A-C MAGNETO HYDRODYNAMIC INSTABILITY

by

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Submitted to the Department of Electrical Engineering, February 7, 1977, in partial fulfillment of the requirements for the Degree of (Doctor of Science.)

ABSTRACT

A theory describing a purely hydromagnetic instability of a dense, viscous, electrically conducting fluid under the action of an alternating magnetic field is presented. Experiments on the onset and growth of instability in a planar layer of fluid are also reported.

In the theory, the effects of fluid motion and magnetic field on one another are found self-consistently, to linear terms. In determining the fluid response to the magnetic force, a "time average" model is used. Predictions of critical field strength and growth rates for the instability, are computed for a variety of boundary conditions. A rotor model which illustrates the physical mechanism of the instability is also developed.

The distinctive visual appearance of the magnetically caused motion is contrasted experimentally with the observed appearance of motion caused by thermal gradients and driven hydrodynamic turbulence. The critical field strength required for the onset of fluid motion is measured as a function of frequency of the applied field. These measurements agree with theoretical predictions over more than an order of magnitude variation in frequency. This frequency range includes low frequencies for which the finite depth of the fluid layer is very important, and higher frequencies for which it is relatively unimportant.

The critical field strengths determined here experimentally and theoretically set an upper limit on the extent to which a fluid layer can be levitated, depressed, or transported magnetically at a given frequency. This sets a practical limit on many forming and metalurgical processes.

The growth times observed experimentally differ greatly from those predicted theoretically. The former are typically between one and ten seconds, while the latter are typically a thousand seconds. No resolution of this conflict is found.

Experiments on the forced motion of a fluid layer caused by horizontal gradients in applied field are reported. A model is developed that includes the effects of turbulence. Qualitative agreement is found for both velocity and growth times.

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1. INTRODUCTION

1.1. Introduction

Alternating magnetic fields are used to heat, transport, and contain liquid metals in a variety of practical applications. In addition, plasma containment has been investigated via the use of liquid metal analogues. In many of these applications a spontaneous convection of the liquid metal occurs, and has a detrimental effect, either directly or through the effects of convection on heat and mass transfer. The subject of this thesis is this instability and its effects on heat transfer. In this first chapter observations of spontaneous motion reported in the literature are discussed. In each case the practical importance of the motion is discussed, along with any explanation for it put forward by the investigators.

The contention of this thesis is that the motion observed in these cases can be accounted for by purely hydromagnetic mechanisms. The pumping of a fluid by gradients in the magnetic field is described. A mechanism for hydromagnetic instability due to the imposed field is shown to exist in cases where no pumping occurs. In the final section of this
introductory chapter the basic mechanism for this instability is discussed qualitatively. A summary of the other chapters in this thesis is given as well.

1.2. **Levitation Melting.**

Induction melting is the most common application of high frequency magnetic fields to the processing of liquid metals. Typically the fields used are audio frequency or somewhat higher. Heating of the metal results from image currents induced in the metal by a surrounding coil. It is commonly observed that the metal is violently agitated by this process.\(^1\) A study of the magnetic pumping that occurs in practical configurations is in the recent literature.\(^2\) Levitation melting is a special case of this process in which the field is used to levitate as well as heat the sample. A schematic diagram of such a device is shown in Fig. 1.1. Alternating current in a coil beneath the sample causes the conducting metal to be repelled via the magnetic field and image currents. Levitation melting is commonly used to obtain samples of extreme purity.

In the first major developmental work on levitation melting Okress et al.\(^3\) reported the levitation of a variety of metal samples of different sizes and compositions. The samples in each case underwent spontaneous motion, described as a swimming of
the surface. This motion had a dramatic effect on the absorption of gas from the atmosphere surrounding the sample. Ordinarily a very slow process, the absorption of oxygen was particularly evident. This change in composition is very important metalurgically. Pfiefer\textsuperscript{4} reviewed the state of the art in levitation melting at a later date and remarked that this augmented gas transfer is very useful in the removal of gas, especially the removal of oxygen from iron. Sunderlam\textsuperscript{5} discussed the effect of the churning in producing samples of very uniform composition.

Much greater attention was directed to the motion itself in the work of Fraser et al.\textsuperscript{7} The idea was that the surface tension of the metal could be deduced from the frequency of its free surface oscillations. In the published analysis, hydromagnetic and other effects were neglected that would change the relation between frequency and surface tension. Experimentally, the frequency of oscillation was observed to be approximately 40 Hz., which is near the expected from the Rayleigh theory for the oscillation of a spherical surface.\textsuperscript{6} The motion described by Fraser, Okress, and others persisted much longer than the decay time associated with viscous damping. The motion would seem to have been driven by some agent.
Fraser et al. did not offer an explanation for this, but dismissed the possibility of electromagnetic forces because of the high frequency of the fields. This ignored the fact that these fields can exert net average forces either steady in time or at the low frequencies associated with fluid motion, for instance the force supporting the sample.

In the same article it was also reported that if the experiment was continued 'for a long time' the motion diminished to an imperceptible level. It is difficult to grasp the implication of this without knowing how long a time, and without knowing if any changes in the composition of the sample or its surface texture took place.

Aside from electromagnetic forces, other forces that could have driven the convection arise from the fact the metal is being heated from below, and unevenly. This could give rise to motion due to density gradients and gradients in surface tension, due to temperature differences. 8, 9

1.3. **Liquid Surface Shaping.**

An application similar to levitation is the shaping of liquid surfaces by magnetic fields. The difference is that here the coils are placed above a bath of fluid. The $J \times B$ force causes a downward
deflection of the fluid beneath the coils.

Colgate et al. reported the investigation of different configurations and surface shapes as analog to plasma confinement devices. The experiments occurred in a sealed chamber, and results were judged by the form the surface takes when cooled and solidified. No signs of instability were reported, though the possibility was discussed.

Deflection of a liquid surface by an alternating magnetic field was also the principle involved in a glass forming process developed by Melcher and Hurwitz. A layer of glass floated on a bath of liquid tin, the depth of the glass being governed by the surface tension and wetting angle of the glass, tin, and nitrogen atmosphere. Denting the tin would allow control of the glass thickness.

When this method was tried, spontaneous convection of the surface occurred, which unfortunately translated itself directly into ripples on the glass surface. This remains a very serious problem in the practical use of this method.

The fluid here was heated from above, but non-uniformly, so that while thermal instability of the Bernard type was not present, thermal pumping might have occurred. The non-uniformity of the fields could
also have caused magnetic pumping.

1.4. Hydromagnetic Surface Waves.

Schaffer,\textsuperscript{12,13} in a study on hydromagnetic surface waves, found that under certain conditions spontaneous motion could occur. The experiment consisted of a fluid resonator in which a field uniform in the horizontal directions was applied parallel to the surface of the liquid. At frequencies high enough so that the system dimensions (and hence wavelengths at resonance) were much greater than the magnetic skin depth, the field acted just as would a D.C. Field above a perfectly conducting fluid. However, with the skin depth comparable to the wavelength, it was found that as the field increased at some point spontaneous motion began to occur.

Thermal instability and pumping appear unlikely to have been present here, as the heating was very nearly uniform and was from above. The side walls enclosing the container may have caused some local heating or distortion of the field. This is because it was necessary for the currents flowing in the fluid to flow into the side walls; depending on the degree of wetting between the fluid and the wall, and the voltage drop due to contact resistance, there was some possibility of thermal pumping.
Schaffer discussed several explanations for this effect. The first was a complicated parametric model involving interactions of velocities generated at harmonics of the imposed frequency of the magnetic field. Also considered was the possibility of parametric acoustic pumping of bulk motion. These two were shown to be too weak to explain the observed motion. Schaffer himself attributed the motion to pumping by non-uniformities in the field caused by edge effects. This view was supported by experiments in which introducing inhomogeneities into the field increased the motion observed.

Left unexplained by this theory are two facts. The motion appeared to start at some critical field strength, and be absent below it. Pumped motion would have increased smoothly as the field was raised, and existed at any finite field strength. Also, the motion observed was oscillatory, not steady. This might be explained by turbulence in the pumped motion, but this possibility was not explored further by Schaffer. It is considered further in Chapter V here.

Some of Schaffer's theoretical results concerning surface waves are extended in Chapter IV here.
1.5. A Hydromagnetic Mechanism for Fluid Instability.

In Fig. 1-2 a thin shell cylindrical rotor is shown. An alternating magnetic field is applied perpendicular to its axis of rotation. In this case, which is analogous to that of a single phase induction machine, under certain conditions the rotor is unstable to small perturbations, that is, a small angular velocity will perturb the field and thereby cause a force that reinforces the original motion. The shell latches onto one of the travelling waves and follows it.14 This interaction is based on a pseudo-time averaging of the electrical forces.

A fluid layer can be pictured as breaking up into a series of such rotors, as in Fig. 1-3. Of course the analysis of such motion includes more than does that of a single rotor since the layer can pick a preferred wavelength and since the frictional losses experienced are determined by the exact velocity distribution. In addition, if the surface of the layer is allowed to move it perturbs the field in a way dissimilar to that of the rotor. Chapter 2 considers the motion of a thin shell rotor and discusses the analogies between that motion and the motions in a layer of fluid.
In Chapter 4 a linear stability theory is developed for the stability of a planar fluid layer. A model similar to that in Chapter 2 is introduced, in that the bulk motions of the fluid perturb the magnetic field. The perturbed field affects the fluid in a ponderable way only by the action of an electrical force averaged over a period long compared to the frequency of the imposed field. The effect of surface coupling to the field is also considered.

The analysis of Chapters 2, 3 and 4 suggests what essential features are required in an experiment to investigate the proposed hydromagnetic instability. In Chapter 5 these requirements and an apparatus designed to satisfy them are discussed, followed by the experimental results themselves.

The experiments are concerned with forced fluid motion, the onset of instability, and the effect of spontaneous convection on heat transfer. The appearance and behavior of the fluid in laminar and turbulent forced motion is contrasted with that of spontaneous convection.
CHAPTER 1 REFERENCES


Fig. 1-1. Levitation Melting Apparatus
\[ \bar{B}_0 = \text{Re} \left[ t_x \, \hat{A}_0 \, e^{i\omega t} \right] \]

Fig. 1-2. Rotor Model
\[ \hat{B}_o = i \sigma \Re(\hat{B}_o e^{i \omega t}) \]

Fig. 1-3. Rotor Picture of Fluid Continuum
2. A ROTOR MODEL FOR HYDROMAGNETIC INSTABILITY

2.1. Introduction.

The rotor model introduced in the previous chapter is developed here to give insight into the basic physical mechanism of hydromagnetic instability and pumping. The relative effects of fluid inertia and viscosity, the fluid response to imposed field pumping, the size scale and time constants of fluid instability are predicted in qualitative agreement with the bulk fluid theory of Chapter 3 (pumping) and Chapter 4 (fluid instability). In discussing experimental results in Chapter 5 the rotor model is also used to examine the roles of field inhomogeneity and turbulence in the experiment.

Section 2.2 of this chapter examines the stability of the rotor of Fig. 1-1. Because of the symmetry of the imposed field it is clear that an equilibrium solution exists in which no rotation occurs. However, under certain conditions this equilibrium is unstable; the rotor, having begun to rotate in one or another direction, continues to do so.

In section 2.3 the transient motion of a rotor is examined. The growth rates associated with the linear
theory of small rotations are developed, along with the non-linear saturation rotations. In addition the effect of field inhomogeneity, which causes an equilibrium torque on the rotor, is examined.

The final section of the chapter, section 2.4, discusses the connection of the previous two chapters to fluid motion. In electro-hydrodynamic problems lumped rotor models have been used previously to explain fluid behavior. Linear and non-linear fluid mechanics have been explained in this way. Thin sheet models for both electric and magnetic interactions are developed by Melcher.

2.2. Stability of a Thin Shell Rotor.

In Fig. 1-1 a thin cylindrical shell of conducting material is shown, subjected to an alternating magnetic field, that in the absence of the shell is uniform. The dimensions and coordinate system pictured there are those used in this section.

The alternating field is applied uniformly and in a direction perpendicular to the axis of rotation of the cylinder, far away from the cylinder. Without loss of generality this can be taken as the x direction:

\[
\bar{B}(r \to \infty) = i_x \text{Re}[B_0 e^{i\omega t}] \quad (2-1)
\]
For analysis in cylindrical coordinates it is convenient to split this field into two traveling wave components:

\[
\bar{B}(r \rightarrow \infty) = \text{Re}[\hat{B}_0 e^{j\omega t}] \text{Re}[(\hat{r} e^{j\theta} + j \hat{\theta} e^{j\theta})] \quad (2-2)
\]

\[
= 1/2 \text{Re}[(\hat{B}_0 e^{j\omega t + j\theta} + \hat{B}_0^* e^{-j\omega t + j\theta})(\hat{r} e^{j\theta} + j \hat{\theta} e^{j\theta})] \quad (2-3)
\]

To find the magnetic field everywhere the motion of the shell must be taken into account. For this purpose the shell is assumed to have a constant angular velocity \( \Omega \). The rotor is assumed to be statically unstable, that is, that instability occurs first as a steady exponential growth in velocity rather than as oscillatory motion. This assumption can be justified by considering the effects of unsteady motion on the magnetic field, and vice versa, are related self-consistently. The analysis is similar to that of this section, but involves considerably more manipulation. The condition for static instability is only that for small rotations the electrical torque just balance the viscous retarding torque.

In finding the fields the two travelling wave components are treated separately. The fields associated
with the \((-j\omega t+j\theta\)) dependence are labeled with a superscript \(\dagger\); those with the \((+j\omega t+j\theta\)) dependence are labeled \(\bar{}\). This convention follows the direction in which the waves travel.

In both cases the fields inside the conducting shell \((r<R)\) are labeled with a subscript \(b\). Those outside are labeled \(a\). Both inside and outside the shell there is no current density so that in both regions \((\bar{J}=0)\):

\[
\nabla \times \vec{H} = 0 \quad (2-4)
\]

\[
\nabla \cdot (\mu_0 \vec{H}) = 0 \quad (2-5)
\]

\(\vec{H}\) is thus the gradient of a scalar potential:

\[
\vec{H} = \nabla \psi \quad (2-6)
\]

\[
\nabla^2 \psi = 0 \quad (2-7)
\]

The potential is separated into its two travelling wave components:

\[
\psi = \text{Re}\{\hat{\psi}^+(r)e^{-j\omega t+j\theta}\} + \text{Re}\{\hat{\psi}^-(r)e^{j\omega t+j\theta}\} \quad (2-8)
\]
The solution applicable in each region for $\hat{\psi}^+$ is

$$\hat{\psi}^+_b = A_1 r$$

$$\hat{\psi}^+_a = A_2 \frac{R}{r} + \frac{B^*_O}{2\mu} r$$

These solutions make the field finite at $r = 0$ and make it approach the imposed field as $r \to \infty$.

The value of the fields immediately inside the cylinder are denoted by a superscript ($^\beta$); those immediately outside are denoted by a superscript ($^\alpha$). The shell is assumed to be thin enough that these values for the fields result by taking the values of the ($_b$) and ($_a$) fields respectively at $r = R$. The fields are, as with the potentials, assumed to be of the form:

$$\psi = \text{Re}\{\hat{\psi}^+ e^{-j\omega t+j\theta}\} + \text{Re}\{\hat{\psi}^- e^{j\omega t+j\theta}\}$$

Direct substitution of the potentials of Eqs. (2-9a) and (2-9b) allows elimination of the potential amplitudes:

$$\hat{B}^{\beta}_r = -j\mu \hat{H}^{\beta}_\theta$$

$$\hat{u}^{\alpha}_H = \hat{B}^{\alpha}_r - 2B^*_O$$
The boundary conditions across the shell are:

\[ \mathbf{n} \cdot \left[ \begin{array}{c} \mathbf{B} \\ \mathbf{H} \end{array} \right] = 0 \]

\[ \mathbf{n} \times \left[ \begin{array}{c} \mathbf{H} \\ \mathbf{H} \end{array} \right] = \mathbf{K} \]

The symbol '[[ ] A [ ]]' indicates \( A^\alpha - A^\beta \).

\( \mathbf{K} \) is the surface current in the shell, which in this case is all in the z direction. For these to be satisfied:

\[ \hat{B}_r^+ - \hat{B}_r = 0 \]

\[ \hat{H}_\theta^+ - \hat{H}_\theta = \hat{K}_+ \quad (2-13) \]

\( \hat{K}_+ \) is the surface current density in the cylinder.

The shell is considered thin enough that the current can be considered as a surface current. The constitutive law for the shell is:

\[ \hat{K}_+ = \sigma \Delta \hat{E}_+ \quad (2-14) \]

\( \hat{E}_+ \) is the electric field in the frame of the rotating shell. It is clear that if the shell rotates at a synchronous speed, that is, if \( \omega = \Omega \), there would be no electric field or current density. This is
because in the frame of the rotor the magnetic field would be constant in time. The transformation law for the electric field in a magnetic field system is:  

\[
\vec{E}' = \vec{E} + \vec{v} \times \vec{B}
\]  

(2-15)

\( \vec{E}' \) is the electric field in the transformed coordinates and \( \vec{v} \) is the material velocity. Using this transformation to go from the frame rotating at synchronous speed to that rotating at an angular velocity gives the electric field in the front of the rotor as:

\[
\hat{E}^+ = (\omega - \Omega) R \hat{B}^+_{r}^\alpha
\]  

(2-16)

The current density follows from Eq. (2-14):

\[
\hat{K}^+ = \sigma \Delta (\omega - \Omega) R \hat{B}^+_{r}^\alpha
\]  

(2-17)

It is now possible to solve for \( \hat{K} \) and \( \hat{B}^\alpha_{r} \) in terms of \( \hat{B}_o \), since Eqs. (2-10), (2-11), (2-12), (2-13), and (2-17) are a complete set of linear equations. The results are:

\[
\hat{B}^\alpha_{r} = \frac{2j}{\mu \sigma \Delta (\omega - \Omega) R + 2} \hat{B}_o
\]  

(2-18)
\[ K^+ = \frac{2j \sigma AR(\omega - \Omega)}{\mu \sigma (\omega - \Omega) R + 2j} \bar{B}_o \] (2-19)

This analysis has been for the \( ^+ \) fields. For the \( ^- \) fields, it is clear that the results will be the same with \( \bar{B}_o \) replaced by \( \bar{B}_o^* \) and \( \omega \) replaced by \( (-\omega) \). Defining a magnetic diffusion time \( \tau_m \), the fields can then be written as:

\[ \tau_m = \frac{\mu \sigma AR}{2} \] (2-20)

\[ \hat{B}_r^+ = \frac{1}{1 - j(\omega - \Omega) \tau_m} \frac{\bar{B}_o^*}{2} \] (2-21)

\[ \hat{K}^+ = \frac{2(\omega - \Omega) \tau_m}{1 - j(\omega - \Omega) \tau_m} \frac{\bar{B}_o}{2\mu} \] (2-22)

\[ \hat{B}_r^- = \frac{1}{1 + j(\omega + \Omega) \tau_m} \frac{\bar{B}_o}{2} \] (2-23)

\[ \hat{K}^+ = \frac{-2(\omega + \Omega) \tau_m}{1 + j(\omega + \Omega) \tau_m} \frac{\bar{B}_o}{2\mu} \] (2-24)

The torque exerted on the rotor by the electrical force has two components, one which is steady and one which has a frequency \( 2\omega \). It is assumed that the rotor responds to the constant or time average component, the
high frequency component being obliterated by the inertia of the shell. The time average torque, \( \Gamma \), results from directly integrating the \( \vec{J} \times \vec{B} \) force on the rotor:

\[
\Gamma^m = \frac{\omega}{2\pi} \int_0^{2\pi} \int_0^{2\pi} (K B_\alpha^r) \, d\theta \, dt \, R^2 \quad (2-25)
\]

\[
= \frac{|\hat{B}_\circ|^2 2\pi R^2}{4\mu} \left( \frac{(\omega - \Omega) \tau_m}{1 + [(\omega - \Omega) \tau_m]^2} - \frac{\ldots (\omega + \Omega) \tau_m \ldots}{1 + [(\omega + \Omega) \tau_m]^2} \right) \quad (2-26)
\]

\[
= \frac{|\hat{B}_\circ|^2 (\omega \tau_m) (2\pi R^2) \Omega}{2\mu \omega} \left( \frac{(\omega \tau_m)^2 - (\Omega \tau_m)^2 - 1}{[1 + (\omega \tau_m)^2 + (\Omega \tau_m)^2]^2} - 4(\omega \tau_m)^2 (\Omega \tau_m)^2 \right)
\]

The viscous torque for steady laminar rotation is established in Appendix E:

\[
\Gamma^v = -2\pi \eta R \Omega \quad (2-27)
\]

This torque applies only if the fluid velocity around the rotor has reached its final, fully developed distribution. For transient motion this amounts to neglecting the mass density of the fluid in comparison to that of the rotor. A more refined view, taking into account the diffusion of vorticity out from the rotor, is given in section 2.4.
The viscosity of the surrounding fluid is $\eta$.

The critical condition is that these two torques balance exactly:

$$0 = \Gamma^v + \Gamma^m$$  \hspace{1cm} (2-28)

$$2 = \frac{|\hat{B}_o|^2(\omega \tau_m)}{\mu \eta \omega} \cdot \frac{(\omega \tau_m)^2 - (\Omega \tau_m)^2 - 1}{[1+(\omega \tau_m)^2+(\Omega \tau_m)^2]^2 - 4(\omega \tau_m)^2(\Omega \tau_m)^2}$$  \hspace{1cm} (2-29)

For the incipience of motion $\Omega \tau_m$ can be taken as zero:

$$\frac{|\hat{B}_o|^2}{\mu \eta \omega} = \frac{2[1+(\omega \tau_m)^2]^2}{\omega \tau_m [(\omega \tau_m)^2 - 1]}$$  \hspace{1cm} (2-30)

The dimensionless group on the left side of Eq. (2-30) is the critical field parameter, and appears in the continuum fluid problem as well:

$$M = \frac{|\hat{B}_o|^2}{\mu \eta \omega}$$  \hspace{1cm} (2-31)

The dimensionless parameter $(\omega \tau_m)$ contains the radius, thickness, and conductivity of the shell and the frequency of the applied field. This dimensionless
number is the only way in which these parameters enter the stability problem. In Fig. 2-1, the critical value of $M$, termed $M^*$, is shown as a function of $(\omega \tau_m)$. The minimum value of $M^*$ is 8 and is reached at $(\omega \tau_m)$ equals $(1+\sqrt{2})$. This is shown by directly finding the zero of the derivative of Eq. (2-30) with respect to $(\omega \tau_m)$. Qualitatively, this result is expected on physical grounds. If $(\omega \tau_m)$ is very large, the shell appears to be a perfect conductor, completely excluding the field. The field can then exert no sheer stress on the shell. At the other extreme, for very low frequency the field completely penetrates the shell and acts essentially as a D.C. field which only damps any motion. At an intermediate frequency the maximum positive effect is achieved.

The steady value finally attained by the rotor for $M$ greater than $M^*$ is shown in Fig. 2-2. For computational convenience, the required value of $M$ is shown as a function of $(\Omega \tau_m)$.

The results of this section can be generalized to include the effects of finite shell thickness by using transfer relations across the thickness rather than directly applying the boundary conditions of Eqs. (2-12) and (2-13) and the constitutive law of Eq. (2-14). These transfer relations are developed by
Melcher, and include Bessel functions of complex arguments. The current distribution in a shell rotating in a uniform D.C. field was previously given by Perry.

The connection between the results of this section and the modelling of a planar layer is taken up in section 2.4 of this chapter. In Appendix E the viscous torque on a rotating cylinder is found, both for steady and exponentially varying angular velocity.

2.3. Transient Rotor Motion and the Effect of Field Inhomogeneity.

In this section the transient motion of the thin shell rotor of section 2.2. is discussed. The purpose is to gain insight into the time constant for acceleration that characterize bulk fluid motion. The effect of an unbalanced or inhomogeneous field, meaning a field in which the magnitude of the one travelling wave component exceeds that of the other, is also studied in order to shed light on the transient and steady response of a fluid layer to an applied inhomogeneous field.

The procedure for finding the fields is the same as in the previous case. The acceleration of the shell is assumed to be small enough that the current and field distribution is the same as if the rotation were steady. The 'time average' model for the action of the magnetic
force is adopted in the same spirit, the angular acceleration being assumed small compared to $\omega$, the frequency of the imposed field.

The imposed field is assumed to be of the form

$$\bar{B}(r \to \infty) = \text{Re}[(\hat{B}_+ e^{-j\omega t + j\theta} + \hat{B}_- e^{+j\omega t + j\theta})(\bar{I}_r + j\bar{I}_\theta)]$$  \hspace{1cm} (2-32)

The analysis is exactly that of the previous section, with $\hat{B}_o^*$ replaced by $\hat{B}_+$ and $\hat{B}_o$ by $\hat{B}_-:

$$\hat{B}_r^+ = \frac{1}{1-j(\omega-\Omega)\tau_m} \hat{B}_+$$  \hspace{1cm} (2-33)

$$\hat{K}_+ = \frac{2(\omega-\Omega)\tau_m}{1-j(\omega-\Omega)\tau_m} \frac{\hat{B}_+}{\mu}$$  \hspace{1cm} (2-34)

$$\hat{B}_r^- = \frac{1}{1+j(\omega+\Omega)\tau_m} \hat{B}_-$$  \hspace{1cm} (2-35)

$$\hat{K}_- = -\frac{2(\omega+\Omega)\tau_m}{1+j(\omega+\Omega)\tau_m} \frac{\hat{B}_-}{\mu}$$  \hspace{1cm} (2-36)

The 'time average' torque is again found by integrating $\bar{K} \times \bar{B}$ over the rotor and over a period of time $2\pi/\omega$. The result is:
\[
\Gamma_m = 2\pi R^2 \frac{\left( |\hat{B}_+|^2 + |\hat{B}_-|^2 \right)}{\mu \omega}.
\]

\[
\Omega(\omega \tau_m) \left[ (\omega \tau_m)^2 - 1 - (\omega \tau_m)^2 (\frac{\Omega}{\omega})^2 \right] + \gamma \omega (\omega \tau_m) \left[ 1 + (\omega \tau_m)^2 - (\omega \tau_m)^2 (\frac{\Omega}{\omega})^2 \right] \\
\left[ 1 + (\omega \tau_m)^2 + (\omega \tau_m)^2 (\frac{\Omega}{\omega})^2 - 4 (\omega \tau_m)^4 (\frac{\Omega}{\omega})^2 \right]
\]

\[
\gamma = \frac{\left( |\hat{B}_+|^2 - |\hat{B}_-|^2 \right)}{\left( |\hat{B}_+|^2 + |\hat{B}_-|^2 \right)}
\]

Again, it is assumed that the viscous drag on the shell is that associated with steady motion. This point is further discussed in the Appendix E. The torques acting on the shell are then:

\[
\Gamma_m = \frac{\left( |\hat{B}_+|^2 + |\hat{B}_-|^2 \right)}{\mu \omega} \{\Omega g(\omega \tau_m, \frac{\Omega}{\omega}) + \gamma \omega L(\omega \tau_m, \frac{\Omega}{\omega})\}
\]

\[
\Gamma^\nu = -2\pi R^2 \frac{\Omega}{\eta}
\]

The moment of inertia of the shell about its axis of rotation is:

\[
I_o = 2\pi \rho R^2 \Delta
\]
The mass density of the shell is \( \rho \).

The equation of motion for the shell is:

\[
2 \pi \rho R^3 \Delta \frac{d\Omega}{dt} = 2 \pi R^2 \frac{(|\hat{B}_+|^2 + |\hat{B}_-|^2)}{\mu \omega} \left\{ \Omega g(\omega, \frac{\Omega}{\omega}) + \gamma \omega L(\omega, \frac{\Omega}{\omega}) \right\} - 2 \pi R^2 \eta \Omega
\]

(2-42)

This equation is then put into dimensionless form:

\[
2 (\omega \tau) \frac{d\Omega}{dt} = (Mg - 2) \Omega + M\gamma L
\]

(2-43)

\[
M = \frac{2 (|\hat{B}_+|^2 + |\hat{B}_-|^2)}{\mu \eta \omega}
\]

(2-44)

\[
\Omega = \frac{\Omega}{\omega}
\]

(2-45)

\[
t = t \left( \frac{\omega \sigma \eta}{2 \rho} \right)
\]

(2-46)

There are four independent parameters to be specified in the equation of motion. The first is \( M \), the dimensionless field strength, and for a symmetric field the parameter that determines instability. The degree of field inhomogeneity is represented by \( \gamma \).
In addition, the initial value of $\Omega$ must be specified, as well as $\omega_m$ which specifies the rotor size.

The behavior of $\Omega$ in time is shown for different values of field strength in Figs. 2-3 and 2-4. For values of $M$ greater than the critical value for instability, $\Omega$ grows until it is comparable to 1, i.e., until a near-synchronous speed is reached. For values of $M$ less than the critical value, the curves asymptote to lower values and are always concave downwards.

The ultimate angular velocity, $\Omega_0$, attained by the shell is shown in Fig. 2-4 as a function of field strength for various values of $\gamma$. Initially the $\Omega_0$ increases relatively slowly with field strength, but as the value of $M$ passes that required for instability $\Omega_0$ increases very rapidly with $M$. Ultimately the approach to synchronism limits $\Omega_0$.

2.4. The Rotor As A Model for Bulk Fluid Behavior.

The rotor model provides an analogue to the perturbation of the imposed field by the current caused by the bulk $\vec{\nu} \times \vec{B}$ electric field. It does not have an analogue to the perturbation of the imposed field by a rippled surface. Thus, the rotor model corresponds most closely to bulk problems in which the surface is held flat.
The motions considered here then are those of Fig. 1-2. In order to apply the results of this chapter the coupling between adjacent cells is ignored. It is also necessary to guess values of \((\Delta/R)\) that hopefully correspond to the actual fluid cells. It is arbitrarily assumed here that \((\Delta/R)\) is 1/3. The optimum value of \(R\) then comes from the results of section 2.2.

\[
\omega \frac{\mu \sigma R A}{2} = 1 + \sqrt{2} \tag{2-47}
\]

\[
R \approx \frac{(3)(2.4)(2)}{\mu \sigma \omega} \tag{2-48}
\]

\[\approx 2.7 \delta\]

\[
\delta = \frac{2}{\omega \mu \sigma} \tag{2-49}
\]

Taken literally, this would imply that the optimum wavelength, corresponding to 4\(R\), would be about 10\(\delta\). In Chapter IV this in fact turns out to be the case. In addition, the rotor model predicts that for rotors below a certain size \((\omega \tau_m < 1)\) instability is impossible. This also agrees qualitatively with the continuum theory, in which it is found that for short wavelengths or thin layers instability does not occur.
The growth rate associated with bulk instability are also related to the bulk theory. The growth rate of an instability is seen from Eq. (2-43) with $\gamma$ set to zero and $\Omega$ set to zero in $g$. This last is to keep only linear terms in $\Omega$. $\Omega$ is then assumed to be exponential in time:

$$\Omega = \text{Re} [\Omega e^{s t}]$$  \hspace{1cm} (2-50)

Then directly from Eq. (2-43)

$$s = \frac{Mg - 2}{2\omega_{m}}$$  \hspace{1cm} (2-51)

$g$ is 0.25 for the optimum choice of $\omega_{m} = 1 + \sqrt{2}$. Thus for values of $M$ within a factor of two of that required for instability $s$ is of order unity, or:

$$s = \omega \left( \frac{\mu \sigma \eta}{2 \rho} \right)$$  \hspace{1cm} (2-52)

Typical bulk fluid properties are:

$\omega \sim 10^4$/sec.

$\mu \sim 10^{-6}$ H./m.

$\sigma \sim 2 \times 10^6$ mho/m.

$\rho \sim 10^7$ kg/m.$^3$
\[
\Gamma^V = - \frac{2\pi R \eta \Omega (\sqrt{\frac{\rho S}{\eta}})}{K_1 (\sqrt{\frac{\rho S}{\eta}})} [K_0 (\sqrt{\frac{\rho S}{\eta}}) + K_2 (\sqrt{\frac{\rho S}{\eta}})]
\]

This can be written for convenience as:

\[
\Gamma^V = -2\pi R^2 \eta \, g(s)
\]  \hspace{1cm} (2-60)

\[
g(s) = \frac{(\sqrt{\frac{\rho S}{\eta}}) [K_0 (\sqrt{\frac{\rho S}{\eta}}) + K_2 (\sqrt{\frac{\rho S}{\eta}})]}{K_1 (\sqrt{\frac{\rho S}{\eta}})}
\]

Assuming that \( \Omega \) takes the form of Eq. (2-56), the equation of motion, E(2-53), becomes:

\[
I_o s = -2\pi R \eta \, g(s) + MF(\omega_{m}) \pi R \eta
\]  \hspace{1cm} (2-62)

It is assumed that \( \omega_{m} \) is chosen to maximize \( F \), in which case:

\[
\omega_{m} = 1 + \sqrt{2}
\]  \hspace{1cm} (2-63)

\[
F = \frac{1}{4}
\]  \hspace{1cm} (2-64)

The critical value of \( M \) then follows from Eq. (2-62):

\[
M = (2g(s) + \frac{I_o s}{2\pi R \eta}) \cdot 4
\]  \hspace{1cm} (2-65)
The value of $s$ is typically $10^{-3}$/sec. and the growth time is approximately $10^3$ sec. The non-linear transient behavior of Figs. (2-3) and (2-4) also takes place on the same time scale. These growth rates are of the same order as those in section 2.2 of Chapter IV. These growth rates are discussed again in Chapter V, where it is noted that the phenomena observed experimentally appear much more quickly ($\sim$1/sec.).

The saturation values of $\Omega$, found in this chapter in many cases approach $\omega$. This is much greater than would be achieved in practice. Long before that rate of rotation, turbulence sets in. The early part of the time history of a rotor is shown in Figs. 2-5 and 2-6 for various values of $M$. It is clear that rotation speeds high in absolute terms can be achieved quickly for various values of $\gamma$, the field inhomogeneity. However, when this occurs there is no effect of a critical field strength.

In summary, the rotor model qualitatively predicts important features of the bulk continuum theory. The nature of the onset of instability, the importance of cell size, and the order of magnitude of growth rates agree. Sharing the same physics, this is perhaps not surprising. In both cases growth rates are a sore point, as both theories are widely divergent from experimental
results. Rotor models are discussed again in Chapter V in connection with the importance of turbulence in the experimental results.

An additional aspect of the fluid mechanics can be added to the rotor model by including the inertia of the surrounding fluid. The fact that the fluid must be accelerated leads to the formation of a boundary layer of fluid moving with the cylinder while the bulk remains stationary.

The rotor is assumed to be hollow and have a moment of inertia \( I_0 \) per unit length. The equation of motion is:

\[
I_0 \frac{d\Omega}{dt} = \Gamma^v + \Gamma^m
\]  

(2-53)

\( \Gamma^m \) is as given by Eq. (2-29):

\[
\Gamma^m = M \cdot F(\omega \tau_m) \cdot \pi R \eta \Omega
\]  

(2-54)

\[
F = \frac{\Gamma(\omega \tau_m)^2 - 1}{[1 + (\omega \tau_m)^2]^2}
\]  

(2-55)

In Appendix D the torque on a rotor whose angular velocity is exponential in time is found:

\[
\Omega = \Omega e^{st}
\]  

(2-56)
Rearranging this and defining a normalized $s$ gives:

$$s = s o R^2 \eta$$  \hspace{1cm} (2-66)

$$I_o = \frac{I_o}{2\pi R^3 \rho}$$  \hspace{1cm} (2-67)

$$M = (2g(s) + \frac{I_o}{2\pi R^3 \rho}) \cdot 4$$  \hspace{1cm} (2-68)

In Fig. 2-7 the $s$ that is a solution of Eq. (2-68) is shown as a function of $M$. $s$ is seen to increase more than linearly is $M$, while when the amount of mass to be accelerated is fixed by ascribing all of the inertia to the rotor the dependence is linear as in Eq. (2-51).

This tendency for the growth rate to at first increase faster than unity with $M$ is also consistent with the results of Chapter IV. In that chapter this issue and the subject of growth rates in general are discussed in more detail. The role of turbulence in attaining steady state more rapidly is discussed again in Chapter V, both for forced pumping and instability.
CHAPTER 2 REFERENCES


Fig. 2-1. Critical Field Parameter, $M$, vs. $\frac{\omega \mu \sigma R \Delta}{2}$
Fig. 2-2. Field Parameter, $M$, vs. Saturation Velocity
Fig. 2-3. Velocity Transient Behavior for various Values of M.
Fig. 2-4. Velocity Transient Behavior with Field Inhomogeneity.
Fig. 2-5. Velocity Transient Behavior with Field Inhomogeneity, at Very Small Times.
Fig. 2-6. Velocity Transient Behavior with Field Inhomogeneity, at Very Small Times.
Fig. 2-7. Field Parameter, M, vs. Growth Rate, S, including effects of fluid inertia, for various values of rotor inertia $I_o$. 
3. DRIVEN FLUID RESPONSE OF A PLANAR LAYER

3.1. Introduction

This chapter studies the motion caused in a planar layer of fluid by inhomogeneity in the imposed magnetic field. In addition to being important in understanding its direct effects on convection, the ideas introduced here are important in the more complicated problem of stability discussed in the next chapter.

In the following sections, the basic equations for fluid velocity and magnetic fields are developed, and then applied to two special cases in which solutions can be found analytically. The first of these is that in which the magnetic force is all in one of the longitudinal directions and a function of only the transverse coordinate. In the coordinate system of Fig. 3-1, the net force is all in the z-direction, but only a function of x. This problem is analogous to the electric problem of the Taylor pump, except that here the forces are distributed through the bulk of the fluid rather than applied at the surface of the fluid.

The second problem considered is that in which the field is sinusoidal in the transverse coordinate z, or a combination of sinusoidal and uniform fields.
This problem can be generalized to include any form of the imposed field via Fourier series or integral.

The final section discusses the role of turbulence, and how it modifies the previous results. Approximate results derived from a simple mixing length model are given.

3.2. Basic Equations

There are two fundamental approximations made in this chapter. The first is that the motion of the fluid does not affect the magnetic. The general equation governing the magnetic field is:

\[ - \frac{1}{\mu_0} \nabla^2 \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \]  

(3-1)

In essence, then, we are neglecting the term on the right hand side of Eq. (3-1). If the frequency of the imposed field is \( \omega \), this is valid if:

\[ \varepsilon = \frac{\mathbf{U}}{\omega \mathbf{l}} \ll 1 \]  

(3-2)

In this dimensionless ratio \( \nu \) is a typical velocity and \( \mathbf{l} \) is a typical length. In practice, values are approximately:
\[ U \sim 10^{-2} \text{ m/sec.} \]
\[ \lambda \sim 10^{-2} \text{ m/sec.} \]
\[ \omega \sim 10^4 \text{ /sec.} \]

With forced pumping present, there is a force causing fluid motion that is of zero order in the parameter \( \epsilon \). In the stability analysis of the next chapter, there is no zero order driving force, and it is necessary to consider the term on the right side of Eq. (3-1) in order to determine stability. It represents the feedback to the field of the fluid motion. In Chapter II it is shown that for the lumped parameter rotor this term is essential in predicting instability, but is unimportant if significant field inhomogeneity is present.

It is assumed then that the field in the fluid obeys Eq. (3-1) with \( \bar{v} \) set to zero. The imposed field can thus be determined by Eq. (3-1) and appropriate boundary conditions and then used to find the fluid response.

The equations governing the fluid response are:

\[ \rho \left( \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} \right) = \eta \nabla^2 \bar{v} - \nabla p + \nabla \cdot \bar{m} \tag{3-3} \]

\[ \nabla \cdot \bar{v} = 0 \tag{3-4} \]
\[ T_{ij}^m = \frac{1}{\mu} B_i B_j - \frac{1}{2\mu} \Delta_{ij} B_k B_k \] (3-5)

The fluid is assumed to be incompressible.

The second major assumption involves the electrical force. The magnetic field is in general of the form

\[ \vec{B} = \text{Re}\{B(x,z)e^{j\omega t}\} \] (3-6)

The stress tensor involves products of the field components:

\[
B_i B_j = \text{Re}\{\hat{B}_i e^{j\omega t}\} \text{Re}\{\hat{B}_j e^{j\omega t}\}
\]

\[
= \frac{1}{2} \text{Re}\{\hat{B}_i \hat{B}_j^*\} + \frac{1}{2} \text{Re}\{B_i B_j e^{2j\omega t}\}
\]

The basic assumption is that the fluid responds to the steady term and that the term with frequency \(2\omega\) is ironed out by the fluid inertia. In the situations studied experimentally here \(\omega\) is typically one to ten kilohertz. This type of model has been used previously in an electrohydrodynamic problem by Devitt and Melcher\(^2\) and in the previously cited study of magnetohydrodynamic surface waves by Schaffer\(^3,4\).
In the absence of turbulence, these steady magnetic stresses produce a steady fluid velocity. In the next two sections additionally the inertial stress, \( \rho \bar{v} \cdot \nabla \bar{v} \), is ignored. In the last section, it is considered approximately as a turbulent stress. Even in the absence of turbulence the inertial stress can make an important contribution. In the sinusoidally driven motion it can be important, and in the quasi-one-dimensional problem end effects lead to an inertial contribution.

3.3. Quasi-One-Dimensional Flow.

The problem considered in this section is one in which the magnetic field is tailored so that the stress on the layer causes a flow that is fully developed, i.e., that is all in the \( z \) direction and a function only of \( x \), again in the coordinates of Fig. 3-2.

It is assumed that \( \Delta \), the layer thickness, and \( s \), the skin depth in the fluid, are much less than the length \( l \) over which the tangential magnetic field, \( B_z' \), varies. In that case, and with the fluid backed as shown in Fig. 3-2, the tangential field is approximately given by:

\[
B_z \approx \Re \{ \hat{B}_z(z) e^{j \omega t} e^{(1+j)X/S} \} 
\]
This follows directly from Eq. (3-1). The normal field follows from Gauss' law for magnetic fields:

$$\nabla \cdot \vec{B} = 0$$  \hspace{1cm} (3-7)

$$\frac{\partial B_z}{\partial z} = - \frac{\partial B_x}{\partial x}$$  \hspace{1cm} (3-8)

$$B_x = \text{Re}[\frac{\partial B_z}{\partial z} e^{j \omega t \delta} \frac{e^{j (1+j) \delta}}{(1+j)^{\delta}}]$$

These field solutions are accurate to first order in ($\delta/\lambda$).

The 'time average' sheer stresses can now be directly computed:

$$T_{xz} = T_{zx} = \frac{1}{2\mu} \text{Re}[\frac{\hat{B}_z \partial B_z}{\partial z} \frac{\delta}{(1+j)^{\delta}} e^{\frac{2\lambda}{\delta}}]$$  \hspace{1cm} (3-9)

$$T_{xx} = - T_{zz} = \frac{1}{4\mu} \text{Re}[\left(\frac{1}{2} \frac{\partial B_z}{\partial z} \right)^2 - \hat{H}_z^2 e^{\frac{2\lambda}{\delta}}]$$  \hspace{1cm} (3-10)

To first order in ($\delta/\lambda$), Eq. (3-10) is just

$$T_{xx} = - \frac{1}{4\mu} \left|\hat{H}_z\right|^2 e^{\frac{2\lambda}{\delta}}$$  \hspace{1cm} (3-11)

The $x$ and $z$ components of Eq. (3-3) are
\[ \eta \frac{\partial^2 \nu_x}{\partial x^2} = \frac{\partial P}{\partial x} - \frac{\partial T_{xx}}{\partial x} - \frac{\partial T_{xz}}{\partial z} \]  
(3-12)

\[ \eta \frac{\partial^2 \nu_z}{\partial z^2} = \frac{\partial P}{\partial z} - \frac{\partial T_{zx}}{\partial x} - \frac{\partial T_{zz}}{\partial z} \]  
(3-13)

The pressure can be redefined, as is common in the analysis of incompressible flow:

\[ p^1 = P - T_{xx} \]  
(3-14)

It is now assumed that \( T_{xz} \) is made to be independent of \( z \). This is essential if a fully developed flow is to result. It is also clear that in that case \( \nu_x \) also vanishes. Eq. (3-12) then implies that:

\[ \frac{\partial p^1}{\partial x} = 0 \]  
(3-15)

For a fully developed flow it also follows from Eq. (3-13) that:

\[ \frac{\partial^2 p^1}{\partial z^2} = 0 \]  
(3-16)
It is now possible to proceed directly to Eq. (3-13)

\[ \eta \frac{\partial^2 \nu_z}{\partial x^2} = \frac{\partial P}{\partial z} + 2 \frac{\partial T_{xx}}{\partial z} - \frac{\partial T_{xz}}{\partial x} \]  

(3-17)

The magnetic stresses are substituted for from Eqs. (3-10) and (3-11):

\[ \eta \frac{\partial^2 \nu_z}{\partial x^2} = \frac{\partial P}{\partial z} + \mu \frac{2x}{\delta} \Re[H_z(z)] \frac{dH_z(z)}{dz} \]  

(3-18)

The magnetic field \( H_z \) must satisfy the condition that:

\[ \frac{\mu}{2} \Re[H_z(z)] \frac{dH_z(z)}{dz} = \text{constant} \]

\[ = K \]  

(3-19)

If \( H_z \) and \( \frac{dH_z}{dz} \) are in temporal phase, then:

\[ \frac{\mu}{2} \frac{d|H_z|^2}{dz} = K \]

(3-20)

Where \( z_o \) is a constant of integration. A physical method to impose such a magnetic field is discussed in
Chapter V.

It is now possible to write Eq. (3-18) as:

$$\eta \frac{\partial^2 v_z}{\partial x^2} = \frac{\partial P^1}{\partial z} + Ke \frac{2^X}{6}$$  \hspace{1cm} (3-21)

This can be integrated directly to give:

$$v_z = \frac{1}{2} \left( \frac{\partial P^1}{\partial z} \right) x^2 + \frac{K \delta^2 e}{4} \frac{2^X}{6} + A_1 x + A_0$$  \hspace{1cm} (3-22)

It is also required that the flow conserve mass:

$$0 = \int_{-\Delta}^{0} v_z(x) \, dx$$  \hspace{1cm} (3-23)

Also, the velocity must vanish at the lower surface:

$$v_z(x=-\Delta) = 0$$  \hspace{1cm} (3-24)

The viscous shear stress must vanish at the upper surface. This implies that:

$$\frac{\partial v_z}{\partial x} \bigg|_{x=0} = 0$$  \hspace{1cm} (3-25)
It is clear that Eqs. (3-23), (3-24), and (3-25) serve to determine the constants \( \frac{dP}{dz}, A_1, \) and \( A_0 \). The end result is an expression for \( v_z \):

\[
v_z(x) = \hat{v} \left\{ -3 + \frac{3}{2} \frac{\delta^2}{\Delta^2} (1-e^{-\frac{2\Delta}{\delta}}) - 3 \frac{\delta e^{-\frac{2\Delta}{\delta}}}{\Delta} \frac{x}{\Delta^2} ight. \\
+ 2 \frac{\delta}{\Delta} e^{\frac{2x}{\delta}} - 4 \frac{x}{\delta} \\
\left. + [-1 + \frac{3}{2} \frac{\delta^2}{\Delta^2} (1-e^{-\frac{2\Delta}{\delta}}) + \frac{\delta e^{-\frac{2\Delta}{\delta}}}{\Delta} \right] \right\} \tag{3-26a}
\]

\[
\hat{v} = \frac{K\Delta\delta}{4\eta} \tag{3-26b}
\]

In Fig. 3-3 this velocity profile is plotted for various values of \((\Delta/\delta)\). As \((\Delta/\delta)\) increases, the profile approaches that of the Taylor pump. This is expected, since in this limit the magnetic force acts essentially as a sheer stress.

In the above analysis it is assumed that the \( \rho \hat{u} \cdot \nabla \hat{u} \) term could be ignored. It is now clear that where the quasi-one-dimensional model applies it is in fact identically zero. It comes into play only when the flow becomes turbulent. However, it is a fact that for any system of finite length \( L \) the fluid has to turn around
at the ends to conserve mass. The relative importance of this can be qualitatively assessed by estimating the volume force densities and multiplying them by the volumes over which they act. If a unit length in the $y$ direction is assumed, the force required to turn the fluid around is approximately:

$$ F_\rho = \rho U \left( \frac{U}{\Delta} \right) \Delta^2 $$  \hspace{1cm} (3-27)

The overall viscous force is similarly:

$$ F_\eta = \eta \left( \frac{U}{\Delta} \right) L \Delta $$  \hspace{1cm} (3-28)

The ratio of these two is

$$ \theta = \frac{F_\rho}{F_\eta} = Re_\Delta \left( \frac{\Delta}{L} \right) $$  \hspace{1cm} (3-29)

$Re_\Delta$ is the Reynold's number based on the length $\Delta$ and is defined as usual by:

$$ Re_\Delta = \frac{\rho U \Delta}{\eta} $$  \hspace{1cm} (3-30)

For the analysis of this section to be applicable, $\theta$ should be much less than unity. In the experimental investigation of this topic this point is further
discussed. Also, in section 3.5. of this chapter the effect of turbulence in this problem is discussed.

3.4. Driven Sinusoidal Fluid Motion.

In this section two types of interactions are studied. In the first the imposed magnetic field is a pure standing wave in the z direction. In the second the field is almost uniform in z, with a slight standing wave added.

The first case can easily be generalized to that of two interacting standing waves. The fluid mechanics considered here are linear due to the neglect of inertial effects. With that provision, it is clearly possible to solve in the case of a magnetic field that is expressible as any sum of sinusoids, and thus the fluid response to an arbitrary field configuration is possible. This is not pursued here because it has no connection with the experiments discussed in Chapter V.

In the first problem, the tangential field at \( z = 0 \) is assumed to be:

\[
B_z(x=0) = \text{Re} [B_0 e^{j \omega t} \sin kz] \tag{3-31}
\]

The solution for the fields as a function of \( x \) results directly from the application of Eq. (3-1).
The fluid is backed by rigid material with identical electrical properties, as in Fig. 3-2. The fields for \( x < 0 \) are then:

\[
B_z = \text{Re}[B_0 e^{i\omega t} e^{\alpha \frac{x}{\delta}} \sin k z] \tag{3-32}
\]

\[
B_x = \text{Re}[\frac{-k \delta}{\alpha} B_0 e^{i\omega t} e^{\alpha \frac{x}{\delta}} \cos k z] \tag{3-33}
\]

\[\alpha = (2j + k^2 \delta^2)^{1/2} \tag{3-34}\]

The two components of Eq. (3-3) are:

\[
\eta \nabla^2 v_x = \frac{\partial P}{\partial x} - \frac{\partial T_{xx}}{\partial x} - \frac{\partial T_{xz}}{\partial z} \tag{3-35}
\]

\[
\eta \nabla^2 v_z = \frac{\partial P}{\partial z} - \frac{\partial T_{zx}}{\partial x} - \frac{\partial T_{zz}}{\partial z} \tag{3-36}
\]

These are combined with Eq. (3-4) to give

\[-\eta \nabla^2 v_x = \frac{\partial^2 T_{xx}}{\partial x \partial z} + \frac{\partial^2 T_{xz}}{\partial z^2} - \frac{\partial^2 T_{xz}}{\partial x^2} - \frac{\partial^2 T_{zz}}{\partial x \partial z} \tag{3-37}\]

The magnetic stresses are computed directly from Eqs. (3-5) and (3-6):
\[ T_{xx} = -T_{zz} = \frac{1}{4\mu} \frac{k^2}{|\alpha|^2} \left| B_0 \right|^2 e^{2\alpha_r^x} e^{2\alpha_r^y} \cos^2 k z - \left| B_0 \right|^2 e^{2\alpha_r^x} e^{2\alpha_r^y} \sin^2 k z \]

(3-38)

\[ T_{xz} = \frac{1}{2\mu} \text{Re} \left[ \frac{k \delta}{\alpha} B_0 \right] e^{2\alpha_r^x} e^{2\alpha_r^y} \cos k z \sin k z \]  

(3-39)

\[ \alpha_r = \text{Re}[\alpha] \]  

(3-40)

It is now possible to write Eq. (3-37) as:

\[ -\eta \nabla^2 v^2 \nu_x = \frac{\partial}{\partial z} \left[ 2 \frac{\partial^2 T_{xx}}{\partial x \partial z} + \left( \frac{\partial}{\partial z} \right)^2 - \left( \frac{\partial}{\partial x} \right)^2 \right] T_{xz} \]  

(3-41)

The term on the right is evaluated directly, taking advantage of trigonometric identities to sum the different products of \( \sin k z \) and \( \cos k z \). Terms that are constant in only augment the pressure, leaving the following:

\[ \eta \nabla^2 v^2 \nu_x = \frac{2 \left| B_0 \right|^2}{\mu \delta} \left[ \alpha_r \left( 1 + \frac{k^2 \delta^2}{|\alpha|^2} \right) - \left( \alpha_r^2 + k^2 \delta^2 \right) \right] \]

\[ \times e^{2\alpha_r^x} e^{2\alpha_r^y} \cos 2k z \]  

(3-42)

This equation can be solved directly for arbitrary layer thickness \( \Delta \), but for simplicity the solution given
here is for \( \Delta \) being infinite. In that case, the solution for the velocity is:

\[
\dot{v}_x = \left[ \dot{v}_{x\rho} e^{2\alpha r \delta} + A_1 e^{2kx} + A_2 kxe^{2kx} \right] \cos 2kx \quad (3-43)
\]

\[
\dot{v}_{x\rho} = \frac{|B_0| k^2 \delta^4}{2\eta \mu (\alpha^2 - k^2 \delta^2)} \left[ \alpha_r (1 + k^2 \delta^2) - (\alpha^2 + k^2 \delta^2) \right] \quad (3-44)
\]

The constant \( \dot{v}_{x\rho} \) is found directly by substitution of a solution of the appropriate form into Eq. (3-42). The constants \( A_1, A_2 \) are determined by the boundary conditions at the upper surface. The two conditions applied here are that the surface be flat and stress free. As commented on above, these two conditions are:

\[
\dot{v}_x(0) = 0 \quad (3-45)
\]

\[
\frac{\partial^2 v_x}{\partial x^2} = 0 \quad (3-46)
\]

These determine \( A_1 \) and \( A_2 \). The velocity is given by:

\[
\dot{v}_x = \dot{v}_{x\rho} \{ e^{2\alpha r \delta} - e^{2kx} + (1 - \frac{\alpha^2}{k^2 \delta^2}) kxe^{2kx} \cos 2kz \} \quad (3-47a)
\]
The quantity observed in an experiment is likely to be the tangential velocity at the surface. This is found by applying Eqs. (3-4) and (3-47). The result is:

\[ \nu_z(x=0) = \nu_{x\rho} \frac{\alpha r}{k\delta} \left( \frac{\alpha r}{k\delta} - 1 \right) \sin 2kz \]  

(3-47b)

In Fig. 3-4 \( \nu_{x\rho} \) is shown as a function of \( k\delta \). It is always negative, so that the flow is always from the regions where the magnitude of the field is greatest. This agrees with the analysis of the previous section.

The importance of the neglected \( \rho \vec{v} \cdot \nabla \vec{v} \) term can be estimated here as in section 3.3. The result is just:

\[ \frac{F_\rho}{F_\eta} \approx \text{Re}_\lambda \]  

(3-48)

\( \text{Re}_\lambda \) is the Reynolds's number based on a wavelength. The criterion for applicability of our results is again that this ratio be less than unity.

3.5. **Turbulent Effects.**

Liquid metals have very low viscosity and high density as compared to other liquids, such as water. For this reason in normal sized experiments in which the dimensions are typical 1-10 centimeters, the flow is turbulent for velocities greater than a few millimeters per second.
Because solutions to the basic fluid equations are unknown, many mathematical models for turbulent processes have been proposed. The simplest of these is the mixing length model, in which the viscosity in an essentially parallel flow is replaced by:

\[ \eta_T = \rho l_m^2 \left| \frac{\partial \nu}{\partial x} \right| \]  

(3-49)

The mixing length, \( l_m \) varies both with position in individual flow and between different types of flow. Prandtl suggested that it be the distance to the nearest wall in boundary layer flow. In general this substitution leads to a non-linear differential equation. In the quasi-one-dimensional flow of section 3.3, Eq. (3-21) is replaced by:

\[ \eta_T(x) \frac{\partial^2 \nu}{\partial x^2} = \frac{\partial P}{\partial z} + \frac{x}{2\delta} \]  

(3-50)

or

\[ \rho l_m^2(x) \frac{\partial \nu}{\partial x} \frac{\partial^2 \nu}{\partial x^2} = \frac{\partial P}{\partial z} + \frac{x}{2\delta} \]  

(3-50)

Direct solution of this equation must be numerical, even if a relatively simple assumption about the variation of \( l_m \) is made. The prospect of a non-linear boundary
value problem is unappealing. With that in mind the following simplification is made:

\[ \eta_T = \rho U \ell \] (3-51)

Again, \( U \) is a typical velocity and \( \ell \) is a typical length. In this case, the viscosity is once again a constant in space, the non-linearity entering in a trivial way. The problem can be solved with an unknown constant. Substituting for \( \eta \) by Eq. (3-51) then determines the actual value of \( U \).

In the case of the quasi-one-dimensional flow of section 3.3., it \( \nu \) is taken to be \( \hat{\nu} \) and \( \ell \) to be \( \Delta \), the layer thickness, the result is that \( \hat{\nu} \) is:

\[ \hat{\nu} = \frac{K \Delta \delta}{4 \eta} \quad \text{(Laminar creep flow)} \] (3-52)

\[ \hat{\nu} = \left( \frac{K \delta}{4 \rho} \right)^{1/2} \quad \text{(Turbulent flow)} \] (3-53)

Thus in the turbulent case the velocity is linearly dependent of the field strength as opposed to being quadratically dependent in laminar creep flow.

The validity of this approximate method is difficult to assess. In actuality the effective viscosity varies approximately linearly in \( x \) near the
lower surface, goes through zero at one point and then increases again. As discussed in reference 5, there is a real question as to the physical reality of such gyrations in effective viscosity. The approximation made here, and Eq. (3-53), are compared to the results of an experiment in Chapter 5, and are discussed further there.

In discussing the forced sinusoidal fluid motion, the applicability of any mixing length model is very questionable because the flow is inherently recirculating. More complicated theories may be applied to such a flow, but the solution is not attempted here. The significance of turbulence in explaining experimental results is taken up in Chapter 5.
CHAPTER 3 REFERENCES


Fig. 3-1. Fluid Layer and Field Configuration.

Fig. 3-2. Fluid Layer with Solid Backing.
Fig. 3-3. Velocity Distribution for Quasi-One-Dimensional Flow and Various Values of $\frac{\delta}{\Delta}$.
Fig. 3-4. Normalized Velocity Parameter vs. kδ for Sinusoidal Flow.
4. STABILITY OF A PLANAR FLUID LAYER UNDER A UNIFORM FIELD

4.1. Introduction

In the previous chapter on forced, or pumped, fluid motion, the idea of a time averaged electrical force producing a steady fluid motion was considered. In this chapter this idea is extended to the consideration of the stability of a planar layer under the action of a uniform AC field. In this case the imposed field causes no motion by itself; it is the perturbation of the imposed field by the fluid motion that causes a magnetic force than can reinforce or oppose that motion.

The second section of this chapter deals with internal modes, that is, modes in which the surface of the fluid is assumed flat, and the coupling of field and fluid velocity is all in the bulk of the fluid. The third section deals with the motion of a layer when all of the perturbation of the magnetic field is caused by disturbances of the fluid interface; these are called surface modes. In the fourth section the case where both effects are present is analyzed.

The equilibrium situation applicable in all three sections is shown in Fig. 4-1. The fluid layer is at rest, and the imposed magnetic field, uniform in the
transverse $y$ and $z$ directions, obeys the diffusion equation in the bulk of the fluid:

$$- \frac{1}{\mu \sigma} \frac{\partial^2 B}{\partial x^2} + \frac{\partial B}{\partial t} = 0$$  \hspace{1cm} (4-1)

$$\bar{B} = i_z B_e (x, t)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

$$B_x = B_y = 0$$

$$B_{ze} = \text{Re} \left[ \hat{B}_e (x) e^{j \omega t} \right]$$  \hspace{1cm} (4-2)

If the fluid layer is infinitely thick the equilibrium field is:

$$B_z = \text{Re} \left[ \hat{B}_o e^{j \omega t} e^{(1+j) \frac{x}{\delta}} \right]$$  \hspace{1cm} (4-3)

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}}$$

As stated above, the fluid is at rest in equilibrium; the electrical forces are balanced by the gradient of a pressure:
\[
\frac{dp}{dx} = \frac{dT_{xx}}{dx} - \rho g \tag{4-4}
\]

\[
P = T_{xx} - \rho x + C
\]

\[
T_{xx} = -\frac{p}{2\mu} z^2
\]

4.2. Internally Coupled Modes.

Bullard's equation and continuity of flux are

\[
- \frac{1}{\mu_0} \nabla^2 B + \frac{\partial B}{\partial t} = \nabla \times (\nabla x B) \tag{4-5}
\]

\[
\nabla \cdot \vec{B} = 0 \tag{4-6}
\]

To linear terms in the perturbation quantities:

\[
- \frac{1}{\mu_0} \nabla^2 B + \frac{\partial \vec{B}}{\partial t} = \nabla \times (\nabla x B e^{-z}) \tag{4-7}
\]

\[
\nabla \cdot \vec{B} = 0
\]

The Navier-Stokes equation and incompressibility condition are

\[
\rho \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \eta \nabla^2 \vec{v} - \nabla p + \nabla \cdot \nabla \vec{v} \tag{4-8}
\]

\[
\nabla \cdot \vec{v} = 0 \tag{4-9}
\]
\[ T_{ij}^m = \frac{1}{\mu} B_i B_j - \frac{\Lambda_{ij}}{2\mu} B_k B_k \]  

(4-10)

Again to linear terms in the perturbation variables:

\[ \rho \frac{\partial \bar{\nu}}{\partial t} = \eta \nabla^2 \bar{\nu} - \nabla p + \nabla \cdot \bar{v}^m \]  

(4-11)

\[ \nabla \cdot \bar{v} = 0 \]  

(4-12)

\[ \bar{v}^m = \begin{bmatrix}
-\frac{1}{\mu} B e b_z & 0 & \frac{1}{\mu} B e b_x \\
0 & -\frac{1}{\mu} B e b_z & \frac{1}{\mu} B e b_y \\
\frac{1}{\mu} B e b_x & \frac{1}{\mu} B e b_y & \frac{1}{\mu} B e b_z
\end{bmatrix} \]  

(4-13)

At this point, the governing bulk equations are coupled partial differential equations with coefficient functions of \( x \) and \( t \). The assumption is now made that the fluid motion is characterized by a complex frequency \( s \), with \(|s| \ll |\omega|\). Since the equations are constant in \( y \) and \( z \), they can be Fourier-analyzed in those variables. Thus the assumed form of \( v \) is:

\[ \bar{v} = \text{Re}[\bar{v}(x)e^{st-jkyy-jkzz}] \]  

(4-14)
Substitution of this form into Eq. (4-7) demands the following form for the perturbation flux density:

\[
\bar{b} = \text{Re} \left\{ b^+(x) e^{-(s+j\omega)t-jk_y y-jk_z z} + b^-(x) e^{-(s-j\omega)t-jk_y y-jk_z z} \right\}
\]  

(4-15)

Eqs. (4-7) can then be written:

\[
[D^2 - k^2 - \mu \sigma (s+j\omega)] \hat{b}_x^+ = \frac{1}{2} j k_z \mu \sigma \hat{B}_e^* (x) \hat{\nu}_x
\]

(4-16)

\[
[D^2 - k^2 - \mu \sigma (s-j\omega)] \hat{b}_x^- = \frac{1}{2} j k_z \mu \sigma \hat{B}_e^* (x) \hat{\nu}_x
\]

\[
[D^2 - k^2 - \mu \sigma (s+j\omega)] \hat{b}_y^+ = \frac{1}{2} j k_z \mu \sigma \hat{B}_e^* (x) \hat{\nu}_y
\]

(4-17)

\[
[D^2 - k^2 - \mu \sigma (s-j\omega)] \hat{b}_y^- = \frac{1}{2} j k_z \mu \sigma \hat{B}_e^* (x) \hat{\nu}_y
\]

\[
D \hat{b}_x^+ - j k_z \hat{b}_z^+ - j k_y \hat{b}_y^+ = 0
\]

(4-18)

\[
D \hat{b}_x^- - j k_z \hat{b}_z^- - j k_y \hat{b}_y^- = 0
\]

\[
D(\ ) = \frac{d}{dx}(\ )
\]

\[
k^2 = k_y^2 + k_z^2
\]
If the assumed form of the flux densities is substituted into Eq. (4-13), the result is, eg.:

\[ t_{xz} = \frac{1}{\mu} e^{b_x} \]

\[ = \frac{1}{\mu} \text{Re}[B_e(x)e^{j\omega t}] \text{Re}[b_x e^{(s+j\omega)z} + b_x e^{-(s-j\omega)z}] \]

\[ = \frac{1}{2\mu} \text{Re}[(b_x B_0^* + b_x B_e) e^{-jk_yz}] \]

\[ + \frac{1}{2\mu} \text{Re}[(b_x B_e e^{2j\omega t} + b_x B_e e^{-2j\omega t}) e^{-jk_yz}] \quad (4-19) \]

Thus, the stresses produced have one component at the fluid frequency \( s \) and another at that frequency plus or minus \( 2j\omega \). Just as in the case of pumped motion, the higher harmonic terms are ignored on the grounds that inertia will 'iron out' any response at such a high frequency.

Once the form of Eq. (4-19) is inserted into Eq. (4-13), with the second harmonic terms dropped, Eqs. (4-8) and (4-13) can be written:
\[ \rho s \ddot{v}_x = \eta (D^2 - k^2) \dot{v}_x - DP + \dot{D}_x - jk_z \dot{t}_{xz} \]

\[ \rho s \ddot{v}_y = \eta (D^2 - k^2) \dot{v}_y + jk_P - jk_t \dot{t}_{yy} - jk_z \dot{t}_{yz} \] (4-20a)

\[ \rho s \ddot{v}_z = \eta (D^2 - k^2) \dot{v}_z + jkP + \dot{D}_z - jk_y \dot{t}_{zy} - jk_{zz} \]

\[ D \ddot{v}_x - jk_y \ddot{v}_y - jk_z \ddot{v}_z = 0 \]

\[ \begin{bmatrix}
-\frac{1}{2\mu} (B e_z + B e_z) & 0 & \frac{1}{2\mu} (B e_x + B e_x) \\
0 & -\frac{1}{2\mu} (B e_z + B e_z) & \frac{1}{2\mu} (B e_y + B e_y) \\
\frac{1}{2\mu} (B e_x + B e_x) & \frac{1}{2\mu} (B e_y + B e_y) & \frac{1}{2\mu} (B e_z + B e_z)
\end{bmatrix} \]

(4-20b)

These equations apply in the bulk of the fluid.

In addition the solutions must also satisfy certain homogeneous boundary conditions. These will be discussed below.

Two types of internal modes can be distinguished.

In the first type, termed longitudinal, the normal velocity \( v_x \) vanishes everywhere. In the second, termed...
depth modes, it does not. The longitudinal modes are considered first. They are shown to be stable under a variety of boundary conditions.

\( \hat{v}_x \) is assumed to be identically zero. This implies \( \hat{b}_x^\pm \) also vanish. The electrical equations are:

\[
[D^2 - k^2 - \mu \sigma (s + j \omega)] \hat{b}_y^+ = \frac{1}{2} j k z B e \hat{v}_y
\]

\[
[D^2 - k^2 - \mu \sigma (s - j \omega)] \hat{b}_y^- = \frac{1}{2} j k z B^* \hat{v}_y
\]

\[
-j k \hat{b}_y^+ - j k \hat{b}_z^+ = 0
\]

\[
-j k \hat{b}_y^- - j k \hat{b}_z^- = 0
\]

Eqs. (4-19) are combined to eliminate \( \hat{P} \), and continuity used to eliminate \( \hat{v}_z \). Substituting in from Eq. (4-20) for the magnetic stresses, and using Eq. (4-22) to eliminate \( \hat{b}_z^\pm \), a final equation including only \( \hat{v}_y \) and \( \hat{b}_y^\pm \) is obtained:

\[
[\rho s - \eta (D^2 - k^2)] \hat{v}_y = -j k \frac{B^* b}{2 \mu} - j k \frac{B b^-}{2 \mu}
\]
The boundary condition applied to $\hat{v}_y$ at the upper $(x=x_1)$ and lower $(x=x_2)$ surfaces of the layer is one of the following:

1. The bound is a rigid wall. In this case the condition is:

$$\hat{v}_y = 0$$  \hspace{1cm} (4-24)

2. The bound is a free surface or slippery wall.
   In this case the sheer stress vanishes, i.e.:

$$S_{xy} = 0$$

$$\frac{\partial \hat{v}_y}{\partial x} = \eta \frac{\partial \hat{v}_y}{\partial x}$$

$$\therefore \: D\hat{v}_y = 0$$  \hspace{1cm} (4-25)

3. The upper surface may be connected to a half-space of insulating fluid with density $\rho_a$ and viscosity $\eta_a$. In this case, stress balance shows that

$$D\hat{v}_y + \frac{\eta_a}{\eta} (\gamma_{va} + k) \hat{v}_y = 0$$

$$\gamma_{va} = k^2 + s \rho_a / \eta_a$$  \hspace{1cm} (4-26)
where the root intended is that with a positive real part.

In each case the use of continuity, Eq. (4-19d) shows that an identical condition holds on velocities and sheer stresses in the z direction.

**Stability of Longitudinal Modes.**

It is now possible to incorporate the boundary conditions and bulk equations into a variational principle to establish the stability of these modes. This technique is often applied in stability problems of the Benard Type.¹

From Eq. (4-23) follows:

\[
\rho \int_{x_2}^{x_1} \frac{\nu}{\nu} \hat{d}x - \eta \int_{x_2}^{x_1} \nu (D^2 - k^2) \hat{\nu} \hat{d}x
\]

\[
= -\frac{jk_z}{2\mu} \int_{x_2}^{x_1} \nu (B^+ b^+ + B^- b^-) \hat{d}x
\]

\[
\int_{x_2}^{x_1} \nu \hat{\nu} \hat{d}x = \int_{x_2}^{x_1} \nu_y |\nu_y|^2 \hat{d}x
\]
\[
\int_{x_2}^{x_1} \nu_y D^* \nu_y \, dx = \left[ \nu_y D \nu_y \right]_{x_2}^{x_1} - \int_{x_2}^{x_1} |D \nu_y|^2 \, dx
\]

\[
= \psi - \int_{x_2}^{x_1} |D \nu_y|^2 \, dx \tag{4-29}
\]

Where \( \psi = 0 \) if Eq. (4-24) or Eq. (4-25) applies at \( x = x \), and if Eq. (4-26) applies there, then:

\[
\psi = \frac{\eta_a}{\eta_f} (\gamma v_a + k) |\nu_y(x_1)|^2 \tag{4-30}
\]

The boundary conditions imposed at \( x=x_1 \) and \( x=x_2 \) on \( \hat{b}_y^\pm \) is one of the following:

1. The bound is a perfect conductor. The tangential electric field at that point must then vanish, and it follows from Ampere's law that:

\[
\hat{D} b_y^+ = \hat{D} b_y^- = 0 \tag{4-31}
\]

2. The bound is a perfect insulator. The normal current must then vanish, and Ampere's law again demands:
\[ \hat{b}^+ = \hat{b}^- = 0 \]  \hspace{1cm} (4-32)

Using Eq. (4-21) it follows that:

\[
\int_{x_2}^{x_1} \nabla_y^{*} \hat{b}^+_y \, dx = \int_{x_2}^{x_1} \frac{2b^+_y}{-jk_z} (D^2 - \gamma^+_*) \hat{b}^{**}_y \, dx
\]

\[
= \frac{2}{-jk_z} [b^+_y D^+ b^+_y] - \int_{x_2}^{x_1} (|D b^+_y|^2 + \gamma^+_* |b^+_y|^2) \, dx
\]

\[
= \frac{2}{jk_z} \int_{x_2}^{x_1} (|D b^+_y|^2 + \gamma^+_* |b^+_y|^2) \, dx \hspace{1cm} (4-33)
\]

where:

\[ \gamma^+_* = k^2 + \mu \sigma (j \omega + \sigma) \]  \hspace{1cm} (4-34)

where the root intended is that with a positive real part. Similarly,

\[
\int_{x_2}^{x_1} \nabla_y^{*} \hat{b}^-_y \, dx = \frac{2}{jk_z} \int_{x_2}^{x_1} (|D b^-_y|^2 + \gamma^-_* |b^-_y|^2) \, dx \hspace{1cm} (4-35)
\]
where:

\[ \gamma_- = k^2 + \mu \sigma (s-j \omega) \]  \hspace{1cm} (4-36)

Inserting Eqs. (4-28), (4-29), (4-33), and (4-35) into Eq. (4-27):

\[
\begin{align*}
\rho s \int_{x_2}^{x_1} |\hat{\nu}_y|^2 dx + \eta \int_{x_2}^{x_1} (|D \hat{\nu}_y|^2 + k^2 |\hat{\nu}_y|^2) dx + \psi \\
= -\frac{1}{\mu} \int_{x_2}^{x_1} (|D \hat{b}_y^+|^2 + |D \hat{b}_y^-|^2 + \gamma_+ |\hat{b}_y^+|^2 + \gamma_- |\hat{b}_y^-|^2) dx \\
= -\frac{1}{\mu} \int_{x_2}^{x_1} (|D \hat{b}_y^+|^2 + |D \hat{b}_y^-|^2 + k^2 |\hat{b}_y^+|^2 + k^2 |\hat{b}_y^-|^2) dx
\end{align*}
\]

\[
-\frac{\mu \sigma s}{\mu} \int_{x_2}^{x_1} (|\hat{b}_y^+|^2 + |\hat{b}_y^-|^2) dx \hspace{1cm} (4-37)
\]

\[
-\frac{\mu \sigma j \omega}{\mu} \int_{x_2}^{x_1} (|\hat{b}_y^-|^2 - |\hat{b}_y^+|^2) dx
\]
Taking the real part of both sides of Eq. (4-37)

\[
\text{Re}[s] \left\{ \int_{x_2}^{x_1} \rho |\hat{v}_y|^2 \, dx + \int_{x_2}^{x_1} \sigma (|b_y^+|^2 + |b_y^-|^2) \, dx \right\} 
\]

\[
= -\text{Re}[\psi] - \eta \int_{x_2}^{x_1} (|\hat{D} \hat{v}_y|^2 + k^2 |\hat{v}_y|^2) \, dx \quad (4-38)
\]

\[
- \int_{x_2}^{x_1} \frac{1}{\mu} (|\hat{D} \hat{b}_y^+|^2 + |\hat{D} \hat{b}_y^-|^2 + k^2 |\hat{b}_y^+|^2 + k^2 |\hat{b}_y^-|^2) \, dx
\]

All of the above integrals are positive definite.

Referring to Eqs. (4-26) and (4-30), it is clear that:

\[
\text{Re}[\psi] \geq 0
\]

Eq. (4-38) thus implies that:

\[
\text{Re}[s] < 0
\]

These longitudinal modes are therefore always stable, at least for the boundary conditions considered.
**Depth Modes.**

Depth modes are those modes in which \( \nu_x \) is not always zero. In this section it is first shown that for boundary conditions leading to purely internal coupling the principle of the exchange of stabilities holds. A description of a numerical technique used to determine the point of instability and growth rates follows, along with the numerical results.

Eqs. (4-16), (4-17), (4-18), (4-20a), and (4-20b) govern the magnetic field and velocity in the bulk. Directly there follows:

\[
\rho s \hat{\nu}_x = \eta (k^2 - D^2) \hat{\nu}_x + D \hat{\rho} + \hat{f}_x
\]

\[
\rho s \hat{\nu}_y = \eta (k^2 - D^2) \hat{\nu}_y + jk \hat{z} + \hat{f}_y
\]  \hspace{1cm} (4-39)

\[
\rho s \hat{\nu}_z = \eta (k^2 - D^2) \hat{\nu}_z + jk \hat{x} + \hat{f}_z
\]

\[
\hat{f}_i = D \hat{t}_i - jk \hat{t}_y - jk \hat{t}_z
\]  \hspace{1cm} (4-40)

Taking the derivative (D) of the second and third parts of Eq. (4-39), multiplying the first by \((jk_y + jk_z)\), summing the results and applying continuity gives:
\[ \rho s (D^2 - k^2) \hat{\nu}_x = \eta (D^2 - k^2) \hat{\nu}_x + jk_y D \hat{f}_y + jk_z D \hat{f}_z - k^2 \hat{f}_x \]

(4-40)

The force terms are evaluated by directly substituting the magnetic fields into the stresses. Following that: (i) Eq. (4-18) is used to eliminate \( D b_x, b_y, \) and \( b_z; \) (ii) Eq. (4-17) is used to eliminate \( D^2 b_x; \) and (iii), Eq. (4-1) is used in the following form to substitute for \( D^2 B_e^e: \)

\[ D^2 B_e = j \omega \mu_0 \hat{B}_e \]

(4-41)

The result is an equation in \( \hat{\nu}_x \) and \( \hat{b}_x^\pm. \)

\[ \rho s (D^2 - k^2) \hat{\nu}_x = \eta (D^2 - k^2) \hat{\nu}_x = \eta (D^2 - k^2) \hat{\nu}_x + k^2 \frac{|B_e(x)|^2}{2} \sigma \hat{\nu}_x \]

\[ -\frac{j k_z}{2} \sigma B_e(x)(s+2j\omega) \hat{b}_x^+ \]

(4-42)

\[ -\frac{j k_z}{2} \sigma B_e(x)(s-2j\omega) \hat{b}_x^- \]

This and the first part of Eq. (4-17) form a complete set of coupled equations. Repeating that equation here for convenience:
\[ D^2 - k^2 - \mu \sigma (s + j \omega) \] \[ b^+_{x} = \frac{jk_z}{2} \mu \sigma B e^x \]

\[ D^2 - k^2 - \mu \sigma (s - j \omega) \] \[ b^-_{x} = \frac{jk_z}{2} \mu \sigma B e^x \]

These equations are now normalized for convenience:

\[ \delta = \frac{2}{\omega \mu \sigma} \]

\[ D = \delta D \]

\[ k = \delta k \]

\[ s = s/\omega Pr_m \]

\[ f(x) = \frac{\hat{B}_e(x)}{\hat{B}_o} \]

\[ \hat{b}^+_{x} = \frac{\hat{b}^+_{x}}{\hat{B}_o} \]

\[ \hat{b}^-_{x} = \frac{\hat{b}^-_{x}}{\hat{B}_o} \]

\[ \gamma^2_+ = k^2 + 2(s Pr_m + j) \]

\[ \gamma^2_- = k^2 + 2(s Pr_m - j) \]

\[ \nu_x = \frac{\nu_x}{(\mu \sigma \delta)} \]

\[ (D^2 - \gamma^2_+) b^+_{x} = \frac{jk_z}{2} f(x) \nu_x \]

\[ (D^2 - \gamma^2_-) b^-_{x} = \frac{jk_z}{2} f^*(x) \nu_x \]
\[\text{SPr}^{-1}_{m}(D^2 - k^2)\hat{\nu}_x = (D^2 - k^2)\hat{\nu}_x + k^2 M |f(x)|^2 \hat{\nu}_x\]

\[-2jk_z M \{ f^* (\text{Pr}_{m} s + 2j) \hat{b}^+_{-x} + f (\text{Pr}_{m} - 2j) \hat{b}^-_{-x} \}\]

(4-46)

\[\text{Pr}_m = \frac{\mu_0 \eta}{2 \rho} \quad M = \frac{|B_o|^2}{\mu_\eta \omega}\]

(4-47)

For the purposes of studying internally coupled modes, it is assumed that the following boundary conditions apply:

1. \(\hat{\nu}_x = 0\). Normal velocity at an interface leads to surface coupling and is the subject of the next two sections of this chapter.

2. Either \(D\hat{\nu}_x = 0\) (rigid wall) or \(D^2 \hat{\nu}_x = 0\) (slippery surface).

3. Either:
   
   a. \(\hat{b}^+_{-x} = \hat{b}^-_{-x} = 0\) (perfectly conducting wall). This follows directly from Faraday's law.
   
   b. \(\hat{D}\hat{b}^+_{-x} = \hat{D}\hat{b}^-_{-x} = 0\) (infinitely permeable wall). This follows directly from Ampere's law.
   
   c. \(\hat{D}\hat{b}^+_{-x} + kb^+_{-x} = \hat{D}\hat{b}^-_{-x} + kb^-_{-x} = 0\) (upper surface connected to vacuum). This follows from matching the magnetic field at the upper surface to a Laplacian field in the upper half-space.
Exchange of Stabilities for Depth Modes.

It will now be shown that under these boundary conditions the principle of exchange of stabilities holds.

Much as in the development of Eq. (4-37), Eq. (4-46) is multiplied on both sides by \( \hat{\psi}_x \) and integrated between the two boundaries:

\[
S \int_{x_2}^{x_1} \hat{\psi}_x (D^2 - \omega^2) \hat{\psi}_x \, dx = \int_{x_2}^{x_1} \hat{\psi}_x (D^2 - \omega^2) \hat{\psi}_x \, dx
\]

\[
-2jk_z M \int_{x_2}^{x_1} \hat{\psi}_x f(s+2j) \hat{\psi}_x \, dx \quad (4-48)
\]

\[
-2jk_z M \int_{x_2}^{x_1} \hat{\psi}_x f(s-2j) \hat{\psi}_x \, dx
\]

In the same manner as in the development of Eqs. (4-28) and (4-29), integration by parts and application of the boundary conditions gives:

\[
\int_{x_2}^{x_1} \hat{\psi}_x \frac{2\hat{\psi}_x}{-\psi_x} \, dx = -\int_{x_2}^{x_1} |D\hat{\psi}_x|^2 \, dx
\]
\begin{align*}
\frac{2}{jk_z} \{-[\hat{b}_x^+ \hat{b}_x^{+ \ast}] + \int_{x_2}^{x_1} (|\hat{b}_x^+|^2 + \gamma_+^* |\hat{b}_x^+|^2) dx\}\end{align*}

\begin{align*}
\frac{2}{jk_z} \{-[\hat{b}_x^+ \hat{b}_x^{+ \ast}] + \int_{x_2}^{x_1} (|\hat{b}_x^+|^2 + \gamma_+^* |\hat{b}_x^+|^2) dx\}
\end{align*}

Where

\begin{align*}
\varepsilon_+ = 0 \quad \text{if boundary condition 3(a) or 3(b)}
\end{align*}

applies

\begin{align*}
\varepsilon_+ = k|\hat{b}_x^+(x_1)|^2 \quad \text{if boundary condition 3(c)}
\end{align*}

applies

Similarly,

\begin{align*}
\int_{x_2}^{x_1} f_{\nu} \hat{b}_x^- dx = \frac{2}{jk_z} \{-[\hat{b}_x^- \hat{b}_x^{- \ast}] + \int_{x_2}^{x_1} (|\hat{b}_x^-|^2 + \gamma_-^* |\hat{b}_x^-|^2) dx\}
\end{align*}

\varepsilon_- \text{ is similarly defined.} \quad (4-50)

In the limit \(Pr_m \to 0\), which is taken here, it is clear that:
\[ \varepsilon_+ = \varepsilon_- \]

\[ |b_x^+|^2 = |b_x^-|^2 \]

\[ |D_{b_x}^+|^2 = |D_{b_x}^-|^2 \]

In experimental situations of interest,

\[ Pr_m \sim 10^{-6} - 10^{-7}. \]

Taking this limit, and inserting Eq. (4-49) and Eq. (4-50) into Eq. (4-48):

\[ -s \int_{x_2}^{x_1} (|D_{\nu_x}^+|^2 + k^2 |\nu_x^-|^2) dx = \int_{x_2}^{x_1} (|D_{\nu_x}^-|^2 + 2k^2 |\nu_x|^2) dx \]

\[ +k^4 |\nu_x|^2 dx + k^2 M \int_{x_2}^{x_1} |f|^2 |\nu_x|^2 dx \]

\[ -4M \int_{x_2}^{x_1} (|b_x^+|^2 + |b_x^-|^2) dx \]  \hspace{1cm} (4-51)
In this limit the \( \varepsilon \)'s and the terms involving \( \hat{b}_x^\pm \) cancel.

\[
\begin{align*}
\Sigma &= 4M \int_{x_2}^{x_1} \left( |\hat{b}_x^+|^2 + |\hat{b}_x^-|^2 \right) dx - k_z^2 M \int_{x_2}^{x_1} \left( f^2 |\hat{\nu}_x|^2 \right) dx \\
&- \int_{x_2}^{x_1} \left( |\hat{D}_{-\nu_x}|^2 + 2k_z^2 |\hat{D}_{\nu_x}|^2 + k_z^4 |\hat{\nu}_x|^2 \right) dx \\
&+ \int_{x_2}^{x_1} \left( |\hat{D}_{\nu_x}|^2 + k_z^2 |\hat{\nu}_x|^2 \right) dx
\end{align*}
\]

(4-52)

Since all of the integrals involved are positive definite, the principle of the exchange of stabilities holds. In fact, \( \Sigma \) is always real, not just at the point of incipience. It is also clear that instability is a real possibility, depending on the particular values of \( M \) and \( k_z \).

**Results of Stability Analysis for Depth Modes.**

The fact that terms involving \( \hat{b}_y^\pm \) do not couple back into the fluid equation causes the suspicion that the most critical situation is that in which \( k \) is all in the \( z \)-direction. This is in fact the case, as can
be shown by eliminating $\hat{b}_{x}^{\pm}$ in Eq. (4-46) using Eqs. (4-45). The result is of little use in itself, but all of the terms involving $M$ are multiplied by $k_z^2$, indicating that for a given value of $k$ the magnetic field will have its greatest effect when $k_z = k$. This situation is the one which is considered numerically.

The numerical problem is to find the values of $s$, $M$, and $k$ which allow non-trivial solutions of Eqs. (4-45), (4-46), and the boundary conditions. The parameters $M$ and $k$ must be real and both will be considered positive; since $s$ is real for problems of interest here. Interest is focused on: (a) the minimum value of $M$ which gives instability for each value of $k$; (b) the value of $k$, called $k^{*}$, which gives the smallest critical value of $M$, called $M^{*}$; and (c), the growth rates when $M$ is greater than $M^{*}$.

The technique used here consists of finding four linearly independent solutions to Eqs. (4-45) and (4-46) which satisfy the boundary conditions at the lower surface, and testing whether it is possible for a non-trivial linear combination of these solutions to satisfy the boundary conditions at the upper surface.

This technique is discussed in the context of hydrodynamic problems in Betchov and Criminale.
Each of the four solutions is determined up to a multiplicative constant, called A. The solutions can thus be written as:

\[
\begin{bmatrix}
\hat{\nu}_x \\
\hat{D}\nu_x \\
D^2\hat{\nu}_x \\
D^3\hat{\nu}_x \\
\hat{B}_x \\
\hat{D}\hat{B}_x \\
\hat{D}^-\hat{B}_x \\
D^-\hat{D}^-\hat{B}_x \\
\end{bmatrix} = [T]
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
\end{bmatrix}
\]

(4-53)

The matrix \([T]\) is a 4 \times 8. The boundary conditions at \(x = x\), in general demand that four linear combinations of \(\hat{\nu}_x \ldots D^-\hat{D}^-\hat{B}_x\) vanish at that point. This can be written as

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} = [R][T]
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
\end{bmatrix}
\]

(4-53a)
For a non-trivial solution to exist, it is clear that:

$$\det\{[R][T]\} = 0$$

The task thus divides itself into the following parts:

Step (1): Generate four independent solutions satisfying the boundary conditions at the lower surface. This is equivalent to computing $[T]$.

Step (2): Directly compute $\det\{[R][T]\}$.

Step (3): Vary the appropriate parameter in such a way as to approach a zero in $\det\{[R][T]\}$. This task is done here via the Secant method, but Newton's method or others could be used.\(^2\)

The method used to generate the four independent solutions in step (1) is different in two cases considered here. In the first, problem A, the fluid forms an infinite (lower) half space bounded above by an infinite half space of vacuum. In the second, problem B, the thickness of the fluid layer is finite.

Under the heading of problem A, two problems are considered. In both the surface is of course assumed flat; in the first the surface is assumed to be free of
sheer stress, while in the second tangential velocity is assumed to vanish at the surface.

For convenience, $X_1$ is taken to be zero in the consideration of problem A. The fluid boundary conditions are then:

$$\frac{\hat{\nu}}{\nu_x}(0) = 0 \quad (4-54)$$

Either

$$\frac{D^2}{\nu_x}(0) = 0 \quad (4-55)$$

or

$$\frac{D^2}{\nu_x}(0) = 0 \quad (4-56)$$

Continuity leads to Eq. (4-56):

$$\frac{D\nu_x}{\nu_x} - jk \frac{j\nu_y}{j\nu_y} - jk \frac{j\nu_z}{j\nu_z} = 0$$

Since $\hat{\nu_y}$ and $\hat{\nu_z}$ vanish, so must $D\nu_x$. Shear stress balance and continuity are combined in Eq. (4-55):

$$S_{xz} = \eta \left( \frac{\partial \nu_x}{\partial z} + \frac{\partial \nu_z}{\partial x} \right) = \eta \frac{\partial \nu_z}{\partial x} = 0$$

$$S_{xy} = \eta \left( \frac{\partial \nu_x}{\partial y} + \frac{\partial \nu_y}{\partial x} \right) = \eta \frac{\partial \nu_y}{\partial x} = 0$$

This implies

$$\frac{\partial}{\partial x} \left( \frac{\partial \nu_y}{\partial y} + \frac{\partial \nu_z}{\partial x} \right) = 0$$
or \[
\frac{\partial^2 \psi}{\partial x^2} = 0
\]

For an infinite lower half space, the equilibrium magnetic field takes the form of Eq. (4-3). Then Eqs. (4-45) and (4-46) become:

\[
(D^2 - \gamma^2) \hat{b}_x^+ = \frac{j k}{2} \frac{\nu}{\gamma} e(1+j) \frac{\hat{x}}{\nu}
\]

\[
(4-57)
\]

\[
(D^2 - \gamma^2) \hat{b}_x^- = \frac{j k}{2} \frac{\nu}{\gamma} e(1-j) \frac{\hat{x}}{\nu}
\]

\[
\therefore (D^2 - k^2) \hat{\psi} = (D^2 - k^2) \frac{\hat{\psi}}{\nu} + k^2 \operatorname{Me}^2 \frac{\hat{x}}{\nu}
\]

\[
+ 4k \frac{\nu}{\gamma} (e(1-j) \frac{\hat{x}}{\nu} + e(1+j) \frac{\hat{x}}{\nu})
\]

\[
(4-58)
\]

The next step is to generate four linearly independent solutions to these equations that vanish as \(x\) goes to minus infinity.

Intuitively, one form these solutions can take can be seen by considering the limit of the above equations as \(M\) vanishes. In that case there are four decaying solutions, exponential in \(x\). The four solutions are:
\[ (1) \quad \hat{v}_x = e^{kx} ; \quad \hat{v}_x = k e^{kx} \]

\[ \hat{b}_x^+ = \frac{jk_z}{2[(k+1+j)^2 - \gamma_+^2]} e^{(k+1+j)x} \]

\[ \hat{b}_x^- = \frac{jk_z}{2[(k+1-j)^2 - \gamma_-^2]} e^{(k+1-j)x} \]

\[ (2) \quad \hat{v}_x = \gamma_v e^{kx} \; ; \; \hat{v}_x = \gamma_v e^{kx} \quad \gamma_v = (k^2 + s)^{1/2} \]

\[ \hat{b}_x^+ = \frac{jk_z}{2[(\gamma_v+1+j)^2 - \gamma_+^2]} e^{(k+1+j)x} \]

\[ \hat{b}_x^- = \frac{jk_z}{2[\gamma_v+j-j)^2 - \gamma_-^2]} e^{(k+1-j)x} \]

\[ (3) \quad \hat{b}_x^+ = \gamma_+ e^{kx} \]

\[ \hat{b}_x^- = \hat{v}_x = 0 \]

\[ (4) \quad \hat{b}_x^- = \gamma_- e^{kx} \]

\[ \hat{b}_x^+ = \hat{v}_x = 0 \]
These are the solutions that exist when $M$ is zero. If $M$ were finite but very small, the terms of first order in $M$ could be found for each mode by putting the zero order magnetic fields into Eq. (4-58), finding the first order velocity, then putting that velocity into Eqs. (4-57) and obtaining the first order magnetic fields. It is clear from the form of Eq. (4-58) that if the zero order velocity is of the form:

$$\hat{v}_x = e^{-\frac{P x}{x_0}}$$

then the first order term will be of the form

$$\hat{v}_{x1} \propto e^{(P+2)x}$$

This procedure can be continued to find the terms of second, third, and higher order in $M$. Explicitly, this can be written as:

$$\hat{v}_x = A_n \sum_{\ell=0}^{\infty} D_\ell e^{-\frac{P x}{x_0}} e^{2\ell x}$$  \hspace{1cm} (4-59)$$

$$\hat{b}_x = A_n \sum_{\ell=0}^{\infty} C_\ell^+ e^{-\frac{P}{n+1+j}x} e^{2\ell x}$$  \hspace{1cm} (4-60)$$
\[
\hat{b}_x^- = A_n \sum_{\ell=0}^{\infty} C_{\ell}^- e^{[P_n+1-j]x} e^{2\ell x} 
\]
(4-61)

\[
C_{\ell}^+ = \frac{j k z}{2[(P_n+1+j)^2-\gamma_+^2]} D_{\ell} 
\]
(4-62)

\[
C_{\ell}^- = \frac{j k z}{2[(P_n+1-j)^2-\gamma_-^2]} D_{\ell} 
\]
(4-63)

\[
D_{\ell+1} = \frac{k_z^2 M D_{\ell} + 4M(C_{\ell}^+-C_{\ell}^-)}{8[(P_n+2\ell)^2-k^2]-[(P_n+2\ell)^2-k^2]^2} 
\]
(4-64)

The four modes are specified by:

Mode (1) \(D_0 = 1, \ P_1 = k\)  

Mode (2) \(D_0 = 1, \ P_2 = \gamma_v\)  

Mode (3) \(D_0 = C_0^- = 0, \ C_0^+ = 1, \ P_3 = \gamma_+-(1+j)\)  

Mode (4) \(D_0 = C_0^+ = 0, \ C_0^- = 1, \ P_4 = \gamma-(1-j)\)  

In this manner the values of \(\hat{v}_x^+\) and \(\hat{b}_x^\pm\) and their derivatives at \(x=0\) can be generated as a simple power series in \(M\). This method is much more economical than numerically integrating the system of equations, Eqs. (4-57) and (4-58). The different series involved
are all strongly convergent, as can be seen by examining the recurrence relations Eqs. (4-63), (4-64), and (4-65). For most cases of interest here, eight to ten terms of the series were adequate to provide nine place accuracy. This rapid convergence, plus the fact that it is not necessary to repeatedly evaluate transcendental functions, give this method its great economy.

Once the four independent solutions are generated, the matrix \([R][T]\) and its determinant are calculated. The determinant is evaluated via a computer program written by Dudley\(^4\) which uses a pivotal condensation scheme.

The four independent solutions generated here are only independent for \(s\) not equal to zero. For this reason the critical value of \(M\) cannot be found directly by setting \(s\) to zero and varying \(M\) to make the determinant zero. It is necessary to consider small values of \(s\), find the appropriate values of \(M\), and interpolate to \(s\) equals zero. In Fig. 4-2 the results of this calculation are shown for the case of the flat, stress free surface. What is shown is the minimum value of \(M\) for each value of \(k\) and \(s\). There are in fact other, higher values of \(M\) that are also solutions. These correspond to solutions in which \(\hat{v}_x\) and \(\hat{b}^+_x\) vary more quickly in \(x\). Similar results are shown in Fig. 4-3
for the case of the flat, rigid surface. In both cases there is a definite value of $k$ at which a minimum value of $M$ is required for instability to occur. In the stress free case this minimum value of $M$ is approximately 106, and occurs at $k$ approximately 0.4.

Other boundary conditions at $x=0$ can be considered. Changes cause differences in the critical values of $M$, but not in the qualitative shape of the $M$ vs. $k$ curve.

The second problem, called problem B, concerns cases in which the thickness of the layer is finite. In that case, the form of the normalized equilibrium magnetic field will no longer be a simple exponential. This precludes the use of the numerical method used in problem A to generate the four independent solutions to the bulk differential equations. It is necessary instead to integrate Eqs. (4-57) and (4-58) numerically. There are a variety of methods available for this purpose. The one used here is the Adams-Bashford-Moulton predictor-corrector algorithm. The boundary conditions considered are simple enough that it is possible to state initial conditions at the lower surface for each of the four independent solutions. Once these are obtained the same methods apply as in problem A.

Two problems in which the layer has finite depth are considered here. In the first, the upper surface,
at $x=0$, is flat and stress free. The lower surface, at $x=-\Delta$, is rigid and perfectly conducting. The results indicate that for $(\Delta/\delta)$ greater than four the results are substantially those of the similar problem considered in problem A.

The second problem considered is that of a layer bounded above and below by rigid, insulating walls. The results are shown in Fig. 4-4 for several values of $(\Delta/\delta)$. Growth rates computed in both problems are of the same order as in the problems considered in problem A.

The results in all the problems of this section have similarities to the results of the rotor stability problem of Chapter II. There is in each case a critical cell size. In most cases the critical wavelength is in fact on the order of $10\delta$. The growth rates are also comparable to those predicted by the rotor theory.

The critical value of $M$ for a flat, stress free surface is 106. For most liquid metals this implies that instability occurs when an attempt is made to deflect a liquid surface more than a few millimeters with an audio frequency field. Typically the maximum depression attainable is

$$h = \frac{B^2}{4\mu g} = M \cdot \left(\frac{n\omega}{\rho g}\right)$$
\[ \mu = 10^{-6} \text{ H/m.} \]
\[ \omega = 10^4 \text{ /sec.} \]
\[ \rho = 10^4 \text{ kg/m}^3 \]
\[ g = 10 \text{ m/sec}^2 \]
\[ \eta = 10^{-3} \text{ kg/m.-sec} \]
\[ h = 2 \times 10^{-3} \text{ m.} \]

In the majority of applications concerned with depressing a liquid surface instability is present.

The results of critical M for layers of finite depth are substantially those of a layer of infinite depth when the thickness, \( \Delta \), is greater than three or four skin depths. As the thickness decreases beyond that point the critical value of M increases and at some point, typically when \( \Delta \) is approximately equal to \( \delta \), it becomes infinite.

4.3. **Surface Coupled Waves.**

In the previous section, the surface of the layer is assumed to be flat, and all of the perturbation magnetic field is caused by bulk \( \vec{v} \times \vec{B} \) electric fields. In this section, the other extreme will be considered. Here, the perturbation field is caused entirely by the rippling of the surface, with bulk coupling ignored. When this approximation is made consistently throughout,
the governing equations result from Eqs. (4-16), (4-19), and (4-20):

\[ [D^2 - k^2 - \mu \sigma (s + j\omega)] \hat{b}_x^+ = 0 \]  \hspace{1cm} (4-66)

\[ [D^2 - k^2 - \mu \sigma (s - j\omega)] \hat{b}_x^- = 0 \]  \hspace{1cm} (4-67)

\[ \rho_s (D^2 - k^2)^\nabla_x = \eta (D^2 - k^2)^2 \nabla_x \frac{jk}{2\mu} e^{(\mu \sigma s + 2j\omega \mu \sigma)} \hat{b}_x^+ \]

\[ - \frac{jk}{2\mu} e^{(\mu \sigma s - 2j\omega \mu \sigma)} \hat{b}_x^- \]  \hspace{1cm} (4-68)

The layer is assumed to be of infinite depth throughout this section.

The boundary conditions applicable when the surface deforms are considerably more complicated than when the surface is flat. The electrical boundary conditions are considered here first.

The upper surface of the layer is assumed to be bounded by a half-space of vacuum. The perturbation magnetic fields in the vacuum are then given by the gradient of a scalar potential which satisfies Laplace's equation. Since the magnetic field is sinusoidal in \( y \) and \( z \), with wavenumbers \( k_y \) and \( k_z \) it follows that:

\[ \frac{\partial b_x}{\partial x} = -kb_x \]
Subscript (a) indicates the area above the fluid. Since there are no surface currents or magnetic materials, the following must be true:

\[
\tilde{\eta} \cdot \left[ B \right]_{x=\xi} = 0 \quad (4-69)
\]

\[
\tilde{\eta} \times \left[ B \right]_{x=\xi} = 0 \quad (4-70)
\]

The ' [ ] [ ] ' indicates the difference in the enclosed quantity between the upper and lower regions, and \( \xi \) is the x coordinate of the surface of fluid. If the surface is flat, \( \xi = 0 \). The unit vector normal to the surface is \( \tilde{\eta} \), and is given to linear terms by:

\[
\tilde{\eta} = \hat{i}_x + \frac{\partial \xi}{\partial y} \hat{i}_y + \frac{\partial \xi}{\partial z} \hat{i}_z
\]

To linear terms in \( \xi \), Eqs. (4-71) and (4-48) are:

\[
\left[ B \right]_{x} = 0 \quad (4-72a)
\]

\[
\left[ B \right]_{y} = 0 \quad (4-72b)
\]

\[
\left[ B \right]_{z} = -\xi \frac{\partial B_{ef}}{\partial x} \bigg|_{x=0} \quad (4-72c)
\]
Subscript $f$ means in the fluid. The Eqs. (4-49) can be combined as:

\[
\begin{align*}
\left[ \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial x} \right] &= \frac{\partial \xi}{\partial z} \frac{\partial B_{ef}}{\partial x} \\
|_{x=0} \\
\left[ \frac{\partial b_x}{\partial x} \right] &= -\frac{\partial \xi}{\partial z} \frac{\partial B_{ef}}{\partial x} \\
|_{x=0} \\
\frac{\partial b_{xf}}{\partial x} + k_b \frac{b_{xf}}{x} &= \frac{\partial \xi}{\partial z} \frac{\partial B_{ef}}{\partial x} \\
|_{x=0}
\end{align*}
\]

(4-73)

In terms of the Fourier-analyzed variables $[]$, the boundary condition at $x=0$ is:

\[
\begin{align*}
\hat{D}b_x^+ + k_b \hat{b}_x^+ &= -jk_z \frac{\nu_x}{s} \frac{d\hat{B}_e}{dx} \\
(4-74) \\
\hat{D}b_x^- + k_b \hat{b}_x^- &= -jk_z \frac{\nu_x}{s} \frac{d\hat{B}_e^+}{dx} \\
(4-75)
\end{align*}
\]

Use is made here of the fact that, to linear terms,

\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= v_x(x=0) \\
\hat{s}_\xi &= \hat{v}_x(0) \\
(4-76)
\end{align*}
\]
The mechanical boundary conditions result from consideration of force balance for the interface:

\[ [S_{ij}] n_j + [T_{ij}] n_j + \eta_1 \gamma \left( \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right) = 0 \]  (4-77)

The surface tension is \( \gamma \), and \( S_{ij} \) is the viscous stress tensor, given by:

\[ S_{ij} = -P \delta_{ij} + \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]  (4-78)

In equilibrium:

\[ T_{xx}^e = T_{yy}^e = -T_{zz}^e = -\frac{1}{2\mu} B e^T e \]  (4-79)

\[ S_{xx}^e = S_{yy}^e = S_{zz}^e = -P^e = -T_{xx}^e + \rho gx \]  (4-80)

\[ T_{ij}^e = 0 \ , \ i \neq j \]  (4-81)

\[ S_{ij}^e = 0 \ , \ i \neq j \]  (4-82)

The \( x \) component of Eq. (4-77) is:

\[ [S_{xx}]_{x=\xi} + [T_{xx}]_{x=\xi} - \gamma k^2 \xi = 0 \]  (4-83)
\[ -T_{xx}^e + \rho g x + T_{xx}^e + t_{xx} + s_{xx} = 0 \]  \hspace{1cm} (4-84)

However, using Eq. (4-73):

\[ t_{xx} = -\frac{1}{\mu} e b_z = -\frac{1}{\mu} e b_z \]

\[ = \frac{1}{\mu} e \frac{\partial B_{ef}}{\partial x} \xi \]  \hspace{1cm} (4-85)

Applying the pseudo-time average method, and using Eq. (4-3):

\[ t_{xx} = -\frac{\left| B_o \right|^2}{2\mu \delta} \xi \]  \hspace{1cm} (4-86)

\( s_{xx} \) is evaluated from Eq. (4-78):

\[ s_{xxf} = -P + 2\eta \frac{\partial \nu_x}{\partial x} \]  \hspace{1cm} (4-87)

These results are inserted into Eq. (4-77):

\[ -\rho g \xi - \frac{\left| B_o \right|^2}{2\mu \delta} \xi - 2\eta D\nu_x + \hat{P} - k^2 \delta \hat{\xi} \]

\( \hat{P} \) is found from Eq. (4-39); combining the last two parts of that equation:
\[ \rho s(jk_y v_y + jk_z v_z) = \eta (D^2 - k^2)(jk_y v_y + jk_z v_z) - k^2 p + jk_y f_y + jk_z f_z \]

(4-87)

In the fluid, from Eq. (4-40):

\[ jk_y \hat{f}_y + jk_z \hat{f}_z = [B_e (-k^2 b_+ + k_z b_y^+) + B_e (-k^2 b_+ + k_z b_y^+)] \]

\[ + DB_e^* (jk_z b_x^+) + DB_e (jk_z b_x^+) \]

(4-88)

\[ b_y^\pm \] is eliminated via Eq. (4-18):

\[ 2\mu_j (jk_y \hat{f}_y + jk_z \hat{f}_z) = B_e^* (k_z Db_x^+ - jk^2 b_y^+) + DB_e^* (-k_z b_x^+) \]

(4-89)

\[ + B_e (k_z Db_x^- - jk^2 b_y^-) + DB_e (-k_z b_x^-) \]

As remarked above, the magnetic fields are the gradient of a scalar potential in the vacuum above the fluid. Explicitly, this can be written as:

\[ \vec{b}_\alpha^\pm = \nabla \psi^\pm \]

(4-90)

\[ \psi^\pm = \text{Re}[\hat{\psi}_o e^{(s\pm j\omega)t-jk_y y-jk_z z}] \]

From this it follows directly that
\[
\hat{b}_{za}^\pm = j\hat{b}_{xa}^\pm
\]

(4-91)

This is combined with Eqs. (4-49) and (4-18) to obtain

\[
2\mu(j_{y}^\pm f_{y}^\pm + j_{z}^\pm f_{z}^\pm) = -B_{\theta j}^* k_{k}^\pm b_{x}^\pm \left(\frac{k_{k}}{k_{z}} - 1\right) B_{\theta j}^* k_{z}^\pm b_{x}^\pm \left(\frac{k_{z}}{k_{z}} - 1\right)
\]

\[
+ \frac{k_{z}^{2} \xi}{2\delta} |B_{O}|^{2} \left(\frac{k_{z}^{2}}{k_{z}^{2}} - 1\right)
\]

\[
+ DB_{e}^{*} j_{k}^z b_{x}^+ + DB_{e}^{*} j_{k}^z b_{x}^-
\]

(4-92)

This is substituted back into Eq. (4-37), and with the use of Eq. (4-19), it follows that:

\[
k^{2} \hat{P} = -\rho s D_{\nu x} \hat{\nu}_{x} + \eta (D^{3}_{\nu x} - k^{2} D_{\nu x})
\]

\[
+ \frac{1}{2\mu} (-B_{O}^{*} j_{k} (k-k_{z}) b_{x}^{+} - B_{O} j_{k} (k-k_{z}) b_{x}^{-})
\]

\[
+ \frac{\xi}{\delta} |B_{O}|^{2} (k^{2} - k_{z}^{2})
\]

\[
+ DB_{e}^{*} j_{k}^z b_{x}^+ + DB_{e}^{*} j_{k}^z b_{x}^-
\]

(4-93)

Combining all of the above into Eq. (4-83):
\[-\rho g \hat{\xi} - 3 n D \hat{v}_x - k^2 \gamma \hat{\xi} - \frac{\rho s}{k^2} D \hat{v}_x + \frac{n}{k^2} D^3 \hat{v}_x\]

\[-\frac{k^2}{k^2} \frac{|B_0|^2}{2\mu_\delta} \hat{\xi} + \frac{1}{2\mu k^2} \{ - \hat{B}_o^* jk (k-k_z) \hat{b}_x^+ + \hat{B}_o (l-j) \frac{\delta}{\delta} jk_z \hat{b}_x^+ + \hat{B}_o (l+j) \frac{\delta}{\delta} jk_z \hat{b}_x^- \} \]

\[= 0 \quad (4-94)\]

Again replacing \( \hat{\xi} \) via Eq. (4-76):

\[0 = (k^2 \gamma + \rho g + \frac{k^2}{k^2} \frac{|B_0|^2}{2\mu_\delta}) \hat{\xi} + 3 n x D \hat{\xi} + \frac{\rho s^2}{k^2} D \hat{\xi} + \frac{n}{k^2} s D^3 \hat{\xi} + \frac{S}{2\mu k^2} \{ \hat{B}_o^* jk (k-k_z) \hat{b}_x^+ + \hat{B}_o^* jk (k-k_z) \hat{b}_x^- \}

\[= \hat{B}_o^* (l-j) \frac{\delta}{\delta} jk_z \hat{b}_x^+ + \hat{B}_o (l+j) \frac{\delta}{\delta} jk_z \hat{b}_x^- \} \quad (4-95)\]

This is one boundary condition on \( \hat{v}_x \). The other results from consideration of the y and z components of Eq. (4-82). To linear terms these are:

\[\eta (\frac{\partial \hat{v}_x}{\partial x} + \frac{\partial \hat{v}_x}{\partial y}) = 0 \quad (4-96a)\]
\[
\eta \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) = 0 \tag{4-96b}
\]

These combine to form:
\[
\frac{\partial v_y}{\partial x} \left( \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} = 0 \tag{4-96c}
\]

Using continuity, Eq. (4-19), and transforming to 'hat' variables:
\[
D^2 \hat{v}_x + k^2 \hat{v}_x = 0 \tag{4-97}
\]

This is the other boundary condition on \( \hat{v}_x \).

These boundary conditions, Eqs. (4-73), (4-74), (4-95), and (4-97), are independent of the approximation made in Eqs. (4-66), (4-67), and (4-68) and apply when bulk coupling is included as well.

The surface-coupled case differs from the volume-coupled case in that the four linearly independent solutions of the bulk differential equations can readily be found analytically. This greatly simplifies finding the natural frequencies.

For this analysis it is assumed that \( k \) is all in the \( z \) direction, i.e.:
\[
k_z = k.
\]
Surface tension is also neglected.

The solutions of Eqs. (4-66) and (4-67) that are appropriate are:

\[ \hat{b}^+_x = A_1 e^{\gamma_+ x} \quad (4-98) \]

\[ \hat{b}^-_x = A_2 e^{\gamma_- x} \quad (4-99) \]

\[ A_1 \text{ and } A_2 \text{ are determined from Eqs. (4-73) and (4-74):} \]

\[ A_1 = -jk(1+j)\hat{B}_x^O \frac{\hat{\nu}_x(0)}{2s\delta(\gamma_+ + k)} \quad (4-100) \]

\[ A_2 = -jk(1-j)\hat{B}_x^* \frac{\hat{\nu}_x(0)}{2s\delta(\gamma_- + k)} \quad (4-101) \]

These magnetic fields are inserted into Eq. (4-68):

\[ \rho s(D^2 - k^2)^2 \hat{\nu}_x - \eta (D^2 - k^2)^2 \hat{\nu}_x \]

\[ = \frac{-jk|\hat{B}_x^O|^2}{2\mu} + \frac{(\mu s + 2j\mu s)(-jk(1+j)(\hat{\nu}_x(0)))}{2s\delta(\gamma_+ + k)} \left( l-j \frac{x}{\delta} + \gamma_+ x \right) \]

\[ + \frac{(\mu s - 2j\mu s)(-jk(1-j)(\hat{\nu}_x(0)))}{2s\delta(\gamma_- + k)} \left( l+j \frac{x}{\delta} + \gamma_- x \right) \quad (4-102) \]
Since this equation has constant coefficients, the solution is straightforward:

\[ \hat{\nu}_x = [A_+ e^{P_+ x} + A_0 e^{-P_0 x} + A_p e^{k x} + A_v e^{\gamma_v x}] \hat{\nu}_x(0) \]  (4-103)

\[ P_+ = \gamma_+ + \frac{(1-j)}{\delta} \]  (4-104)

\[ P_- = \gamma_- + \frac{(1+j)}{\delta} \]  (4-105)

\[ \gamma_v = (k^2 + s \rho/\eta)^{1/2} \]  (4-106)

\[ A_+ = \frac{-k^2 (\mu_0 s + 2j \mu_0 \omega) (1+j) |\hat{B}_0|^2}{2s \delta (\gamma_+ + k) [\rho s (P_+ - k^2) - \eta (P_+ - k^2)^2]} \]  (4-107)

\[ A_- = \frac{-k^2 (\mu_0 s - 2j \mu_0 \omega) (1-j) |\hat{B}_0|^2}{2s \delta (\gamma_- + k) [\rho s (P_- - k^2) - \eta (P_- - k^2)^2]} \]  (4-108)

\[ A_p \text{ and } A_v \text{ have to be determined from the boundary conditions, Eqs. (4-95) and (4-97). By definition:} \]

\[ l = A_+ + A_- + A_p + A_v \]

Combining this with Eq. (4-75) gives:
\[ A_P = \frac{[A_+(P_+^2 - \gamma^2) + A_-(P_-^2 - \gamma^2) + (\gamma^2 + k^2)]}{(\gamma^2 - k^2)} \]  \hspace{1cm} (4-109)

\[ A_\nu = \frac{[2k^2 + A_+(P_+^2 - k^2) + A_- (P_-^2 - k^2)]}{(\gamma^2 - k^2)} \]  \hspace{1cm} (4-110)

\[ 0 = (1+\theta)k^2 + (s^2 + 3k^2 s \psi) (A_+ p_+ + A_- p_- + A_p k + A_\nu \gamma_\nu) \]

\[ -\psi_s (A_+ p_+^2 + A_- p_-^2 + A_p k^3 + A_\nu \gamma_\nu^3) \]

\[ -k^2 \theta \frac{1}{\gamma_+ + k} + \frac{1}{\gamma_- + k} \]  \hspace{1cm} (4-111)

\[ s = s/\sqrt{\frac{g}{\delta}} \]  \hspace{1cm} (4-112)

\[ k = k \delta \]  \hspace{1cm} (4-113)

\[ \psi = \eta/\sqrt{\rho \gamma_3} \]  \hspace{1cm} (4-114)

\[ (\gamma_+, \gamma_-, \gamma_\nu, P_+, P_-) = (\gamma_+, \gamma_-, \gamma_\nu, P_+, P_-) \delta \]  \hspace{1cm} (4-115)

\[ \theta = \frac{|B_0|^2}{2 \mu \rho g \delta} \]  \hspace{1cm} (4-116)
The normalization was chosen so that for $k\delta$ of order unity, $s$ is also of order unity.

The numerical problem has thus been reduced to finding $s$ that satisfy Eq. (4-101) for specified $k$, $\Psi$, and $\theta$. The form of the equation allows the construction of many rapidly converging numerical sequences for finding $s$.

The solution that is of physical interest is that connected with the surface Alven wave. $s$ is almost purely imaginary; the imaginary part of $s$ is termed here $\Omega$.

Surface Alven waves are the subject of extensive past research. If the fluid is assumed perfectly conducting and viscosity is neglected, it is easy to show that:

$$\Omega = (k + \theta k^2)^{1/2}$$

(4-117)

Schaffer\textsuperscript{6,7} derived a correction to allow for finite conductivity. His result is correct to order $k\delta$:

$$\Omega = (k + \theta k^2 (1 - \frac{k}{2}))^{1/2}$$

(4-118)

These two formulas are compared with the numerical
results considered here in Fig. 4-5, where the results of Eq. (4-117) are denoted by (a), those of Eq. (4-118) by (c) and the numerical results by (b).

At small values of $k\delta$ all three agree; as $k\delta$ is increased, Eq. (4-117) progressively diverges from the others, as expected, while Eq. (4-118) is quite accurate up to $k\delta$ of about 0.8.

The error inherent in neglecting the bulk coupling can be approximated from Eq. (4-10). The velocity always varies in $x$ at least as:

$$\hat{v}(x) \propto e^{kx}$$

This can be seen, for instance, from Eqs. (4-38) and (4-44). Inserting this into Eq. (4-10):

$$|b_x|_{\text{internal}} \approx \frac{\mu \sigma \delta \hat{v}_x(0)}{4\sqrt{2} \hat{\nu}_x(0)}$$

From Eq. (4-78):

$$|b_x|_{\text{surface}} \approx \frac{k\sqrt{2} \hat{\nu}_x(0)}{2\delta s[\gamma_+ + k]^+}$$

Combining these, and explicitly substituting for $\gamma_+$:
\[ \frac{|b^x|_{\text{internal}}}{|b^x|_{\text{surface}}} \approx \left| \frac{s}{\omega} \right| \] (4-119)

The numerical results indicate that the imaginary part of \( s \) is much larger than the real \( s \) found in the section on internally coupled waves. For this reason, and in view of Eq. (4-119) it can be asserted that the imaginary part of \( s \) is found here accurately, and that these results are a generalization of those of Schaffer. The real part of \( s \) found here, however, is of the same order as that found in bulk-coupled waves under similar conditions. This means, unfortunately, that growth rates and the question of instability cannot be settled from the surface-coupled problem.

4.4. Modes Coupled Both at the Surface and Internally.

The application of bulk equations of section 4.2 and the boundary conditions of section 4.3 involves no new difficulties. The effect of adding bulk effects to the modes associated with surface Alven waves is particularly important since it is necessary to do so in order to know whether or not they are stable. The bulk modes of section 4.3 are also modified by the addition of free surface effects.
In section 4.3 the boundary conditions are condensed into four conditions on \( \hat{b}_x^\pm, \hat{v}_x \) and their derivatives. This involves combining the actual physical boundary conditions and effectively reduces the number of boundary conditions by three, involving \( \hat{b}_y \) and \( \hat{v}_y \). The question is left open there as to whether these conditions are satisfied by solutions of the condensed set of boundary conditions, if \( k_y \) is finite.

This dilemma of satisfying an extra three boundary conditions is met by considering the fact that there is an additional set of modes independent from those described by Eqs. (4-42) and (4-43). These additional modes are those for which \( \hat{v}_x \) is identically zero, and are described by Eqs. (4-21) and (4-22). Without specifying explicitly the form of the additional boundary conditions, it is clear that the conditions that must be met at \( x=0 \) are of the form:

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
x \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
= [T][A]
\]

(4-120)
The matrix \([X]\) is a 4 x 4 matrix representing the boundary conditions on \(\hat{u}_x\), \(\hat{b}^+_x\), and \(\hat{b}^-_x\) and their derivatives. This set of boundary conditions is deduced from normal and sheer stress balance and the boundary conditions on normal and tangential fields. These conditions are Eqs. (4-74), (4-75), (4-95), and (4-97). While these four conditions certainly must be satisfied, there are in addition three other boundary conditions that also must be satisfied, represented by Eqs. (4-72a,b) and (4-96a).

To satisfy three additional boundary conditions there are necessarily three additional modes. In fact, these are modes satisfying Eqs. (4-21) and (4-23), for which \(\hat{u}_x\), \(\hat{b}^+_x\), and \(\hat{b}^-_x\) are all identically zero. These are the longitudinal modes. There are three solutions to Eqs. (4-21) and (4-23) that decay as \(x\) goes to minus infinity, and these are the three solutions of interest here. Since for these three solutions \(\hat{u}_x\) and \(\hat{b}^+_x\) vanish, they make no contribution to the four boundary conditions involving only \(\hat{u}_x\) and \(\hat{b}^+_x\). They contribute only to the three addition boundary conditions involving \(\hat{b}^-_x\), and \(\hat{u}_x\). This contribution is represented by the matrix \([Y]\), which is 3 x 3. The matrix \([S]\) represents the contribution of the four solutions of the 'depth' equations to the boundary conditions on \(\hat{u}_y\) and \(\hat{b}^+_y\).
It is a theorem from linear algebra that:

\[ \det[T] = \det[X] \cdot \det[Y] \]

The final condition is then that either \( \det[X] \) or \( \det[Y] \) vanish. In the three solutions represented in \( [Y] \), \( \hat{v}_x \) and \( \hat{b}_x^+ \) are always zero. Under those conditions, the boundary conditions must return to those of section 4.2 for purely longitudinal modes, because if \( \hat{v}_x \) is zero the surface is flat. Thus any value of \( s \) that makes \( \det[Y] \) vanish is a solution to the problem of purely longitudinal motions. In that section it is shown that all such solutions have negative real parts. The main interest here is in unstable solutions, so the possibility of \( \det[Y] \) being zero can be ignored, leaving as the condition for a solution:

\[ \det[X] = 0 \]  \hspace{1cm} (4-121)

It is interesting to note that since the three longitudinal modes make no contribution to the four boundary conditions on \( \hat{v}_x \) and \( \hat{b}_x^+ \), it is unnecessary to find the contribution of the four depth solutions to the boundary conditions on \( \hat{v}_y \) and \( \hat{b}_y^+ \). Indeed, it is unimportant even if that contribution is undiscoverable from the four solutions, which seems to be the case.
The first set of natural frequencies considered here is that of the surface waves of the previous section. There the imaginary part of the natural frequency is determined from the surface-coupled theory, but that theory is shown to be unable to determine stability. The results of applying the general theory to that problem are shown in Figs. 4-5 and 4-6. For all wave numbers considered the effect of the field is to increase the imaginary part of $s$, and also to increase the rate of decay. These effects are strongest when $k_z = k$ and diminish as $k_z$ decreases. When $k_y = k$ the field has no effect whatsoever.

In no case considered does the application of the field lead in the direction of instability for these surface waves.

The waves that are essentially the internally coupled modes of section 4.3 are more unstable in the case of a free surface than if the surface is flat. In Figs. 4-7 and 4-8 the growth rates associated with the case of a flat surface, which corresponds to that when gravity is infinite, and the case in which gravity has a value typifying actual experimental conditions. The normalization used in Figs. 4-7 and 4-8 is that of section 4.3. When that normalization is introduced into Eq. (4-73) the natural dimensionless gravity constant is
\[ g = \frac{\rho g \delta}{\eta \omega} \]  

(4-100)

For actual experimental conditions \( g \) is approximately 300; for the flat surface it is infinite. The inclusion of the free surface only moderately changes the stability condition, as shown in Fig. 4-9.

As stated above, the effect of introducing the effects of finite gravity into the consideration of essentially internally complex waves is that they become more unstable. A complete consideration of the effects of bulk coupling shows that surface waves are damped by the field.
CHAPTER 4 REFERENCES


Fig. 4-1. Schematic Diagram of Fluid Layer.
Fig. 4-2. Critical Field Parameter, M, vs. kδ for a Flat, Stress-Free Surface.
Fig. 4-3. Critical Field Parameter, $M$, vs. $k_0$ for a Flat, Rigid Upper Boundary.
Fig. 4-4. Critical Field Parameter, $M$, vs. $k\delta$ for Layers Bounded by Rigid Walls.
Fig. 4-5. Natural Frequency, $\Omega$, for Free Surface, vs. Field Parameter $\theta$. 

$k = 0.2$

$k = 0.8$

$k = 1.5$
Fig. 4-6. Decay Rate vs. Field Parameter, $M$, for Free Surface Oscillations.
Fig. 4-8. Growth Rate, $S$, vs. Field Parameter, $M$, for Free and Rigid Surfaces.
Fig. 4-8. Growth Rate, $\dot{S}$, vs. Field Parameter, $M$, for Free and Rigid Surfaces.
Fig. 4-9. Critical Field Parameter, $M$, vs. $k\delta$ for Rigid and Free Boundaries.
5. Experimental Results

5.1 Introduction

Experimental evidence, and models developed in previous chapters, are now examined in an effort to understand the 'unstable' fluid motion discussed in chapter 1. The apparatus of Fig. 5-1 is used to apply a nearly uniform audio frequency magnetic field to a container of liquid metal. An isolation cell is suspended inside the container to prevent end or edge effects from influencing the fluid inside the cell. These edge effects include inhomogeneity in the magnetic field, contact heating at the side walls, and the wetting angle of the fluid to the container walls. These effects cannot be completely eliminated by the isolation cell, but are considerably reduced.

The physical properties of the liquid metal used here are given in Appendix A. Details of the experimental apparatus are given in Appendix B.

As the magnetic field is increased in this apparatus, at some point the fluid begins to undergo a distinctive type of motion. The surface is corrugated in a chaotic, unsteady manner. The motion, shown in the photographs of Fig. 5-2, is turbulent in appearance. The undulations of the surface have a frequency of approximately 0.5 - 2.0 Hz. When the field is turned on suddenly, the oscillations appear to take 1-10 seconds to reach their final amplitude.
The instability is characterized by an apparent point of incipience. Below a certain field strength there is only low level random convection. Once this field strength is passed, the motion described above ensues. With care taken to avoid edge effects that also have their origin in the magnetic field, there is apparently a sharp change from a relatively quiescent state to one of random motion.

This motion is distinguishable from convection caused by heating the fluid unevenly or from below. When an amount of heat, essentially the same as that dissipated by the field in the above experiments, is injected into the bottom of the fluid layer, the result is very slow, steady convection. In an experiment in a copper container, the fluid rises towards the edges of the container and sinks in the middle.

The oscillations of the surface are also different in appearance from gravity-capillary surface waves. At typical observed wavelengths, the observed frequencies are much less than those of a gravity-capillary wave. Gravity-capillary waves can be excited by striking the fluid container; the difference is very apparent.

In chapter 3 the motion of a fluid layer caused by a non-uniform field is considered. From those results it is clear that a small inhomogeneity in the field can drive large fluid motions, in many cases leading to
turbulence. In section 5.3 an experiment involving forced pumping of the fluid is described. The flow in this experiment is almost certainly turbulent, yet the appearance of the fluid surface is not at all like the chaotic boiling described above. The surface seems smooth, almost glassy.

The surface motion described in the first part of this section seems distinct in appearance from thermal convection, hydrodynamic turbulence, or any kind of gravity-capillary wave. It also shows a sharp point of incipience. In section 5.2 measurements of this point of incipience are compared to the predictions of chapter 4. There is good agreement on the dependence of critical field strength on the frequency of the field. The magnitude of the field strength measured also agrees with predictions within errors in measuring fluid viscosity and the exact point of incipience.

As discussed in section 5.2, observed and predicted critical field strengths nearly agree; however, there is a great disparity between observed and predicted growth rates. The growth times predicted in chapter 4 are approximately $10^3$ seconds, two to three orders of magnitude longer than those observed. Possibly, turbulent momentum transfer from field inhomogeneities plays a key role in explaining this discrepancy. For that reason, the results of an experiment, in which a field gradient is imposed,
are given in section 5.3. The ultimate velocities achieved and the time taken to reach them are correlated qualitatively with the turbulent mixing length model of chapter 3. Growth rates are considered further in section 5.4. The time in which the observed motion decays is seen to be qualitatively what is predicted from a mixing length turbulent viscosity model. The growth time is on the same order, but it is not plausible that the turbulence could increase the growth rate for an instability of the type modelled.

The coincidence of the decay time due to turbulence, and the observed growth and decay times leads to the conjecture that in some way the proposed instability mechanism acts as a kind of gate, allowing another source of motion access to the energy in the magnetic field. In itself, however, the instability mechanism cannot explain the observed growth rates, and no rigorous theory is proposed here that can.

Visual observation is necessarily confined to surface motions. Evidence of bulk motion is given by the results of the thermal experiments of section 5.5. In that experiment, the onset of instability, as observed visually, corresponded to a change in observed thermal transient response. This indicates a bulk convective effect.

A summary of the results is given in section 5.6. The points of agreement and disparity between theory
and experiment are listed, and further possibilities for experimental and theoretical investigation are given.

5.2 Onset of Instability

With the frequency of the magnetic field fixed, there is a definite value of field strength at which the surface motion begins. Measurements of this critical field strength are reported in this section for experiments with four different fluid containers, and for various depths of fluid. In each case the results predicted from the application of the stability analysis of depth modes for layers of finite thickness of chapter 4. These theoretical results are derived with the assumption that the lower bound on the fluid is a perfectly conducting plate.

The different containers are shown in Fig. 5-3. The first is a glass baking pan, while the second is constructed with copper on two sides and the bottom, with endpieces of transite, an insulator. Inside each of these an isolation cell, also shown in Fig. 5-2, can be suspended.

In the glass container the observed disturbances seen longer and more slowly undulating than in the copper container. The results for experiments in the glass container with and without the isolation cell are shown on Fig. 5-4 and Fig. 5-5 respectively. In each case two different fluid depths are considered.
In Fig. 5-6 the results of experiments in the copper container with isolation cell are shown. When the field strength is below that required to generate motion inside the cell there is considerable motion in the fluid between the isolation cell and the copper walls. The reason for this is unknown. It is conjectured that the fact that the liquid metal does not really wet the copper causes local field distortion and heating. This could lead to magnetic or thermal pumping.

In view of these observed edge effects, the meaning of experiments in the copper container without the isolation cell is compromised. However, the thermal experiments are performed under these conditions. In addition, for the thermal experiments the upper surface is not kept clear of oxide film. In the experiments reported above the surface is kept clean by a thin acid layer, as discussed in Appendix B. The results for an experiment in the copper dish, without an isolation cell, and with an oxide layer present, are shown in Fig. 5-6. The dependence of critical field strength on frequency is completely different from that seen in other experiments.

In Fig. 5-7 the incipience data of Fig. 5-3 and Fig. 5-5 is collected in normalized form.

The agreement of these controlled experiments with the theory is the strongest evidence that the hydromagnetic linear stability theory has bearing on these experiments.
5.3 Quasi-One-Dimensional Magnetic Pumping

The experiment undertaken in this section allows a comparison of the appearance of forced motion leading to hydrodynamic turbulence, with the unstable motion considered in the previous section. It also constitutes an example in which turbulence and its affects on steady and transient motion are readily observed. The quasi-one-dimensional motion is chosen for study because the appropriate form for the field can be attained experimentally.

The quasi-one-dimensional pumping of a fluid layer is discussed in section 3.3. It is shown there that in order for this type of motion to occur the magnetic field at the fluid surface must be of the form:

\[ H_z = [K(z-z_0)]^k \]  \hspace{1cm} (5-1)

The constants K and z_0 are arbitrary.

Experimentally this form of the field is attained by suspending a conducting wedge of appropriate shape above the fluid surface. In that case, if the frequency of the field is high enough that in both the wedge and the fluid the magnetic skin depth is relatively short, then the flux is trapped between the fluid and the wedge. If the slope of the wedge is small, the field strength is inversely proportional to the distance between the fluid and the wedge. A wedge shape that tailors the field properly is
shown in Fig. 5-8. The actual wedge is machined from a block of solid aluminum.

In this experiment the fluid is held in the glass container, with the wedge suspended over the surface. With the isolation cell in place the depth of the fluid is 1 cm.; without the cell it is 1.8 cm.

The coil is driven at 4 kHz. The fluid does undergo motion, but it is different than the motion described in section 3.3. As shown in Fig. 5-9, the fluid forms cells rotating in the plane of the surface. In the center the motion is in the direction predicted by the theory of section 3.3, that is, opposite the gradient in field strength. The fluid returns, however, at the edges of the container rather than near the bottom. This type of behavior, in which fluid motion occurs as cells rather than achieving a true quasi-one-dimensional profile, has been previously reported in an electrohydrodynamic experiment.¹ When narrow isolation cell is introduced the cellular motion is still found, though from the appearance of the surface perhaps more of the fluid returns at the bottom of the container as it is made narrower.

The mean steady velocity of the flow at the center of the flow at the center of the channel is found by measuring the time taken by a brass nut floating on the surface of the field to traverse 4 cm. directly underneath the wedge.
The mean velocity is defined as

$$V = \frac{1}{t} = \frac{(4 \times 10^{-2})}{t} \text{ (m./sec.)} \quad (5-2)$$

The measured velocity is shown in Fig. 5-10. The velocity is nearly linear in the field strength, following the predictions of the turbulent theory of section 3.5 rather than the laminar theory of section 3.3. In fact, the typical Reynolds's number can be found for velocities and lengths typical of the experiment:

$$U = 10^{-2} \text{ m./sec.} \quad (5-3)$$

$$L = 3 \times 10^{-2} \text{ m.} \quad (5-4)$$

$$v = 6 \times 10^{-8} \text{ m}^2/\text{sec.} \quad (5-5)$$

$$\text{Re} = 5 \times 10^3 \quad (5-6)$$

The observed flow is thus probably turbulent. The line shown in Fig. 5-11 results from taking 1 cm. as in Eq. 3-53. The observed and predicted velocities agree within a factor of two. Factors possibly contributing to the error are: (1) the observed motion is not really quasi-one-dimensional, since it involves cellular fluid motion; (ii) the length of the wedge and hence the tapered field is only six times the depth of the fluid; and (iii) the turbulence model of section 3.5 is itself very approximate and untested.
In view of these sources of uncertainty, a factor of two agreement, and agreement on the basic form of the curve of \( \nu \) vs. \( B \), leads to the assumption that field stresses and turbulent stresses are balanced in determining the steady velocity.

The time taken for the fluid to reach its steady velocity is also of interest. In Fig. 5-12 the mean velocity measured by dropping the float to the metal surface at different times is shown. The field strength is held fixed. The velocity seems to reach its steady value after about 3 seconds.

The time taken to reach steady state for laminar flow is:

\[
\tau_\nu \propto \frac{\rho l^2}{T} 
\]

(5-7)

\[
\propto \frac{8 \times 10^3 (2 \times 10^{-2})^2}{(5 \times 10^{-4})} 
\]

(5-8)

\[
\propto 6 \times 10^3 \text{ sec.}
\]

If instead the turbulent viscosity is used,

\[
\tau_\rho \propto \frac{\rho l^2}{\rho v l} 
\]

(5-9)

\[
\propto \frac{1}{\nu}
\]
This turbulent viscous diffusion time is much more in line with experimental observations.

The sinusoidal driven motion is not the subject of extensive experiments. By driving the end coils of the apparatus 180° out of phase, the field at the surface can be made basically sinusoidal. The observed motion is in fact in the direction predicted in section 3.4.

5.4 Characteristic Times for Observed Motion

The two attributes of the observed motion that are most disparate from the theory are the fact that the motion is oscillatory and that its growth time is on the order of seconds. The theory predicts static instability and growth times on the order of a thousand seconds.

The observed decay times for the instability when the field is turned off is also typically a few seconds. An explanation for this is seen from the viscous decay time:

\[
\tau_v \sim \frac{\rho l^2}{\eta}
\]  \hspace{1cm} (5-11)

For these experiments, because of the low value of \( \eta \), this time is about a thousand seconds. If, however, the turbulent
viscosity of section 3.5 is introduced instead of 7:

\[
\tau_\rho \sim \frac{1}{\nu} \quad (5-12)
\]

This time is typically on the order of seconds for observed lengths and velocities and agrees with observations.

This argument leads to the conclusion that the observed motion is indeed turbulent, and that over the course of the first few seconds the field is turned on it is driven into that state.

Turbulent time constants seemingly characterize the motion, but it is hard to see how turbulence could increase the original growth rates.

The pressures associated with surface deflection, bulk velocity, bulk acceleration, and the magnetic field can be estimated:

\[
E_1 \sim \rho g \xi
\]

\[
\sim 10^2 \text{ NT/m}^2
\]

\[E_2 \sim \rho v^2\]

\[
\sim 5 \text{ NT/m}^2 \quad (5-14)
\]

\[E_3 \sim \rho \frac{V}{\tau}\]

\[
\sim 10^2 \text{ NT/m}^2 \quad (5-15)
\]
\[ E_4 \gtrapprox \frac{B^2}{\mu} \gtrapprox 10^3 \text{ NT/m}^2 \] (5-16)

The energy associated with the corrugation of the surface is far greater than the free stream kinetic energy. It represents a large fraction of the field energy. However, the part of the field stress available from bulk coupling depends on currents caused by fluid motion. It is proportional to the magnetic Reynold's number:

\[ R_m \gtrapprox \mu \nu v l \gtrapprox 10^{-4} - 10^{-3} \] (5-17)

Possible explanations for the anomalous growth rates are:

(1) Turbulence in some way allows a much larger amount of the field stress to be applied to increasing the fluid motion. This is clearly a non-linear effect. At saturation the observed flow is almost certainly turbulent, as evidenced by the relatively short time required for the motion to stop when the field is turned off.

(2) A pumping mechanism from edge effects is enabled to make itself felt throughout the volume by the instability mechanism.
(3) The motion caused by the instability mechanism discussed here is itself unstable, that instability having a much higher growth rate.

(4) The nature of the surface of the liquid is not fully understood. It clearly influences surface motion here, and the possibility of a surface instability such as the Marigoni instability cannot be dismissed. The observed motion, however, does not resemble that of standard thermally induced instabilities.

5.5 Effects of Instability on Heat Transfer

The advent of motion in the fluid can have a drastic effect on heat transfer. The ratio of convected to conducted heat is the Peclet number:

\[ P_e = \frac{\rho v}{\frac{c_p}{k}} \]  \hspace{1cm} (5-18)

\[ k = \text{thermal conductivity} \]

\[ c_p = \text{heat capacity} \]

\[ \rho \sim 8 \times 10^3 \text{ kg/m}^3 \]  \hspace{1cm} (5-19)

\[ l \sim 10^{-2} \text{ m.} \]

\[ c_p \sim 150 \text{ J/kg.} - \text{°C} \]
\[ k \sim 20 \text{ W/m. - } ^\circ\text{C} \]

\[ P_e \sim 800 \text{ V (m./sec.)} \quad (5-20) \]

For even small velocities, then, convection overwhelms conduction.

The apparatus used in the thermal experiments is described in Appendix B. The liquid metal is held in a copper dish. To the bottom of this dish a coil of copper tubing is soldered, allowing water to be circulated in order to cool or heat the bottom of the fluid layer.

The thermal experiments consist of monitoring the temperature of the effluent cooling water to obtain the thermal history of the bottom plate in the copper container. The isolation cell is absent. In this experiment a 1.7 kHz field is turned on abruptly and maintained constant throughout an experiment. The temperature of the effluent water rises steadily to a steady-state value. This transient response is shown for several values of M in Fig. 5-13. Heat is put into the fluid at the upper surface via inductive heating by the magnetic field. In Fig. 5-13 temperature has been scaled linearly so that \( T(t = 0) \) is zero and \( T(t \to \infty) \) is one. Simple dimensional considerations demand that if conduction is the only mechanism for heat transfer and fluid properties \((\rho, k, c_p)\) are constant, the resulting curves are independent of the magnitude of the temperature
excursion, and would lie on top of one another. In fact, the temperature measured approaches its steady state value more quickly as \( M \) increases. Between \( M = 100 \) and \( M = 130 \) a rapid transition in thermal behavior is evident.

The aim of these experiments is to establish that the magnetically induced convection has an effect on heat transfer. Supporting the conclusion that the convection causes the change in thermal transient response, are two experiments in which motion of the liquid is mechanically controlled. In the first, the fluid is vigorously stirred by a paddle. It is observed that at low field strength the thermal transient resembles that associated with higher field strengths and field induced convection. In the second a set of flat slats is inserted into the fluid. The slats which are thin insulators, are all parallel to the \( x-y \) plane so as not to disturb the equilibrium current. It is then seen that even at high field strengths the thermal profile follows those previously associated with low field strengths.

The experiments suggest that as the field is raised beyond a certain point the fluid changes its state from slight motion to turbulent motion, and that this point is essentially determined by the parameter \( M \).

The point at which the thermal behavior changes most rapidly corresponds well to the point at which the onset
of motion is observed at 1.7 kH (M ≈ 110), in Fig. 5-7. The occurrence of surface motion therefore seems to be coincident with bulk stirring. The uncontrolled experimental conditions allow some doubt about whether the surface motion observed in this case is the same as that in other experiments. Its appearance is similar.

5.6 Summary and Recommendations

Measurements of critical field strength required for instability are in good agreement with the theoretical prediction of the linear hydromagnetic stability theory of chapter 4. Thermal experiments give evidence that the observed motion is distributed throughout the bulk. However, the predicted growth times are much longer than those observed.

For a given frequency and fluid, the critical value of M determines the maximum stable depth of fluid that can be levitated or depressed magnetically. This depth, L, and the critical value of M, M*, can be related directly:

\[ M^* = \frac{|\hat{B}|^2}{\mu \eta \omega} \quad (5-21) \]

\[ L = \frac{|\hat{B}|^2}{2 \mu \rho g} \quad (5-22) \]

\[ \therefore L = \frac{\eta \omega M^*}{2 \rho g} \quad (5-23) \]
If the thickness of the fluid layer is greater than two or three skin depths, $M^*$ is essentially independent of $\omega$. In the case of a typical free surface, $M^*$ becomes approximately 68.

With the exception of rheocast alloys, the viscosity and density of most liquid metals is not a strong function of temperature or any other easily controlled variable. Therefore the only way to increase the depth $l$ is to increase $\omega$. Since the power dissipated here is proportional to $\sqrt{\omega}$ there is an optimum value of $\omega$, which is in most cases the smallest value that allows stable levitation.

Typical fluid parameters are:

$$\eta = 5 \times 10^{-4} \text{ kg/m-sec.}$$

$$\rho = 8 \times 10^3 \text{ kg/m}^2$$

The value of $M^*$ varies with the boundary conditions. Taking it as 60:

$$l \approx 1.2 \cdot f(\text{kHz}) \text{ mm.} \quad (5-24)$$

The disagreement is predicted and observed growth rates are the key to further research directly along the lines of this thesis. Assessment of the effects of edges and field inhomogeneities is certainly the next step experimentally. Further removal of the influence of the edges,
and further efforts to make the field more uniform, could cause the motion to grow more slowly. This would imply that in some way that the driven motion of the fluid played a key role. Motion at the boundaries could also be imposed mechanically to ascertain the effect of the magnetic field on the propagation of edge effects into the fluid.

A self-consistent theory of the effects of turbulence on the magnetic field, and vice versa, could possibly be developed analogous to the Reynolds stresses development of hydrodynamics. With such a theory the possible effect of turbulence as an agent for increasing growth rates could be evaluated.

This thesis has two major parts. The theoretical part, principally chapter 4, establishes an upper limit on the amount of field stress that can be applied to a dense conducting fluid without causing spontaneous motion. This constraint applies even when the field is perfectly uniform. The theory given here also tells how one instability mechanism can be circumvented by appropriately choosing the frequency of the applied field.

The experimental part of this thesis, chapter 5, gives the critical field strength necessary to cause a spontaneous fluid motion, as the frequency of the imposed field is varied over an order of magnitude. These results indicate an upper limit on the amount of uniform field stress that can be
applied to a liquid metal. The theoretical and experimental values of this limit are in substantial agreement.

The description of motion driven by field inhomogeneity and its rough correlation with a simple turbulence model offer a method by which the magnitude of driven motions can be estimated.

The major problem in joining the two halves of the thesis together is the wide disparity in predicted and observed growth rates. Resolution of this problem is the basis for the recommendations for further research made in this section.
CHAPTER 5 REFERENCES


Fig. 5-1. Experimental Apparatus.
Fig. 5-2  Photograph of Instability:  a) without field;  
   b) with field.
Fig. 5-3. Fluid Containers.

a.) GLASS CONTAINER

b.) COPPER CONTAINER

c.) ISOLATION CELL
Fig. 5-4. Critical Current, $i$, vs. Frequency, for Glass Container with Isolation Cell.
Fig. 5-5. Critical Current, i, vs. Frequency for Glass Container without Isolation Cell.
Fig. 5-6. Critical Current, i, vs. Frequency for Copper Container with Isolation Cell.
Fig. 5-7. Critical Current, \( i \), vs. Frequency for Copper Container without Isolation Cell.
Fig. 5-8. Collected Results of Critical Field Parameter, $M$, vs. Normalized Frequency, $\omega \mu_0 \sigma \Delta^2$.  

- GLASS DISH, W/I.C. $\Delta = 1.5$ cm
- GLASS DISH, W/I.C. $\Delta = 2.2$ cm
- COPPER DISH, W/I.C. $\Delta = 1.5$ cm
Fig. 5-9. Wedge for Shaping Field.
Fig. 5-10. Flow Pattern in 'Quasi-One-Dimensional' Flow Experiment.
Fig. 5-11. Velocity, \( v \), vs. Current, \( i \), in 'Quasi-One-Dimensional' Experiment. Line is from Turbulent Theory.
Fig. 5-12. Transient Velocity Behavior in Quasi-One-Dimensional Experiment.
Fig. 5-13. Thermal Transient for Various Values of Field Parameter M.
APPENDIX A

MATERIALS PROPERTIES OF CERELOW-117 ALLOY

Cerelow-117 alloy is a low melting point alloy designed for testing molds and for making temporary forms for epoxy casting. It melts at $117^\circ$F and expands upon solidification, making it ideal for these purposes and as a thermal bath. For these reasons its thermal properties are well known and publicized,\(^1\) while mechanical and electrical properties are less available. The published thermal properties are shown in Table A-1; they have been accepted here untested. The values in Table A-1 are conversions to MKS units from the published data.

The two additional fluid properties needed for this thesis are bulk electrical conductivity and viscosity. The electrical conductivity is measured inductively as a function of temperature in the cell shown in Fig. A-1. The method used depends on the fact that the inductance measured at the terminals of the coil shown depends in large part on the degree to which the metal core excludes magnetic flux. In general, this relationship can be written as:
\[ L = f(\omega \mu_0 \sigma \ell^2) \quad (A-1) \]

\( \ell \) is a length scale, here taken as the diameter of the rod. The material is assumed to be non-magnetic, i.e., \( \mu = \mu_0 \). The function \( f \) is then found by measuring the inductance as a function of frequency with the core made of a material with known conductivity. The experiment is then repeated with the material whose conductivity is to be measured. The conductivity is taken as that which comes closest to causing the curve to be the previously measured \( f(\omega \mu_0 \sigma \ell^2) \).

In Fig. A-2 the results are shown for aluminum, copper, and Cerelow-117. The values for the conductivity of copper and aluminum are from Woodson and Melcher.\(^2\) The deduced value of the conductivity of Cerelow-117 is \( 2 \times 10^6 \) mho/m. No significant change in conductivity is evident as the temperature changes.

The viscosity is measured by a ball dropping technique. A ball is dropped through the fluid and its velocity measured. If the Reynolds number of the sphere is small enough, the viscosity can be deduced from Stoke's law. Experimentally, the ball consisted of a small piece of lead shot tied to a string.
The deduced value of viscosity is approximately \(5 \times 10^{-4}\) kg/m-sec. Various factors contribute to the uncertainty of this figure. The lead balls tend to dissolve in the liquid metal, the string seems to stick to the surface, and the Reynold's number in some cases is near unity.

**TABLE A-1**

**PROPERTIES OF CEREWOW-117 ALLOY**

\[(\text{Bi 44.7\%, Pb 22.6\%, In 19.1\%, Sn 8.3\%, Cd 5.3\%})\]

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_m), Melting Point</td>
<td>47°C</td>
</tr>
<tr>
<td>(\rho), Mass Density</td>
<td>(8.8 \times 10^3) kg/m(^3)</td>
</tr>
<tr>
<td>(c_p), Specific Heat</td>
<td>150 J/°C-kg</td>
</tr>
<tr>
<td>(H_f), Heat of Fusion</td>
<td>(1.4 \times 10^4) J/kg</td>
</tr>
<tr>
<td>(k), Thermal Conductivity</td>
<td>16.5 W/m-°C</td>
</tr>
</tbody>
</table>
APPENDIX A REFERENCES


Fig. A-1. Conductivity Measurement Apparatus.
Fig. A-1. Conductivity Measurement Apparatus.
Fig. A-2. Relative Inductance of Coil vs. Frequency for Different Core Materials.
APPENDIX B

EXPERIMENTAL APPARATUS

In Fig. 5-1 the coils used to produce the magnetic field are shown. The container used in some of the experiments is also shown. The coils are wound of Litz wire, each wire consisting of 30 separate strands of insulated #36 wire. This minimizes coil resistance at audio frequencies. The end coils have 200 turns each, and the middle coil 175. When the coils are driven in series they produce a nearly uniform field in the axial direction. Field measurements made with the copper dish and fluid present are shown in Fig. B-1.

The total inductance of the three coils, driven in series, is 25 mH. The coil resistance, with the copper dish and fluid absent, is approximately 0.5Ω at 1.0 kHz and 1.2Ω at 10.0 kHz. Much of this resistance is probably due to contact resistance in connections.

With the liquid metal present, the coil resistance is typically 8Ω at 1 kHz, and is nearly inversely proportional to the square root of frequency. Some variation in resistance naturally occurs when the container or amount of fluid is changed. It is clear that the great bulk (>90%)
of the power dissipated is in the fluid rather than the coil wire.

The different containers and isolation cells are shown in Fig. 5-1 and Fig. 5-3. The glass dish and the isolation cells require no special comment.

The copper container used in the thermal experiments of section 5.5, shown in Fig. 5-1, is ¼" copper plate on three sides, with transite ends. A cooling coil of copper tube (1/8" O.D., 1/16" I.D.) is soft soldered to the bottom. Water is circulated through the tubing by a small "tube" pump. The flow rate is 200 ml./min.

A schematic diagram of the audio power supply is shown in Fig. 8-2. The two output transistors, T1, T2, act as switches, connecting the output alternately to the positive and negative D.C. supplies. The diode arrangement D1, D2, D3 prevents both output transistors from being turned on at the same time. The diodes D4, D5 prevent the output voltage from swinging past the D.C. supply voltages. Without them this could happen with an inductive load.

The load coil is run, in series with a capacitor bank, at resonance so that the load appears purely resistive to the A.C. power supply. The capacitors are A.C. oil or mica capacitors.
Thermal measurements are made by iron-constantan thermocouples. The signals are amplified and filtered internally by a Tektronix oscilloscope, and recorded by a Heathkit chart recorder.
APPENDIX B REFERENCES

Fig. B-1. Field Distribution Inside Coil.
Fig. B-2. Power Supply.
APPENDIX C

COMPUTER PROGRAMS AND NUMERICAL METHODS

Three problems in the next required numerical solution. This appendix is a description of the methods used, including a text of the actual FORTRAN computer codes used.

In Chapter IV, section 4.2 and 4.4 the stability analysis of modes involving bulk coupling in an infinite layer is described. The difference between the two cases is in the boundary conditions imposed at the surface. In Program Listing C-1, both boundary conditions are present; applying the results to either problem involves only switching the order of the cards.

In this first program, the bulk of the computation occurs in the function subroutine 'FUN'. In the subroutine, the four independent solutions to Eqs. (4-45) and (4-46) are generated as described by Eqs. (4-60), (4-61), (4-62), (4-63), (4-64), and (4-65). As each solution is generated, the appropriate entry in the matrix [R][T] of Eq. (4-53a) is computed. This matrix is called 'A' in the program. The determinant of 'A' is then computed by the function subroutine 'MATDET'. The value of the determinant is then returned
to the main program.

The main program varies in order to approach a root of the determinant. The method used for this is the secant method.¹

The second program solves the same problem as the first, but in the case where the depth is finite, it is similar, except that the four solutions are generated by direct numerical solution of the differential equations. In the program shown in Program Listing C-2 the method used is the Runge-Kutta method; other methods were also used.

The third program, listed in Program Listing C-3, finds the frequency of a surface wave. This program is simpler than the previous two, since it involves only the solution of Eq. (4-111). The method used to find the root is functional iteration with Aiken extrapolation.¹
PROGRAM LISTING C-1

IMPLICIT REAL*8(A-H,M,0-Z)
COMPLEX*16 S,S1,S2,J,CDSQRT,G1,G2,FUN
COMPLEX*16 G
COMMON XK,XKZ,M,GBAR,PRM
NL11=21
NL12=15
NLOOK=6
DSR=.02D-6
PRM=1.D-6
DSR=DSR/10.D0
PRM=PRM/10.D0
DSI=2.D-5
J=(0.D0,1.D0)
M=200.D0
M=M*.81D0
GBAR=3.D2
XK=0.D0
DO 3001 IMAD=1,NL12
XK=XK+.2D0
XKZ=0.D0
S1=DQSQT(XK*GBAR*PRM+XKZ*XKZ*M*PRM)
S1=J*S1
S2=S1-XK*XK*PRM
G1=FUN(S1)
PRINT,S1,G1
G2=FUN(S2)
PRINT,S2,G2
DO 3001 ILOOK=1,NLOOK
S=S2*G1/(G1-G2)+S1*G2/(G2-G1)
G=FUN(S)
WRITE(6,1)XK,M,S,G
1 FORMAT('XK=','D13.6,' M=','D13.6,' S=','2D13.6,'
D=','
1 2D13.6)
S1=S2
S2=S
G1=G2
G2=G
3001 CONTINUE
STOP
END
COMPLEX FUNCTION MATDET*16(A,IA,JA,MA,IR,IC)
COMPLEX*16 A(IA,JA),PIV,PIV1,PIV2  
REAL*8 CDABS  
DIMENSION IR(MA),IC(MA)  
COMPLEX*16 ZONE  
ZONE = (1.D0,0.D0)  
DO 1 I=1,MA  
IR(I)=0  
IC(I)=0  
1 CONTINUE  
KS=0  
KR=MA  
MATDET=1.0D0  
2 CONTINUE  
CALL MAXFND(A,IA,JA,MA,MA,IR,IC,I,J)  
PIV=A(I,J)  
MATDET=MATDET*PIV  
IF(CDABS(PIV).EQ.0.0D0) GO TO 6  
P IV2=1.0D0/PIV  
P IV2=ZONE/PIV  
IR(I)=J  
IC(J)=I  
DO 4 K=1,MA  
IF(K.EQ.I) GO TO 4  
PIV1=A(K,J)*PIV2  
DO 3 L=1,MA  
IF(IC(L).EQ.0)A(K,L)=A(K,L)-A(I,L)*PIV1  
IF(K.EQ.1.AND.L.EQ.7) WRITE (6,97)K,L,A(K,L)  
97 FORMAT ('***DEBUG 97', I5,I5,2E20.10)  
3 CONTINUE  
4 CONTINUE  
KS=KS+1  
IF(KS.LT.KR) GO TO 2  
DO 5 I=1,MA  
J=IR(I)  
M=IC(I)  
IF(I.FQ.J) GO TO 5  
IC(J)=M  
IR(M)=J  
MATDET=-MATDET  
5 CONTINUE  
6 CONTINUE  
RETURN  
END  
SUBROUTINE MAXFND(A,IA,JA,MA,NA,IR,IC,I,J)  
COMPLEX*16 A(KA,JA)  
REAL*8 TEST,X,CDABS  
DIMENSION IR(MA),IC(NA)  
I=0  
J=0  
TEST=0.0D0
DO 2 K=1,MA
IF(IR(K).NE.0) GO TO 2
DO 1 L=1,NA
IF(IC(L).NE.0) GO TO 1
X=CDABS(A(K,L))
IF(X.LT.TEST) GO TO 1
I=K
J=L
TEST=X
1 CONTINUE
2 CONTINUE
RETURN
END
COMPLEX FUNCTION FUN*16(S)
 IMPLICIT REAL*8(A-H,M,O-Z)
 INTEGER*4 I,N
 INTEGER*4 U4I
 COMPLEX*16 P,S,V,VHAT,BP,BPHAT,VRAT,GAMAM2,
 GAMAP2,GAMAM,GAMAP
 COMPLEX*16 CDSQRT,J
 CC:PLEX*16 BM,BMHAT
 COMPLEX*16 FROOT,A(4,4)
 COMPLEX*16 CDEXP,NURD,TENSE,D3V
 COMPLEX*16 DV,DBM,MADET,XDET,DBP
 COMPLEX*16 C(2,4,4),HOLD(4,4),XHOLD,EMLOOK,
 CON9,CON2,CON3
 COMPLEX*16 D2V
 DIMENSION IR(4),IC(4)
 COMMON XK,XKZ,M,GBAR,PRM
 EMLOOK=M
 N=10
 TOL=1.D-32
 J=(0.D0,1.D0)
 T1=1.D0
 T2=2.D0
 D1=-1190477156020638D02
 D2=1440355260362196D01
 GAMAP2=XK*XK+2.D0*(S+J)
 GAMAP=CDSQRT(GAMAP2)
 GAMAM2=XK*XK+2.D0*(S-J)
 GAMAM=CDSQRT(GAMAM2)
 FROOT=CDSQRT(XK*XK+S/PRM)
 C///<
 PRESSURE MODE
 IMODE=1
 VHAT=1.D0
 V=01.D0
 DV=XK
 BP=0.D0
 BM=0.D0
DBP=0.0
DBM=0.0
D(1,IMODE,1)=VHAT
C(1,IMODE,2)=DV
P=KK
D2V=P*DV
D3V=P*D2V
DO 100 L=1,N
I=I-1
BMHAT=VHAT*J*XKZ/(2.00*((P+1.00-J)**2-GAMAM2))
BPHAT=VHAT*J*XKZ/(2.00*((P+1.00+J)**2-GAMAP2))
IF(I.GE.2)GO TO 105
C(L,IMODE,3)=(P-KK)*BPHAT
C(L,IMODE,4)=(P-KK)*BMHAT
105 CONTINUE
BP=BP+BPHAT*M**I
BM=BM+BMHAT*M**I
DBP=NPB+(P+1.00+J)*BPHAT*M**I
DBM=NBM+(P+1.00-J)*BMHAT*M**I
P=P+2.00
VHAT=VRAT(S,P,KK,XKZ,PRM,BPHAT,BMHAT,VHAT)
IF(I.GE.2)GO TO 101
C(2,IMODE,1)=VHAT
C(2,IMODE,2)=P*VHAT
101 CONTINUE
V=V+VHAT*M**(I+1)
DV=DV+P*VHAT*M**(I+1)
D2V=D2V+P*P*M**(I+1)*VHAT
D3V=D3V+P*P*P*M**(I+1)*VHAT
IF(CDABS(VHAT).LT.TOL)GO TO 188
100 CONTINUE
188 CONTINUE
A(1,IMODE)=DBP+KK*BP
A(2,IMODE)=DBM+KK*BM
A(3,IMODE)=V
A(4,IMODE)=D2V
A(1,IMODE)=4.00*(DBP+KK*BP)-J*(1.00+J)*XKZ*
((S/PRM+3.00*KK*KK)*DV
2-D3V*M*(J*KK*(KK-XKZ)*(BP+BM)-J*XKZ*((1.00-J)*BP
3+(1.00+J)*BM)))/((GBAR*KK*XX+.500*MXKZ*XX)
A(2,IMODE)=(1.00-J)*(DBP+KK*BP)-(1.00+J)*
(DBM+KK*BM)
A(3,IMODE)=(GBAR*KK*XX+.500*MXKZ*XX)*V+
(S/S/PRM+3.00*S*KK*KK)*DV
2-S*D3V*M*(J*KK*(KK-XKZ)*(BP+BM)-J*XKZ*
3((1.00-J)*BP+(1.00+J)*BM))*S
A(4,IMODE)=D2V+KK*KK*V
C///
VISCOUS MODE
IMODE=2
VHAT=1.0D0
V=0.1D0
DV=FROOT
C(1,IMODE,1)=VHAT
C(1,IMODE,2)=DV
BP=0.0D0
BM=0.0D0
DBP=0.0D0
DBM=0.0D0
P=FROOT
D2V=P*DV
D3V=P*D2V
DO 200 L=1,N
I=I-1
BMHAT=VHAT*J*XKZ/(2.0D0*((P+1.0D0-J)**2-GAMAM2))
BPHT=VHAT*J*XKZ/(2.0D0*((P+1.0D0+J)**2-GAMAP2))
IF (I.GE.2) GO TO 205
C(L,IMODE,3)=(P-XK)*BPHT
C(L,IMODE,4)=(P-XK)*BMHAT
205 CONTINUE
BP=BP+BPHT*M**I
BM=BM+BMHAT*M**I
DBP=DBP+(P+1.0D0+J)*BPHT*M**I
DBM=DBM+(P+1.0D0-J)*BMHAT*M**I
P=P+2.00
VHAT=VRAT(S,P,XK,XKZ,PRM,BPHT,BMHAT,VHAT)
IF (I.GE.2) GO TO 201
C(2,IMODE,1)=VHAT
C(2,IMODE,2)=P*VHAT
201 CONTINUE
V=V+VHAT*M**(I+1)
DV=DV+P*VHAT*M**(I+1)
D2V=D2V+P*P*M**(I+1)*VHAT
D3V=D3V+P*P*P*M**(I+1)*VHAT
IF (CABS(VHAT).LT.TOL) GO TO 288
200 CONTINUE
288 CONTINUE
A(1,IMODE)=DBP+XK*BP
A(2,IMODE)=DBM+XK*BM
A(3,IMODE)=V
A(4,IMODE)=D2V
A(1,IMODE)=4.0D0*DBP+XK*BP-J*(1.0D0+J)*XKZ*
((S/PRM+3.0D0*XK*XK)*DV
2 -D3V+M*(J*XK*(XK-XKZ)*(BP+BM)-J*XKZ*((1.0D0-J)*BP
3 +(1.0D0+J)*BM))/((GBAR*XK*XK+5D0*M*XKZ*XKZ)
A(2,IMODE)=(1.0D0-J)*DBP+XK*BP-(1.0D0+J)*
(GBM+XK*BM)
A(3,IMODE)=(GBAR*XK*XK+5D0*M*XKZ*XKZ)*V+
(S*S/PRM+3.0D0*S*XK*XK)*DV
2  \(-S*D3V+M*(J*XK*(XK-XKZ)*(BP+BM)-J*XKZ*
3  ((1.D0-J)*BP+(1.D0+J)*BM))*S
A(4,IMODE)=D2V+XK*XK*V

C//M
B+ROOT
IMODE=3
dV=0.D0
VHAT=0.D0
V=0.D0
BM=0.D0
BP=0.D0
DBP=0.D0
DBM=0.D0
C(1,IMODE,1)=VHAT
C(1,IMODE,2)=DV
p=GAMAP-1.D0-J
D2V=p*DV
D3V=p*D2V
D0 300 L=1,N
I=L-1
BMHAT=VHAT*J*XKZ/(2.D0*((P+1.D0-J)**2-GAMAM2))
IF(I.GT.0)GO TO 301
BPHAT=1.D0
GO TO 302
301 BPHAT=VHAT*J*XKZ/(2.D0*((P+1.D0+J)**2-GAMAP2))
302 CONTINUE
IF(I.GE.2)GO TO 305
C(L,IMODE,3)=(P-XK)*BPHAT
C(L,IMODE,4)=(P-XK)*BMHAT
305 CONTINUE
BM=BM+BMHAT*M**I
BP=BP+BPHAT*M**I
DBP=DBP+(P+1.D0+J)*BPHAT*M**I
DBM=DBM+(P+1.D0-J)*BMHAT*M**I
P=P+2.D0
VHAT=VRAT(S,P,XK,XKZ,PRM,BPHAT,BMHAT,VHAT)
IF(I.GE.2)GO TO 304
C(2,IMODE,1)=VHAT
C(2,IMODE,2)=P*VHAT
304 CONTINUE
V=V+VHAT*M**(I+1)
DV=DV+P*VHAT*M**(I+1)
D2V=D2V+P*P*M**(I+1)*VHAT
D3V=D3V+P*P*P*M**(I+1)*VHAT
IF(CDABS(BPHAT).LT.TOL)GO TO 388
300 CONTINUE
388 CONTINUE
A(1,IMODE)=DBP+XK*BP
A(2,IMODE)=DBM+XK*BM
A(3,IMODE)=V
A(4,IMODE)=D2V
A(1,IMODE)=4.D0*(DBP+XK*BP)-J*(1.D0+J)*XKZ* 
((S/PRM+3.D0*XK*XK)*DV
2-D3V+M*(J*XK*(XK-XKZ)*(BP+BM)-J*XKZ*((1.D0-J)*BP
3+(1.D0+J)*BM))/(GBAR*XK*XK+.5D0*M*XKZ*XKZ)
A(2,IMODE)=(1.D0-J)*(DBP+XK*BP)-(1.D0+J)*(DBM+XK*BM)
A(3,IMODE)=(GBAR*XK*XK+.5D0*M*XKZ*XKZ)*V+
(S*S/PRM+3.D0*S*XK*XK)*DV
2-S*D3V+M*(J*XK*(XK-XKZ)*(BP+BM)-J*XKZ*
3((1.D0-J)*BP+(1.D0+J)*BM))*S
A(4,IMODE)=D2V+XK*XK*V

C:/// B-ROOT
IMODE=4
VHAT=0.D0
V=0.D0
DV=0.D0
BM=0.D0
BP=0.D0
DBM=0.D0
C(1,IMODE,2)=DV
C(1,IMODE,1)=VHAT
DBP=0.D0
P=GAMAM-1.D0+J
D2V=P*DV
D3V=P*D2V
D0 400 L=1,N
I=L-1
BPHAT=VHAT*J*XKZ/(2.D0*((P+1.D0+J)**2-GAMAM2))
IF(I.GT.0)GOTO 401
BMHAT=1.D0
GOTO 402
401 BMHAT=VHAT*J*XKZ/(2.D0*((P+1.D0-J)**2-GAMAM2))
402 CONTINUE
IF(I.GE.2)GOTO 405
C(L,IMODE,3)=(P-XK)*BPHAT
C(L,IMODE,4)=(P-XK)*BMHAT
405 CONTINUE
BP=BP+BPHAT*M**I
BM=BM+BMHAT*M**I
DBP=DBP+(P+1.D0+J)*BPHAT*M**I
DBM=DBM+(P+1.D0-J)*BMHAT*M**I
P=P+2.D0
VHAT=VRAT(S,P,XK,XKZ,PRM,BPHAT,BMHAT,VHAT)
IF(I.GE.2)GOTO 404
C(2,IMODE,1)=VHAT
C(2,IMODE,2)=P*VHAT
404 CONTINUE
V=V+VHAT*M**(I+1)
DV=DV+P*VHAT*M**(I+1)
D2V=D2V+P*P*M**(I+1)*VHAT
D3V=D3V+P*P*M**(I+1)*VHAT
IF(CDABS(BMHA1L).LT.TOL)GO TO 488
400 CONTINUE
488 CONTINUE
A(1,IMODE)=DBP+XK*BP
A(2,IMODE)=DBM+XK*BM
A(3,IMODE)=V
A(4,IMODE)=D2V
A(1,IMODE)=4.D0*(DBP+XK*BP)-J*(1.D0+J)*XKZ*
  ((S/PRM+3.D0*XK*XK)*DV
2 -D3V+M*(J*XK*(XK-XKZ)*(BP+BM)-J*XKZ*((1.D0-J)*BP
3 +(1.D0+J)*BM)))/(GBAR*XK*XK+.5D0*M*XKZ*XKZ)
A(2,IMODE)=(1.D0-J)*(DBP+XK*BP)-(1.D0+J)*
  (DBM+XK*BM)
A(3,IMODE)=(GBAR*XK*XK+.5D0*M*XKZ*XKZ)*V+
  (S/PRM+3.D0*S*XK*XK)*DV
2 -S*D3V+M*(J*XK*(XK-XKZ)*(BP+BM)-J*XKZ*
3 ((1.D0-J)*BP+(1.D0+J)*BM))*S
A(4,IMODE)=D2V+XK*XK*V
XDET=MATDET(A,4,4,IR,IC)
YDET=CDABS(XDET)
6666 CONTINUE
FUN=XDET
RETURN
END
COMPLEX FUNCTION VRAT16(S,P,XK,XKZ,PRM,BP,BM,V)
IMPLICIT REAL*8(A-H,M-O-Z)
COMPLEX*16 S,BP,BM,V,P,CON1,CON2,CON3,CON6,J
COMPLEX*16 CON5
J=(0.D0,1.D0)
CON4=XK*XK
CON5=S/PRM
CON1=P*P-XK*XK
CON2=CON1*CON1
CON3=S+2.D0*J
CON6=S-2.D0*J
CON8=XKZ*XKZ
VRAT=CON8*V-2.D0*J*XKZ*(CON3*BP+CON6*BM)
VRAT=VRAT/(CON5*CON1-CON2)
RETURN
END
$ENTRY
$STOP
//C.SYSIN DD *
$ENTRY
$STOP
/*EOJ */****
PROGRAM LISTING C-2

COMPLEX*16 MATDET, V(12), DV(6,12), VPNEW(12),
VCNEW(12), DVNEW(12)
COMPLEX*16 DET, S
COMPLEX*16 A(4,4)
REAL*8 T, TNEW, H
REAL*8 X, DELTA, M, XK, XKZ, DFLOAT, M0, M1, DEXP
REAL*8 DREAL, DIMAG, PRM
REAL*8 DABS, D0, D1
DIMENSION IR(4), IC(4)
COMMON M, XK, XKZ, S, PRM
S=0. D0
N=8
NUM=10
DELTA=2. D0
H=DELTA/DFLOAT(NUM)
PRM=1. D-6
M0=0. D0
M1=200. D0
XK= - .1D0
DO 3000 ILOOK2=1, 7
PRINT, XK
XK=XK+.2D0
XKZ=XK
M0=M0*.5D0
M1=M1*1.5D0
DO 2000 ILOOK=1, 7
IF (ILOOK.GT.2) GO TO 403
IF (ILOOK.GT.1) GO TO 402
M=M0
GO TO 404
402 M=M1
GO TO 404
403 CONTINUE
M=M1*D0-M0*D1
M=M/(D0-D1)
404 CONTINUE
DO 1000 IMODE=1, 4
X=0
DO 1111 KLM=1, 8
1111 V(KLM)=0.D0
IF (IMODE.EQ.2) GO TO 200
IF (IMODE.EQ.3) GO TO 300
IF (IMODE.EQ.4) GO TO 400
V(1)=1. D0
GO TO 500
200 V(3)=1. D0
GO TO 500
300 V(5)=1.D0
GO TO 500
400 V(7)=1.D0
500 CONTINUE
   DO 100 I=1 NUM
   CALL RUNGE(X,V,DVNEW,H,N)
100 CONTINUE
   A(1,IMODE)=V(2)
   A(2,IMODE)=V(1)
   A(3,IMODE)=V(5)
   A(4,IMODE)=V(7)
1000 CONTINUE
   DET=MATDET(A,4,4,4,IR,IC)
   PRINT,M,DET
   IF(ILook.GT.2)GO TO 503
   IF(ILook.GT.1)GO TO 502
   D0=DREAL(DET)
   GO TO 2000
502 D1=DREAL(DET)
   GO TO 2000
503 CONTINUE
   M0=M1
   D0=D1
   M1=M
   D1=DREAL(DET)
2000 CONTINUE
3000 CONTINUE
   STOP
   END
   SUBROUTINE RUNGE(T,V,DVNEW,H,N)
   COMPLEX*16 DVNEW(12),V(12),XK0(12),XK1(12),
   XK2(12),XK3(12)
   REAL*8 T,H
   CALL DERIV(T,V,DVNEW)
   D0 1 I=1,N
   XK0(I)=H*DVNEW(I)
   1 V(I)=V(I)+.5D0*XK0(I)
   T=T+.5D0*H
   CALL DERIV(T,V,DVNEW)
   D0 3 I=1,N
   XK1(I)=H*DVNEW(I)
   3 V(I)=V(I)+.5D0*(XK1(I)-XK0(I))
   CALL DERIV(T,V,DVNEW)
   D0 5 I=1,N
   XK2(I)=H*DVNEW(I)
   5 V(I)=V(I)+XK2(I)-.5D0*XK1(I)
   T=T+H*.5D0
   CALL DERIV(T,V,DVNEW)
DO 7 I=1,N
XK3(I)=H*DVEW(I)
7 V(I)=V(I)-XK2(I)+(XK0(I)+2.D0*(XK1(I)+XK2(I))
      +XK3(I))/6.D0
CALL DERIV(T,V,DVEW)
RETURN
END

SUBROUTINE DERIV(T,VPNEW,DVEW)
COMPLEX*16 F,FSTART,FF,J,VPNEW(12),DVEW(12)
REAL*8 T,PRM
COMPLEX*16 CDSQRT,S
REAL*8 M,XK,XKZ
REAL*8 DREAL,DIMAG
COMMON M,XK,XKZ,S,PRM
J=(0.D0,1.D0)
FF=F(T)
FSTART=DREAL(FF)-J*DIMAG(FF)
DVEW(1)=VPNEW(2)
DVEW(2)=VPNEW(3)
DVEW(3)=VPNEW(4)
DVEW(4)=-(XX**4+XKZ*XXZ*M*FF*FSTART)*VPNEW(1)
      +2.D0*XXZ*XXZ*VPNEW(3)
1 -4.D0*XXZ*M*(FSTART*VPNEW(5)-FF*VPNEW(7))
2 +S*VPNEW(1)/PRM+2.D0*J*XXZ*M*(FSTART*S*VPNEW(5)
3 +FF*S*VPNEW(7))
DVEW(5)=VPNEW(6)
DVEW(7)=VPNEW(8)
DVEW(6)=J*XXZ*FF*VPNEW(1)/2.D0+(XXZ*XXZ+2.D0*
      J+S)*VPNEW(5)
DVEW(8)=J*XXZ*FSTART*VPNEW(1)/2.D0+(XXZ*XXZ-2.D0*
      J+S)*VPNEW(7)
RETURN
END

COMPLEX FUNCTION F*17(X)
REAL*8 X
REAL*8 DELTA,DCOS,DCOSH
COMPLEX*16 J,CDKCS
COMPLEX*16 CDEXP
REAL*8 U
J=(0.D0,1.D0)
DELTA=2.D0
F=CDKCS((1.D0-J)*X)/DCOS((1.D0-J)*DELTA)
RETURN
END

COMPLEX FUNCTION MATDET*16(A,IA,JA,MA,IR,IC)
COMPLEX*16 A(IA,JA),PIV,PIV1,PIV2
REAL*8 CDABS
DIMENSION IR(MA),IC(MA)
COMPLEX*16 ZONE
ZONE=(1.D0,0.D0)
DO 1 I=1,MA
   IR(I)=0
   IC(I)=0
1 CONTINUE
   KS=0
   KR=MA
   MATDET=1.0D0
2 CONTINUE
   CALL MAXFND(A,IA,JA,MA,MA,IR,IC,I,J)
   PIV=A(I,J)
   MATDET=MATDET*PIV
   IF(CDABS(PIV).EQ.0.D0) GO TO 6
   C PIV2=1.0D0/PIV
      PIV=ZONE/PIV
   IR(I)=J
   IC(J)=I
   DO4K=1,MA
      IF(K.EQ.I) GO TO 4
      PIV1=A(K,J)*PIV2
   DO 3 L=1,MA
      IF(IC(L).EQ.0) A(K,L)=A(K,L)-A(I,L)*PIV1
      IF(K.EQ.1.AND.L.EQ.7) WRITE(6,97) K,L,A(K,L)
97 FORMAT('***DEBUG 97',I5,I5,2E20.10)
3 CONTINUE
4 CONTINUE
   KS=KS+1
   IF(KS.LT.KR) GO TO 2
   DO 5 I=1,MA
      J=IR(I)
      M=IC(I)
      IF(I.EQ.J) GO TO 5
      IC(J)=M
      IR(M)=J
      MATDET=-MATDET
5 CONTINUE
6 CONTINUE
   RETURN
END

SUBROUTINE MAXFND(A,IA,JA,MA,NA,IR,IC,I,J)
COMPLEX*16 A(IA,JA)
REAL*8 TEST,X,CDABS
DIMENSION IR(MA),IC(NA)
I=0
J=0
TEST=0.0D0
DO 2 K=1,MA
   IF(IR(K).NE.0) GO TO 2
   DO 1 L=1,NA
      IF(IC(L).NE.0) GO TO 1
X=CDABS(A(K,L))
IF(X.LT.TEST)GO TO 1
I=K
J=L
TEST=X
1 CONTINUE
2 CONTINUE
RETURN
END

$ENTRY
$STOP
//C.SYSIN DD *
$ENTRY
$STOP
/*/EOJ *******
PROGRAM LISTING C-3

IMPLICIT REAL*8 (A-H,O-Z)
COMPLEX*16 S,F,S1,S2,F1,F2,CSQRT
COMPLEX*16 S01,S02,H
COMPLEX*16 J
COMMON XK,THETA,PSI
PSI=1.E-5
DTH=.1
J=(0.,1.)
NLOOK=2
NL2=16
XK=0.
DO 11 IART=1,20
   XK=XK+.1
11 WRITE(6,7)
7 FORMAT(/////////)
   S=J*DSQRT(XK)-XK*XK*PSI*2.
   THETA=-DTH
   DO 11 K=1,NL2
      THETA=THETA+DTH
      WRITE(6,2) XK,THETA,PSI
   WRITE(6,2) XK,THETA,PSI
2 FORMAT('OXK=',E13.6,' THETA=',E13.6,' PSI=',E13.6)
   IF (K.EQ.1) GO TO 30
   IF (K.EQ.2) GO TO 31
   S01=S02
   S02=S
   S=2.*S02-S01
   GO TO 30
31   S02=S
30   CONTINUE
   WRITE(6,1)S
1 FORMAT('S=',E13.6,2X,E13.6)
   DO 10 I=1,NLOOK
      S=H(S)
      S=H(S)
      S1=H(S)
      S2=H(S1)
      S=H(S2)
      F2=S-S2
      R=CABS(F2)
      IF (R.EQ.0.) GO TO 12
      F1=S-2.*S2+S1
      R=CABS(F1)
      IF (R.EQ.0.) GO TO 10
      S=S-F2*F2/F1
      WRITE(6,4)S
10  CONTINUE
12 CONTINUE
WRITE (6, 3) S
3 FORMAT ('+', 40X, 'S=', ',E13.6,2X, 13.6)
11 CONTINUE
STOP
END
COMPLEX FUNCTION F*16(S)
IMPLICIT REAL*8 (A-H, O-Z)
COMPLEX*16 J, OPJ, OMJ, GP2, GP, GM2, GM, PP, PP2, PM,
    PM2, GN2, GN, AP
COMPLEX*16 AM, APR, AN, CSQRT, S, SUM1, SUM2, SUM3
COMMON XK, THETA, PSI
J=(0., 1.)
XK2=XK**2
BETA=3.D-3
OPJ=1.+J
OMJ=1.-J
GP2=XK2+2.*J
GP2=GP2+S*BETA**2.
GP=CSQRT (GP2)
GM2=XK2-2.*J
GM2=GM2+S*BETA**2.
GM=CSQRT (GM2)
PP=GP+OMJ
PP2=PP*PP
PM=GM+OPJ
PM2=PM*PM
GN2=XK2+S/PSI
GN=CSQRT (GN2)
AP=-XK2*2.*J*OPJ*THETA/(S*(GP+XK)*(S*(PP2-XK2)-
    PSI*(PP2-XK2)**2))
AP=AP*(1.+S*BETA/(2.*J))
AM=XK2*2.*J*OMJ*THETA/(S*(GM+XK)*(S*(PM2-XK2)
    -PSI*(PM2-XK2)**2))
AM=AM*(1.-S*BETA/(2.*J))
APR=(AP*(PP2-GN2)+AM*(PM2-GN2)+(GN2+XK2))/(GN2-XK2)
AN=-(2.*XK2+AP*(PP2-XK2)+AM*(PM2-XK2))/(GN2-XK2)
SUM1=AP*PP+AM*PM+APR*XK+AN*GN
SUM2=AP*PP2*PP+AM*PM2*PM+APR*XK*2*XK+AN*GN2*GN
SUM3=(1.//(GP+XK))+(1.//(GM+XK))
F=-3.*XK2*S*PSI+(PSI*S*SUM2-XK2*(1.+THETA)+
    XK2*THETA*SUM3)/SUM1
RETURN
END
COMPLEX FUNCTION H*16(S)
IMPLICIT REAL*8 (A-H, O-Z)
COMPLEX*16 S, S1, F, CSQRT
S1=CSQRT (F(S))
T1=CABS (S1-S)
T2 = CABS(S1+S)
IF(T1.LT.T2) GO TO 11
S1 = -S1

11 H = S1
RETURN
END

REAL FUNCTION CABS*8(S)
COMPLEX*16 S
REAL*8 CDABS
CABS = CDABS(S)
RETURN
END

COMPLEX FUNCTION CSQRT*16(S)
COMPLEX*16 CDSQRT, S
CSQRT = CDSQRT(S)
RETURN
END
APPENDIX C REFERENCES

APPENDIX D

TORQUE ON A ROTATING CYLINDER
IN A VISCOUS FLUID

A cylinder rotating in an infinite space of viscous fluid is subject to a viscous sheer stress at its surface:

\[ \tau = \eta \frac{\partial \nu_{\theta}}{\partial r} \]  \hspace{1cm} (E-1)

It is assumed that the flow is laminar. By symmetry, then, \( \nu \) is all in the \( \theta \) direction, and a function only of \( r \). In that case the Navier-Stokes equation reduces to:

\[ \frac{\rho}{\eta} \frac{1}{r} \frac{\partial \nu_{\theta}}{\partial t} = \nu^2 (\nu_{\theta} i_{\theta}) \]  \hspace{1cm} (E-2)

\[ \frac{\rho}{\eta} \frac{\partial \nu_{\theta}}{\partial t} = \frac{\partial^2}{\partial r^2} \nu_{\theta} + \frac{1}{r} \frac{\partial \nu_{\theta}}{\partial r} - \frac{1}{r^2} \nu_{\theta} \]  \hspace{1cm} (E-3)

In steady motion, \( \frac{\partial}{\partial t} = 0 \) and the solution for that remains finite at infinity is:

\[ \nu_{\theta} = Ar^{-1} \]  \hspace{1cm} (E-4)
The cylinder of radius $R$ rotates at a speed $\Omega$, thus determining $v_\theta$:

$$v_\theta = \frac{\Omega R^2}{r} \quad (E-5)$$

The shear stress follows directly by substituting Eq. (E-5) into Eq. (E-1):

$$\tau = \eta \frac{\partial v_\theta}{\partial r}$$

$$= -\eta \frac{\Omega R}{r}$$

The torque per unit length is:

$$\Gamma = 2\pi R^2 \tau \quad (E-6)$$

$$= -2\pi \eta R \Omega \quad (E-7)$$

In many stability problems the motion of the shell is exponential in time:

$$\dot{\Omega} = \text{Re}[\dot{\Omega}e^{\lambda t}] \quad (E-8)$$

$$v_\theta = \text{Re}[v_\theta(r)e^{\lambda t}] \quad (E-9)$$

This form introduced into Eq. (E-3) results in:
\[ 0 = r^2 \frac{d^2 \hat{\nu}_\theta}{dr^2} + r \frac{d \hat{\nu}_\theta}{dr} - \left(1 + \frac{\rho^2 r^2}{\eta^2}\right) \hat{\nu}_\theta \]  

(E-10)

The solution to this equation that satisfies the boundary conditions is:

\[ \hat{\nu}_\theta = \Omega R \frac{K_1(\sqrt{\frac{\rho s}{\eta}} R)}{K_1(\sqrt{\frac{\rho s}{\eta}} R)} \]  

(E-11)

\( K_1 \) is a modified Bessel function (Kelvin function).\(^1\)

\( Y_1 \) is the modified Bessel function (Weber function).\(^1\)

The torque follows directly from Eq. (E-1) and the recurrence relations for the Bessel functions

\[ \Gamma = - \frac{2 \pi R^2 \Omega \rho s \eta \left[ K_0(\sqrt{\frac{\rho s}{\eta}} R) + K_2(\sqrt{\frac{\rho s}{\eta}} R) \right]}{2 K_1(\sqrt{\frac{\rho s}{\eta}} R)} \]  

(E-12)

This solution for the torque is shown in Fig. E-1.

It can be shown that in the limit of \( s \to 0 \) the answer of Eq. (E-7) is attained:

\[ \lim_{x \to 0} K_1(x) = -\ln x \]  

(E-13)

\[ \lim_{x \to 0} K_2(x) = 2x^{-2} \]  

(E-14)
\[
\ln K_1(x) = x^{-1}
\]
\[
x \to 0
\]

(E-15)

Inserting these into Eq. (E-12):

\[
\ln \Gamma = -2\pi \eta \hat{R}\Omega
\]
\[
s \to 0
\]

(E-16)
APPENDIX D REFERENCES