Multipath Aided Rapid Acquisition

by

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Abstract

Spread-spectrum systems with large transmission bandwidth present significant challenges from the standpoint of achieving synchronization before communication commences. This research investigates a rapid acquisition procedure that exploits the multipath to aid the synchronization. In particular, we consider a class of serial search strategies and determine the optimal search procedure for the uncertainty space consisting of N_{S} total cells and N_{Q} correct cells. We derive closed-form expressions for both the minimum and maximum mean acquisition times (MATs) and the conditions for achieving these limits. We prove that the fixed-step serial search (FSSS), with the step size N_{Q}, achieves the near-optimal MAT. We also prove that the conventional serial search, in which consecutive cells are tested sequentially, and the FSSS with the step size $N_{\mathrm{S}} - 1$ should be avoided as they result in the maximum MAT. Analytical tools used in the research include Markov chain diagrams, the transformation of feasible spaces, and convexity theory. Our results apply to all signal-to-noise ratio values, regardless of the detection-layer decision rule and the fading distribution. The impact of this research is significant for the design, implementation, and deployment of spread-spectrum systems.

Thesis Supervisor: Moe Z. Win
Title: Charles Stark Draper Assistant Professor

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Chapter 1

Introduction

This thesis is divided into five chapters. Chapter 1 discusses the motivation, the objectives, the scope, the commonly used notations, and the contributions of the research. Chapter 2 reviews the previous work that is related to this thesis. The steps that are used to accomplish the research objectives are outlined in Chapter 3. Chapter 4 presents the research results. Chapter 5 summarizes the important findings.

1.1 Motivation

Spread-spectrum systems present significant challenges from the standpoint of achieving synchronization at the receivers. Before communication commences, the receiver must search for the location of sequence phase within a required accuracy. This task is not easy to achieve, especially when the transmission bandwidth is large. This research investigates a method that exploits the multipath to aid the acquisition.

The receiver performs hypothesis testing in the acquisition stage. The number of phases or cells to test is proportional to the transmission bandwidth, while the number of correct cells is proportional to the number of resolvable paths. Because the resolvable paths arrive at the receiver near one another in the dense multipath channel, the correct cells are consecutive in the uncertainty index set. A flow chart in Fig. 1-1 depicts the steps that the receiver performs in the acquisition stage. The goal of designing an acquisition receiver is to minimize the mean acquisition time (MAT), the average duration required for the receiver to perform the acquisition stage.

In general, there are two approaches to improve the MAT. The first approach uses the

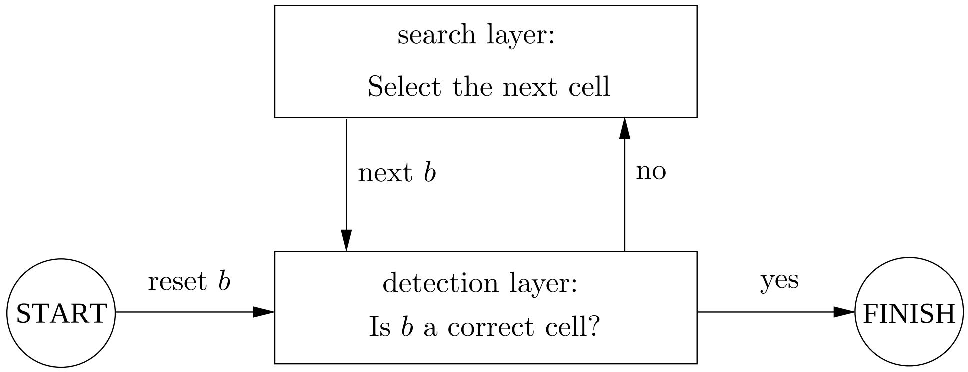

Figure 1-1: An acquisition receiver searches for a correct cell. The goal is to minimize the mean acquisition time (MAT), the average time to transit from START to FINISH.

optimal decision rule to improves the MAT at the detection layer. The second approach uses the optimal search order to improve the MAT at the search layer. Under certain conditions, one can optimize separately the two layers and still have the optimal acquisition receiver. This thesis improves the MAT at the search layer under the conditions that allow the separation between the search and detection layers. Section 1.3 will discuss the scope of the research and outline these conditions.

After selecting a search order, one can evaluate the MAT. Some search orders will yield a shorter MAT than does the others. Therefore, the central questions for this research are the following:

- What are the fundamental limits of the achievable MATs? In other words, what are the minimum and maximum MATs?
- What are the search orders that achieve the minimum MAT?
- What are the search orders that result in the maximum MAT?
- What are the benefits of using intelligent search strategies?

This thesis will investigate those questions.

1.2 Objectives of the Research

1. To find the minimum MAT.
2. To find the maximum MAT.

3. To find the search orders that achieve the minimum or the near-optimal MATs.

4. To avoid the search orders that exhibit the maximum MAT.

5. To quantify the benefits of using intelligent search strategies.

1.3 Scope of the Research

This research investigates a technique that improves the MAT and studies any spread-spectrum receiver with the following characteristics:

- The receiver employs a serial search.

- The receiver does not know the location of a correct cell.

- The receiver is equally likely to select any cell to test when it begins the search.

- The receiver tests the cells in the predetermined search order.

- If two distinct cells i and j are both correct cells, the decision variable Z_i for cell i and the decision variable Z_j for cell j are independent and identically distributed (i.i.d).

- If two distinct cells i and j are both incorrect cells, the decision variable Z_i for cell i and the decision variable Z_j for cell j are i.i.d.

Section 4.1 will model the receiver by a *non-preferential* flow diagram.

1.4 Notations and Definitions

1. N_{S}

   The number of total cells. This parameter satisfies $1 \leq N_{\mathrm{S}}$.

2. N_{Q}

   The number of correct cells. This parameter satisfies $1 \leq N_{\mathrm{Q}} \leq N_{\mathrm{S}}$.

3. $\mathbf{x}$

   A vector. If the dimension n of the vector is clear from the context, vector $\mathbf{x}$ may be referred to as $(x_1, x_2, \ldots, x_n)$.

4. $x \oplus y$

   A modulo N_{S} addition, which is defined to equal $x + y - lN_{\mathrm{S}}$, for the unique integer l such that $1 \leq x + y - lN_{\mathrm{S}} \leq N_{\mathrm{S}}$.

5. $x \triangleq y$

   Equal by definition.

6. The uncertainty index set

   The set of cells to test. This set is denoted by

$$\mathcal{U} \triangleq \{1, 2, 3, \ldots, N_{\mathrm{S}}\}.$$

7. B

   The first correct cell in the uncertainty index set. This random variable has a uniform distribution over $\mathcal{U}$.

8. $\mathcal{H}_{\mathrm{C}}$(b)

   The index set of the correct cells, conditioned on $B = b$:

$$\mathcal{H}_{\mathrm{C}}(b) \triangleq \{b, b \oplus 1, b \oplus 2, \ldots, b \oplus (N_{\mathrm{Q}} - 1)\}.$$

9. K

   The first cell that the receiver tests. This random variable is statistically independent of B and has a uniform distribution over $\mathcal{U}$.

10. An in-phase cell

    A cell in which the timing error between the received signal and the locally generated reference resides within a fraction of chip duration. This term is also referred to as a correct cell, a correct phase, or an H_1-state.

11. A non-in-phase cell

    An incorrect cell. This term is also referred to as an incorrect phase or an H_0-state.

12. The mean acquisition time (MAT)

    The average duration from the start of the acquisition stage until the end, in which the receiver finds a correct cell.

13. A search order

    An order of cells in which the receiver tests. The set of search orders is denoted by

$$\mathcal{P} \triangleq \left\{\pi \,\middle|\, \pi\colon \mathcal{U} \to \mathcal{U} \text{ is a permutation function and } \pi(1) = 1\right\}.$$

    One may refer to a search order π by the tuple $[\pi(1), \pi(2), \ldots, \pi(N_\mathrm{S})]$. A receiver that uses the search order π begins the search at any cell $\pi(k)$, for $1 \le k \le N_\mathrm{S}$, and tests the cells in the order

$$\pi(k), \pi(k+1), \ldots, \pi(N_\mathrm{S}), \pi(1), \pi(2), \ldots, \pi(N_\mathrm{S}), \pi(1), \pi(2), \ldots .$$

14. $\mathbb{E}\left\{T_\mathrm{ACQ}(\pi)\right\}$

    The mean acquisition time associated with the search order π.

15. The serial search

    A technique to search for a correct cell in the uncertainty index set. In particular, a receiver that uses a serial search must accept or reject a cell before it tests the next cell.

16. The conventional serial search (CSS)

    The serial search with the search order $\pi^1 \triangleq [1, 2, 3, \ldots, N_\mathrm{S}]$. In particular, a receiver uses the CSS if it tests the consecutive cells sequentially.

17. The fixed-step serial search with the step size N_J (FSSS-N_J)

    A serial search with the search order $\pi^{N_\mathrm{J}} \triangleq [1, 1 \oplus N_\mathrm{J}, 1 \oplus 2N_\mathrm{J}, \ldots, 1 \oplus (N_\mathrm{S} - 1)N_\mathrm{J}]$. In particular, a receiver uses the FSSS-N_J if it skips N_J cells after a test in one cell is complete. The step size N_J must be relatively prime with N_S, so that π^{N_J} is an element of the set $\mathcal{P}$ of search orders.

18. An η-optimal search order

    A search order π_η that satisfies

$$\frac{\mathbb{E}\left\{T_\mathrm{ACQ}(\pi_\eta)\right\} - \min_{\pi\in\mathcal{P}} \mathbb{E}\left\{T_\mathrm{ACQ}(\pi)\right\}}{\min_{\pi\in\mathcal{P}} \mathbb{E}\left\{T_\mathrm{ACQ}(\pi)\right\}} \le \eta(N_\mathrm{S}, N_\mathrm{Q}), \tag{1.1}$$

    where $\eta(\cdot)$ is some function only of N_S and N_Q, with the limit $\eta(N_\mathrm{S}, N_\mathrm{Q}) \to 0$ as

$N_Q/N_S \to 0$.

19. A non-preferential flow diagram

    A Markov flow diagram which has the following properties:

    (a) The probability of entering any non-absorbing state is uniform.

    (b) Every path going into the absorbing state has the same path gain.

    (c) Every path going out of an H_0-state has the same path gain.

    (d) Every path going out of an H_1-state to the adjacent non-absorbing state has the same path gain.

20. A description

    A tuple (π, b) of the search order and the location of the first in-phase cell. This tuple describes the arrangement of H_1-states and H_0-states in a flow diagram: the states are ordered according to π and the H_1-states are the elements of the set $\mathcal{H}_C(b)$. The set of descriptions is denoted by

$$\mathcal{D} \triangleq \left\{ (\pi, b) \,\middle|\, \pi : \mathcal{U} \to \mathcal{U} \text{ is a permutation function, } \pi(1) = 1, \text{ and } 1 \le b \le N_S \right\}.$$

21. A spacing rule

    A tuple $(m_1, m_2, \ldots, m_{N_Q})$, which describes the arrangement of H_1-states and H_0-states in a non-preferential flow diagram: an in-phase cell is followed by m_1 non-in-phase cells, which are then followed by another in-phase-cell, which is then followed by m_2 non-in-phase cells, and so on. Therefore, the non-negative integer m_i is the number of H_0-states between two neighboring H_1-states. The set of spacing rules is denoted by

$$\mathcal{S} \triangleq \left\{ (m_1, m_2, \ldots, m_{N_Q}) \,\middle|\, \sum_{i=1}^{N_Q} m_i = N_S - N_Q; \forall i, \text{integer } m_i \ge 0 \right\}.$$

22. The absorption time

    The average time to arrive at an absorbing state of a Markov flow diagram.

23. $v(\mathbf{m})$

    The absorption time for the flow diagram associated with the spacing rule $\mathbf{m}$.

24. An η-optimal subset of the set of spacing rules

A subset $\mathcal{S}_\eta \subset \mathcal{S}$ that has the following property: for every $\mathbf{m}_\eta \in \mathcal{S}_\eta$,

$$\frac{v(\mathbf{m}_\eta) - \min_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})}{\min_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})} \leq \eta(N_\text{S}, N_\text{Q}), \tag{1.2}$$

where $\eta(\cdot)$ is some function only of N_S and N_Q, with the limit $\eta(N_\text{S}, N_\text{Q}) \to 0$ as $N_\text{Q}/N_\text{S} \to 0$.

25. Clustering states in the flow diagram with the search order $[\pi(1), \pi(2), \ldots, \pi(N_\text{S})]$

Elements $s_1, s_2, \ldots, s_n$ of the uncertainty index set, for some $n \geq 1$, such that

$$\{s_1, s_2, \ldots, s_n\} = \Big\{\pi(i), \pi(i \oplus 1), \pi(i \oplus 2), \ldots, \pi(i \oplus (n-1))\Big\},$$

for some index $i \in \mathcal{U}$.

26. A decreasing function

A multi-variate function $h\colon \mathcal{W}^n \to \mathbb{R}$, for some set $\mathcal{W}$ and some positive integer $n \geq 1$, which has the following property:

For any $\mathbf{x} \in \mathcal{W}^n$ and $\mathbf{y} \in \mathcal{W}^n$, if $\forall 1 \leq i \leq n, x_i \leq y_i$, then $h(\mathbf{x}) \geq h(\mathbf{y})$.

27. An increasing function

A multi-variate function $h\colon \mathcal{W}^n \to \mathbb{R}$, for some set $\mathcal{W}$ and some positive integer $n \geq 1$, which has the following property:

For any $\mathbf{x} \in \mathcal{W}^n$ and $\mathbf{y} \in \mathcal{W}^n$, if $\forall 1 \leq i \leq n, x_i \leq y_i$, then $h(\mathbf{x}) \leq h(\mathbf{y})$.

28. Bounded from below

A property of a function $f\colon \mathcal{W} \to \mathbb{R}$, for some set $\mathcal{W}$. In particular, f is bounded from below, if there is a real number a such that

$$a \leq f(w),$$

for all elements $w \in \mathcal{W}$.

29. Bounded from above

A property of a function $f\colon \mathcal{W} \to \mathbb{R}$, for some set $\mathcal{W}$. In particular, f is bounded from above, if there is a real number a such that

$$f(w) \leq a,$$

for all elements $w \in \mathcal{W}$.

1.5 Contributions of the Research

- We introduce a concept of a *spacing rule*, which describes the structure of the flow diagram. Then, we derive an explicit expression for the absorption time as a function of a *spacing rule*, and derive the optimal *spacing rule* by using convexity theory.
- We derive the explicit expression for the minimum MAT over all possible search orders, and then prove that the search order $\pi^{N_{\mathrm{Q}}}$ is η-optimal.
- We derive the explicit expression for the maximum MAT, and then show that the conventional serial search and the fixed-step serial search with the step size $N_{\mathrm{S}} - 1$ yield this maximum.
- We show that the MAT of the optimal serial search is approximately constant, when the uncertainty index set is refined by a factor of r. In contrast, the MAT of the conventional serial search increases approximately r times.
- We show that the optimal serial search is approximately N_{Q} times faster than the conventional serial search in a high signal-to-noise ratio (SNR) environment.
- We show that the optimal serial search is approximately one to two times faster than the conventional serial search in a low SNR environment.

Chapter 2

Literature Review

Sequence synchronization is an important task of the spread spectrum receiver. Before communication commences, the receiver must search for the location of the sequence phase within the required accuracy, which depends on the autocorrelation properties of the spreading waveforms and is typically less than one chip duration. The synchronization process occurs in two stages: the acquisition stage and the tracking stage [8, 16]. The acquisition stage is the focus of this thesis.

The receiver performs a series of tasks during the acquisition stage. It coarsely aligns the sequence of the locally generated reference (LGR) with the sequence of the received signal. If the LGR phase does not correspond to the phase of the received signal, the receiver will move the LGR to a new phase position according to some strategy. If the receiver finds a correct sequence phase, it will enter the tracking stage to finely align the two sequences and maintain the synchronization throughout the communication. Therefore, the acquisition receiver is faced with the hypothesis testing problem [19].

There are two key parameters associated with the acquisition stage. The first parameter is the number N_{S} of phases or cells to test. The second parameter is the number N_{Q} of correct phases (in-phase cells), which is proportional to the number of resolvable paths. The uncertainty index set

$$\mathcal{U} = \{1, 2, 3, \ldots, N_{\mathrm{S}}\} \tag{2.1}$$

denotes the collection of cells to test. The signal acquisition is difficult to achieve when the number N_{S} of cells is large.

The parameter N_{S} depends on several factors, including the sequence period (T_{period}), the relative clock uncertainty (T_{clock}) between the transmitter and the receiver, and the accuracy (T_{res}) within which the acquisition system must resolve its uncertainty. The expression for N_{S} is given by

$$N_{\text{S}} = \frac{\min\{T_{\text{period}}, T_{\text{clock}}\}}{T_{\text{res}}}. \tag{2.2}$$

The parameter $1/T_{\text{res}}$ is proportional to the transmission bandwidth, and so is N_{S}. Therefore, the number of cells to test can be very large for the wide bandwidth transmission system.

Designing an acquisition system involves two broad design aspects. One aspect deals with how the decision is made at the detection layer. Examples of the relevant issues at the detection layer include combining methods for decision variables and the evaluations of the detection and false-alarm probabilities in the multipath channel. The other aspect deals with how the search for a correct cell is performed at the search layer. Examples of the relevant issues in the search layer include the following:

- What is the search strategy (fully parallel search, hybrid search, or serial search) to use?
- What is the search order (the sequence of cells to test)?

The performance of the acquisition system is measured by the mean acquisition time, the average duration required for the receiver to perform the acquisition stage.

A common approach for finding the mean acquisition time (MAT) is the use of flow diagram. For an additive white Gaussian noise (AWGN) channel, a flow diagram simply has one in-phase cell [9–11]. The expression of the MAT for an AWGN channel is given in [10]. For a multipath fading channel, the flow diagram has multiple in-phase cells, which correspond to the multiple resolvable paths [6,15,17,20–22]. The MATs are derived for the Rayleigh [15,20], Rician [17], and Nakagami-m fading channels [21]. A MAT expression as a function of parameters at the detection layer is given in [6]. Analysis based on a flow diagram that takes into account the dependence among the decision variables representing different cells is given in [14]. In general, the goal of the acquisition receiver is to find a correct sequence phase as fast as possible.

There are a number of approaches to improve the MAT. One approach is to dedicate more resources, such as correlators or matched filters, to improve the decision at the detection layer. The receiver in [20] uses the serial search with two correlators, while that in [15] uses the serial search with three correlators. Toward this end, [17] uses the hybrid search with an arbitrary number of correlators and [12] uses a fully parallel search with N_S correlators. The decision variables are formed by appropriately combining the correlator outputs [15, 21]. For frequency-selective Rayleigh and Rician fading channels, the optimal decision rules are given in [13]. For a hybrid search, each receiver in [3–5] partitions the uncertainty index set into smaller subsets, selects from a subset the cell with the maximum decision variable, and accepts the cell if its decision variable exceeds the threshold.

Another approach to improve the acquisition time is to use an intelligent search procedure. For example, the receivers in [6, 15] skip a fixed number of cells after a test in one cell is completed. The focus of this thesis is to improve the MAT by using an intelligent search procedure.

In a dense multipath channel, an intelligent search procedure will improve the MAT. If there is only one resolvable path, the number N_Q of correct cells is also one. Assume that every cell in the uncertainty index set is equally likely to be the correct cell. Then every search order gives the same MAT. By improving the MAT with an intelligent search order, we take advantage of the multipath, which has long been considered deleterious for efficient communication.

The set of all possible search orders is denoted by

$$\mathcal{P} = \left\{ \pi \,\middle|\, \pi : \mathcal{U} \rightarrow \mathcal{U} \text{ is a permutation function and } \pi(1) = 1 \right\}. \tag{2.3}$$

An element π of $\mathcal{P}$ is called a search order or a permutation function. One can denote π by the N_S-tuple $[\pi(1), \pi(2), \ldots, \pi(N_S)]$ to emphasize that the receiver tests the cells in the order

$$\pi(k), \pi(k+1), \ldots, \pi(N_S), \pi(1), \pi(2), \ldots, \pi(N_S), \pi(1), \pi(2), \ldots .$$

Here, $\pi(k) \in \mathcal{U}$ is the first cell that the receiver tests. Some common search orders that have been used in the literature are shown in Fig. 2-1.

The conventional serial search (CSS) [20, 21], where the consecutive cells are tested sequentially, yields the search order $[1, 2, 3, \ldots, N_S]$. The permutation function for this search order is

$$\pi^1(k) = k. \tag{2.4}$$

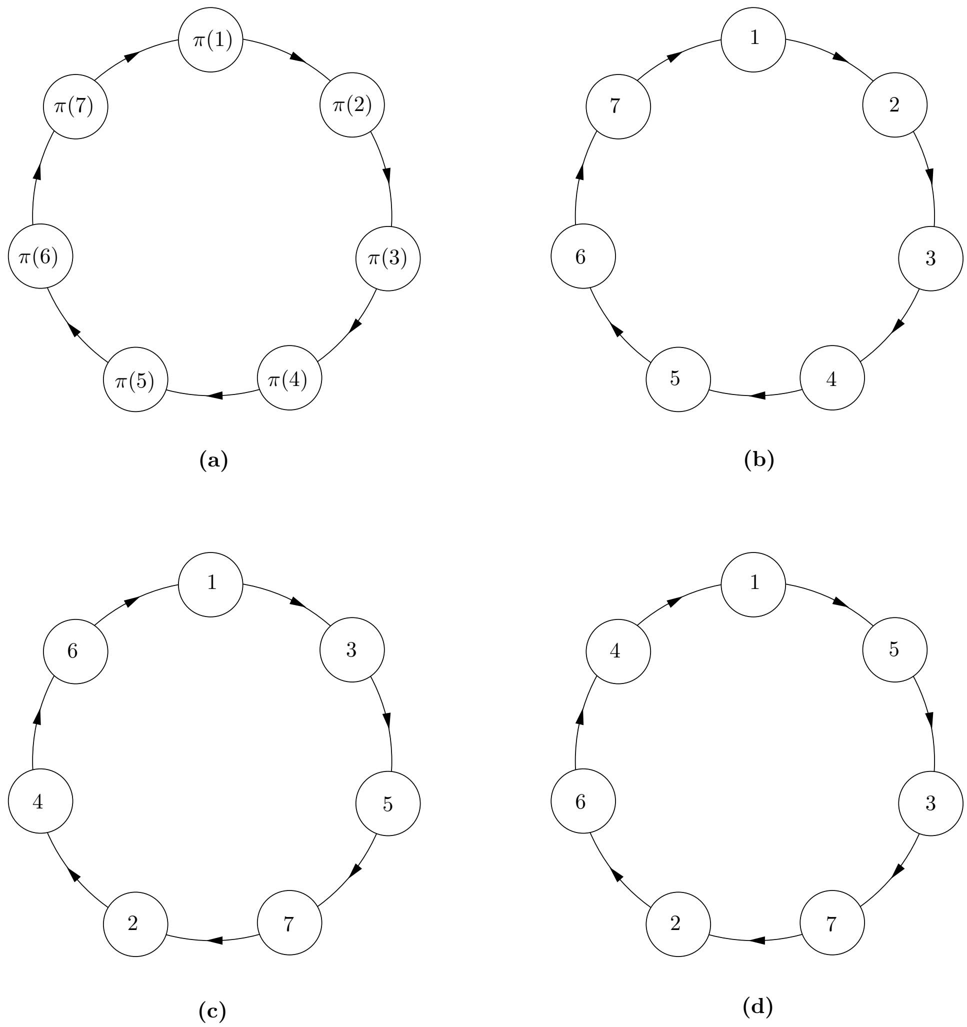

Figure 2-1: A receiver tests the cells according to the search order: **(a)** a generic search order π; **(b)** the search order π^1 of the CSS; **(c)** the search order π^2 of the FSSS with the step size $N_{\mathrm{J}} = 2$; **(d)** the search order π_{R} of the bit-reversal serial search.

The fixed-step serial search (FSSS) [6, 15], where the receiver skips $N_{\mathrm{J}} \geq 1$ cells before it performs the next test, corresponds to the search order[1]

$$[1, 1 \oplus N_{\mathrm{J}}, 1 \oplus 2N_{\mathrm{J}}, \ldots, 1 \oplus (N_{\mathrm{S}} - 1)N_{\mathrm{J}}] .$$

The permutation function for this search order is

$$\pi^{N_{\mathrm{J}}}(k) = 1 \oplus (k-1)N_{\mathrm{J}}. \tag{2.5}$$

[1]The symbol $\oplus$ denotes the modulo N_{S} addition defined by $x \oplus y \triangleq x + y - lN_{\mathrm{S}}$, for some unique integer l such that $x + y - lN_{\mathrm{S}} \in \mathcal{U}$.

To ensure that the mapping $\pi^{N_\text{J}}\colon \mathcal{U} \to \mathcal{U}$ is a bijection, we require that N_J and N_S be relatively prime. Clearly, the CSS π^1 is a special case of the FSSS π^{N_J} with the step size $N_\text{J} = 1$.

The bit-reversal serial search is proposed in [6] and corresponds to the search order $[\pi_\text{R}(1) = 1, \pi_\text{R}(2), \ldots, \pi_\text{R}(N_\text{S})]$, where elements $\pi_\text{R}(\cdot)$ are defined relatively to one another. For $i \neq j$,

$$\pi_\text{R}(i) < \pi_\text{R}(j) \Leftrightarrow \text{rev}(i) < \text{rev}(j), \tag{2.6}$$

where $\text{rev}(i)$ is the reversal of the $\lceil \log_2 N_\text{S} \rceil$ binary digit representation of the integer $i - 1$. Equation (2.6) specifies the unique order of N_S cells in the uncertainty index set: assign the cost $\text{rev}(i)$ to cell i and arrange the cells in the ascending order according to their costs. In general, there are $(N_\text{S} - 1)!$ different search orders, and it is imperative to find the one that minimizes the MAT.

There are a few works that compare different search orders. For a Rayleigh fading channel, the MAT of the FSSS with $N_\text{J} = N_\text{Q}$ is shorter than that of the CSS for certain signal-to-interference ratio (SIR) values [15]. It is unclear, however, if the result of [15] is valid for all SIR values and all detection schemes. The search order π^{N_Q} is shown to result in a shorter MAT than π^1 does, in [6], for the specific example that fixes the probability of detection to 0.95, the probability of false-alarm to 0.10, the dwell-time to 1 time-unit, the penalty time to 10 time-units, and the number of total cells to 16. It is again unclear if this conclusion is valid for other values of the detection probability, the false-alarm probability, the dwell-time, the penalty time, and the number of total cells.

In this thesis, we find a search order that is optimal or near-optimal for all values of SIR, regardless of the decision rules at the detection layer or the operating environments. Our goal is to investigate the following questions:

- What are the fundamental limits of the achievable MATs? In other words, what are the minimum and maximum MATs?
- What are the search orders that achieve the minimum MAT?
- What are the search orders that result in the maximum MAT?
- What are the benefits of using intelligent search strategies?

We focus on the most commonly used search strategy, the serial search [6,9–11,14,15,20,21]. The key contributions of this thesis are as follows:

- We introduce a concept of a *spacing rule*, which describes the structure of the flow diagram. Then, we derive an explicit expression for the absorption time[2] as a function of a *spacing rule*, and derive the optimal *spacing rule* by using convexity theory.
- We derive the explicit expression for the minimum MAT over all possible search orders, and then prove that the search order π^{N_Q} yields the near-optimal MAT.
- We derive the explicit expression for the maximum MAT, and then show that the CSS π^1 and the FSSS with the step size $N_\mathrm{S} - 1$ yield this maximum.
- We derive the performance gain, which is a ratio of the MAT of the CSS and that of the optimal serial search.

In the next chapter, we present the steps for achieving the objectives of this research.

[2]The average time to arrive at the absorption state in a Markov flow diagram.

Chapter 3

Methods

We outline the steps that are used to achieve the thesis' objectives, which are listed in Section 1.2.

3.1 Approach to Find the Minimum MAT

1. We formulate an optimization problem

$$\min_{\pi \in \mathcal{P}} \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\}, \tag{3.1}$$

whose solution corresponds to the minimum MAT.

2. We express the MAT as the average of the absorption times:

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\} = \frac{1}{N_{\mathrm{S}}} \sum_{b=1}^{N_{\mathrm{S}}} v(\mathbf{s}(\pi, b)). \tag{3.2}$$

Here, $v(\mathbf{s}(\pi, b))$ is the absorption time of the flow diagram with the spacing rule $\mathbf{s}(\pi, b)$.

3. We derive the explicit closed-form expression for the absorption time $v(\cdot)$.

4. We show that the minimum MAT in (3.1) is lower-bounded by the solution of the convex optimization problem $\min_{\mathbf{m} \in \mathcal{Q}} \overline{v}(\mathbf{m})$:

$$\min_{\mathbf{m} \in \mathcal{Q}} \overline{v}(\mathbf{m}) \leq \min_{\pi \in \mathcal{P}} \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\}, \tag{3.3}$$

where $\mathcal{Q}$ is the convex hull of the set $\mathcal{S}$ of spacing rules, and $\overline{v}(\cdot)$ is the natural extension of $v(\cdot)$.

5. We derive the unique solution $T^{\mathrm{L}}_{\mathrm{min}}$ of the convex optimization problem $\min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})$ and conclude that the optimal MAT satisfies the bound

$$T^{\mathrm{L}}_{\mathrm{min}} \leq \min_{\pi\in\mathcal{P}} \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\}. \tag{3.4}$$

3.2 Approach to Find the Maximum MAT

1. We formulate an optimization problem

$$\max_{\pi\in\mathcal{P}} \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\}, \tag{3.5}$$

whose solution corresponds to the maximum MAT.

2. We show that the maximum MAT in (3.5) is equal to the solution of the convex optimization problem $\max_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})$:

$$\max_{\pi\in\mathcal{P}} \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\} = \max_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}). \tag{3.6}$$

3. We derive the explicit expression $T_{\mathrm{max}} = \max_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})$ and conclude that the maximum MAT satisfies

$$\max_{\pi\in\mathcal{P}} \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\} = T_{\mathrm{max}}. \tag{3.7}$$

3.3 Approach to Find an η-Optimal Search Order

1. We prove that the lower-bound in (3.4) is satisfied with equality if and only if $N_{\mathrm{Q}} = 1$ or $N_{\mathrm{Q}} = N_{\mathrm{S}}$.

2. We prove that the search order $\pi^{N_{\mathrm{Q}}}$ is η-optimal with $\eta = \left(\frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}}-N_{\mathrm{Q}}}\right)$. Thus, the MAT of the FSSS $\pi^{N_{\mathrm{Q}}}$ satisfies the following bounds:

$$T^{\mathrm{L}}_{\mathrm{min}} \leq \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^{N_{\mathrm{Q}}})\right\} \leq \left(1 + \frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}} - N_{\mathrm{Q}}}\right) T^{\mathrm{L}}_{\mathrm{min}}.$$

3.4 Approach to Avoid the Worst Search Orders

1. We prove that the complete solutions to the maximization problem $\max_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})$ are the elements of the set

$$\mathcal{E} \triangleq \Big\{(0,0,\ldots,0,N_{\mathrm{S}}-N_{\mathrm{Q}}),(0,0,\ldots,0,N_{\mathrm{S}}-N_{\mathrm{Q}},0),\ldots,(N_{\mathrm{S}}-N_{\mathrm{Q}},0,0,\ldots,0)\Big\}.$$

2. We prove that the CSS and the FSSS with the step size $N_{\mathrm{S}}-1$ result in the maximum MAT:

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^{1})\right\} = \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^{N_{\mathrm{S}}-1})\right\} = T_{\max}.$$

3. We prove that for N_{S} and N_{Q} satisfying $2 \leq N_{\mathrm{Q}} \leq N_{\mathrm{S}}-2$, the CSS and the FSSS with the step size $N_{\mathrm{S}}-1$ are the only two search orders that result in the maximum MAT.

3.5 Approach to Quantify the Benefits of Using Intelligent Search Strategies

1. We use the explicit expressions of $T_{\min}^{\mathrm{L}}$ and $T_{\max}$ to show that the MAT of the optimal serial search is approximately constant, when the uncertainty index set is refined by a factor of r. In contrast, the MAT of the CSS increases approximately r times.

2. To show that the optimal serial search is approximately N_{Q} times faster than the CSS in a high SNR environment, we perform the following steps:

   (a) We derive the explicit expression for the MAT of the CSS with the optimal thresholds at the detection layer in a high SNR environment,

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^{1})\right\}\Big|_{\text{high SNR, optimal thresholds}}. \tag{3.8}$$

   (b) We derive the upper and lower-bounds for the MAT of the optimal serial search with the optimal thresholds at the detection layer in a high SNR environment,

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^{*})\right\}\Big|_{\text{high SNR, optimal thresholds}}. \tag{3.9}$$

(c) We define the performance gain G_{H} in a low SNR environment to be the ratio of (3.8) to (3.9).

(d) We use the expression of (3.8) and the bounds of (3.9) to show that the performance gain G_{H} is approximately equal to N_{Q}.

3. To show that the optimal serial search is approximately one to two times faster than the CSS in a low SNR environment, we perform the following steps:

(a) We derive the upper and lower-bounds for the MAT of the CSS with the optimal thresholds at the detection layer in a low SNR environment,

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^1)\right\}\Big|_{\text{low SNR, optimal thresholds}}. \tag{3.10}$$

(b) We derive the upper and lower-bounds for the MAT of the optimal serial search with the optimal thresholds at the detection layer in a low SNR environment,

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^*)\right\}\Big|_{\text{low SNR, optimal thresholds}}. \tag{3.11}$$

(c) We define the performance gain G_{L} in a low SNR environment to be the ratio of (3.10) to (3.11).

(d) We use the bounds of (3.10) and (3.11) to show that the performance gain G_{L} is in the approximated range from one to two.

In the next chapter, we present the details from the approaches outlined in this chapter.

Chapter 4

Research Results

This chapter contains the research results and the details from the approaches, which are outlined in the last chapter. This chapter is divided into six sections. In Section 4.1, we present the system model for the acquisition system and define the terms *description* and *spacing rule*. In Section 4.2, we derive the absorption time as a function of the spacing rule. In Section 4.3, we derive the explicit expression of the lower-bound for the MAT and prove that the search order $\pi^{N_{\mathrm{Q}}}$ yields the near-optimal MAT. In Section 4.4, we derive the explicit expression for the maximum MAT and prove that the CSS and the FSSS $\pi^{N_{\mathrm{S}}-1}$ result in the maximum MAT. In Section 4.5 and Section 4.6, we compare the MAT of the CSS and the MAT of the optimal serial search when the uncertainty index set is refined, when the signal-to-noise ratio (SNR) is high, and when the SNR is low.

4.1 System Model

We consider the flow diagram in Fig. 4-1, which models a serial search with the permutation function π. There are $N_{\mathrm{S}}+1$ states totally: 1 absorbing state (ACQ), N_{Q} states of type H_1, and $N_{\mathrm{S}}-N_{\mathrm{Q}}$ states of type H_0. The ACQ state represents the event of successful acquisition. Each of the N_{Q} states of type H_1 corresponds to an in-phase cell, while each of the remaining $N_{\mathrm{S}}-N_{\mathrm{Q}}$ states of type H_0 corresponds to a non-in-phase cell. The disjoint union of the in-phase and non-in-phase cells forms an uncertainty set that can be represented by the index set $\mathcal{U}=\{1,2,3,\ldots,N_{\mathrm{S}}\}$.

The location B of the first in-phase cell is unknown to the receiver. We consider the uniform distribution for random variable B, which takes the value in $\mathcal{U}$. Conditioned on

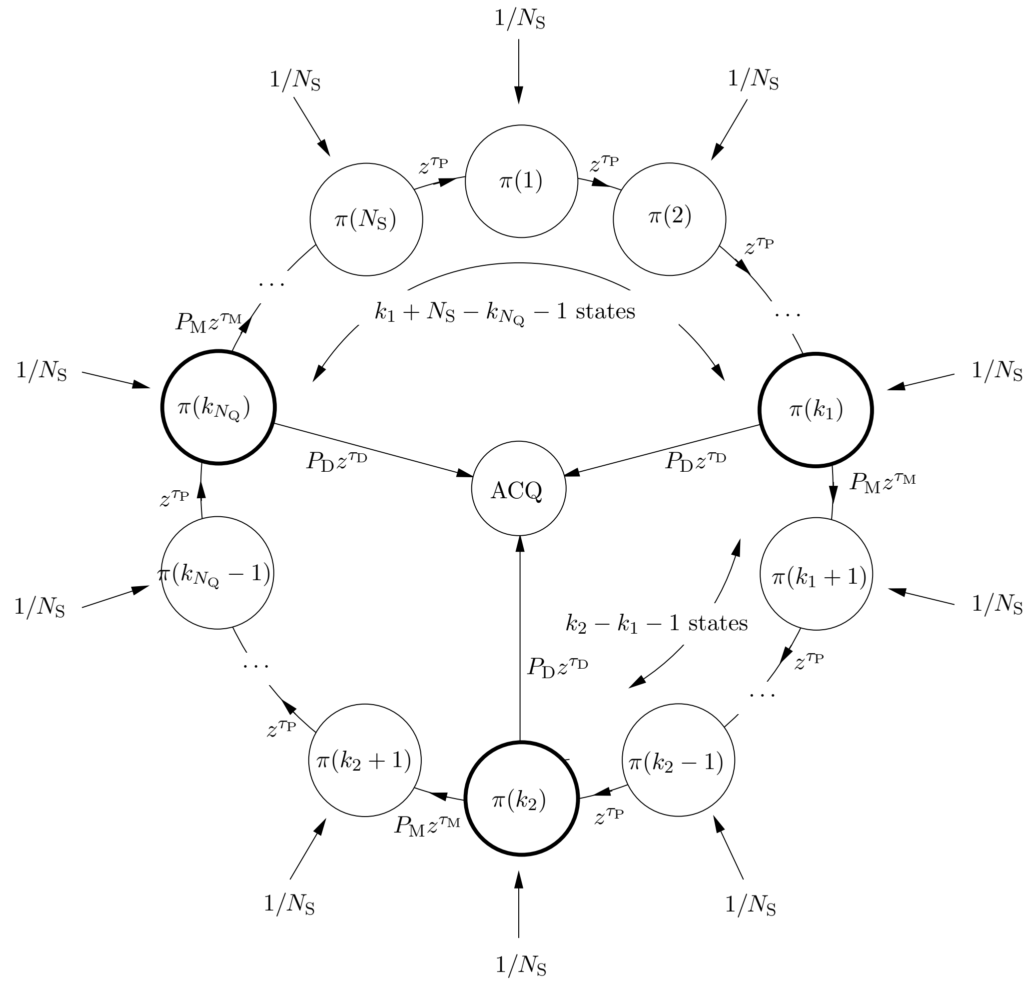

Figure 4-1: A flow diagram for the serial search with the permutation function π. The state labeled ACQ is the absorbing state. The states in thick circles are H_1-states. The remaining states are H_0-states.

$B = b_o$, the index set corresponding to the in-phase cells $\mathcal{H}_C(b_o) \subset \mathcal{U}$ is then

$$\mathcal{H}_C(b_o) \triangleq \{b_o, b_o \oplus 1, \ldots, b_o \oplus (N_Q - 1)\}. \tag{4.1}$$

The probability $\Pr\{K = k\}$ that the receiver begins the search at cell k is also uniform and equal to $1/N_S$.

Since π is bijective, there are exactly N_Q values of $k \in \mathcal{U}$ such that the state $\pi(k)$ is of type H_1. Let $k_1 < k_2 < k_3 < \cdots < k_{N_Q}$ denote the unique integers such that the set

$$\{\pi(k_1), \pi(k_2), \ldots, \pi(k_{N_Q})\} = \mathcal{H}_C(b_o)$$

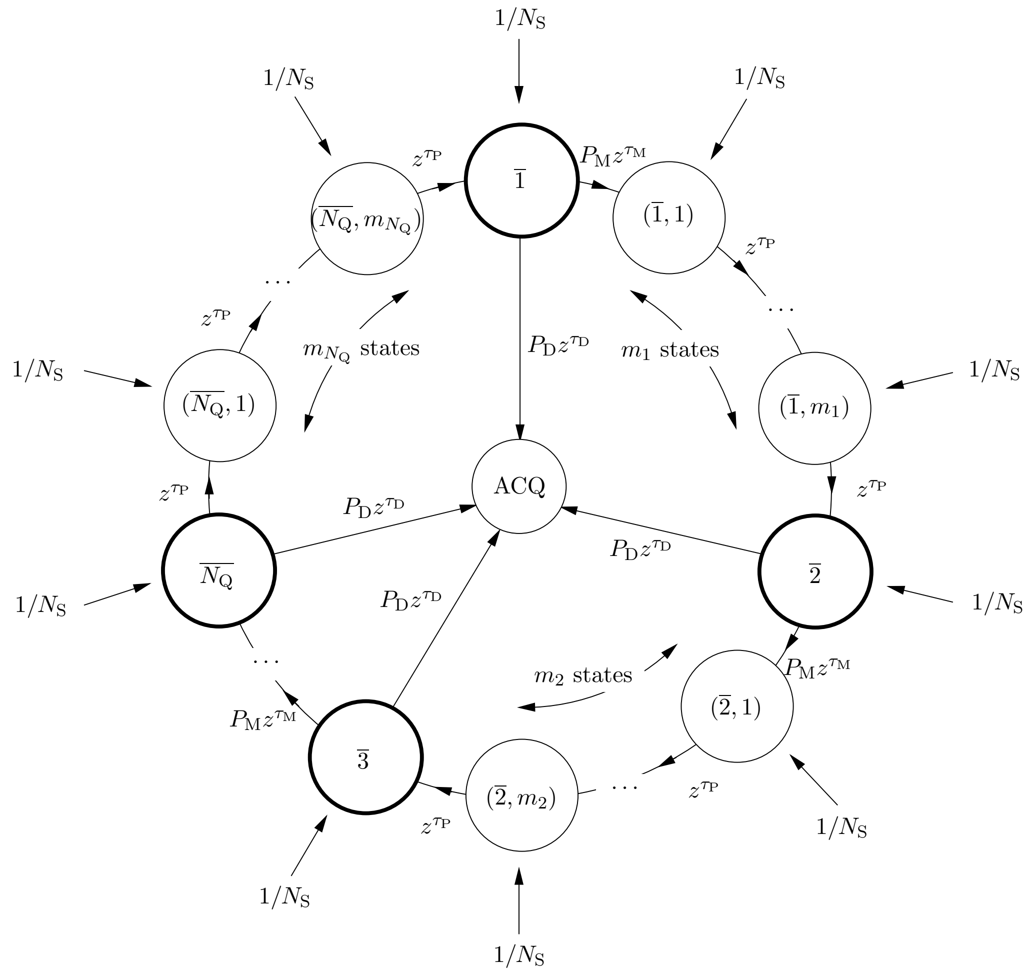

Figure 4-2: The spacing rule $\mathbf{m} = (m_1, m_2, \ldots, m_{N_\mathrm{Q}})$ characterizes the structure of the flow diagram.

is the index set of in-phase cells. In a flow diagram, those in-phase cells $\pi(k_i)$ have paths to the absorbing state.

The path gains $P_\mathrm{D} z^{\tau_\mathrm{D}}$, $P_\mathrm{M} z^{\tau_\mathrm{M}}$, and z^{τ_P} depend on the probabilities and durations of the detection process. The receiver spends τ_D time-units and makes the correct decision with probability P_D when testing an in-phase cell. The receiver spends τ_M time-units and makes the incorrect decision with probability $P_\mathrm{M} = 1 - P_\mathrm{D}$ when testing an in-phase cell. The receiver spends τ_P time-units to eventually make the correct decision when testing a non-in-phase cell. The variable z on a path gain represents a unit-delay. The quantities P_D, P_M, τ_D, τ_M, and τ_P are parameters of the detection layer.

Path gains of any flow diagram with one absorbing state (ACQ) can be reduced to the forms $P_\mathrm{D} z^{\tau_\mathrm{D}}$, $P_\mathrm{M} z^{\tau_\mathrm{M}}$, and z^{τ_P}. Let $H_\mathrm{D}(z)$ denote a generic path gain from an H_1-state to

ACQ, $H_{\mathrm{M}}(z)$ denote a generic path gain from an H_1-state to the adjacent non-absorbing state, and $H_0(z)$ denote a generic path gain from an H_0-state to the adjacent non-absorbing state. These path gains $H_{\mathrm{D}}(z)$, $H_{\mathrm{M}}(z)$, and $H_0(z)$ can be represented respectively by the simplified forms $P_{\mathrm{D}}z^{\tau_{\mathrm{D}}}$, $P_{\mathrm{M}}z^{\tau_{\mathrm{M}}}$, and $z^{\tau_{\mathrm{P}}}$, where

$$\begin{aligned}
P_{\mathrm{D}} &= H_{\mathrm{D}}(1), \\
\tau_{\mathrm{D}} &= \begin{cases} H'_{\mathrm{D}}(1)/H_{\mathrm{D}}(1) & H_{\mathrm{D}}(1) \neq 0 \\ 1 & H_{\mathrm{D}}(1) = 0, \end{cases} \\
P_{\mathrm{M}} &= H_{\mathrm{M}}(1), \\
\tau_{\mathrm{D}} &= \begin{cases} H'_{\mathrm{M}}(1)/H_{\mathrm{M}}(1) & H_{\mathrm{D}}(1) \neq 0 \\ 1 & H_{\mathrm{M}}(1) = 0, \end{cases} \\
\tau_{\mathrm{P}} &= H'_0(1).
\end{aligned} \tag{4.2}$$

The absorption time depends only on $H_i(1)$ and $H'_i(1)$, for $i \in \{\mathrm{D}, \mathrm{M}, 0\}$ [6,9–11]. Therefore, the flow diagram with path gains $H_i(z)$, $i \in \{\mathrm{D}, \mathrm{M}, 0\}$, and the flow diagram with path gains $P_{\mathrm{D}}z^{\tau_{\mathrm{D}}}$, $P_{\mathrm{M}}z^{\tau_{\mathrm{M}}}$, and $z^{\tau_{\mathrm{P}}}$ have the same absorption time.

We note that the flow diagram under consideration in Fig. 4-1 has one absorbing state and is *non-preferential*.

Definition 4.1 (Non-preferential flow diagram). The flow diagram is *non-preferential* if it has the following properties:

1. Probability of entering any non-absorbing state is equally likely.
2. Every path going into the absorbing state has the same path gain.
3. Every path going out of an H_0-state has the same path gain.
4. Every path going out of an H_1-state to the adjacent non-absorbing state has the same path gain.

The structure of the flow diagram describes the arrangement of the in-phase and non-in-phase cells. This structure is important because it strongly influences the absorption time and the MAT. One method to describe the structure of any flow diagram, including the non-preferential one, is by its *description*.

Definition 4.2 (Description). A *description* is a tuple (π, b) of the permutation function π and the location b of the first in-phase cell. The set of descriptions is

$$\mathcal{D} = \Big\{(\pi, b) \,\Big|\, \pi\colon \mathcal{U} \to \mathcal{U} \text{ is a permutation function, } \pi(1) = 1, \text{ and } 1 \le b \le N_{\mathrm{S}}\Big\}. \tag{4.3}$$

The description (π, b) characterizes the structure of a flow diagram. In particular, π constrains the order $[\pi(1), \pi(2), \ldots, \pi(N_{\mathrm{S}})]$ of the non-absorbing state, while b indicates the set $\mathcal{H}_{\mathrm{C}}(b)$ of states that have transition edges to the absorbing state. When a flow diagram is non-preferential, one has an alternative method to describe the flow diagram structure.

The structure of a non-preferential flow diagram can be specified by the number of non-in-phase cells between the two neighboring in-phase cells. For the flow diagram in Fig. 4-1, there are $(k_{i+1} - k_i - 1)$ non-in-phase-cells between the in-phase cells $\pi(k_i)$ and $\pi(k_{i+1})$, for $1 \le i \le N_{\mathrm{Q}} - 1$, and $(k_1 + N_{\mathrm{S}} - k_{N_{\mathrm{Q}}} - 1)$ non-in-phase-cells between the in-phase cells $\pi(k_{N_{\mathrm{Q}}})$ and $\pi(k_1)$. These N_{Q} integers collectively form a *spacing rule*.

Definition 4.3 (Spacing rule). A *spacing rule* of a non-preferential flow diagram with N_{Q} H_1-states and $(N_{\mathrm{S}} - N_{\mathrm{Q}})$ H_0-states is an element $\mathbf{m}$ of the set

$$\mathcal{S} = \Big\{(m_1, m_2, \ldots, m_{N_{\mathrm{Q}}}) \,\Big|\, \sum_{i=1}^{N_{\mathrm{Q}}} m_i = N_{\mathrm{S}} - N_{\mathrm{Q}};\, \forall i, \text{integer } m_i \ge 0\Big\}. \tag{4.4}$$

The spacing rule $\mathbf{m} = (m_1, m_2, \ldots, m_{N_{\mathrm{Q}}})$ characterizes the structure of a non-preferential flow diagram. In particular, the flow diagram has an H_1-state, which are followed by m_1 H_0-states, which is followed by another H_1-state, which is followed by m_2 H_0-states, and so on. The sum $\sum_{i=1}^{N_{\mathrm{Q}}} m_i$ must equal the number $N_{\mathrm{S}} - N_{\mathrm{Q}}$ of H_0-states. Fig. 4-2 is the flow diagram with the spacing rule $\mathbf{m} = (m_1, m_2, \ldots, m_{N_{\mathrm{Q}}})$.

Given the description (π, b), one can find the spacing rule

$$\mathbf{s}(\pi, b) \triangleq (m_1, m_2, \ldots, m_{N_{\mathrm{Q}}}) \tag{4.5}$$

corresponding to (π, b). Here, $m_i \triangleq k_{i+1} - k_i - 1$, for the unique integers $k_1 < k_2 < \cdots < k_{N_{\mathrm{Q}}} < k_{N_{\mathrm{Q}}+1} \triangleq k_1 + N_{\mathrm{S}}$ that satisfy $\{\pi(k_1), \pi(k_2), \ldots, \pi(k_{N_{\mathrm{Q}}})\} = \mathcal{H}_{\mathrm{C}}(b)$. Equation (4.5) establishes the mapping $\mathbf{s}\colon \mathcal{D} \to \mathcal{S}$.

Fig. 4-3 shows the flow diagrams of the CSS π^1 when the first in-phase cells are $B =$

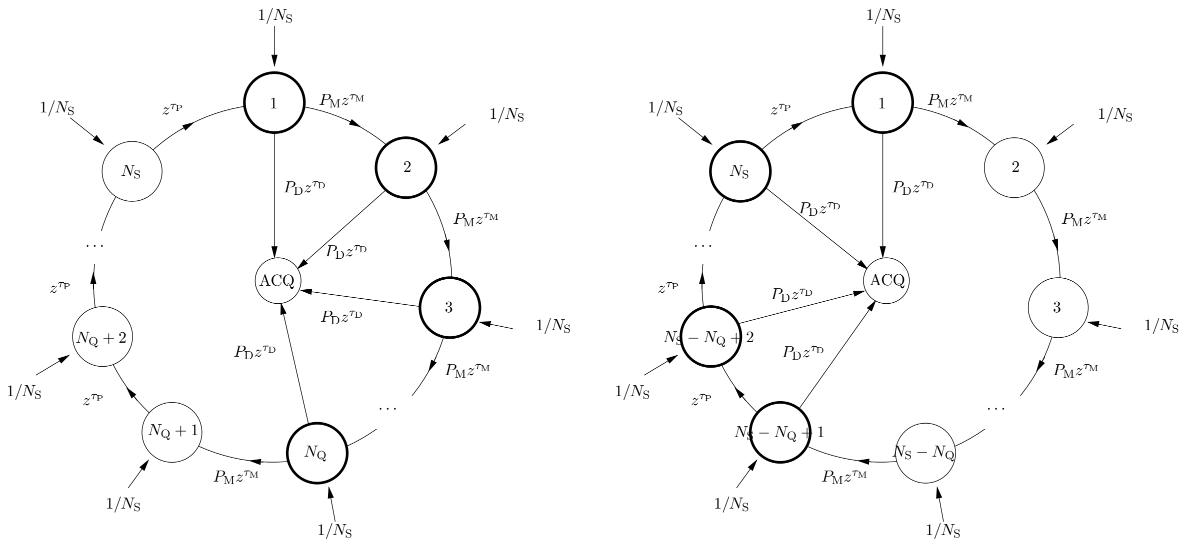

Figure 4-3: Flow diagrams for the conventional serial search correspond to the different locations B of the first in-phase cell: **(a)** $B = 1$; **(b)** $B = N_{\mathrm{S}} - N_{\mathrm{Q}} + 1$.

1 and $B = N_{\mathrm{S}} - N_{\mathrm{Q}} + 1$. The spacing rule corresponding to the description $(\pi^1, 1)$ is $(0, 0, \ldots, 0, N_{\mathrm{S}} - N_{\mathrm{Q}})$, while the spacing rule corresponding to the description $(\pi^1, N_{\mathrm{S}} - N_{\mathrm{Q}} + 1)$ is $(N_{\mathrm{S}} - N_{\mathrm{Q}}, 0, 0, \ldots, 0)$. For different values of B, the spacing rule of the CSS becomes an element of the set

$$\mathcal{E} \triangleq \Big\{(0, 0, \ldots, 0, N_{\mathrm{S}} - N_{\mathrm{Q}}), (0, 0, \ldots, 0, N_{\mathrm{S}} - N_{\mathrm{Q}}, 0), \ldots, (N_{\mathrm{S}} - N_{\mathrm{Q}}, 0, 0, \ldots, 0)\Big\}. \quad (4.6)$$

Every spacing rule in $\mathcal{E}$ corresponds to the flow diagram with consecutive H_1-states and consecutive H_0-states.

The properties of set $\mathcal{D}$, set $\mathcal{S}$, and their mapping $\mathbf{s}(\cdot, \cdot)$ are as follows. First, the mapping $\mathbf{s}\colon \mathcal{D} \to \mathcal{S}$ is surjective:

$$\mathcal{S} = \{\mathbf{s}(\pi, b) \,|\, (\pi, b) \in \mathcal{D}\}. \quad (4.7)$$

Second, the cardinalities $|\mathcal{D}|$ and $|\mathcal{S}|$ satisfy

$$|\mathcal{D}| = N_{\mathrm{S}}! \quad (4.8)$$

and

$$|\mathcal{S}| = \binom{N_\mathrm{S}-1}{N_\mathrm{Q}-1}. \tag{4.9}$$

We note that $|\mathcal{S}|$ is the number of ways to distribute $N_\mathrm{S} - N_\mathrm{Q}$ undistinguishable objects into N_Q distinguishable boxes. Third, $\mathbf{s}\colon \mathcal{D} \to \mathcal{S}$ is not injective, because $|\mathcal{D}| > |\mathcal{S}|$. In other words, many descriptions are redundant because they correspond to the same flow diagram structure. Fourth, the mapping $\mathbf{s}$ is non-invertible. Therefore, we cannot always find the unique description corresponding to a given spacing rule. In the next section, we use the description and the spacing rule to calculate the absorption time.

4.2 The Absorption Time

For a given search order π, the description and the spacing rule provide two possible approaches to calculate the absorption time. The first approach directly finds the absorption time from the description (π, b):

$$\begin{aligned} \mathbb{E}\{T_\mathrm{ACQ}(\pi)\} &= \sum_{b=1}^{N_\mathrm{S}} \sum_{k=1}^{N_\mathrm{S}} \mathbb{E}\{T_\mathrm{ACQ}(\pi) \mid B=b, K=k\} \Pr\{B=b\} \Pr\{K=k\} \\ &= \frac{1}{N_\mathrm{S}} \sum_{b=1}^{N_\mathrm{S}} f(\pi, b). \end{aligned} \tag{4.10}$$

Here, $f(\pi, b)$ is the absorption time for the flow diagram corresponding to the description (π, b):

$$\begin{aligned} f(\pi, b) &\triangleq \sum_{k=1}^{N_\mathrm{S}} \mathbb{E}\{T_\mathrm{ACQ}(\pi) \mid B=b, K=k\} \Pr\{K=k\} \\ &\overset{(a)}{=} \frac{1}{N_\mathrm{S}} \frac{d}{ds} \left(\frac{\sum_{k=1}^{N_\mathrm{S}} \sum_{i=1}^{N_\mathrm{S}} H_{\pi(i\oplus k)}^{b}(z) \prod_{j=1}^{i} H_{\pi(j\oplus k)}^{b}(z)}{1 - \prod_{i=1}^{N_\mathrm{S}} G_i^b(z)} \right) \Bigg|_{z=1}, \end{aligned} \tag{4.11}$$

where

$$H_i^b(z) = \begin{cases} P_\mathrm{D} z^{\tau_\mathrm{D}} & i \in \mathcal{H}_\mathrm{C}(b) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G_i^b(z) = \begin{cases} P_{\mathrm{M}} z^{\tau_{\mathrm{M}}} & i \in \mathcal{H}_{\mathrm{C}}(b) \\ z^{\tau_{\mathrm{P}}} & \text{otherwise.} \end{cases}$$

The equality (a) follows from a loop-reduction technique, which is used to find the MATs in [6, 10, 11, 15]. Note that (4.11) is an implicit expression of the search order π.

The second approach finds the absorption time from the spacing rule $\mathbf{s}(\pi, b)$:

$$\begin{aligned} \mathbb{E}\{T_{\mathrm{ACQ}}(\pi)\} &= \sum_{b=1}^{N_{\mathrm{S}}} \sum_{k=1}^{N_{\mathrm{S}}} \mathbb{E}\{T_{\mathrm{ACQ}}(\pi) \,|\, B = b, K = k\} \Pr\{B = b\} \Pr\{K = k\} \\ &= \frac{1}{N_{\mathrm{S}}} \sum_{b=1}^{N_{\mathrm{S}}} v(\mathbf{s}(\pi, b)). \end{aligned} \tag{4.12}$$

Here, $v(\mathbf{m}) \triangleq v(\mathbf{s}(\pi, b))$ is the absorption time when the spacing rule is $\mathbf{m} = \mathbf{s}(\pi, b)$:

$$\begin{aligned} v(\mathbf{s}(\pi, b)) &\triangleq \sum_{k=1}^{N_{\mathrm{S}}} \mathbb{E}\{T_{\mathrm{ACQ}}(\pi) \,|\, B = b, K = k\} \Pr\{K = k\} \\ &= f(\pi, b). \end{aligned} \tag{4.13}$$

The explicit form of $v(\mathbf{m})$ is given in the next subsection.

Although both approaches can be used to find the MAT in principle when the search order π is given, the second approach is more suitable for finding the minimum and maximum MATs. In particular, any search order π will result in the MAT that satisfies

$$\min_{(\pi,b)\in\mathcal{D}} f(\pi, b) = \min_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \leq \mathbb{E}\{T_{\mathrm{ACQ}}(\pi)\} \leq \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) = \max_{(\pi,b)\in\mathcal{D}} f(\pi, b). \tag{4.14}$$

Note that $\min_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$ and $\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$ are integer programming problems [1, 2]. Because the expression of $f(\pi, b)$ does not reveal how the absorption time $f(\pi, b)$ explicitly depends on the search order π, it is unclear how one can solve efficiently—if at all possible—the optimization problems $\min_{(\pi,b)\in\mathcal{D}} f(\pi, b)$ and $\max_{(\pi,b)\in\mathcal{D}} f(\pi, b)$. Therefore, we transform the *descriptions* into the *spacing rules* and then employ the convex optimization theory.

To accentuate the need for optimization over the set of spacing rules, we note that the direct approach, which searches exhaustively over $\mathcal{P}$ for the best and the worst search orders, is impractical. Evaluation of the right-hand side of (4.10) or (4.12) requires at least N_{S} arithmetic operations. As a result, the exhaustive search of the permutation function

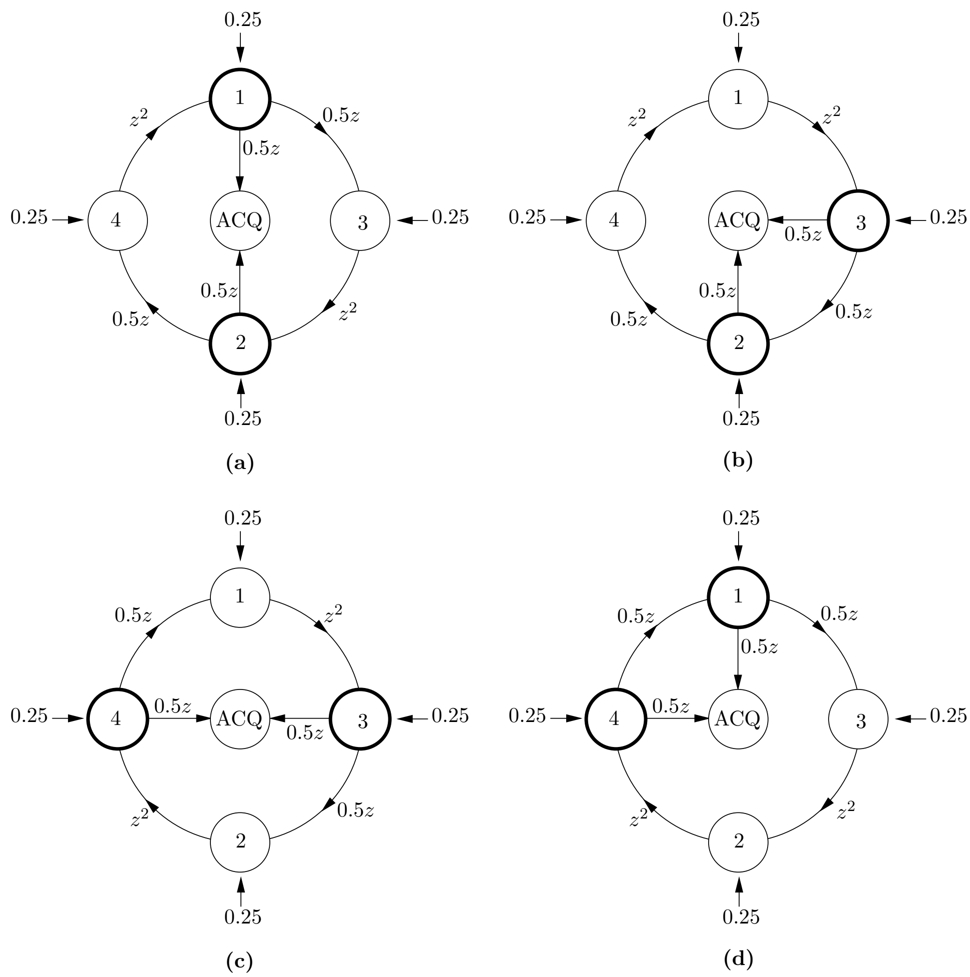

Figure 4-4: The structure of the flow diagram for the bit-reversal serial search π_R varies with the location B of the first in-phase cell: **(a)** $B = 1$; **(b)** $B = 2$; **(c)** $B = 3$; **(d)** $B = 4$.

requires at least $N_S \cdot |\mathcal{P}| = N_S!$ arithmetic operations. For a small size $N_S = 100$ of the uncertainty index set and a fictional machine that has a clock speed of 10^{20} Hz and performs 1 arithmetic operation per cycle, the exhaustive search requires more than 10^{130} years to complete. The direct approach is clearly inefficient.

In general, we note that averaging over b in (4.10) and (4.12) is required. Fig. 4-4 depicts flow diagrams, which correspond to different values of the first in-phase cell B. All flow diagrams model the bit-reversal search π_R and has the following parameters: $N_S = 4$, $N_Q = 2$, $P_D = 0.5$, $\tau_D = \tau_M = 1$, and $\tau_P = 2$. When $B = 1$ or $B = 3$, the absorption time is

$$f(\pi_R, 1) = f(\pi_R, 3) = 5.$$

When $B = 2$ or $B = 4$, the absorption time is

$$f(\pi_{\mathrm{R}}, 2) = f(\pi_{\mathrm{R}}, 4) = 5\tfrac{1}{6}.$$

Averaging over b, we have the MAT

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi_{\mathrm{R}})\right\} = 5\tfrac{1}{12},$$

which is not equal to $f(\pi_{\mathrm{R}}, 1)$, $f(\pi_{\mathrm{R}}, 2)$, $f(\pi_{\mathrm{R}}, 3)$, or $f(\pi_{\mathrm{R}}, 4)$. In general, the absorption time is a function of a particular value of B. Therefore, the MATs in [6, 15], which assumes that $B = 1$, are only the approximations.

In a few special cases, averaging over b is not necessary. When $N_{\mathrm{Q}} = 1$ [9–11], when the CSS [15, 20] is used, or when the FSSS with the step size $N_{\mathrm{S}} - 1$ is used, the absorption time does not depend on B. Therefore, one can assume that $B = 1$ in those cases. In the next subsection, we derive explicit form of $v(\cdot)$.

4.2.1 Closed-Form Expression of $v(\mathbf{m})$

The goal of this subsection is to derive the explicit absorption time expression $v(\mathbf{m})$ for $\mathbf{m} \in \mathcal{S}$. Because the flow diagram has one absorbing state, finding the MAT reduces to simply solving a system of linear equations. The closed-form expression of $v(\mathbf{m})$ is given explicitly by the following theorem.

Theorem 4.1 (Absorption Time). *The absorption time of the flow diagram with the spacing rule* $\mathbf{m} \in \mathcal{S}$ *is given by*

$$v(\mathbf{m}) = A \sum_{i=1}^{N_{\mathrm{Q}}} m_i^2 + \sum_{i=1}^{N_{\mathrm{Q}}} \sum_{j=i+1}^{N_{\mathrm{Q}}} B_{ij} m_i m_j + C \tag{4.15a}$$

$$= \frac{1}{2} \mathbf{m}^T \mathbf{H} \mathbf{m} + C, \tag{4.15b}$$

where

$$A = \frac{\tau_{\mathrm{P}}\left(1 + P_{\mathrm{M}}^{N_{\mathrm{Q}}}\right)}{2N_{\mathrm{S}}\left(1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}\right)}, \tag{4.16}$$

$$B_{ij} = \frac{\tau_{\mathrm{P}}\left(P_{\mathrm{M}}^{N_{\mathrm{Q}}-(j-i)} + P_{\mathrm{M}}^{j-i}\right)}{N_{\mathrm{S}}\left(1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}\right)}, \tag{4.17}$$

$$C = \left(1 - \frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\right) \cdot \left(\frac{1 + P_{\mathrm{M}}}{1 - P_{\mathrm{M}}}\right) \frac{\tau_{\mathrm{P}}}{2} + \frac{P_{\mathrm{M}}}{1 - P_{\mathrm{M}}}\tau_{\mathrm{M}} + \tau_{\mathrm{D}}, \tag{4.18}$$

$$\mathbf{H} = \frac{\tau_{\mathrm{P}}}{N_{\mathrm{S}}\left(1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}\right)} \left[P_{\mathrm{M}}^{N_{\mathrm{Q}}-|i-j|} + P_{\mathrm{M}}^{|i-j|}\right]_{ij}, \tag{4.19}$$

with $0^0 \triangleq 1$ *and* $\sum_{j=1}^{0} \triangleq 0$.

Proof. Let T_i denote the conditional absorption time, conditioned on the start location of the search at the H_1-state $\bar{i}, 1 \le i \le N_{\mathrm{Q}}$. The states are labeled according to the convention in Fig. 4-2. Define $\alpha \triangleq P_{\mathrm{M}}\tau_{\mathrm{P}}$ and $\beta \triangleq P_{\mathrm{D}}\tau_{\mathrm{D}} + P_{\mathrm{M}}\tau_{\mathrm{M}}$. We have the relationship

$$\begin{aligned}
T_1 &= P_{\mathrm{D}}\tau_{\mathrm{D}} + P_{\mathrm{M}}(\tau_{\mathrm{M}} + m_1\tau_{\mathrm{P}} + T_2) \\
&= \beta + \alpha m_1 + P_{\mathrm{M}}T_2 \\
T_2 &= \beta + \alpha m_2 + P_{\mathrm{M}}T_3 \\
T_3 &= \beta + \alpha m_3 + P_{\mathrm{M}}T_4 \\
&\vdots \\
T_{N_{\mathrm{Q}}} &= \beta + \alpha m_{N_{\mathrm{Q}}} + P_{\mathrm{M}}T_1.
\end{aligned}$$

Solving the above system of equations yields

$$T_i = \frac{\alpha}{1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}} \cdot \left(\sum_{j=1}^{i-1} P_{\mathrm{M}}^{N_{\mathrm{Q}}+j-i} m_j + \sum_{j=i}^{N_{\mathrm{Q}}} P_{\mathrm{M}}^{j-i} m_j\right) + \frac{\beta}{1 - P_{\mathrm{M}}},$$

for $1 \le i \le N_{\mathrm{Q}}$ and where $\sum_{i=1}^{0} \triangleq 0$.

For $1 \le i \le N_{\mathrm{Q}}, 1 \le j \le m_j$, let T_{ij} denote the conditional absorption time, conditioned on the start location of the search at the H_0-state $(\bar{i}, j)$. Then,

$$T_{ij} = T_{i+1} + (m_i - j + 1)\tau_{\mathrm{P}},$$

with $T_{N_{\mathrm{Q}}+1} \triangleq T_1$.

Once we have the expressions for T_i and T_{ij}, the expression of the absorption time is available:

$$
\begin{aligned}
v(\mathbf{m}) &= \frac{1}{N_{\mathrm{S}}}\Bigg(\sum_{i=1}^{N_{\mathrm{Q}}} T_i + \sum_{i=1}^{N_{\mathrm{Q}}}\sum_{j=1}^{m_i} T_{ij}\Bigg) \\
&= \frac{1}{N_{\mathrm{S}}}\Bigg[\sum_{i=1}^{N_{\mathrm{Q}}} T_i + m_i T_{i+1} + \frac{m_i(m_i+1)}{2}\tau_{\mathrm{P}}\Bigg] \\
&= \frac{1}{N_{\mathrm{S}}}\Bigg[\sum_{i=1}^{N_{\mathrm{Q}}}\Big(\frac{\alpha P_{\mathrm{M}}^{N_{\mathrm{Q}}-1}}{1-P_{\mathrm{M}}^{N_{\mathrm{Q}}}} + \frac{\tau_{\mathrm{P}}}{2}\Big)m_i^2 \\
&\qquad + \sum_{i=1}^{N_{\mathrm{Q}}}\sum_{j=i+1}^{N_{\mathrm{Q}}}\Big(\frac{\alpha P_{\mathrm{M}}^{j-i-1}}{1-P_{\mathrm{M}}^{N_{\mathrm{Q}}}} + \frac{\alpha P_{\mathrm{M}}^{N_{\mathrm{Q}}-j+i-1}}{1-P_{\mathrm{M}}^{N_{\mathrm{Q}}}}\Big)m_i m_j \\
&\qquad + \sum_{i=1}^{N_{\mathrm{Q}}}\Big(\frac{\alpha+\beta}{1-P_{\mathrm{M}}} + \frac{\tau_{\mathrm{P}}}{2}\Big)m_i + \frac{\beta N_{\mathrm{Q}}}{1-P_{\mathrm{M}}}\Bigg] \\
&\overset{(a)}{=} A\sum_{i=1}^{N_{\mathrm{Q}}} m_i^2 + \sum_{i=1}^{N_{\mathrm{Q}}}\sum_{j=i+1}^{N_{\mathrm{Q}}} B_{ij} m_i m_j + C \\
&= \frac{1}{2}\mathbf{m}^T\mathbf{H}\mathbf{m} + C.
\end{aligned}
$$

The simplification in (a) uses the constraint $\sum_{i=1}^{N_{\mathrm{Q}}} m_i = N_{\mathrm{S}} - N_{\mathrm{Q}}$. The proof is completed.

$\square$

In the subsequent analysis, we will allow the right-hand side of (4.15) to take the argument $\mathbf{m}$, which contains a non-integer component. In particular, let

$$
\mathcal{Q} = \Bigg\{(m_1, m_2, \ldots, m_{N_{\mathrm{Q}}})\,\Bigg|\,\sum_{i=1}^{N_{\mathrm{Q}}} m_i = N_{\mathrm{S}} - N_{\mathrm{Q}}; \forall i, m_i \geq 0\Bigg\} \tag{4.20}
$$

denote the convex hull of $\mathcal{S}$ and consider the function $\overline{v}\colon \mathcal{Q} \to \mathbb{R}$ to be the natural extension of $v\colon \mathcal{S} \to \mathbb{R}$. That is, we evaluate $\overline{v}(\mathbf{m})$ by simply allowing $v(\mathbf{m})$ in (4.15) to take the value in $\mathbf{m} \in \mathcal{Q}$. Because $\mathcal{S} \subset \mathcal{Q}$, the MAT for the search order π satisfies the following bounds:

$$
\min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}) \leq \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\} \leq \max_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}). \tag{4.21}
$$

Before delving into the derivations of the bounds in (4.21) explicitly, we first examine the properties of $\overline{v}(\mathbf{m})$. In the next subsection, we use the explicit expression of the absorption time in Thm. 4.1 to prove the convexity and some other important properties of $\overline{v}(\cdot)$. Evidently, the properties of $\overline{v}(\mathbf{m})$ with $\mathbf{m} \in \mathcal{Q}$ also hold for $v(\mathbf{m})$ with $\mathbf{m} \in \mathcal{S}$.

4.2.2 Properties of $\overline{v}(\mathbf{m})$

In this subsection we prove three important properties of $\overline{v}(\mathbf{m})$ for $\mathbf{m} \in \mathcal{Q}$. The first and second properties will be crucial for the development of the forthcoming sections. The three properties are the results of the theorem below.

Theorem 4.2 (Convexity, Rotational Invariance, and Reversal Invariance). *Assume that $P_{\mathrm{M}} < 1$, so that $\overline{v}(\cdot)$ is finite.*

1. *Function $\overline{v}(\cdot)$ is strictly convex on $\mathcal{Q}$.*

2. *$\overline{v}(m_1, m_2, \ldots, m_{N_{\mathrm{Q}}}) = \overline{v}(m_2, m_3, \ldots, m_{N_{\mathrm{Q}}}, m_1), \forall (m_1, m_2, \ldots, m_{N_{\mathrm{Q}}}) \in \mathcal{Q}$.*

3. *$\overline{v}(m_1, m_2, \ldots, m_{N_{\mathrm{Q}}}) = \overline{v}(m_{N_{\mathrm{Q}}}, m_{N_{\mathrm{Q}}-1}, \ldots, m_2, m_1), \forall (m_1, m_2, \ldots, m_{N_{\mathrm{Q}}}) \in \mathcal{Q}$.*

Proof. 1. Let any elements $\mathbf{x} \in \mathcal{Q}$ and $\mathbf{y} \in \mathcal{Q}$ be given. For any $\lambda \in (0, 1)$, we want to show that

$$\overline{v}(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < \lambda\overline{v}(\mathbf{x}) + (1-\lambda)\overline{v}(\mathbf{y}).$$

Because $(\lambda^2 - \lambda) < 0$ and $\mathbf{H}$ is a positive definite matrix (see Appendix A), we conclude that

$$(\lambda^2 - \lambda)(\mathbf{x} - \mathbf{y})^T\mathbf{H}(\mathbf{x} - \mathbf{y}) < 0.$$

We expand the appropriate terms in the above inequality and have the following results:

$$\begin{aligned}
(\lambda^2 - \lambda)\Big(\mathbf{x}^T\mathbf{H}\mathbf{x} - 2\mathbf{x}^T\mathbf{H}\mathbf{y} + \mathbf{y}^T\mathbf{H}\mathbf{y}\Big) &< 0 \\
\lambda^2\mathbf{x}^T\mathbf{H}\mathbf{x} + 2\lambda(1-\lambda)\mathbf{x}^T\mathbf{H}\mathbf{y} + (1-\lambda)^2\mathbf{y}^T\mathbf{H}\mathbf{y} &< \lambda\mathbf{x}^T\mathbf{H}\mathbf{x} + (1-\lambda)\mathbf{y}^T\mathbf{H}\mathbf{y} \\
\Big(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\Big)^T\mathbf{H}\Big(\lambda\mathbf{x}^T + (1-\lambda)\mathbf{y}\Big) &< \lambda\mathbf{x}^T\mathbf{H}\mathbf{x} + (1-\lambda)\mathbf{y}^T\mathbf{H}\mathbf{y} \\
\overline{v}(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &< \lambda\overline{v}(\mathbf{x}) + (1-\lambda)\overline{v}(\mathbf{y}).
\end{aligned}$$

Therefore, $\overline{v}(\cdot)$ is strictly convex on $\mathcal{Q}$.

2. Let $(m_1, m_2, \ldots, m_{N_\mathrm{Q}}) \in \mathcal{Q}$ be given.

$$\begin{aligned}
&\overline{v}(m_2, m_3, \ldots, m_{N_\mathrm{Q}}, m_1) \\
&= A \sum_{i=1}^{N_\mathrm{Q}} m_i^2 + \sum_{i=1}^{N_\mathrm{Q}-1} \sum_{j=i+1}^{N_\mathrm{Q}-1} B_{ij} m_{i+1} m_{j+1} + \sum_{i=1}^{N_\mathrm{Q}-1} B_{iN_\mathrm{Q}} m_{i+1} m_1 + C \\
&\overset{(a)}{=} A \sum_{i=1}^{N_\mathrm{Q}} m_i^2 + \sum_{i=1}^{N_\mathrm{Q}-1} \sum_{j=i+1}^{N_\mathrm{Q}-1} B_{(i+1)(j+1)} m_{i+1} m_{j+1} + \sum_{i=1}^{N_\mathrm{Q}-1} B_{1(i+1)} m_{i+1} m_1 + C \\
&= A \sum_{i=1}^{N_\mathrm{Q}} m_i^2 + \sum_{i=2}^{N_\mathrm{Q}} \sum_{j=i+1}^{N_\mathrm{Q}} B_{ij} m_i m_j + \sum_{j=2}^{N_\mathrm{Q}} B_{1j} m_1 m_j + C \\
&= \overline{v}(m_1, m_2, \ldots, m_{N_\mathrm{Q}}).
\end{aligned}$$

The equality (a) follows from $B_{ij} = B_{(i+1)(j+1)}$ and $B_{iN_\mathrm{Q}} = B_{1(i+1)}$.

3. Let $(m_1, m_2, \ldots, m_{N_\mathrm{Q}}) \in \mathcal{Q}$ be given.

$$\begin{aligned}
\overline{v}(m_{N_\mathrm{Q}}, &m_{N_\mathrm{Q}-1}, \ldots, m_2, m_1) \\
&= A \sum_{i=1}^{N_\mathrm{Q}} m_i^2 + \sum_{i=1}^{N_\mathrm{Q}} \sum_{j=i+1}^{N_\mathrm{Q}} B_{ij} m_{N_\mathrm{Q}-i+1} m_{N_\mathrm{Q}-j+1} + C \\
&\overset{(a)}{=} A \sum_{i=1}^{N_\mathrm{Q}} m_i^2 + \sum_{i=1}^{N_\mathrm{Q}} \sum_{j=i+1}^{N_\mathrm{Q}} B_{(N_\mathrm{Q}-j+1)(N_\mathrm{Q}-i+1)} m_{N_\mathrm{Q}-i+1} m_{N_\mathrm{Q}-j+1} + C \\
&= A \sum_{i=1}^{N_\mathrm{Q}} m_i^2 + \sum_{i=1}^{N_\mathrm{Q}} \sum_{j=i+1}^{N_\mathrm{Q}} B_{ij} m_i m_j + C \\
&= \overline{v}(m_1, m_2, \ldots, m_{N_\mathrm{Q}}).
\end{aligned}$$

The equality (a) follows from $B_{ij} = B_{(N_\mathrm{Q}-j+1)(N_\mathrm{Q}-i+1)}$.

That completes the proof. □

Before ending this section, we provide an interpretation of the second and third properties associated with non-preferential flow diagrams. Consider a case when $\mathbf{m} \in \mathcal{S}$. The second property then states that the absorption time is invariant when every state in the flow diagram is rotated to the left. Applying the second property to the flow diagram several times, we can show that the absorption time is also invariant when the flow diagram is rotated to the right. Thus, the absorption time is rotationally invariant. The third property states that the absorption time is invariant when the flow diagram is viewed in a

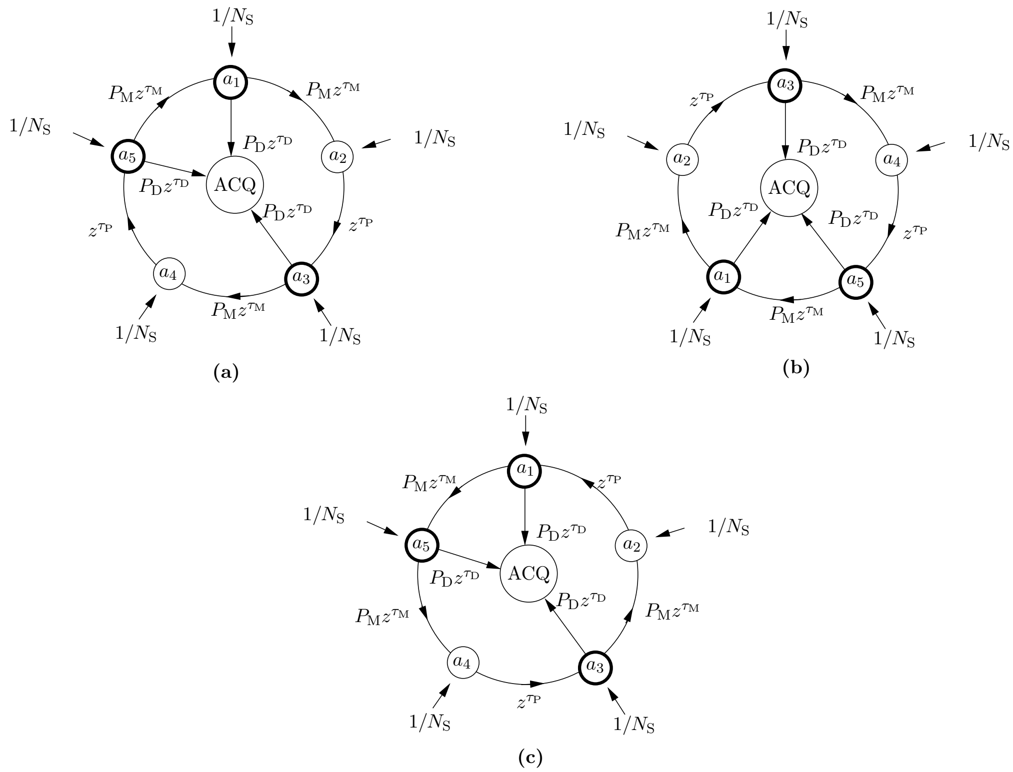

Figure 4-5: The three flow diagrams have the same absorption time, but they have different spacing rules: **(a)** a flow diagram with the spacing rule $(m_1, m_2, \ldots, m_{N_\mathrm{Q}})$; **(b)** a rotated flow diagram with the spacing rule $(m_2, m_3, \ldots, m_{N_\mathrm{Q}}, m_1)$; **(c)** a reversed flow diagram with the spacing rule $(m_{N_\mathrm{Q}}, m_{N_\mathrm{Q}-1}, \ldots, m_1)$. To simplify the drawing, we show the case when $N_\mathrm{S} = 5$, $N_\mathrm{Q} = 3$, and $(m_1, m_2, m_3) = (1, 1, 0)$.

reverse direction. These assertions are valid because the flow diagram is non-preferential. See Fig. 4-5 for an illustration. In the next section, we will use the explicit expression $\overline{v}(\cdot)$ and its properties to bound the minimum MAT.

4.3 The Minimum MAT

In this section, we find the upper and lower-bounds for the minimum MAT

$$T_{\min} \triangleq \min_{\pi \in \mathcal{P}} \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\}.$$

We will show that for certain values of N_Q, there *exists* a search order that achieves the lower-bound. Furthermore, we will obtain a "near-optimal" search order that results in the MAT reasonably close to the minimum one. The lower-bound of $T_{\min}$ is given in the following theorem.

Theorem 4.3 (Minimum MAT). *The optimal mean acquisition time T_{min} satisfies*

$$T^{\mathrm{L}}_{min} \le T_{min}, \tag{4.22}$$

where T^{L}_{min} is given by

$$T^{\mathrm{L}}_{min} = \left(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1\right)\left(\frac{1+P_{\mathrm{M}}}{1-P_{\mathrm{M}}}\right)\frac{\tau_{\mathrm{P}}}{2} + \frac{P_{\mathrm{M}}}{1-P_{\mathrm{M}}}\tau_{\mathrm{M}} + \tau_{\mathrm{D}}. \tag{4.23}$$

Moreover, the equality in (4.22) is achieved if and only if $N_{\mathrm{Q}} = 1$ or $N_{\mathrm{Q}} = N_{\mathrm{S}}$.

Proof. $T_{\min}$ is lower-bounded by

$$\begin{aligned}
T_{\min} &= \frac{1}{N_{\mathrm{S}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{S}}} v(\mathbf{s}(\pi, b)) \\
&\ge \frac{1}{N_{\mathrm{S}}} \sum_{b=1}^{N_{\mathrm{S}}} \min_{\pi \in \mathcal{P}} v(\mathbf{s}(\pi, b)) \\
&\ge \frac{1}{N_{\mathrm{S}}} \sum_{b=1}^{N_{\mathrm{S}}} \min_{(\pi,i) \in \mathcal{D}} v(\mathbf{s}(\pi, i)) \\
&= \min_{(\pi,i) \in \mathcal{D}} v(\mathbf{s}(\pi, i)) \\
&= \min_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}) \\
&\ge \min_{\mathbf{m} \in \mathcal{Q}} \overline{v}(\mathbf{m}) \\
&\overset{(a)}{=} T^{\mathrm{L}}_{\min}.
\end{aligned} \tag{4.24}$$

The equality (a) follows from part two of Lemma B.1 in Appendix B, which shows that

$$\begin{aligned}
\min_{\mathbf{m} \in \mathcal{Q}} \overline{v}(\mathbf{m}) &= \overline{v}\Big(\tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1, \tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1, \ldots, \tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1\Big) \\
&= T^{\mathrm{L}}_{\min}.
\end{aligned} \tag{4.25}$$

Therefore, we have the bound $T_{\min} \ge T^{\mathrm{L}}_{\min}$.

Now, we show that the equality in (4.22) is achieved if and only if $N_{\mathrm{Q}} = 1$ or $N_{\mathrm{Q}} = N_{\mathrm{S}}$.

Assume that there is one in-phase cell ($N_{\mathrm{Q}} = 1$). Then, for any description $(\pi, b) \in \mathcal{D}$, the spacing rule satisfies $\mathbf{s}(\pi, b) = (N_{\mathrm{S}} - 1)$, and the absorption time $v(\mathbf{s}(\pi, b)) = v(N_{\mathrm{S}} - 1)$

is a constant.[1] The optimal MAT satisfies

$$\begin{aligned}
T_{\min} &= \frac{1}{N_{\mathrm{S}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{S}}} v(\mathbf{s}(\pi, b)) \\
&= \frac{1}{N_{\mathrm{S}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{S}}} v(N_{\mathrm{S}} - 1) \\
&= v(N_{\mathrm{S}} - 1) \\
&= T_{\min}^{\mathrm{L}}.
\end{aligned}$$

Next, assume that all cells are in-phase cells ($N_{\mathrm{Q}} = N_{\mathrm{S}}$). Then, for any description $(\pi, b) \in \mathcal{D}$, the spacing rule satisfies $\mathbf{s}(\pi, b) = (0, 0, \ldots, 0)$, and the absorption time $v(\mathbf{s}(\pi, b)) = v(0, 0, \ldots, 0)$ is a constant. The optimal MAT satisfies

$$\begin{aligned}
T_{\min} &= \frac{1}{N_{\mathrm{S}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{S}}} v(\mathbf{s}(\pi, b)) \\
&= \frac{1}{N_{\mathrm{S}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{S}}} v(0, 0, \ldots, 0) \\
&= v(0, 0, \ldots, 0) \\
&= T_{\min}^{\mathrm{L}}.
\end{aligned}$$

Therefore, if $N_{\mathrm{Q}} = 1$ or $N_{\mathrm{Q}} = N_{\mathrm{S}}$, the equality in (4.22) is achieved.

To show that the equality in (4.22) implies $N_{\mathrm{Q}} = 1$ or $N_{\mathrm{Q}} = N_{\mathrm{S}}$, we consider a contrapositive proof. Assume that $N_{\mathrm{Q}} \neq 1$ and $N_{\mathrm{Q}} \neq N_{\mathrm{S}}$. Lemma C.1 in Appendix C shows that for all $\pi \in \mathcal{P}$, there exists $b_{\mathrm{o}} \in \mathcal{U}$ such that

$$\mathbf{s}(\pi, b_{\mathrm{o}}) \neq \left(\tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1, \tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1, \ldots, \tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1 \right). \tag{4.26}$$

In particular, (4.26) holds for the optimal search order

$$\pi^* \triangleq \arg\min_{\pi \in \mathcal{P}} \mathbb{E} \left\{ T_{\mathrm{ACQ}}(\pi) \right\}.$$

$$\pi^* \triangleq \arg\min_{\pi \in \mathcal{P}} \mathbb{E} \left\{ T_{\mathrm{ACQ}}(\pi) \right\}$$

[1]When $N_{\mathrm{Q}} = 1$, the spacing rule contains only one element.

be the optimal search order.

Part one of Lemma B.1 shows that the right-hand side is the unique minimizer of $\overline{v}(\cdot)$. As a result, the absorption time $v(\mathbf{s}(\pi^*, b_\text{o}))$ satisfies the following bound:

$$\begin{aligned} v(\mathbf{s}(\pi^*, b_\text{o})) &> \overline{v}\left(\tfrac{N_\text{S}}{N_\text{Q}} - 1, \tfrac{N_\text{S}}{N_\text{Q}} - 1, \ldots, \tfrac{N_\text{S}}{N_\text{Q}} - 1\right) \\ &= \min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}). \end{aligned} \tag{4.27}$$

Then, the minimum MAT is strictly greater than its lower-bound:

$$\begin{aligned} T_{\min} &= \mathbb{E}\left\{T_\text{ACQ}(\pi^*)\right\} \\ &= \frac{1}{N_\text{S}}\left[v(\mathbf{s}(\pi^*, b_\text{o})) + \sum_{\substack{b=1 \\ b\neq b_\text{o}}}^{N_\text{S}} v(\mathbf{s}(\pi^*, b))\right] \\ &\overset{(a)}{>} \frac{1}{N_\text{S}}\left[\min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}) + \sum_{\substack{b=1 \\ b\neq b_\text{o}}}^{N_\text{S}} v(\mathbf{s}(\pi^*, b))\right] \\ &\geq \frac{1}{N_\text{S}}\left[\min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}) + \sum_{\substack{b=1 \\ b\neq b_\text{o}}}^{N_\text{S}} \min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})\right] \\ &= \min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}) \\ &= T_{\min}^\text{L}. \end{aligned} \tag{4.28}$$

The inequality (a) follows from (4.27). Therefore, the equality in (4.22) is not achieved. That completes the proof. □

Next, we will show that if N_Q and N_S are relatively prime, the MAT $\mathbb{E}\left\{T_\text{ACQ}(\pi^{N_\text{Q}})\right\}$ achieved by the search order π^{N_Q} is near optimal.

Definition 4.4 (η-Optimal Search Order). Let $\eta(N_\text{S}, N_\text{Q})$ be a function only of N_Q and N_S, and let π^* be the optimal search order. A search order π is η-optimal, if

$$\frac{\mathbb{E}\left\{T_\text{ACQ}(\pi)\right\} - \mathbb{E}\left\{T_\text{ACQ}(\pi^*)\right\}}{\mathbb{E}\left\{T_\text{ACQ}(\pi^*)\right\}} \leq \eta(N_\text{S}, N_\text{Q}), \tag{4.29}$$

and $\eta(N_\text{S}, N_\text{Q}) \to 0$ as the ratio $N_\text{Q}/N_\text{S} \to 0$.

We point out that the rapid acquisition is crucial especially when the total number of cells is significantly larger than the number of correct cells: $N_\text{S} \gg N_\text{Q} \gg 1$. In this case, the ratio $N_\mathrm{Q}/N_\mathrm{S}$ is small and η-optimal solutions are almost as good as the optimal one.

Definition 4.5 (η-Optimal Subset of Set of Spacing Rules). Let $\eta(N_\mathrm{S}, N_\mathrm{Q})$ be a function only of N_Q and N_S, and let $\mathbf{m}^*$ be the optimal spacing rule. A subset $\mathcal{S}_\eta \subset \mathcal{S}$ is η-optimal, if for every $\mathbf{m} \in \mathcal{S}_\eta$

$$\frac{v(\mathbf{m}) - v(\mathbf{m}^*)}{v(\mathbf{m}^*)} \leq \eta(N_\mathrm{S}, N_\mathrm{Q}), \tag{4.30}$$

and $\eta(N_\mathrm{S}, N_\mathrm{Q}) \to 0$ as the ratio $N_\mathrm{Q}/N_\mathrm{S} \to 0$.

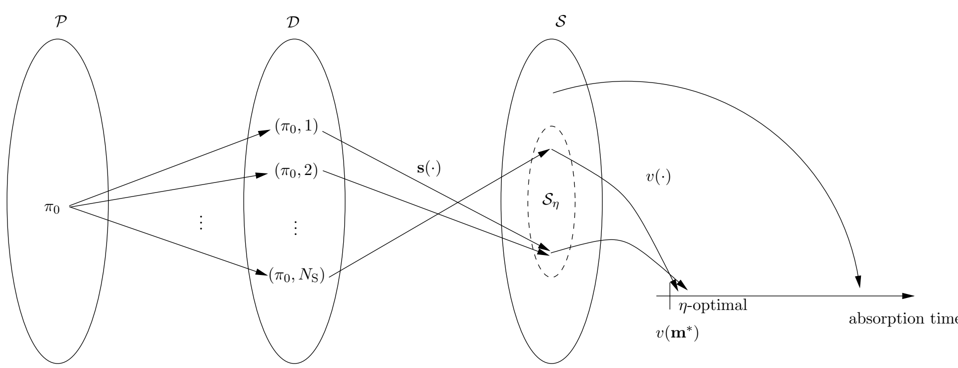

Figure 4-6: The search order π_0 is η-optimal, because the spacing rules $\mathbf{s}(\pi_0, 1)$, $\mathbf{s}(\pi_0, 2)$, $\ldots$, $\mathbf{s}(\pi_0, N_\mathrm{S})$ are members of an η-optimal subset $\mathcal{S}_\eta \subset \mathcal{S}$.

Lemma D.1 in Appendix D establishes the relationship between the η-optimal search order and the η-optimal spacing rules (see Fig. 4-6). In particular, the lemma states that if $\mathcal{S}_\eta \subset \mathcal{S}$ is η-optimal, and if the search order π satisfies

$$\mathbf{s}(\pi, b) \in \mathcal{S}_\eta, \quad b = 1, 2, \ldots, N_\mathrm{S},$$

then π is η-optimal. In the next theorem, we use the relationship between the η-optimal search order and the η-optimal subset of the set of spacing rules to prove that the search order π^{N_Q} is η-optimal with $\eta = \left(\frac{2N_\mathrm{Q}}{N_\mathrm{S} - N_\mathrm{Q}}\right)$.

Theorem 4.4 (Near-Optimality). *If N_Q and N_S are relatively prime, then the search order π^{N_Q} is η-optimal, with*

$$\eta = \frac{2N_\mathrm{Q}}{N_\mathrm{S} - N_\mathrm{Q}}.$$

Proof. Let

$$\mathcal{R} \triangleq \left\{(m_1, m_2, \ldots, m_{N_\mathrm{Q}}) \middle| \sum_{i=1}^{N_\mathrm{Q}} m_i = N_\mathrm{S} - N_\mathrm{Q};\ \forall i, \text{integer } 0 \le m_i \le \left\lfloor \tfrac{N_\mathrm{S}}{N_\mathrm{Q}} \right\rfloor \right\} \tag{4.31}$$

be a subset of $\mathcal{S}$. Lemma E.1 in Appendix E shows that $\mathcal{R}$ is η-optimal with $\eta = \left(\frac{2N_\mathrm{Q}}{N_\mathrm{S}-N_\mathrm{Q}}\right)$. For any $b \in \mathcal{U}$, Lemma F.1 in Appendix F shows that

$$\mathbf{s}(\pi^{N_\mathrm{Q}}, b) \in \mathcal{R}.$$

Therefore, by Lemma D.1, the search order π^{N_Q} is η-optimal with $\eta = \left(\frac{2N_\mathrm{Q}}{N_\mathrm{S}-N_\mathrm{Q}}\right)$. That completes the proof. □

Using Thm. 4.3 and Thm. 4.4, we immediately have the following corollary:

Corollary 4.1 (Narrow bounds). *Let π^* be the optimal search order. If N_Q and N_S are relatively prime, then*

$$T^\mathrm{L}_{min} \le \mathbb{E}\left\{T_\mathrm{ACQ}(\pi^*)\right\} \le \mathbb{E}\left\{T_\mathrm{ACQ}(\pi^{N_\mathrm{Q}})\right\} \le \left(1 + \frac{2N_\mathrm{Q}}{N_\mathrm{S} - N_\mathrm{Q}}\right) T^\mathrm{L}_{min},$$

in which T^L_{min} is given in (4.23).

In the next section, we derive the search orders which result in the maximum MAT.

4.4 The Maximum MAT

In this section we show that the CSS and the FSSS with the step size $N_\mathrm{S} - 1$ should be avoided because they yield the maximum MAT. This result is followed from the theorem below.

Theorem 4.5 (Maximum MAT).

1. *The expression for the maximum MAT* $\max_{\pi\in\mathcal{P}} \mathbb{E}\left\{T_{\text{ACQ}}(\pi)\right\}$ *is given by*

$$\begin{aligned} T_{max} = &\frac{(N_{\text{S}}-N_{\text{Q}})^2}{N_{\text{S}}} \cdot \left(\frac{1+P_{\text{M}}^{N_{\text{Q}}}}{1-P_{\text{M}}^{N_{\text{Q}}}}\right)\frac{\tau_{\text{P}}}{2} \\ &+ \left(1-\frac{N_{\text{Q}}}{N_{\text{S}}}\right)\cdot\left(\frac{1+P_{\text{M}}}{1-P_{\text{M}}}\right)\frac{\tau_{\text{P}}}{2} \\ &+ \frac{P_{\text{M}}}{1-P_{\text{M}}}\tau_{\text{M}} + \tau_{\text{D}}. \end{aligned} \tag{4.32}$$

If the receiver uses the CSS π^1 *or the FSSS* $\pi^{N_{\text{S}}-1}$*, it will result in the maximum MAT.*

2. *If the number* N_{Q} *of in-phase cells satisfies* $2 \le N_{\text{Q}} \le N_{\text{S}}-2$*, and the receiver's MAT is equal to* T_{max}*, then the receiver must use the CSS* π^1 *or the FSSS* $\pi^{N_{\text{S}}-1}$*.*

Proof. 1. The permutation functions π^1 and $\pi^{N_{\text{S}}-1}$ correspond to the search orders $[1,2,3,\ldots,N_{\text{S}}]$ and $[1,N_{\text{S}},N_{\text{S}}-1,\ldots,3,2]$, respectively. For any $b\in\mathcal{U}$, a careful thought will reveal that the spacing rules $\mathbf{s}(\pi^1,b)$ and $\mathbf{s}(\pi^{N_{\text{S}}-1},b)$ satisfy

$$\mathbf{s}(\pi^1,b)\in\mathcal{E}, \tag{4.33}$$

$$\mathbf{s}(\pi^{N_{\text{S}}-1},b)\in\mathcal{E}, \tag{4.34}$$

in which

$$\mathcal{E} = \Big\{(0,0,\ldots,0,N_{\text{S}}-N_{\text{Q}}),(0,0,\ldots,0,N_{\text{S}}-N_{\text{Q}},0),\ldots,(N_{\text{S}}-N_{\text{Q}},0,0,\ldots,0)\Big\}.$$

As a result,

$$\begin{aligned} \mathbb{E}\left\{T_{\text{ACQ}}(\pi^1)\right\} &= \frac{1}{N_{\text{S}}}\sum_{b=1}^{N_{\text{S}}} v(\mathbf{s}(\pi^1,b)) \\ &\overset{(a)}{=} \frac{1}{N_{\text{S}}}\sum_{b=1}^{N_{\text{S}}} \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \\ &= \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \\ &\overset{(b)}{=} T_{\max}. \end{aligned}$$

The equality (a) follows from equation (4.33) and part one of Lemma G.1, which shows that elements of $\mathcal{E}$ are solutions of $\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$. The equality (b) follows from part two of Lemma G.1, which gives the explicit closed-form expression for the maximum absorption time.

Similarly, since

$$\mathbf{s}(\pi^{N_\mathrm{S}-1}, b) \in \mathcal{E}, \quad \text{for all } b \in \mathcal{U},$$

we also have

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^{N_\mathrm{S}-1})\right\} = T_{\max}. \tag{4.35}$$

Therefore, the search orders π^1 and $\pi^{N_\mathrm{S}-1}$ maximize the MAT.

2. Let N_S and N_Q such that $2 \leq N_\mathrm{Q} \leq N_\mathrm{S} - 2$ be given. Assume that the receiver uses the search order π_w that results in the maximum MAT: $\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi_\mathrm{w})\right\} = T_{\max}$.

   We show in Subsection H.1 that the absorption time $v(\mathbf{s}(\pi_\mathrm{w}, b))$ for each $b \in \mathcal{U}$ is equal to the maximum absorption time:

$$v(\mathbf{s}(\pi_\mathrm{w}, 1)) = v(\mathbf{s}(\pi_\mathrm{w}, 2)) = \cdots = v(\mathbf{s}(\pi_\mathrm{w}, N_\mathrm{S}-1)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}), \tag{4.36}$$

$$v(\mathbf{s}(\pi_\mathrm{w}, N_\mathrm{S})) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}). \tag{4.37}$$

   By Lemma H.2 in Appendix H, the conditions (4.36) imply that

$$\begin{aligned}\pi_\mathrm{w} &\in \Big\{[1, 2, 3, \ldots, N_\mathrm{S}], [1, N_\mathrm{S}, N_\mathrm{S}-1, \ldots, 3, 2]\Big\}\\ &= \{\pi^1, \pi^{N_\mathrm{S}-1}\}.\end{aligned}$$

   The search orders π^1 and $\pi^{N_\mathrm{S}-1}$ both satisfy the remaining condition (4.37):

$$\begin{aligned}v(\mathbf{s}(\pi^1, N_\mathrm{S})) &= v(0, 0, \ldots, 0, N_\mathrm{S}-N_\mathrm{Q}, 0)\\ &\overset{(a)}{=} \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}),\end{aligned}$$

   and

$$\begin{aligned}v(\mathbf{s}(\pi^{N_\mathrm{S}-1}, N_\mathrm{S})) &= v(0, N_\mathrm{S}-N_\mathrm{Q}, 0, \ldots, 0, 0)\\ &\overset{(b)}{=} \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}),\end{aligned}$$

in which the equalities (a) and (b) follow from part one of Lemma G.1. As a result, the worst search order π_w is either π^1 or π^{N_S-1}. Therefore, the receiver must use the CSS or the FSSS with the step size $N_S - 1$.

That completes the proof. $\square$

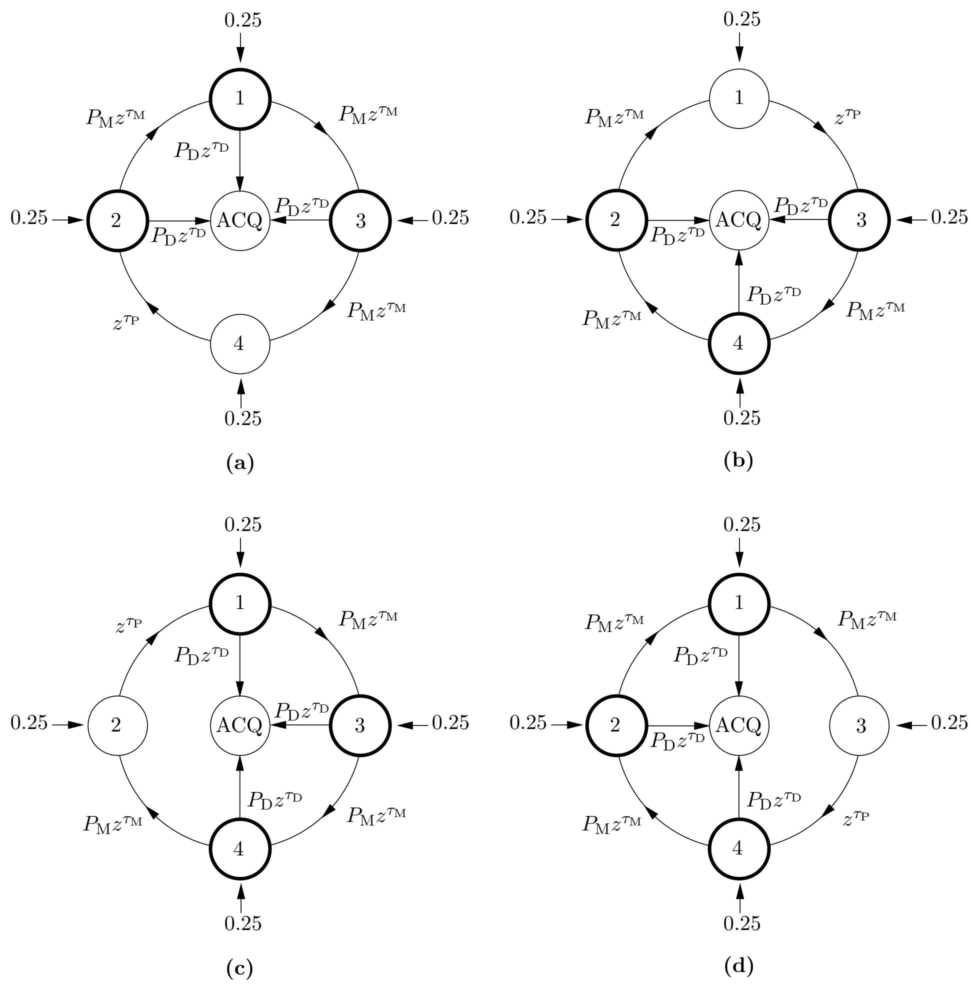

Figure 4-7: When $N_Q = 3$ and $N_S = 4$, the search order $[1, 3, 4, 2]$ maximizes the MAT because the H_1-states in $\mathcal{H}_C(b)$ are *clustering* for every location $B = b$ of the first in-phase cell: **(a)** $B = 1$ and $\mathcal{H}_C(1) = \{1, 2, 3\}$; **(b)** $B = 2$ and $\mathcal{H}_C(2) = \{2, 3, 4\}$; **(c)** $B = 3$ and $\mathcal{H}_C(3) = \{3, 4, 1\}$; **(d)** $B = 4$ and $\mathcal{H}_C(4) = \{4, 1, 2\}$.

Before we end this section, we note that the range $2 \leq N_Q \leq N_S - 2$ in Thm. 4.5 cannot be expanded. In particular, for $N_Q \in \{1, N_S - 1, N_S\}$, the search order π_w that maximizes the MAT is not necessarily the search order π^1 or π^{N_S-1}. This is trivial when $N_Q = 1$ or $N_Q = N_S$, because every search order results in the same MAT. For $N_Q = N_S - 1$, we provide a simple counterexample, in which $N_Q = 3$, $N_S = 4$, and $\pi_w = [1, 3, 4, 2]$. As shown in Fig. 4-7, the corresponding flow diagram for each B gives

$$\mathbf{s}(\pi_{\rm w}, 1) = (0,1,0),$$
$$\mathbf{s}(\pi_{\rm w}, 2) = (0,0,1),$$
$$\mathbf{s}(\pi_{\rm w}, 3) = (0,0,1), \text{ and}$$
$$\mathbf{s}(\pi_{\rm w}, 4) = (1,0,0).$$

Note that Lemma G.1 implies that these spacing rules result in the maximum absorption time, and thus

$$v(\mathbf{s}(\pi_{\rm w}, 1)) = v(\mathbf{s}(\pi_{\rm w}, 2)) = v(\mathbf{s}(\pi_{\rm w}, 3)) = v(\mathbf{s}(\pi_{\rm w}, 4)) = T_{\max}.$$

As a result, π yields the maximum MAT. Evidently, this search order π is not the search order π^1 of the CSS or the search order $\pi^{N_{\rm S}-1}$ of the FSSS.

In a typical scenario, $N_{\rm Q}$ is in the range $2 \leq N_{\rm Q} \leq N_{\rm S} - 2$. As a result, the receiver exhibits the maximum MAT if and only if it uses the CSS or the FSSS with the step size $N_{\rm S} - 1$. Therefore the receiver can immediately improve the MAT by choosing another search order other than the worst search orders π^1 and $\pi^{N_{\rm S}-1}$. In the next section, we compare the MAT of the CSS to that of the optimal serial search, when the uncertainty index set is refined.

4.5 Refining of the Uncertainty Index Set

In this section, we investigate the change in the MAT when the uncertainty index set is refined. We will assume that the number of total cells is much larger than the number of correct cells,

$$N_{\rm S} \gg N_{\rm Q},$$

so that, from Corollary 4.1 of Section 4.3, the optimal MAT is approximately equal to its lower-bound:

$$T_{\min} \approx T_{\min}^{\rm L}.$$

Using the closed-form expression of $T_{\min}^{\rm L}$ in (4.23), we note that $T_{\min}^{\rm L}$ depends on the ratio $N_{\rm S}/N_{\rm Q}$, as opposed to the individual terms $N_{\rm S}$ and $N_{\rm Q}$. When the uncertainty index set $\mathcal{U}$ is refined by a factor of r, the number of total states N_{S} and the number of H_1-states N_{Q} change to rN_{S} and rN_{Q}, respectively. Here, $r \geq 2$ controls the refinement resolution. If r is small enough, the new resulting flow diagram will still have approximately the same path gains $P_{\mathrm{D}}z^{\tau_{\mathrm{D}}}$, $P_{\mathrm{M}}z^{\tau_{\mathrm{M}}}$, and $z^{\tau_{\mathrm{P}}}$. Clearly, the ratio of the number of total cells to the number of correct cells is unchanged,

$$(rN_{\mathrm{S}})/(rN_{\mathrm{Q}}) = N_{\mathrm{S}}/N_{\mathrm{Q}}.$$

As a result, the MAT of the system with the refined uncertainty index set will not change significantly. Therefore, refinement of the uncertainty index set does not significantly increase the MAT of the system that employs the optimal search order.

In contrary, the MAT $T_{\max}$ of the CSS in (4.32) depends on the individual terms N_{S} and N_{Q}. As a result, the MAT of the CSS is affected by the refinement of the uncertainty index set. Let $T_{\max}(r)$ denote the MAT of the system that employs the CSS and refines its uncertainty index set by a factor of r. When r is small enough, the path gains are approximately unchanged. Then, we have the relationship

$$\begin{aligned}\frac{T_{\max}(r)}{T_{\max}} &\approx \frac{\frac{(rN_{\mathrm{S}}-rN_{\mathrm{Q}})^2}{rN_{\mathrm{S}}} \cdot \left(\frac{1+P_{\mathrm{M}}^{N_{\mathrm{Q}}}}{1-P_{\mathrm{M}}^{N_{\mathrm{Q}}}}\right)\frac{\tau_{\mathrm{P}}}{2}}{\frac{(N_{\mathrm{S}}-N_{\mathrm{Q}})^2}{N_{\mathrm{S}}} \cdot \left(\frac{1+P_{\mathrm{M}}^{N_{\mathrm{Q}}}}{1-P_{\mathrm{M}}^{N_{\mathrm{Q}}}}\right)\frac{\tau_{\mathrm{P}}}{2}} \\ &= r.\end{aligned}$$

Thus, the MAT of the CSS is increased approximately r times when the uncertainty index set is refined by a factor of r. Because the MAT of the optimal serial search does not change significantly when the uncertainty index set is refined, the optimal serial search is more robust than the CSS. In the next section, we compare the MAT of the optimal serial search to that of the CSS when the SNR is asymptotically high or low.

4.6 Asymptotic MATs of a Multi-Dwell Detector

In this section, we compare the CSS and the optimal serial search in terms of the MAT in low and high SNR environments. At the detection layer, we consider the multi-dwell detector.

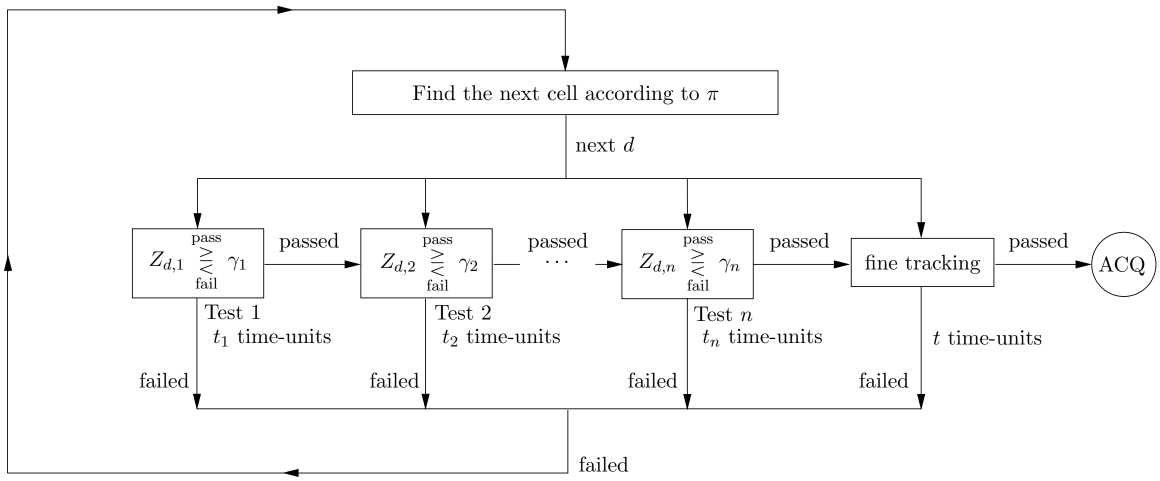

Figure 4-8: Cell d must pass n independent tests before the receiver begins the tracking stage, which decides without the error whether d is a correct cell.

Fig. 4-8 shows the multi-dwell architecture with several decision stages. An acquisition receiver tests whether d is a correct cell by employing n independent tests, $n \geq 1$, one test after the other. The receiver starts by employing test 1. After cell d passes test 1, the receiver employs test 2. After cell d passes test 2, the receiver employs test 3 and so on. If cell d passes *all* n tests, the receiver will enter the tracking stage, which decides without the error after t time-units whether d is a correct cell. On the other hand, if cell d fails any test, the receiver discards cell d and tests the next cell according to the search order π.

Test i involves a threshold test and lasts for a fixed duration of t_i time-units. If a decision variable $Z_{d,i}$ equals or exceeds γ_i, cell d passes test i. Otherwise, cell d fails test i. The receiver selects the thresholds $\gamma_1, \gamma_2, \ldots, \gamma_n$ to minimize the acquisition time.

We consider the following four cases:

- The CSS π^1 with the optimal thresholds at the detection layer in a high SNR environment with the corresponding MAT

$$\mathbb{E}\left\{T_{\text{ACQ}}(\pi^1)\right\}\Big|_{\text{high SNR, optimal thresholds}}. \tag{4.38}$$

- The optimal serial search π^* with the optimal thresholds at the detection layer in a high SNR environment with the corresponding MAT

$$\mathbb{E}\left\{T_{\text{ACQ}}(\pi^*)\right\}\Big|_{\text{high SNR, optimal thresholds}}. \tag{4.39}$$

- The CSS π^1 with the optimal thresholds at the detection layer in a low SNR environment with the corresponding MAT

$$\mathbb{E}\left\{T_{\text{ACQ}}(\pi^1)\right\}\Big|_{\text{low SNR, optimal thresholds}}. \tag{4.40}$$

- The optimal serial search π^* with the optimal thresholds at the detection layer in a low SNR environment with the corresponding MAT

$$\mathbb{E}\left\{T_{\text{ACQ}}(\pi^*)\right\}\Big|_{\text{low SNR, optimal thresholds}}. \tag{4.41}$$

We want to bound the above four MATs by the expressions that depend only on the number n of tests, the durations $t_1, t_2, \ldots, t_n, t$, the number N_{S} of total cells, and the number N_{Q} of correct cells.

We consider a receiver that does not know the locations of the correct cells. As a result, the probability that the receiver begins the test at each cell is uniform and equal to $1/N_{\text{S}}$. We assume that all correct cells $d \in \mathcal{H}_{\text{C}}(B)$ have the same probability $p_{\text{D}i}$ of passing test i:

$$\Pr\left\{Z_{d,i} \geq \gamma_i \,|\, d \in \mathcal{H}_{\text{C}}(B)\right\} = p_{\text{D}i}. \tag{4.42}$$

Similarly, all incorrect cells $d \notin \mathcal{H}_{\text{C}}(B)$ have the same probability $p_{\text{F}i}$ of incorrectly passing test i:

$$\Pr\left\{Z_{d,i} \geq \gamma_i \,|\, d \notin \mathcal{H}_{\text{C}}(B)\right\} = p_{\text{F}i}. \tag{4.43}$$

Equations (4.42) and (4.43) imply that the acquisition procedure can be modelled by a non-preferential flow diagram. In the next subsection, we derive the path gains of this flow diagram.

4.6.1 Path Gains

The path gain from an H_1-state to the acquisition state is given by

$$\begin{aligned} H_{\text{D}}(z) &= (p_{\text{D}1} p_{\text{D}2} \cdots p_{\text{D}n}) \cdot z^{(t_1+t_2+\ldots+t_n)} \\ &= \left(\prod_{i=1}^{n} p_{\text{D}i}\right) \cdot z^{\sum_{l=1}^{n} t_l}, \end{aligned}$$

while the path gain from an H_1-state to the adjacent non-absorbing state is given by

$$\begin{aligned} H_{\mathrm{M}}(z) &= (1-p_{\mathrm{D}1}) \cdot z^{t_1} + p_{\mathrm{D}1}(1-p_{\mathrm{D}2}) \cdot z^{(t_1+t_2)} + \cdots \\ &\quad + (p_{\mathrm{D}1}p_{\mathrm{D}2}\cdots p_{\mathrm{D}n-1})(1-p_{\mathrm{D}n}) \cdot z^{(t_1+t_2+\cdots+t_n)} \\ &= \sum_{i=1}^{n}\Big[\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j}\Big) \cdot (1-p_{\mathrm{D}i}) \cdot z^{\sum_{l=1}^{i} t_l}\Big], \end{aligned}$$

with $\prod_{i=1}^{0} \triangleq 1$. The path gain from an H_0-state to the adjacent non-absorbing state is given by

$$\begin{aligned} H_0(z) &= (1-p_{\mathrm{F}1}) \cdot z^{t_1} + p_{\mathrm{F}1}(1-p_{\mathrm{F}2}) \cdot z^{(t_1+t_2)} + \cdots \\ &\quad + (p_{\mathrm{F}1}p_{\mathrm{F}2}\cdots p_{\mathrm{F}n-1})(1-p_{\mathrm{F}n}) \cdot z^{(t_1+t_2+\cdots+t_n)} \\ &\quad + (p_{\mathrm{F}1}p_{\mathrm{F}2}\cdots p_{\mathrm{F}n}) \cdot z^{(t_1+t_2+\cdots+t_n+t)} \\ &= \sum_{i=1}^{n}\left[\Big(\prod_{j=1}^{i-1} p_{\mathrm{F}j}\Big) \cdot (1-p_{\mathrm{F}i}) \cdot z^{\sum_{l=1}^{i} t_l}\right] \\ &\quad + \Big(\prod_{i=1}^{n} p_{\mathrm{F}i}\Big) \cdot z^{(t+\sum_{l=1}^{n} t_l)}. \end{aligned}$$

Using (4.2), we can reduce the path gains $H_{\mathrm{D}}(z)$, $H_{\mathrm{M}}(z)$, and $H_0(z)$ into the forms $P_{\mathrm{D}} z^{\tau_{\mathrm{D}}}$, $P_{\mathrm{M}} z^{\tau_{\mathrm{M}}}$, and $z^{\tau_{\mathrm{P}}}$, respectively. The path gain parameters become the following:

$$\begin{aligned}
P_{\mathrm{D}} &= \prod_{i=1}^{n} p_{\mathrm{D}i} \\
P_{\mathrm{M}} &= 1 - \prod_{i=1}^{n} p_{\mathrm{D}i} \\
\tau_{\mathrm{D}} &= \sum_{l=1}^{n} t_l \\
\tau_{\mathrm{M}} &= \frac{1}{1-P_{\mathrm{D}}} \cdot \sum_{i=1}^{n} \Big[\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j} \Big) \cdot (1 - p_{\mathrm{D}i}) \cdot \sum_{l=1}^{i} t_l \Big] \\
&\overset{(a)}{=} \frac{1}{1-P_{\mathrm{D}}} \cdot \Big[\sum_{i=1}^{n} \Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j} \Big) \cdot t_i - \prod_{i=1}^{n} p_{\mathrm{D}i} \cdot \sum_{l=1}^{n} t_l \Big] \\
\tau_{\mathrm{P}} &= \sum_{i=1}^{n} \Big[\Big(\prod_{j=1}^{i-1} p_{\mathrm{F}j} \Big) \cdot (1 - p_{\mathrm{F}i}) \cdot \sum_{l=1}^{i} t_l \Big] + \prod_{i=1}^{n} p_{\mathrm{F}i} \cdot \Big(t + \sum_{l=1}^{n} t_l \Big) \\
&\overset{(b)}{=} \sum_{i=1}^{n} \Big(\prod_{j=1}^{i-1} p_{\mathrm{F}j} \Big) \cdot t_i + \Big(\prod_{i=1}^{n} p_{\mathrm{F}i} \Big) \cdot t,
\end{aligned} \tag{4.44}$$

where (a) follows from Corollary I.1, and (b) follows from Lemma I.1 of Appendix I.

In the subsequent sections, we find $p_{\mathrm{D}i}$ and $p_{\mathrm{F}i}$ for a receiver that selects the thresholds γ_i optimally when the SNR is asymptotically high or low.

4.6.2 The MAT of the CSS in a High SNR Environment

When the SNR is high, the probability of detection $p_{\mathrm{D}i}$ approaches one for any value of the probability of false-alarm $p_{\mathrm{F}i} \in (0, 1]$ (see Fig. 4-9):

$$p_{\mathrm{D}i} \to 1, \quad i = 1, 2, 3, \ldots, n.$$

Equation (4.44) implies that $P_{\mathrm{M}} \to 0$. Therefore, Thm. 4.5 gives

$$\begin{aligned}
&\mathbb{E}\left\{ T_{\mathrm{ACQ}}(\pi^1) \right\} \Big|_{\text{high SNR}} \\
&\qquad = \Big[\frac{(N_{\mathrm{S}} - N_{\mathrm{Q}})^2}{N_{\mathrm{S}}} + 1 - \frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}} \Big] \cdot \frac{\tau_{\mathrm{P}}}{2} + \sum_{l=1}^{n} t_l.
\end{aligned}$$

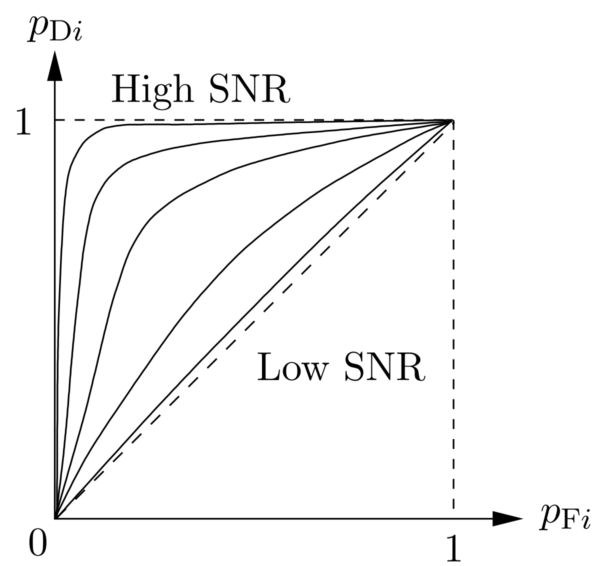

Figure 4-9: In a high SNR environment and in test i, $p_{Di} \to 1$ for every positive value of the probability of false-alarm p_{Fi}. In a low SNR environment, $p_{Di} \to p_{Fi}$.

The MAT can be reduced by selecting the optimal thresholds (or equivalently the optimal probabilities of false-alarm) at the detection layer. Thus,

$$\begin{aligned}
&\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^1)\right\}\Big|_{\text{high SNR, optimal thresholds}} \\
&\qquad = \Big[\frac{(N_\mathrm{S}-N_\mathrm{Q})^2}{N_\mathrm{S}} + 1 - \frac{N_\mathrm{Q}}{N_\mathrm{S}}\Big] \cdot \inf_{\substack{p_{\mathrm{F}i}\in(0,1],\\ 1\le i\le n}} \frac{\tau_\mathrm{P}}{2} + \sum_{l=1}^{n} t_l \\
&\qquad \overset{(a)}{=} \Big[\frac{(N_\mathrm{S}-N_\mathrm{Q})^2}{N_\mathrm{S}} + 1 - \frac{N_\mathrm{Q}}{N_\mathrm{S}}\Big] \cdot \frac{t_1}{2} + \sum_{l=1}^{n} t_l.
\end{aligned} \tag{4.45}$$

Equality (a) follows from the third part of Lemma J.1, which finds the infimum of τ_P (see Appendix J).

4.6.3 The MAT of the Optimal Serial Search in a High SNR Environment

Corollary 4.1 gives the lower-bound and the upper-bound on the optimal MAT. A similar argument to the previous subsection shows that when SNR is small, $P_\mathrm{M} \to 0$. As a consequence,

$$\begin{aligned}\Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}}-1\Big)\cdot\frac{t_1}{2}+\sum_{l=1}^{n}t_l \\ \leq \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^*)\right\}\Big|_{\text{high SNR, optimal thresholds}} \\ \leq \Big(1+\frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}}-N_{\mathrm{Q}}}\Big)\cdot\Big[\Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}}-1\Big)\cdot\frac{t_1}{2}+\sum_{l=1}^{n}t_l\Big].\end{aligned} \tag{4.46}$$

4.6.4 The MAT of the CSS in a Low SNR Environment

When the SNR is low, the probability of detection $p_{\mathrm{D}i}$ approaches the probability of false-alarm $p_{\mathrm{F}i}$ (see Fig. 4-9):

$$p_{\mathrm{D}i}\to p_{\mathrm{F}i},\quad i=1,2,3,\ldots,n.$$

Equation (4.44) implies that $P_{\mathrm{M}}\to(1-\prod_{i=1}^{n}p_{\mathrm{F}i})$. Thus, the MAT is

$$\begin{aligned}&\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^1)\right\}\Big|_{\text{low SNR}}\\ &=\frac{(N_{\mathrm{S}}-N_{\mathrm{Q}})^2}{2N_{\mathrm{S}}}\cdot\left[\frac{1+(1-\prod_{i=1}^{n}p_{\mathrm{F}i})^{N_{\mathrm{Q}}}}{1-(1-\prod_{i=1}^{n}p_{\mathrm{F}i})^{N_{\mathrm{Q}}}}\right]\cdot\left[\sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1}p_{\mathrm{F}j}\Big)\cdot t_i+\Big(\prod_{i=1}^{n}p_{\mathrm{F}i}\Big)\cdot t\right]\\ &\quad+\Big(2-\frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\Big)\cdot\sum_{i=1}^{n}\Big(\frac{t_i}{\prod_{j=i}^{n}p_{\mathrm{F}j}}\Big)\\ &\quad-\Big(\frac{N_{\mathrm{S}}-N_{\mathrm{Q}}}{2N_{\mathrm{S}}}\Big)\cdot\left[\sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1}p_{\mathrm{F}j}\Big)\cdot t_i+\Big(\prod_{i=1}^{n}p_{\mathrm{F}i}\Big)\cdot t\right]\\ &\quad+\Big(\frac{N_{\mathrm{S}}-N_{\mathrm{Q}}}{N_{\mathrm{S}}}\Big)\cdot t\\ &\triangleq f(p_{\mathrm{F}1},p_{\mathrm{F}2},\ldots,p_{\mathrm{F}n}).\end{aligned} \tag{4.47}$$

When the receiver selects the optimal thresholds (or equivalently the optimal probabilities of false-alarm), the above MAT becomes

$$\begin{aligned}&\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^1)\right\}\Big|_{\text{low SNR, optimal thresholds}}\\ &\qquad=\inf_{\substack{p_{\mathrm{F}i}\in(0,1],\\ 1\leq i\leq n}} f(p_{\mathrm{F}1},p_{\mathrm{F}2},\ldots,p_{\mathrm{F}n}).\end{aligned} \tag{4.48}$$

Using the lower-bound in Lemma K.1 and the upper-bound in Lemma K.3 in Appendix K, we have the following result:

$$\begin{aligned}\frac{(N_{\text{S}}-N_{\text{Q}})^2}{2N_{\text{S}}}\cdot t_1+\Big(\frac{N_{\text{S}}-N_{\text{Q}}}{2N_{\text{S}}}\Big)\cdot\Big(t+\sum_{l=2}^{n}t_l\Big)&\\ \le \mathbb{E}\left\{T_{\text{ACQ}}(\pi^1)\right\}\Big|_{\text{low SNR, optimal thresholds}}&\\ \le (N_{\text{S}}+3N_{\text{Q}})\cdot t_1+\Big(\frac{\ln 3\cdot N_{\text{S}}}{N_{\text{Q}}}+2\Big)\sum_{l=2}^{n}t_l+\Big(\frac{\ln 3\cdot N_{\text{S}}}{N_{\text{Q}}}+1\Big)\cdot t,&\end{aligned} \tag{4.49}$$

in which $\sum_{l=2}^{1}\triangleq 0$.

4.6.5 The MAT of the Optimal Serial Search in a Low SNR Environment

Corollary 4.1 gives the lower-bound and the upper-bound on the optimal MAT. A similar argument to the previous subsection shows that when SNR is small, $P_{\text{M}}\to(1-\prod_{i=1}^{n}p_{\text{F}i})$. Because the receiver selects the optimal thresholds, we have the following bounds:

$$\begin{aligned}\inf_{\substack{p_{\text{F}i}\in(0,1],\\ 1\le i\le n}} g(p_{\text{F}1},p_{\text{F}2},\ldots,p_{\text{F}n}) \le \mathbb{E}\left\{T_{\text{ACQ}}(\pi^*)\right\}\Big|_{\text{low SNR, optimal thresholds}}&\\ \le\Big(1+\frac{2N_{\text{Q}}}{N_{\text{S}}-N_{\text{Q}}}\Big)\cdot\inf_{\substack{p_{\text{F}i}\in(0,1],\\ 1\le i\le n}} g(p_{\text{F}1},p_{\text{F}2},\ldots,p_{\text{F}n}),&\end{aligned} \tag{4.50}$$

in which

$$\begin{aligned}g(p_{\text{F}1},p_{\text{F}2},\ldots,p_{\text{F}n})\triangleq T_{\min}^{\text{L}}\Big|_{p_{\text{D}i}=p_{\text{F}i},\ i=1,2,\ldots,n}&\\ =\frac{N_{\text{S}}}{N_{\text{Q}}}\cdot\sum_{i=1}^{n}\Big(\frac{t_j}{\prod_{j=i}^{n}p_{\text{F}j}}\Big)&\\ -\Big(\frac{N_{\text{S}}-N_{\text{Q}}}{2N_{\text{Q}}}\Big)\cdot\Big[\sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1}p_{\text{F}j}\Big)\cdot t_i+\Big(\prod_{i=1}^{n}p_{\text{F}i}\Big)\cdot t\Big]&\\ +\Big(\frac{N_{\text{S}}-N_{\text{Q}}}{N_{\text{Q}}}\Big)\cdot t\,.&\end{aligned} \tag{4.51}$$

The infimum of $g(\cdot)$, evaluated in Lemma J.4 in Appendix J, together with (4.50), gives

$$\begin{aligned}
&\left(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}}-1\right)\cdot\left(\frac{t+\sum_{l=1}^{n}t_l}{2}\right)+\sum_{l=1}^{n}t_l \\
&\qquad\le \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^*)\right\}\Big|_{\text{low SNR, optimal thresholds}} \\
&\qquad\le \left(1+\frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}}-N_{\mathrm{Q}}}\right)\cdot\left[\left(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}}-1\right)\cdot\left(\frac{t+\sum_{l=1}^{n}t_l}{2}\right)+\sum_{l=1}^{n}t_l\right].
\end{aligned} \tag{4.52}$$

4.6.6 Asymptotic Gains

We compare the MAT of the optimal serial search to that of the CSS in high and low SNR environments. Let

$$G_{\mathrm{H}} \triangleq \frac{\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^1)\right\}\Big|_{\text{high SNR, optimal thresholds}}}{\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^*)\right\}\Big|_{\text{high SNR, optimal thresholds}}}$$

denote the gain in a high SNR regime. Using (4.45) and (4.46), we bound the performance gain G_{H} by

$$\begin{aligned}
&\frac{\left[\frac{(N_{\mathrm{S}}-N_{\mathrm{Q}})^2}{N_{\mathrm{S}}}+1-\frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\right]\cdot\frac{t_1}{2}+\sum_{l=1}^{n}t_l}{\left(1+\frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}}-N_{\mathrm{Q}}}\right)\cdot\left[\left(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}}-1\right)\cdot\frac{t_1}{2}+\sum_{l=1}^{n}t_l\right]} \le G_{\mathrm{H}} \\
&\qquad\qquad\le \frac{\left[\frac{(N_{\mathrm{S}}-N_{\mathrm{Q}})^2}{N_{\mathrm{S}}}+1-\frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\right]\cdot\frac{t_1}{2}+\sum_{l=1}^{n}t_l}{\left(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}}-1\right)\cdot\frac{t_1}{2}+\sum_{l=1}^{n}t_l}.
\end{aligned} \tag{4.53}$$

If the number N_{S} of total cells is much larger than the number N_{Q} of correct cells,

$$N_{\mathrm{S}} \gg N_{\mathrm{Q}}, \tag{4.54}$$

then both the lower and upper-bounds are approximately equal to

$$\begin{aligned}
\frac{\frac{N_{\mathrm{S}}t_1}{2}+\sum_{l=1}^{n}t_l}{\frac{N_{\mathrm{S}}t_1}{2N_{\mathrm{Q}}}+\sum_{l=1}^{n}t_l} &\approx \frac{\frac{N_{\mathrm{S}}t_1}{2}}{\frac{N_{\mathrm{S}}t_1}{2N_{\mathrm{Q}}}} \\
&= N_{\mathrm{Q}}.
\end{aligned} \tag{4.55}$$

Thus, in a high SNR regime the serial search with the optimal search order is approximately N_{Q} times faster than the CSS:

$$G_{\mathrm{L}} \approx N_{\mathrm{Q}}. \tag{4.56}$$

This clearly indicates that the multipath improves the acquisition in a high SNR regime.

For a low SNR regime, we let

$$G_{\mathrm{L}} \triangleq \frac{\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^1)\right\}\Big|_{\text{low SNR, optimal thresholds}}}{\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi^*)\right\}\Big|_{\text{low SNR, optimal thresholds}}}$$

denote the gain. Using (4.49) and (4.52), we upper-bound the performance gain G_{L} by

$$G_{\mathrm{L}} \leq \frac{(N_{\mathrm{S}} + 3N_{\mathrm{Q}}) \cdot t_1 + \left(\frac{\ln 3 \cdot N_{\mathrm{S}}}{N_{\mathrm{Q}}} + 2\right) \sum_{l=2}^{n} t_l + \left(\frac{\ln 3 \cdot N_{\mathrm{S}}}{N_{\mathrm{Q}}} + 1\right) \cdot t}{\left(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1\right) \cdot \left(\frac{t + \sum_{l=1}^{n} t_l}{2}\right) + \sum_{l=1}^{n} t_l}.$$

When $N_{\mathrm{S}} \gg N_{\mathrm{Q}}$ and the tracking duration is much longer than the testing duration for test i,

$$t \gg t_i, \; i = 1, 2, \ldots, n, \tag{4.57}$$

the upper-bound can be approximated by

$$\begin{aligned} \frac{\left(\frac{\ln 3 \cdot N_{\mathrm{S}}}{N_{\mathrm{Q}}} + 1\right) \cdot t}{\frac{N_{\mathrm{S}}}{2N_{\mathrm{Q}}} t} &\approx 2 \ln 3 \\ &\approx 2. \end{aligned}$$

Because π^* is the optimal search order in a low SNR regime, the performance gain is at least one:

$$G_{\mathrm{L}} \geq 1.$$

Therefore, the performance gain G_{L} is constrained in the approximated range

$$1 \leq G_{\mathrm{L}} \lesssim 2, \tag{4.58}$$

in which the symbol "$\lesssim$" means "approximately less than." In other words, the performance gain in a low SNR environment is a small number that does not depend strongly on N_{Q}.

Chapter 5

Conclusion

In this thesis, we propose a technique that exploits the multipath to aid the sequence acquisition. This technique improves the mean acquisition time (MAT) by utilizing an intelligent search procedure. We consider a serial search and model a search procedure by a *non-preferential* flow diagram. The uncertainty index set contains N_{S} total cells and N_{Q} correct cells, and the rapid acquisition is a crucial problem when $N_{\mathrm{S}} \gg N_{\mathrm{Q}} \gg 1$.

To alleviate the difficulty associated with the direct derivation of the MAT from the flow diagram's *descriptions*, we transform the *descriptions* into the *spacing rules* and then evaluate the MAT from the spacing rules. In this new framework, finding the fundamental limits of the achievable MATs is equivalent to solving the convex optimization problems. Solutions to those optimization problems give the analytical expressions for the minimum and maximum MATs.

We derive the lower and the upper-bounds on the minimum MAT. The lower-bound is achieved with equality if and only if there is one correct cell ($N_{\mathrm{Q}} = 1$), or there are N_{S} correct cells ($N_{\mathrm{Q}} = N_{\mathrm{S}}$). We introduce a notation of η-optimality and prove that the fixed-step serial search (FSSS) with the step size N_{Q} is η-optimal. As a consequence, the FSSS $\pi^{N_{\mathrm{Q}}}$ can be effectively used to achieve the near-optimal MAT in a wide-band transmission system.

We also investigate the search orders that result in the maximum MAT. It turns out that the search order π^1 of the conventional serial search (CSS) and the search order $\pi^{N_{\mathrm{S}}-1}$ of the FSSS exhibit the maximum MAT. For a typical range $2 \leq N_{\mathrm{Q}} \leq N_{\mathrm{S}} - 2$, we further show that only those two search orders result in the maximum MAT. Therefore, the receiver can immediately improve the MAT by avoiding the CSS or the FSSS with the step size $N_{\mathrm{S}} - 1$.

The benefits of selecting the intelligent search order are evident. Unlike the CSS, the optimal serial search is robust with respect to the refinement of the uncertainty region. Our results show that the optimal serial search is approximately N_{Q} times faster than the CSS in a high SNR environment. This thesis provides the methodology for exploiting the multipath, typically considered deleterious for efficient communications, to aid the acquisition and quantifies the performance gain due to the intelligent utilization of the multipath.

Appendix A

Positive Definiteness of the Hessian Matrix H

The goal of this appendix is to show that an $N_{\mathrm{Q}} \times N_{\mathrm{Q}}$ matrix

$$\mathbf{H} \triangleq \frac{\tau_{\mathrm{P}}}{N_{\mathrm{S}}\left(1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}\right)} \left[P_{\mathrm{M}}^{N_{\mathrm{Q}}-|i-j|} + P_{\mathrm{M}}^{|i-j|}\right]_{ij}$$

is positive definite, where $0^0 \triangleq 1$ and $P_{\mathrm{M}} < 1$. The result in this appendix is used in Thm. 4.2 to prove the strict convexity of function $\overline{v}(\cdot)$.

When $P_{\mathrm{M}} = 1$, the absorption time in Thm. 4.1 becomes infinite, and the receiver will never find a correct phase. When $P_{\mathrm{M}} = 0$, the matrix $\tilde{\mathbf{H}}$ becomes

$$\mathbf{H} = \frac{\tau_{\mathrm{P}}}{N_{\mathrm{S}}}\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. Clearly, $\mathbf{H}$ is positive definite when $P_{\mathrm{M}} = 0$. Therefore, we will consider the case when $0 < P_{\mathrm{M}} < 1$.

We rewrite $\mathbf{H}$ as

$$\mathbf{H} = \frac{P_{\mathrm{M}}^{N_{\mathrm{Q}}/2}\tau_{\mathrm{P}}}{N_{\mathrm{S}}\left(1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}\right)}\tilde{\mathbf{H}},$$

where

$$\tilde{\mathbf{H}} \triangleq \left[P_{\mathrm{M}}^{N_{\mathrm{Q}}/2-|i-j|} + P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2+|i-j|}\right]_{ij}$$

$$= \begin{bmatrix} P_{\mathrm{M}}^{N_{\mathrm{Q}}/2} + P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2} & P_{\mathrm{M}}^{N_{\mathrm{Q}}/2-1} + P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2+1} & \dots & P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2+1} + P_{\mathrm{M}}^{N_{\mathrm{Q}}/2-1} \\ P_{\mathrm{M}}^{N_{\mathrm{Q}}/2-1} + P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2+1} & P_{\mathrm{M}}^{N_{\mathrm{Q}}/2} + P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2} & \dots & P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2+2} + P_{\mathrm{M}}^{N_{\mathrm{Q}}/2-2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2+1} + P_{\mathrm{M}}^{N_{\mathrm{Q}}/2-1} & P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2+2} + P_{\mathrm{M}}^{N_{\mathrm{Q}}/2-2} & \dots & P_{\mathrm{M}}^{N_{\mathrm{Q}}/2} + P_{\mathrm{M}}^{-N_{\mathrm{Q}}/2} \end{bmatrix}. \tag{A.1}$$

Note that the coefficient

$$\frac{P_{\mathrm{M}}^{N_{\mathrm{Q}}/2}\tau_{\mathrm{P}}}{N_{\mathrm{S}}\left(1-P_{\mathrm{M}}^{N_{\mathrm{Q}}}\right)}$$

is positive. Therefore, it is sufficient to show the positive definiteness of $\mathbf{H}$ from the positive definiteness of $\tilde{\mathbf{H}}$.

Lemma A.1 (Positive Definite Matrix). *For any $N_{\mathrm{Q}} \geq 1$ and any $P_{\mathrm{M}} \in (0, 1)$, matrix $\tilde{\mathbf{H}}$ defined in (A.1) is positive definite.*

Proof. For $N_{\mathrm{Q}} \geq 1$ and $P_{\mathrm{M}} \in (0, 1)$, the matrix $\tilde{\mathbf{H}}$ is circulant, symmetric, and Toeplitz. The $N_{\mathrm{Q}} \times N_{\mathrm{Q}}$ Fourier matrix

$$\mathbf{F} \triangleq \left[\omega^{(i-1)(j-1)}\right]_{ij}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N_{\mathrm{Q}}-1} \\ \vdots & \vdots & \vdots & \ddots & \omega^{2(N_{\mathrm{Q}}-1)} \\ 1 & \omega^{N_{\mathrm{Q}}-1} & \omega^{2(N_{\mathrm{Q}}-1)} & \dots & \omega^{(N_{\mathrm{Q}}-1)(N_{\mathrm{Q}}-1)} \end{bmatrix}. \tag{A.2}$$

diagonalizes the circulant matrix $\tilde{\mathbf{H}}$ [18, p.268]. Here, $\omega \triangleq \mathrm{e}^{2\pi\sqrt{-1}/N_{\mathrm{Q}}}$. Therefore, columns of $\mathbf{F}$ are eigenvectors of $\tilde{\mathbf{H}}$.

Note that the first element of every eigenvector is one. Therefore, the inner product of the first row of $\tilde{\mathbf{H}}$ and the k-th column of $\mathbf{F}$ is the eigenvalue corresponding to the k-th eigenvector:

$$\lambda_k = \sum_{j=1}^{N_{\mathrm{Q}}} \tilde{\mathbf{H}}_{1j}\mathbf{F}_{jk}, \quad k = 1, 2, \dots, N_{\mathrm{Q}}.$$

Substituting $\tilde{\mathbf{H}}_{1j}$ and $\mathbf{F}_{jk}$ and simplifying terms, we have

$$\begin{aligned}
\lambda_k &= \sum_{i=0}^{N_\mathrm{Q}-1} \left(P_\mathrm{M}^{N_\mathrm{Q}/2-i} + P_\mathrm{M}^{-N_\mathrm{Q}/2+i}\right) \cdot \omega^{i(k-1)} \\
&= P_\mathrm{M}^{N_\mathrm{Q}/2} \cdot \sum_{i=0}^{N_\mathrm{Q}-1} \left(P_\mathrm{M}^{-1}\omega^{k-1}\right)^i + P_\mathrm{M}^{-N_\mathrm{Q}/2} \cdot \sum_{i=0}^{N_\mathrm{Q}-1} \left(P_\mathrm{M}\omega^{k-1}\right)^i \\
&\stackrel{(a)}{=} P_\mathrm{M}^{N_\mathrm{Q}/2}\left[\frac{1-\left(P_\mathrm{M}^{-1}\omega^{k-1}\right)^{N_\mathrm{Q}}}{1-P_\mathrm{M}^{-1}\omega^{k-1}}\right] + P_\mathrm{M}^{-N_\mathrm{Q}/2}\left[\frac{1-\left(P_\mathrm{M}\omega^{k-1}\right)^{N_\mathrm{Q}}}{1-P_\mathrm{M}\omega^{k-1}}\right] \\
&\stackrel{(b)}{=} P_\mathrm{M}^{N_\mathrm{Q}/2}\left[\frac{1-P_\mathrm{M}^{-N_\mathrm{Q}}}{1-P_\mathrm{M}^{-1}\omega^{k-1}}\right] + P_\mathrm{M}^{-N_\mathrm{Q}/2}\left[\frac{1-P_\mathrm{M}^{N_\mathrm{Q}}}{1-P_\mathrm{M}\omega^{k-1}}\right] \\
&\stackrel{(c)}{=} \frac{P_\mathrm{M}^{N_\mathrm{Q}/2}\left(1-P_\mathrm{M}\omega^{k-1}\right)\left(1-P_\mathrm{M}^{-N_\mathrm{Q}}\right) + P_\mathrm{M}^{-N_\mathrm{Q}/2}\left(1-P_\mathrm{M}^{-1}\omega^{k-1}\right)\left(1-P_\mathrm{M}^{N_\mathrm{Q}}\right)}{-P_\mathrm{M}^{-1}\omega^{k-1}\left(1-P_\mathrm{M}\omega^{-(k-1)}\right)\left(1-P_\mathrm{M}\omega^{k-1}\right)} \\
&\stackrel{(d)}{=} \frac{\omega^{k-1}\left(P_\mathrm{M}^{-(N_\mathrm{Q}+2)/2} - P_\mathrm{M}^{-(N_\mathrm{Q}-2)/2} + P_\mathrm{M}^{(N_\mathrm{Q}+2)/2} - P_\mathrm{M}^{(N_\mathrm{Q}-2)/2}\right)}{-P_\mathrm{M}^{-1}\omega^{k-1}\left|1-P_\mathrm{M}\omega^{k-1}\right|^2} \\
&= \frac{\left(1-P_\mathrm{M}^{N_\mathrm{Q}}\right)(1-P_\mathrm{M}^2)}{P_\mathrm{M}^{N_\mathrm{Q}/2}\left|1-P_\mathrm{M}\omega^{k-1}\right|^2}.
\end{aligned} \tag{A.3}$$

The equality (a) follows from the geometric sum. The equality (b) follows from the fact that $\omega^{N_\mathrm{Q}} = 1$. The equality (c) follows from the combination of the two sums and the factoring of the denominator. The equality (d) follows from the fact that the denominator contains the product of a complex conjugate pair.

It is clear from (A.3) that for $P_\mathrm{M} \in (0,1)$,

$$\lambda_k > 0, \quad k = 1, 2, \ldots, N_\mathrm{Q}.$$

Since every eigenvalue of $\tilde{\mathbf{H}}$ is positive, the matrix $\tilde{\mathbf{H}}$ is positive definite [7, Thm 7.2.1, p.402]. That completes the proof. □

Appendix B

Solution to the Minimization Problem

A lemma in this appendix is used to justify the proof statement of Thm. 4.3. The lemma implies that the components m_i^* of the optimal spacing rule $\mathbf{m}^*$ are close to one another. The precise statement of the lemma is given below.

Lemma B.1 (Minimum Extended Absorption Time).

1. *The unique solution* $\mathbf{m}^*$ *to the optimization problem* $\min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})$ *is*

$$m_1^* = m_2^* = m_3^* = \cdots = m_{N_\mathrm{Q}}^* = \tfrac{N_\mathrm{S}}{N_\mathrm{Q}} - 1. \tag{B.1}$$

2. *The optimal cost* $\min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})$ *satisfies*

$$\begin{aligned} \min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}) &= \overline{v}\Big(\tfrac{N_\mathrm{S}}{N_\mathrm{Q}} - 1, \tfrac{N_\mathrm{S}}{N_\mathrm{Q}} - 1, \ldots, \tfrac{N_\mathrm{S}}{N_\mathrm{Q}} - 1\Big) && \text{(B.2a)} \\ &= \left(\frac{N_\mathrm{S}}{N_\mathrm{Q}} - 1\right)\left(\frac{1 + P_\mathrm{M}}{1 - P_\mathrm{M}}\right)\frac{\tau_\mathrm{P}}{2} + \frac{P_\mathrm{M}}{1 - P_\mathrm{M}}\tau_\mathrm{M} + \tau_\mathrm{D} && \text{(B.2b)} \\ &\triangleq T^\mathrm{L}_{min}. && \text{(B.2c)} \end{aligned}$$

3. *If* $N_\mathrm{S}/N_\mathrm{Q}$ *is an integer,* $\mathbf{m}^*$ *is also the unique solution to the integer programming problem* $\min_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$.

Proof.

1. By Weierstrass' theorem [7, p.541], there exists $\mathbf{m}^* \in \mathcal{Q}$ such that $\overline{v}(\mathbf{m}^*) \leq \overline{v}(\mathbf{m})$, for all $\mathbf{m} \in \mathcal{Q}$. By strict convexity of $\overline{v}(\cdot)$, $\mathbf{m}^*$ is the unique optimal solution to the relaxation problem. Furthermore, by property 2 of Thm. 4.2 any $\mathbf{m} \in \mathcal{Q}$ satisfies

$$\begin{aligned}\overline{v}(m_1, m_2, \ldots, m_{N_\mathrm{Q}}) &= \overline{v}(m_2, m_3, \ldots, m_{N_\mathrm{Q}}, m_1) \\ &= \overline{v}(m_3, m_4, \ldots, m_{N_\mathrm{Q}}, m_1, m_2) \\ &\vdots \\ &= \overline{v}(m_{N_\mathrm{Q}}, m_1, m_2, \ldots, m_{N_\mathrm{Q}-1}).\end{aligned}$$

   Applying the above property to the unique solution $\mathbf{m}^*$, we have $m_1^* = m_2^* = m_3^* = \cdots = m_{N_\mathrm{Q}}^*$. Since the sum of its components is $(N_\mathrm{S} - N_\mathrm{Q})$, the optimal solution satisfies

$$m_1^* = m_2^* = m_3^* = \cdots = m_{N_\mathrm{Q}}^* = N_\mathrm{S}/N_\mathrm{Q} - 1.$$

2. Equation (B.2a) follows immediately from part one of this lemma. Equation (B.2b) follows from the explicit expression of $v(\cdot)$ in Thm. 4.1.

3. Since $\mathcal{S} \subset \mathcal{Q}$, we have the relationship $\min_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \geq \min_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}) = \overline{v}(\mathbf{m}^*)$. If $N_\mathrm{S}/N_\mathrm{Q}$ is an integer, then $\mathbf{m}^* \in \mathcal{S}$, and the above lower-bound is satisfied with equality, i.e.

$$\min_{\mathbf{m}\in\mathcal{S}} = v(\mathbf{m}^*).$$

   Therefore, $\mathbf{m}^*$ is the unique solution to the integer programming problem.

That completes the proof. $\square$

Appendix C

Anti-Symmetric Property

The lemma in this appendix is used to justify the proof statement of Thm. 4.3. The MAT $T^{\mathrm{L}}_{\min}$ occurs when the receiver employs the search order that uniformly distributes the non-in-phase cells in the search sequence. However, the lemma in the appendix will show that the receiver cannot always distribute uniformly the non-in-phase cells in the search sequence. Thus, the MAT $T^{\mathrm{L}}_{\min}$ is impossible to achieve in general.

Lemma C.1 (Anti-Symmetric Property). *If $2 \le N_{\mathrm{Q}} \le N_{\mathrm{S}} - 1$, then for all $\pi \in \mathcal{P}$, there is $b_{\mathrm{o}} \in \mathcal{U}$ such that*

$$\mathbf{s}(\pi, b_{\mathrm{o}}) \neq \left(\tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1, \tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1, \ldots, \tfrac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1\right).$$

Proof. Let any $\pi \in \mathcal{P}$ be given. We consider three cases.

1. $\mathbf{s}(\pi, 1) \neq (N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, \ldots, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1)$.
   In this case, $b_{\mathrm{o}} = 1$ and the claim is completed.

2. $\mathbf{s}(\pi, 2) \neq (N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, \ldots, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1)$.
   In this case, $b_{\mathrm{o}} = 2$ and the claim is completed.

3. $\mathbf{s}(\pi, 1) = (N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, \ldots, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1)$ and
   $\mathbf{s}(\pi, 2) = (N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, \ldots, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1)$.
   We want to show that this case is impossible. Assume to the contrary that

$$\mathbf{s}(\pi, 1) = (N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1, \ldots, N_{\mathrm{S}}/N_{\mathrm{Q}} - 1) \tag{C.1}$$

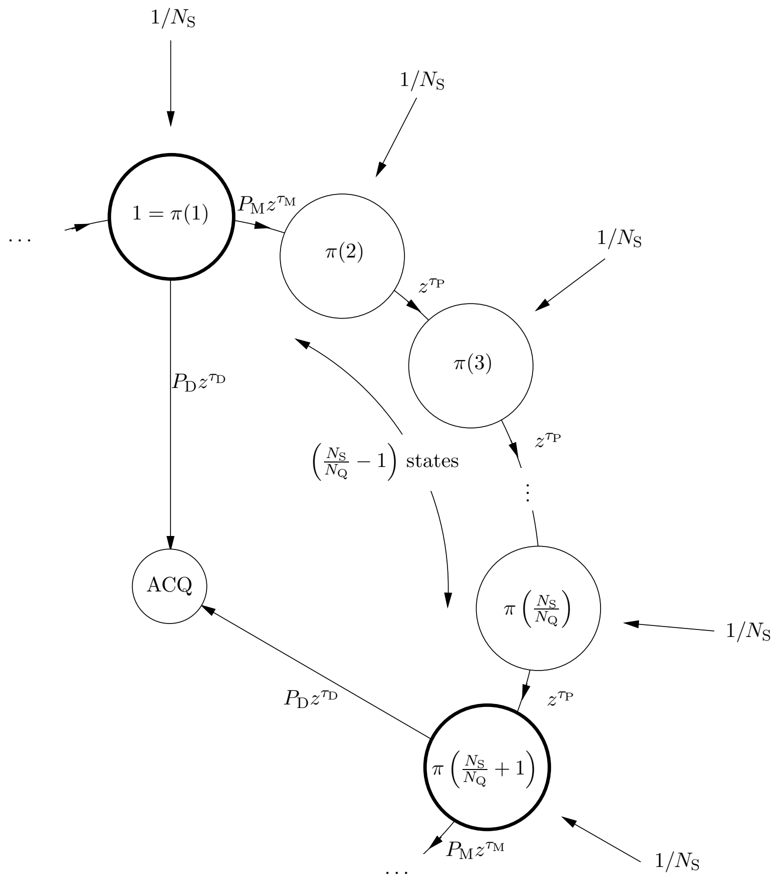

Figure C-1: The set of in-phase cells is $\mathcal{H}_C(1) = \{1, 2, \ldots, N_Q\}$, and there are $\left(\frac{N_S}{N_Q} - 1\right)$ H_0-states between two neighboring in-phase cells.

and

$$\mathbf{s}(\pi, 2) = (N_S/N_Q - 1, N_S/N_Q - 1, \ldots, N_S/N_Q - 1). \tag{C.2}$$

Equation (C.1) implies that elements of

$$\mathcal{H}_C(1) \triangleq \{1, 2, 3, \ldots, N_Q\}$$

are equally spaced in the flow diagram (see Fig. C-1). Similarly, equation (C.2) implies that elements of

$$\mathcal{H}_C(2) \triangleq \{2, 3, 4, \ldots, N_Q, N_Q \oplus 1\}.$$

are equally spaced in the flow diagram.

Because $2 \le N_Q$, we have $2 \in \mathcal{H}_C(1) \cap \mathcal{H}_C(2)$, and $\mathcal{H}_C(1) \cap \mathcal{H}_C(2) \neq \emptyset$. Select any $k_0 \in \mathcal{H}_C(1) \cap \mathcal{H}_C(2)$. Then, equation (C.1) implies that the N_Q elements of $\mathcal{H}_C(1)$

are as follows:

$$\pi\Big(k_0 \oplus i\tfrac{N_S}{N_Q}\Big), \quad i = 0, 1, 2, \ldots, N_Q - 1.$$

Similarly, equation (C.1) implies that the N_Q elements of $\mathcal{H}_C(2)$ are as follows:

$$\pi\Big(k_0 \oplus i\tfrac{N_S}{N_Q}\Big), \quad i = 0, 1, 2, \ldots, N_Q - 1.$$

Therefore,

$$\mathcal{H}_C(1) = \mathcal{H}_C(2). \tag{C.3}$$

Because $N_S \leq N_Q - 1$, state $1 \oplus N_Q$ and state 1 are distinct, and

$$\mathcal{H}_C(1) \neq \mathcal{H}_C(2). \tag{C.4}$$

Using (C.3) and (C.4), we have a contradiction: $\mathcal{H}_C(1) \neq \mathcal{H}_C(1)$. Therefore, the third case is impossible.

That completes the proof. □

Appendix D

Relationship Between η-Optimal Search Orders and η-Optimal Spacing Rules

The rapid acquisition is crucial when $N_{\mathrm{S}} \gg N_{\mathrm{Q}} \gg 1$. In that case, an η-optimal search order will achieve the near-optimal MAT. This appendix investigates one approach for proving that a search order is η-optimal. This approach exploits the relationship $\mathbf{s}\colon \mathcal{D} \to \mathcal{S}$. The result in this appendix is used to justify the proof statement of Thm. 4.4.

Lemma D.1 (Inheritance Property). *If the subset $\mathcal{S}_\eta \subset \mathcal{S}$ of the set of spacing rules is η-optimal, and if the search order π satisfies*

$$\mathbf{s}(\pi, b) \in \mathcal{S}_\eta, \quad b = 1, 2, \ldots, N_{\mathrm{S}},$$

then π is η-optimal.

Proof. Let $\mathcal{S}_\eta \subset \mathcal{S}$ be η-optimal, and $\mathbf{m}^*$ be the optimal spacing rule. Then, for any $\mathbf{m} \in \mathcal{S}_\eta$,

$$\frac{v(\mathbf{m}) - v(\mathbf{m}^*)}{v(\mathbf{m}^*)} \leq \eta(N_{\mathrm{S}}, N_{\mathrm{Q}}),$$

and $\eta(N_{\mathrm{S}}, N_{\mathrm{Q}}) \to 0$ as $N_{\mathrm{Q}}/N_{\mathrm{S}} \to 0$.

Let π be a search order that satisfies

$$\mathbf{s}(\pi, b) \in \mathcal{S}_\eta, \quad b = 1, 2, \ldots, N_{\mathrm{S}}.$$

If π^* is the optimal search order, then

$$v(\mathbf{m}^*) \leq \mathbb{E}\left\{T_{\text{ACQ}}(\pi^*)\right\}.$$

Thus,

$$\begin{aligned}
\frac{\mathbb{E}\left\{T_{\text{ACQ}}(\pi) - \mathbb{E}\left\{T_{\text{ACQ}}(\pi^*)\right\}\right\}}{\mathbb{E}\left\{T_{\text{ACQ}}(\pi^*)\right\}} &\leq \frac{\frac{1}{N_{\text{S}}}\left[\sum_{b=1}^{N_{\text{S}}} v(\mathbf{s}(\pi, b))\right] - v(\mathbf{m}^*)}{v(\mathbf{m}^*)} \\
&= \frac{1}{N_{\text{S}}} \sum_{b=1}^{N_{\text{S}}} \left[\frac{v(\mathbf{s}(\pi, b)) - v(\mathbf{m}^*)}{v(\mathbf{m}^*)}\right] \\
&\leq \frac{1}{N_{\text{S}}} \sum_{b=1}^{N_{\text{S}}} \eta(N_{\text{S}}, N_{\text{Q}}) \\
&\leq \eta(N_{\text{S}}, N_{\text{Q}}).
\end{aligned}$$

Therefore, the search order π is η-optimal. That completes the proof. □

Appendix E

η-Optimal Spacing Rules

The goal of this appendix is to show that the subset $\mathcal{R}$ of $\mathcal{S}$, in which

$$\mathcal{R} \triangleq \left\{ (m_1, m_2, \ldots, m_{N_\mathrm{Q}}) \middle| \sum_{i=1}^{N_\mathrm{Q}} m_i = N_\mathrm{S} - N_\mathrm{Q}; \forall i, \text{integer } 0 \leq m_i \leq \left\lfloor \tfrac{N_\mathrm{S}}{N_\mathrm{Q}} \right\rfloor \right\}, \tag{E.1}$$

is η-optimal. Each spacing rule $\mathbf{m} \in \mathcal{R}$ has components that are "almost equal to one another." As a result, the absorption time $v(\mathbf{m})$ for $\mathbf{m} \in \mathcal{R}$ is close to the minimum absorption time $T^{\mathrm{L}}_{\min}$. The lemma below provides a precise proof of this statement. The result in this appendix is used in the proof of Thm. 4.4.

Lemma E.1 (η-Optimal Spacing Rules). *Let $\mathbf{m}^*$ be the optimal spacing rule. For any $\mathbf{m} \in \mathcal{R}$,*

$$\frac{v(\mathbf{m}) - v(\mathbf{m}^*)}{v(\mathbf{m}^*)} < \frac{2N_\mathrm{Q}}{N_\mathrm{S} - N_\mathrm{Q}}.$$

Thus, $\mathcal{R}$ is η-optimal with $\eta = \left(\frac{2N_\mathrm{Q}}{N_\mathrm{S} - N_\mathrm{Q}}\right)$.

Proof. Let any $\mathbf{m} = (m_1, m_2, \ldots, m_{N_\mathrm{Q}}) \in \mathcal{R}$ be given. By part two of Lemma B.1, the optimal spacing rule satisfies

$$T^{\mathrm{L}}_{\min} \leq v(\mathbf{m}^*).$$

Using the definition of $T^{\mathrm{L}}_{\min}$ in (4.23), we have

$$\begin{aligned}
\frac{v(\mathbf{m}) - v(\mathbf{m}^*)}{v(\mathbf{m}^*)} &\leq \frac{v(\mathbf{m}) - T^{\mathrm{L}}_{\min}}{T^{\mathrm{L}}_{\min}} \\
&= \frac{1}{T^{\mathrm{L}}_{\min}} \bigg\{ A \sum_{i=1}^{N_{\mathrm{Q}}} \bigg[m_i^2 - \Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1 \Big)^2 \bigg] \\
&\quad + \sum_{i=1}^{N_{\mathrm{Q}}} \sum_{j=i+1}^{N_{\mathrm{Q}}} B_{ij} \bigg[m_i m_j - \Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1 \Big)^2 \bigg] \bigg\} \\
&\overset{(a)}{<} \frac{1}{T^{\mathrm{L}}_{\min}} \bigg\{ A \sum_{i=1}^{N_{\mathrm{Q}}} \bigg[\Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} \Big)^2 - \Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1 \Big)^2 \bigg] \\
&\quad + \sum_{i=1}^{N_{\mathrm{Q}}} \sum_{j=i+1}^{N_{\mathrm{Q}}} B_{ij} \bigg[\Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} \Big)^2 - \Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1 \Big)^2 \bigg] \bigg\} \\
&= \frac{1}{T^{\mathrm{L}}_{\min}} \cdot \frac{2N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{Q}}} \Big(N_{\mathrm{Q}} A + \sum_{i=1}^{N_{\mathrm{Q}}} \sum_{j=i+1}^{N_{\mathrm{Q}}} B_{ij} \Big) \\
&= \frac{1}{T^{\mathrm{L}}_{\min}} \cdot \Big(\frac{2N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{S}}} \Big) \cdot \Big(\frac{1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}}{1 + P_{\mathrm{M}}^{N_{\mathrm{Q}}}} \Big) \cdot \Big(\frac{1 + P_{\mathrm{M}}}{1 - P_{\mathrm{M}}} \Big) \cdot \frac{\tau_{\mathrm{P}}}{2} \\
&< \frac{\Big(\frac{2N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{S}}} \Big) \cdot \Big(\frac{1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}}}{1 + P_{\mathrm{M}}^{N_{\mathrm{Q}}}} \Big) \cdot \Big(\frac{1 + P_{\mathrm{M}}}{1 - P_{\mathrm{M}}} \Big) \cdot \frac{\tau_{\mathrm{P}}}{2}}{\Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{Q}}} \Big) \cdot \Big(\frac{1 + P_{\mathrm{M}}}{1 - P_{\mathrm{M}}} \Big) \cdot \frac{\tau_{\mathrm{P}}}{2}} \\
&\overset{(b)}{\leq} \frac{N_{\mathrm{Q}}(2N_{\mathrm{S}} - N_{\mathrm{Q}})}{N_{\mathrm{S}}(N_{\mathrm{S}} - N_{\mathrm{Q}})} \\
&< \frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}} - N_{\mathrm{Q}}}.
\end{aligned}$$

The strict inequality (a) follows from the fact that every $m_i \leq N_{\mathrm{S}}/N_{\mathrm{Q}}$ and some $m_k < N_{\mathrm{S}}/N_{\mathrm{Q}}$. The inequality (b) follows from the fact that the expression $(1 - P_{\mathrm{M}}^{N_{\mathrm{Q}}})/(1 + P_{\mathrm{M}}^{N_{\mathrm{Q}}})$ is maximum when $P_{\mathrm{M}} = 0$.

We note that

$$\frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}} - N_{\mathrm{Q}}} \to 0,$$

as $N_{\mathrm{Q}}/N_{\mathrm{S}} \to 0$. Thus, the set $\mathcal{R}$ is η-optimal with $\eta = \Big(\frac{2N_{\mathrm{Q}}}{N_{\mathrm{S}} - N_{\mathrm{Q}}} \Big)$. That completes the proof. $\square$

Appendix F

The Search Order π^{N_Q} and the Corresponding Spacing Rules

The result in this appendix is used to justify the proof statement of Thm. 4.4. The goal here is to prove that for any $b \in \mathcal{U}$, the description (π^{N_Q}, b) maps to the spacing rule

$$\mathbf{s}(\pi^{N_\mathrm{Q}}, b) \in \mathcal{R},$$

where $\mathcal{R}$ is defined in (4.31).

Lemma F.1 (Spacing Rules of π^{N_Q}). *If N_Q and N_S are relatively prime, then*

$$\mathbf{s}(\pi^{N_\mathrm{Q}}, b) \in \mathcal{R}, \quad \textit{for all } b = 1, 2, \ldots, N_\mathrm{Q}. \tag{F.1}$$

Proof. Let any $b \in \mathcal{U}$ be given. Let $\mathbf{s}(\pi^{N_\mathrm{Q}}, b) = (m_1, m_2, \ldots, m_{N_\mathrm{Q}})$. We want to show that $m_i \leq \lfloor N_\mathrm{S}/N_\mathrm{Q} \rfloor$ for all $i = 1, 2, \ldots, N_\mathrm{Q}$.

Let

$$l_1 \triangleq \left\lfloor \frac{N_\mathrm{S}}{N_\mathrm{Q}} \right\rfloor N_\mathrm{Q}, \tag{F.2}$$

$$l_2 \triangleq \left\lceil \frac{N_\mathrm{S}}{N_\mathrm{Q}} \right\rceil N_\mathrm{Q}. \tag{F.3}$$

Because N_S and N_Q are relatively prime, the ratio N_S/N_Q is fractional. As a result,

$$\left\lfloor \frac{N_S}{N_Q} \right\rfloor = \left\lceil \frac{N_S}{N_Q} \right\rceil - 1, \tag{F.4}$$

$$\left\lfloor \frac{N_S}{N_Q} \right\rfloor < \frac{N_S}{N_Q}, \tag{F.5}$$

$$\frac{N_S}{N_Q} < \left\lceil \frac{N_S}{N_Q} \right\rceil. \tag{F.6}$$

The equality (F.4) implies that

$$l_2 - l_1 = N_Q. \tag{F.7}$$

The strict inequalities (F.5) and (F.6) imply that

$$l_1 < N_S < l_2.$$

Let an H_1-state $a \in \mathcal{H}_C(b)$ be given. Consider the sequence

$$a \oplus l_1, a \oplus (l_1 + 1), a \oplus (l_1 + 2), \ldots, a \oplus N_S, \ldots, a \oplus l_2. \tag{F.8}$$

Because $a \oplus N_S = a$, the state a is a member of the sequence. The number of elements in the sequence is $l_2 - l_1 + 1$, which satisfies

$$\begin{aligned} l_2 - l_1 + 1 &= N_Q + 1 \\ &\le N_S. \end{aligned} \tag{F.9}$$

The above inequality follows from that fact that N_S and N_Q are relatively prime, so that $N_Q \neq N_S$ and $N_Q \le N_S - 1$. The inequality (F.9) implies that elements of the sequence (F.8) are distinct.

We want to show that $a \oplus l_1 \in \mathcal{H}_C(b)$ or $a \oplus l_2 \in \mathcal{H}_C(b)$. Assume to the contrary that

$$a \oplus l_1 \notin \mathcal{H}_C(b) \text{ and} \tag{F.10}$$

$$a \oplus l_2 \notin \mathcal{H}_C(b). \tag{F.11}$$

The set

$$\mathcal{H}_C(b) \triangleq \{b, b \oplus 1, b \oplus 2, \ldots, b \oplus N_S\}$$

contains a by the assumption that $a \in \mathcal{H}_{\mathrm{C}}(b)$. Conditions (F.10) and (F.11), in addition to the fact that $a \in \mathcal{H}_{\mathrm{C}}(b)$, imply that

$$\mathcal{H}_{\mathrm{C}}(b) \subset \{a \oplus (l_1+1), a \oplus (l_1+2), \ldots, a \oplus (l_2-1)\} \triangleq \mathcal{A}.$$

As a consequence,

$$|\mathcal{H}_{\mathrm{C}}(b)| \leq |\mathcal{A}|. \tag{F.12}$$

The cardinalities $|\mathcal{H}_{\mathrm{C}}(b)|$ and $|\mathcal{A}|$ are given by

$$|\mathcal{H}_{\mathrm{C}}(b)| = N_{\mathrm{Q}},$$

and

$$\begin{aligned} |\mathcal{A}| &\overset{(a)}{=} l_2 - l_1 - 1 \\ &= N_{\mathrm{Q}} - 1, \end{aligned}$$

where the equality (a) follows from the fact that elements of the sequence (F.8) are distinct. Thus, the cardinality relationship in (F.12) implies that

$$N_{\mathrm{Q}} \leq N_{\mathrm{Q}} - 1,$$

and we have a contradiction: $0 \leq -1$. Therefore, $a \oplus l_1 \in \mathcal{H}_{\mathrm{C}}(b)$ or $a \oplus l_2 \in \mathcal{H}_{\mathrm{C}}(b)$.

If $a \oplus l_1 \in \mathcal{H}_{\mathrm{C}}(b)$, let m denote the number of H_0-states between the two neighboring H_1-states a and $a \oplus l_1$ (see Fig. F-1). Then, there exists some k_1 such that

$$a \oplus (k_1 N_{\mathrm{Q}}) = a \oplus l_1.$$

Thus,

$$\begin{aligned} m &= k_1 - 1 \\ &= \frac{l_1}{N_{\mathrm{Q}}} - 1 \\ &= \left\lfloor \frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} \right\rfloor - 1. \end{aligned}$$

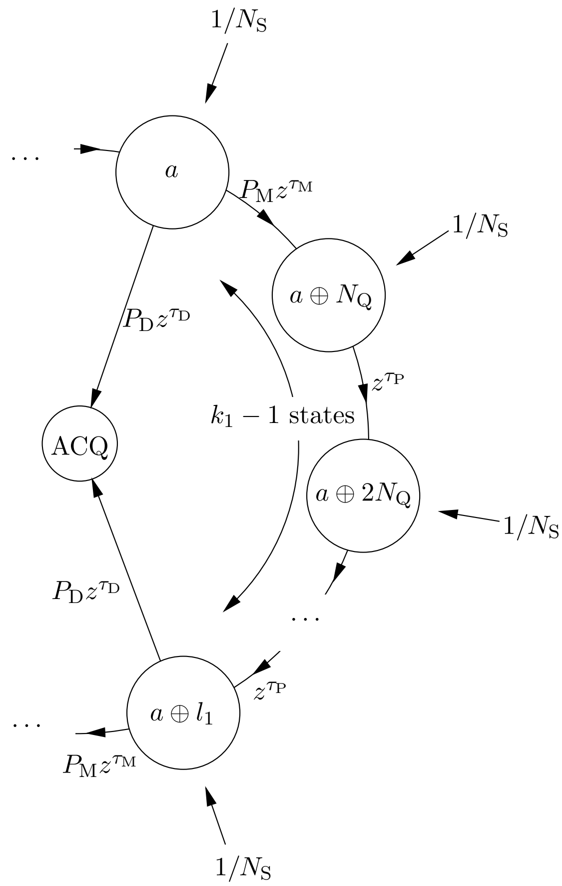

Figure F-1: States a and $a \oplus l_1$ are two neighboring H_1-states. There are $(k_1 - 1)$ H_0-states between states a and $a \oplus l_1$.

If $a \oplus l_2 \in \mathcal{H}_\mathrm{C}(b)$, let m denote the number of H_0-states between the two neighboring H_1-states a and $a \oplus l_2$ (see Fig. F-2). Then, there exists some k_2 such that

$$a \oplus (k_2 N_\mathrm{Q}) = a \oplus l_2.$$

Thus,

$$\begin{aligned} m &= k_2 - 1 \\ &= \frac{l_2}{N_\mathrm{Q}} - 1 \\ &= \left\lceil \frac{N_\mathrm{S}}{N_\mathrm{Q}} \right\rceil - 1 \\ &= \left\lfloor \frac{N_\mathrm{S}}{N_\mathrm{Q}} \right\rfloor. \end{aligned}$$

Because $a \in \mathcal{H}_\mathrm{C}(b)$ is arbitrary, the number m_i of H_0-states between any two neighboring

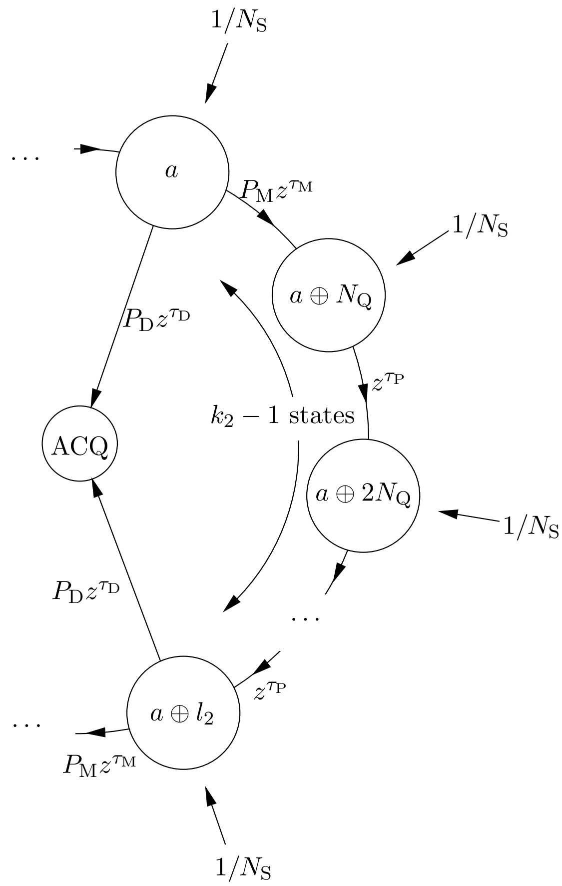

Figure F-2: States a and $a \oplus l_2$ are two neighboring H_1-states. There are $(k_2 - 1)$ H_0-states between states a and $a \oplus l_2$.

H_1-states satisfies

$$m_i \leq \left\lfloor \frac{N_S}{N_Q} \right\rfloor, \quad i = 1, 2, \ldots, N_Q.$$

That completes the proof. $\square$

Appendix G

Solution to the Maximization Problem

A lemma in this appendix is used to justify the proof statement of Thm. 4.5. Intuitively, the lemma states that the maximum absorption time occurs if and only if the non-in-phase cells are consecutive in the flow diagram. The precise statement of the lemma is given below.

Lemma G.1 (Maximum Absorption Time).

1. *The complete solutions of the integer programming problem* $\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$ *are given by*

$$\mathcal{E} \triangleq \Big\{(0,0,\ldots,0,N_{\mathrm{S}}-N_{\mathrm{Q}}),(0,0,\ldots,0,N_{\mathrm{S}}-N_{\mathrm{Q}},0),\ldots,(N_{\mathrm{S}}-N_{\mathrm{Q}},0,0,\ldots,0)\Big\}. \tag{G.1}$$

2. *The maximum absorption time is equal to*

$$\begin{aligned}\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) &= \frac{(N_{\mathrm{S}}-N_{\mathrm{Q}})^2}{N_{\mathrm{S}}}\cdot\left(\frac{1+P_{\mathrm{M}}^{N_{\mathrm{Q}}}}{1-P_{\mathrm{M}}^{N_{\mathrm{Q}}}}\right)\frac{\tau_{\mathrm{P}}}{2}\\ &\quad+\left(1-\frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\right)\cdot\left(\frac{1+P_{\mathrm{M}}}{1-P_{\mathrm{M}}}\right)\frac{\tau_{\mathrm{P}}}{2}\\ &\quad+\frac{P_{\mathrm{M}}}{1-P_{\mathrm{M}}}\tau_{\mathrm{M}}+\tau_{\mathrm{D}}\\ &\triangleq T_{max}.\end{aligned} \tag{G.2}$$

Proof. 1. Consider the relaxation problem $\max_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m})$. For $1 \le i \le N_{\mathrm{Q}}$, let

$$\mathbf{m}^i \triangleq (0, 0, \ldots, 0, N_{\mathrm{S}} - N_{\mathrm{Q}}, 0, 0, \ldots, 0),$$

where only the i-th component of $\mathbf{m}^i$ is non-zero. Therefore $\mathcal{E} = \{\mathbf{m}^1, \mathbf{m}^2, \ldots, \mathbf{m}^{N_{\mathrm{Q}}}\}$.

For any $\mathbf{m} \notin \mathcal{E}$, there exist i and j such that $i \neq j$, $m_i > 0$, and $m_j > 0$. Expressing $\mathbf{m}$ as a convex combination of $\mathbf{m}^1$, $\mathbf{m}^2$, ..., and $\mathbf{m}^{N_{\mathrm{Q}}}$, we obtain

$$\mathbf{m} = \sum_{k=1}^{N_{\mathrm{Q}}} \left(\frac{m_k}{N_{\mathrm{S}} - N_{\mathrm{Q}}} \right) \mathbf{m}^k.$$

Thus,

$$\begin{aligned}
\overline{v}(\mathbf{m}) &\overset{(a)}{<} \sum_{k=1}^{N_{\mathrm{Q}}} \left(\frac{m_k}{N_{\mathrm{S}} - N_{\mathrm{Q}}} \right) \overline{v}(\mathbf{m}^k) \\
&\overset{(b)}{=} \sum_{k=1}^{N_{\mathrm{Q}}} \left(\frac{m_k}{N_{\mathrm{S}} - N_{\mathrm{Q}}} \right) \overline{v}(\mathbf{m}^1) \\
&= \overline{v}(\mathbf{m}^1).
\end{aligned}$$

The inequality (a) follows from strict convexity of $\overline{v}(\cdot)$, positivity of m_i, and positivity of m_j. The equality (b) follows from the second property of Thm. 4.2, $\overline{v}(\mathbf{m}^1) = \cdots = \overline{v}(\mathbf{m}^{N_{\mathrm{Q}}})$. Thus, for any $\mathbf{m} \notin \mathcal{E}$,

$$\overline{v}(\mathbf{m}) < \overline{v}(\mathbf{m}^1) = \overline{v}(\mathbf{m}^2) = \cdots = \overline{v}(\mathbf{m}^{N_{\mathrm{Q}}}). \tag{G.3}$$

Therefore, $\mathcal{E}$ contains all solutions to the relaxation problem.

Consider the integer programming problem $\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$. Since

$$\mathcal{E} \subset \mathcal{S} \subset \mathcal{Q},$$

we have

$$\max_{\mathbf{m}\in\mathcal{E}} v(\mathbf{m}) \le \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \le \max_{\mathbf{m}\in\mathcal{Q}} \overline{v}(\mathbf{m}). \tag{G.4}$$

Note that

$$\arg_{\mathbf{m}\in\mathcal{Q}} \max \overline{v}(\mathbf{m}) = \tilde{\mathbf{m}} \in \mathcal{E}.$$

As a consequence,

$$\max_{\mathbf{m}\in\mathcal{E}} v(\mathbf{m}) = \max_{\mathbf{m}\in\mathcal{Q}} v(\mathbf{m}). \tag{G.5}$$

Therefore, (G.4) and (G.5) imply that

$$\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) = \max_{\mathbf{m}\in\mathcal{Q}} v(\mathbf{m}).$$

2. This part of the lemma follows immediately from part one and the explicit expression of $v(\cdot)$ in Thm. 4.1.

That completes the proof. □

Appendix H

Search Orders that Exhibit the Maximum MAT

In this appendix, we show that the CSS π^1 and the FSSS with the step size $N_{\mathrm{S}}-1$ are the *only* two search orders that result in the maximum MAT, when N_{Q} is constrained in the range $2 \leq N_{\mathrm{Q}} \leq N_{\mathrm{S}}-2$. For the other range of N_{Q}, $N_{\mathrm{Q}} \in \{1, N_{\mathrm{S}}-1, N_{\mathrm{S}}\}$, the other search orders can also achieve the maximum MAT.

Throughout this appendix, we assume that the durations τ_{D}, τ_{M}, and τ_{P} are finite and the probability P_{M} is in the range $P_{\mathrm{M}} \in [0, 1)$. Therefore, the absorption time $v(\cdot)$ in Thm. 4.1 is finite for every spacing rule and so is the MAT.

This appendix is divided into three parts. In the first part, we define the term *clustering* and provide the physical interpretation of the flow diagram with the worst search order π_{w}. In the second part, we consider a simple example, which will help to understand the proofs in this appendix. In the last part, we presents the lemmas and their proofs.

H.1 Clustering States in the Flow Diagram

In this section, we explain the nature of spacing rules $\mathbf{s}(\pi_{\mathrm{w}}, 1), \mathbf{s}(\pi_{\mathrm{w}}, 2), \ldots, \mathbf{s}(\pi_{\mathrm{w}}, N_{\mathrm{S}})$ that correspond to the worst search order π_{w}.

Let the search order π_{w} results in the maximum MAT,

$$\mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi_{\mathrm{w}})\right\} = T_{\max}.$$

Then we have

$$\begin{aligned}
T_{\max} &= \mathbb{E}\left\{T_{\mathrm{ACQ}}(\pi)\right\} \\
&= \frac{1}{N_{\mathrm{S}}}\sum_{b=1}^{N_{\mathrm{S}}} v(\mathbf{s}(\pi_{\mathrm{w}}, b)) \\
&\overset{(a)}{\leq} \frac{1}{N_{\mathrm{S}}}\sum_{b=1}^{N_{\mathrm{S}}} \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \\
&= \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \\
&\overset{(b)}{=} T_{\max}.
\end{aligned}$$

The equality (b) follows from part two of Lemma G.1, which shows that the maximum absorption time is $T_{\max}$. Therefore, that inequality (a) is satisfied with equality, and the absorption times are equal to $\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$:

$$v(\mathbf{s}(\pi_{\mathrm{w}}, 1)) = v(\mathbf{s}(\pi_{\mathrm{w}}, 2)) = \ldots = v(\mathbf{s}(\pi_{\mathrm{w}}, N_{\mathrm{S}})) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}). \tag{H.1}$$

By part one of Lemma G.1, the condition $v(\mathbf{s}(\pi, b)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$ is equivalent to the condition $\mathbf{s}(\pi, b) \in \mathcal{E}$, where

$$\mathcal{E} \triangleq \Big\{(0, 0, \ldots, 0, N_{\mathrm{S}} - N_{\mathrm{Q}}), (0, 0, \ldots, 0, N_{\mathrm{S}} - N_{\mathrm{Q}}), \ldots, (N_{\mathrm{S}} - N_{\mathrm{Q}}, 0, 0, \ldots, 0)\Big\}.$$

Thus, each constraint in (H.1) has a physical interpretation that the H_1-states in the flow diagram are clustered together. The next definition makes this statement precise.

Definition H.1 (Clustering property). For $1 \leq n \leq N_{\mathrm{S}}$, elements $s_1, s_2, \ldots, s_n \in \mathcal{U}$ of the uncertainty index set are *clustering* in the flow diagram with the permutation function π, if there is an index $i \in \mathcal{U}$, such that

$$\{s_1, s_2, \ldots, s_n\} = \Big\{\pi(i), \pi(i \oplus 1), \pi(i \oplus 2), \ldots, \pi(i \oplus (n-1))\Big\}.$$

For example, in Fig. 4-4b, states 2 and 3 are *clustering* in the flow diagram with the search order $[1, 3, 2, 4]$, while in Fig. 4-4d, states 1 and 4 are *clustering* in the flow diagram with the search order $[1,3,2,4]$. Therefore, for any $b \in \mathcal{U}$, the condition

$$v(\mathbf{s}(\pi, b)) = \max_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m})$$

implies that the H_1-states in $\mathcal{H}_\mathrm{C}(b) \triangleq \{b, b \oplus 1, \ldots, b \oplus (N_\mathrm{Q} - 1)\}$ are *clustering*.

H.2 An Illustrative Example

We consider an example, in which $N_\mathrm{S} = 5$ and $N_\mathrm{Q} = 3$. Assume that the search order π_w results in the maximum MAT. Therefore,

$$v(\mathbf{s}(\pi_\mathrm{w}, 1)) = v(\mathbf{s}(\pi_\mathrm{w}, 2)) = \ldots = v(\mathbf{s}(\pi_\mathrm{w}, 5)) = \max_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}), \tag{H.2}$$

and the sets of *clustering* states are the following:

$$\{1,2,3\},$$
$$\{2,3,4\},$$
$$\{3,4,5\},$$
$$\{4,5,1\}, \text{ and}$$
$$\{5,1,2\}.$$

Without using any constraints in (H.2), there are $(N_\mathrm{S} - 1)! = 24$ possibilities of π_w. Because the states in $\{1,2,3\}$ are *clustering*, the possible form of π_w becomes one of the following $N_\mathrm{Q}! = 3!$ possibilities:

$$[1,2,3,-,-],$$
$$[1,3,2,-,-],$$
$$[1,2,-,-,3],$$
$$[1,3,-,-,2],$$
$$[1,-,-,2,3], \text{ or}$$
$$[1,-,-,3,2].$$

Here, the symbol "$-$" denotes an unknown number. See Fig. H-1 for an illustration.

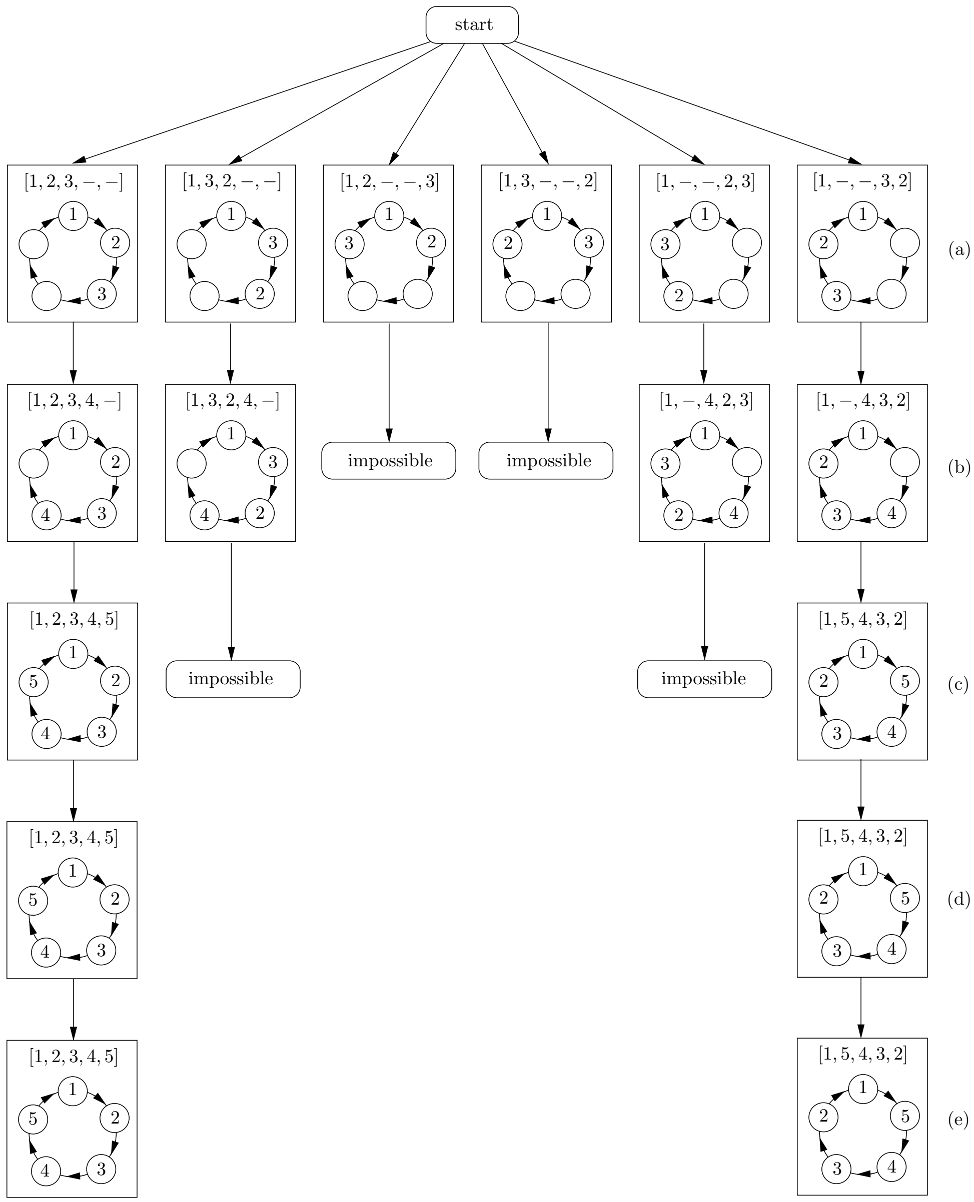

Figure H-1: The constraints $v(\mathbf{s}(\pi, \cdot)) = \max_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m})$ reduce the possible forms of the worst search order π_{w}. The constraints are imposed in the following order: **(a)** the states in $\{1, 2, 3\}$ must be *clustering*; **(b)** in addition, the states in $\{2, 3, 4\}$ must be *clustering*; **(c)** in addition, the states in $\{3, 4, 5\}$ must be *clustering*; **(d)** in addition, the states in $\{4, 5, 1\}$ must be *clustering*; **(e)** in addition, the states in $\{5, 1, 2\}$ must be *clustering*.

When the states in $\{2,3,4\}$ are additionally required to be *clustering*, the possible form of π_{w} becomes one of the following:

$$[1,2,3,4,-],$$
$$[1,3,2,4,-],$$
$$[1,-,4,2,3], \text{ or}$$
$$[1,-,4,3,2].$$

When the states in $\{3,4,5\}$ are additionally required to be *clustering*, the possible form of π_{w} becomes one of the following:

$$[1,2,3,4,5], \text{ or}$$
$$[1,5,4,3,2].$$

When the states in $\{4,5,1\}$ are additionally required to be *clustering*, the possible form of π_{w} becomes one of the following:

$$[1,2,3,4,5], \text{ or}$$
$$[1,5,4,3,2].$$

When the states in $\{5,1,2\}$ are additionally required to be *clustering*, the possible form of π_{w} becomes one of the following:

$$[1,2,3,4,5], \text{ or}$$
$$[1,5,4,3,2].$$

When all five constraints in (H.2) are imposed, we conclude that π_{w} is one of the two forms,

$$\pi_{\mathrm{w}} \in \{[1,2,3,4,5],[1,5,4,3,2]\} = \{\pi^1, \pi^4\}.$$

Therefore, the receiver uses the CSS or the FSSS with the step size $N_{\mathrm{S}} - 1 = 4$.

H.3 Reduction of Possible Forms of $\pi_{\mathbf{w}}$

We prove two lemmas in this section. The first lemma shows that if the search order $\pi_{\rm w}$ satisfies the conditions

$$v(\mathbf{s}(\pi_{\rm w},1)) = v(\mathbf{s}(\pi_{\rm w},2)) = \cdots = v(\mathbf{s}(\pi_{\rm w},N_{\rm Q})) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}), \tag{H.3}$$

the search order $\pi_{\rm w}$ must be one of the following forms:

$$[1,2,3,\ldots,N_{\rm Q},-,-,-,\ldots,-], \text{ or}$$
$$[1,-,-,\ldots,-,N_{\rm Q},\ldots,3,2].$$

The second lemma shows that if the search order $\pi_{\rm w}$ satisfies conditions (H.3) and conditions

$$v(\mathbf{s}(\pi_{\rm w},N_{\rm Q}+1)) = v(\mathbf{s}(\pi_{\rm w},N_{\rm Q}+2)) = \cdots = v(\mathbf{s}(\pi_{\rm w},N_{\rm S}-1)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}), \tag{H.4}$$

the search order $\pi_{\rm w}$ must be one of the following forms:

$$[1,2,3,\ldots,N_{\rm Q},\ldots,N_{\rm S}], \text{ or}$$
$$[1,N_{\rm S},N_{\rm S}-1,\ldots,N_{\rm Q},\ldots,3,2].$$

Therefore, $\pi_{\rm w} \in \pi^1, \pi^{N_{\rm S}-1}$.

The two lemmas in this section will use the notation

$$\Upsilon(n,m) \triangleq \left\{\epsilon \,\middle|\, \epsilon\colon \{n,n+1,n+2,\ldots,m\} \rightarrow \{n,n+1,n+2,\ldots,m\} \text{ is a bijection}\right\}, \tag{H.5}$$

to denote a set of bijections, for some integers $n \leq m$.

Lemma H.1 (Front Reduction). *For any $N_{\rm S}$, $N_{\rm Q}$, and b that satisfy $2 \leq N_{\rm Q} \leq N_{\rm S}-2$ and $2 \leq b \leq N_{\rm Q}$, if the search order π_w satisfies*

$$v(\mathbf{s}(\pi_w,1)) = v(\mathbf{s}(\pi,2)) = \cdots = v(\mathbf{s}(\pi_w,b)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}),$$

then the search order π_w belongs to the set

$$\pi_w \in \Big\{ \Big[1, 2, \ldots, b-1, \epsilon(b), \epsilon(b+1), \ldots, \epsilon(N_\mathrm{Q}), \xi(N_\mathrm{Q}+1), \xi(N_\mathrm{Q}+2), \ldots, \xi(N_\mathrm{S})\Big], \Big[1, \hat{\xi}(N_\mathrm{S}), \hat{\xi}(N_\mathrm{S}-1), \ldots, \hat{\xi}(N_\mathrm{Q}+1), \hat{\epsilon}(N_\mathrm{Q}), \hat{\epsilon}(N_\mathrm{Q}-1), \ldots, \hat{\epsilon}(b), b-1, b-2, \ldots, 2\Big] \Big\},$$

for some bijections $\hat{\epsilon}, \epsilon \in \Upsilon(b, N_\mathrm{Q})$ and $\hat{\xi}, \xi \in \Upsilon(N_\mathrm{Q}+1, N_\mathrm{S})$.

Proof. Let N_S and N_Q such that $2 \leq N_\mathrm{Q} \leq N_\mathrm{S} - 2$ be given. We prove the lemma by an induction on b.

- Base case ($b = 2$):

  Let any search order $\pi_\mathrm{w} \in \mathcal{P}$ such that

  $$v(\mathbf{s}(\pi_\mathrm{w}, 1)) = \max_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}) \tag{H.6}$$

  $$v(\mathbf{s}(\pi_\mathrm{w}, 2)) = \max_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}) \tag{H.7}$$

  be given. Condition (H.6), together with part one of Lemma G.1, implies that the states in $\{1, 2, \ldots, N_\mathrm{Q}\}$ are *clustering* in the flow diagram with the search order π. Therefore, there exists $i \in \mathcal{U}$ such that

  $$\Big\{\pi_\mathrm{w}(i), \pi_\mathrm{w}(i \oplus 1), \pi_\mathrm{w}(i \oplus 2), \ldots, \pi_\mathrm{w}(i \oplus (N_\mathrm{Q} - 1))\Big\} = \{1, 2, 3, \ldots, N_\mathrm{Q}\}. \tag{H.8}$$

  Similarly, (H.7) implies that the states in $\{2, 3, \ldots, N_\mathrm{Q}, N_\mathrm{Q}+1\}$ are *clustering* in the flow diagram with the search order π. Therefore, there exists $j \in \mathcal{U}$ such that

  $$\Big\{\pi_\mathrm{w}(j), \pi_\mathrm{w}(j \oplus 1), \pi_\mathrm{w}(j \oplus 2), \ldots, \pi_\mathrm{w}(j \oplus (N_\mathrm{Q} - 1))\Big\} = \{2, 3, \ldots, N_\mathrm{Q}, N_\mathrm{Q}+1\}. \tag{H.9}$$

  We want to show that conditions (H.8) and (H.9) imply that $\pi_\mathrm{w}(i) = 1$ or $\pi_\mathrm{w}(i \oplus (N_\mathrm{Q} - 1)) = 1$.

  For $N_\mathrm{Q} = 2$, condition (H.8) requires that

  $$\Big\{\pi_\mathrm{w}(i), \pi_\mathrm{w}(i \oplus 1)\Big\} = \{1, 2\}.$$

  Therefore, $\pi_\mathrm{w}(i) = 1$ or $\pi_\mathrm{w}(i \oplus 1) = 1$.

  For $N_\mathrm{Q} \geq 3$, we assume to the contrary that $\pi_\mathrm{w}(i) \neq 1$ and $\pi_\mathrm{w}(i \oplus (N_\mathrm{Q} - 1)) \neq 1$.

Because π is a bijection, there is an index k, $1 \leq k \leq N_{\mathrm{Q}} - 2$, such that

$$\pi_{\mathrm{w}}(i \oplus k) = 1. \tag{H.10}$$

Clearly,

$$\pi_{\mathrm{w}}(i \oplus (k-1)) \in \Big\{\pi_{\mathrm{w}}(i), \pi_{\mathrm{w}}(i \oplus 1), \pi_{\mathrm{w}}(i \oplus 2), \ldots, \pi_{\mathrm{w}}(i \oplus (N_{\mathrm{Q}} - 1))\Big\}, \text{ and} \tag{H.11}$$
$$\pi_{\mathrm{w}}(i \oplus (k+1)) \in \Big\{\pi_{\mathrm{w}}(i), \pi_{\mathrm{w}}(i \oplus 1), \pi_{\mathrm{w}}(i \oplus 2), \ldots, \pi_{\mathrm{w}}(i \oplus (N_{\mathrm{Q}} - 1))\Big\}. \tag{H.12}$$

Condition (H.10) and the fact that π_{w} is bijection imply that

$$\pi_{\mathrm{w}}(i \oplus (k-1)) \neq 1, \text{ and}$$
$$\pi_{\mathrm{w}}(i \oplus (k+1)) \neq 1.$$

Therefore,

$$\pi_{\mathrm{w}}(i \oplus (k-1)) \in \{2, 3, \ldots, N_{\mathrm{Q}}\} \subset \{2, 3, \ldots, N_{\mathrm{Q}}, N_{\mathrm{Q}} + 1\}, \text{ and} \tag{H.13}$$
$$\pi_{\mathrm{w}}(i \oplus (k+1)) \in \{2, 3, \ldots, N_{\mathrm{Q}}\} \subset \{2, 3, \ldots, N_{\mathrm{Q}}, N_{\mathrm{Q}} + 1\}. \tag{H.14}$$

Conditions (H.13) and (H.9) imply that there exists some l, $0 \leq l \leq N_{\mathrm{Q}} - 1$, such that

$$\pi_{\mathrm{w}}(j \oplus l) = \pi_{\mathrm{w}}(i \oplus (k-1)). \tag{H.15}$$

Similarly, conditions (H.14) and (H.9) imply that there exists some $\tilde{l} \neq l$, $0 \leq \tilde{l} \leq N_{\mathrm{Q}} - 1$, such that

$$\pi_{\mathrm{w}}(j \oplus l) = \pi_{\mathrm{w}}(i \oplus (k-1)). \tag{H.16}$$

If $l < \tilde{l}$, condition (H.9) implies that

$$\Big\{\pi_{\mathrm{w}}(j \oplus (l+1)), \pi_{\mathrm{w}}(j \oplus (l+2)), \ldots, \pi_{\mathrm{w}}(j \oplus (\tilde{l} - 1))\Big\} \subset \{2, 3, \ldots, N_{\mathrm{Q}}, N_{\mathrm{Q}} + 1\}.$$

From (H.15), we have

$$\begin{aligned}\pi_{\mathrm{w}}(j \oplus (l+1)) &= \pi_{\mathrm{w}}(j \oplus l \oplus 1) \\ &= \pi_{\mathrm{w}}(j \oplus (k-1) \oplus 1) \\ &= \pi_{\mathrm{w}}(j \oplus k) \\ &= 1.\end{aligned}$$

Thus, $1 \in \{2, 3, \ldots, N_{\mathrm{Q}}, N_{\mathrm{Q}}+1\}$ and we have a contradiction.

If $\tilde{l} < l$, condition (H.9) implies that

$$\left\{\pi_{\mathrm{w}}(j \oplus \tilde{l}, \pi_{\mathrm{w}}(j \oplus (\tilde{l}+1)), \pi_{\mathrm{w}}(j \oplus l)\right\} \subset \{2, 3, \ldots, N_{\mathrm{Q}}, N_{\mathrm{Q}}+1\}. \tag{H.17}$$

Condition (H.10) and the definition the set of search orders imply that $\pi_{\mathrm{w}}(i \oplus k) = 1 = \pi_{\mathrm{w}}(1)$. Thus,

$$\pi_{\mathrm{w}}(i \oplus (k-1)) = \pi_{\mathrm{w}}(N_{\mathrm{S}}), \text{ and} \tag{H.18}$$

$$\pi_{\mathrm{w}}(i \oplus (k+1)) = \pi_{\mathrm{w}}(2). \tag{H.19}$$

Since $\pi_{\mathrm{w}}(i \oplus (k-1)) = \pi_{\mathrm{w}}(j \oplus l)$ and $\pi_{\mathrm{w}}(i \oplus (k+1)) = \pi_{\mathrm{w}}(j \oplus \tilde{l})$, condition (H.17) is equivalent to the condition

$$\left\{\pi_{\mathrm{w}}(2), \pi_{\mathrm{w}}(3), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{S}}-1), \pi_{\mathrm{w}}(N_{\mathrm{S}})\right\} \subset \{2, 3, \ldots, N_{\mathrm{Q}}, N_{\mathrm{Q}}+1\},$$

which implies that

$$\begin{aligned}\left|\left\{\pi_{\mathrm{w}}(2), \pi_{\mathrm{w}}(3), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{S}}-1), \pi_{\mathrm{w}}(N_{\mathrm{S}})\right\}\right| &\leq \left|\{2, 3, \ldots, N_{\mathrm{Q}}, N_{\mathrm{Q}}+1\}\right| \\ N_{\mathrm{S}}-1 &\leq N_{\mathrm{Q}}.\end{aligned} \tag{H.20}$$

Using the assumption $N_{\mathrm{Q}} \leq N_{\mathrm{S}} - 2$ on the rage of N_{Q} and N_{S} and the cardinality relationship in (H.20), we have a contradiction: $N_{\mathrm{S}} - 1 \leq N_{\mathrm{S}} - 2$.

Therefore, $\pi_{\mathrm{w}}(i) = 1$ or $\pi_{\mathrm{w}}(i \oplus (N_{\mathrm{Q}}-1)) = 1$.

If $\pi_{\mathrm{w}}(i) = 1$, condition (H.8) and the fact that $\pi_{\mathrm{w}}(1) = 1$ imply that

$$\{\pi_{\mathrm{w}}(2), \pi_{\mathrm{w}}(3), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{Q}})\} = \{2, 3, \ldots, N_{\mathrm{Q}}\}.$$

As a result,

$$\Big\{\pi_{\mathrm{w}}(N_{\mathrm{Q}}+1), \pi_{\mathrm{w}}(N_{\mathrm{Q}}+2), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{S}})\Big\} = \{N_{\mathrm{Q}}+1, N_{\mathrm{Q}}+2, \ldots, N_{\mathrm{S}}\},$$

and π_{w} is of the form

$$\begin{aligned}\Big[\pi_{\mathrm{w}}(1), \pi_{\mathrm{w}}(2), \pi_{\mathrm{w}}(3), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{Q}}), \\ \pi_{\mathrm{w}}(N_{\mathrm{Q}}+1), \pi_{\mathrm{w}}(N_{\mathrm{Q}}+2), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{S}})\Big] \\ =\Big[1, \epsilon(2), \epsilon(3), \ldots, \epsilon(N_{\mathrm{Q}}), \\ \xi(N_{\mathrm{Q}}+1), \xi(N_{\mathrm{Q}}+2), \ldots, \xi(N_{\mathrm{S}})\Big],\end{aligned}$$

for some bijections $\epsilon \in \Upsilon(2, N_{\mathrm{Q}})$ and $\xi \in \Upsilon(N_{\mathrm{Q}}+1, N_{\mathrm{S}})$.

On the other hand, if $\pi_{\mathrm{w}}(i \oplus (N_{\mathrm{Q}}-1)) = 1$, condition (H.8) and the fact that $\pi_{\mathrm{w}}(1) = 1$ imply that

$$\Big\{\pi_{\mathrm{w}}(N_{\mathrm{S}}-N_{\mathrm{Q}}+2), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{S}}-1), \pi_{\mathrm{w}}(N_{\mathrm{S}})\Big\} = \{2, 3, \ldots, N_{\mathrm{Q}}\}.$$

As a result,

$$\Big\{\pi_{\mathrm{w}}(2), \pi_{\mathrm{w}}(3), \ldots \pi_{\mathrm{w}}(N_{\mathrm{S}}-N_{\mathrm{Q}}+1)\Big\} = \{N_{\mathrm{Q}}+1, N_{\mathrm{Q}}+2, \ldots, N_{\mathrm{S}}\},$$

and π_{w} is of the form

$$\begin{aligned}\Big[\pi_{\mathrm{w}}(1), \pi_{\mathrm{w}}(2), \pi_{\mathrm{w}}(3), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{S}}-N_{\mathrm{Q}}+1), \\ \pi_{\mathrm{w}}(N_{\mathrm{S}}-N_{\mathrm{Q}}+2), \pi_{\mathrm{w}}(N_{\mathrm{S}}-N_{\mathrm{Q}}+3), \ldots, \pi_{\mathrm{w}}(N_{\mathrm{S}})\Big] \\ = \Big[1, \hat{\xi}(N_{\mathrm{S}}), \hat{\xi}(N_{\mathrm{S}}-1), \ldots, \hat{\xi}(N_{\mathrm{Q}}+1), \\ \hat{\epsilon}(N_{\mathrm{Q}}), \hat{\epsilon}(N_{\mathrm{Q}}-1), \ldots, \hat{\epsilon}(2)\Big],\end{aligned}$$

for some bijections $\hat{\epsilon} \in \Upsilon(2, N_\mathrm{Q})$ and $\hat{\xi} \in \Upsilon(N_\mathrm{Q}+1, N_\mathrm{S})$.

We complete the base case.

- Inductive step:

Assume the inductive hypothesis for the case of some b, $2 \le b \le N_\mathrm{Q} - 1$. Consider the case of $b+1$.

Let the search order $\pi_\mathrm{w} \in \mathcal{P}$, such that

$$v(\mathbf{s}(\pi, 1)) = v(\mathbf{s}(\pi, 2)) = \cdots = v(\mathbf{s}(\pi, b)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \tag{H.21}$$

$$v(\mathbf{s}(\pi, b+1)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \tag{H.22}$$

be given. The condition (H.21) and the inductive hypothesis imply that the search order π_w belongs to the set

$$\begin{aligned}\pi_\mathrm{w} \in \Big\{ \Big[&1, 2, \dots, b-1, \epsilon(b), \epsilon(b+1), \dots, \epsilon(N_\mathrm{Q}), \\ &\quad \xi(N_\mathrm{Q}+1), \xi(N_\mathrm{Q}+2), \dots, \xi(N_\mathrm{S}) \Big], \\ \Big[&1, \hat{\xi}(N_\mathrm{S}), \hat{\xi}(N_\mathrm{S}-1), \dots, \hat{\xi}(N_\mathrm{Q}+1), \\ &\quad \hat{\epsilon}(N_\mathrm{Q}), \hat{\epsilon}(N_\mathrm{Q}-1), \dots, \hat{\epsilon}(b), b-1, b-2, \dots, 2 \Big] \Big\},\end{aligned}$$

for some bijections $\hat{\epsilon}, \epsilon \in \Upsilon(b, N_\mathrm{Q})$ and $\hat{\xi}, \xi \in \Upsilon(N_\mathrm{Q}+1, N_\mathrm{S})$. We want to show that $\epsilon(b) = b$ or $\hat{\epsilon}(b) = b$.

Using the condition (H.22), we conclude that the states in

$$\mathcal{C} \triangleq \{b+1, b+2, b \oplus 3, b \oplus 4, \dots, b \oplus N_\mathrm{Q}\} \tag{H.23}$$

are *clustering* in the flow diagram with the search order π_w.

First, we argue that $b-1 \notin \mathcal{C}$. Assume to the contrary that $b-1 \in \mathcal{C}$. Therefore, there is an integer k in the range $1 \le k \le N_\mathrm{Q}$, such that

$$b - 1 = b \oplus k. \tag{H.24}$$

Because $b-1$ is strictly less than $b+1$, condition (H.24) implies that

$$b-1 = b \oplus k = b + k - lN_{\mathrm{S}}, \tag{H.25}$$

for some positive integer $l \geq 1$. Hence,

$$\begin{aligned} 1 &= lN_{\mathrm{S}} - k \\ &\overset{(a)}{\geq} lN_{\mathrm{S}} - N_{\mathrm{Q}} \\ &\overset{(b)}{\geq} N_{\mathrm{S}} - N_{\mathrm{Q}} \\ &\overset{(c)}{\geq} 2. \end{aligned} \tag{H.26}$$

The inequality (a) follows from the fact that $k \leq N_{\mathrm{Q}}$. The inequality (b) follows from the fact that $l \geq 1$. The inequality (c) follows from the fact that $N_{\mathrm{Q}} \leq N_{\mathrm{S}} - 2$.

Using (H.26), we have a contradiction: $1 \geq 2$. Therefore, $b-1 \notin \mathcal{C}$.

Consider two possible forms of π_{w}.

1. π_{w} is equal to

$$\Big[1, 2, \ldots, b-1, \epsilon(b), \epsilon(b+1), \ldots, \epsilon(N_{\mathrm{Q}}), \\ \xi(N_{\mathrm{Q}}+1), \xi(N_{\mathrm{Q}}+2), \ldots, \xi(N_{\mathrm{S}})\Big]. \tag{H.27}$$

   We want to show that $\epsilon(b) = b$.

   Assume to the contrary that $\epsilon(b) \neq b$. Since

$$\epsilon \colon \{b, b+1, \ldots, N_{\mathrm{Q}}\} \to \{b, b+1, \ldots, N_{\mathrm{Q}}\}$$

   is a bijection, there exists the index i, $b+1 \leq i \leq N_{\mathrm{Q}}$, such that

$$\epsilon(i) = b.$$

   The assumption that $\epsilon(b) \neq b$ implies that

$$\epsilon(b) \in \{b+1, b+2, \ldots, N_{\mathrm{Q}}\}. \tag{H.28}$$

We note that

$$\{b+1, b+2, \ldots, N_{\mathrm{Q}}\} \subset \mathcal{C}, \tag{H.29}$$

because of the definition of $\mathcal{C}$ in (H.23) and the cardinality relationship below:

$$\begin{aligned} \Big|\{b+1, b+2, \ldots, N_{\mathrm{Q}}\}\Big| &= N_{\mathrm{Q}} - b \\ &\le N_{\mathrm{Q}} - 2 \\ &< N_{\mathrm{Q}} \\ &= |\mathcal{C}|. \end{aligned}$$

Therefore, conditions (H.28) and (H.29) imply that $\epsilon(b) \in \mathcal{C}$. Furthermore, $(b-1) \notin \mathcal{C}$ implies that $b \notin \mathcal{C}$, or equivalently, $\epsilon(i) \notin \mathcal{C}$.

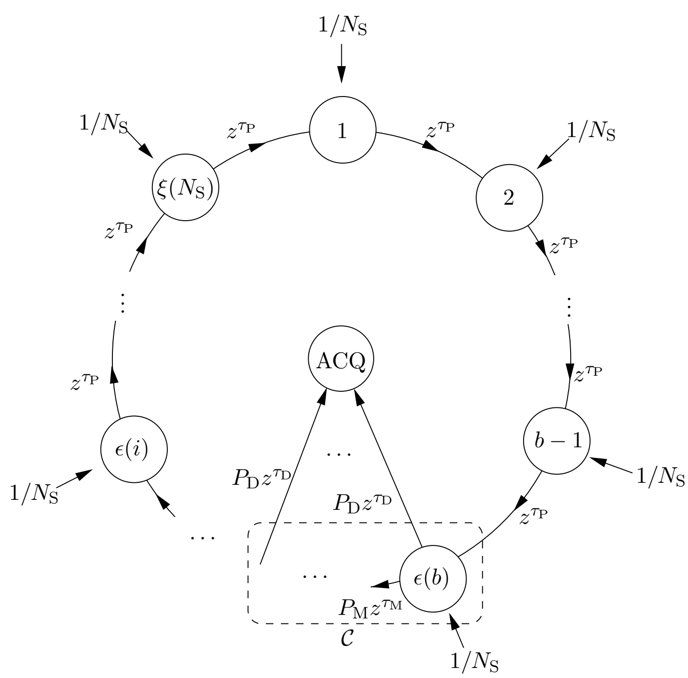

Figure H-2: The states in $\mathcal{C} \subset \{\epsilon(b), \epsilon(b+1), \ldots, \epsilon(i-1)\}$ are clustering.

Because $(b-1) \notin \mathcal{C}$, $\epsilon(b) \in \mathcal{C}$, and $\epsilon(i) \notin \mathcal{C}$ (see Fig. H-2), the set $\mathcal{C}$ of clustering states must satisfy

$$\mathcal{C} \subset \{\epsilon(b), \epsilon(b+1), \ldots, \epsilon(i-1)\}.$$

As a consequence, the cardinalities of two sets above satisfies

$$\begin{aligned} N_{\mathrm{Q}} &= |\mathcal{C}| \\ &\leq \Big|\{\epsilon(b), \epsilon(b+1), \ldots, \epsilon(i-1)\}\Big| \\ &\leq i - b \\ &\leq N_{\mathrm{Q}} - 2, \end{aligned} \tag{H.30}$$

in which the last inequality follows from the fact that $i \leq N_{\mathrm{Q}}$ and $2 \leq b$. From (H.30), we have a contradiction: $0 \leq -2$. Therefore, $\epsilon(b) = b$.

Substituting $\epsilon(b) = b$ into (H.27), the search order π_{w} is equal to

$$\Big[1, 2, \ldots, b-1, b, \delta(b+1), \delta(b+2), \ldots, \delta(N_{\mathrm{Q}}), \\ \xi(N_{\mathrm{Q}}+1), \xi(N_{\mathrm{Q}}+2), \ldots, \xi(N_{\mathrm{S}})\Big],$$

where a bijection $\delta \in \Upsilon(b+1, N_{\mathrm{Q}})$ is given by

$$\delta(i) \triangleq \epsilon(i), \quad i = b+1, b+2, \ldots, N_{\mathrm{Q}}.$$

2. π_{w} is equal to

$$\Big[1, \hat{\xi}(N_{\mathrm{S}}), \hat{\xi}(N_{\mathrm{S}}-1), \ldots, \hat{\xi}(N_{\mathrm{Q}}+1), \\ \hat{\epsilon}(N_{\mathrm{Q}}), \hat{\epsilon}(N_{\mathrm{Q}}-1), \ldots, \hat{\epsilon}(b), b-1, b-2, \ldots, 2\Big]. \tag{H.31}$$

We want to show that $\hat{\epsilon}(b) = b$.

Assume to the contrary that $\hat{\epsilon}(b) \neq b$. Since

$$\hat{\epsilon} \colon \{b, b+1, \ldots, N_{\mathrm{Q}}\} \to \{b, b+1, \ldots, N_{\mathrm{Q}}\}$$

is a bijection, there exists the index j, $b+1 \leq j \leq N_{\mathrm{Q}}$, such that

$$\hat{\epsilon}(j) = b.$$

The assumption that $\hat{\epsilon}(b) \neq b$ implies that

$$\hat{\epsilon}(b) \in \{b+1, b+2, \ldots, N_{\mathrm{Q}}\}. \tag{H.32}$$

By (H.29), the set in the right-hand side of (H.32) is a subset of $\mathcal{C}$. Therefore, $\hat{\epsilon}(b) \in \mathcal{C}$.

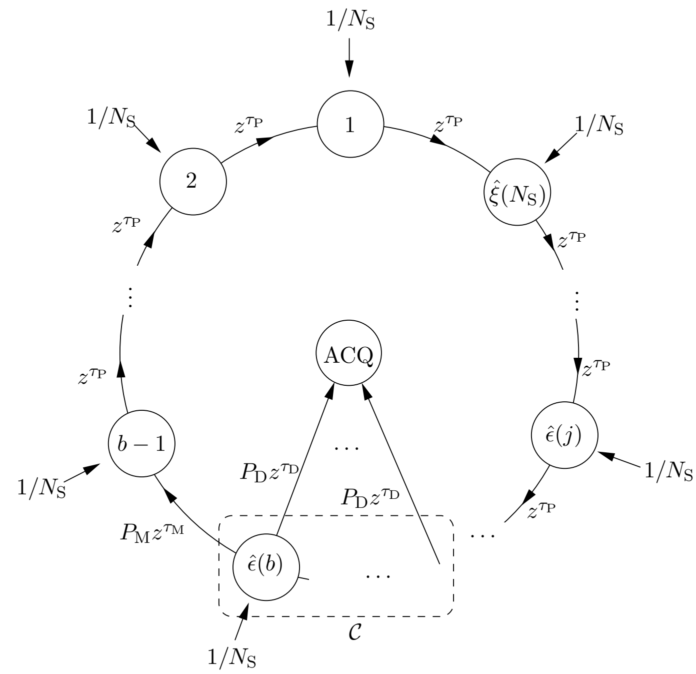

Figure H-3: The states in $\mathcal{C} \subset \{\hat{\epsilon}(b), \hat{\epsilon}(b+1), \ldots, \hat{\epsilon}(j-1)\}$ are clustering.

Because $(b-1) \notin \mathcal{C}$, $\hat{\epsilon}(b) \in \mathcal{C}$, and $\hat{\epsilon}(j) \notin \mathcal{C}$ (see Fig. H-3), the set $\mathcal{C}$ of clustering states must satisfy

$$\mathcal{C} \subset \{\hat{\epsilon}(b), \hat{\epsilon}(b+1), \ldots, \hat{\epsilon}(j-1)\}.$$

As a consequence, the cardinalities of two sets above satisfies

$$\begin{aligned} N_{\mathrm{Q}} &= |\mathcal{C}| \\ &\leq \Big|\{\epsilon(b), \epsilon(b+1), \ldots, \epsilon(i-1)\}\Big| \\ &\leq N_{\mathrm{Q}} - 2. \end{aligned} \tag{H.33}$$

Then, we have a contradiction: $0 \leq -2$. Therefore, $\hat{\epsilon}(b) = b$.

Substituting $\hat{\epsilon}(b) = b$ into (H.31), the search order π_{w} is equal to

$$\Big[1, \hat{\xi}(N_{\mathrm{S}}), \hat{\xi}(N_{\mathrm{S}}-1), \ldots, \hat{\xi}(N_{\mathrm{Q}}+1), \\ \hat{\delta}(N_{\mathrm{Q}}), \hat{\delta}(N_{\mathrm{Q}}-1), \ldots, \hat{\delta}(b+1), b, b-1, \ldots, 2\Big],$$

where a bijection $\hat{\delta} \in \Upsilon(b+1, N_{\mathrm{Q}})$ is given by

$$\hat{\delta}(i) \triangleq \hat{\epsilon}(i), \quad i = b+1, b+2, \ldots, N_{\mathrm{Q}}.$$

That completes the inductive proof. □

Lemma H.2 (Tail Reduction). *For any N_{S}, N_{Q}, and b that satisfy $2 \leq N_{\mathrm{Q}} \leq N_{\mathrm{S}} - 2$ and $N_{\mathrm{Q}} \leq b \leq N_{\mathrm{S}} - 1$, if the search order π_w satisfies*

$$v(\mathbf{s}(\pi_w, 1)) = v(\mathbf{s}(\pi_w, 2)) = \cdots = v(\mathbf{s}(\pi_w, b)) = \max_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}),$$

then the search order π_w belongs to the set

$$\pi_w \in \Big\{ \Big[1, 2, \ldots, b, \xi(b+1), \xi(b+2), \ldots, \xi(N_{\mathrm{S}})\Big], \\ \Big[1, \hat{\xi}(N_{\mathrm{S}}), \hat{\xi}(N_{\mathrm{S}}-1), \ldots, \hat{\xi}(b+1), b, b-1, \ldots, 3, 2\Big] \Big\}$$

for some bijections $\xi, \hat{\xi} \in \Upsilon(b+1, N_{\mathrm{S}})$.

Proof. Let N_{S} and N_{Q} such that $2 \leq N_{\mathrm{Q}} \leq N_{\mathrm{S}} - 2$ be given. We prove the lemma by an induction on b.

- Base case ($b = N_{\mathrm{Q}}$):
  Let any search order π_{w} such that

  $$v(\mathbf{s}(\pi_{\mathrm{w}}, 1)) = v(\mathbf{s}(\pi_{\mathrm{w}}, 2)) = \cdots = v(\mathbf{s}(\pi_{\mathrm{w}}, N_{\mathrm{Q}})) = \max_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}),$$

  be given. By Lemma H.1, the search order π_{w} belongs to the set

  $$\pi_{\mathrm{w}} \in \Big\{ \Big[1, 2, \ldots, N_{\mathrm{Q}}, \xi(N_{\mathrm{Q}}+1), \xi(N_{\mathrm{Q}}+2), \ldots, \xi(N_{\mathrm{S}})\Big], \\ \Big[1, \hat{\xi}(N_{\mathrm{S}}), \hat{\xi}(N_{\mathrm{S}}-1), \ldots, \hat{\xi}(N_{\mathrm{Q}}+1), N_{\mathrm{Q}}, N_{\mathrm{Q}}-1, \ldots, 3, 2\Big] \Big\},$$

where $\xi, \hat{\xi} \in \Upsilon(N_{\mathrm{Q}}+1, N_{\mathrm{S}})$.

- Inductive step:

  Assume the inductive hypothesis for the case of some b, $N_{\mathrm{Q}} \leq b \leq N_{\mathrm{S}}-2$. Consider the case of $b+1$.

  Let any search order π_{w} such that

$$v(\mathbf{s}(\pi_{\mathrm{w}},1)) = v(\mathbf{s}(\pi_{\mathrm{w}},2)) = \cdots = v(\mathbf{s}(\pi_{\mathrm{w}},b)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \tag{H.34}$$

$$v(\mathbf{s}(\pi_{\mathrm{w}},b+1)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m}) \tag{H.35}$$

  be given.

  The condition (H.34) and the inductive hypothesis imply that the search order π_{w} belongs to the set

$$\begin{aligned}\pi_{\mathrm{w}} \in \Big\{ &\Big[1,2,\ldots,b,\xi(b+1),\xi(b+2),\ldots,\xi(N_{\mathrm{S}})\Big],\\ &\Big[1,\hat{\xi}(N_{\mathrm{S}}),\hat{\xi}(N_{\mathrm{S}}-1),\ldots,\hat{\xi}(b+1),b,\ldots,3,2\Big]\Big\},\end{aligned}$$

  for some bijections $\xi, \hat{\xi} \in \Upsilon(b+1, N_{\mathrm{S}})$. Since $N_{\mathrm{Q}} \leq b$, we have

$$\begin{aligned}&0 \leq b - N_{\mathrm{Q}}, \text{ and}\\ &2 \leq b - (N_{\mathrm{Q}}-2)\\ &\quad\leq b,\end{aligned}$$

  where the last inequality follows from $2 \leq N_{\mathrm{Q}}$. Because $2 \leq (b - N_{\mathrm{Q}} + 2) \leq b$, the condition

$$v(\mathbf{s}(\pi_{\mathrm{w}}, b - N_{\mathrm{Q}} + 2)) = \max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$$

  appears in (H.34). Thus, the states $\{b - N_{\mathrm{Q}} + 2, b - N_{\mathrm{Q}} + 3, \ldots, \ldots, b, b+1\}$ are *clustering* in the flow diagram with the search order π_{w}. Notice that the smallest element of the clustering set is $(b - N_{\mathrm{Q}} + 2) \geq 2$ and the largest element is $b+1$.

  If π_{w} is equal to

$$\Big[1,2,\ldots,b,\xi(b+1),\xi(b+2),\ldots,\xi(N_{\mathrm{S}})\Big],$$

we must have $\xi(b+1) = b+1$. On the other hand, if π_{w} is of the form

$$\Big[1, \hat{\xi}(N_{\mathrm{S}}), \hat{\xi}(N_{\mathrm{S}}-1), \ldots, \hat{\xi}(b+1), b, \ldots, 3, 2\Big],$$

we must have $\hat{\xi}(b+1) = b+1$. Therefore, the search order belongs to the set

$$\begin{aligned}\pi_{\mathrm{w}} \in \Big\{ &\Big[1, 2, \ldots, b, b+1, \delta(b+2), \delta(b+3), \ldots, \delta(N_{\mathrm{S}})\Big], \\ &\Big[1, \hat{\delta}(N_{\mathrm{S}}), \hat{\delta}(N_{\mathrm{S}}-1), \ldots, \hat{\delta}(b+2), b+1, b, \ldots, 3, 2\Big]\Big\},\end{aligned}$$

in which

$$\begin{aligned} \delta(i) &\triangleq \xi(i), && i = b+2, b+3, \ldots, N_{\mathrm{S}}, \text{ and} \\ \hat{\delta}(i) &\triangleq \hat{\xi}(i), && i = b+2, b+3, \ldots, N_{\mathrm{S}}. \end{aligned}$$

Because ξ and $\hat{\xi}$ are bijections, $\delta, \hat{\delta} \in \Upsilon(b+2, N_{\mathrm{S}})$ are also bijections.

That completes the inductive proof. □

Appendix I

Simplification of the Parameters τ_{P} and τ_{M}

In this appendix, we simplify the expressions of τ_{P} and τ_{M} in equation (4.44).

Lemma I.1 (Simplification of τ_{P}). *Let a positive integer $n \geq 1$ be given. Let real numbers t, t_i, and $p_{\mathrm{F}i}$, for $i = 1, 2, \ldots, n$, be given. Then,*

$$\sum_{i=1}^{n} \left[\left(\prod_{j=1}^{i-1} p_{\mathrm{F}j} \right) \cdot (1 - p_{\mathrm{F}i}) \cdot \sum_{l=1}^{i} t_l \right] + \prod_{i=1}^{n} p_{\mathrm{F}i} \cdot \left(t + \sum_{l=1}^{n} t_l \right) = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} p_{\mathrm{F}j} \right) \cdot t_i + \left(\prod_{i=1}^{n} p_{\mathrm{F}i} \right) \cdot t.$$

Proof. We prove the lemma by an induction on n.

- Base case ($n = 1$):

  Let real numbers t, t_1, and $p_{\mathrm{F}1}$ be given. Then,

  $$(1 - p_{\mathrm{F}1})t_1 + p_{\mathrm{F}1}(t_1 + t) = t_1 + p_{\mathrm{F}1}t.$$

- Inductive step:

  Assume the inductive hypothesis for the case of n, for some $n \geq 1$. Consider the case of $n + 1$.

Let real numbers t, t_i, and $p_{\mathrm{F}i}$, for $i = 1, 2, \ldots, n+1$, be given. Then,

$$\begin{aligned}
&\sum_{i=1}^{n+1}\Bigg[\Big(\prod_{j=1}^{i-1} p_{\mathrm{F}j}\Big)\cdot(1-p_{\mathrm{F}i})\cdot\sum_{l=1}^{i} t_l\Bigg]+\prod_{i=1}^{n+1} p_{\mathrm{F}i}\cdot\Big(t+\sum_{l=1}^{n+1} t_l\Big)\\
&\qquad = (1-p_{\mathrm{F}1})t_1\\
&\qquad\quad + p_{\mathrm{F}1}\cdot\Bigg\{\sum_{i=2}^{n+1}\Bigg[\Big(\prod_{j=2}^{i-1} p_{\mathrm{F}j}\Big)\cdot(1-p_{\mathrm{F}i})\cdot\Big(t+\sum_{l=1}^{n+1} t_l\Big)\Bigg]\\
&\qquad\qquad + \prod_{i=2}^{n+1} p_{\mathrm{F}i}\cdot\Big(t+\sum_{l=1}^{n+1} t_l\Big)\Bigg\}\\
&\qquad = (1-p_{\mathrm{F}1})t_1\\
&\qquad\quad + p_{\mathrm{F}1}\cdot\Bigg\{\sum_{i=1}^{n}\Bigg[\Big(\prod_{j=1}^{i-1}\rho_j\Big)\cdot(1-\rho_i)\cdot\sum_{l=1}^{i}\tau_l\Bigg]+\prod_{i=1}^{n}\rho_i\cdot\Big(t+\sum_{l=1}^{n}\tau_l\Big)\Bigg\},
\end{aligned} \tag{I.1}$$

in which

$$\tau_i \triangleq \begin{cases} t_1 + t_2 & i = 1 \\ t_{i+1} & i = 2, 3, \ldots, n \end{cases}$$

$$\rho_i \triangleq p_{\mathrm{F}i+1}, \quad i = 1, 2, 3, \ldots, n.$$

Applying the inductive hypothesis to the last step of (I.1), we have

$$\begin{aligned}
\text{Equation (I.1)} &= (1-p_{\mathrm{F}1})t_1 + p_{\mathrm{F}1}\cdot\Bigg[\sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1}\rho_j\Big)\cdot\tau_i + \Big(\prod_{i=1}^{n}\rho_i\Big)\cdot t\Bigg]\\
&= \sum_{i=1}^{n+1}\Big(\prod_{j=1}^{i-1} p_{\mathrm{F}j}\Big)\cdot t_i + \Big(\prod_{i=1}^{n+1} p_{\mathrm{F}i}\Big)\cdot t.
\end{aligned}$$

That completes the proof.

□

One immediate result from the last lemma is given in the next corollary.

Corollary I.1 (Simplification of τ_{D}). *Let a positive integer $n \geq 1$ be given. Let real* *numbers t, t_i, and $p_{\mathrm{D}i}$, for $i = 1, 2, \ldots, n$, be given. Then,*

$$\sum_{i=1}^{n}\Bigg[\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j}\Big) \cdot (1 - p_{\mathrm{D}i}) \cdot \sum_{l=1}^{i} t_l\Bigg] = \sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j}\Big) \cdot t_i - \prod_{i=1}^{n} p_{\mathrm{D}i} \cdot \sum_{l=1}^{n} t_l.$$

Proof. Let a positive integer $n \geq 1$ be given. Let real numbers t, t_i, and $p_{\mathrm{D}i}$, for $i = 1, 2, \ldots, n$, be given. Then,

$$\begin{aligned}
&\sum_{i=1}^{n}\Bigg[\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j}\Big) \cdot (1 - p_{\mathrm{D}i}) \cdot \sum_{l=1}^{i} t_l\Bigg] \\
&\qquad = \Bigg\{\sum_{i=1}^{n}\Bigg[\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j}\Big) \cdot (1 - p_{\mathrm{D}i}) \cdot \sum_{l=1}^{i} t_l\Bigg] + \prod_{i=1}^{n} p_{\mathrm{D}i} \cdot \Big(t + \sum_{l=1}^{n} t_l\Big)\Bigg\} \\
&\qquad\quad - \prod_{i=1}^{n} p_{\mathrm{D}i} \cdot \Big(t + \sum_{l=1}^{n} t_l\Big) \\
&\qquad \overset{(a)}{=} \Bigg[\sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j}\Big) \cdot t_i + \Big(\prod_{i=1}^{n} p_{\mathrm{D}i}\Big) \cdot t\Bigg] \\
&\qquad\quad - \prod_{i=1}^{n} p_{\mathrm{D}i} \cdot \Big(t + \sum_{l=1}^{n} t_l\Big) \\
&\qquad = \sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1} p_{\mathrm{D}j}\Big) \cdot t_i - \prod_{i=1}^{n} p_{\mathrm{D}i} \cdot \sum_{l=1}^{n} t_l.
\end{aligned}$$

The equality (a) follows from Lemma I.1. That completes the proof. □

Appendix J

Important Decreasing and Increasing Functions

In Section 4.6, we consider a receiver that selects the optimal thresholds. To derive the MATs in that section, we need to find the solution of the form

$$\inf_{\substack{p_{\mathrm{F}i}\in(0,1],\\ 1\le i\le n}} h(p_{\mathrm{F}1}, p_{\mathrm{F}2}, \ldots, p_{\mathrm{F}n}) \tag{J.1}$$

or

$$\sup_{\substack{p_{\mathrm{F}i}\in(0,1],\\ 1\le i\le n}} h(p_{\mathrm{F}1}, p_{\mathrm{F}2}, \ldots, p_{\mathrm{F}n}). \tag{J.2}$$

Here, $n \geq 1$ is the number of statistical tests that the receiver employs, and $h(\cdot)$ is some objective function that is defined on the space $(0,1]^n$.

Definition J.1 (Decreasing function). A function $h\colon (0,1]^n \to \mathbb{R}$ is *decreasing* on $(0,1]^n$, if for any $(x_1, x_2, \ldots, x_n) \in (0,1]^n$ and $(y_1, y_2, \ldots, y_n) \in (0,1]^n$,

$$h(x_1, x_2, \ldots, x_n) \geq h(y_1, y_2, \ldots, y_n)$$

whenever

$$x_i \leq y_i, \quad \text{for all } i = 1, 2, \ldots, n.$$

Definition J.2 (Increasing function). A function $h\colon (0,1]^n \to \mathbb{R}$ is *increasing* on $(0,1]^n$, if for any $(x_1, x_2, \dots, x_n) \in (0,1]^n$ and $(y_1, y_2, \dots, y_n) \in (0,1]^n$,

$$h(x_1, x_2, \dots, x_n) \le h(y_1, y_2, \dots, y_n)$$

whenever

$$x_i \le y_i, \quad \text{for all } i = 1, 2, \dots, n.$$

we have

Lemma J.1 (Increasing Sum). *Let a positive integer $n \ge 1$, non-negative real number $t \ge 0$, and non-negative real numbers $t_i \ge 0$, for $i = 1, 2, \dots, n$, be given. Let*

$$h_{sum}(x_1, x_2, \dots, x_n) \triangleq \sum_{i=1}^{n} \Big(\prod_{j=1}^{i-1} x_j \Big) \cdot t_i + \Big(\prod_{i=1}^{n} x_i \Big) \cdot t \tag{J.3}$$

be a function on $(0,1]^n$. Then,

1. *function $h_{sum}(\cdot)$ is increasing on $(0,1]^n$,*

2. *function $h_{sum}(\cdot)$ is bounded from above and below,*

3. $\inf_{(x_1, x_2, \dots, x_n) \in (0,1]^n} h_{sum}(x_1, x_2, \dots, x_n) = t_1$*, and*

4. $\sup_{(x_1, x_2, \dots, x_n) \in (0,1]^n} h_{sum}(x_1, x_2, \dots, x_n) = t + \sum_{i=1}^{n} t_i$.

Proof. Let any tuples $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ be given. Assume that

$$0 < x_i \le y_i \le 1, \quad \text{for all } i = 1, 2, \dots, n.$$

Then, we have the relationships

$$0 \le \Big(\prod_{j=1}^{n} x_j \Big) \cdot t \le \Big(\prod_{j=1}^{n} y_j \Big) \cdot t \le t$$

and

$$0 \le \Big(\prod_{j=1}^{i-1} x_j \Big) \cdot t_i \le \Big(\prod_{j=1}^{i-1} y_j \Big) \cdot t_i \le t_i,$$

in which the index i is in the range $1 \le i \le n$. Adding the appropriate terms, we have the bounds

$$\begin{aligned}
0 &\le \sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1} x_j\Big)\cdot t_i + \Big(\prod_{j=1}^{n} x_j\Big)\cdot t \\
&\le \sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1} y_j\Big)\cdot t_i + \Big(\prod_{j=1}^{n} y_j\Big)\cdot t \\
&\le \Big(\sum_{i=1}^{n} t_i\Big) + t,
\end{aligned}$$

which imply that

$$\begin{aligned}
0 &\le h_{\text{sum}}(x_1, x_2, \ldots, x_n) \\
&\le h_{\text{sum}}(y_1, y_2, \ldots, y_n) \\
&\le \Big(\sum_{i=1}^{n} t_i\Big) + t
\end{aligned}$$

Therefore, function $h_{\text{sum}}(\cdot)$ is increasing on $(0,1]^n$ and bounded from above and below. As a consequence, the infimum and supremum exist and equal

$$\begin{aligned}
\inf_{(x_1,x_2,\ldots,x_n)\in(0,1]^n} h_{\text{sum}}(x_1, x_2, \ldots, x_n) &= \lim_{x_1\to0^+}\lim_{x_2\to0^+}\cdots\lim_{x_n\to0^+} h(x_1, x_2, \ldots, x_n) \\
&= t_1
\end{aligned}$$

and

$$\begin{aligned}
\sup_{(x_1,x_2,\ldots,x_n)\in(0,1]^n} h_{\text{sum}}(x_1, x_2, \ldots, x_n) &= h(1, 1, \ldots, 1) \\
&= \Big(\sum_{i=1} t_i\Big) + t,
\end{aligned}$$

respectively. That completes the proof. □

Lemma J.2 (Decreasing Ratio). *Let positive integers $n \ge 1$ and $N_{\text{Q}} \ge 1$ be given. Let*

$$h_{rat}(x_1, x_2, \ldots, x_n) \triangleq \frac{1 + (1 - \prod_{i=1}^{n} x_i)^{N_{\text{Q}}}}{1 - (1 - \prod_{i=1}^{n} x_i)^{N_{\text{Q}}}}. \tag{J.4}$$

be a function on $(0,1]^n$. Then,

1. *function $h_{rat}(\cdot)$ is decreasing on $(0,1]^n$,*

2. *function* $h_{rat}(\cdot)$ *is bounded from below, and*

3. $\inf_{(x_1,x_2,\ldots,x_n)\in(0,1]^n} h_{rat}(x_1, x_2, \ldots, x_n) = 1$.

Proof. Let any tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be given. Assume that

$$0 < x_i \le y_i \le 1, \quad \text{for all } i = 1, 2, \ldots, n.$$

For any i in the range $1 \le i \le n$, we have the relationships

$$0 \le \Big(1 - \prod_{i=1}^{n} y_i\Big)^{N_\mathrm{Q}} \le \Big(1 - \prod_{i=1}^{n} x_i\Big)^{N_\mathrm{Q}} < 1.$$

Since $(1 + x)/(1 - x)$ is increasing on $[0, 1)$, we have the bounds

$$1 \le \frac{1 + (1 - \prod_{i=1}^{n} y_i)^{N_\mathrm{Q}}}{1 - (1 - \prod_{i=1}^{n} y_i)^{N_\mathrm{Q}}} \le \frac{1 + (1 - \prod_{i=1}^{n} x_i)^{N_\mathrm{Q}}}{1 - (1 - \prod_{i=1}^{n} x_i)^{N_\mathrm{Q}}},$$

which imply that

$$1 \le h_\mathrm{rat}(y_1, y_2, \ldots, y_n) \le h_\mathrm{rat}(x_1, x_2, \ldots, x_n).$$

Therefore, function $h_\mathrm{rat}(\cdot)$ is decreasing on $(0, 1]^n$ and bounded from below. As a consequence, the infimum exists and equals

$$\begin{aligned} \inf_{(x_1,x_2,\ldots,x_n)\in(0,1]^n} h_\mathrm{rat}(x_1, x_2, \ldots, x_n) &= h_\mathrm{rat}(1, 1, \ldots, 1) \\ &= 1. \end{aligned}$$

That completes the proof. □

Lemma J.3 (Decreasing Reciprocal). *Let a positive integer* $n \ge 1$ *and non-negative real numbers* $t_i \ge 0$*, for* $i = 1, 2, \ldots, n$*, be given. Let*

$$h_{rec}(x_1, x_2, \ldots, x_n) \triangleq \sum_{i=1}^{n} \Big(\frac{t_i}{\prod_{j=i}^{n} x_i}\Big) \tag{J.5}$$

be a function on $(0, 1]^n$*. Then,*

1. *function* $h_{rec}(\cdot)$ *is decreasing on* $(0, 1]^n$*,*

2. *function* $h_{rec}(\cdot)$ *is bounded from below, and*

3. $\inf_{(x_1,x_2,\ldots,x_n)\in(0,1]^n} h_{rec}(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} t_i$.

Proof. Let any tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be given. Assume that

$$0 < x_i \le y_i \le 1, \quad \text{for all } i = 1, 2, \ldots, n.$$

For any i in the range $1 \le i \le n$, we have the relationships

$$t_i \le \frac{t_i}{\prod_{j=i}^{n} y_j} \le \frac{t_i}{\prod_{j=i}^{n} x_j}.$$

Summing over the index i, we have the bounds

$$\sum_{i=1}^{n} t_i \le \sum_{i=1}^{n}\Big(\frac{t_i}{\prod_{j=i}^{n} y_i}\Big) \le \sum_{i=1}^{n}\Big(\frac{t_i}{\prod_{j=i}^{n} x_i}\Big),$$

which imply that

$$\sum_{i=1}^{n} t_i \le h_{\text{rec}}(y_1, y_2, \ldots, y_n) \le h_{\text{rec}}(x_1, x_2, \ldots, x_n).$$

Therefore, function $h_{\text{rec}}(\cdot)$ is decreasing on $(0,1]^n$ and bounded from below. As a consequence, the infimum exists and equals

$$\begin{aligned}\inf_{(x_1,x_2,\ldots,x_n)\in(0,1]^n} h_{\text{rec}}(x_1, x_2, \ldots, x_n) &= h_{\text{rec}}(1, 1, \ldots, 1) \\ &= \sum_{i=1}^{n} t_i.\end{aligned}$$

That completes the proof. □

Lemma J.4 (Infimum of g). *Let positive integer $n \ge 1$, positive integer $N_{\mathrm{Q}} \ge 1$, positive integer $N_{\mathrm{S}} \ge N_{\mathrm{Q}}$, non-negative real number $t \ge 0$, and non-negative real numbers $t_i \ge 0$, for $i = 1, 2, \ldots, n$, be given. Let*

$$\begin{aligned}g(x_1, x_2, \ldots, x_n) \triangleq{}& \frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} \cdot \sum_{i=1}^{n}\Big(\frac{t_j}{\prod_{j=i}^{n} x_j}\Big) \\ &- \Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{2N_{\mathrm{Q}}}\Big) \cdot \Big[\sum_{i=1}^{n}\Big(\prod_{j=1}^{i-1} x_j\Big) \cdot t_i + \Big(\prod_{i=1}^{n} x_i\Big) \cdot t\Big] \\ &+ \Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{Q}}}\Big) \cdot t\end{aligned}$$

be a function on $(0,1]^n$*. Then, the infimum is equal to*

$$\inf_{(x_1,x_2,\ldots,x_n)\in(0,1]^n} g(x_1,x_2,\ldots,x_n) = \Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1\Big) \cdot \Big(\frac{t + \sum_{l=1}^{n} t_l}{2}\Big) + \sum_{i=1}^{n} t_i.$$

Proof. We rewrite $g(\cdot)$ in terms of $h_{\mathrm{sum}}(\cdot)$ and $h_{\mathrm{rec}}(\cdot)$, which are respectively given in (J.3) and (J.5):

$$g(\mathbf{x}) = \Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}}\Big) \cdot h_{\mathrm{rec}}(\mathbf{x}) - \Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{2N_{\mathrm{Q}}}\Big) \cdot h_{\mathrm{sum}}(\mathbf{x}) + \Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{Q}}}\Big) \cdot t,$$

for $\mathbf{x} \in (0,1]^n$. By Lemma J.1 and Lemma J.3, the functions $-h_{\mathrm{sum}}(\cdot)$ and $h_{\mathrm{rec}}(\cdot)$ are decreasing on $(0,1]^n$ and bounded from below. Therefore, $g(\cdot)$ is decreasing on $(0,1]^n$ and bounded from below. As a consequence, the infimum exists and equals

$$\begin{aligned}\inf_{\mathbf{x}\in(0,1]^n} g(\mathbf{x}) &= g(1,1,\ldots,1)\\ &= \Big(\frac{N_{\mathrm{S}}}{N_{\mathrm{Q}}} - 1\Big) \cdot \Big(\frac{t + \sum_{l=1}^{n} t_l}{2}\Big) + \sum_{i=1}^{n} t_i.\end{aligned}$$

That completes the proof. □

Appendix K

Bounds for the MAT of the CSS in a Low SNR Regime

We consider in Subsection 4.6.4 the MAT of the multi-dwell detector, which employs the CSS and operates in a low SNR environment. Equations (4.47) and (4.48) imply that the MAT, which is optimized over all thresholds, is given by

$$\inf_{\substack{p_{\mathrm{F}i}\in(0,1],\\ 1\le i\le n}} f(p_{\mathrm{F}1}, p_{\mathrm{F}2}, \ldots, p_{\mathrm{F}n}), \tag{K.1}$$

where $f(\cdot)$ can be written in terms of the functions $h_{\mathrm{sum}}(\cdot)$ in (J.3), $h_{\mathrm{rat}}(\cdot)$ in (J.4), and $h_{\mathrm{rec}}(\cdot)$ in (J.5):

$$\begin{aligned} f(\mathbf{x}) = \frac{(N_{\mathrm{S}}-N_{\mathrm{Q}})^2}{2N_{\mathrm{S}}} \cdot h_{\mathrm{rat}}(\mathbf{x}) \cdot h_{\mathrm{sum}}(\mathbf{x}) + \left(2-\frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\right) \cdot h_{\mathrm{rec}}(\mathbf{x}) \\ -\left(\frac{N_{\mathrm{S}}-N_{\mathrm{Q}}}{2N_{\mathrm{S}}}\right) \cdot h_{\mathrm{sum}}(\mathbf{x}) + \left(\frac{N_{\mathrm{S}}-N_{\mathrm{Q}}}{N_{\mathrm{S}}}\right) \cdot t, \end{aligned} \tag{K.2}$$

for $\mathbf{x} \in (0,1]^n$. We want to find the infimum in (K.1).

The explicit closed-form expression of (K.1) is difficult to derive. A direct approach, which sets the partial derivative of $f(\cdot)$ to zero,

$$\frac{\partial}{\partial p_{\mathrm{F}i}} f(p_{\mathrm{F}1}, p_{\mathrm{F}2}, \ldots, p_{\mathrm{F}n}) = 0,$$

and finds the global minimum $(p^*_{\mathrm{F}1}, p^*_{\mathrm{F}2}, \ldots, p^*_{\mathrm{F}n})$, will require a method for solving the system of polynomial equations. Because the degrees of those polynomials depend on N_{Q}, the explicit closed-form expression for $(p^*_{\mathrm{F}1}, p^*_{\mathrm{F}2}, \ldots, p^*_{\mathrm{F}n})$ is unknown in general. In this appendix, we find an upper-bound and a lower-bound of the infimum in (K.1).

To avoid repetition in the statements of the lemmas, we state the following ranges for n, N_S, N_Q, t_1, t_2, $\ldots$, t_n, and t:

$$\begin{aligned}
&n \geq 1\\
&N_\mathrm{S} \geq N_\mathrm{Q} \geq 1\\
&t_i \geq 0,\ \ i = 1, 2, \ldots, n\\
&t \geq 0.
\end{aligned}$$

In the next subsection, we derive a lower-bound.

K.1 A Lower-Bound

In this subsection, we find the lower-bound of the infimum in (K.1) by using the results from Appendix J.

Lemma K.1 (Low-SNR-CSS Lower-Bound). *The infimum is lower-bounded by*

$$\frac{(N_\mathrm{S}-N_\mathrm{Q})^2}{2N_\mathrm{S}} \cdot t_1 + \left(\frac{N_\mathrm{S}-N_\mathrm{Q}}{2N_\mathrm{S}}\right) \cdot \left(t + \sum_{l=2}^{n} t_l\right) \leq \inf_{\mathbf{x}\in(0,1]^n} f(\mathbf{x}).$$

Proof. Clearly,

$$\begin{aligned}
\inf_{\mathbf{x}\in(0,1]^n} f(\mathbf{x}) \geq{}& \frac{(N_\mathrm{S}-N_\mathrm{Q})^2}{2N_\mathrm{S}} \cdot \inf_{\mathbf{x}\in(0,1]^n} h_{\mathrm{rat}}(\mathbf{x}) \cdot \inf_{\mathbf{x}\in(0,1]^n} h_{\mathrm{sum}}(\mathbf{x})\\
&+ \left(2 - \frac{N_\mathrm{Q}}{N_\mathrm{S}}\right) \cdot \inf_{\mathbf{x}\in(0,1]^n} h_{\mathrm{rec}}(\mathbf{x})\\
&- \left(\frac{N_\mathrm{S}-N_\mathrm{Q}}{2N_\mathrm{S}}\right) \cdot \sup_{\mathbf{x}\in(0,1]^n} h_{\mathrm{sum}}(\mathbf{x})\\
&+ \left(\frac{N_\mathrm{S}-N_\mathrm{Q}}{N_\mathrm{S}}\right) \cdot t\\
\overset{(a)}{=}{}& \frac{(N_\mathrm{S}-N_\mathrm{Q})^2}{2N_\mathrm{S}} \cdot 1 \cdot t_1 + \left(2 - \frac{N_\mathrm{Q}}{N_\mathrm{S}}\right) \cdot \sum_{l=1}^{n} t_l\\
&- \left(\frac{N_\mathrm{S}-N_\mathrm{Q}}{2N_\mathrm{S}}\right) \cdot \left(t + \sum_{l=1}^{n} t_l\right) + \left(\frac{N_\mathrm{S}-N_\mathrm{Q}}{N_\mathrm{S}}\right) \cdot t
\end{aligned}$$

$$= \Big(\frac{(N_\mathrm{S}-N_\mathrm{Q})^2}{2N_\mathrm{S}} + \frac{3N_\mathrm{S}-N_\mathrm{Q}}{2N_\mathrm{S}}\Big)\cdot t_1$$
$$+ \Big(\frac{3N_\mathrm{S}-N_\mathrm{Q}}{2N_\mathrm{S}}\Big)\cdot \sum_{l=2}^{n} t_l + \Big(\frac{N_\mathrm{S}-N_\mathrm{Q}}{2N_\mathrm{S}}\Big)\cdot t$$
$$\geq \frac{(N_\mathrm{S}-N_\mathrm{Q})^2}{2N_\mathrm{S}}\cdot t_1 + \Big(\frac{N_\mathrm{S}-N_\mathrm{Q}}{2N_\mathrm{S}}\Big)\cdot \Big(t + \sum_{l=2}^{n} t_l\Big).$$

The equality (a) follows from the infimums and supremum in Lemma J.1, Lemma J.2, and Lemma J.3. That completes the proof. □

K.2 An Upper-Bound

In this subsection, we find the upper-bound of the infimum in (K.1) by evaluating its objective function $f(\cdot)$ at some feasible solution $\mathbf{p} \triangleq (p_1, p_2, \ldots, p_n) \in (0,1]^n$. To derive a reasonable upper-bound, we need to select intelligently the feasible solution $\mathbf{p}$. In the next lemma, we will investigate a property of function

$$c(x) \triangleq x - \frac{x}{3^{1/x}}, \quad x \in [1,\infty). \tag{K.3}$$

Then, we will select one component of $\mathbf{p}$ to be proportional to $c(N_\mathrm{Q})$. The plot of $c(x)$ is shown in Fig. K-1.

Lemma K.2 (Almost-Constant Function). *For all $x \geq 1$, the function $c(x)$ in (K.3) is constrained in the range*

$$\frac{2}{3} \leq c(x) < \ln 3.$$

Proof. First, we show that $c(x)$ is strictly increasing on $[1,\infty)$ by considering the first derivative of $c(x)$:

$$c'(x) = 1 - \frac{1}{3^{1/x}} - \frac{\ln 3}{x \cdot 3^{1/x}}.$$

See Fig. K-2 for the plots of $c'(x)$.

Since the second derivative

$$c''(x) = -\frac{(\ln 3)^2}{x^3 \cdot 3^{1/x}}$$

is negative for all $x \in [1,\infty)$, the function $c'(x)$ is strictly decreasing on $[1,\infty)$.

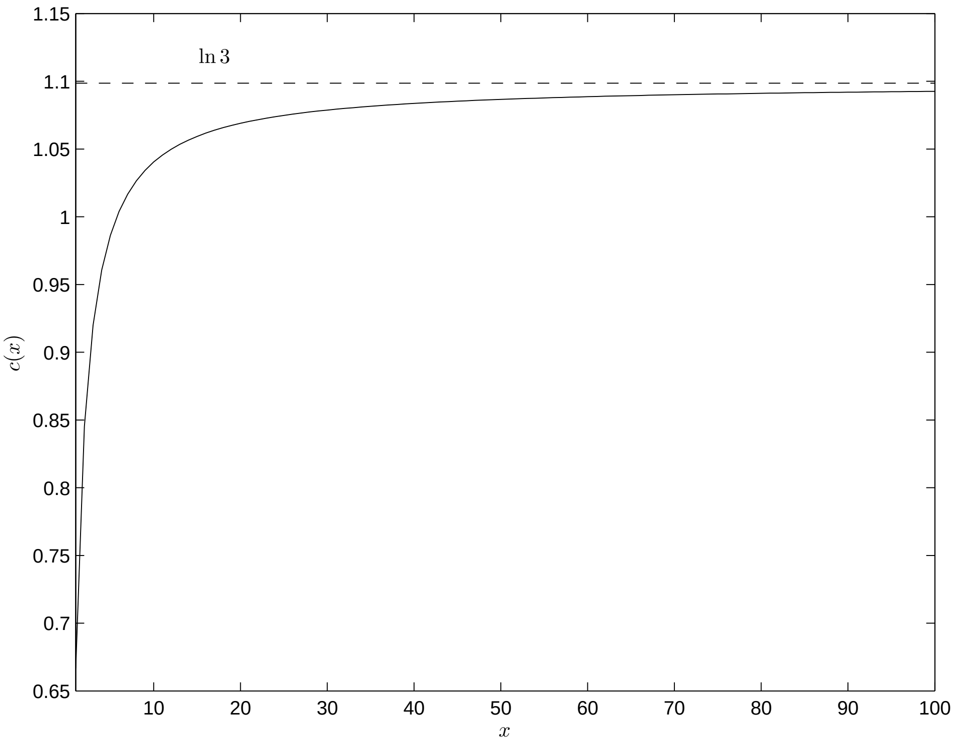

Figure K-1: For $x \in [1, \infty)$, the function $c(x) \triangleq x - x/3^{1/x}$ is upper-bounded by $\ln 3$.

The limit of $c'(x)$ when x approaches infinity is zero:

$$\lim_{x\to\infty} c'(x) = 0. \tag{K.4}$$

Condition (K.4), together with the fact that $c'(x)$ is strictly decreasing on $[1, \infty)$, implies that $c'(x) > 0$, for all $x \in [1, \infty)$. Therefore, $c(x)$ is strictly increasing on $[1, \infty)$.

The limit of $c(x)$, as x approaches infinity, is

$$\begin{aligned}\lim_{x\to\infty} c(x) &= \lim_{x\to\infty} \frac{3^{1/x} - 1}{x^{-1} \cdot 3^{1/x}} \\ &\overset{(a)}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(3^{1/x} - 1)}{\frac{d}{dx}(x^{-1} \cdot 3^{1/x})} \\ &= \lim_{x\to\infty} \frac{\ln 3}{\frac{\ln 3}{x} + 1} \\ &= \ln 3,\end{aligned}$$

where the equality (a) follows from L' Hôspital's rule. The limit of $c(x)$ and the strictly-

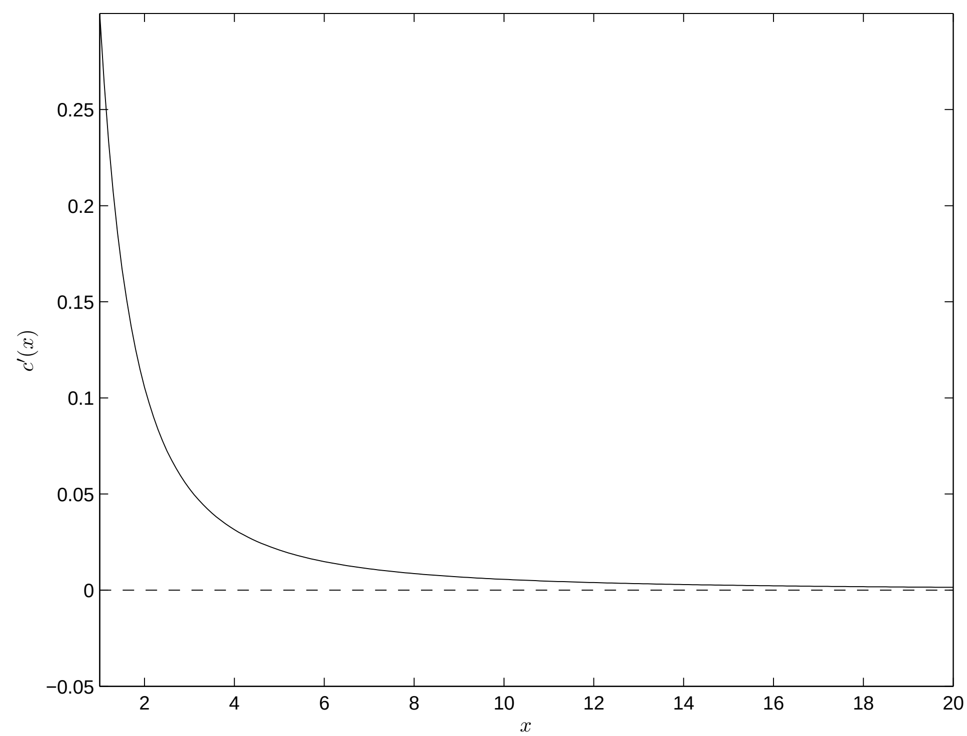

Figure K-2: The derivative $c'(x)$ is strictly decreasing on $[1, \infty)$.

increasing property of $c(x)$ imply that

$$c(x) \leq \ln 3, \quad \text{for all } x \in [1, \infty). \tag{K.5}$$

Because $c(x)$ is strictly increasing, $c(x)$ is lower-bounded by

$$\frac{2}{3} \leq c(1) \leq c(x).$$

That completes the proof. □

We use the results in the last lemma to prove the upper-bound in the next lemma.

Lemma K.3 (Low-SNR-CSS Upper-Bound). *The infimum is upper-bounded by*

$$\inf_{\mathbf{x} \in (0,1]^n} f(\mathbf{x}) \leq (N_\text{S} + 3N_\text{Q}) \cdot t_1 + \Big(\frac{\ln 3 \cdot N_\text{S}}{N_\text{Q}} + 2\Big) \sum_{l=2}^{n} t_l + \Big(\frac{\ln 3 \cdot N_\text{S}}{N_\text{Q}} + 1\Big) \cdot t.$$

Proof. Consider the tuple $\mathbf{p} \triangleq (p_1, p_2, \ldots, p_n)$, in which

$$p_1 \triangleq \frac{c(N_{\mathrm{Q}})}{N_{\mathrm{Q}}} = 1 - \frac{1}{3^{1/N_{\mathrm{Q}}}}$$
$$p_i \triangleq 1, \qquad i = 2, 3, \ldots, n.$$

Clearly, $\mathbf{p} \in [0,1)^n$ is a feasible solution. Therefore,

$$\begin{aligned}
\inf_{\mathbf{x}\in(0,1]^n} f(\mathbf{x}) &\le f(\mathbf{p}) \\
&= \frac{(N_{\mathrm{S}} - N_{\mathrm{Q}})^2}{2N_{\mathrm{S}}} \cdot h_{\mathrm{rat}}\Big(\frac{c(N_{\mathrm{Q}})}{N_{\mathrm{Q}}}, 1, 1, \ldots, 1\Big) \cdot h_{\mathrm{sum}}\Big(\frac{c(N_{\mathrm{Q}})}{N_{\mathrm{Q}}}, 1, 1, \ldots, 1\Big) \\
&\quad + \Big(2 - \frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\Big) \cdot h_{\mathrm{rec}}\Big(\frac{c(N_{\mathrm{Q}})}{N_{\mathrm{Q}}}, 1, 1, \ldots, 1\Big) \\
&\quad - \Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{2N_{\mathrm{S}}}\Big) \cdot h_{\mathrm{sum}}\Big(\frac{c(N_{\mathrm{Q}})}{N_{\mathrm{Q}}}, 1, 1, \ldots, 1\Big) \\
&\quad + \Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{S}}}\Big) \cdot t \\
&= \Big[\frac{(N_{\mathrm{S}} - N_{\mathrm{Q}})^2}{N_{\mathrm{S}}} - \frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{2N_{\mathrm{S}}}\Big] \cdot \Big[t_1 + \frac{c(N_{\mathrm{Q}})}{N_{\mathrm{Q}}} \cdot \Big(t + \sum_{l=2}^{n} t_l\Big)\Big] \\
&\quad + \Big(2 - \frac{N_{\mathrm{Q}}}{N_{\mathrm{S}}}\Big) \cdot \Big(\frac{N_{\mathrm{Q}} t_1}{c(N_{\mathrm{Q}})} + \sum_{l=2}^{n} t_l\Big) \\
&\quad + \Big(\frac{N_{\mathrm{S}} - N_{\mathrm{Q}}}{N_{\mathrm{S}}}\Big) \cdot t \\
&\overset{(a)}{<} N_{\mathrm{S}}\Big[t_1 + \frac{\ln 3}{N_{\mathrm{Q}}}\Big(t + \sum_{l=2}^{n} t_l\Big)\Big] + 2\Big(\frac{N_{\mathrm{Q}} t_1}{2/3} + \sum_{l=2}^{n} t_l\Big) + t \\
&= (N_{\mathrm{S}} + 3N_{\mathrm{Q}}) \cdot t_1 + \Big(\frac{\ln 3 \cdot N_{\mathrm{S}}}{N_{\mathrm{Q}}} + 2\Big) \sum_{l=2}^{n} t_l + \Big(\frac{\ln 3 \cdot N_{\mathrm{S}}}{N_{\mathrm{Q}}} + 1\Big) \cdot t.
\end{aligned}$$

The inequality (a) follows from the bound $2/3 \le c(N_{\mathrm{Q}}) < \ln 3$ in Lemma K.2 and the fact that $N_{\mathrm{S}} \ge N_{\mathrm{Q}}$. That completes the proof. □

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