effect of the temperature on the fluid viscosity and conductivity can be accounted for.

For filtration by electroosmosis and pressure, the following steps are taken instead of 9a and 11a:

9b. The volume of filtrate is measured using the graduate. Periodically, valve 11 is set so that the filtrate flow is directed into the pipette
and the flow rate recorded on the strip chart recorder via the pressure transducer. At the end of the flow rate measurement, the valve is reset so that the fluid accumulated in the pipette is dumped into the graduate.

10b. In experiments III-1 and III-3 we attempt to maintain the temperature of the slurry constant by only applying the electric field periodically for a minute or so. The slurry is essentially filtered under pressure alone. The instantaneous increase in the flow rate due to electroosmosis and the electric power required are measured. These experiments therefore do not simulate exactly the model in Section 4.9, where the electric field is applied continuously along with the pressure gradient. However, the data we obtain is useful, as shown in Section 5.3.

11b. In experiment III-4, a constant current density of 13.6 amp/m² is applied continuously after \( t_0 = 187 \text{ min} \).

The following steps are carried out whether pressure is applied or not:

12. Periodically, the cake thickness is measured roughly with a probe consisting of a vertical stainless steel wire (0.16 cm in diameter) with a horizontal loop at the bottom end. More accurate methods are possible, such as pressure taps;⁰⁰ they are, however, more complex, and could not be installed in the time span available for this series of experiments.

13. At the end of the experiment, the test cell is disassembled and the remaining slurry above the cake is reserved. The cake is sectioned
horizontally. The mass fractions of fluid in the slurry, before and after the experiment, and in the sections of the cake, are determined first by weighing the moist samples (to .01 gram accuracy), then drying at 103°C for 12 hours and re-weighing. The porosity is calculated by using Eq. (3-1).
SECTION 5.3. EXPERIMENTAL RESULTS

EXPERIMENT III-1

In this experiment we filter a slurry of Kaolin clay under a pressure of $6.89 \times 10^6$ N/m² (10 psi). Periodically the electric field is applied for a minute or so, and the increase in the filtrate flow rate due to electroosmosis is measured. The power source is regulated to supply either a constant current density ($i_0 = 10$ amp/m²) or a constant voltage difference between the anode ($\Delta \phi_0 = 6.65$ V) and the stainless steel screen supporting the filter medium. The anode is movable so as to maintain contact with the surface of the slurry (Fig. 4-8). The results are summarized in Tables 5-1, 5-2, and C-1 through C-4 (Appendix C).

We cannot use the equations derived in Section 4.9, since the electric field is not applied continuously. The volume of filtrate in this experiment is essentially due to pressure filtration alone, so that we can use Eq. (4-65), repeated below:

$$v_1 = \left[ \frac{2k \Delta p (n_0 - n_k) t}{(1 - n_0)} \right]^{1/2} \tag{5-1}$$

By mass conservation, the thickness of the cake is related to $v_1$ by Eq. (4-10):

$$h = \left( \frac{1 - n_0}{n_0 - n_k} \right) v_1 = \left[ \frac{2k \Delta p (1 - n_0) t}{\mu (n_0 - n_k)} \right]^{1/2} \tag{5-2}$$

When the electric field is applied momentarily, the following equations apply, regardless of how the cake is formed:
(1) For $\Delta \phi = \Delta \phi_0$:

$$i = \frac{\Delta \phi_0 - \Delta \phi_e}{(b_2 + b_3 v_1)}$$  \hspace{1cm} (5-3)

$$q_e = \frac{b_1}{(b_2 + b_3 v_1)}.$$  \hspace{1cm} (5-4)

$b_1$, $b_2$, and $b_3$ are constants given by Eqs. (4-44) and (4-45).

(2) For $i = i_0$:

$$q_e = -\frac{n_p k e \psi_k}{\mu \tau_k^2 \kappa_k} (1 - 2 \lambda_p) i_0$$ \hspace{1cm} (5-5)

$$\Delta \phi = i_0 (b_2 + b_3 v_1) + \Delta \phi_e.$$  \hspace{1cm} (5-6)

$\kappa_k$ is given by Eqs. (2-159).

We can determine $\Delta \phi_e$ from the voltages at the measuring electrodes (referred to hereafter as "probes") as long as they are in contact with the slurry (see Fig. 5-15). Mistakenly, the first measurement was made when the probe closest to the cathode was already immersed in the cake, so that there is no data available on $\Delta \phi_e$ for this experiment.

Figure 5-3 shows the filtrate flow rate as a function of time, with and without electroosmosis. The sudden increase at $180 \text{ min} < t < 260 \text{ min}$ is probably due to a momentary crack in the cake or leakage between the cake and the wall of the cell. The electroosmotic flow rate is obtained by subtracting the pressure-driven flow rate from the total flow rate, and is given in Table C-1. The flow rate due to pressure alone is given
in Table C-3, and can be matched by the time-derivative of Eq. (5-1) with \( k = 1.87 \times 10^{-16} \text{ m}^2 \).

Since we could not obtain a value for \( e_{co} \) from the experiments on Kaolin in Chapter 3, we will make a first approximation that \( e_{co} = 0.43 \) (as in Fig. 3-15 and in Olsen\(^3\)) in the calculation for the pore radius and the Debye length ratio (Table 5-2).

For \( \lambda_p \ll 1 \) and \( \psi_s = \psi_k \), the group \((\mu q_e \sigma_0 / i)\) is given by Eq. (2-169), and repeated below:

\[
\frac{\mu q_e \sigma_0}{i} = -\frac{e \psi_k (1 - 2 \lambda_p)}{1 + \cosh \psi_k \frac{n c}{n_p} \left( \frac{\tau_k}{\tau_c} \right)^2 + \psi_k^2 \lambda_p \left[ \frac{\varepsilon}{\mu D} + 2 \left( \frac{Z F}{R T} \right)^2 \right]}.
\]

Equation (5-7) is used to calculate \( \psi_k \), since it does not depend on the variable \( \Delta \phi_e \). The resulting average value of \( \psi_k \) is -10.5 mV.

Figure 5-4 shows the instantaneous values of \( q_e \) vs. time, along with the theoretical curve from Eq. (5-5), which does not depend on \( \Delta \phi_e \). Figure 5-5 shows \( P_e \) vs. time.

From Fig. 5-3 and Table C-2, we see that for \( t > 300 \text{ min} \) (\( h > 1.6 \text{ Cm} \), \( H_s < 1.8 \text{ Cm} \)), we can use electroosmosis (\( \Delta \phi_e = 7.0 \text{ V} \)) to more than double the flow rate under pressure alone (\( \Delta P = 6.55 \times 10^4 \text{ N/m}^2 \)), with an energy expenditure of \((P_e / q_e) \approx 23 \text{ kW-hr/m}^3 \).

The "efficiency" of electroosmosis can best be measured by the variable \( \eta \), as defined in Eq. (2-118). If the cake can be approximated as being incompressible, then \( \eta \) can be expressed in the form:
\[
\eta = \frac{q_e^2 \Delta P}{\rho_e q_h}.
\]  
(5-8)

Figure 5-6 and Table C-4 give \( \eta \) as a function of time. We see that for a given flow rate through the Kaolin filter cake, the energy required by electroosmotic filtration would be about \( 10^3 \) - \( 10^6 \) times that required for pressure filtration.

**EXPERIMENT III-2**

This experiment is an attempt to simulate the electroosmotic filtration model in Section 4.7. A cake is first formed on the filter by gravity drainage, and then the electric field is applied continuously. The slurry is unconditioned anaerobically digested sewage sludge. The anode is fixed, and the voltage difference between the anode and the stainless steel screen supporting the filter is maintained constant at \( \Delta \phi_0 = 3.4 \) V. The results are given in Tables 5-1, 5-2, C-5, and C-6 (Appendix C).

\( e_{co} \) for sewage sludge is assumed to be 1.22, as given by the experiments in Chapter 3. The calculated values for pore radius and Debye length ratio are presented in Table 5-2.

Figure 5-7 shows \( v_1 \) as a function of time. The electric field is applied after \( t_0 = 48.6 \) min. The theoretical curve for \( t < t_0 \) is given by Eq. (4-20) with \( k = 1.80 \times 10^{-15} \) m\(^2\). The curve for \( t > t_0 \) is based on Eq. (4-54), with an attempt made to include the effect of temperature rise on the viscosity by using step increases in viscosity.
The surface potential used to fit the data is -12.9 mV.

Figure 5-8 shows $P_e$ as a function of time, along with the theoretical curve based on Eq. (4-55) with stepwise changes in the conductivity with the increase in measured temperature.

$\tau_k^2$ can be calculated from the data for $i$ and Eqs. (4-50), (2-159), and (4-12); the average value is $\tau_k^2 = 1.1$ (see Table C-2).

Figures 5-7 and 5-8 show the rapid increase in temperature due to ohmic losses. An attempt to cool the test cell with an external fan was not successful.

The temperature rise does not seem to account completely for the increase in $v_1$ and $P_e$ with time. Other possible causes are:

1. changes in $\psi_k$ with temperature;
2. change in $\psi_k$ and $\sigma_0$ due to pH change and products from anode reactions.

The average energy required per unit volume of fluid removed from the sludge is:

$$\frac{W_e}{\Delta v_1} = \frac{\int_{t_0}^{t} P_e \, dt}{v_1(t) - v_1(t_0)}.$$  \hspace{1cm} (5-9)

$W_e$ can be approximated by integrating $P_e$ graphically over time. $(w_e/\Delta v_1)$ for the experiment $\approx 76$ kW-hr/m$^3$. 
EXPERIMENT III-3

In this experiment the sewage sludge is filtered under a pressure of $1.37 \times 10^5$ N/m² (20 psi). Periodically a constant voltage difference of 3.4 V is applied between the anode and the stainless steel screen supporting the filter, and the increase in flow rate due to electroosmosis is measured, along with the power required. The anode is fixed, and Eqs. (5-1), (4-54), and (4-55) are applicable. The results are given in Tables 5-1, 5-2, and C-7 through C-9.

Figure 5-9 shows $(\mu q_e \sigma_0 / i)$ as a function of time. The data for $t < 173$ min can be fitted by Eq. (5-7) with $\psi_k = -11.4$ mV.

The polarization voltage $\Delta \phi_e$ is determined from the voltages measured at the two probes (Fig. 5-15 and Table C-7), and is relatively constant with an average value of 0.65 V for $t < 194$ min.

Figure 5-10 shows $q_h$ and $q_e$ vs. time, along with the theoretical curves from Eq. (5-1) ($k = 3.22 \times 10^{-17}$ m²) and Eq. (4-54) ($\tau_k^2 = 1.22$, $\psi_k = -11.4$ mV, and $\Delta \phi_e = 0.65$ V). Figure 5-11 shows $P_e$ vs. time, along with the curve using Eqs. (4-55) and (5-1). The changes in $\mu$ and $\sigma_0$ with temperature are accounted for, using the measured temperature.

We see in Fig. 5-10 that $q_e$ increases rapidly and unexpectedly for $t > 173$ min; there is no such increase in $P_e$ (Fig. 5-11). Consequently, $(P_e / q_e)$, the instantaneous energy required per unit volume of fluid removed, decreases rapidly (Table C-7). There seems then to be some variable, other than temperature change, which affects $\psi_k$ but not $\sigma_0$. 
Table C-9 and Fig. 5-12 give \((\eta)^{-1}\) vs. time. \((\eta)^{-1}\) is the ratio of the electric to the hydraulic energy required to produce a given flow rate.

**EXPERIMENT III-4**

This experiment with Kaolin is an attempt to simulate the model in Section 4.9, with \(i_0 = 13.6\) amp/m² and \(\Delta P = 6.55 \times 10^4\) N/m² (9.5 psi). The results are given in Tables 5-1, 5-2, C-10, and C-11.

Figure 5-13 shows \(v_1\) vs. time. From the data for \(t < 183\) min and Eq. (4-65), we determine \(k\) to be \(3.15 \times 10^{-16}\) m². The data for \(t > 183\) min. can be best matched by a curve using Eqs. (4-66) and (4-22) with \(\psi_k = -22\) mV.

Figure 5-14 shows \(P_e\) vs. time. From Eq. (4-27) we would expect \(P_e\) to decrease as \(t\) and \(v_1\) increase, since

\[
\left[ \frac{(1 - n_0)}{\kappa_k} - \frac{(1 - n_k)}{\kappa_s} \right] < 0
\]

We do not have any data on \(\Delta \phi_e\) because the probes were already immersed in the cake when the electric field was applied. We have therefore plotted Eq. (4-27) in Fig. 5-14 with \(\Delta \phi_e = 0\). Equation (4-27) over-estimates the power required, and its disagreement with the data will be worse when a nonzero value of \(\Delta \phi_e\) (which must exist) is included.

The value of \(\psi_k\) calculated from data is higher than the previous values from Experiments Series II and Experiment III-1. The calculations for \(\psi_k\) with Eq. (4-22) and \(P_e\) with Eq. (4-27) rely on the measured value of \(\sigma_0\), which may be the source of error.
SECTION 5.4. SUMMARY AND CONCLUSION

The Kaolin slurry in Experiment III-1 and the sewage sludge in Experiment III-3 are essentially filtered under pressure alone; the electric field is applied only briefly and intermittently. From the data we can compare pressure-driven and electroosmotic flow rates, and calculate the wall potential $\psi_k$ and the "efficiency" $\eta$. For both materials, $\eta$ is low ($\sim 10^{-4}$ - $10^{-3}$).

The sewage sludge in Experiment III-2 is filtered by electroosmosis and gravity, as modeled in Section 4.7. We were unable to maintain the model assumptions of constant temperature; $T$ increased by about 15° C due to ohmic losses. Both the flow rate and the power required increased more rapidly than predicted by theory, even when the effects of temperature on the fluid viscosity and conductivity were taken into account. Most likely $\psi_k$ and $\sigma_0$ were also affected by the products of anode reactions. The average energy required per unit volume of fluid filtered was 76 kW-hr/m³.

The Kaolin slurry in Experiment III-4 is filtered under both pressure and electroosmosis, as modeled in Section 4.9. The value of $\psi_k$ calculated from the data (-22 mV) is much higher than those from previous experiments on Kaolin.

For each experiment, the comparison with theory is not absolute, since there are two constants $\psi_k$ and $\tau_k^2$ which can be adjusted so that the theoretical curves match the data. This flexibility would not be a
problem if the values for $\psi_k$ and $\tau_k^2$ (Table 5-2) so determined were consistent, or could be supported by independent measurements. To my chagrin, neither is true for these experiments.

(a) The values for $\psi_k$ for sewage sludge in Table 5-2 are close enough to each other, given the variability of the properties of sludge. They are higher than the values determined in previous experiments (Table 3-4).

(b) The two values for $\psi_k$ for Kaolin clay are very different and higher than those from previous experiments (Table 3-6). The two samples used came from the same batch of Kaolin, so that the reasons for the difference are not known.

$\psi_k$ can be independently measured by electrophoresis. $\tau_k^2$ must be inferred from data on $i$ or $q_e$ and depends partially on the porosity of the cake. We would expect it to increase as the porosity decreases. The assumed values of $\tau_k^2$ for the Kaolin clay in Experiments III-1 and III-4 are similar to those calculated from the data of Olsen$^{33}$ and Gray$^{34}$ (see Section 3.5).

The voltage drops at the electrodes due to polarization ($\Delta \phi_e$) must be included in the theory if it is a sizable fraction of the voltage difference applied.† $\Delta \phi_e$ for these experiments must be determined from the

†In our experiments, only the polarization potential at the anode is included, since the cathode is not in contact with the slurry.
measured voltages at probes immersed in the slurry. To my knowledge there
is no published data on polarization potentials of stainless steel elec-
trodes in NaCl solutions, much less in sewage sludge.

We can use Eq. (4-137) to estimate the relative costs of sludge
dewatering by pressure filtration vs. electroosmosis. For simplicity,
we will neglect \( k_r \) and \( \Delta \phi_e \), so that Eq. (4-137) can be written as:

\[
R_c = \frac{1}{\varepsilon |\psi_k| (1 - n_0)} \left\{ \frac{\mu \sigma_0 k (1-n_k) K_{ce}}{K_{cp}} \times \left[ \frac{(1-n_k) \tau_s^2}{n_{ps}} + \frac{(1-n_0) \tau_k^2}{n_{pk}} \right] \right\}^{\frac{1}{2}}
\]

For the following typical values:

\[
\begin{align*}
\sigma_0 &= 1.17 \text{ (}\Omega\text{-m})^{-1} & \psi_k &= -12 \text{ mV} \\
k &= 10^{-17} \text{ m}^2 & \mu &= 10^{-3} \text{ kg/m-sec} \\
n_0 &= 0.97 & n_{ps} &= 0.94 \\
n_k &= 0.94 & n_{pk} &= 0.85 \\
\tau_s^2 &= 1 & \tau_k^2 &= 1.5
\end{align*}
\]

\[
R_c = 36 \times \left( \frac{K_{ce}}{K_{cp}} \right)^{\frac{1}{2}}.
\]

\( K_{ce} \) and \( K_{cp} \) are the sums of capital amortization, interest, and labor
costs for electroosmosis and pressure filtration, respectively. For elec-
troosmosis to be favorable, \( (K_{ce}/K_{cp}) \) must be less than \( 8 \times 10^{-4} \). This
ratio is most likely impossible, even if inexpensive, corrosion-resistant
electrodes were available.
<table>
<thead>
<tr>
<th>Experiment</th>
<th>III-2</th>
<th>III-3</th>
<th>III-1</th>
<th>III-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>sewage sludge</td>
<td>sewage sludge</td>
<td>Kaolin clay</td>
<td>Kaolin clay</td>
</tr>
<tr>
<td>Fluid conductivity ($25^\circ C$) $\sigma_0$ ((\Omega\cdot m))$^{-1}$</td>
<td>1.17</td>
<td>1.17</td>
<td>0.118</td>
<td>0.118</td>
</tr>
<tr>
<td>Equivalent concentration of NaCl (N)</td>
<td>0.11</td>
<td>0.11</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>Mass fraction of fluid in slurry</td>
<td>0.954</td>
<td>0.95</td>
<td>0.818</td>
<td>0.80</td>
</tr>
<tr>
<td>Mass fraction of fluid in cake</td>
<td>0.923</td>
<td>0.85</td>
<td>0.373</td>
<td>0.421</td>
</tr>
<tr>
<td>Solid specific density</td>
<td>1.91</td>
<td>1.91</td>
<td>2.60</td>
<td>2.60</td>
</tr>
<tr>
<td>$n_0$</td>
<td>.975</td>
<td>.973</td>
<td>.921</td>
<td>.912</td>
</tr>
<tr>
<td>$n_k$</td>
<td>.958</td>
<td>.914</td>
<td>.607</td>
<td>.654</td>
</tr>
<tr>
<td>Hydraulic permeability $k$ (m$^2$)</td>
<td>$1.8 \times 10^{-15}$</td>
<td>$3.22 \times 10^{-17}$</td>
<td>$1.87 \times 10^{-16}$</td>
<td>$3.15 \times 10^{-16}$</td>
</tr>
<tr>
<td>$\Delta \phi_e$ average (volt)</td>
<td>0.82</td>
<td>0.65</td>
<td>N.A.</td>
<td>N.A.</td>
</tr>
<tr>
<td>($\mu q_e \sigma_0 / i$) average, used in computing $\psi_k$ in $10^{-12}$ (coul/m)</td>
<td>8.80</td>
<td>7.30</td>
<td>7.00</td>
<td>13.9</td>
</tr>
<tr>
<td>Power supply set at:</td>
<td>constant voltage</td>
<td>constant voltage</td>
<td>either</td>
<td>constant current density</td>
</tr>
<tr>
<td></td>
<td>3.4 V</td>
<td>3.4 V</td>
<td>1) constant voltage 7.0 V, or</td>
<td>13.6 amp/m$^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2) constant current density 10 amp/m$^2$</td>
<td></td>
</tr>
<tr>
<td>Pressure applied in $10^4$ (N/m$^2$)</td>
<td>0</td>
<td>13.8</td>
<td>6.89</td>
<td>6.55</td>
</tr>
</tbody>
</table>
### TABLE 5-2. Experiments Series III

#### Calculated Data

<table>
<thead>
<tr>
<th>Experiment</th>
<th>III-2</th>
<th>III-3</th>
<th>III-1</th>
<th>III-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debye length 25°C (Å)</td>
<td>8.2</td>
<td>8.2</td>
<td>27.4</td>
<td>27.4</td>
</tr>
<tr>
<td>$e_{co}$ assumed</td>
<td>1.22</td>
<td>1.22</td>
<td>0.43</td>
<td>0.43</td>
</tr>
<tr>
<td>$n_{ps}$</td>
<td>0.944</td>
<td>0.94</td>
<td>0.887</td>
<td>0.874</td>
</tr>
<tr>
<td>$n_{pk}$</td>
<td>0.907</td>
<td>0.809</td>
<td>0.438</td>
<td>0.505</td>
</tr>
<tr>
<td>$a_p$ (Å)</td>
<td>2100</td>
<td>282</td>
<td>925</td>
<td>1117</td>
</tr>
<tr>
<td>$\lambda_p = \lambda_D/a_p$</td>
<td>$3.89 \times 10^{-3}$</td>
<td>$2.91 \times 10^{-2}$</td>
<td>$2.96 \times 10^{-2}$</td>
<td>$2.45 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\psi_k$ (mV) from (uq_eσ_o/i)</td>
<td>-12.9</td>
<td>-11.4</td>
<td>-10.5</td>
<td>-22</td>
</tr>
</tbody>
</table>
FIGURE 5-1. Test cell.

1. PROBE FOR CAKE THICKNESS
2. THERMOMETER
3. ANODE
4. ELECTRODES
5. STAINLESS SCREEN

COMPRESSED GAS

CIRCULATING FLOW
FIGURE 5-2. Experimental apparatus.
FIGURE 5-3. Experiment III-1. Filtrate flow velocity, with and without electroosmosis, as a function of time.
FIGURE 5-5. Experiment III-1. Electrical energy required per unit area vs. time.
FIGURE 5-6. Experiment III-1. $\eta$ vs. time.

- $i_0 = 10$ amp/m$^2$
- $\Delta \phi_0 = 7.0$ V
LINE: THEORY, ASSUMING

\[ k = 1.80 \times 10^{-15} \text{ m}^2 \]
\[ \psi_k = -12.9 \text{ mV} \]
\[ \tau_k = 1.05 \]
FIGURE 5-8. Experiment III-2. Energy required per unit area vs. time.
FIGURE 5-9. Experiment III-3.
LINE: THEORY, ASSUMING

\[ \tau_k = 1.22 \]
FIGURE 5.12. (n) -1 vs. time.
FIGURE 5-13. Experiment III-4. Filtrate volume per unit area vs. time.

LINE: THEORY, ASSUMING

\[ k = 3.15 \times 10^{-16} \text{ m}^2 \]

\[ \psi_k = -19 \text{ mV} \]
FIGURE 5-14. Experiment III-4. Power required per unit area vs. time.
FIGURE 5-15. Determination of the polarization voltage $\Delta \phi_e$ at the anode by extrapolation from voltages measured at two probes in the slurry.
CHAPTER 6:

DEWATERING BY DRYING BEDS

A 1968 report by the Federal Water Quality Administration\textsuperscript{52} surveyed 12,565 sewage treatment plants, serving a population of 131 million (out of a total U.S. population of 197 million). We can derive from the survey the following distribution of sludge treatment methods:

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of Plants</th>
<th>Population Served (millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>septic tanks, Imhoff tanks, stabilization ponds</td>
<td>4,456</td>
<td>24.8</td>
</tr>
<tr>
<td>lagoons</td>
<td>191</td>
<td>8.3</td>
</tr>
<tr>
<td>mechanical dewatering</td>
<td>656</td>
<td>26.2</td>
</tr>
<tr>
<td>drying beds</td>
<td>6,046</td>
<td>52.0</td>
</tr>
<tr>
<td>other methods</td>
<td>1,216</td>
<td>19.7</td>
</tr>
</tbody>
</table>

The mechanical dewatering and drying beds are followed by ultimate disposal of the sludge solid, in landfills, as fertilizers, or by incineration.

If we assume that the average mass of sludge produced per capita per day is relatively constant throughout the country, then the population figures can indicate the distribution of sludge mass processed by the various methods:

- septic tanks, etc. 19%
- lagoons 6.3%
- mechanical 20%
- drying beds 40%
- other methods 14.7%
Sludge dewatering by drying beds, then, appears to be the major method of sludge treatment, despite its antiquity. Certainly it is simple and economic when ample land and hot dry weather are available.

Septic tanks, stabilization ponds, and lagoons also require large land areas.

Disposal of sludge into the ocean is used by many cities on the coast, including Boston, New York, and Los Angeles. This method is increasingly being restricted by pollution regulations.

Mechanical dewatering has the advantages of relatively low land requirements, but at higher operating costs. The most common mechanical devices are centrifuges, vacuum filters, and filter presses.

Typical costs for various sludge handling methods are given below:

<table>
<thead>
<tr>
<th>System</th>
<th>Capital and Operating Cost /ton dry solid (Burd²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incineration</td>
<td>$30</td>
</tr>
<tr>
<td>Landfill dewatered sludge</td>
<td>25</td>
</tr>
<tr>
<td>Land disposal as liquid</td>
<td>15</td>
</tr>
<tr>
<td>Lagooning</td>
<td>12</td>
</tr>
<tr>
<td>Barging to sea</td>
<td>12</td>
</tr>
<tr>
<td>Heat drying</td>
<td>42 (does not account for money received from sale as fertilizer)</td>
</tr>
</tbody>
</table>
This chapter analyzes the dewatering of sewage sludges on drying beds, by drainage and evaporation. The existing theories are reviewed, and further developed when necessary. The effects of sludge cake cracking on the evaporation rate is included with the aid of a simple model. Based on the theories, a cost equation is derived, with the important parameters listed. Optimization of the cost equation results in equations for the optimum loading of sludge solids per unit area of drying bed and for the optimum cake moisture content at removal.

There is an abundance of papers on sludge dewatering. The experimental data presented, however, are usually valid only for the particular sludge being investigated. The theories given in this paper include sludge parameters which must be determined experimentally: some of these parameters are at this moment not available; for example, no previous research has been done on the possible increase of evaporation rate due to the cracking of sludge cakes upon drying.

A typical design of a sand drying bed is shown in Fig. 6-1a. A network of perforated drainage pipes is covered by a layer of pebbles, then layers of progressively finer sand. Sludge of typically 3-8% solid by weight is applied onto the bed, where the liquid is removed by drainage through the porous bed, and by evaporation into the air. The dried sludge solid remaining is removed manually or mechanically.

Figure 6-1b shows another design. The drying bed is paved with concrete or asphalt, with shallow ditches filled with sand for limited drainage. This design has the advantages of easier maintenance and
mechanical removal of dried solids.

The dewatering characteristics of sewage sludges are extremely variable from sludge to sludge. The fraction of sludge liquid which can be removed and the rate of removal depend upon the initial sources of the waste and the treatment prior to dewatering. For example, anaerobically digested sludges are probably the most difficult to dewater, because the solid content is mostly in form of microorganisms, which retain water within the cells and in the gelatinous structure formed by the flocculated cells.\(^1\)

The following dewatering curve, Fig. 6-2, is therefore meant only to be illustrative. The moisture content is plotted as a function of time. Initially, dewatering is primarily by drainage, which stops after a few hours and leaves a sludge cake of about 85-91% moisture.\(^\dagger\) Further dewatering by evaporation is much slower. Sewage sludge typically has to be dewatered to about 75-80% moisture before it can be removed with ease from the bed. Figure 6-3 shows a plot of the ratio \(U\) to the moisture content in \(\%\). It is clear that the bulk of the liquid is removed by drainage. Paved drying beds therefore may require longer dewatering time than sand beds, although the labor and maintenance costs will be lower.

\(^{\dagger}\) The moisture content of the sludge is conventionally expressed in two ways:

\[
\text{wet basis} = \frac{\text{weight liquid}}{\text{total sludge weight}} \quad \text{in } \% \\
\text{dry basis} = \frac{\text{weight liquid}}{\text{weight solid}} = U .
\]

In dewatering, the solid is hopefully completely retained, so that the second ratio has a constant denominator, and will be referred to as \(U\). The first will be given as \(\%\).
SECTION 6.1. DEWATERING BY DRAINAGE

During drainage, a major portion of the sludge solid is hopefully retained above the bed. The remainder of the solid consists of a fraction which penetrates into the porous bed, and is held up within the bed, and a fraction carried through the entire bed by the filtrate. The former can cause clogging; the latter may require further treatment or recycling of the filtrate. The relative fractions depend of course on the solid particle size distribution in the sludge and the pore size distribution in the bed.

The solid retained forms a highly compressible porous layer, or cake, on the bed. Solid is added continuously to the cake as the remaining liquid sludge is drained. Under the weight of the liquid sludge and the cake solid, the cake compacts, especially near the cake-bed interface. The profile of the fluid pressure through the bed would look like Fig. 6-4. The major resistance to drainage flow would be near the cake-bed interface. Tiller et al.\textsuperscript{53} and Halff,\textsuperscript{54} among others, have investigated the filtration process. They have applied Darcy's law locally through the bed, taking into account the local cake porosity and permeability. The solutions involve numerical integrations, with empirical formulas relating the variations of porosity and permeability with the local fluid pressure.

Drainage stops when the height $H_S$ of the liquid sludge goes to zero. The surface tension in the pores at the air-cake interface may then resist further drainage of liquid.
Nebiker et al.\textsuperscript{55} have given a simplified analysis of the drainage process. Darcy's law is applied across the entire cake, with an average flow resistance per unit thickness of the cake, which depends only on the overall pressure drop across the cake.

The solution presented below is similar to Nebiker's; the final equation, however, differs by a constant multiplier.

Darcy's law relates the flow rate per unit bed area to the overall pressure drop across the cake:

\[
q_1 = \frac{dV_1}{dt} = \frac{\rho gh_f}{\mu (r_k h + r_b h_b)}, \quad (6-1)
\]

where \( q_1 \) = filtrate volume flow rate per unit area [L/T];

\( V_1 \) = filtrate volume per unit area [L];

\( g \) = gravitational constant;

\( h_f \) = head loss across cake [L];

\( h \) = cake thickness [L];

\( r_k \) = resistance of cake per unit thickness [L\(^{-2}\)];

\( h_b \) = thickness of bed [L];

\( r_b \) = resistance of bed per unit thickness [L\(^{-2}\)].

Assuming that the liquid and solid are incompressible, the conservation of mass then gives:

\[
(1 - n_o)H_0 = (1 - n_o)H_S + (1 - n_k)h \quad (6-2)
\]

\[
n_oH_0 = n_oH_S + n_k h + v_1 \quad , \quad (6-3)
\]
where \( n_0 \) = average volume fraction of fluid in the sludge;
\( n_k \) = average volume fraction of fluid in cake;
\( H_0 \) = initial height of sludge;
\( H_s \) = height of sludge above the cake.

From Eqs. (6-2) and (6-3) we can derive the following relationships:

\[
h = \left( \frac{1 - n_0}{1 - n_k} \right) (H_0 - H_s) \quad (6-4)
\]

\[
\frac{dv_1}{dt} = - \left( \frac{n_0 - n_k}{1 - n_k} \right) \frac{dH_s}{dt} \quad . \quad (6-5)
\]

Substitute Eqs. (6-4) and (6-5) into Eq. (6-1). If we neglect the resistance of the bed, we have the governing differential equation:

\[
\frac{dH_s}{dt} = - \frac{\rho g H_s (1 - n_k)^2}{\mu r_k (n_0 - n_k)(1 - n_0)(H_0 - H_s)} \quad . \quad (6-6)
\]

Experimentally, \( r_k \) is found to be a function of the total pressure across the cake:

\[
r_k = r^*(\frac{H_s}{H_s^*})^\beta \quad , \quad (6-7)
\]

where \( \beta \) = compressibility coefficient;
\( H_s^* \) = same reference head;
\( r^* \) = resistance at \( H^* \).

We can convert the variables \( n_0 \) and \( n_k \) in Eq. (6-6) to the usually measured variables \( S_0 \) and \( S_k \), where
\[ S_0 = \text{solid mass fraction in the sludge}; \]
\[ S_k = \text{solid mass fraction in the cake}. \]

The governing differential equation is then:
\[
\frac{dH_s}{dt} = -\frac{\rho g H_s}{\mu r_k (H_0 - H_s) f(S_0, S_k)} \tag{6-8}
\]

\[
f(S_0, S_k) = \frac{\rho_s (S_k - S_0)}{\left[ \frac{\rho_s S_k + \rho S_k (1 - S_0)}{\rho_s S_k + \rho S_k (1 - S_k)} \right]^2}. \tag{6-9}
\]

Substituting for \( r_k \) into Eq. (6-8) and using the initial conditions that \( H_s = H_0 \) at \( t = 0 \), we have
\[
t = \frac{\mu f(S_0, S_k) r^*}{\beta (\beta + 1) (H_0^*)^\beta} \left[ H_0^{\beta+1} + \beta H_s^{\beta+1} - (\beta + 1) H_0 H_s^\beta \right]. \tag{6-10}
\]

If the solid content \( S_k \) of the cake is constant, then the thickness of the cake is proportional to the weight of the cake solid per unit area. In filtration literature, the resistance of the cake is given as "specific resistance," or the resistance per unit mass of cake solid per unit area. The specific resistance \( \alpha_T \) is related to \( r_k \) by the equation:\(^1\)
\[
\alpha_T = \frac{r_k}{\rho_s (1 - n_k)} \quad . \tag{6-11}
\]

\(^1\)The units of \( \alpha_T \) are [length/mass].
We can non-dimensionalize Eq. (6-10) and arrive at a simple final form:

$$
\bar{t} = 1 + \beta \bar{H}^{\beta+1} - (\beta + 1) \bar{H}^\beta
$$

(6-12)

$$
\bar{t} = \frac{\beta (\beta + 1) (H_s^*)^\beta t}{\mu r^* f(S_0, S_k)(H_0)^{\beta+1}}, \quad \bar{H} = \frac{H_s}{H_0}.
$$

(6-13)

At the end of the filtration, \( \bar{H} = 0 \) and \( \bar{t} = 1 \). The characteristic time for the drainage is:

$$
t_{\text{char}} = \frac{\mu r^*(H_0)^{\beta+1} f(S_0, S_k)}{\beta (\beta + 1) (H_s^*)^R},
$$

(6-14)

where \( t_c \) is a function of the following parameters:

- sludge characteristics: \( \mu, \beta, r^*, H_s^*, \rho, \rho_s \)
- operating parameters: \( S_0, H_0 \)
- the solid content of the cake: \( S_k \)

The solid content \( S_k \) appears to depend primarily on the sludge, and is weakly dependent on the applied initial height \( H_0 \) or on \( S_0 \). The assumption of constant \( S_k \) is probably a good one.

Nebiker has shown good agreement of the theory with experimental data, with the help of a fudge factor to allow for the media resistance.
SECTION 6.2. DEWATERING BY EVAPORATION; GENERAL REMARKS

At the end of the drainage period, the sludge is in the form of a moist cake of typically 85-91% moisture. Further dewatering by evaporation involves the following mechanisms:

(1) The transfer of heat from the atmosphere into the sludge cake by conduction, convection, and radiation.

(2) The evaporation of liquid water at the sludge surface, and within the cake itself.

(3) The mass transfer of moisture in two stages: the internal mass transfer of moisture to the air-sludge interface, followed by external mass transfer of water vapor into the air. The slower of the two rates would be the limiting one.

Heat transfer into the sludge depends upon:

(1) air and sludge temperature;

(2) wind velocity;

(3) radiation intensity;

(4) emissivity of the sludge surface;

(5) physical characteristics of the sludge surface.

The external mass transfer of vapor into the air depends upon:

(1) the air temperature and humidity;

(2) the wind velocity;

(3) the physical characteristics of the sludge surface.
The internal transfer of vapor is by:\textsuperscript{56}

(1) molecular effusion in microcapillaries, radius \( < 10^{-5} \) cm. The mean free path of water vapor is \( 0.42 \times 10^{-5} \) cm at 15°C and 760 mm Hg;

(2) molar diffusion in macrocapillaries;

(3) molar motion under a pressure gradient within the porous cake, due to:

(a) expansion of water vapor-air mixture upon heating, and

(b) rapid evaporation of liquid water.

The internal transfer of liquid water is influenced by four potentials:\textsuperscript{40}

(1) osmotic potential, due to the presence of dissolved substances in the water;

(2) gravitational potential;

(3) matrix, or capillary-sorption potential;

(4) pneumatic potential due to a gas pressure gradient.

In open air drying of sewage sludges, the liquid water moves predominantly by capillary suction, and the vapor by diffusion.

The drying portion of the "typical" dewatering curve is shown in Fig. 6-5, with the rate plotted as a function of \( U \). Constant atmospheric conditions are assumed. We can divide the curve into three parts: a constant rate period, followed by two distinct falling rate periods.

**Constant Rate Period**

After the end of drainage, the sludge cake dries at a constant rate
to a moisture content of about 75-84% (U = 3-5). The rate is close to that of a free water surface. The cake temperature remains constant, near the wet bulb temperature. Evaporation takes place at the surface of the cake, and internal transfer is mostly in the form of water.

Figure 6-6 shows a model of the cake. The solid forms a loose structure. The voids are initially completely filled with liquid, forming a network of interconnecting capillaries of a distribution of sizes. As the water evaporates from the menisci of the capillaries at the surface the curvature of the menisci exerts enough suction to draw water up from the large pores within the cake. The capillary tension can be sufficient to cause shrinkage of cake thickness. Air enters into the cake through a few large pores to replace the liquid removed.

As drying proceeds, capillaries of successively smaller diameters are depleted. Breaks appear in the liquid network. Eventually the air-filled capillaries within the cake are of comparable diameters to those at the surface. The flow rate decreases, and the surface menisci begin to retreat into the cake. This does not happen simultaneously over the entire surface, because of the variation in capillary sizes. The overall effects are that the wetted fraction of the surface area and the internal transfer rates decrease. The evaporation rate then begins to fall.

**First Falling Rate Period**

The evaporation rate decreases because of the decrease in wetted surface area and liquid transfer rate. Some evaporation takes place at
the menisci which have retreated into the cake; the water vapor then moves to the surface by diffusion.

The rate decreases approximately linearly with $U$.

**Second Falling Rate Period**

When all of the menisci have receded from the surface, then evaporation takes place only within the cake. Diffusion of the vapor through the porous cake to the surface is the limiting transfer rate.

The drying rate does not vary linearly with $U$.

**Transition Points**

There are two transition points in the curve, generally called the first and second "critical moisture contents" $U_{cr1}$ and $U_{cr2}$. Their dependence on the drying parameters will be discussed later in the analysis.

In closing the general description of the drying process, there are two important notes:

1. The drying curve in Fig. 6-5 is only illustrative. Depending on the type of sludge and the drying conditions, there may not be any clear transition points in the curve, and any of the three periods may be missing.

2. Sludge has to be dewatered to about 20-25% solid content ($U = 3 + 4$) to be removed and handled with ease. This moisture content can sometimes be reached in the constant rate period.
Initial Conditions for Evaporative Drying

Figure 6-7 shows that for an aerobically digested sludge, the drainage period ends after the first few hours. The sludge is then in the form of a moist cake of typically 85-91% moisture \((U = 5.7 - 10)\). Figure 6-7 also shows the thickness of the same sludge as a function of time. The initially applied thickness of the liquid sludge was 8 inches. The thickness decreases rapidly during drainage to about 2 inches, and then remains relatively constant until day 5. To simplify the subsequent theoretical analysis, we will assume that the cake thickness during the drying period is constant. A mass balance then gives

\[
H = \frac{(\rho + \rho_s U_c) H_0}{(\rho + \rho_s U_0)}, \tag{6-15}
\]

where \(U_0\) = moisture content of initial liquid sludge;

\(U_d\) = moisture content at end of drainage;

\(H\) = thickness of cake during evaporative drying.

This can be rewritten in the form:

\[
w = \gamma_0 H, \tag{6-16}
\]

where \(w\) = mass of cake solid per unit area;

\[
\gamma_0 = \text{mass of cake solid per unit cake volume} = \frac{\rho \rho_s}{(\rho + \rho_s U_d)}.\]

The cake is removed from the drying bed when it has reached a moisture content \(U_d\); it is then dry enough to be handled. The mass of the cake per unit area at removal is \(w_D\),

\[
w_D = w(1 + U_d). \tag{6-17}
\]
SECTION 6.3. EVAPORATION; THEORY

Governing Equations

Luikov$^{56}$ has derived a set of equations describing mass and heat transfer in a porous body. There are three coupled non-linear differential equations relating $T$, $p$, and $\Theta$ (Luikov, p. 260), where $T$ is the temperature, $p$ is the pressure, and $\Theta$ is a "mass transfer potential," defined with respect to a reference material.

In most drying situations the relaxation time for a perturbation in pressure in the porous medium is much shorter than those for perturbations in temperature and moisture potential, so that the pressure can be assumed to be uniform (Luikov, p. 289).

$\Theta$ is then a function of $T$ and of $u$, the local moisture content (distinct from $U$, the average moisture content for the cake). The mass of vapor is usually negligible compared to that of liquid water, so that $u$ is assumed equal to the liquid content. Usually $\Theta$ exhibits hysteresis with respect to $u$, depending on whether moisture is being added to or removed from the porous body (Fig. 6-8). During drying, we are on the dewatering curve, so that $\Theta$ is then a single-valued function of $u$, expressed as

$$u = c_m \Theta$$

(6-18)

where $c_m$ is the "specific mass capacity."

The governing equations can then be rewritten as:
\[ \frac{\partial T}{\partial t} = \nabla \cdot (a_v \nabla T) + \frac{\varepsilon_v \Lambda}{c_v} \frac{\partial u}{\partial t} \]  \hspace{1cm} (6-19) \\
\[ \frac{\partial u}{\partial t} = \nabla \cdot (a_m \nabla u + \delta_v \nabla T) \]  \hspace{1cm} (6-20)

where \( a_m \) = "coefficient of mass diffusivity;"
\( a_v \) = coefficient of thermal diffusivity;
\( \varepsilon_v \) = phase conversion factor = fraction of liquid converted to vapor;
\( \Lambda \) = latent heat of vaporization;
\( c_v \) = thermal heat capacity;
\( \delta_v \) = temperature gradient coefficient.

It must be emphasized that Eq. (6-20) is of the diffusional form only in a mathematical sense. In sludge drying, liquid water moves predominantly by capillary suction, and vapor by diffusion.

The parameter \( \varepsilon_v \) accounts for evaporation within the porous body. When \( \varepsilon_v = 1 \), all changes in liquid content are due to phase change. When \( \varepsilon_v = 0 \), then all liquid content changes are due to liquid transfer. \( \varepsilon_v \) is determined experimentally from drying data (see p.

The coefficient \( \delta_v \) accounts for:

1. the effect of temperature on the moisture transfer potential; for example, surface tension is a function of \( T \); and
2. the movement of liquid due to the expansion of entrapped air and vapor upon heating.

The coefficients \( a_m, a_v, c_v \) and \( \delta_v \) are all functions of both \( u \)
and \( T \), to be found experimentally. Data is not available for wastewater sludges; Fig. 6-9 shows data for other materials as illustration.

Figure 6-9c is typical of most capillary-porous bodies. \( a_m \) is constant for \( u \) down to a certain value (0.33 in this figure) and then decreases rapidly. This feature is important later in the discussion of \( U_{cr1} \).

An energy balance for the sludge cake would be:

\[
J_q = dE_T + \Lambda \, dM_L, \quad (6-21)
\]

where \( E_T \) = thermal energy stored in cake;
\( dM_L \) = mass of liquid evaporated;
\( J_q \) = net heat input.

Heat input is by radiation, convection, and conduction. The conduction heat transfer, with the air and the drying bed, is usually neglected compared to the other two.

\[
J_{qs} = h_c(T_a - T_s) + \epsilon_r \sigma_B(T_r^4 - T_s^4); \quad (6-22)
\]

where \( J_{qs} \) = heat transfer rate at surface;
\( h_c \) = convective heat transfer coefficient;
\( T_a \) = atmospheric temperature;
\( T_s \) = surface temperature;
\( \epsilon_r \) = emissivity of sludge surface;
\( \sigma_B \) = Stefan-Boltzmann constant;
\( T_r \) = temperature of radiation source.
For a large bed area, we can neglect the edge effects and assume that the differential equations are one-dimensional. If we further assume that during drying, the heat and mass transfer across the sludge-bed interface are negligible, then the boundary conditions are:

At the air-sludge cake interface, $x = H$:

$$-k_v(\nabla T)_s - (1 - \epsilon_v)\Lambda j_{ms}(t) + j_{qs}(t) = 0 \quad (6-23)$$

$$a_m\gamma_0(\nabla u)_s + \delta_v(\nabla T)_s + j_{ms}(t) = 0 \quad (6-24)$$

$k_v =$ coefficient of thermal conductivity;

$j_{ms} =$ mass transfer rate at surface.

At the sludge cake-bed interface, $x = 0$:

$$j_q = j_m = 0 \quad . \quad (6-25)$$

Luikov classified the boundary conditions at $x = H$ into three kinds:

1. First kind: the surface temperature and/or moisture content $u_s$ are given as functions of time.

2. Second kind: the transfer rates $j_{ms}$ and $j_{qs}$ are given as functions of time.

3. Third kind: the transfer rates are linear functions of the surface temperature and moisture content.

$$j_{qs}(t) = h_c(T_a - T_s) \quad (6-26)$$

$$j_{ms}(t) = h_m\gamma_0(u_s - u_a) \quad . \quad (6-27)$$
Constant Rate Period

Luikov\(^{56}\) has solved the set of equations (6-19) and (6-20) for the following conditions (Luikov, p. 342):

1. constant coefficients \(a_m, a_v, c_v,\) and \(\delta_v;\)
2. constant surface transfer rates \(j_{ms} = q_m; j_{qs} = q_c.\)

The solution is "quasi-stationary" after a definite time interval, defined by \(Fo_m > 0.54,\) where \(Fo_m\) is the Fourier mass transfer number:

\[
Fo_m = \frac{a_m t}{H^2} .
\]

In the quasi-stationary regime,

\[
\frac{\partial T}{\partial t} \sim 0 ; \quad \frac{\partial u}{\partial t} = -\frac{q_m}{\gamma_0 H} ; \quad (6-28)
\]

the differential equations are then

\[
a_v \frac{\partial^2 T}{\partial x^2} - \frac{\epsilon_v \Delta q_m}{c_v \gamma_0 H} = 0 \quad (6-29)
\]

\[
a_m \frac{\partial^2 u}{\partial x^2} + a_m \delta_v \frac{\partial^2 T}{\partial x^2} - \frac{q_m}{\gamma_0 H} = 0 \quad (6-30)
\]

Solution of Eqs. (6-29) and (6-30) shows that the distributions of temperature and moisture content are parabolic,

\[
u = u_c - \frac{x^2}{H^2} (u_c - u_s) ; \quad T = T_c + \frac{x^2}{H^2} (T_s - T_c) \quad (6-31)
\]

and that

\[
\frac{u_c - u_s}{u_0 - u_e} = \frac{\frac{\epsilon_v \Delta v \Lambda}{c_v}}{1 + \frac{\epsilon_v \Delta v \Lambda}{c_v}} \quad (6-32)
\]
\[ \frac{T_S - T_c}{T_a - T_0} = \frac{1}{2} K_i m \varepsilon_v K_0 L_u, \quad (6-33) \]

where the subscript "c" refers to the bottom cake surface \( x = 0 \), and subscript "s" refers to the surface. \( u_e \) is the final equilibrium moisture content if the constant rate period continued until \( t \to \infty \). The subscripts "o" refer to the initial conditions at \( t = 0 \). The non-dimensional parameters are:

\[ K_i m = \frac{q_m H}{a_m \gamma_0 (u_o - u_e)} \quad K_0 = \frac{\Lambda (u_o - u_e)}{c_v (T_a - T_0)} \quad L_u = \frac{a_m}{a_v}. \quad (6-34) \]

If the moisture transfer is entirely in the liquid form \( \varepsilon_v = 0 \), then \( T_S = T_c = \) constant; i.e., then the temperature is uniform throughout the cake. \( \varepsilon_v \) can be determined experimentally, with the use of Eqs. (6-33) and (6-34):

\[ \Delta T = T_S - T_c = \frac{1}{2} \frac{\Lambda q_m H \varepsilon_v}{K_v}. \quad (6-35) \]

Luikov shows that for gypsum, \( \varepsilon_v = 0.045 \). The approximation of \( \varepsilon_v = 0 \) is therefore probably not a bad one for sludge.

If the cake temperature is constant, then the mass transfer equation is uncoupled from temperature effects, and the moisture distribution is given by the following equation:

\[ u(x, t) = u_0 + \frac{q_m H}{a_m \gamma_0} \left[ \frac{1}{6} - \frac{1}{2} \frac{x^2}{H^2} - \frac{a_m t}{H^2} + \right. \]

\[ \left. + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n \pi}{n^2} \cos \left( \frac{n \pi x}{H} \right) \exp \left( - \frac{n^2 \pi^2 a_m t}{H^2} \right) \right]. \quad (6-36) \]
The exponential terms approach zero for $F_{0_m} > 0.54$. The surface moisture content is then given as:

$$u_s(t) = u_0 - \frac{q_m H}{a_m \gamma_0} \left( \frac{1}{3} + \frac{d_m t}{H^2} \right), \quad (6-37)$$

and $U_s$, the average moisture content, is given by:

$$U = \frac{1}{H} \int_0^H u(x,t) \, dx = u_0 - \frac{q_m t}{H \gamma_0}. \quad (6-38)$$

The constant rate period ends when $u_s$ decreases to a certain moisture content $(u_s)_{cr}$. The critical moisture content, $U_{cr}$, is the average cake moisture content at that point, or:

$$U_{cr} = (u_s)_{cr} + \frac{1}{3} \frac{q_m H}{\gamma_0 a_m}. \quad (6-39)$$

**Critical Moisture Content**

There are three descriptions so far for the transition point from constant rate into the falling rate period.

1. Luikov$^{56}$ and Nebiker$^{60}$ both assumed that $(u_s)_{cr}$ is a characteristic of the particular sludge, and is therefore constant for a given drying process. Luikov proposed that $(u_s)_{cr}$ corresponds to the maximum sorptional moisture content; the maximum moisture content which can be retained by adsorptional forces alone. Equation (6-39) then shows that $U_{cr}$ is a linear function of the product $(q_m H)$.

Figure (6-10) from Nebiker$^{60}$ presents $U_{cr}$ as a function of
\( (q_m \gamma_o H) \). For some reason, Nebiker correlated the data with the equation:

\[
U_{cr} = 5 \left(q_m H \gamma_o \right)^{1/2}.
\]

It seems that a linear curve would fit the data as well, and give \((u_s)_{cr} = 2.0\).

In soil literature, a term called moisture of rupture of capillary bond is used (MRC). This is the moisture content below which there is no capillary network continuously filled with water; i.e., the liquid phase is no longer continuous and cannot move by capillary suction. I presume that \( u = 0.33 \) in Fig. 6-9c corresponds to the MRC for the diatomite slab; \( a_m \), the moisture diffusivity coefficient, decreases abruptly below this point.

The value of 0.33 is, however, very different from \((u_s)_{cr} = 2.0\) from Nebiker's data. Certainly some difference is expected between diatomite slab and sewage sludge, but not of this magnitude.

(2) Drying data by other authors—for example, Quon and Ward\(^{58}\) and Randall and Koch\(^{59}\)—show that \( U_{cr} \) seems to be relatively constant, independent of the cake thickness \( H \) or of the drying rate \( q_m \). No explanations were given.

(3) If we examine the boundary conditions of the third kind, Eq. (6-27), we see that for a given rate \( q_m \), there is a value of \( u_s \) below which

\[
j_{ms}(t) = h_m \gamma_o (u_s - u_e) < q_m.
\]
If we assume that this transition value of \( u_s \) is \((u_s)_{cr}\) (Fig. 6-11), then:

\[
(u_s)_{cr} = u_e + \frac{q_m}{h_m \gamma_0}.
\]  

(6-41)

In other words, we assume that the maximum possible evaporation rate at the surface is proportional to the wetted fraction of the surface area. The transfer rate from the surface to the air depends on the mass transfer resistance of the air boundary layer. The slower of the two rates is the limiting one.

Substituting for \((u_s)_{cr}\) into Eq. (6-36), we have

\[
U_{cr} = u_e + \frac{q_m}{h_m \gamma_0} + \frac{1}{3} \frac{q_m^H}{\sigma_m \gamma_0}.
\]  

(6-42)

The raw data from Nebiker,\(^{60}\) however, could not be correlated with this equation.

**First Falling Rate Period**

Luikov\(^{56}\) solved the differential equations (6-19) and (6-20) for the following conditions:

1. Initial conditions: parabolic distribution of temperature and moisture content, from the end of the constant rate period. \( \varepsilon_v \), however small, is assumed to be nonzero, so that \( T_S \neq T_C \).

\[
T(x,0) = T_C + \frac{x^2}{H^2} (T_S - T_C);
\]  

(6-43)

\[
u(x,0) = u_c - \frac{x^2}{H^2} \left[ u_c - (u_s)_{cr} \right];
\]  

(6-44)

\[
U_s = (u_s)_{cr} = \text{constant}.
\]  

(6-45)
(2) Boundary conditions of the third kind:

\[- k_y \frac{\partial T(H,t)}{\partial x} + h_c \left[ T_a - T(H,t) \right] - \Lambda (1 - \epsilon_v) h_m \gamma_0 \left[ u(H,t) - u_e \right] = 0 \]  
(6-46)

\[ a_m \gamma_0 \left[ \frac{\partial u(H,t)}{\partial x} + \delta_v \frac{\partial T(H,t)}{\partial x} \right] + h_m \gamma_0 \left[ u(H,t) - u_e \right] = 0 \]  
(6-47)

\[ \frac{\partial u(0,t)}{\partial x} = \frac{\partial T(0,t)}{\partial x} = 0 \]  
(6-48)

The solution is in the form:

\[ T^* = \frac{T(x,t) - T_s}{T_c - T_s} = 1 - \sum_{n=1}^{\infty} \left\{ \left[ c_{n_2} \cos \frac{\nu_1 \mu_n x}{H} - c_{n_1} \cos \frac{\nu_2 \mu_n x}{H} \right] \times \exp \left( -\mu_n^2 F_o \right) \right\} \]  
(6-49)

\[ u^* = \frac{u_s - u(x,t)}{u_s - u_e} = 1 - \sum_{n=1}^{\infty} \left\{ \left[ c_{n_1}^* (1 - \nu_2^2) \cos \frac{\nu_2 \mu_n x}{H} - c_{n_2}^* (1 - \nu_1^2) \cos \frac{\nu_1 \mu_n x}{H} \right] e^{-\mu_n^2 F_o} \right\}, \]  
(6-50)

where \( F_o \) is the Fourier heat transfer number \( = a_v t / H^2 \). For definitions of the non-dimensional parameters in the summations, see Luikov, p. 293-294, and p. 346.

The effect of the initial parabolic distributions is important only at the beginning of the falling rate period. Afterwards, the solution enters a regular regime, independent of the initial distributions, and the non-dimensional average moisture content \( U \) can be expressed in the following form:
\[ U^* = \frac{U - u_e}{U_{cr} - u_e} = f(Fo_m, Bi_m/Pn) \quad \text{for } U^* < 0.7 \quad (6-51) \]

where \( Fo_m = \) Fourier mass transfer number \( = \alpha_m t/H^2 \);
\( Bi_m = \) Biot mass transfer number \( = h_m H/\alpha_m \);
\( Pn = \) Posnov number \( = \delta_v (\partial T/\partial u) \).

Figure 6-12 from Luikov (p. 348) shows the variation of the non-dimensional drying rate \( dU^*/dFo_m \) vs. \( U^* \). For clay, we see that the drying rate decreases linearly with the moisture content; i.e., in Eq.(6-50), the exponential terms in the summation all become negligible except for the first term.

This linear decrease of drying rate with moisture content is observed for sewage sludges as well (see Ref. 10 and 12).

**Second Falling Rate Period**

In the final period of drying, evaporation takes place entirely within the porous cake, at the menisci which have receded from the surface and at the surface of isolated pockets of water.

Morgan and Yerazunis\(^61\) have presented a theoretical analysis, with the following major assumptions:

1. The evaporative interface is a plane parallel to and located at a fixed distance \( \delta_e \) below the surface of the porous body.
2. The heat and mass transfers are one-dimensional and steady state.
3. Laminar and turbulent boundary layers are considered.
The coordinate system is defined in Figure 6-13.

A dimensionless heat flux is defined:

\[
\bar{q}(\Gamma) = \frac{q_{av}}{(q_{av})_0},
\]

where \( \Gamma \) is the dimensionless variable

\[
\Gamma = \frac{k_{\text{eff}}}{\delta h_L} \frac{T_{eo} - T_a}{T_{so} - T_a},
\]

where \( q_{av} \) = average heat flux over the length \( \ell \) of the body;

\( (q_{av})_0 \) = average heat flux when evaporation takes place at the surface;

\( k_{\text{eff}} \) = effective thermal conductivity of the porous medium above the evaporative interface;

\( h_L \) = heat transfer coefficient (from surface to air) at \( z = \ell \);

\( T_{eo} \) = temperature of evaporative plane at \( z = 0 \); depends only on the ratio of heat to mass transfer coefficients for porous medium in accordance with energy balance;

\( T_a \) = free stream temperature;

\( T_{so} \) = surface temperature when evaporation takes place at surface.

From energy balance, we have \( q_{av} = -\Lambda j_{av} \), so that

\[
\bar{q}(\Gamma) = \frac{j_{av}}{(j_{av})_0},
\]
where \( J_{av} \) = average mass transfer flux;
\[(J_{av})_0 = \text{average mass flux when evaporation takes place at the surface.}\]

Figure 6-14 shows a plot of \( \overline{q}(\Gamma) \) vs. \( \Gamma \) for four types of boundary layers.

Assumptions (1) and (2) on p. 240 are reasonable for drying situations when the evaporation rate, and consequently the downward motion of the evaporative interface, is slow enough.

To derive the drying time required, we first assume that at the beginning of the second falling rate period the sludge cake is at a uniform moisture content of \( U_{c2} \). Under atmospheric conditions, the cake cannot be dried to zero moisture content; it would retain a minimum moisture content of \( U_F \) in equilibrium with the humidity of the air. \( U_F \) is typically 0.1 - 0.18, and can be higher for anaerobically digested sludges, because of the water held within the biological cells.

If
\[
\begin{align*}
  u &= U_F & y < \delta_e \\
  u &= U_{c2} & \delta_e < y < H
\end{align*}
\]

then a mass balance gives:

\[
U = U_{c2} - (U_{c2} - U_F) \frac{\delta_e}{H} \quad (6-56)
\]

or
\[
\frac{dU}{dt} = -\frac{(U_{c2} - U_F)}{H} \frac{d\delta_e}{dt} = - \frac{J_{av}}{w} \quad (6-57)
\]
\( J_{av} \) is given by Eq. (6-54). We can non-dimensionalize Eq. (6-57) using Eq. (6-53), and integrate:

\[
\int_\Gamma \frac{d\Gamma}{\Gamma^2 \bar{q}(\Gamma)} = \frac{(J_{av})_e H h_L (T_s - T_a) t}{w (U_{c2} - U_F) k_e [T_e(0) - T_a]}.
\] (6-58)

\( \bar{q}(\Gamma) \) is a complex infinite series in powers of \( \Gamma \); the reader is referred to the original paper. Equation (6-58) gives \( \delta_e \) numerically as a function of \( t \), from which we could calculate \( U \) vs. \( t \) using Eq. (6-56). We will not do so in our analysis, since the cake is usually removed before the second falling rate period begins.

**The Effects of Cracks Upon the Evaporation Rate**

During drainage, the sludge solid is deposited in the form of a moist cake upon the bed. The cake is compressible and plastic, and shrinks upon dewatering. If the shrinkage is constrained, then cracks appear in the cake.

Cracking increases the surface area exposed to the air, and could therefore increase the evaporation rate. Drying cracks are common with other materials, such as clay and soils, and seem to appear in a somewhat regular pattern.

When cracks appear will depend on the particular material and the drying conditions. Martin,\(^{62}\) from field work with clay drying beds,
and Randall and Koch,\textsuperscript{59} from experiments with aerobically digested sludges, have observed cracks appearing at the end of the drainage period. Other authors,\textsuperscript{60,63} however, have not observed cracks until well into the drying period.

To determine the effects of cracks on the evaporation rate, we need to know:

1. if a characteristic pattern width, and depth of cracks could be determined for each particular sludge and drying condition; and
2. how the evaporation rate from the surface area exposed by the cracks would depend on crack depth and width, wind velocity, and crack orientation with respect to the wind direction.

There has been very little research on drying cracks. The results available are:

1. From Keey\textsuperscript{64} and Wakeman,\textsuperscript{65} we have analyses of stress distribution in the cake, based on the assumption of some relationship between volume shrinkage and decrease in moisture content.
2. From Adams et al.,\textsuperscript{66} we have experimental results on evaporation from a simulated crack oriented perpendicularly to the wind velocity, as a function of crack width, depth, and of wind velocity.

No research, however, has been done on cracking patterns. The problem is further complicated by the fact that cracks appear with all possible angles to the wind direction.

We will therefore propose a most simple and crude model. The cracks are assumed to appear in circular patterns, with a characteristic diameter
\( D_c \), as shown in Fig. 6-15.

For each cylinder, let \( q_m \) be the evaporation flux from the top surface, and \( f q_m \) be the flux from the side. Then the total evaporation rate from the cylinder is

\[
Q = - \frac{dm_F}{dt} = q_m \frac{\pi D_c^2}{4} + f q_m \pi D_c H ,
\]

(6-59)

where \( m_F \) = mass of water in the cylinder.

The mass of solid in the cylinder is \( m_s = \gamma_0 \frac{\pi D_c^2}{4} H \). Therefore,

\[
\frac{dU}{dt} = \frac{1}{m_s} \frac{dm_F}{dt} = - \frac{q_{me}}{\gamma_0 H} = - \frac{q_m}{\gamma_0 H} \left( 1 + \frac{4fH}{D_c} \right) ,
\]

(6-60)

where \( q_{me} \) = effective evaporation rate per unit area of the bed.

There are two extreme cases for the dimensions of the cracks, assuming that \( f = \text{constant} \).

1. If cracks occur with a constant aspect ratio \( D_c/H \), then:

\[
q_{me} = q_m (1 + b) \quad ; \quad b = \frac{4fH}{D_c} .
\]

(6-61)

2. If the diameter \( D_c \) is constant regardless of cake thickness, then:

\[
q_{me} = q_m \left( 1 + \frac{4fH}{D_c} \right) = q_m (1 + cW) ;
\]

(6-62)

\[
c = \frac{4f}{D_c \gamma_0} .
\]

(6-63)
SECTION 6.4. TOTAL Dewatering TIME

The drainage time is given by the following equation:

$$t = \frac{\mu f(S_0, S_k) r^*(H_0)^{\beta+1}}{\beta(\beta + 1)(H_0^*)^\beta}.$$  \hfill (6-14)

The derivation of the equation and the definition of the symbols are given in Section 6.1.

For a particular sludge, the drying curves of \( dU/dt \) vs. \( U \) would be a function of both \( q_{me} \), the effective evaporation rate during the constant-rate period, and of \( H \), the cake thickness, as shown in Fig. 6-16.

The critical point \( U_{cr} \) is given by Eq. (6-39):

$$U_{cr} = U_s + \frac{1}{3} \frac{q_{me} H}{a_m \gamma_0},$$ \hfill (6-39)

where \( U_s = (u_s)_{cr} = \) moisture content of surface at critical point.

\( U_e = u_e \) is the equilibrium moisture content which can be reached during the first falling rate period. It is introduced by the boundary condition:

$$j_{ms} = \gamma_0 h_m (u_s - u_e).$$ \hfill (6-27)

\( U_e \) therefore would depend on the drying conditions, as shown in Fig. 6-16.

A constant \( U_{cr} \), independent of \( q_{me} \) and \( H \), has been observed by several authors. It can be explained with the use of Eq. (6-39) for \( U_{cr} \) and Eq. (6-36) for the moisture content distribution within the cake during the constant rate period [the exponential term in Eq. (6-36) is
neglected; it decays rapidly with time].

\[ u(z,t) = U_c + \frac{q_{me}^H}{a_m \gamma_0} \left( \frac{1}{6} - \frac{1}{2} \frac{x^2}{H^2} \right) - \frac{q_{me} t}{\gamma_0 H} \]  \hspace{1cm} (6-64)

For \( (q_m/\gamma_0)/(a_m/H) \ll 1 \), the internal mass transfer rate is potentially much higher than the evaporation rate; the moisture content is therefore uniform throughout the cake, and \( U_{cr} = U_s = \text{constant} \) in Eq. (6-39).

Suppose that we start with a sludge cake of moisture content \( U_c \) at the end of the drainage period. We want to dry the sludge to \( U_d \). We will assume that \( U_d < U_{cr} \); that it cannot be reached during the constant rate period.

The drying time is then:

**Constant Rate Period:**

\[ t_1 = \frac{\gamma_0 H}{q_{me}} (U_c - U_{cr}) \]  \hspace{1cm} (6-65)

**First Falling Rate Period:**

\[ t_2 = \frac{\gamma_0 H}{q_{me}} (U_{cr} - U_e) \ln \frac{U_{cr} - U_e}{U_d - U_e} \]  \hspace{1cm} (6-66)

**Total Drying Time:** \( t_{\text{drying}} = t_1 + t_2 \)

\[ t_{\text{drying}} = \frac{\gamma_0 H}{q_{me}} \left[ (U_c - U_{cr}) + (U_{cr} - U_e) \ln \left( \frac{U_{cr} - U_e}{U_d - U_e} \right) \right] \]  \hspace{1cm} (6-67)

The total dewatering time, including drainage and drying, is given
by the sum of Eqs. (6-14) and (6-67):

\[ t_{\text{dewatering}} = \frac{\mu f(S_0, S_k) r^* (H_0)^{\beta+1}}{\beta (\beta + 1) (H_\ast^*)^\beta} + \frac{\gamma \phi H}{q_{\text{me}}} \left[ (U_c - U_{cr}) \right. \]

\[ + \left. (U_{cr} - U_e) \ln \left( \frac{U_{cr} - U_e}{U_d - U_e} \right) \right]. \]
SECTION 6.5. COST ANALYSIS

The cost of dewatering sewage sludge on drying beds can be divided into four parts:

(1) cost of amortization of capital investment and interest;
(2) cost of labor, fuel, and mechanical operation for sludge application, removal, and disposal;
(3) cost of drying bed maintenance, including labor and materials;
(4) cost of chemicals for sludge conditioning, if any.

The cost of sludge treatment is usually expressed in dollars per ton of sludge solid.

We will begin with a definition of a dewatering cycle. The function for dewatering time will then be introduced, along with a "down" time. Equations for the mass of dewatered cake per unit area will be given. We will then discuss each of the four contributions to the cost. Finally, we will present a cost equation and discuss the means of optimizing the cost.

Dewatering Cycle

We will define a cycle to include the following steps, in order:

(1) Application of liquid sludge to the drying bed;
(2) Dewatering of sludge to a cake with a selected moisture content $U_D$;
(3) Removal of sludge cake;
(4) Maintenance and preparation of bed for the following cycle.
Dewatering Time

The dewatering time, including drainage and evaporative drying, is given by Eq. (6-68); the atmospheric conditions are assumed to be representable by a constant average evaporation rate $q_{me}$. Drainage removes the bulk of the liquid, and ends during the first few hours. The rest of the sludge moisture must be removed by evaporation, which can require several days. We can therefore neglect the drainage time, and assume that optimization of the total dewatering time involves only the time required for evaporation. The dewatering time is then approximated by the equation:

$$t_{dew} = \frac{\gamma d H}{q_{me}} \left[ U_c - U_{cr} + (U_{cr} - U_e) \ln \left( \frac{U_{cr}}{U_d} \right) \right]. \quad (6-69)$$

We have not neglected the importance of the drainage period; the effectiveness of drainage affects the dewatering time and cost through $U_c$ in Eq. (6-69). $U_c$ is strongly dependent on the properties and conditioning of the sludge, and only weakly dependent on $w$, the mass of solid applied per unit area. We will assume that the value of $U_c$ and the corresponding cost of chemicals required has been determined empirically.

Down Time

We must also include in the cycle time a "down" time, when the drying bed is not available for dewatering. This time, $t_d$, includes:
(1) Time required for application of liquid sludge to the bed—proportional to the mass of the sludge applied per bed;

(2) Time required for removal of dewatered cake—proportional to the total mass of cake per bed;

(3) Time required for maintenance and preparation—proportional to the total bed area.

If we assume that the mass of liquid sludge applied per unit area and the mass of cake per unit area are both relatively constant, then the "down" time would be proportional to the total area of the drying bed:

$$ t_d = K_t A , \quad (6-70) $$

where $K_t$ = down time per unit area per cycle;

$A$ = bed area.

---

**Total Cycle Time $t_c$**

$$ t_c = t_{dew} + t_d \quad . \quad (6-71) $$

Mass of solid per unit area per cycle:

$$ w = \frac{\rho \rho_S H_0}{\left(\rho + \rho_S U_0\right)} . \quad (6-72) $$

Mass of cake (solid and fluid) per unit area per cycle at removal:

$$ w_D = \frac{\rho \rho_S H_0}{\left(\rho + \rho_S U_0\right)} \left(1 + U_d\right) = w(1 + U_d) . \quad (6-73) $$
Cost of Capital Amortization and Interest

The major capital cost for sludge drying beds is usually the land cost. We can therefore assume that the total capital cost is proportional to the bed area. If the cost is amortized linearly over a fixed period of time, then we have a "specific cost" in the form:

\[ K_c = \text{capital amortization cost/unit area/unit time} \]

To obtain the amortization cost in dollars per unit mass of sludge solid, we multiply \( K_c \) by \( t_c \), the cycle time, and divide by \( w \), the mass of solid per unit area per cycle.

\[ c_c = \frac{K_c t_c}{w} \quad (6-74) \]

Labor, Fuel, and Machine Operating Costs

These costs are associated with the movement of the sludge: onto the bed, off the bed, and to the final disposal site. A reasonable assumption would be that the costs are proportional to the mass of the sludge handled.

Liquid sludge is usually pumped onto the bed. The pumping cost is assumed to be negligible compared with the subsequent cost of cake removal and disposal. We can then introduce a cost coefficient:

\[ K_{rd} = \frac{\text{cost of removal and disposal}}{\text{unit mass of cake}} \]
Sludge cakes can usually be handled when dewatered to 75-80% moisture content. The mass of the cake at removal therefore includes a large fraction of water. The mass of the cake solids is related to the total mass of the cake by Eq. (6-73) so that the cost per unit mass of solid would be:

\[ C_{rd} = (1 + U_d) K_{rd} \]  \hspace{1cm} (6-75)

**Drying Bed Maintenance and Preparation Cost**

This cost would cover the labor and material costs for:

1. resurfacing of paved beds;
2. addition of makeup sand to sand beds;
3. repair to drainage systems; etc.

We can assume that the cost can be expressed by a coefficient:

\[ K_m = \text{cost of bed maintenance/unit area/cycle}. \]

The cost per unit mass of solid would then be:

\[ C_b = \frac{K_m}{w} \]  \hspace{1cm} (6-76)

**Chemical Cost**

We assume that the cost of the chemicals required to obtain a given \( U_c \) is determined empirically as a cost per unit mass of sludge solid.
Total Cost of Dewatering by Drying Beds

The total cost per unit weight of sludge solid, $C_T$, would be

$$C_T = C_c + C_{rd} + C_b + C_s \quad \text{(dollars/unit weight of solid)}$$

$$= \frac{K_c}{w} t_c + (1 + U_d)K_{rd} + \frac{K_m}{w} + C_s \quad , \quad (6-77)$$

where

$$t_c = \frac{v_0 H}{q_{me}} \left[ (U_c - U_{cr}) + (U_{cr} - U_e) \ln \left( \frac{U_{cr} - U_e}{U_d - U_e} \right) \right] + t_d \quad (6-78)$$

$$w = \frac{\rho S}{(\rho + \rho_s U_0)} H_0 \quad , \quad (6-72)$$

$$H = \frac{w}{\gamma_0} \quad . \quad (6-16)$$

Equation (6-77) can be used as an aid in the initial design of the drying beds or in the selection of optimum operating values of $w$ and $U_d$ once the beds are installed.

We will discuss below the application of Eq. (6-77) to determine the minimum cost and bed area required to dewater a given rate of sludge solid input.
SECTION 6.6 DESIGN APPLICATION OF COST EQUATION

Using Eq. (6-16) and (6-78), the cost equation (6-79) can be written as:

\[ C_T = \frac{K_c}{q_{me}} \left( (U_c - U_{cr}) + (U_{cr} - U_e) \ln\left(\frac{U_{cr} - U_e}{U_d - U_e}\right) \right) \]

\[ + \frac{(K_c t_d + K_m)}{w} + (1 + U_d)K_{rd} \]  \hspace{1cm} (6-79)

For a fixed bed area, \( t_d = K_t A \) is a constant. The parameters in the equation can then be classified into four types:

(1) Cost parameters: \( K_c, K_d, K_m, C_s \);

(2) Sludge characteristics: \( q_m, U_c, U_s \);

(3) Drying conditions: \( q_{me} \). The evaporation rate during the constant rate period can be controlled if the bed is covered. If the cake cracks during drying, then \( q_{me} \) would depend on the thickness \( H \) of the cake (or on \( w \));

(4) Operating parameters: \( w, U_d \).

Let us first optimize the cost with respect to \( U_d \).

\[ \frac{\partial C_T}{\partial U_d} = -\frac{K_c}{q_{me}} \frac{(U_{cr} - U_e)}{(U_d - U_e)} + K_{rd} = 0 \]

(6-80)

\[ (U_d)_{\text{optimum}} = U_e + \frac{K_c(U_{cr} - U_e)}{q_{me}K_{rd}} \]
Substituting for \((U_d)_{\text{optimum}}\) into Eq. (6-79), we have \(C_T\) as a function of \(w\), \(U_{cr}\), and \(q_{me}\).

\[
C_T = \frac{K_c}{q_{me}}(U_c - U_{cr}) + \frac{(K_c t_d + K_m)}{w} + (1 + U_e)K_{rd} + \frac{K_c}{q_{me}}(U_{cr} - U_e) \left[ 1 + \ln \left( \frac{q_{me} K_{rd}}{K_c} \right) \right].
\] (6-81)

Before we can optimize the equation with respect to \(w\), we must take into account the possible dependence of \(q_{me}\) and \(U_{cr}\) on \(w\). We will consider two cases.

Case 1. (a) \(U_{cr}\) is given by Eq. (6-39):

\[
U_{cr} = U_s + \frac{1}{3} \frac{q_{me} w}{\gamma_0 a_m}.
\]

(b) \(q_{me}\) is independent of \(w\); i.e., either \(q_{me} = q_m = \text{constant}\), when the cake does not crack during drying;

or

\(q_{me} = q_m (1 + b) = \text{constant}\), when the cake cracks with a constant aspect ratio.

\(C_T\) can then be put into the form:

\[
C_T = (1 + U_e)K_{rd} + \frac{K_c}{q_{me}} \left[ (U_c - U_e) + (U_s - U_e) \ln \left( \frac{q_{me} K_{rd}}{K_c} \right) \right]
+ \frac{(K_c t_d + K_m)}{w} + \frac{1}{3} \frac{K_c w}{\gamma_0 a_m} \ln \left( \frac{q_{me} K_{rd}}{K_c} \right).
\] (6-82)

\[
\frac{\partial C_T}{\partial w} = 0 \quad \text{at} \quad (w)_{\text{opt}} = \left[ \frac{(K_c t_d + K_m) 3\gamma_0 a_m}{K_c \ln \left( \frac{q_{me} K_{rd}}{K_c} \right)} \right]^{1/2}.
\] (6-83)
Substituting for \((w)_{\text{optimum}}\) into Eq. (6-81), we have the final cost optimized with respect to both \(U_d\) and \(w\):

\[
(C_T)_{\text{optimum}} = (1 + u_e)K_{rd} + \frac{K_c}{q_{me}} \left[ (U_c - U_e) + (U_s - U_e) \ln \left( \frac{q_{me} K_{rd}}{K_c} \right) \right] \\
+ 2 \left[ \frac{K_c t_d + K_m}{3} \frac{K_c}{\gamma_0^2 a_m} \ln \left( \frac{q_{me} K_{rd}}{K_c} \right) \right]^{\frac{1}{2}} \tag{6-84}
\]

Using \((w)_{\text{opt}}\) in the equation for \(U_{cr}\), and then substituting for \(U_{cr}\) in Eq. (6-80), we have the optimum value of \(U_d\) in terms of the fixed parameters:

\[
(U_{d})_{\text{optimum}} = \frac{K_c}{q_{me} K_{rd}} \left\{ (U_s - U_e) + q_{me} \left[ \frac{K_c t_d + K_m}{3 a_m \gamma_0^2 K_c} \ln \left( \frac{q_{me} K_{rd}}{K_c} \right) \right]^{\frac{1}{2}} \right\} \tag{6-85}
\]

The cycle time required with \((w)_{\text{opt}}\) and \((U_{d})_{\text{opt}}\) is:

\[
t_{\text{opt}} = \frac{w}{q_{me}} \left\{ (U_c - U_s) + (U_s - U_e) \ln \left( \frac{K_{rd} q_{me}}{K_c} \right) + \frac{1}{3} \frac{q_{me} w}{a_m \gamma_0^2} \left[ \ln \left( \frac{K_{rd} q_{me}}{K_c} \right) - 1 \right] \right\} + t_d \tag{6-86}
\]

**Case 2.**

(a) \(U_{cr} = U_s + \frac{1}{3} \frac{q_{me} w}{\gamma_0^2 a_m}\).

(b) \(q_{me}\) is dependent on the thickness \(H\), or on \(w\): 

\[
q_{me} = q_m (1 + cw_s)
\]

The exact definition of \(U_{cr}\) and \(q_{me}\) do not affect the optimization of the initial cost equation (6-79) with respect to \(U_d\), so we can
substitute for $U_{cr}$ and $q_{me}$ directly into Eq. (6-81) and obtain:

$\left( C_T \right)_{\text{optimized}} = \frac{K_c}{q_m(1 + cw)} \left\{ \ln \left( \frac{q_m(1 + cw)K_{rd}}{K_c} \right) (U_c - U_e) + (U_s - U_e) \right\} \ln \left( \frac{q_m(1 + cw)K_{rd}}{K_c} \right) + \frac{(K_c t_d + K_m)}{w} + \frac{wK_c}{3a_m \gamma_0^2} \ln \left( \frac{q_m(1 + cw)K_{rd}}{K_c} \right)

w/\text{respect to } U_d

+ \left( 1 + U_e \right) K_{rd} . \quad (6-87)

Equation (6-87) must be optimized with respect to $w$ by numerical methods.

**Bed Area Required**

Suppose that the drying beds must dewater an average of $M_0$ mass of sludge per unit time. The bed area required is then given by the equation:

$$A = \frac{M_0 w}{t_c} . \quad (6-88)$$

**SAMPLE CALCULATION FOR DESIGN APPLICATION**

In the following examples we will calculate the optimum values of $w$, $U_d$, $t_c$, and $C_T$ as $U_c$, $K_c$, and $q_{me}$ are varied, with all other parameters constant. The area of the drying beds required is then given by the following equation:
\[ A = \frac{M_0 t_c}{w} \]

where \( A \) = area required;

\( M_0 \) = average input rate of sludge solids to be dewatered.

The values of the parameters are taken from a number of sources, and are used only for illustrative purposes.

For the lower values of \( K_c \), the optimum cycle times are extremely long, so that the dewatering process may be more appropriately described as "lagooning" rather than "sand bed drying."

**Data Used**

**Cost Coefficients:**

\[ K_{rd} = 5.0 \text{$/\text{metric ton} = 5.0 \times 10^{-3}$}$/\text{kg} \]

\[ K_m = 4.94 \times 10^{-2}$/m²/cycle \]

\[ = 200$/acre/cycle \]

\[ K_c: \begin{array}{c|c}
($/m^2/sec) & ($/acre/yr) \\
\hline
1.96 \times 10^{-9} & 250 \\
3.92 \times 10^{-9} & 500 \\
7.84 \times 10^{-9} & 1000 \\
1.17 \times 10^{-8} & 1500 \\
1.57 \times 10^{-8} & 2000 \\
1.96 \times 10^{-8} & 2500 \\
2.35 \times 10^{-8} & 3000 \\
3.14 \times 10^{-8} & 4000 \\
\end{array} \]

**Evaporation Rate, \( q_{me} \) (constant rate period):**

\[ q_{me} = \begin{cases} 
0.05 \text{ kg/m²/hr} = 1.39 \times 10^{-5} \text{ kg/m²/sec} \\
or \\
0.1 \text{ kg/m²/hr} = 2.78 \times 10^{-5} \text{ kg/m²/sec} 
\end{cases} \]
Sludge Properties:

\[ U_0 = 19 \text{ (95\% moisture content)} \]

\[ U_c = \begin{cases} 
9 + \gamma_0 = 105 \text{ kg/m}^3 \\
\text{or} \\
5.67 + \gamma_0 = 161 \text{ kg/m}^3 
\end{cases} \]

\[ \rho = 10^3 \text{ kg/m}^3 \]

\[ \rho_s = 1.9 \times 10^3 \text{ kg/m}^3 \]

\[ U_s = 2.0 \]

\[ U_e = 0.1 \]

\[ a_m = 3.5 \times 10^{-9} \text{ m}^2/\text{sec} \]

"Down" Time: \[ t_d = 2 \text{ days} \]

Example 1

Suppose that we wish to determine the cost-effectiveness of chemical conditioning. We assume that \( U_c \), the moisture content at the end of the drainage period, can be reduced from 9 (90\% moisture content) to 5.67 (85\% moisture content) by addition of chemicals at a cost of $6/ton of solid. The results of the calculations are then shown in Tables 6-1 and 6-2 and in Fig. 6-17.

If \( w \), the mass of solid per unit area per cycle, were kept constant, then chemical conditioning would decrease \( t_c \), the cycle time. The results, however, show that the optimum cost is obtained by keeping \( t_c \) relatively the same and increasing \( w \) instead.
We note that chemical conditioning, for the particular data used in this example, is cost effective for $K_c > 2472/\text{ha/yr} (1000/\text{acre/year})$, and that it can substantially decrease the area required.

**Example 2**

The results for two different values of $q_{me}$, the evaporation rate, are given in Tables 6-3 and 6-4 and in Fig. 6-18.

We note that doubling $q_{me}$ does not halve the cycle time. There are two reasons:

1. Part of the drying occurs in the falling rate period, which is not inversely proportional to $q_{me}$.
2. $U_{cr}$, the critical moisture content, increases as $q_{me}$ increases, so that a larger fraction of the water must then be evaporated during the falling rate period.
\( q_{me} = 0.05 \text{ kg/m}^2/\text{hr} \)

\( K_m = 2.47 \times 10^{-2} \text{ \$/m}^2/\text{cycle} = 100/\text{acre/ cycle} \)

\( U_c = 9.0 \)  no conditioning

\( C_s = 0 \)

<table>
<thead>
<tr>
<th>( K_c ) ( \text{($$/ha/yr)} )</th>
<th>( (w)_{opt} ) ( \text{(kg/m}^2)</th>
<th>( (U_d)_{opt} ) ( \text{(m)} )</th>
<th>( H_o ) ( \text{(m)} )</th>
<th>( t_c ) ( \text{(days)} )</th>
<th>( C_T ) ( \text{($$/ton)} )</th>
<th>Area Req'd for M_0 = 909 kg/day</th>
</tr>
</thead>
<tbody>
<tr>
<td>618</td>
<td>21.4</td>
<td>0.13</td>
<td>0.42</td>
<td>365</td>
<td>10.17</td>
<td>1.55</td>
</tr>
<tr>
<td>1236</td>
<td>17.0</td>
<td>0.22</td>
<td>0.33</td>
<td>232</td>
<td>12.69</td>
<td>1.24</td>
</tr>
<tr>
<td>2472</td>
<td>14.0</td>
<td>0.40</td>
<td>0.27</td>
<td>155</td>
<td>16.87</td>
<td>1.01</td>
</tr>
<tr>
<td>3690</td>
<td>12.8</td>
<td>0.58</td>
<td>0.25</td>
<td>125</td>
<td>20.22</td>
<td>0.89</td>
</tr>
<tr>
<td>4951</td>
<td>12.2</td>
<td>0.76</td>
<td>0.24</td>
<td>109</td>
<td>23.44</td>
<td>0.81</td>
</tr>
<tr>
<td>6181</td>
<td>12.0</td>
<td>0.94</td>
<td>0.23</td>
<td>100</td>
<td>26.34</td>
<td>0.75</td>
</tr>
<tr>
<td>7411</td>
<td>12.0</td>
<td>1.13</td>
<td>0.23</td>
<td>94</td>
<td>29.06</td>
<td>0.71</td>
</tr>
<tr>
<td>9902</td>
<td>12.3</td>
<td>1.53</td>
<td>0.24</td>
<td>86</td>
<td>34.11</td>
<td>0.63</td>
</tr>
</tbody>
</table>
\( q_{me} = 0.05 \text{ kg/m}^2/\text{hr} \)

\( K_m = 2.47 \times 10^{-2} \text{ $/m}^2/\text{cycle} = 100/\text{acre}/\text{cycle} \)

\( U_c = 5.67 \)

\( C_s = 6/\text{ton} \)

<table>
<thead>
<tr>
<th>( K_C ) ($/ha/yr)</th>
<th>( (w)_{opt} ) (kg/m²)</th>
<th>( (U_d)_{opt} ) (m)</th>
<th>( H_0 ) (m)</th>
<th>( C_T ) ($/ton)</th>
<th>( t_c ) (days)</th>
<th>Area Req'd For M₀ = 909 kg/day</th>
</tr>
</thead>
<tbody>
<tr>
<td>618</td>
<td>32.8</td>
<td>0.10</td>
<td>0.64</td>
<td>13.24</td>
<td>405</td>
<td>1.12</td>
</tr>
<tr>
<td>1236</td>
<td>26.0</td>
<td>0.18</td>
<td>0.51</td>
<td>14.61</td>
<td>254</td>
<td>0.89</td>
</tr>
<tr>
<td>2472</td>
<td>21.4</td>
<td>0.34</td>
<td>0.42</td>
<td>16.98</td>
<td>164</td>
<td>0.70</td>
</tr>
<tr>
<td>3690</td>
<td>19.6</td>
<td>0.49</td>
<td>0.38</td>
<td>19.04</td>
<td>130</td>
<td>0.60</td>
</tr>
<tr>
<td>4951</td>
<td>18.7</td>
<td>0.64</td>
<td>0.36</td>
<td>20.99</td>
<td>110</td>
<td>0.54</td>
</tr>
<tr>
<td>6181</td>
<td>18.4</td>
<td>0.80</td>
<td>0.36</td>
<td>22.75</td>
<td>99</td>
<td>0.49</td>
</tr>
<tr>
<td>7411</td>
<td>18.3</td>
<td>0.96</td>
<td>0.36</td>
<td>24.40</td>
<td>90</td>
<td>0.45</td>
</tr>
<tr>
<td>9902</td>
<td>18.9</td>
<td>1.29</td>
<td>0.37</td>
<td>27.50</td>
<td>80</td>
<td>0.39</td>
</tr>
</tbody>
</table>
TABLE 6-3

\[ q_{me} = 0.05 \text{ kg/m}^2/\text{hr} \]
\[ K_m = 4.94 \times 10^{-2} \text{ $/m}^2/\text{cycle} = 200/\text{acre/cycle} \]
\[ U_c = 5.67 \]
\[ C_s = 6/\text{ton} \]

<table>
<thead>
<tr>
<th>( K_c ) ($/ha/yr)</th>
<th>( (w)_{opt} ) (kg/m(^2))</th>
<th>( (U_d)_{opt} ) (m)</th>
<th>( H_0 ) (days)</th>
<th>( t_c ) (days)</th>
<th>( C_T ) ($/ton)</th>
<th>Area Req'd For M(_0) = 909 kg/day</th>
</tr>
</thead>
<tbody>
<tr>
<td>618</td>
<td>28.7</td>
<td>0.15</td>
<td>0.56</td>
<td>463</td>
<td>16.70</td>
<td>1.47</td>
</tr>
<tr>
<td>1236</td>
<td>22.7</td>
<td>0.26</td>
<td>0.44</td>
<td>271</td>
<td>19.02</td>
<td>1.08</td>
</tr>
<tr>
<td>2472</td>
<td>18.5</td>
<td>0.46</td>
<td>0.36</td>
<td>163</td>
<td>22.45</td>
<td>0.80</td>
</tr>
<tr>
<td>3690</td>
<td>16.9</td>
<td>0.66</td>
<td>0.33</td>
<td>123</td>
<td>25.12</td>
<td>0.66</td>
</tr>
<tr>
<td>4951</td>
<td>16.1</td>
<td>0.86</td>
<td>0.31</td>
<td>102</td>
<td>27.46</td>
<td>0.57</td>
</tr>
<tr>
<td>6181</td>
<td>15.7</td>
<td>1.07</td>
<td>0.31</td>
<td>88</td>
<td>29.46</td>
<td>0.51</td>
</tr>
<tr>
<td>7411</td>
<td>15.6</td>
<td>1.28</td>
<td>0.30</td>
<td>78</td>
<td>31.25</td>
<td>0.46</td>
</tr>
<tr>
<td>9902</td>
<td>15.8</td>
<td>1.72</td>
<td>0.31</td>
<td>65</td>
<td>34.36</td>
<td>0.37</td>
</tr>
</tbody>
</table>
TABLE 6-4

$q_{me} = 0.1 \text{ kg/m}^2/\text{hr}$

$K_m = 4.94 \times 10^{-2} \$/$\text{m}^2/\text{cycle} = $200/\text{acre/} \text{cycle}$

$U_c = 5.07$

$C_s = $6/ton

<table>
<thead>
<tr>
<th>$K_c$ ($$/\text{ha/yr})$</th>
<th>$(w)$ opt (Kg/m$^2$)</th>
<th>$(U_d)$ opt (m)</th>
<th>$H_0$ (m)</th>
<th>$t_c$ (days)</th>
<th>$C_T$ ($$/\text{ton})$</th>
<th>Area Req'd For $M_o = 909 \text{ kg/day}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>618</td>
<td>26.3</td>
<td>0.12</td>
<td>0.51</td>
<td>356</td>
<td>16.24</td>
<td>1.23</td>
</tr>
<tr>
<td>1236</td>
<td>20.4</td>
<td>0.19</td>
<td>0.40</td>
<td>197</td>
<td>18.16</td>
<td>0.88</td>
</tr>
<tr>
<td>2472</td>
<td>16.2</td>
<td>0.32</td>
<td>0.31</td>
<td>113</td>
<td>20.89</td>
<td>0.63</td>
</tr>
<tr>
<td>3690</td>
<td>14.3</td>
<td>0.45</td>
<td>0.28</td>
<td>82</td>
<td>22.99</td>
<td>0.52</td>
</tr>
<tr>
<td>4951</td>
<td>13.3</td>
<td>0.57</td>
<td>0.26</td>
<td>66</td>
<td>24.83</td>
<td>0.45</td>
</tr>
<tr>
<td>6181</td>
<td>12.6</td>
<td>0.69</td>
<td>0.25</td>
<td>56</td>
<td>26.41</td>
<td>0.40</td>
</tr>
<tr>
<td>7411</td>
<td>12.2</td>
<td>0.81</td>
<td>0.24</td>
<td>49</td>
<td>27.84</td>
<td>0.37</td>
</tr>
<tr>
<td>9902</td>
<td>11.7</td>
<td>1.06</td>
<td>0.23</td>
<td>40</td>
<td>30.38</td>
<td>0.31</td>
</tr>
</tbody>
</table>
(a) Sand drying beds

(b) Paved drying beds

FIGURE 6-1.
FIGURE 6-2. Typical dewatering curve.
FIGURE 6-3.

FIGURE 6-4. Profile of fluid pressure during drainage.
Figure 6-5. Dewatering by evaporation.

Figure 6-6. Idealized drawing of the movement of water by capillarity in a porous medium (from Sherwood).
FIGURE 6-7. Dewatering by Drainage: filtrate volume and sludge depth vs. time.
FIGURE 6-8. Schematic illustrating hysteresis in the relationship of moisture potential to the moisture content.
FIGURE 6-10. Critical moisture content $U_{cr}$ vs. $q_m W$ (from Nebiker).
FIGURE 6-12. Non-dimensional drying rate vs. $U^*$ (from Luikov$^{56}$).
FIGURE 6-13. Model of Evaporation During Second Falling Rate Period.

Flow Over Flat Plate
1. Laminar, Pr << 1
2. Laminar, Pr > 1
3. Turbulent, Pr > 1

FIGURE 6-14. Non-dimensional heat flux as a function of dimensionless variable $\Gamma$. 

$$\bar{q}(\Gamma) = \frac{q_{av}}{q_{av0}}$$

$$\Gamma = \frac{K_{eff}}{\frac{T_{ev} - T_a}{T_{so} - T_a}}$$
FIGURE 6.15. Model of cracking in the sludge cake during drying.
FIGURE 6-16. Dependence of drying rate on $q_{me}$ and $H$.
FIGURE 6-17. Sample calculation of cost-effectiveness of chemical conditioning.
FIGURE 6-18. Sample calculation of the effect of evaporation rate \( q_{me} \) on the dewatering cost.
CHAPTER 7:
VACUUM FILTRATION

SECTION 7.1. INTRODUCTORY REMARKS

Vacuum filtration is the most widely used method of mechanical dewatering. The sludge liquid is forced to flow through a porous filter medium under an applied pressure differential (a vacuum) and the sludge solid is deposited in the form of a cake upon the medium. The filter cake is usually highly compressible, and its hydraulic permeability depends on its state of compaction under pressure. After the filtration, or forming, period, the cake is removed from the liquid sludge and further dewatered by vacuum to the desired moisture content before removal.

In this chapter we will present the models for the filtration and dewatering processes and derive cost equations for vacuum filter operations. The limitations of our models will be discussed in the summary, as well as suggestions for future research.

There are many types of vacuum filters, such as rotary drum, precoat, continuous belt, and top loading. Our analysis applied to a continuous belt filter, although it could be easily modified to apply to other types as well.

A sketch of a belt filter is shown in Fig. 7-1. A rotating drum supports a continuous belt of filter medium. The filter medium is usually made from weaves of natural or synthetic filaments, or of closely wound stainless steel coils. The circumferential surface of the drum is perforated
and divided into sections. A multiport valve connects each section to the vacuum source. Vacuum is applied to each section as it is immersed into the feed sludge, at (A). Filtration and cake formation then proceeds as shown in the figure. At (B), the cake is removed from the sludge and dewatered. Vacuum is released at (C) and the cake is discharged by a sharp change in belt curvature at (D).

The ease with which a wastewater sludge can be filtered and dewatered on a given vacuum filter will depend on the properties of the sludge and the operating variables of the filter.

The sludge properties include:

(1) shape and size distribution of the solid particles;
(2) concentration of suspended solids;
(3) chemical composition and surface chemistry, such as surface charge and potential;
(4) viscosity of sludge and filtrate;
(5) age and temperature;
(6) compressibility and hydraulic permeability of the filter cake.

These properties depend on the source of the raw wastewater and the sludge and on the pretreatment and conditioning of the sludge before filtration by physical, chemical, or biological means.

The operating variables of the filter include:

(1) level of vacuum during filtration and dewatering;
(2) time of filtration and of dewatering;
(3) speed of drum rotation;
(4) fraction of drum surface submerged in the feed sludge;
(5) filter medium.

Experience with vacuum filter operations has shown that the yield (mass of solid dewatered/unit time/unit filter area) and the final moisture content of the cake are highly affected by the conditioning of the sludge. The physical chemistry of chemical conditioning is not yet well understood, and the optimum chemical dosages must be determined empirically.

We will therefore assume that the sludge properties are given inputs, and that the controllable variables are:

1. applied pressure difference, assumed to be the same for the filtration and dewatering;
2. the rotational speed of the drum \( \Omega \);
3. the fraction of the drum surface submerged \((\Theta_f/2\pi)\). The active fraction of the surface \((\Theta_0/2\pi)\) is assumed to be constant (see Fig. 7-1).
SECTION 7.2. FILTRATION

We have modeled the initial filtration period following the theory of Tiller et al.\textsuperscript{53} The reader is referred to Tiller\textsuperscript{68} and Skubnik\textsuperscript{69} for detailed discussion and bibliography.

Darcy's law can be written in the following form for one-dimensional flow through the porous cake at a point \( x \) (see Fig. 7-2):

\[
q_x = -\frac{1}{\mu \alpha_x} \frac{\partial p_x}{\partial w_x},
\]

where \( q_x \) = superficial flow rate (volume flow rate/filter area);
\( \mu \) = fluid viscosity;
\( p_x \) = local fluid pressure;
\( \partial w_x \) = incremental element in material coordinate \( w_x \);
\( w_x = \int_0^x \rho_s(1 - n_x) \, dx \);
\( \alpha_x \) = specific resistance;
\( n_x \) = local porosity;
\( \rho_s \) = solid density.

The filtrate flow rate can be described by the following equation:

\[
q_1 = \frac{dv_1}{dt} = \frac{P}{\mu(\alpha_T w + R_m)},
\]

where \( q_1 \) = filtrate volume flow rate/filter area;
\( v_1 \) = volume of filtrate/filter area;
\( \alpha_T \) = average specific resistance for the filter cake = \( f(P - p_1) \);
\[ R_m = \text{flow resistance of the filter medium}; \]
\[ P = \text{total pressure drop across the cake and the medium}; \]
\[ p_1 = \text{pressure drop across the filter medium}; \]
\[ w = \text{mass of solid in the cake/filter area} = \int_0^h \rho_s(1 - n_x)dx; \]
\[ h = \text{thickness of cake}. \]

In our discussion we will consider only constant pressure filtration, where \( P \) is held constant. Other modes of filtration (for example, variable pressure-variable flow rate) are discussed by Tiller.\(^{68}\)

Initially all of the applied pressure difference \( P \) occurs across the filter medium. As the cake thickness increases, its flow resistance increases relative to that of the medium, and the pressure drop across the cake approaches the constant \( P \). In vacuum filtration, the filtration is only about a minute, and the medium resistance may not be negligible. Equation (7-2) then has to be solved numerically, such as by Haynes.\(^{70}\)

An analytic solution is possible under the following assumptions:

1. \( R_m = 0 \), i.e., the pressure drop across the cake is constant and equal to \( P \).
2. The fluid flow is one-dimensional, with Reynolds number \( \ll 1 \).
3. The solid particles in the filter cake are perfectly elastic, incompressible, and in point contact. A force balance over an element of the cake then results in the equation (see Fig. 7-3):

\[ p_s + p_x = P, \tag{7-3} \]

where \( p_s \) is the "effective solid stress" = force on the solid due to fluid drag/filter area.
(4) The specific resistance is related to $p_s$ by

$$\alpha_x = \alpha_0 \left( \frac{p_s}{p^*} \right)^\beta \quad p_s \geq p_i$$

$$\alpha_x = \alpha_0 \left( \frac{p_i}{p^*} \right)^\beta \quad p_s < p_i$$

(7-4)

where $p^*$ is some reference pressure and $p^*$, $p_i$, $\alpha_0$ and $\beta$ are constants to be determined experimentally. This assumption accounts for the compression of the cake under pressure, with subsequent decrease in hydraulic permeability.

(5) The distribution of the porosity and the ratios $q_x/q_i$ and $p_x/p$ are only functions of $x/h$, where $h$ is the instantaneous thickness of the cake. The average porosity and the average specific resistance of the cake then remains constant throughout the filtration period.

Tiller et al.\textsuperscript{53} have experimentally verified assumptions (4) and (5) for a variety of materials under constant pressure filtration.

Using Eqs. (7-3) and (7-4), we can integrate Eq. (7-1) as follows:

$$\int_0^w q_x \, dw_x = \frac{1}{\mu} \int_0^p \frac{dp_s}{\alpha_x}$$

$$= \frac{1}{\mu} \int_0^{p_i} \frac{(p^*)^\beta}{\alpha_0 \, p_i^\beta} \, dp_s + \frac{1}{\mu} \int_{p_i}^p \frac{(p^*)^\beta}{\alpha_0 \, p_s^\beta} \, dp_s$$

$$= \frac{(p_i^{1-\beta} - \beta p_i^{1-\beta}) (p^*)^\beta}{\mu \alpha_0 (1 - \beta)}.$$
If we define
\[ J_R \equiv \frac{\int_0^w q_x \, dw_x}{q_1 w} = \int_0^1 \left( \frac{q_x}{q_1} \right) \, d\left( \frac{w_x}{w} \right) \quad , \quad (7-5) \]
then
\[ q_1 = \frac{\left( p^{1-\beta} - \beta p_i^{1-\beta} \right) (p^*)^\beta}{\mu J_R \alpha_0 (1 - \beta) w} \quad . \quad (7-6) \]

From assumption (5), \( J_R \) = constant for constant pressure filtration, and comparing equations (7-6) with (7-2), we see that:
\[ \alpha_T = \frac{J_R \alpha_0 (1 - \beta) p}{\left( p^{1-\beta} - \beta p_i^{1-\beta} \right) (p^*)^\beta} \quad . \quad (7-7) \]

For constant average porosity, \( w \) is related to \( v_1 \) by:
\[ w = \frac{p S_0 S_c}{(S_c - S_0)} \quad v_1 \quad , \quad (7-8) \]
where \( \rho \) = fluid density;
\( S_0 \) = solid content of feed sludge (mass fraction);
\( S_c \) = solid content of cake (mass fraction).

Using Eqs. (7-7) and (7-8), we can integrate Eq. (7-2) as follows:
\[ \frac{dv_1}{dt} = \frac{1}{\rho} \left( \frac{S_c - S_0}{S_0 S_c} \right) \frac{dw}{dt} = \frac{\left( p^{1-\beta} - \beta p_i^{1-\beta} \right) (p^*)^\beta}{\mu J_R \alpha_0 (1 - \beta) w} \]
\[ w = \xi t_f^{\frac{1}{\beta}} \]
\[
\xi = \left[ \frac{2\rho(p^*)^\beta S_0S_C(p^{1-\beta} - \beta p_i^{1-\beta})}{J_Ru a_0(1 - \beta)(S_C - S_0)} \right]^{\frac{1}{2}} \tag{7-9}
\]

\[t_f = \text{filtration time}.\]

Equation (7-9) is used in the cost analysis for vacuum filtration (p. 303) with the understanding that it is only a first approximation to the actual process where \( R_m \) is not negligible.

Equation (7-1) applies only when the solid concentration in the feed sludge is low enough so that the solid velocity towards the cake can be neglected. Shirato et al.\textsuperscript{71} have investigated the filtration of concentrated slurries and have included the effect of solid velocity in a modified definition of \( J_R \), which also remains constant if assumption (5) holds.
SECTION 7.3. DRYING PERIOD

Figure 7-4a shows a typical fluid-filled pore at the air-cake interface at the beginning of the drying period. To force the liquid-air meniscus into the porous cake, the applied pressure difference \( P = P_{\text{atm}} - P_{\text{vacuum}} \) acting over the pore area must exceed the surface tension force. For a perfectly wetting fluid in a circular pore, the pressure drop across the meniscus due to surface tension is:

\[
P_c = \frac{2\gamma}{r},
\]

(7-10)

where \( \gamma = \) surface tension (joule/m\(^2\));

\( r = \) pore radius.

<table>
<thead>
<tr>
<th>TABLE 7-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>For ( \gamma = 7.2 \times 10^{-2} ) joule/m(^2) (water-air)</td>
</tr>
<tr>
<td>( r ) (m)</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>( 10^{-8} )</td>
</tr>
<tr>
<td>( 10^{-7} )</td>
</tr>
<tr>
<td>( 10^{-6} )</td>
</tr>
</tbody>
</table>

Some sample calculations are presented in Table 7-1. The wetting characteristics and the surface tension for the actual sludge must be determined experimentally.

We can postulate two mechanisms for the dewatering of the filter
cake. If the pore radii are small enough, then the applied vacuum cannot draw the meniscus away from the air-cake surface, and the cake must remain saturated with fluid and compact by consolidation (Fig. 7-4b). On the other hand, if the pore radii are large enough so that \( P > \frac{2\gamma}{r} \), then the vacuum applied will draw the fluid out of the porous cake, and the cake will become unsaturated progressively (Fig. 7-4c).

In practice the cake is not uniform and contains a distribution of pore sizes, so that dewatering could occur by a combination of the two mechanisms. However, from experimental data on the pore radii in filter cakes from sewage sludge (Table 2-4), we can assume that for most sewage sludges, the predominant mechanism would be consolidation. The mechanism of vacuum drying would be important in dewatering of filter cakes from coarse, granular materials, such as coal slurries.

Porous media contract upon drying, and the resulting stresses can cause cracks in the cake. The subsequent air leakage rate accounts for the major portion of the power required for the vacuum pump. Wakeman has investigated the drying stresses; however, at present there is no theory to predict the size and distribution of the cracks. We will therefore account for the air leakage rate by an empirical approximation (see p. 300.

We will present the models for the two mechanisms.

Consolidation

The theory for consolidation of porous media under applied pressure
is extensively developed in soil mechanics. The reader is referred to Biot\textsuperscript{72} and Gibson.\textsuperscript{77}

Our discussion is based on the following assumptions (see also Fig. 7-2):

(1) The consolidation and the fluid flow are one-dimensional, with Reynolds number \( \ll 1 \).

(2) The solid in the cake is modeled as a porous matrix of incompressible particles in point contact. The matrix is perfectly elastic.

(3) The pores of the matrix are completely saturated with fluid.

A force balance over an element of the cake results in the following equation:

\[ p_s + p_x = P \quad , \tag{7-11} \]

where \( p_x \) = local fluid pressure;

\( p_s \) = "effective solid stress" = net force on the solid matrix per filter area;

\( P \) = total applied pressure differential across the cake = constant.

The fluid flow is assumed to obey Darcy's law:

\[ q_x = n_x (v_f - v_s) = - \frac{k_x}{\mu} \frac{\partial p_x}{\partial x} \quad , \tag{7-12} \]

where \( v_f \) = fluid velocity, absolute;

\( v_s \) = solid velocity, absolute;

\( k_x \) = local hydraulic permeability;

\( x \) = spatial coordinate.
The spatial coordinate can be transformed to a material coordinate by the following relation:

\[ ds = (1 - n_x) dx = \frac{dx}{1 + e} \quad ; \quad (7-13) \]

\[ s = \int_0^x (1 - n_x) dx = \int_0^x \frac{dx}{1 + e} \quad ; \quad (7-14) \]

\[ s_0 = \int_0^L (1 - n_x) dx = \int_0^L \frac{dx}{1 + e} \quad , \quad (7-15) \]

where \( s_0 \) = volume of solid per unit filter area within a distance \( x \) from the filter medium;

\( e \) = void ratio = volume of pore fluid per volume of solid;

\( L \) = instantaneous thickness of cake.

Using Eqs. (7-11) and (7-13), we can rewrite Eq. (7-12) as:

\[ q_x = -\frac{k_x}{\mu(1 + e)} \frac{\partial p_x}{\partial s} = \frac{k_x}{\mu(1 + e)} \frac{\partial p_s}{\partial x} = \frac{k_x}{\mu(1 + e)} \left( \frac{\partial p_s}{\partial e} \right) \left( \frac{\partial e}{\partial s} \right) \quad (7-16) \]

Mass conservation for an element \( ds \) of the cake then results in the equation

\[ \frac{\partial e}{\partial t} = \frac{\partial}{\partial s} \left( c_F \frac{\partial e}{\partial s} \right) \quad \quad (7-17) \]

\[ c_F = -\frac{k_x}{\mu(1 + e)} \left( \frac{\partial p_s}{\partial e} \right) \quad . \]

\( c_F \) can be approximated as constant for the consolidation of soils,\(^{47}\) clays,\(^{51}\) and digested sewage sludges.\(^{54}\) Equation (7-17) can then be
rewritten as:

\[ \frac{\partial e}{\partial t} = c_F \frac{\partial^2 e}{\partial s^2} . \] (7-18)

The boundary conditions are:

\[ e(0,t) = e_1 = f(P) = \text{constant, from Eq. (7-11) with } p_x(0,t) = 0 \] (7-19)

\[ \frac{\partial e}{\partial s}(s_0,t) = 0 , \text{ from Eq. (7-16), with } q_x(s_0,t) = 0 . \]

If the distribution of the void ratio can be assumed to be sinusoidal at \( t = 0 \) (immediately after the end of the filtration period), then the solution to Eq. (7-18) with boundary conditions (7-19) is:

\[ e(s,0) = e_1 + (e_2 - e_1) \sin \left( \frac{\pi s}{2s_0} \right) \text{ initial condition} \] (7-20)

\[ e(s,t) = e_1 + (e_2 - e_1) \sin \left( \frac{\pi s}{2s_0} \right) \exp \left( - \frac{\pi^2 c_F t}{4s_0^2} \right) . \] (7-21)

At the end of the drying period \( (t = t_d) \), the average void ratio \( \bar{e}_d \) in the cake is:

\[ \bar{e}_d = e_1 + \frac{2(e_2 - e_1)}{\pi} \exp \left( - \frac{\pi^2 c_F t_d}{4s_0^2} \right) \] (7-22)

Equation (7-22) can be rewritten as:

\[ \frac{\bar{e}_d - e_1}{e_c - e_1} = \exp \left( - \frac{\pi^2 c_F t_d}{4s_0^2} \right) \] (7-23)

or

\[ t_d = \frac{4s_0^2}{\pi^2 c_F} \ln \left( \frac{e_c - e_1}{\bar{e}_d - e_1} \right) = \frac{4w^2}{\rho s^2 \pi^2 c_F} \ln \left( \frac{e_c - e_1}{\bar{e}_d - e_1} \right) . \] (7-24)
where \( e_d \) = average void ratio at \( t_d = 0 \) (end of filtration period);

\[
e_c = e_1 + \frac{2(e_2 - e_1)}{\pi}
\]  \( (7-25) \)

From Fig. 7-1, we have:

\[
t_d = \frac{\Theta_0 - \Theta_f}{\Omega} \quad t_f = \frac{\Theta_f}{\Omega}
\]  \( (7-26) \)

Substituting for \( w \) from Eq. (7-9), we can rearrange Eq. (7-24) in the form:

\[
\Theta_f = \frac{\Theta_0}{\left[ 1 + \frac{4c^2}{\rho_s^2 \pi^2 c_F} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \right]}
\]  \( (7-27) \)

\( e_c \) is the average void ratio in the filter cake during the filtration period. We have assumed that \( e_c \) is constant throughout a constant pressure filtration. Due to the compressibility of the sludge filter cake, \( e_c \) and \( e_1 \) are functions of \( P \). In Eq. (7-17), \( (\partial p_s/\partial e) \) is the modulus of elasticity of the solid matrix. To date, there is little data available for wastewater sludges, so we must resort to approximations.

\( e_1 \) is the void ratio in the filter cake adjacent to the filter medium, where the effective solid stress is a maximum. \( e_1 \) is also the final equilibrium void ratio at the end of consolidation. Under assumption (3) on p. 284, \( e_1 \) is a function of \( P \) only, which can be experimentally determined with an oedometer (widely used in soil mechanics).

We can make an assumption for \( e_1(P) \), using the Kozeny-Carman equation for permeability and the empirical relationship for specific resistance, Eq. (7-4). Darcy's law is usually written in the form:
\[ q_x = - \frac{k_x \partial p_x}{\mu \partial x} = - \frac{k_x \rho_s (1 - n_x)}{\mu \partial w_x} \cdot \]

Comparison with Eq. (7-1) shows that

\[ k_x = \frac{1}{\alpha_x \rho_s (1 - n_x)} = \frac{1 + e}{\alpha_x \rho_s} . \quad (7-28) \]

From the widely used Kozeny-Carman equation: \(^7\)

\[ k_x = \frac{n_x^3}{c_k \tau^2 S^2 (1 - n_x)^2} = \frac{e^3}{c_k \tau^2 S^2 (1 + e)} , \quad (7-29) \]

where \( c_k \) = Kozeny coefficient;
\( \tau \) = tortuosity;
\( S \) = specific surface area (surface area per volume of solid).

For \( P > P_i \), \( \alpha_x \) is given by Eq. (7-4).

Using Eqs. (7-4), (7-28), and (7-29), we have

\[ \frac{(1 + e)^2}{e^3} = \alpha_o \left( \frac{p_s}{p^*} \right) \frac{\rho_s}{c_k \tau^2 S^2} \quad (7-30) \]

or

\[ \frac{(1 + e)^2}{e^3} = \frac{(1 + e_n)^2}{e_n^3} \left( \frac{p_s}{p_n} \right)^\beta , \quad (7-31) \]

where \( e_n(P_n) \) and \( P_n \) are reference values determined experimentally.

We can differentiate Eq. (7-30) and obtain

\[ \frac{\partial p_s}{\partial e} = \frac{p_s (3 + e)}{\beta e (1 + e)} \quad . \quad (7-32) \]
Experimental data from Kos and Adrain\textsuperscript{73} seem to support the approximation of Eq. (7-30) for water treatment sludges.

To our knowledge there is no experimental data available for wastewater sludges for the dependence of $e_c$ on $P$, so that we can only assume that the porosity distribution in the sludge cake is similar to that of highly compressible materials such as polystyrene latex and fine silica. Data from Tiller and Cooper\textsuperscript{75} show that in filter cakes of latex spheres the porosity is relatively uniform through most of the cake but decreases rapidly in a thin layer next to the filter medium. Most of the flow resistance in the cake is due to this thin layer (see Fig. 7-5). Under an increase in applied pressure, the decrease in porosity may be confined to this thin layer, and the average porosity of the cake would be insensitive to $P$. We will therefore assume that:

$$\frac{3e_c}{3P} = \frac{3n_c}{3P} = \frac{3U_c}{3P} = 0 \quad . \quad (7-33)$$

**Vacuum Drying**

The second possible mechanism of cake dewatering is by withdrawal of the pore fluid under vacuum (Fig. 7-4c). Our model for this mechanism is similar to that for filtration. We make the following assumptions:

(1) Assumptions (1) to (4) for filtration on p. 284 hold;

(2) the applied pressure difference occurs across the saturated portion of the cake;

(3) within the saturated portion of the cake, the distribution of
porosity and the ratios \( q_x/q_1 \) and \( p_x/p \) are assumed to be functions only of \( x/L_s \), where \( L_s \) is the instantaneous thickness of the saturated portion.

(4) the dried portion of the cake has a residual moisture content of \( U_r \) (mass of fluid per mass of solid).

Applying Darcy's law and mass conservation to the saturated portion, we get:

\[
\rho q_1 = \frac{\rho p_d}{\mu \alpha_T w_s} = -\frac{\partial}{\partial t} \left[ w_s (U_c - U_r) \right] \quad (7-34)
\]

\[
P_d = P - P_c = P - \frac{2Y}{r} \quad , \quad (7-35)
\]

where \( q_1 = \) filtrate flow rate per filter area;

\( U_c = \) average moisture content in saturated portion (mass of fluid per mass of solid); assumed to be the same as that during filtration;

\[ w_s = \int_0^{L_s} \rho_s (1 - n_x) \, dx \quad ; \]

\( P_c = \) pressure drop across the air-fluid meniscus in the pore [see Eq. (7-10)].

Equation (7-34) can be integrated to give

\[
w^2 - w_s^2 = \frac{2\rho p_d t}{\mu \alpha_T (U_c - U_r)} \quad . \quad (7-36)
\]

The average specific resistance \( \alpha_T \) is given by Eq. (7-7). At the end of the drying period \( t = t_d \), the average moisture content of the
cake is $U_d$:

$$U_d = \frac{U_c w_S + (w - w_S) U_r}{w}$$

or

$$w_S = w \left( \frac{U_d - U_r}{U_c - U_r} \right) . \quad (7-37)$$

Substituting for $\alpha_T$ and $w_S$ in Eq. (7-36), we have

$$w^2 \left[ 1 - \left( \frac{U_d - U_r}{U_c - U_r} \right)^2 \right] = \frac{2 \rho (P_{d1-\beta} - \beta \phi_{d1-\beta}) (p^*) \beta t_d}{u \lambda R \alpha_0 (1 - \beta)(U_c - U_r)} . \quad (7-38)$$

If we substitute for $w$ from filtration theory, Eq. (7-9), then

$$1 - \left( \frac{U_d - U_r}{U_c - U_r} \right)^2 = \frac{t_d (S_c - S_0) (P_{d1-\beta} - \beta \phi_{d1-\beta})}{t_f s_0 S_c (U_c - U_r) (p^{1-\beta} - \beta \phi_{d1-\beta})} . \quad (7-39)$$

From Fig. 7-1 we see that

$$\frac{t_d}{t_f} = \frac{\Theta_0 - \Theta_f}{\Theta_f} ,$$

so that Eq. (7-39) can be rewritten in the form:

$$\Theta_f = \frac{\Theta_0}{\left[ 1 + R_p \frac{(U_c^2 - U_d^2 + 2U_d U_r - 2U_c U_r)}{(U_0 - U_c)(U_c - U_r)} \right]} . \quad (7-40)$$

$^\dagger$ $S_0$ and $S_c$ have been replaced by their equivalents in $U_0$ and $U_c$ ($U$ is moisture content as mass of fluid per mass of solid).
\[ \Theta_0 - \Theta_f = \frac{R_p \left( U_c^2 - U_d^2 + 2U_dU_r - 2U_cU_r \right) \Theta_0}{(U_0 - U_c)(U_c - U_r) - R_p \left( U_c^2 - U_d^2 + 2U_dU_r - 2U_cU_r \right)} \tag{7-41} \]

\[ R_p = \frac{p^{1-\beta} - \beta p_i^{1-\beta}}{p_d^{1-\beta} - \beta p_i^{1-\beta}}. \]
SECTION 7.4. ENERGY REQUIREMENT

We will derive the energy required per unit mass of solid dewatered.

\[ E = E_a + E_v \quad , \quad (7-42) \]

where \( E \) = total energy required/unit mass solid;
\( E_v \) = energy required/unit mass for vacuum pump;
\( E_a \) = energy required/unit mass for accessory equipment, such as sludge and filtrate motors, drive motor for the drum, and agitators.

We will assume that the power required for accessory equipment is constant for a given filter, so that

\[ E_a = \frac{\text{constant}}{M} \quad , \quad (7-43) \]

where \( M \) = mass of solid dewatered per unit operating time.

\( E_v \) is a sum of three parts:

\[ E_v = E_f + E_d + E_b \quad ; \quad (7-44) \]

where \( E_f \) = energy required per unit mass solid for filtration;
\( E_d \) = energy required per unit mass solid for dewatering;
\( E_b \) = energy required per unit mass solid due to air leakage during dewatering period.

For incompressible fluid and solid, the filtration energy required per unit filter area is simply \( P v_1 \), where \( v_1 \) is the volume of filtrate per unit filter area. From mass balance,
\[ U_0w = U_w + v_1 \rho , \]

where \( w = \text{mass of solid per unit area;} \)

\( U_0 = \text{moisture content of feed sludge (mass of fluid per mass of solid);} \)

\( U_w = \text{average moisture content of cake (mass of fluid per mass of solid),} \)

so that

\[ E_f = \frac{p v_1}{w} = \frac{P(U_0 - U_w)}{\rho} . \quad (7-45) \]

Similarly, for dewatering, the energy required per unit area is

\( p v_2 , \) where \( v_2 \) is the filtrate volume per unit area, and

\[ U_w = U_d w + p v_2 , \]

where \( U_d \) is the average moisture content of cake at end of dewatering period (mass of fluid per mass of solid).

\[ E_d = \frac{p v_2}{w} = \frac{P(U_w - U_d)}{\rho} . \quad (7-46) \]

We then have

\[ E_f + E_d = \frac{P(U_0 - U_d)}{\rho} . \quad (6-47) \]

We will assume that the air leakage rate is directly proportional to \( P \); the proportionality constant is to be determined empirically. The air leakage is present only during dewatering, so that the power required is:

\[ b p^2 A_d = b p^2 \left( \frac{\theta_0 - \theta_f}{2\pi} \right) A , \]
where $A_d$ = area of drum used for dewatering;

$A$ = total area of drum;

$$\frac{(\theta_0 - \theta_f)}{2\pi} = \text{fraction of drum area used for dewatering (see Fig. 7-1).}$$

$E_b$ is then

$$E_b = \frac{bp^2(\theta_0 - \theta_f)A}{2\pi M}.$$  \hfill (7-48)

$M$ is given by

$$M = \frac{w\Omega A}{2\pi},$$  \hfill (7-49)

where $\Omega$ is rotational speed of the drum, so that

$$E_b = \frac{bp^2(\theta_0 - \theta_f)}{w\Omega}.$$  \hfill (7-50)

To compare the relative magnitudes of $E_f$, $E_d$, and $E_b$, we will use the following numbers, typical of vacuum filter operation for dewatering sewage sludge:

- $P = 11.5 \text{ psi} = 20 \text{ in. Hg} = 7.95 \times 10^4 \text{ kg/m-sec}^2$
- $bp = 2 \frac{\text{CFM}}{\text{ft}^2} = 10^{-2} \text{ m/sec}^2$
- $\theta_0 = \frac{3\pi}{2}$, $\theta_f = \frac{\pi}{2}$
- $\Omega = 2.62 \times 10^{-2} \frac{\text{radians}}{\text{sec}} = 0.25 \frac{\text{revolutions}}{\text{minute}}$
- $w = 2.8 \text{ kg/m}^2$
- $U_0 = 15.7 \ (6\% \text{ solid by weight})$
- $U_d = 4.0 \ (20\% \text{ solid by weight})$
- $\rho = 1 \text{ g/cm}^3 = 10^3 \text{ kg/m}^3$
The results are:

\[ E_b = 34,100 \text{ joules/kg} \]

\[ E_f + E_d = 930 \text{ joules/kg} . \]

It is clear that \( E_b \gg E_f + E_d \), so that we can assume that

\[ E_v \approx E_b = \frac{b \beta^2 (\Theta_0 - \Theta_f)}{w \Omega} = \frac{b \beta^2 (\Theta_0 - \Theta_f) A}{2 \pi M} . \]  \hspace{1cm} (7-51)
SECTION 7.5. COST ANALYSIS

We assume that the vacuum filter facility must dewater an overall average of $M_0$ mass of solid per unit time. The filter is operating a fraction $f$ of the time, so that the mass of solid to be dewatered per unit operating time is:

$$ M = \frac{M_0}{f} = \frac{w\Omega A}{2\pi} \quad , $$  \hspace{1cm} (7-52)

where $A =$ filter area;

$\Omega =$ rotational speed of drum (radians/sec);

$w =$ mass of solid per unit filter area.

Let

$$ Y \equiv \frac{M}{A} = \frac{M_0}{Af} = \frac{w\Omega}{2\pi} \quad . $$  \hspace{1cm} (7-53)

$Y$ is referred to in filtration practice as the "yield" (mass of solid dewatered per unit area per unit operating time).

We derive the cost equation on the basis of dollars per unit mass of sludge solid. In the total cost, we include

(1) capital amortization and interest cost per unit mass of solid = $C_c$. We assume that the total equipment and construction cost scales as some power of the filter area,$^7$ and that the amortization and interest cost per unit time, $K_c$, can be written as:

$$ K_c = IB_c A^c \quad , $$  \hspace{1cm} (7-54)

where $I$ is the amortization and interest rate and $B_c$ and $c$ are constants. On a per mass basis:
\[ C_c = \frac{K_c}{M_0} = \frac{IB_c A^c}{M_0} \tag{7-55} \]

or, substituting for \( A \) from Eq. (7-52),

\[ C_c = \frac{IB_c}{M_0^{1-c}} \left( \frac{2\pi}{f} \right)^c \left( \frac{1}{\omega \Omega} \right)^c \tag{7-56} \]

(2) Labor cost per unit mass of solid = \( C_L \). We assume that the labor cost per unit time \( K_L \) scales also with the filter area,\(^7\), so that

\[ C_L = \frac{K_L}{M_0} = \frac{B_m A^m}{M_0} \tag{7-57} \]

or

\[ C_L = \frac{B_m}{M_0^{1-m}} \left( \frac{2\pi}{f} \right)^m \left( \frac{1}{\omega \Omega} \right)^m \tag{7-58} \]

(3) Chemical and supply cost per unit mass of solid = \( C_S \). We assume that chemical addition occurs before filtration, and that the chemical dosage has been optimized empirically. \( C_S \) is assumed to be constant.\(^7\)

(4) Energy cost per unit mass of solid = \( C_e \).

\[ C_e = K_e E \tag{7-59} \]

where \( K_e \) = cost per unit energy;

\[ E = \text{energy required per unit mass of solid}. \]

\( E \) is the sum of \( E_v \) and \( E_a \), where \( E_v \) is the energy required for the vacuum pump and \( E_a \) is the energy required for accessory equipment, such as sludge and filtrate pumps, drive motor for the drum,
agitators, etc. The power required for the accessory equipment can be assumed to scale with the filter area, so that

\[ E_a = \frac{B_a A^a}{M}, \]  

(7-60)

where \( B_a \) and \( a \) are constants.

\( E_v \) is given by Eq. (7-51), so that

\[ C_e = \frac{K e B_a A^a f}{M_0} + \frac{K e b p^2 (\Theta_0 - \Theta_f) A f}{2\pi M_0} \]  

(7-61)

or

\[ C_e = K e B_a \left( \frac{f}{M_0} \right)^{1-a} \left( \frac{2\pi}{w\Omega} \right)^a + \frac{K e b p^2 (\Theta_0 - \Theta_f)}{w\Omega}. \]  

(7-62)

We have used Eq. (7-52) to relate \( M \), \( M_0 \), and \( w\Omega \).

(5) Disposal Cost = \( C_d \). The disposal cost will depend on the method of ultimate disposal. Two examples are given below:

(a) Landfill - the cost of transportation to the site is assumed to be proportional to the mass transported, which includes water in the moist cake. The disposal cost per unit mass solid will therefore decrease with decreasing final cake moisture content,

\[ C_d = K_d (1 + U_d), \]  

(7-63)

where \( K_d \) = disposal cost per unit mass transported;
\( U_d \) = moisture content of cake at disposal = mass of fluid per mass of solid.

(b) Incineration - the cost of additional fuel required will depend on the fuel value of the final cake. If the final moisture content
$U_d$ is low enough so that the fuel value exceeds that required to self-sustain combustion, the energy recovery is possible. The fuel value $E_C$ per unit mass of cake is given by the equation:

$$E_C = w(H_C - H_V U_d) \quad ,$$  \hspace{1cm} (7-64)

where $H_C$ = heat of combustion per unit mass solid (energy per unit mass);

$H_V$ = heat of vaporization of sludge liquid (energy per unit mass).

The cost of additional fuel per unit mass of solid = $C_f$:

$$C_f = \frac{K_f E_C}{w} = K_f (H_s - H_V U_d) \quad ,$$  \hspace{1cm} (7-65)

where $K_f$ = cost per unit energy of fuel.

In practice, the cost of disposal by incineration will have to take into account the complete operating costs of the incineration system, including capital amortization, maintenance, interest, labor, etc., and the fuel costs required for startup, heat losses to the environment, etc.

The total dewatering cost per unit mass of solid, $C_T$, is the sum:

$$C_T = C_V + C_d \quad ,$$  \hspace{1cm} (7-66)

where $C_V$ = filter cost per unit mass of solid.

$$C_V = C_c + C_L + C_s + C_e \quad .$$  \hspace{1cm} (7-67)

Using Eqs. (7-55), (7-57), and (7-61), we can write $C_V$ as
\[ C_V = \frac{I_B A^a}{M_0} + \frac{B A^m}{M_0} + C_s + \frac{K_e B A^f}{M_0} + \frac{K_e b p^2 A(\Theta_0 - \Theta_f)}{2\pi M_0} \]  

(7-68)

or, using Eqs. (7-56), (7-58), and (7-62),

\[ C_V = \frac{I_B C}{M_0^{1-c}} \left( \frac{2\pi}{f} \right)^c \left( \frac{1}{\epsilon_\Omega} \right)^c + \frac{B}{M_0^{1-m}} \left( \frac{2\pi}{f} \right)^m \left( \frac{1}{\epsilon_\Omega} \right)^m + C_s 
+ \frac{K_e B a}{(M_0^f)}^{1-a} \left( \frac{2\pi}{\epsilon_\Omega} \right)^a + \frac{K_e b p^2 (\Theta_0 - \Theta_f)}{\epsilon_\Omega} \]  

(7-69)

We can substitute for \( \omega \) from Eq. (7-9), so that

\[ C_V = \frac{I_B C}{M_0^{1-c}} \left( \frac{2\pi}{f} \right)^c \left[ \frac{1}{\xi(\Theta_f \Omega)^{\frac{1}{2}}} \right]^c + \frac{B}{M_0^{1-m}} \left( \frac{2\pi}{f} \right)^m \left[ \frac{1}{\xi(\Theta_f \Omega)^{\frac{1}{2}}} \right]^m + C_s 
+ \frac{K_e B a}{(M_0^f)}^{1-a} \left[ \frac{2\pi}{\xi(\Theta_f \Omega)^{\frac{1}{2}}} \right]^a + \frac{K_e b p^2 (\Theta_0 - \Theta_f)}{\xi(\Theta_f \Omega)^{\frac{1}{2}}} \]  

(7-70)

where

\[ \xi = \left[ \frac{2p (p^{1-\beta} - \beta p^{1-\beta})}{J_R u x_0 (1 - \beta) (U_0 - U_c)} \right]^{\frac{1}{2}} \]

Equation (7-70) can be optimized with respect to the controllable variables \( P, \Theta_f \), and \( \Omega \), subject to the constraints that:

\[ \frac{b p^2 (\Theta_0 - \Theta_f) A}{2\pi} \leq P_m \]  

(7-71)

\[ w(1 + U_d) = \xi \left( \frac{\Theta_f}{\Omega} \right)^{\frac{1}{2}} (1 + U_d) \geq w_m \]  

(7-72)

\[ P \leq P_{\text{atmospheric}} \]  

(7-73)
\[
\Omega \leq \Omega_m \quad (7-74)
\]
\[
\Theta_f \leq \Theta_m \quad (7-75)
\]

where \( P_m \) = maximum power output of the vacuum pump;
\( w_m \) = minimum allowable mass of cake per unit filter area for satisfactory discharge;
\( \Omega_m, \Theta_m \) = mechanical limitations on the maximum values of \( \Omega \) and \( \Theta_f \), respectively.

From Fig. 7-1 it is clear that the release of the cake from the filter medium at (D) depends on the relative magnitudes of two forces: the adhesive force of the cake to the medium and the gravitational force on the cake. For a given sludge and medium, we assume that a minimum \( w_m \) can be determined experimentally.
SECTION 7.6 APPLICATIONS OF THE COST EQUATION

The cost equation can be used as an aid in the initial design of a vacuum filter facility or in selecting optimum values of the controllable variables \( P \), \( Q_f \) and \( \Omega \) once a filter is installed and operating.

Each application is described in detail below.

**Design Application**

The necessary inputs are:

1. \( M_0 \) and \( f \);
2. sludge properties: \( S_0 \), \( S_c \), \( \mu \), \( \alpha_0 \), \( p^* \), \( \beta \), \( p_f \);
3. the dewatering mechanism which applies to the particular sludge; for consolidation, we need to know \( c_f \), \( e_1(p) \);
   for vacuum drying: \( e_r \);
4. the dependence of \( K_c \), \( K_L \), and \( E_a \) on the filter area, in terms of the constant \( B_c \), \( c \), \( B_m \), \( m \), \( B_a \), \( a \);
5. other constants \( C_S \), \( I \);
6. the method of disposal.

\( C_v \) is given by Eq. (7-70), and the total cost \( C_T \) is optimized with respect to the variables \( P \), \( Q_f \), and \( \Omega \), subject to the constraints of Eqs. (7-71) to (7-75). The optimum values are then used to calculate the optimum yield and the filter area required.

The optimization procedure is as follows:
(1) From either Eq. (7-27) or (7-40), we have the final moisture content \( U_d \) in the form:

\[
U_d = U_d(\Theta_f, \Omega, P)
\]

which can be rewritten as

\[
\Theta_f = \Theta_f(U_d, \Omega, P)
\]

For algebraic convenience, we substitute for \( \Theta_f \) in Eq. (7-70) and obtain \( C_V \) as a function of \( P, U_d, \) and \( \Omega \).

(2) We then optimize \( C_V \) with respect to \( \Omega \), and obtain an expression for \( \Omega_{\text{optimum}} \) in terms of \( \Omega_m \) or of \( U_d \) and \( P \).

(3) We optimize \( C_V \) with respect to \( P \). This optimization must be carried out numerically. The result, with \( \Omega_{\text{optimum}} \) and \( P_{\text{optimum}} \) substituted into the cost equation, is a functional dependence of \( C_V \) on \( U_d \).

(4) Further optimization of \( C_V \) with respect to \( U_d \) is meaningless without some constraint on \( U_d \). This constraint is furnished by the disposal cost \( C_d \), which depends on \( U_d \).

(5) We must therefore choose a method of disposal, and then optimize the total cost \( C_T \) with respect to \( U_d \).

From Eq. (7-70), we see that the optimum value for \( \Omega \) is the maximum possible under the constraint of Eqs. (7-72) and (7-74), so that:
\[ \Omega_{\text{optimum}} = \Omega_{\text{maximum}} = \text{smaller of} \left\{ \frac{\xi^2 \Theta_f (1 + U_d)^2}{w_m^2} \right\} \]

Equation (7-70) can then be rewritten as:

For \( \frac{\xi^2 \Theta_f (1 + U_d)^2}{w_m^2} \leq \Omega_m \):

\[ C_v = \frac{IBc}{M_0^{1-c}} \left( \frac{2\pi}{f} \right)^c \left[ \frac{w_m}{\xi^2 \Theta_f (1 + U_d)} \right]^c + \frac{B_m}{M_0^{1-m}} \left( \frac{2\pi}{f} \right)^m \]

\[ \times \left[ \frac{w_m}{\xi^2 \Theta_f (1 + U_d)} \right]^m + C_s + K_e B_a \left( \frac{f}{M_0} \right)^{1-a} \left[ \frac{2\pi w_m}{\xi^2 \Theta_f (1 + U_d)} \right]^a \]

\[ + \frac{K_e b P^2 w_m (\Theta_0 - \Theta_f)}{\xi^2 \Theta_f (1 + U_d)} \] ; \hspace{1cm} (7-77)

For \( \Omega_m < \frac{\xi^2 \Theta_f (1 + U_d)^2}{w_m^2} \):

\[ C_v = \frac{IBc}{M_0^{1-c}} \left( \frac{2\pi}{f} \right)^c \left[ \frac{1}{\xi(\Theta_f \Omega_m)^{\frac{3}{2}}} \right]^c + \frac{B_m}{M_0^{1-m}} \left( \frac{2\pi}{f} \right)^m \]

\[ \times \left[ \frac{1}{\xi(\Theta_f \Omega_m)^{\frac{3}{2}}} \right]^m + C_s + K_e B_a \left( \frac{f}{M_0} \right)^{1-a} \left[ \frac{2\pi}{\xi(\Theta_f \Omega_m)^{\frac{3}{2}}} \right]^a \]

\[ + \frac{K_e b P^2 (\Theta_0 - \Theta_f)}{\xi(\Theta_f \Omega_m)^{\frac{3}{2}}} \] . \hspace{1cm} (7-78)
For dewatering by consolidation, $\Theta_f$ is related to $U_d$ and $P$ by Eq. (7-27), and for vacuum drying, by Eq. (7-40).

By substituting for $\Theta_f$ into Eqs. (7-71) and (7-75), the remaining constraint equations can be expressed in terms of $P$. For example, for consolidation:

$$P \leq P_{\text{atmospheric}}$$  \hspace{1cm} (7-73)

$$1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \geq \frac{\Theta_0}{\Theta_m}$$  \hspace{1cm} (7-79)

$$\frac{\alpha P^2 \Theta_0 \xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \leq P_m$$  \hspace{1cm} (7-80)

$$2\pi \left[ 1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \right]$$

where

$$\alpha \equiv \frac{\rho_s^2 \pi^2 c_F}{4}$$

The transition point between the applicability of Eqs. (7-77) and (7-78) when $\xi^2 \Theta_f (1 + U_d)^2 / \omega_m^2 = \Omega_m$, can be expressed in terms of $P$.

Let

$$\Omega_b \equiv \frac{\xi^2 \Theta_f (1 + U_d)^2}{\omega_m^2}$$  \hspace{1cm} (7-81)

Substituting for $\Theta_f$, we have

$$\Omega_b = \frac{(1 + U_d)^2 \Theta_0 \xi^2}{\omega_m^2 \left[ 1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \right]} \leq \frac{(1 + U_d)^2 \Theta_0}{\omega_m^2 \left[ \frac{1}{\xi^2} + \frac{1}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \right]}$$  \hspace{1cm} (7-82)
The qualitative behavior of $\Omega_b$ as a function of $P$ is shown in Fig. 7-6. Let $P_b$ be the pressure at which $\Omega_b = \Omega_m$.

For $P < P_b$, $\Omega_b < \Omega_m$ and $\Omega_{\text{optimum}} = \Omega_b$, Eq. (7-77) applies; \hspace{1cm} (7-83)

For $P > P_b$, $\Omega_b > \Omega_m$, $\Omega_{\text{optimum}} = \Omega_m$, Eq. (7-78) applies.

Substituting for $\Theta_f$ from either Eq. (7-27) or (7-40) into Eq. (7-77) or (7-78), we obtain $C_v$ in the form:

$$C_v = C_v(P, U_d). \hspace{1cm} (7-84)$$

The optimum $P$ which would minimize $C_v$ is found by numerical means\(^\dagger\) and substituted back into Eq. (7-84). We then end up with

$$C_v = C_v(U_d).$$

We plot $C_v$ and $C_d$ as functions of $U_d$ and find the value of $U_d$ which would minimize $C_T$ by graphical means.

Once the optimum values of $P$ and $U_d$ are found, we can calculate $(\Theta_f)_{\text{optimum}}$ by Eq. (7-27) or (7-40), $\Omega_{\text{optimum}}$ by Eq. (7-76), and $Y_{\text{optimum}}$ and filter area required by Eq. (7-53).

In practice, the optimum values may not be realizable, the optimization procedures must then be repeated with the additional constraints imposed.

\(^\dagger\)The dependence of $e_1$ and $e_c$ on $P$ is discussed on p. 294-295.
A numerical example for a digested sewage sludge is given on p.316 and the result plotted in Fig. 7-7.

**Operating Application**

Once a filter is installed and operating, the cost equation can still be useful in minimizing the total cost \( C_T \) as the inputs to the filter vary. In particular, \( M_0 \) and the sludge properties would vary from day to day.

For a given filter, \( K_c \), \( K_L \), and \( E_a \) are determined and constant, so that Eq. (7-78) can be written as:

\[
C_v = \frac{(I B_c A^c + B_m A^m + K_e b A^f)}{M_0} + C_s + \frac{K_e b P^2 A_f (\Theta_0 - \Theta_f)}{2\pi M_0} \quad (7-68)
\]

The numerator of the first term is constant, and \( M_0 \) is a given input, so that optimization of \( C_v \) only involves the last term. Let

\[
C_{ef} = \frac{K_e b P^2 A_f (\Theta_0 - \Theta_f)}{2\pi M_0} = \frac{K_e b P^2 (\Theta_0 - \Theta_f)}{w \Omega} \quad (7-85)
\]

\( \Theta_f \) is again given by either Eq. (7-27) or (7-40).

All of the constraint equations (7-71) to (7-75) can be rewritten in terms of \( P \) and \( U_d \) (or \( e_d \)). We will do so assuming consolidation as the dewatering mechanism.

From Eq. (7-53) we have
\[ w \Omega = \xi (\Theta_f \Omega)^{1/2} = \frac{2\pi M_0}{Af}. \]

Substituting for \( \Theta_f \) from Eq. (7-27), and squaring both sides, we have

\[ \frac{\xi^2 \Theta_0 \Omega}{1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right)} = \left( \frac{2\pi M_0}{Af} \right)^2. \]  

(7-86)

The upper limits on \( \Omega \) are given by Eqs. (7-72) and (7-75), and we can apply the inequalities to Eq. (7-86), so that

\[ \frac{\xi^2}{1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right)} \geq \left( \frac{2\pi M_0}{Af} \right)^2 \frac{1}{\Theta_0 \Omega_m}, \]  

(7-87)

and

\[ \frac{\xi^2}{1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right)} \geq \frac{2\pi M_0 w_m}{Af \Theta_0 (1 + U_d)}. \]  

(7-88)

Using Eq. (7-27) again, we can express Eqs. (7-71) and (7-75) as

\[ 1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \geq \frac{\Theta_0}{\Theta_m}, \]  

(7-89)

\[ \frac{\Theta_0 AbP^2}{2\pi \alpha} \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \leq \frac{P_m}{\mu}. \]  

(7-90)

Equations (7-87) through (7-90), along with Eq. (7-73), are the constraints on \( P \). \( (\Theta_0 - \Theta_f) \) can be written as
\[
(\Theta_0 - \Theta_f) = \frac{\Theta_0 \xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right) \frac{1}{1 + \frac{\xi^2}{\alpha} \ln \left( \frac{e_c - e_1}{e_d - e_1} \right)} \quad \text{(7-91)}
\]

As \( P \) decreases, \( e_1 \) increases and \( \xi^2 \) decreases, so that \( (\Theta_0 - \Theta_f) \) decreases monotonically. By inspection of Eq. (7-68), we see that \( C_{ef} \) is a minimum for the minimum \( P \) which satisfies Eqs. (7-87) - (7-89).

A numerical example is given on p. 320. Figure 7-8 illustrates the constraints on \( P \). The multiple constraints would not be a problem in computation if a digital computer is used. It is likely that some of the constraints would be unnecessary. For example, in Fig. 7-8 the upper limit on \( P \) is always \( p_{\text{atmospheric}} \), and the lower limits are defined by Eqs. (7-88) and (7-89).

For \( U_d < U_{d*} \), a solution for \( P \) is not possible; i.e., the filter cannot dewater the sludge to a moisture content less than \( U_{d*} \) and still maintain an output rate of \( M_0 \).

The results are presented in Table 7-4 and in Fig. 7-9.

For dewatering by vacuum drying, the optimization procedures are similar to those described above.

**SAMPLE CALCULATIONS FOR DESIGN APPLICATION**

The data used are typical of vacuum filters and of anaerobically digested sewage sludge. The main sources of data are:
(1) cost coefficients--for 1971--from Patterson and Bonker;\textsuperscript{76}
(2) vacuum filter characteristics--from Komline-Sanderson;\textsuperscript{77}
(3) sludge properties--from Halff.\textsuperscript{54}

Dewatering is assumed to be by consolidation, as observed by Halff. All units are SI, except where noted. In particular, ton refers to metric ton.

**Data Used**

Solid mass input rate to filter facility:

- Population served: one million
- lb. solid/capita/day: 0.2
- lb. solid/day: 200,000
- english tons/year: 36,500
- metric tons/year: 33,182

Total construction cost = $B_CM^C$

- $c = 0.67$
- $B_C = 30,375 \text{ } ($)(m^2)^{-0.67}$
- $I = .07823 / \text{year}$

Annual labor cost = $B_M^M$

- $m = 0.67$
- $B_M = 4392 \text{ } ($)(m^2)^{-0.67} \text{ (yr)}^{-1}$
Chemical and supply cost per ton = \( C_s = 9.8 \) /ton

Energy cost = \( K_e = 1.5\)¢/kWH = \( 4.17 \times 10^{-9} \) $/joule

Disposal cost: three values used ($/wet ton): 1.20, 2.40, and 3.60

Vacuum filter:
\[
\text{maximum output of vacuum pump} = P_m = B_V A^V
\]

\[
B_V = 0.359 \text{ (HP)(ft}^2\text{)}^{-0.85}
\]

\[
B_V = 2019 \text{ (watt)(m}^2\text{)}^{-0.85}
\]

\[
v = 0.85
\]

Power required by accessory equipment = \( B_a A^a \)

\[
a = 0.5
\]

\[
B_a = 4168 \text{ (watt)(m}^2\text{)}^{-0.5}
\]

Active fraction of drum = \( \Theta_0/2\pi = 0.75 \)

Maximum rotational speed = \( \Omega_m \)

\[
\Omega_m = \pi \text{ radians/minute} = 5.236 \times 10^{-2} \text{ radians/sec}
\]

Maximum fraction of drum submerged = \( \Theta_m/2\pi = 0.375 \)

Air leakage rate = \( bP = 2 \) CFM/ft\(^2\) at \( P = 20 \) in. Hg or

\[
b = 1.26 \times 10^{-7} \text{ m}^2\text{sec/kg}
\]
Sludge Properties

Type: digested sewage sludge

\[ \rho = 1 \text{ g/cm}^3 = 10^3 \text{ kg/m}^3 \quad \rho_s = 1.9 \text{ g/cm}^3 = 1.9 \times 10^3 \text{ kg/m}^3 \]

\[ \mu = 10^{-3} \text{ kg/m-sec} \]

\[ S_0 = 0.06 \quad U_0 = 15.67 \quad e_0 = 29.77 \]

\[ S_c = 0.106 \quad U_c = 8.4 \quad e_c = 15.97 \]

\[ e_n = 5.97 \quad \text{for} \quad p_n = 6.67 \times 10^4 \text{ new/m}^2 \]

\[ \frac{J_R^{\alpha_0}}{(p^*)^\beta} = 3.042 \times 10^9 \text{ m/kg}, \quad \beta = 0.4 \]

\[ c_f = 5.2 \times 10^{-9} \text{ m}^2/\text{sec} \]

\[ p_i = 6.9 \times 10^3 \text{ newton/m}^2 \]

Results of Calculations

\( C_v \) is optimized with respect to \( \Omega \) and \( P \) and is tabulated as \( C_v(U_d) \) in Table 7-2. \( C_v(U_d) \) is plotted in Fig. 7-7 along with \( C_d(U_d) \).

The optimum values of \( U_d \), \( Y \), and \( A \) are given on Table 7-3 for three different \( K_d \) (disposal cost/wet ton).

For each \( U_d \), the optimum values of \( P \), \( \Omega \), and \( \Theta_f \) are tabulated in Table 7-2. The optimum \( P \) in all cases is found to be the maximum allowed by Eq. (7-73). \( P_{\text{optimum}} \) is set at \( 9.83 \times 10^4 \text{ new/m}^2 \), about 3% less than \( P_{\text{atmospheric}} \). The corresponding \( e_1(P) \) is 5.3.

The optimum \( \Omega \) is \( \Omega_b \) [Eq. (7-81)] for \( U_d \leq 5.5 \) and is \( \Omega_m \) for \( U_d \geq 6.0 \).
SAMPLE CALCULATIONS FOR OPERATING APPLICATION

The sources of the data used are the same as those for the sample calculation for design application, p. 317. The cost coefficients are for 1971.

Dewatering is again assumed to be by consolidation. All units are SI, except where noted.

Data Used

Equipment

3 vacuum belt filters, each of filter area 500 ft² (139 m²)

Cost of construction = $830,000

Interest and amortization rate = .07823

$C = $ 64,931/year = $2.059 \times 10^{-3} \$/sec

Labor Cost

19 shifts/week, 52 weeks/year

3.95 men/shift

$8,000/man-shift/year

$L = $3.805 \times 10^{-3}$/sec

$f = 0.90$

Normal input to dewatering facility:

$M_0 = 200,000 \text{ lb/day} = 1.052 \text{ kg/sec}$

We consider three cases:
\[
M_0 \text{ (kg/sec)} = \begin{cases} 
1.578 & \text{(50\% above normal)} \\
1.052 & \text{(normal)} \\
0.526 & \text{(50\% below normal)} 
\end{cases}
\]

and calculate the costs for each.

All other data are identical to those used in the design example, p. 317.

The results are plotted in Fig. 7-9 for \( K_d = $2.40/\text{wet ton} \). The arrows indicate \( U_d^* \), the minimum possible \( U_d \) which would still permit the filter to maintain an output rate of \( M_0 \). In all the cases considered in our example, \( (U_d)_{\text{optimum}} = U_d^* \).
SECTION 7.7. SUMMARY

In this chapter we have derived the cost equation for vacuum filtration based on a simplified version of the filtration theory as developed by Tiller.\textsuperscript{53}

The results of sample calculations are shown in Figs. 7-7 and 7-9. In actual practice it may be impractical, if not impossible, to "tune" the vacuum filter system to operate continuously at the optimum (minimum) cost, for the following reasons:

(1) The dynamics of the system may be too sluggish to follow the input variations. In general the rate and the properties of the input sludge would vary from hour to hour, unless storage tanks and regulating equipment are available and effective. Certain sludge properties required in the cost equations cannot be measured rapidly, such as the solid fraction, and the vacuum filter itself can require as long as two hours to equilibrate to a change in a machine parameter (P, $\Theta_F$, or $\Omega$).\textsuperscript{78}

(2) The total dewatering cost may be insensitive to variations in the controllable parameters. For example, in Fig. 7-7, for $K_d = $1.20/wet ton, a variation of 25% in $U_d$ about ($U_d)_{\text{optimum}}$ would result in a change of only 5% in $C_T$. It would be difficult then to determine the exact value of ($U_d)_{\text{optimum}}$.

The other shortcomings in our analysis are due to the limitations of the present filtration theory. We will mention them briefly; the
reader is referred to Skubnik\textsuperscript{69} for a more detailed discussion.

(1) The flow resistance of the filter medium may not be negligible compared to that of the cake, for the short filtration times ($\sim 1$ minute) encountered in vacuum filtration. Numerical solutions which account for the medium resistance are possible; however, they would be applicable only to the particular combination of sludge, filter medium, and operating conditions.

(2) The model of the solid in the filter cake as a perfectly elastic matrix of particles in point contact is most likely inaccurate. The solid particles in a wastewater sludge would actually be in the form of deformable flocs, held together by surface forces and exhibiting elastic and plastic behavior. The net result is that the porosity and the permeability at any point in the filter cake are not solely dependent on the fluid pressure, but are in fact time-dependent as well, with possibly several time constants involved.

(3) Equation (7-9) predicts that $w \propto t_f^{0.5}$. Experimental results have shown that the exponent may be smaller than 0.5. This may be due to the migration of fine particles within the cake. A sludge is actually composed of particles of a distribution of sizes, with different mechanisms of capture. The larger particles are retained by the filter medium by blockage, and are subsequently deposited in the form of a porous matrix whose thickness increases with time. The smaller particles are convected through
the pores and are captured by the matrix or by the filter medium through blockage and van der Waals and electrostatic forces. The flow passages are then progressively plugged up by the finer particles.

The theory for cake dewatering by consolidation also relies on the questionable assumption of a perfectly elastic solid matrix. The consolidation coefficient $c_F$ is approximated as a constant, which may be valid only for small deformations. In practice the variation of $c_F$ with the porosity should be determined experimentally with an oedometer, as was done by Kos and Adrian.\textsuperscript{73}

Most of the research in vacuum filtration has been concerned only with the initial period of filtration and cake formation; the subsequent period of cake dewatering has been largely neglected. There has not been, to my knowledge, any experimental studies of the mechanisms of cake dewatering, with the exception of the work by Halff.\textsuperscript{54}

Finally, it is generally acknowledged that air leakage through the cake during the dewatering period is responsible for the major portion of the power required by the vacuum pump. However, the mechanisms of crack formation in a drying cake are still not well understood.
<table>
<thead>
<tr>
<th>$U_d$</th>
<th>$C_{ea}$</th>
<th>$C_{ef}$</th>
<th>$C_{c}$</th>
<th>$C_L$</th>
<th>$C_v$</th>
<th>$P$</th>
<th>$\Omega$</th>
<th>$\Theta_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>0.30</td>
<td>1.26</td>
<td>1.56</td>
<td>4.09</td>
<td>7.56</td>
<td>23.01</td>
<td>9.83</td>
<td>0.50</td>
</tr>
<tr>
<td>3.5</td>
<td>0.23</td>
<td>0.70</td>
<td>0.93</td>
<td>0.93</td>
<td>5.27</td>
<td>9.80</td>
<td>18.85</td>
<td>0.96</td>
</tr>
<tr>
<td>4.0</td>
<td>0.19</td>
<td>0.46</td>
<td>0.65</td>
<td>0.36</td>
<td>3.13</td>
<td>4.13</td>
<td>9.80</td>
<td>1.54</td>
</tr>
<tr>
<td>4.5</td>
<td>0.17</td>
<td>0.33</td>
<td>0.50</td>
<td>0.82</td>
<td>3.78</td>
<td>9.80</td>
<td>15.48</td>
<td>2.31</td>
</tr>
<tr>
<td>5.0</td>
<td>0.14</td>
<td>0.24</td>
<td>0.38</td>
<td>1.51</td>
<td>2.78</td>
<td>9.80</td>
<td>14.47</td>
<td>3.35</td>
</tr>
<tr>
<td>5.5</td>
<td>0.13</td>
<td>0.17</td>
<td>0.30</td>
<td>1.26</td>
<td>2.33</td>
<td>9.80</td>
<td>13.69</td>
<td>4.72</td>
</tr>
<tr>
<td>6.0</td>
<td>0.11</td>
<td>0.14</td>
<td>0.25</td>
<td>1.14</td>
<td>2.11</td>
<td>9.80</td>
<td>13.31</td>
<td>1.08</td>
</tr>
<tr>
<td>6.5</td>
<td>0.11</td>
<td>0.11</td>
<td>0.22</td>
<td>1.07</td>
<td>1.98</td>
<td>9.80</td>
<td>13.08</td>
<td>1.57</td>
</tr>
<tr>
<td>7.0</td>
<td>0.11</td>
<td>0.08</td>
<td>0.19</td>
<td>1.00</td>
<td>1.85</td>
<td>9.80</td>
<td>12.84</td>
<td>1.90</td>
</tr>
</tbody>
</table>

**Units:** All costs in $/\text{metric ton}$, $P$ in $10^4 \text{ N/m}^2$, $C_{ea}$ and $C_{ef}$ are energy costs for accessories and vacuum pump, respectively. $\Omega$ in $10^{-2}$ radians/sec, $\Omega_m = 5.23 \times 10^{-2}$ rad/sec, $\Theta_f$ in radians.
<table>
<thead>
<tr>
<th>$k_d$</th>
<th>$U_d$</th>
<th>$(C_T)_\text{optimum}$</th>
<th>$Y$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>5.5</td>
<td>21.49</td>
<td>$1.61 \times 10^{-2}$</td>
<td>72.3</td>
</tr>
<tr>
<td>2.40</td>
<td>4.5</td>
<td>28.68</td>
<td>$9.33 \times 10^{-3}$</td>
<td>125</td>
</tr>
<tr>
<td>3.60</td>
<td>$\sim 4.0$</td>
<td>34.81</td>
<td>$6.87 \times 10^{-3}$</td>
<td>170</td>
</tr>
</tbody>
</table>

$A$ is the filter area required to handle the solid input rate of 1.052 kg/sec (200,000 lb/day).
TABLE 7-4. Sample Calculations for Operating Application of Cost Equation with $K_d = $2.4/wet tonne.

<table>
<thead>
<tr>
<th>$M_0$ tonne/day</th>
<th>$U_d$ (in $10^6$ N/m²)</th>
<th>$P_{opt}$</th>
<th>$(\Omega)_{opt}$ (in $10^{-2}$ rad/sec)</th>
<th>$(\Theta_F)_{opt}$ (rad)</th>
<th>$C_v$ $$$/tonne</th>
<th>$C_d$ $$$/tonne</th>
<th>$C_T$ $$$/tonne</th>
</tr>
</thead>
<tbody>
<tr>
<td>45.4</td>
<td>3.4</td>
<td>9.13</td>
<td>0.87</td>
<td>0.52</td>
<td>22.02</td>
<td>10.56</td>
<td>32.58</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>7.88</td>
<td>0.85</td>
<td>0.56</td>
<td>21.83</td>
<td>10.8</td>
<td>32.63</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>4.31</td>
<td>0.94</td>
<td>0.76</td>
<td>21.47</td>
<td>12.0</td>
<td>33.47</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>2.72</td>
<td>1.03</td>
<td>0.96</td>
<td>21.37</td>
<td>13.2</td>
<td>34.57</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>1.85</td>
<td>1.13</td>
<td>1.17</td>
<td>21.34</td>
<td>14.4</td>
<td>35.74</td>
</tr>
<tr>
<td></td>
<td>5.5</td>
<td>1.34</td>
<td>1.22</td>
<td>1.40</td>
<td>21.33</td>
<td>15.6</td>
<td>36.93</td>
</tr>
<tr>
<td></td>
<td>6.0</td>
<td>1.0</td>
<td>1.35</td>
<td>1.03</td>
<td>21.32</td>
<td>16.8</td>
<td>38.12</td>
</tr>
<tr>
<td>90.9</td>
<td>4.3</td>
<td>10.1</td>
<td>1.99</td>
<td>0.81</td>
<td>15.96</td>
<td>12.72</td>
<td>28.68</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>7.40</td>
<td>2.07</td>
<td>0.96</td>
<td>15.76</td>
<td>13.20</td>
<td>28.96</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>4.03</td>
<td>2.25</td>
<td>1.33</td>
<td>15.61</td>
<td>14.40</td>
<td>30.01</td>
</tr>
<tr>
<td></td>
<td>5.5</td>
<td>2.51</td>
<td>3.92</td>
<td>1.72</td>
<td>15.58</td>
<td>15.60</td>
<td>31.18</td>
</tr>
<tr>
<td></td>
<td>6.0</td>
<td>1.72</td>
<td>6.34</td>
<td>2.14</td>
<td>15.57</td>
<td>16.80</td>
<td>32.37</td>
</tr>
<tr>
<td>136</td>
<td>5.0</td>
<td>10.1</td>
<td>3.40</td>
<td>1.07</td>
<td>13.89</td>
<td>14.2</td>
<td>28.29</td>
</tr>
<tr>
<td></td>
<td>5.5</td>
<td>5.21</td>
<td>3.67</td>
<td>1.55</td>
<td>13.70</td>
<td>15.6</td>
<td>29.30</td>
</tr>
<tr>
<td></td>
<td>6.0</td>
<td>3.12</td>
<td>3.96</td>
<td>2.05</td>
<td>13.66</td>
<td>16.8</td>
<td>30.46</td>
</tr>
<tr>
<td></td>
<td>6.5</td>
<td>3.85</td>
<td>2.96</td>
<td>2.36</td>
<td>13.65</td>
<td>18.0</td>
<td>31.65</td>
</tr>
<tr>
<td></td>
<td>7.0</td>
<td>9.41</td>
<td>1.62</td>
<td>2.36</td>
<td>13.78</td>
<td>19.2</td>
<td>32.98</td>
</tr>
</tbody>
</table>
FIGURE 7-1. Schematic of a continuous belt vacuum filter.
FIGURE 7-2. Schematic diagram of cake illustrating internal variables and compaction of cake near the filter medium (from Tiller\textsuperscript{68}).

FIGURE 7-3. Frictional drag and compressive forces on the solid particles in the cake (from Tiller\textsuperscript{68}).
(a) Fluid saturated pores at beginning of de-watering period.

(b) Dewatering by consolidation.

(c) Dewatering by vacuum drying.

FIGURE 7.4.
(a) Data from Tiller and Cooper.\textsuperscript{75}

(b) Schematic illustrating the effect of increasing filtration pressure.

FIGURE 7-5. Porosity variation in highly compressible filter cakes.
FIGURE 7-6. $\Omega_b$ vs. $P$ [see Eqs. (7-81) and (7-82)].
FIGURE 7-7. Effect of $U_d$, the moisture content at disposal, on the filtration cost ($C_v$), disposal cost ($C_d$), and the total dewatering cost ($C_T$).
FIGURE 7-8. Constraints on $P$, the level of vacuum.
FIGURE 7-9. Total dewatering cost $C_T$ as a function of moisture content at removal $U_d$, for three different solid input rates.
CHAPTER 8:
SUMMARY AND CONCLUSIONS

In Chapter 2 we have reviewed the theory of electroosmosis through a single capillary tube of circular cross-section, and then applied the theory to two models of porous media:

(1) the uniform tube model, in which the fluid volume in the medium is uniformly distributed among identical, parallel capillary tubes;

(2) the cluster model, in which a fraction of the fluid in the medium is bound within clusters of solid particles and unavailable for flow.

For the uniform tube model, the equations for the superficial flow velocity and the current density (with $\nabla p = 0$) can be non-dimensionalized as follows:

$$\frac{q_e}{\left(\frac{n}{\tau^2} \frac{eRT}{\mu zF} \frac{\partial \phi}{\partial x}\right)} = f(\lambda, \overline{\psi}_k \text{ or } \overline{\sigma}_k)$$

$$\frac{i}{\sigma_n \frac{\partial \phi}{\partial x}} = f\left(\lambda, \overline{\psi}_k \text{ or } \overline{\sigma}_k, \frac{e}{\mu D} \left(\frac{RT}{zF}\right)^2\right).$$

The parameters on the right hand side are dimensionless.

$$\lambda \equiv \frac{\lambda_D}{\alpha} \quad \text{Debye length ratio}$$

$$\overline{\psi}_k \equiv \frac{ZF \psi_K}{RT} \quad \text{dimensionless surface potential}$$
\[
\bar{\sigma}_k \equiv \frac{ZF\sigma_k a}{\varepsilon RT}
\]
dimensionless surface charge density

\[
\frac{\varepsilon}{\mu D} \left( \frac{RT}{ZF} \right)^2
\]
usually negligible (~10^{-3}).

For soils, clays, and wastewater sludges, the cluster model is expected to be more accurate than the uniform tube model.

The "efficiency" of electroosmosis can be expressed by \( \eta \), the ratio of the hydraulic to the electrical energy required to force a given flow rate through a porous medium. The equations for \( \eta \) for the limits of \( \lambda \ll 1 \) and \( \lambda \gg 1 \) indicate that there exists a maximum value for \( \eta \) at an intermediate value of \( \lambda \approx 0.5 \).

Chapter 3 describes experiments in electroosmotic flow through plugs of Kaolin clay and anaerobically digested sewage sludge. The parameters varied are the porosity and the ionic concentration of the external solution. The cluster model seems to agree well with the experimental data, as well as with previous data on Kaolin from Olsen\(^{33}\) and Gray.\(^{34}\) The calculated value of the surface potential \( \psi_k \) is -6.9 mV for both the Kaolin and the sludge. The data confirms the theoretical dependence of \( \eta \) and \( (\mu q_\alpha \sigma_0 / i) \) on \( \lambda \) in the special case of \( \lambda \ll 1 \) (there is no data available for the case of \( \lambda \gg 1 \)). Furthermore, the data for Kaolin indicate that there does exist a maximum for \( \eta \). For both Kaolin and sludge, the maximum would be about \( 10^{-2} \) at \( \lambda \approx 0.3 - 0.7 \) (see Figs. 3-10 and 3-12).

Chapter 4 presents models for electroosmotic dewatering, with or
without the aid of applied pressure. There are two dewatering mechanisms:

(1) filtration, in which the fluid in a slurry is forced through a filter medium and the solid particles are deposited in the form of a cake;

(2) consolidation, in which the fluid is forced out of the cake (or any compressible porous medium).

A simple cost analysis is given in Section 4.11 for use with the models.

Chapter 5 describes experiments in electroosmotic filtration. The models in Chapter 4 cannot account for the rapid increase in the electroosmotic flow rate and power consumption observed in some of the experiments. Possibly $\psi_k$ and the fluid conductivity $\sigma_0$ are affected by temperature rise (from ohmic losses) and by products of electrode reactions.

A major criteria for any dewatering process is the cost required. We have therefore derived cost equations for two conventional methods, sand bed drying and vacuum filtration, for comparison with electroosmosis.

In Chapter 6 we first review the existing theories for dewatering on sand beds by drainage and evaporation. We then formulate a cost equation in which the controllable parameters are the mass of sludge solid applied per unit area of the bed, the moisture content at the end of drainage, and the moisture content at removal. The equation can be optimized with respect to the parameters. We include sample calculations on the effects of capital cost, evaporation rate, and chemical conditioning on the optimum cost per unit mass of solid dewatered.

In vacuum filtration a cake is first formed on the filter medium by filtration, and then removed from the sludge and further dewatered. In
Chapter 7 we model a continuous belt filter by applying the filtration theory of Tiller et al.\textsuperscript{53} to the filtration period and consolidation theory to the dewatering period. We then present a cost equation in which the controllable parameters are the level of vacuum applied, the speed of rotation of the drum, the fraction of the drum immersed in the sludge, and the moisture content at removal. The equation can be optimized with respect to the parameters, subject to various mechanical constraints, and used as an aid in the initial design of a vacuum filter facility or in the operation of an existing filter.

It is unlikely that electroosmosis will ever be an economically feasible method for ordinary dewatering wastewater sludge. A sample calculation in Section 4.11 for the sewage sludge used in our experiments shows that the capital and labor costs for electroosmotic filtration would have to be less than \( \sim 10^{-3} \) times the costs for pressure filtration for it to be competitive.

The reason, of course, is that the ionic concentration in the sludge fluid is high (equivalent to \( \sim 0.1 \text{ N NaCl} \)). Consequently, \( \sigma_0 \) is also high (\( \sim 1.7 \Omega^{-1}\text{m}^{-1} \)) and \( \lambda \ll 1 \). Most of the electrical energy is therefore wasted in ohmic losses in the bulk of the pore fluid.

As explained in Section 4.1, in electroosmotic filtration there is no net force on the solid matrix in the filter cake, whereas in pressure filtration there is a compressive stress on the matrix which tends to reduce the pore sizes and increase the resistance to hydraulic flow. Electroosmosis therefore may be useful for sludges with highly compressible
cakes of low hydraulic permeability, which would be difficult to dewater
by other methods. Its efficiency would improve with increasing surface
potential/surface charge density and decreasing electrical conductivity.

Equation (4-137) can be used to estimate whether electroosmosis is
economically attractive for dewatering a particular sludge. The cost
equations in Sections 4.11, 6.5, and 7.5 should be used if more exact
calculations are necessary.
REFERENCES


65. Wakeman, R.J., Filtration and Separation, July/August, 1974, p. 357.


75. Tiller, F.M. and Cooper, H., AIChe J. 8, 445 (1962).


APPENDIX A:

CONCENTRATION POLARIZATION

When a current is passed through an electrolyte solution, the frac-
tion of the current that is carried by a specie of ions is called its
transport number. In the experiments with Kaolin clay in Chapter 3,
there are two species of ions, Na$^+$ and Cl$^-$. Since the clay has negative
surface charge, the Na$^+$ ions are the counterions, referred to with sub-
script "1," and the Cl$^-$ ions are the co-ions, with subscript "2."

In the neutral bulk solution on both sides of the plug,

$$C_1 = C_2 = C_0$$

so that the transport numbers are

$$\text{Na}^+ : \quad t_{10} = \frac{Z_1j_1}{Z_1j_1 + Z_2j_2} = \frac{D_1}{D_1 + D_2}$$

$$\text{Cl}^- : \quad t_{20} = \frac{Z_2j_2}{Z_1j_1 + Z_2j_2} = \frac{D_2}{D_1 + D_2}$$

Within the clay plug there is an excess of Na$^+$ over Cl$^-$ ions, due
to the double layer. If we designate the transport number of Na$^+$ in the
pores by $t_{1p}$, and that of Cl$^-$ by $t_{2p}$, then

$$t_{1p} > t_{10} \quad \text{and} \quad t_{2p} < t_{20}$$

Concentration polarization occurs when the transport number of the ion
species changes from the external solution to the pores within; there is a depletion of ions on one side of the plug and an accumulation on the other (Fig. A-1). The reader is referred to Newman for a detailed discussion.

The simplest example is a plug (or membrane) which is perfectly selective; i.e., the co-ions are completely excluded from the pores. This occurs when \( \lambda >> 1 \) and the surface charge is high enough so that

\[
C_m = -\frac{2\sigma_k}{aFZ} >> C_0
\]

and Eq. (2-45) reduces to

\[
C_1 = C_m \quad \text{and} \quad C_2 = 0
\]

The flux densities and the ion concentrations are then shown in Fig. A-2. If the convective components of the ion flux densities are much smaller than the diffusive and migratory components, i.e.,

\[
P_{e\lambda D} = \frac{\vec{V}_\lambda D}{D} \ll 1
\]

then using Eqs. (2-4) and (2-8), with \( j_2 = 0 \) and \( C_1 = C_2 = C \) (electro-neutrality outside of the pores), we have

\[
\nabla C = \frac{i}{2FD}
\]

at the boundary between the double layer and the external solution. As the current increases, \( \nabla C \) increases until \( C \to 0 \) at the boundary; the current is then at its maximum possible value (limiting current). Figure
A-3 shows the non-linear relationship between \( i \) and \( \Delta V \), the voltage difference applied between the power electrodes.

The problem with polarization can be minimized by stirring or circulating the external solution, so that concentration gradients do not occur at the boundary. The circulating grooves in the backup plates of the test cell described in Section 3.2 are for this purpose.

Polarization is not expected to be a problem when the ion concentrations are high. The Debye length ratio is then much smaller than 1, and the bulk of the fluid in the pore is electrically neutral. The transport numbers in the pore then do not differ appreciably from those in the external solution. On the other hand, the problem is severe at low ion concentrations and \( \lambda \gg 1 \).

The experimental results confirm the above expectations. In experiments with Kaolin with \( C_0 \sim 10^{-4} \) N, the current varies as shown in Fig. A-4 when a constant voltage is suddenly imposed. The non-linear behavior persists even when the maximum circulation velocity transverse to the plug (\( \sim 1.5 \) cm/sec) was used. The experiments were therefore conducted in the following manner. A constant current density was applied instead; the corresponding electroosmotic flow rate was also constant. The voltage across the plug, however, would now rise rapidly with time. The voltage recorded as data was chosen to be the initial voltage, with a possible error of \( \pm 5\% \).

In experiments with sewage sludge, \( C_0 \sim 0.1 \) N, and no appreciable polarization was observed. For a constant voltage across the plug, the current would vary less than 1% for current densities up to \( \sim 25 \text{ amp/m}^2 \).
FIGURE A-1 Concentration Polarization at a partially-selective membrane.

FIGURE A-2. Concentration polarization at a perfectly-selective membrane.
FIGURE A-3. Limiting current due to concentration polarization.

FIGURE A-4. Current vs. time for an applied step voltage.
APPENDIX B

1. PORE RADIUS $a_p$

Both the Kozeny-Cormany equation for hydraulic permeability [Eq. (2-126)] and Olsen's correction [Eq. (2-125)] rely on the concept of "hydraulic radius," which we will define as

$$\text{hydraulic radius} = 2 \times \left( \frac{\text{volume of pore fluid}}{\text{wetted surface}} \right) . \quad (B-1)$$

For circular cylindrical pores, the hydraulic radius is identical to the pore radius.

The derivation of the Kozeny-Cormany equation assumes that the pore volume is uniformly distributed among pores with hydraulic radius of

$$a_h = \frac{2n}{S(1 - n)} , \quad (B-2)$$

where $n = \text{total porosity}$;

$S = \text{wetted surface area per unit volume of solid}$.

In analogy with Poiseuille flow in a circular tube, the average fluid velocity $\bar{v}$ is assumed to be

$$\bar{v} \propto - \frac{a_h^2}{\mu} \frac{\partial p}{\partial z} = - \frac{a_h^2}{\mu \tau^2} \frac{\partial p}{\partial x} . \quad (B-3)$$

$z$ is the coordinate along the axis of the pore, $x$ is the coordinate along the mean direction of the pressure gradient, and $\tau$ is a tortuosity factor.
relating \( z \) to \( x \) (Fig. 2-2). The fluid velocity per unit cross-sectional area of the porous medium is then

\[
q_0 = \frac{-4n^3}{\mu \tau^2 S^2 (1 - n)^2} \left( \frac{\partial p}{\partial x} \right) \quad (B-4)
\]

or

\[
q_0 = -\frac{n^3}{\mu C_k \tau^2 S^2 (1 - n)^2} \left( \frac{\partial p}{\partial x} \right) \quad , \quad (B-5)
\]

where \( C_k \) is an empirical constant (for circular cylinders, \( C_k = 2 \)). By Darcy's law,

\[
q_0 = -\frac{k_0}{\mu} \frac{\partial p}{\partial x} \quad , \quad (B-6)
\]

so that the permeability \( k_0 \) is

\[
k_0 = \frac{n^3}{C_k \tau^2 S (1 - n)^2} = \frac{e_T^3}{C_k \tau^2 S^2 (1 + e_T)} = \frac{na_h^2}{4C_k \tau^2} \quad (B-7)
\]

or

\[
a_h = \left( \frac{4C_k \tau^2}{n} \right)^{1/2} \quad . \quad (B-8)
\]

In the cluster model we assume that the fluid flows only in the pores between the clusters; we can also define a hydraulic pore radius \( a_p \) as:

\[
a_p \equiv 2 \frac{V_p}{A_{sc}} = 2 \left( \frac{V_p}{V_c} \right) \left( \frac{V_c}{A_{sc}} \right) \quad , \quad (B-9)
\]

where \( V_c = \) volume of clusters (solid and liquid);

\( V_p = \) volume of pores between clusters;
\[ A_{sc} = \text{surface of clusters exposed to the pore fluid.} \]

Since the clusters are assumed to be spherical with radius \( r_c \),

\[ \frac{V_c}{A_{sc}} = \frac{\frac{4}{3} N_c \pi r_c^3}{4 N_c \pi r_c^2} = \frac{r_c}{3} \quad , \tag{B-10} \]

where \( N_c \) is the number of clusters in the porous medium. By definition,

\[ V_c = \frac{4}{3} N_c \pi r_c^3 = (1 + e_c)V_s \tag{B-11} \]

or

\[ \frac{4}{3} \pi r_c^3 = (1 + e_c) \frac{V_s}{N_c} \tag{B-12} \]

where \( V_s \) is the volume of solid.

Let \( N = \) number of particles per cluster;

\( r_s = \) radius of a particle;

\( S = \) surface area of a particle per volume of particle \( = 3/r_s \);

then,

\[ \frac{V_s}{N_c} = N \frac{4}{3} \pi r_s^3 = N \left( \frac{4}{3} \pi \right) \left( \frac{27}{S^3} \right) \quad , \tag{B-13} \]

and, substituting into Eq. (B-12),

\[ \frac{r_c}{3} = \left[ \frac{N(1 + e_c)^{1/3}}{S} \right] \quad . \tag{B-14} \]

Also, by definition,

\[ \frac{V_p}{V_c} = \frac{(V_p/V_s)}{(V_c/V_s)} = \frac{e_p}{(1 + e_c)} = \left( \frac{e_T - e_c}{1 + e_c} \right) \quad , \tag{B-15} \]

so that
\[ a_p = \frac{2(e_T - e_c) N^{4/3}}{S(1 + e_c)^{2/3}} \]  \hspace{1cm} (B-16)

Assuming again laminar flow, we have

\[ \bar{v}_p \propto -\frac{a_p^2}{\mu \tau^2} \frac{\partial p}{\partial x} \]  \hspace{1cm} (B-17)

\[ q_h = -\frac{k}{\mu} \frac{\partial p}{\partial x} \propto n_p \bar{v}_p \]  \hspace{1cm} (B-18)

where \( \bar{v}_p \) = average velocity in the pores;
\( q_h \) = superficial velocity in the cluster model.

For a given pressure gradient,

\[ \frac{q_h}{q_0} = \frac{k}{k_0} = \frac{n_p a_p^2}{n a_h^2} \]  \hspace{1cm} (B-19)

After some algebra, we obtain the equations:

\[ \frac{k}{k_0} = \frac{N^{2/3} (e_T - e_c)^3}{e_T(1 + e_c)^{4/3}} \]  \hspace{1cm} (B-20)

and

\[ a_p = \left( \frac{4C_k \tau^2 k}{n_p} \right)^{1/2} \]  \hspace{1cm} (B-21)

2. DETERMINING \( N, e_c \), and \( e_{cr} \) FROM EXPERIMENTAL DATA

Equations (B-7) and (B-20) can be combined to give
\[ k = \frac{N^3(e_T - e_c)^3}{C_k \tau^2 S^2 (1 + e_c)^{4/3}(1 + e_T)} \]  \hspace{1cm} (B-22)

\( C_k \tau^2 \) is usually taken to be 5,22,24 and \( S \) can be determined from vapor-adsorption data.22,24 From experimental data of \( k \) as a function of \( e_T \), we can then find \( N \), which should be a constant, and \( e_c \), which should be a function of \( e_T \).

3. **ELETRONEUTRALITY WITHIN THE CLUSTERS**

Under the assumption that \( \lambda_c \gg 1 \) within the clusters, the charge density in the cluster fluid will be uniform. By electroneutrality, we have for each cluster:

\[ ZF(C_1 - C_2)V_{fc} = -\sigma_s N4\pi r_s^2 = -\sigma_s S V_{cs}, \]  \hspace{1cm} (B-23)

where \( V_{fc} = \) fluid volume in a cluster;

\( V_{sc} = \) solid volume in a cluster.

By definition, \( V_{fc} = e_c V_{sc} \), so that

\[ C_1 - C_2 = -\frac{\sigma_s S}{ZFe_c} = C_n. \]  \hspace{1cm} (B-24)

For a fully dissociated salt,

\[ C_1 C_2 = C_0^2. \]  \hspace{1cm} (B-25)
Combining Eqs. (B-24) and (B-25), we have

\[
C_1 = \frac{C_n}{2} \left[ 1 + \left( 1 + \frac{4C_0^2}{C_n^2} \right)^{\frac{1}{2}} \right] \tag{R-26}
\]

\[
C_2 = \frac{C_n}{2} \left[ -1 + \left( 1 + \frac{4C_0^2}{C_n^2} \right)^{\frac{1}{2}} \right] \tag{B-27}
\]

The conductivity of the fluid in the cluster would be

\[
\sigma_c = \frac{(ZF)^2 D}{RT} \left( C_1 + C_2 \right) = \frac{2(ZF)^2 D}{RT} \left[ 1 + \left( \frac{\sigma_s S}{2C_0 ZFe_c} \right)^2 \right]^{\frac{1}{2}} \tag{B-28}
\]

\[
= \sigma_0 \left[ 1 + \left( \frac{\sigma_s S}{2C_0 ZFe_c} \right)^2 \right]^{\frac{1}{2}}
\]
APPENDIX C:
DATA AND CALCULATIONS FOR EXPERIMENT SERIES III
TABLE C-1. Experiment III-1

With Constant Current Density $i_0 = 10$ amp/m² (nominal)

Probe 1: $L_1 = 2.2 \times 10^{-2}$ m
Probe 2: $L_2 = 0.9 \times 10^{-2}$ m

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>119</th>
<th>134</th>
<th>151</th>
<th>182</th>
<th>242</th>
<th>247</th>
<th>271</th>
<th>308</th>
<th>382</th>
<th>399</th>
</tr>
</thead>
</table>

Voltage measured at (volts):

<table>
<thead>
<tr>
<th></th>
<th>Anode</th>
<th>Probe 1</th>
<th>Probe 2</th>
<th>$i$ (amp/m²)</th>
<th>$P_e$ (watt/m²)</th>
<th>$q_e$ (m/sec × 10⁷)</th>
<th>$(\mu q_e \sigma_0 / i)$ (coul/m × 10¹²)</th>
<th>$H_s + h$ (cm)</th>
<th>$h$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7.18</td>
<td>6.88</td>
<td>6.58</td>
<td>6.27</td>
<td>5.78</td>
<td>5.83</td>
<td>5.28</td>
<td>5.08</td>
<td>4.78</td>
</tr>
<tr>
<td></td>
<td>2.82</td>
<td>2.72</td>
<td>2.77</td>
<td>2.82</td>
<td>3.22</td>
<td>3.30</td>
<td>3.02</td>
<td>3.12</td>
<td>3.42</td>
</tr>
<tr>
<td></td>
<td>1.82</td>
<td>1.77</td>
<td>1.75</td>
<td>1.77</td>
<td>2.02</td>
<td>2.22</td>
<td>1.87</td>
<td>1.92</td>
<td>1.87</td>
</tr>
<tr>
<td></td>
<td>71.4</td>
<td>68.6</td>
<td>65.8</td>
<td>62.4</td>
<td>57.6</td>
<td>57.9</td>
<td>52.6</td>
<td>50.7</td>
<td>47.8</td>
</tr>
<tr>
<td></td>
<td>4.95</td>
<td>6.56</td>
<td>5.94</td>
<td>5.94</td>
<td>5.82</td>
<td>7.43</td>
<td>6.80</td>
<td>7.89</td>
<td>8.30</td>
</tr>
<tr>
<td></td>
<td>5.23</td>
<td>6.93</td>
<td>6.28</td>
<td>6.28</td>
<td>6.15</td>
<td>7.85</td>
<td>7.18</td>
<td>8.34</td>
<td>8.77</td>
</tr>
<tr>
<td></td>
<td>5.7</td>
<td>5.4</td>
<td>5.2</td>
<td>4.7</td>
<td>3.7</td>
<td>3.45</td>
<td>3.2</td>
<td>2.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.1</td>
<td>1.1</td>
<td>1.2</td>
<td>1.4</td>
<td>1.6</td>
<td>1.7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**TABLE C-2. Experiment III-1**

With Constant Voltage Difference $\Delta\phi_0 = 7.0$ V

<table>
<thead>
<tr>
<th>Probe 1: $L_1 = 2.2 \times 10^{-2}$ m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probe 2: $L_2 = 9.0 \times 10^{-3}$ m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>119</th>
<th>136</th>
<th>187</th>
<th>201</th>
<th>218</th>
<th>274</th>
<th>306</th>
<th>395</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage measured at (volts):</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probe 1</td>
<td>2.82</td>
<td>2.72</td>
<td>3.27</td>
<td>3.37</td>
<td>3.57</td>
<td>4.22</td>
<td>4.52</td>
<td>5.62</td>
</tr>
<tr>
<td>Probe 2</td>
<td>1.82</td>
<td>1.77</td>
<td>2.07</td>
<td>2.12</td>
<td>2.17</td>
<td>2.62</td>
<td>2.57</td>
<td>3.12</td>
</tr>
<tr>
<td>$i$ (amp/m$^2$)</td>
<td>9.95</td>
<td>9.97</td>
<td>11.1</td>
<td>11.4</td>
<td>12.4</td>
<td>13.9</td>
<td>15.2</td>
<td>18.6</td>
</tr>
<tr>
<td>$P_e$ (watt/m$^2$)</td>
<td>71.4</td>
<td>68.6</td>
<td>77.5</td>
<td>79.6</td>
<td>86.5</td>
<td>97.2</td>
<td>106</td>
<td>130</td>
</tr>
<tr>
<td>$H_s + h$ (cm)</td>
<td>5.7</td>
<td>5.4</td>
<td>4.7</td>
<td>4.7</td>
<td>4.4</td>
<td>3.7</td>
<td>3.4</td>
<td>3.1</td>
</tr>
<tr>
<td>$h$ (cm)</td>
<td>1.0</td>
<td>1.1</td>
<td>1.2</td>
<td>1.3</td>
<td>1.35</td>
<td>1.4</td>
<td>1.6</td>
<td></td>
</tr>
<tr>
<td>$q_e$ (m/sec $\times 10^7$)</td>
<td>4.95</td>
<td>6.56</td>
<td>6.93</td>
<td>6.31</td>
<td>7.43</td>
<td>10.6</td>
<td>11.7</td>
<td>15.3</td>
</tr>
<tr>
<td>$\mu q_e \sigma_0 / i$ (coul/m $\times 10^{12}$)</td>
<td>5.73</td>
<td>7.58</td>
<td>7.19</td>
<td>6.38</td>
<td>6.90</td>
<td>8.79</td>
<td>8.87</td>
<td>9.48</td>
</tr>
</tbody>
</table>
TABLE C-3. Experiment III-1

Flow Rate due to Pressure Alone

$\Delta P = 6.89 \times 10^6 \text{ N/m}^2$

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>$q_h \times 10^6 \text{ (m/s)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>57</td>
<td>2.79</td>
</tr>
<tr>
<td>73</td>
<td>2.28</td>
</tr>
<tr>
<td>81</td>
<td>2.17</td>
</tr>
<tr>
<td>84</td>
<td>2.07</td>
</tr>
<tr>
<td>104</td>
<td>2.07</td>
</tr>
<tr>
<td>109</td>
<td>2.00</td>
</tr>
<tr>
<td>115</td>
<td>1.98</td>
</tr>
<tr>
<td>123</td>
<td>1.99</td>
</tr>
<tr>
<td>134</td>
<td>1.81</td>
</tr>
<tr>
<td>138</td>
<td>1.94</td>
</tr>
<tr>
<td>150</td>
<td>1.76</td>
</tr>
<tr>
<td>186</td>
<td>1.76</td>
</tr>
<tr>
<td>195</td>
<td>2.82</td>
</tr>
<tr>
<td>203</td>
<td>2.67</td>
</tr>
<tr>
<td>217</td>
<td>2.12</td>
</tr>
<tr>
<td>233</td>
<td>1.68</td>
</tr>
<tr>
<td>248</td>
<td>1.41</td>
</tr>
<tr>
<td>268</td>
<td>1.22</td>
</tr>
<tr>
<td>274</td>
<td>1.27</td>
</tr>
<tr>
<td>278</td>
<td>1.39</td>
</tr>
<tr>
<td>304</td>
<td>1.15</td>
</tr>
<tr>
<td>311</td>
<td>1.17</td>
</tr>
<tr>
<td>329</td>
<td>1.13</td>
</tr>
<tr>
<td>392</td>
<td>1.07</td>
</tr>
</tbody>
</table>
TABLE C-4. Experiment III-1.

Calculated values of $\eta$

(1) $i_0 = 10$ amp/m$^2$

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>$\eta \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>1.26</td>
</tr>
<tr>
<td>134</td>
<td>2.52</td>
</tr>
<tr>
<td>151</td>
<td>2.22</td>
</tr>
<tr>
<td>182</td>
<td>2.35</td>
</tr>
<tr>
<td>242</td>
<td>2.86</td>
</tr>
<tr>
<td>247</td>
<td>4.98</td>
</tr>
<tr>
<td>271</td>
<td>4.77</td>
</tr>
<tr>
<td>308</td>
<td>7.36</td>
</tr>
<tr>
<td>332</td>
<td>8.79</td>
</tr>
<tr>
<td>399</td>
<td>6.75</td>
</tr>
</tbody>
</table>

(2) $\Delta\phi_0 = 7.0$ V

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>$\eta \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>1.26</td>
</tr>
<tr>
<td>136</td>
<td>2.44</td>
</tr>
<tr>
<td>187</td>
<td>2.57</td>
</tr>
<tr>
<td>201</td>
<td>1.37</td>
</tr>
<tr>
<td>218</td>
<td>2.19</td>
</tr>
<tr>
<td>274</td>
<td>6.55</td>
</tr>
<tr>
<td>306</td>
<td>8.04</td>
</tr>
<tr>
<td>395</td>
<td>12.1</td>
</tr>
</tbody>
</table>
**TABLE C-5. Experiment III-2**

Constant Voltage Difference $\Delta \phi_0 = 3.4$ V

Fixed Anode: $L_a = 3.6 \times 10^{-2}$ m  
Probe 1: $L_1 = 2.3 \times 10^{-2}$ m  
Probe 2: $L_2 = 5.5 \times 10^{-3}$ m

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>Voltage (volts) at:</th>
<th>$\Delta \phi_e$ (volt)</th>
<th>i (amp/m²)</th>
<th>$P_e$ (watt/m²)</th>
<th>$V_1$ (m×10³)</th>
<th>T (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.3</td>
<td></td>
<td></td>
<td>76.8</td>
<td>265</td>
<td>2.07</td>
<td>25</td>
</tr>
<tr>
<td>49.7</td>
<td></td>
<td></td>
<td>77.3</td>
<td>263</td>
<td>2.14</td>
<td>25</td>
</tr>
<tr>
<td>51.3</td>
<td></td>
<td></td>
<td>77.3</td>
<td>264</td>
<td>2.23</td>
<td>25.5</td>
</tr>
<tr>
<td>53.0</td>
<td></td>
<td></td>
<td>79.6</td>
<td>271</td>
<td>2.30</td>
<td>26.0</td>
</tr>
<tr>
<td>55.0</td>
<td>1.64</td>
<td>0.44</td>
<td>0.88</td>
<td>82.2</td>
<td>279</td>
<td>2.45</td>
</tr>
<tr>
<td>57.0</td>
<td></td>
<td></td>
<td>82.0</td>
<td>279</td>
<td>2.56</td>
<td>27</td>
</tr>
<tr>
<td>60.4</td>
<td>1.59</td>
<td>0.36</td>
<td>0.90</td>
<td>81.7</td>
<td>278</td>
<td>2.77</td>
</tr>
<tr>
<td>62.7</td>
<td>1.59</td>
<td>0.37</td>
<td>0.90</td>
<td>82.2</td>
<td>279</td>
<td>2.92</td>
</tr>
<tr>
<td>65.4</td>
<td></td>
<td></td>
<td>83.8</td>
<td>285</td>
<td>3.08</td>
<td></td>
</tr>
<tr>
<td>72.4</td>
<td>1.59</td>
<td>0.34</td>
<td>0.88</td>
<td>86.5</td>
<td>294</td>
<td>3.46</td>
</tr>
<tr>
<td>76.1</td>
<td></td>
<td></td>
<td>88.5</td>
<td>301</td>
<td>3.68</td>
<td>30</td>
</tr>
<tr>
<td>82.1</td>
<td>1.54</td>
<td>0.33</td>
<td>0.97</td>
<td>91.0</td>
<td>309</td>
<td>4.04</td>
</tr>
<tr>
<td>89.3</td>
<td>1.54</td>
<td>0.33</td>
<td>0.97</td>
<td>93.8</td>
<td>319</td>
<td>4.50</td>
</tr>
<tr>
<td>93.4</td>
<td>1.54</td>
<td>0.33</td>
<td>0.97</td>
<td>95.5</td>
<td>325</td>
<td>4.77</td>
</tr>
<tr>
<td>99.3</td>
<td>1.52</td>
<td>0.29</td>
<td>0.98</td>
<td>97.5</td>
<td>331</td>
<td>5.15</td>
</tr>
<tr>
<td>106</td>
<td>1.54</td>
<td>0.34</td>
<td>0.97</td>
<td>100</td>
<td>340</td>
<td>5.61</td>
</tr>
<tr>
<td>112</td>
<td>1.55</td>
<td>0.33</td>
<td>0.95</td>
<td>102</td>
<td>347</td>
<td>5.98</td>
</tr>
<tr>
<td>116</td>
<td>1.59</td>
<td>0.34</td>
<td>0.88</td>
<td>107</td>
<td>364</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>1.64</td>
<td>0.34</td>
<td>0.80</td>
<td>109</td>
<td>371</td>
<td>6.55</td>
</tr>
<tr>
<td>124</td>
<td>1.69</td>
<td>0.36</td>
<td>0.73</td>
<td>113</td>
<td>384</td>
<td>38</td>
</tr>
<tr>
<td>132</td>
<td>1.74</td>
<td>0.36</td>
<td>0.64</td>
<td>99</td>
<td>337</td>
<td>7.45</td>
</tr>
<tr>
<td>139</td>
<td>1.79</td>
<td>0.39</td>
<td>0.57</td>
<td>100</td>
<td>340</td>
<td>40</td>
</tr>
<tr>
<td>146</td>
<td>1.84</td>
<td>0.43</td>
<td>0.51</td>
<td>102</td>
<td>347</td>
<td>8.62</td>
</tr>
</tbody>
</table>
### TABLE C-6. Experiment III-2

Filtrate Volume per Unit Area and Temperature

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>$V_1$ ($m \times 10^3$)</th>
<th>$T$ ($^\circ$C)</th>
<th>Time (min)</th>
<th>$V_1$ ($m \times 10^3$)</th>
<th>$T$ ($^\circ$C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.6</td>
<td>1.05</td>
<td>25</td>
<td>58.6</td>
<td>2.65</td>
<td>27</td>
</tr>
<tr>
<td>14.3</td>
<td>1.11</td>
<td>25</td>
<td>63.4</td>
<td>2.97</td>
<td>28</td>
</tr>
<tr>
<td>15.9</td>
<td>1.18</td>
<td>25</td>
<td>66.7</td>
<td>3.16</td>
<td>28</td>
</tr>
<tr>
<td>18.0</td>
<td>1.25</td>
<td>25</td>
<td>71.5</td>
<td>3.41</td>
<td>29</td>
</tr>
<tr>
<td>21.8</td>
<td>1.36</td>
<td>25</td>
<td>76.1</td>
<td>3.68</td>
<td>30</td>
</tr>
<tr>
<td>25.9</td>
<td>1.49</td>
<td>25</td>
<td>79.2</td>
<td>3.87</td>
<td>30</td>
</tr>
<tr>
<td>30.7</td>
<td>1.62</td>
<td>25</td>
<td>83.1</td>
<td>4.10</td>
<td>31</td>
</tr>
<tr>
<td>36.2</td>
<td>1.77</td>
<td>25</td>
<td>86.9</td>
<td>4.33</td>
<td>32</td>
</tr>
<tr>
<td>43.3</td>
<td>1.93</td>
<td>25</td>
<td>93.4</td>
<td>4.77</td>
<td>32</td>
</tr>
<tr>
<td>46.6</td>
<td>2.02</td>
<td>25</td>
<td>99.3</td>
<td>5.15</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>106</td>
<td>5.61</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>109</td>
<td>5.83</td>
<td>34</td>
</tr>
</tbody>
</table>

$t_0 = 48.6 \text{ min}: \text{Start electroosmosis}$

$v_{10} = 2.05 \times 10^{-3} \text{ m}$

$T = 25^\circ\text{C}$

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>$V_1$ ($m \times 10^3$)</th>
<th>$T$ ($^\circ$C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.7</td>
<td>2.14</td>
<td>25</td>
</tr>
<tr>
<td>51.3</td>
<td>2.23</td>
<td>25</td>
</tr>
<tr>
<td>52.6</td>
<td>2.30</td>
<td>25</td>
</tr>
<tr>
<td>54.4</td>
<td>2.41</td>
<td>25</td>
</tr>
<tr>
<td>56.0</td>
<td>2.51</td>
<td>27</td>
</tr>
</tbody>
</table>

112 5.98 34
116 6.28 34
120 6.55 36
124 6.90 38
132 7.45 39
139 8.07 40
143 8.43 40
146 8.62 41
152 9.14 41
TABLE C-7. Experiment III-3
Constant Voltage Difference $\Delta \phi_0 = 3.4$ V

Fixed Anode: $L_a = 3.6 \times 10^{-2}$ m
Probe 1: $L_1 = 2.3 \times 10^{-2}$ m
Probe 2: $L_2 = 6.0 \times 10^{-3}$ m

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>Probe 1 (volt)</th>
<th>Probe 2 (volt)</th>
<th>$\Delta \phi_e$ (volt)</th>
<th>$i$ amp m$^2$</th>
<th>$P_e$ watt m$^2$</th>
<th>$q_e$ m/sec $\times 10^7$</th>
<th>$P_e/q_e$ kW-hr m$^{-3}$</th>
<th>$\mu q_e \sigma_0$ coul/m $\times 10^{12}$</th>
<th>$v_1$ m x $10^3$</th>
<th>$T$ °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>63.5</td>
<td>1.84</td>
<td>0.62</td>
<td>0.62</td>
<td>77.0</td>
<td>261</td>
<td>5.55</td>
<td>131</td>
<td>7.35</td>
<td>.89</td>
<td>26</td>
</tr>
<tr>
<td>76.5</td>
<td>1.81</td>
<td>0.59</td>
<td>0.68</td>
<td>78.6</td>
<td>267</td>
<td>6.07</td>
<td>123</td>
<td>7.78</td>
<td>.98</td>
<td>26</td>
</tr>
<tr>
<td>89.5</td>
<td>1.80</td>
<td>0.59</td>
<td>0.68</td>
<td>78.6</td>
<td>267</td>
<td>5.94</td>
<td>125</td>
<td>7.54</td>
<td>1.06</td>
<td>27</td>
</tr>
<tr>
<td>103</td>
<td>1.82</td>
<td>0.62</td>
<td>0.68</td>
<td>79.3</td>
<td>270</td>
<td>5.51</td>
<td>136</td>
<td>6.79</td>
<td>1.13</td>
<td>28</td>
</tr>
<tr>
<td>124</td>
<td>1.85</td>
<td>0.62</td>
<td>0.62</td>
<td>79.5</td>
<td>270</td>
<td>5.93</td>
<td>126</td>
<td>7.28</td>
<td>1.23</td>
<td>28</td>
</tr>
<tr>
<td>139</td>
<td>1.83</td>
<td>0.61</td>
<td>0.64</td>
<td>83.6</td>
<td>284</td>
<td>6.43</td>
<td>123</td>
<td>7.42</td>
<td>1.31</td>
<td>28.5</td>
</tr>
<tr>
<td>149</td>
<td>1.82</td>
<td>0.60</td>
<td>0.66</td>
<td>84.6</td>
<td>287</td>
<td>6.22</td>
<td>128</td>
<td>7.02</td>
<td>1.35</td>
<td>29</td>
</tr>
<tr>
<td>162</td>
<td>1.84</td>
<td>0.62</td>
<td>0.64</td>
<td>81.7</td>
<td>277</td>
<td>6.24</td>
<td>123</td>
<td>7.29</td>
<td>1.40</td>
<td>29</td>
</tr>
<tr>
<td>173</td>
<td>1.83</td>
<td>0.62</td>
<td>0.64</td>
<td>81.8</td>
<td>278</td>
<td>6.24</td>
<td>124</td>
<td>7.21</td>
<td>1.45</td>
<td>29</td>
</tr>
<tr>
<td>194</td>
<td>1.84</td>
<td>0.60</td>
<td>0.64</td>
<td>81.7</td>
<td>277</td>
<td>6.61</td>
<td>116</td>
<td>8.45</td>
<td>1.54</td>
<td>30</td>
</tr>
<tr>
<td>219</td>
<td>1.91</td>
<td>0.68</td>
<td>0.56</td>
<td>80.7</td>
<td>275</td>
<td>7.17</td>
<td>106</td>
<td>9.18</td>
<td>1.64</td>
<td>30</td>
</tr>
<tr>
<td>273</td>
<td>1.94</td>
<td>0.67</td>
<td>0.50</td>
<td>77.1</td>
<td>262</td>
<td>8.33</td>
<td>87</td>
<td>11.2</td>
<td>1.82</td>
<td>30</td>
</tr>
<tr>
<td>292</td>
<td>1.86</td>
<td>0.56</td>
<td>0.56</td>
<td>76.0</td>
<td>259</td>
<td>8.69</td>
<td>83</td>
<td>11.8</td>
<td>1.88</td>
<td>30</td>
</tr>
<tr>
<td>334</td>
<td>2.02</td>
<td>0.56</td>
<td>0.26</td>
<td>71.8</td>
<td>244</td>
<td>9.46</td>
<td>72</td>
<td>13.6</td>
<td>2.01</td>
<td>30</td>
</tr>
<tr>
<td>356</td>
<td>1.94</td>
<td>0.49</td>
<td>0.36</td>
<td>69.6</td>
<td>236</td>
<td>10.5</td>
<td>62</td>
<td>15.6</td>
<td>2.10</td>
<td>30</td>
</tr>
<tr>
<td>378</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>234</td>
<td>13.6</td>
<td>48</td>
<td>20.4</td>
<td>2.20</td>
<td>30</td>
</tr>
<tr>
<td>409</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>225</td>
<td>13.4</td>
<td>47</td>
<td></td>
<td></td>
<td>30</td>
</tr>
</tbody>
</table>
## TABLE C-8. Experiment II-3

**Filtrate Volume and Flow Rate Under Pressure Alone**

Filter area = 80.75 cm²

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>Filtrate Volume (cc)</th>
<th>q₂₀ (m/sec × 10⁷)</th>
<th>Time (min)</th>
<th>Filtrate Volume (cc)</th>
<th>q₂₀ (m/sec × 10⁷)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.50</td>
<td>6.6</td>
<td>10.6</td>
<td>64.5</td>
<td>72.6</td>
<td>9.9</td>
</tr>
<tr>
<td>1.26</td>
<td>10.6</td>
<td>85.5</td>
<td>83.6</td>
<td>9.1</td>
<td>8.3</td>
</tr>
<tr>
<td>1.86</td>
<td>12.6</td>
<td>103</td>
<td>91.6</td>
<td>9.1</td>
<td>8.3</td>
</tr>
<tr>
<td>2.47</td>
<td>14.6</td>
<td>121</td>
<td>98.6</td>
<td>8.3</td>
<td>7.8</td>
</tr>
<tr>
<td>3.17</td>
<td>16.6</td>
<td>139</td>
<td>106</td>
<td>7.8</td>
<td>7.52</td>
</tr>
<tr>
<td>3.88</td>
<td>18.6</td>
<td>147</td>
<td>109</td>
<td>7.52</td>
<td>7.50</td>
</tr>
<tr>
<td>4.80</td>
<td>20.6</td>
<td>160</td>
<td>113</td>
<td>7.50</td>
<td>7.13</td>
</tr>
<tr>
<td>6.01</td>
<td>22.6</td>
<td>171</td>
<td>117</td>
<td>7.13</td>
<td>7.01</td>
</tr>
<tr>
<td>7.03</td>
<td>24.6</td>
<td>192</td>
<td>124</td>
<td>7.01</td>
<td>6.08</td>
</tr>
<tr>
<td>9.40</td>
<td>28.6</td>
<td>217</td>
<td>132</td>
<td>6.08</td>
<td>5.93</td>
</tr>
<tr>
<td>10.8</td>
<td>30.6</td>
<td>246</td>
<td>140</td>
<td>5.93</td>
<td>5.41</td>
</tr>
<tr>
<td>12.1</td>
<td>32.6</td>
<td>271</td>
<td>147</td>
<td>5.41</td>
<td>5.30</td>
</tr>
<tr>
<td>21.2</td>
<td>42.6</td>
<td>290</td>
<td>152</td>
<td>5.30</td>
<td>4.78</td>
</tr>
<tr>
<td>32.7</td>
<td>52.6</td>
<td>334</td>
<td>163</td>
<td>4.78</td>
<td>4.61</td>
</tr>
<tr>
<td>40.9</td>
<td>58.6</td>
<td>352</td>
<td>168</td>
<td>4.61</td>
<td>4.64</td>
</tr>
<tr>
<td>47.5</td>
<td>63.6</td>
<td>13.9</td>
<td>376</td>
<td>177</td>
<td>4.64</td>
</tr>
<tr>
<td>57.5</td>
<td>68.6</td>
<td>12.2</td>
<td>406</td>
<td>182</td>
<td>4.11</td>
</tr>
</tbody>
</table>
TABLE C-9. Experiment III-3.

Calculated values of \( \eta \) and \( (\eta)^{-1} \)

<table>
<thead>
<tr>
<th>Time</th>
<th>( \eta \times 10^b )</th>
<th>( (\eta)^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>89.5</td>
<td>1.84</td>
<td>5434</td>
</tr>
<tr>
<td>103</td>
<td>1.70</td>
<td>5882</td>
</tr>
<tr>
<td>124</td>
<td>2.16</td>
<td>4630</td>
</tr>
<tr>
<td>139</td>
<td>2.57</td>
<td>3891</td>
</tr>
<tr>
<td>149</td>
<td>2.47</td>
<td>4048</td>
</tr>
<tr>
<td>162</td>
<td>2.59</td>
<td>3861</td>
</tr>
<tr>
<td>173</td>
<td>2.71</td>
<td>3690</td>
</tr>
<tr>
<td>194</td>
<td>3.10</td>
<td>3225</td>
</tr>
<tr>
<td>219</td>
<td>4.24</td>
<td>2358</td>
</tr>
<tr>
<td>273</td>
<td>6.75</td>
<td>1481</td>
</tr>
<tr>
<td>292</td>
<td>7.59</td>
<td>1317</td>
</tr>
<tr>
<td>334</td>
<td>10.6</td>
<td>943</td>
</tr>
<tr>
<td>352</td>
<td>14.0</td>
<td>714</td>
</tr>
<tr>
<td>378</td>
<td>23.5</td>
<td>425</td>
</tr>
<tr>
<td>409</td>
<td>26.8</td>
<td>373</td>
</tr>
</tbody>
</table>
TABLE C-10. Experiment III-4

Constant Current Density $i_0 = 13.6$ amp/m$^2$

Probe 1: $L_1 = 0.2$ Cm

Probe 2: $L_2 = 2.8$ Cm

Voltages at:

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>$q = q_e + q_h$ $\times 10^6$ (m/s)</th>
<th>Anode (V)</th>
<th>Probe 1 (V)</th>
<th>Probe 2 (V)</th>
<th>$i_0$ (A/m$^2$)</th>
<th>$P_e$ (W/m$^2$)</th>
<th>T (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>191</td>
<td>2.44</td>
<td>6.35</td>
<td>.64</td>
<td>3.54</td>
<td>9.9</td>
<td></td>
<td>27</td>
</tr>
<tr>
<td>194</td>
<td>2.79</td>
<td>8.55</td>
<td>.84</td>
<td>4.94</td>
<td>13.6</td>
<td>116</td>
<td>27</td>
</tr>
<tr>
<td>200</td>
<td>3.01</td>
<td>8.58</td>
<td>.89</td>
<td>5.09</td>
<td>13.6</td>
<td>117</td>
<td>27</td>
</tr>
<tr>
<td>208</td>
<td>2.98</td>
<td>8.35</td>
<td>.89</td>
<td>5.19</td>
<td>13.6</td>
<td>114</td>
<td>27</td>
</tr>
<tr>
<td>215</td>
<td>3.71</td>
<td>7.90</td>
<td>.84</td>
<td>5.14</td>
<td>13.6</td>
<td>107</td>
<td>27.5</td>
</tr>
<tr>
<td>225</td>
<td>4.12</td>
<td>7.55</td>
<td>.74</td>
<td>4.89</td>
<td>13.6</td>
<td>103</td>
<td>28</td>
</tr>
<tr>
<td>240</td>
<td>3.14</td>
<td>7.60</td>
<td>.94</td>
<td>5.24</td>
<td>13.6</td>
<td>103</td>
<td>28</td>
</tr>
<tr>
<td>250</td>
<td>3.09</td>
<td>7.50</td>
<td>1.24</td>
<td>5.54</td>
<td>13.6</td>
<td>102</td>
<td>28.5</td>
</tr>
<tr>
<td>260</td>
<td>3.55</td>
<td>8.9</td>
<td>2.04</td>
<td>6.24</td>
<td>13.6</td>
<td>121</td>
<td>28.5</td>
</tr>
<tr>
<td>271</td>
<td>3.53</td>
<td>8.05</td>
<td>2.79</td>
<td>7.34</td>
<td>13.6</td>
<td>118</td>
<td>28.5</td>
</tr>
<tr>
<td>280</td>
<td>3.78</td>
<td>9.9</td>
<td>3.34</td>
<td>8.74</td>
<td>13.6</td>
<td>135</td>
<td>29</td>
</tr>
<tr>
<td>290</td>
<td>3.26</td>
<td>10.0</td>
<td>3.94</td>
<td>9.04</td>
<td>13.6</td>
<td>136</td>
<td>29</td>
</tr>
<tr>
<td>298</td>
<td>3.17</td>
<td>10.7</td>
<td>5.24</td>
<td>9.94</td>
<td>13.6</td>
<td>145</td>
<td>29</td>
</tr>
<tr>
<td>305</td>
<td>3.09</td>
<td>11.3</td>
<td>6.14</td>
<td>10.7</td>
<td>13.6</td>
<td>154</td>
<td>29</td>
</tr>
</tbody>
</table>
TABLE C-11. Experiment III-4
Filtrate Volume per Unit Area and Temperature

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>$v_1$ (m x 10$^3$)</th>
<th>T (°C)</th>
<th>Time (min)</th>
<th>$v_1$ (m x 10$^3$)</th>
<th>T (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.40</td>
<td>1.86</td>
<td>26</td>
<td>180</td>
<td>41.6</td>
<td>26</td>
</tr>
<tr>
<td>0.90</td>
<td>2.60</td>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.68</td>
<td>3.59</td>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.05</td>
<td>4.83</td>
<td>26</td>
<td>187</td>
<td>Start electroosmosis</td>
<td></td>
</tr>
<tr>
<td>5.60</td>
<td>6.81</td>
<td>26</td>
<td></td>
<td>$i_0 = 9.91$ amp/m$^2$</td>
<td></td>
</tr>
<tr>
<td>7.80</td>
<td>8.05</td>
<td>26</td>
<td>190</td>
<td>43.1</td>
<td>27</td>
</tr>
<tr>
<td>10.7</td>
<td>9.53</td>
<td>26</td>
<td>194</td>
<td>44.0</td>
<td>27</td>
</tr>
<tr>
<td>14.2</td>
<td>11.0</td>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17.3</td>
<td>12.3</td>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.1</td>
<td>13.2</td>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23.9</td>
<td>14.5</td>
<td>26</td>
<td>202</td>
<td>45.2</td>
<td>27</td>
</tr>
<tr>
<td>28.8</td>
<td>16.0</td>
<td>26</td>
<td>210</td>
<td>46.7</td>
<td>27.5</td>
</tr>
<tr>
<td>33.1</td>
<td>17.2</td>
<td>26</td>
<td>220</td>
<td>48.9</td>
<td>27.5</td>
</tr>
<tr>
<td>56</td>
<td>22.7</td>
<td>26</td>
<td>230</td>
<td>51.3</td>
<td>27.5</td>
</tr>
<tr>
<td>67</td>
<td>25.0</td>
<td>26</td>
<td>240</td>
<td>53.2</td>
<td>27.5</td>
</tr>
<tr>
<td>85</td>
<td>28.1</td>
<td>26</td>
<td>255</td>
<td>56.2</td>
<td>28.5</td>
</tr>
<tr>
<td>105</td>
<td>31.3</td>
<td>26</td>
<td>269</td>
<td>59.2</td>
<td>28.5</td>
</tr>
<tr>
<td>128</td>
<td>34.9</td>
<td>26</td>
<td>280</td>
<td>61.7</td>
<td>29.0</td>
</tr>
<tr>
<td>167</td>
<td>40.0$\infty$</td>
<td>26</td>
<td>305</td>
<td>67.0</td>
<td>29.0</td>
</tr>
</tbody>
</table>
APPENDIX D

Some recent patents on equipment for electroosmotic dewatering:


"Dewatering of Slurries," J.G. Sunderland and C.L. Cacicedo; Britain #1,414,564, 19 November 1975; from Chem. Abs. 84, 11, 1328 (1975).
NOMENCLATURE

\[ a \] radius of capillary (L)

\[ a_p \] pore radius in cluster model (L)

\[ a_m \] coefficient of mass diffusivity (L^2T^{-1})

\[ a_v \] coefficient of thermal diffusivity (L^2T^{-1})

\[ A \] cross-sectional area in porous medium (Chap. 2)

\[ A \text{ or} \] dewatering surface area (Chap. 6 and 7)

\[ b \] constant relating the air leakage rate during vacuum dewatering to the level of vacuum (L^2T^{-1})

\[ c_F \] consolidation constant (L^2/T)

\[ c_K = C_k \] Kozeny constant

\[ c_m \] specific isothermal mass capacity (moisture capacity)

The following set of variables are molar concentrations (subscript "1" refers to cation, "2" to anion)

\[ C \] cation or anion in an electrically neutral fluid

\[ C_i \] ion species \( i \)

\[ C_i^0 \] ion species \( i \) at center of pore

\[ C_m = (C_1 - C_2) \] within a cluster

\[ C_0 \] single symmetrical binary salt in bulk fluid external to porous medium

The following set of variables are costs expressed in $/unit mass of sludge solid

\[ C_b \] maintenance in sand bed drying

\[ C_c \] capital amortization and interest
\( C_d \) disposal
\( C_e \) energy
\( C_f \) fuel for disposal by incineration
\( C_L \) labor
\( C_r \) maintenance:  
  a) \( C_{r_1} \) = maintenance and replacement of electrodes
  b) \( C_{r_2} \) = maintenance of other equipment
\( C_{rd} \) removal and disposal in sand bed drying
\( C_S \) chemical conditioning
\( C_V \) vacuum filtration

\( d = q_e/c_F \)

\( d_c \) average distance between particles in a cluster
\( D \) diffusivity of ions in a single symmetrical binary salt
\( D_c \) characteristic diameter of crack patterns in a drying sludge cake (Figure 6-15)

The following set of variables are void ratios (ratio of fluid volume to solid volume):

\( e_{av} \) average during consolidation
  in a cluster (Chap. 2)
\( e_c \) or
  at end of filtration period (Chap. 7)

\( e_{co} \) defined in Figure 2-8
\( e_{cr} \)

\( e_d \) at disposal
\( e_0 \) at beginning of consolidation
\( e_p \) pores in cluster model
\( e_1 \) adjacent to filter medium
E  total energy required per unit mass of solid during vacuum filtration and dewatering
f  fraction of time when vacuum filter is operating
fr  =  Cr2/Cc
F  Faraday's constant
Fo  Fourier heat transfer number  =  av·H2
Fo_m  Fourier mass transfer number  =  am·H2
g  gravitational constant
h  \[
      \begin{cases} 
        \text{thickness of porous medium (Chap. 2 and 3)} \\
        \text{or} \\
        \text{thickness of filter cake (Chap. 4-7)} 
      \end{cases}
\]
H  cake thickness during evaporative drying
Ho  initial height of sludge (or slurry) on the filter medium
H_S  height of sludge (or slurry) above the filter cake
H_S*  reference value of H_S during drainage

The following group of variables are current densities (current/unit area)
i  average in a porous medium
io  constant value of i from a regulated power supply
i†  within capillary

\[
   I \begin{cases} 
   \text{current through a capillary (Chap. 2)} \\
   \text{or} \\
   \text{amortization and interest rate (Chaps. 4,6,7)} 
   \end{cases}
\]
\[j_i\]  flux density of ion species
k  hydraulic permeability (L²)
k_e  electroosmotic permeability (C·m⁻¹)
\( k_0 \)  hydraulic permeability as predicted by Kozeny-Cormen equation
\( K_c \)  capital amortization and interest cost per unit area per unit time
\( K_{ce} = K_c(1 + f_r) + K_L \) for electroosmotic filtration
\( K_{cp} = K_c(1 + f_r) + K_L \) for pressure filtration
\( K_d \)  disposal cost per unit mass of cake
\( K_e \)  energy cost per unit energy
\( K_L \)  labor cost per unit area per unit time
\( K_m \)  cost of sand bed maintenance per unit area per cycle
\( K_{rd} \)  removal and disposal cost per unit area per unit time
\( \lambda \)  length of capillary
\( L \)  thickness of porous medium during consolidation
\( L_a \)  distance of fixed anode from filter medium
\( L_{11} \)  distance of measuring electrodes from filter medium (Figure 5-15)
\( L_{21} \)
\( m \)  number of capillaries per unit cross-sectional area of porous medium
\( M \)  mass of sludge solid dewatered per unit operating time
\( M_0 \)  average mass of sludge solid dewatering per unit time

The following set of variables are porosities (volume fraction of fluid)
\( n \)  total
\( n_c \)  in clusters
\( n_k \)  total in filter cake
\( n_o \)  in feed sludge (slurry)
\( n_p \)  in pores between clusters
\( n_{pk} \)  \( n_p \) in filter cake
\[ n_{ps} \quad n_p \text{ in sludge (slurry)} \]
\[ n_s \quad \text{total in sludge (slurry)} \]

\[ N \quad \text{number of particles in a cluster} \]
\[ p \quad \text{fluid pressure} \]
\[ p_s \quad \text{effective stress on solid matrix in a porous medium} \]
\[ P = \Delta p \quad \text{applied pressure during filtration} \]
\[ P_e \quad \text{electric power per unit area} \]
\[ P_i \quad \text{defined in Eq. (7-4)} \]
\[ P_0 \quad \text{applied pressure in consolidation} \]
\[ P_m \quad \text{maximum power output from vacuum pump} \]
\[ P_p \quad \text{hydraulic power per unit area} \]

The following set of variables are flow rates per unit cross-sectional area of a porous medium

\[ q \quad \text{total} \]
\[ q_e \quad \text{due to electroosmosis} \]
\[ q_h \quad \text{due to applied pressure} \]
\[ q_r \quad \text{due to gravity drainage} \]
\[ q_1 \quad \text{at the filter medium} \]

\[ q_m \quad \text{evaporation rate from surface of cake during constant rate period (ML}^{-2}T^{-1}) \]
\[ q_{me} \quad \text{effective value of } q_m \text{ which may include effect of cake cracking} \]
\[ r \quad \text{radial coordinate in a capillary} \]
\[ \bar{r} = r/a \]
radius of a cluster
flow resistance per unit length in cake during drainage (L^-2)
reference value of \( r_k \) at some reference head \( H_s^* \)
gas constant
ratio of the dewatering cost of electroosmotic filtration to that of pressure filtration (excluding disposal cost)
material coordinate (volume of solid per unit filter area) (L)
total solid volume per unit filter area (L)
specific surface area; surface area per unit volume of solid (L^-1)
= \( S_k \), mass fraction of solid in filter cake
mass fraction of solid in feed sludge
time
\( c_F t/s_0^2 \)
cycle time in sand bed drying
"down" time in sand bed drying
filtration time
time during which a cake is initially formed on the filter medium by gravity drainage or pressure filtration, before application of electric field
temperature
atmospheric temperature
surface temperature of sludge cake
local moisture content (mass of fluid per mass of solid)

The following group of variables are average moisture contents (mass of fluid per mass of solid)
general
at end of drainage period (Chap. 6)

or

at end of filtration period (Chap. 7)

\( U_{cr} = U_{cr1} \) = first critical moisture content

\( U_{cr2} \) second critical moisture content

\( U_d \) at disposal

\( U_e \) defined in Eq. (6-40) and Figure 6-16

\( U_f \) at equilibrium with ambient humidity

The following set of variables refer to fluid velocity

\( v \) within a capillary

\( \bar{v} \) average in capillary

\( v_p \) within the pores between clusters

\( \bar{v}_p \) average in pores

\( v_{1f} \) filtrate volume per unit filter area

\( v_{1f} = v_1(t_f) \)

\( v_{10} = v_1(t_0) \)

\( v_{se} \) velocity of a solid particle relative to fluid

\( v_{sc} \) velocity of a cluster relative to fluid

\( V \) volume flow rate in a capillary

\( V_p \) volume flow rate in a pore

\( w \) mass of sludge solid per unit area

\( w_d \) mass of cake per unit area at disposal

\( w_m \) minimum limit on \( w \) for discharge from vacuum filter belt
$W_e$  electrical energy per unit area
$W_p$  hydraulic energy per unit area
$W_f$  total per unit area during filtration
$x$  coordinate along mean direction of applied electric and/or pressure gradient
$y = a - r$
$Y = M/A = "yield" = mass of solid dewatered per unit area per unit time (ML$^{-2}$T$^{-1}$)
$z$  axial coordinate in capillary
$Z$  volume of ions in a single symmetrical binary salt
$\alpha = \rho_s \pi^2 c_F / 4$  (M$^2$L$^{-4}$T$^{-1}$)
$\alpha_x$  local specific resistance (LM$^{-1}$)
$\alpha_T$  average specific resistance for the filter cake (LM$^{-1}$)
$\alpha_0$  reference value of specific resistance at a pressure $P^*$ (LM$^{-1}$)
$\beta$  compressibility coefficient
$\gamma$  surface tension (MLT$^{-2}$)
$\gamma_0$  mass of solid per unit volume of cake
$\delta_v$  thermal gradient coefficient
$\varepsilon$  permittivity of fluid
$\zeta$  zeta potential
$\eta$  dimensionless measure of electroosmotic efficiency [Eq. (2-118)]
$\theta$  tangential coordinate (Chap. 2)
$\Theta_f$  portion of the filter drum immersed in the sludge (radians) (Fig.7-1)
$\Theta_f^*_f$  or moisture potential (Chap. 6)
\( \Theta_0 \) active portion of filter drum (radians) (Fig. 7-1)

\( \Theta_m \) upper limit on \( \Theta_f \) due to mechanical constraints

\( \kappa \) average electrical conductivity \( (\Omega^{-1}m^{-1}) \)

\( \kappa_k \) \( \kappa \) in a filter cake

\( \kappa_s \) \( \kappa \) in a slurry

\( \lambda \equiv \lambda_D / \alpha \) Debye length ratio

\( \lambda_c = \lambda_D / d_c \)

\( \lambda_D \) Debye length [Eq. (2-21)]

\( \lambda_p = \lambda_D / \alpha_p \)

\( \Lambda \) latent heat of vaporization

\( \mu \) fluid viscosity

\( \nu \) fluid kinematic viscosity

\( \rho \) fluid mass density \( (ML^{-3}) \)

\( \rho_s \) solid mass density \( (ML^{-3}) \)

\( \sigma_s \) effective surface charge density at shear plane in double layer

\( \sigma_k \) assumed constant value of \( \sigma_s \)

The following set of variables are electrical conductivities in the fluid \( (\Omega^{-1}m^{-1}) \)

\( \sigma_0 \) external solution

\( \sigma_c \) within clusters

\( \sigma_p \) in pores between clusters

All \( \tau \)'s refer to tortuosities

\( \tau \) in porous medium

\( \tau_c \) in cluster
τₖ in pores in filter cake
τₛ in pores in sludge

φ a component of the electric field which is independent of the radial coordinate in a capillary

\[ \Delta \phi = \begin{cases} \text{potential difference across a porous medium (Chap. 2 and 3)} \\ \text{or} \\ \text{potential difference between anode and filter medium (Chap. 4 and 5)} \end{cases} \]

Δφ₀ constant value of Δφ from a regulated power supply
Δφₖ voltage drop across filter cake
Δφₛ voltage drop across sludge (slurry)
Δφₑ voltage drop at electrodes due to polarization

The following group of variables refer to a component of the electric potential in a capillary which is radially dependent

ψ general variable
ψ₀ at the particle surface
ψₛ at the Stern plane

ψₛ effective surface potential, at the shear layer
ψₖ constant value of ψₛ --a special case

\[ \bar{\psi}, \bar{\psi}_s, \bar{\psi}_k = \frac{ZF}{RT} \times (\psi, \psi_s, \psi_k) \]

Ω speed of rotation of vacuum filter drum
Ωₜ defined in Eq. (7-81)
Ωₘ upper limit on Ω
LI-SHIANG LIANG
Biographical Note

Birthdate: 25 July 1948
Birthplace: Chao-Chou, Taiwan
Education: Phillips Academy, Andover, Massachusetts: 1963 - 1966
Massachusetts Institute of Technology, Cambridge, Mass.
1970  B.S. in Mechanical Engineering
1971  S.M. in Mechanical Engineering

Honors:  Pi Tau Sigma
Languages:  Knowledge of Spanish, French, Chinese
Personal Interests: Pottery, sculpture, sailing, camping