The Instability of Time-Dependent Jets

by

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Abstract

The central focus of my thesis is to study the instability jets of various complexity by analyzing the linear and nonlinear dynamics. We applied this methodology to four different situations in order to learn the following. First, what asymmetries develop between cyclones and anticyclones because of finite variations in the free surface? Second, how is the stability of a jet flowing along a topographic step altered by the topography beneath? Third, can parametric instability arise in shear flows? Fourth, can an idealized model of a tidally and topographically forced coastal jet develop instabilities, and if so, can these instabilities become turbulent?

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Chapter 1

Introduction

Off the coast of Provincetown, in the Cape Cod Bay, there is an oscillatory current with the same period as that of the ambient tides. The location of the current and its periodicity suggest that the topography and tides play fundamental roles in generating the jet. Even though the periodicity and structure of this current may be regular, it is possible that the jet destabilizes to generate vortical motions. If so, a series of interesting questions arise. What are the structures and parity of the eddies? Do the eddies oscillate? Can the instabilities become turbulent?

The literature on the stability of oscillatory shear flows is rather sparse in comparison to the stability of steady basic states. These time-dependent flows are inherently more complicated to analyze. There do not exist any theorems or general criteria to indicate when, and what type of, instabilities develop. It is unclear as to whether the oscillations stabilize or destabilize the flow. It is conceivable that they can stabilize the flow by impeding the phase locking between the resonant waves that is required for instability. Conversely, oscillations can destabilize the flow through parametric instability as arises in gravity waves (see McEwan and Robinson (1975)) or more pertinently, in shear flow (see Rosenblat (1968)). By studying a simple model we hope to better understand what effects oscillations in geophysical systems have on the stability of shear flows. In addition to the interesting questions that this time-dependent current inspires, its stability characteristics will have a significant impact on the ambient biology. During the spring months, the Cape Cod Bay is a roaming ground for the North Atlantic Right Whale, perhaps the most endangered whale species.
in the world. In the spring, they are observed to travel along the topographic steps eating plankton patches that form in the coastal water. It is likely that the local surface convergences and divergences in the tidal flows and vortices are related to the aggregation of the copepods (Calanus Finmarchicus) which are their primary food source. Understanding the dynamics of this jet is essential to predicting the aggregation of codepods which then may help us understand the likely locations and feeding history of the Right Whales. In this thesis, we investigate the physics which must be understood before one can adequately begin to model the biology.

The oscillatory jet in Cape Cod Bay is rather complicated since it involves complex topography and coastlines, bottom and lateral friction, stratification and numerous other effects. Our goal is to study an idealized model that captures some of the essential features rather than model this phenomena precisely. In particular, we study a barotropic fluid that is forced by simple one-dimensional topography and monochromatic tidal frequencies. This simplification allows us to focus on the physical processes at work; we hope to extrapolate from this idealization to learn about the instability of the current in the Cape Cod Bay.

Since the geometry in question is shallow we assume that the dynamics are governed by the Shallow Water (SW) equations. This requires not only that the fluid be shallow but that the motion be uniformly independent of depth. The inviscid shallow water model has several features which are inconsistent with this region of the ocean: it assumes topographic slopes are small, whereas they can be order one; it neglects bottom friction which can be important in such shallow water and which injects horizontal vorticity into the flow; and it ignores effects of stratification. Nevertheless, we feel both that it offers the opportunity to study the phenomenon of interest – instabilities and turbulence – in a relatively simple system and that it can bring out aspects of the dynamics which can still be operating in the more complex flow.

The goal in this thesis is to understand the instability of oceanic currents similar to that previously mentioned in the Cape Cod Bay. This problem is quite complicated for several reasons. First, the flow is not unidirectional but flows in orbits because of the tides and Coriolis force. Second, the flow is time-dependent. Third, large topography strongly influences the dynamics of the jet. Fourth, the free-surface amplitude changes significantly
because of the tidal forcing. How each of these affects the instability of a jet is not well understood. This is why we divide our analysis into four components; each of which gives us a better understanding of the full problem. First, we investigate how free-surface effects can alter the instability of a jet. Second, we study the instability of a jet flowing along shelf-like topography to learn how the stability characteristics are altered by the topography. Third, we analyze simple oscillatory jets to better understand parametric shear instability. Fourth, we study an idealized model of the Cape Cod Bay that accounts for the Coriolis force, tidal forcing and large step-like topography.

1. Free-Surface Effects

Vortices are frequently generated in the world’s atmosphere and oceans through the instability of large-scale currents such as the Gulf-Stream, the North Atlantic current, the Azores current and the Aghulas current. If a jet is unstable, meanders may develop that grow in amplitude and then pinch-off. This generates vortices that are also known as rings. Maps of the Gulf Stream (see Richardson et al. (1978)) present cyclonic and anticyclonic rings that are generated by such a meandering process. This view of a dynamic Gulf Stream has been supported by numerous experimental studies (see Hansen (1970), Robinson et al. (1974), Watts and Olson (1978), Halliwell and Mooers (1979), Richardson (1980), Watts and Johns (1982), Lee and Cornillon (1996a) and Lee and Cornillon (1996b)). The vortices generated by large-scale oceanic jets are important since they transport physical, biological and chemical properties that are contained within. As the vortex decays it mixes with the ambient ocean and deposits some of its properties in the surrounding environment (see Flierl and Meid (1985)). This is why vortices are important in transporting properties throughout the world’s oceans.

The Gulf Stream contains both horizontal and vertical shear flow. The instabilities that develop due to these particular types of shears are referred to as barotropic and baroclinic instability, respectively. Early attempts to model the Gulf Stream used a linear two-layer Quasi-Geostrophic (QG) model (see Hart (1974), Flierl (1975) and Holland and Haidvogel (1980)). They determined that the major source of energy for the meanders is due to baro-
clinic instability. However, Flierl (1975), Killworth (1980) and Talley (1983) all stated the importance of the horizontal shear. Barotropic instability is the simpler of the two possible mechanisms and is the process that is studied in this thesis.

The linear stability problem in the context of the QG model has been studied in many one and two layer flows. Some examples can be found in Blumen (1970), Blumen et al. (1975), Tung (1981), Maslowe (1991), Pratt and Pedlosky (1991), Boss et al. (1996), Flierl et al. (1999) and Da Silveira and Flierl (2002). A result from these analyzes is that a positive $\beta$ parameter, the ambient vorticity gradient, stabilizes the flow. A negative $\beta$ destabilizes for small magnitudes but then stabilizes for larger values.

In linear SW theory the analysis is more complicated since it not only allows for shear instability, as in QG, but for inertial instability as well. Specific criteria for stability have been obtained for the one and multilayer problems (see Ripa (1983) and Ripa (1991)). Several studies have solved example linear stability problems by calculating the wavelengths and growth rates of the instability, as can be found in Hayashi and Young (1987), Pratt and Pedlosky (1991), Boss et al. (1996), Paldor and Ghil (1997), Baey et al. (1999) and Balmforth (1999). We have not been able to find any published works that solved the linear stability problem in general for both finite and nonzero Rossby and Froude numbers. This is something we shall do in Chapter 2 by using a spectral scheme.

The linear dynamics is useful for determining the length scale and growth rates of the instabilities. However, its range of applicability is limited since the formation of vortices is an inherently nonlinear process. Some earlier studies that investigated this process are Zabusky and Deem (1971), Christiansen and Zabusky (1973) and Aref and Siggia (1981). The nonlinear evolution of the barotropic instability of jets in QG was studied in Flierl et al. (1987), Ford (1994), Pratt and Pedlosky (1991) and Balmforth (1999). The generation of vortex streets is not limited to geophysical contexts since numerical experiments have shown that these structures can develop in plasma shear flows (see Yamamoto (1988)). The stability of these vortex streets is addressed in Saffman and Schatzman (1981), Saffman and Szeto (1981) and Saffman and Schatzman (1982). This is not something that we will address directly.

One common occurrence of barotropic instability is that fluid may be transported across
the jet. This is of fundamental importance since it affects the distribution of physical, biological and chemical properties in the ambient fluid. One means of transport is through the detachment of vortices and subsequent injection of eddies across the jet. Another mechanism by which exchange and stirring may occur is simply by the meandering of the stream without any detachments (see Bower (1991) and Bower and Lozier (1994)); meanders in the surface layer can cause mixing in the deeper layer. The detailed analysis of the instability of a jet in QG has been studied in Rogerson et al. (1999) and Yuan et al. (2002) with $\beta$ zero and non-zero, respectively. It was determined that the exchange of fluid across the Gulf Stream due to meanders is comparable to that of eddy detachment. In this work we will not focus on the means of transport but simply investigate how certainly physical parameters can induce or inhibit transport.

The instability of oceanic jets has often been modeled using QG since they typically have Rossby numbers less than one. In the context of QG, Flierl et al. (1987) demonstrated that when cyclones and anticyclones were generated they are symmetric in their size, strength, and shape: they only differ in polarity. This artificial symmetry is imposed by the fact that geostrophic balance holds to leading order. Changing the sign of a particular pressure field produces a velocity field that is of the same magnitude but only of a different sign. However, ocean currents can have Rossby numbers (see Olson (1991)) larger than those required for QG to be applicable.

We study the linear stability and nonlinear dynamics of these currents in order to understand what asymmetries originate between cyclones and anticyclones in the non-QG regime. This study will also have relevance to other vortex phenomena that are beyond the QG regime such as hurricanes, tornadoes and mesoscale storms since these are examples of vortices with order one or larger Rossby numbers. To explore what asymmetries arise in non-QG regimes, we study the instability of a jet in the Shallow Water (SW) model. This system is richer than QG since it allows for large deformations in height, variable rotation rates and the centrifugal forces to appear in the leading order balance. By increasing the size of the free surface deformations, we discover that asymmetries develop in all three properties.

The asymmetries between vortices in the SW model have already been studied to some
extent. Cyclone-anticyclone asymmetry was addressed in Stegner and Dritschel (2000) where it was determined that ageostrophic effects stabilized cyclones and destabilized anticyclones. It was shown that large Froude numbers always stabilized the eddies.

Polvani et al. (1994) and Arai and Yamagata (1994) studied two-dimensional SW turbulence and observed asymmetries between the evolution of interacting vortices; anticyclones are favored and merge to take coherent circular forms. In contrast, the cyclones become elongated because of the enstrophy cascade. This is because in two-dimensional turbulence enstrophy, the mean of the square of the vorticity, must move to smaller scales to compensate for the energy moving to larger scales (see Rhines (1979)). Cho and Polvani (1996) studied two-dimensional SW turbulence on a sphere and noticed that bands were generated similar to those that are generated on Jupiter. These generated vortices with a preference for anticyclones.

Baey et al. (1999) studied the instability of a two-layer SW jet which allowed for baroclinic instability. However, they did not consider Rossby numbers larger than 0.3, which is why they did not notice the asymmetries reported in this work.

Chapter 2 will explore how finite amplitude deformations of the free surface can alter the barotropic instability of a jet. We first solve the linear stability problem in order to see how the growth rates are affected by varying certain parameters. Then, we numerically integrate the fully nonlinear dynamics to observe the structures that develop. By considering different parameters we discover that asymmetries in size, strength and shape can all occur.

2. Topographic Effects

The topography beneath the ocean is highly variable and can have height that are of the same order as the depth of the ambient fluid. It is a very important mechanism in forcing the evolution of the ocean dynamics. Topography guides the large-scale ocean circulation but it also influences the small scale motions as well. One such example is that coastal topography can induce wave breaking and thereby cascade energy to smaller length scales (see Thorpe (2001)). Another important process, and one we focus on, is that topography may alter the stability characteristics of an oceanic jet. The presence of variable topography
establishes a mean gradient in the background potential vorticity. This provides a mechanism through which vortex tubes can be stretched or contracted (see Pedlosky (1987)). Depending on the particular flow and topography, this can destabilize or stabilize the jet.

There are two topographic idealizations that have been studied to determine how they alter the stability of the overlying shear flow. One is a continental shelf and the other is a ridge or a trough. These two categories are mathematically different in that the depth of the former is monotonically increasing away from the shore whereas the gradient of the depth of the latter changes sign. An example of the former is the South Pacific mid-ocean ridge as illustrated in Webb et al. (1982) and Webb and others (The FRAME Group). Schmidt and Johnson (1997) modeled shear flow over ridges in the context of the barotropic SW model. Their analysis indicated that the ridge topography destabilized the jet. Laboratory experiments of shear flow along a ridge in Hreinsson et al. (1997) indicated that this type of topography can stabilize the flow. Clearly, further investigations are required in order to determine what parameter choices stabilize and destabilize the flow respectively. It is clear from reviewing the literature that continental shelves underneath shear flows have received more attention and these are the conditions that we choose to investigate.

Barotropic slope currents may be topographically steered from beneath by two different types of topography. It is classified as prograde if the cyclonic flank is in shallow water, or retrograde if the cyclonic flank is in deep water. One fundamental question that has yet to be adequately answered is, what effect does step-like topography have in stabilizing or destabilizing the jet? In this chapter we build upon previous studies to gain a clearer picture as to how topography influences coastal jets in both linear and nonlinear regimes.

Several examples of continental shelves that underlie shear flows are the Middle Atlantic Bight (see Beardsley et al. (1985)), the Svalbard Bank (see Johansen et al. (1989)) and the deep western boundary current in the Atlantic (see Warren (1981)). Analytical studies of the barotropic SW model in Pedlosky (1980), Collings (1986), Collings and Grimshaw (1980a), Collings and Grimshaw (1980b) and Bidlot and Stern (1994) have found examples where coastal shelves destabilize the overlying shear flow.

Li and Mcclimans (2000) performed a theoretical analysis of barotropic prograde and retrograde jets along a bottom slope in the Shallow Water (SW) model, for the case with a
rigid lid. They were mostly concerned with the Svalbard Bank but their analysis applies to other similar non-equatorial jets as well. In contrast to the above mentioned results, they concluded that both types of topography stabilized the current. In addition, they found that retrograde currents had a larger range of stable modes, which they attributed to the difficulty in the Rossby waves phase locking with the current that flows anti-parallel; observations of the retrograde current in the Svalbard Bank supported their conclusions that topography stabilizes the flow. They came to this conclusion by determining the neutral curves that are adjacent to both the stable and unstable regions. Since these curves were symmetric, they concluded that the topography had the same effect. However, they also discovered that the prograde topography tended to have more unstable modes. This is suggestive that the prograde topography is more unstable. To determine whether this is true one needs to calculate the growth rates in the unstable regions, something that was not done in Li and McClimans (2000).

The analysis of Li and McClimans (2000) is limited in its applicability. One assumption that they made, which we remove, is the rigid lid approximation. This has the drawback that it requires that we use numerical instead of analytical techniques to solve the linear stability problem. However, its advantage is that it more accurately describes scenarios of non-zero Froude numbers.

Lozier et al. (2002) studied the linear stability problem of a surface current over the Middle Atlantic Bight in the context of the stratified hydrostatic primitive equations. They determined the shelfbreak front is unstable. Moreover, they calculated the temporal and spatial scales and the structure of the instability of this front. They varied different background density and velocity fields. However, they did not thoroughly explore parameter space in order to observe whether there are instances where the jet can be stabilized. This is why their investigation does not reveal the qualitative dependence on the relevant nondimensional parameters. This is something we investigate in the context of the SW model, where the fluid is homogeneous and therefore easier to study.

To begin, we extend the linear stability analysis of Chapter 2 to incorporate topography. We consider a broad range of parameters in order to determine how the stability of currents is affected by the magnitude and direction of the underlying topography. We restrict our
attention to the case of Froude number equal to 0.1 since it is sufficient to prove the point that the Froude number need not be order one for differences to occur in comparison with the barotropic limit. This is not very appropriate for surface currents but more ideal for deep water currents like the deep western boundary current that runs beneath the Gulf Stream (see Warren (1981)).

The growth rates calculated from the linear theory are asymmetric with respect to prograde and retrograde topography of the same size. As in Li and McClimens (2000), retrograde topography is always stabilizing, and for small enough Rossby numbers, there is a critical height past which the jet is entirely stable. This is because the waves generated by topography travel too quickly in the opposite direction of the mean flow to phase lock and thereby generate a resonance.

In contrast to this previous work however, prograde topography can be stabilizing or destabilizing depending on the topographic height and Rossby number. For small Rossby numbers, the current is destabilized for small topographic heights and then destabilized beyond a critical value. The predictions from the linear stability analysis are tested by performing a series of numerical simulations. These experiments capture the intricate dependence of growth rate on topography and verify the linear predictions. Therefore, we conclude that both the magnitude and orientation of topography are important factors in determining the stability of a jet. We have determined that as the width scale of the topography increases, the stability increases, but we do not investigate this issue in great detail. In terms of our basic motivation of studying oscillatory flow along the shelf, a change in topography is equivalent to changing the flow velocity; the width of the topography is invariant throughout the period.

The nonlinear evolution of unstable jets overlying continental shelves in the SW model has not been studied directly. The main contribution is in the series of papers Allen et al. (1990a), Barth et al. (1990) and Allen et al. (1990b). They observed the generation of a single row of vortices not a vortex street, as occurs in the Bickley Jet. Their motivation was not to study the physical problem but instead to compare the intermediate models between the QG and SW models. Other works that studied the instability of a jet overlying plane or barred beaches are Allen et al. (1996), Ozkan-Haller and Kirby (1999) and Slinn et al.
(1998). These are all in a non-rotating environment and are far removed from our regions of interest. The numerical simulations in Chapter 3 are generated with three goals in mind: one, to observe what types of asymmetries develop between cyclones and anticyclones; two, see for what topographic heights fluid is transported across the jet by eddy detachment; three, to support the results from the linear stability analysis.

3. Time-Dependent Effects

The stability of time-dependent basic states has received relatively little attention in comparison to the stability of time-independent basic states. One reason is that the analysis is more complicated. The time-dependence of the basic state is important since an imposed modulation may destabilize an otherwise stable state, inducing the transport of heat, momentum and mass (see Davis (1976)). Conversely, stabilizations by oscillations would inhibit such transport and therefore would also be significant.

The definition of stability for time-dependent basic states is not unique. We take the definitions as stated in Rosenblat (1968) and Davis (1976). A periodic basic state is unstable if there exists a perturbation that experiences net growth over each period. If the perturbations decay at every instant the basic state is said to be stable or monotonically stable. The third possibility, there is no net growth after one cycle but throughout the cycle disturbances both grow and decay, is said to be transiently stable. To determine the stability of a particular basic state one can use either energy theory or linear theory (see Davis (1976)); we chose to use the later.

Oscillations can generate new instabilities; the simplest example arises in the context of a forced pendulum. It is well known that the pendulum has two stationary positions: one stable and the other unstable. When the suspension point oscillates vertically, the equation governing the linear stability problem is Mathieu’s equation. Solutions of this equation demonstrate that the stable (unstable) stationary position can be made unstable (stable) with sufficiently strong forcing (see Stoker (1950)). This change of stability is called parametric resonance or parametric instability.

Early experiments by Faraday of a square tank oscillated in the vertical direction proved
that parametric instability can arise in the context of surface gravity waves. This phenomenon also produces interesting shapes, which have received much attention in studies of pattern formation (see Benjamin and Ursell (1954)). Subsequent works have discovered that parametric instability can occur in internal gravity waves in terms of a resonant wave triad (see McEwan and Robinson (1975)). In this resonant triad the energy is transferred from the two larger waves to the smallest wave. This is therefore a means through which energy is cascaded to smaller scales.

The first article to discover parametric instability in oscillatory shear flows was Greenspan and Benney (1963). The profile they considered was a two-contour model where the width of the middle region oscillated as well. Since then, more thorough analyses have been done in the context of vertical shear instability, (see Kelly (1965), Kelly (1967) and Rosenblat (1968)). Kelly (1965) studied parametric shear instability in oscillatory Kelvin-Helmholtz instability. In contrast, Kelly (1967) studied the interaction of a non-zero continuous mean flow with an oscillatory component, but occurring as a secondary instability. Finally, Rosenblat (1968) studied parametric instability in the context of Taylor-Couette flow. They only considered azimuthally symmetric disturbances which is why the instabilities appear as convective over-turning. Their analysis of this inertial instability has many characteristics in common with the research presented in Chapter 4.

Rosenblat (1968) analyzed the stability of time-periodic azimuthal flows between coaxial, circular cylinder; what is typically referred to as Modulated Taylor-Couette flow. By considering the inviscid dynamics with axisymmetric disturbances, several interesting properties were discovered due to the oscillations of the basic state. The first case they studied is called rigid-body oscillations, where the azimuthal velocity is a function of the radial coordinate multiplied by a periodic function of time with zero mean. It was determined that all of these shear profiles were transiently stable. The second type is a superposition of two different profiles that oscillate out of phase and where each has a zero time average. This situation always yielded instability save for three exceptional circumstances. The third case is for flows with non-zero mean flow. By assuming that the oscillations were infinitesimal it was determined that the stability problem for the case of small amplitude oscillations is described by Mathieu’s equation which therefore produces subharmonic resonance. This
implies that if the mean component of the flow is stable (unstable), oscillations may cause a bifurcation in the stability so that the flow becomes unstable (stable).

Since Rosenblat (1968), there have been theoretical and experimental studies to the same problem but with viscosity and for special shear profiles (see Lopez and Marques (2002) Ern and Wesfreid (1999), Normand (200), Walsh and Donnelly (1988) Hu (1995)). They support the conclusions of Rosenblat (1968) that the oscillations can either stabilize or destabilize the flow because of a parametric resonance. There have been other related studies of Taylor-Couette flow where the each cylinder rotates azimuthally at a fixed rate but the inner cylinder oscillates in the radial direction (see Meseguer and Marques (2000), Marques and Lopez (1997), Marques and Lopez (2000)). They have determined that the oscillation in the axial direction always has a stabilizing effect.

Another study of oscillatory shear flow in a pipe flow by Von Kerczek (1982) has revealed only mild effects of parametric instability. They investigated oscillatory Poiseuille flow and determined that oscillations either slightly stabilize or destabilize the flow, but there was no evidence of bifurcations in the stability properties.

Chapter 4 studies parametric shear instability in some simple shear flows in a planar geometry. It is determined that pure oscillatory shear flows are always stable; this is analogous to the rigid-body rotation in Rosenblat (1968). For the case of mixed oscillatory profiles we find instances where oscillations can either stabilize or destabilize the flow by calculating the growth rates and the stability boundaries. Rosenblat (1968) found similar conclusions for shear flows of non-zero mean but only determined that the governing equation was Mathieu's equation for the case of small amplitude oscillations. We instead solve the problem numerically proving that parametric shear instability can occur in this context and also determining quantitatively how the growth rates are affected. Also, by studying a four-contour example we deduce that a smoothing out of the profile produces more stability tongues and reduces the strength of the instabilities.
4. Tidally Forced Coastal Jet

The tidal forcing of the oceans generates oscillatory motions on many different time scales. Perhaps the two most important are one on the order of half a day and the other on the order of a month. The studies of tidal currents in the literature have focused on studying tidal rectification. This looks at the average of the tides over some shorter times scales. It is the mean current over long time scales that is usually considered (see Huthnance (1973), Loder (1980), Verron et al. (1995) and Mazé (1998)). However, what we consider here is the non-averaged oscillations over the short time scales. There appear to be no studies of the stability of such states.

In Chapter 5 we investigate the instability of an oscillatory jet that exists because of the tidal forcing and the shelf-like topography beneath. We solve the linear instability problem in order to determine the growth rate and structure of the unstable modes. This analysis illustrates how the stability of the time-dependent jet is affected by the numerous nondimensional parameters. This aids in exploring the nonlinear dynamics since it identifies the unstable regions. Our numerical experiments indicate that instabilities may develop that oscillate in tidal orbits, which indicates that the mechanism is essentially shear instability and not parametric instability. By exploring the strongly unstable regions, we discover the development of turbulence which acts to transport shallow water into the deep waters. This has an important impact on the transport of biological, chemical and physical properties along coastlines.
Chapter 2

The Effects of a Free Surface on the Instability of Jets

In §2.2 we discuss the SW model and the numerical method used in our simulations. Then, §2.3 describes the linear stability analysis of the geostrophic jets to illustrate how the growth rates vary with Rossby and Froude numbers. Finally, §2.4 presents the numerical simulations which demonstrate what asymmetries may arise in the instability processes.

1. The Model

The model we study is a reduced gravity model, also known as a $1\frac{1}{2}$-layer model. This means we have one dynamic layer of fluid that overlays topography and is situated beneath an infinitely deep, motionless layer that is homogeneous of density $\rho$. If the density of the dynamic layer is $\rho + \Delta \rho$, the effect of the upper layer is to reduce the gravity in the dynamic layer from $g$ to $g' = \frac{\Delta \rho}{\rho} g$. Figure 2-1 illustrates this pictorially. Note that a reduced gravity model could also be upside down with the dynamic layer overlying an infinitely deep layer. This is the scenario that is appropriate when discussing the Gulf Stream.
Figure 2-1: This depicts the essential geometry of a reduced gravity model. The dynamic lower layer is bounded below by bottom topography and above by an infinitely deep upper layer. The effect of the upper layer is to reduce the gravity that acts on the dynamic layer. The variables $\eta$ and $h$ are the free surface displacements and total heights of the dynamic layer.

a. **Shallow Water Equations**

The inviscid, reduced gravity, shallow water model in dimensional form is governed by the following set of partial differential equations,

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f \mathbf{k} \times \mathbf{u} = -g' \nabla (h - h_B) \tag{2.1}
\]

\[
\frac{\partial h}{\partial t} + \nabla \cdot (hu) = 0. \tag{2.2}
\]

The two parameters, $g'$ and $f$, are the reduced gravity and the Coriolis parameter, both we assume are constant. The variables $\mathbf{u} = (u, v)$, $h$ and $h_B$ denote the horizontal velocity field, total depth and bottom topography respectively. From these we define $\eta = h - h_B$ to be the free surface height. In addition, $L$ and $U$ are the scales of the horizontal motion and velocity based on the width and peak velocity of the jet.

The two conventional nondimensional parameters are the Rossby number $Ro = U/(fL)$ and rotational Froude number $Fr = (fL)^2/(g' H)$. To solve the problem numerically necessarily introduces friction into the problem to stabilize the solution; the dimensional friction parameter is $\nu$. This introduces a third parameter, a numerical Reynolds number $Re = UL/\nu$.

In this article we study the instability of an isolated jet by setting $h_B = 0$. In the fol-
lowing chapters we investigate the effect of topographic and tidal forcing on barotropic instability, but for the moment we consider the simplest possible case, that of an unforced system with a free surface. We will discover that even this simple system is rich enough in its dynamics to produce instabilities that are dramatically different from what has previously been observed.

b. Asymmetry between cyclones and anticyclones

To begin our discussion on asymmetry, let us first consider the potential vorticity (PV) of the SW model. Nondimensionalized, it is

\[
\frac{Ro \nabla \times u + 1}{1 + RoFr \eta} \quad (2.3)
\]

The special case where \( Ro << 1 \) and \( Fr = O(1) \) allows us to Taylor-expand the SW equations about \( Ro = 0 \). The resulting model is what is referred to as QG. There governing equation is simply the statement that the QG PV is conserved along trajectories of fluid parcels

\[
\nabla^2 \eta - Fr \eta. \quad (2.4)
\]

This means that any increase (decrease) in the relative vorticity, the first term above, must be accompanied by an increase (decrease) of the free surface height. In other words, as the relative vorticity increases (decrease) there must be a corresponding vertical stretching (contraction) of the vortex tubes (see Pedlosky (1987)). To derive QG it is necessary to assume that there are no order-one variations in the free surface, unlike the SW model that can have order-one deformations in fluid depth. Interestingly, this vertical structure has implications on the horizontal motions. The horizontal scale of a geostrophic vortex is the radius of deformation,

\[
L_R = \sqrt{gH/f}, \quad (2.5)
\]

which clearly increases with increasing depth. The infinitely deep, motionless upper layer
determines the reduced gravity and therefore the length scale of the motion. Since anticyclones and cyclones are generated in deep and shallow water respectively, the anticyclones tend to be wider. This has been observed in SW turbulence Polvani et al. (1994). We will show that this is also true in the formation of vortices through the instability of a jet.

Another assumption needed to derive QG is that the velocity is geostrophic to leading order. Thus, the vortices that exist in QG are sustained due to a balance of pressure and Coriolis forces. This is unlike SW theory that allows for eddies that exhibit gradient-wind balance:

\[ \frac{v^2}{r} + f v = g' \frac{\partial h}{\partial r}. \]

Olson (1991) showed that in this balance relation, the quadratic appearance of velocity does not permit arbitrarily strong negative pressure gradients, hence, there is a cutoff past which anticyclones cannot exist. The reason being that the right hand side cannot be arbitrarily large and negative since the left hand side has a minimum negative value, because of its quadratic structure. This is why for large Rossby number flows there is a preference for strong cyclones in atmospheric phenomenon such as mesoscale storms.

c. Numerical Method to solve the SW model

We study the nonlinear evolution of a jet by numerically integrating the SW equations. Instead of solving the equations in the canonical form (2.1) and (2.2) we formulate them in terms of the momentum transport functions. This form is preferred since the nonlinearities are in conservation form.

If the inviscid SW equations are written in terms of the momentum transport functions, \( U = uh \) and \( V = vh \), the equations are as follows:

\[ \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left( \frac{U^2}{h} + \frac{g' h^2}{2} \right) + \frac{\partial}{\partial y} \left( \frac{U V}{h} \right) = f V \quad (2.6) \]

\[ \frac{\partial V}{\partial t} + \frac{\partial}{\partial x} \left( \frac{U V}{h} \right) + \frac{\partial}{\partial y} \left( \frac{V^2}{h} + \frac{g' h^2}{2} \right) = -f U \quad (2.7) \]
The geometry we consider is that of a channel where the across and along channel coordinates are \( x \) and \( y \) respectively. We assume there are rigid boundaries at \( x = \pm 1 \) and that the channel is periodic so that \( y = \pm 1 \) are coincident. The flat bottom is located at \( z = 0 \), the free surface is denoted by \( \eta \) and therefore the total depth is \( h = 1 + \eta \). See Figure 2-1 for a schematic of the geometry.

The equations are solved on an unstaggered grid with a finite difference scheme. The advection and pressure terms are discretized by third order upwinding (downwinding) and fourth order center differencing schemes, respectively. The equations are evolved forward in time by the third order Adams-Bashford method (see Fletcher (1991)). The square domain is a \( 200 \times 200 \) regular grid and the state variables are defined at each node of the grid. There are many other possible methods that we could have used such as spectral, Godunov, essentially non-oscillatory (ENS) and flux limited schemes. However, these are all more complicated to implement and we have instead chosen to use the simpler method described above.

In order to guarantee numerical stability, two measures were taken. First, the time step was chosen so that the Courant-Friedrichs-Lewy condition was satisfied:

\[
\Delta t < \frac{\Delta x}{|u| + \sqrt{g H}}.
\]

Second, numerical friction was introduced into the momentum transport equations and the continuity equation as a Laplacian operator with a viscosity coefficient of \( \nu \). The numerical Reynolds number is on the order of \( 10^6 \).

The boundary conditions are no-normal flow, free-slip and \( \partial \eta / \partial x = 0 \) in the \( x \) direction, and periodic boundary conditions in the \( y \) direction. We have chosen free-slip instead of no-slip because our grid is not fine enough to resolve the boundary layers that may develop. Physically, this makes the solid boundaries non-sticky which allows for the fluid to flow along the wall. Even though the no-slip condition is valid at the microscopic scales, there is no reason to believe that it is the appropriate one when describing large-scale fluid
dynamics, like the ones we are interested (see Pedlosky (1996)).

This numerical code is used with two objectives in mind. First, we compare the growth rates from the simulations to those predicted by the linear stability analysis in order to determine whether linear theory predicts the initial developments of the instability. Second, we study parameter choices in the non-QG regime to better understand what asymmetries may arise between cyclones and anticyclones. We consider mainly monochromatic perturbations, not because the physical world behaves in this simple manner, but because it presents a cleaner picture of the instability processes; monochromatic perturbations are those that contain only one wavelength in the along channel direction.

d. Basic State

We take the jet to be a geostrophic Bickley Jet, see Figure 2-2,

\[
\tilde{\eta} = -\Delta \eta \tanh \left( \frac{x}{L} \right) 
\]

\[
\tilde{u} = 0 
\]

\[
\tilde{v} = -\frac{g' \Delta \eta}{fL} \text{sech}^2 \left( \frac{x}{L} \right) . 
\]

Note that the fluid is deeper in the left portion of the channel; this is true in all of the numerical simulations throughout this chapter. The parameter \(\Delta \eta\) denotes the maximum amplitude of the surface deformations from the mean and \(L\) is the length scale of the width of the jet. In the presence of viscosity the geostrophic state is no longer an exact solution; however, since the friction coefficient is very small, the geostrophic state is a very good approximate solution.

In our simulations, \(L = 1/10\) and \(f = 10\). Therefore, the Rossby and Froude numbers simplify to \(Ro = \frac{U}{\Delta \eta/F \sigma} = 1/g'\) and their product is equal to the amplitude of the surface variations \(\Delta \eta\).
2. Linear Stability Analysis

a. Ripa’s Theorem

Two criteria are derived in Ripa (1983) which together are sufficient to guarantee the linear stability of a purely zonal flow in a reduced-gravity shallow water model on a $\beta$-plane; they are referred to as Ripa’s theorem. A weak version of Ripa’s theorem, customized to flows with $\bar{u} = 0$ and $\bar{v}$ semi-definite in sign, states that the flow will be stable when

$$\frac{d}{dx} \left( \frac{\bar{v}_x + f}{H} \right) \neq 0$$

(2.13)

and

$$|\bar{v}| \leq \sqrt{g' H}$$

(2.14)

where $H = 1 + \bar{\eta}$ is the total depth of the basic state. The first criterion requires that the PV not possess a local extrema. This is equivalent to the classical Rayleigh stability condition of QG dynamics (see Pedlosky (1987)). The second criteria requires that the velocity at every point be bounded above by the surface gravity wave speed at that same point. In the QG limit, the surface gravity wave speed is in effect infinite, and therefore (2.14) is trivially satisfied. It is conjectured in Ripa (1991) that if only the first of these statements is violated, the PV gradient of the mean flow generates the instability and hence Rossby
waves are generated. Conversely, if only the second condition is violated the unstable modes that grow are inertio-gravity waves. These two types of instability are shear and inertial instability, respectively. In all of our simulations, (2.13) is violated and therefore Rossby waves are expected to develop. The second condition, (2.14), is satisfied for most of our simulations but is violated when $\Delta \eta$ is sufficiently large. Hence, highly non-QG instabilities should contain both shear and inertial instabilities.

b. Linearized Perturbation Problem

Initially, the numerical simulations perturb the basic state with waves of amplitude at least two orders of magnitude smaller than the basic state itself; this insures that the initial dynamics should be governed by the linearized equations. We use these equations to derive predictions of the growth rates in the linear regime and then compare them with those calculated from the numerical simulations. A correspondence between the two indicates that the linear theory accurately describe the initial stages of instability.

In Flierl et al. (1987) it was determined that linear instability indeed describes the initial process of growth for a barotropic QG jet. To demonstrate that this is true for the SW equations, it is necessary to solve the associated linear stability problem. This has been solved for various parameter regimes, see for example Hayashi and Young (1987), Paldor and Ghil (1997), Balmforth (1999), Li and Mcclimans (2000), Stegner and Dritschel (2000) and Baey et al. (1999). Unlike previous work, we design a method by which to solve the linear stability problem for the reduced gravity SW equations for arbitrary Rossby, Froude and Reynolds numbers.

c. Method to Solve the Linear Stability Problem

The equations that govern the evolution of linear perturbations are obtained by first perturbing the basic state.
The geostrophic solution depends only on the cross channel coordinate, but we study its stability relative to two dimensional perturbations. For our convenience, instead of linearizing the SW equations in transport form, those being the form we used in the numerical simulations, we linearize them in their canonical form, (2.1) and (2.2) with Laplacian friction included in each equation with a coefficient of $\nu$. Using a different form of friction should not alter the results significantly since frictional effects are small due to the large Reynolds numbers we used in the simulations.

We substitute (2.15) into the viscous SW equations and linearize to obtain the following linear equations:

$$
\begin{align*}
\frac{\partial \eta'}{\partial t} + \frac{\partial (Hu')}{\partial x} + \bar{v} \frac{\partial \eta'}{\partial y} + H \frac{\partial u'}{\partial y} &= \nu \nabla^2 \eta' \tag{2.16} \\
\frac{\partial u'}{\partial t} + \bar{v} \frac{\partial u'}{\partial y} - fu' + g' \frac{\partial \eta'}{\partial x} &= \nu \nabla^2 u' \tag{2.17} \\
\frac{\partial v'}{\partial t} + \frac{d\bar{v}}{dx} u' + \bar{v} \frac{\partial v'}{\partial y} + fu' + g' \frac{\partial \eta'}{\partial y} &= \nu \nabla^2 v'. \tag{2.18}
\end{align*}
$$

The perturbations are decomposed into a sum of normal mode solutions, each of the form

$$
\begin{bmatrix}
\eta' \\
u' \\
v'
\end{bmatrix} = \text{Real} \left\{ \begin{bmatrix}
\hat{\eta}(x) \\
iki \hat{u}(x) \\
\hat{v}(x)
\end{bmatrix} \exp(iki(y-ct)) \right\} \tag{2.19}
$$

The wavenumber and phase speed corresponding to motion in the $y$ direction are denoted by $k$ and $c$ respectively and the amplitude variables, $\hat{\eta}$, $\hat{u}$ and $\hat{v}$, are complex functions of $x$. The factor $iki$ is incorporated into the decomposition for $u'$ since the resulting system is real for the inviscid case.

We determine the equations that govern the evolution of $\eta'$, $u'$ and $v'$, by substituting (2.19) into (2.16) to (2.18), and then simplify to obtain
This is the Rayleigh equation for the viscous shallow water model for a one-dimensional geostrophic basic state (see Pedlosky (1987)). Equation (2.20) is written in matrix form where the phase speed appears linearly as the eigenvalue of the differential matrix operator. We are unable to determine the eigenvalues of the matrix analytically and alternatively, we apply a spectral collocation method to discretize the system (see Trefethen (2000)). The eigenvalues of this resulting system are then computed quite simply in MATLAB.

The boundary conditions are \( \hat{u} = \frac{d\hat{v}}{dx} = \frac{d\hat{\eta}}{dx} = 0 \) at \( x = -1 \) and \( x = 1 \). It was determined that as the number of collocation points increased, the growth rates converged. There were spurious roots as is typical of spectral schemes but they did not have any positive growth rates and hence they did not affect our calculations. By using 200 collocation points we usually obtain convergence to within three significant digits.

d. Results of the Linear Analysis

We determine the functional dependence of growth rates on Rossby and Froude numbers by solving the linear stability problem and then calculating the growth rates for a wide range of parameters. The perturbations were of the form \( \sin(kx) \) with wavenumbers \( k = \pi, 2\pi, ..., 6\pi \) since they are the first six modes that fit in the channel and tended to be the most unstable. Figure 2-3 is a contour plot of the growth rate normalized by the Rossby number, what we refer to as the relative growth rate. For Froude numbers near the barotropic limit, \( Fr < 0.1 \), the relative growth rate is nearly independent of Rossby Number. For larger Froude numbers the relative growth rate decreases with increasing Rossby number, the effect of which is strengthened as the Froude number increases. It is clear that the QG model yields the largest relative growth rates with respect to the Rossby number. Moreover, the barotropic limit is always the most unstable as found in the stability of vor-
tics (see Stegner and Dritschel (2000)). This implies that the QG model overestimates the growth rates; as does the barotropic assumption. Therefore, the free surface acts to stabilize the unstable jet.

Figure 2-3: A contour plot of the relative growth rate (growth rate/Ro) with respect to the two nondimensional parameters, \( Ro = U/(fL) \) and \( Fr = (fL)^2/(gH) \).

3. Numerical Simulations

In this section we present the results of the nonlinear evolution of a perturbed unstable jet, (2.10), (2.11) and (2.12), in the SW model for small, intermediate and large Rossby numbers to explore cyclone-anticyclone asymmetry. The simulations of small Rossby number reproduce the QG results. The other cases have implications to jets in the ocean that do not have small Rossby numbers, examples of which can be found in Olson (1991). The intermediate and large Rossby number simulations are also important in other vortex phenomenologies such as hurricanes, tornadoes and mesoscale storms.

We plot six snap shots of the relative vorticity of each simulation where the solid and dashed lines are positive and negative vortices, respectively. In all simulations the flow is down the page with the shallow fluid on the right. For several particular cases we also plot
the maximum of the cyclonic and anticyclonic vorticity as well as the position in the across channel direction. These two combined illustrate when the vortex tubes are stretched or contracted.

a. Small Rossby Number

Ro = 0.1 and Fr = 0.1 Figure 2-4 is similar to the f-plane simulations in Flierl et al. (1987). The perturbations on the unstable jet grow in amplitude until the nonlinearities become significant, and the waves roll up and break. This nonlinear breaking generates vortices in the center which form into a vortex street but also injects fluid across the jet that form into dipoles. This injection is an important means through which fluid is transported across the jet between the deep and shallow waters. The presence of dipoles reflects that transport has occurred. Typically, a more unstable jet will transport more fluid in comparison with a more stable one.

Until subharmonic instabilities develop (see Saffman and Schatzman (1981)), the vortices are symmetric in size, strength and shape. There are two types of structures that are observed. The first are called outer pools; they are the circular shapes that originate on the outskirts of the jet but then settle down to form the circular vortices in the vortex street. The second are inner pools, which form near the jet but are injected as filaments across the street.

Figure 2-5 plots the extrema of vorticity and their cross-channel positions versus time. These curves track the strength and position of the outer pools since they are the strongest sources of vorticity. Initially, the vorticity decays but then oscillations develop where the extrema increase and decrease in unison. This stretching and contracting of vortex-tubes occurs because of the horizontal translation across the jet where the free surface is not uniform. Observe that as the vortices increase (decrease) in strength both the cyclones and anticyclones approach (recede from) the center of the jet. The symmetry in size, strength and shapes of the cyclones and anticyclones indicates that the QG approximation is a very good one for this choice of parameters.
Figure 2-4: $Ro = 0.1$ and $Fr = 0.1$: this evolution is very similar to the $f$-plane simulations of Flierl et al. (1987). The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$ overlying a flat bottom. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 
Figure 2-5: a) amplitude and b) position of the vorticity extrema for $Ro = 0.1$ and $Fr = 0.1$. The symmetry in strength of the cyclones and anticyclones is apparent.

$Ro=0.1$ and $Fr=0.5$ Figure 2-6 presents snapshots of vorticity where free surface effects are more pronounced. In contrast to Figure 2-4, the inner and outer pools do not manage to pinch off and separate. Instead, the inner and outer pools merge to form a very compact triangular shaped vortex street; this is usually referred to as a cats-eye pattern (see Drazin and Reid (1995)). The similarity of this street to those obtained in Flierl et al. (1987) with a nonzero $\beta$ parameter indicates that an increase in Froude number stabilizes the flow in a manner similar to an increase in the $\beta$ parameter. This stabilization prevents the injection of the inner pools and therefore strongly reduces, if not entirely eliminates, the transport of fluid across the jet.

Figure 2-7 depicts that asymmetries between positive and negative vortices are apparent even for small Rossby numbers. The strengths of the cyclones and anticyclones differ and their translations about the center of the jet are not symmetric.
Figure 2-6: $Ro = 0.1$ and $Fr = 0.5$: the stabilizing effect of the Froude number is similar to that of the $\beta$-plane since they both act to make vortices triangular in shape. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 

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Figure 2-7: a) amplitude and b) position of the vorticity extrema for $Ro = 0.1$ and $Fr = 0.5$. Asymmetries in vortex strength are clear.

b. **Intermediate Rossby Number**

$Ro=1.0$ and $Fr=0.1$ The simulation in Figure 2-8 is barotropic, the Froude number is small, but no longer in the QG parameter regime since the Rossby number is not small. The anticyclonic and cyclonic structures resemble Figures 2-4 and 2-6, respectively. This suggests that the cyclonic fluid is preferentially stabilized more than the anticyclonic one; this asymmetry in the stability of the jet is in contrary to the stability of the vortices themselves (see Stegner and Dritschel (2000)). As in Figure 2-4, both the cyclonic and anticyclonic fluids separate into inner and outer pools. The cyclonic outer pools re-attach temporarily by frame five and then finally manage to achieve final separation by frame six. This re-attachment is additional evidence that the cyclones are weaker than the anticyclones hence more stabilized. The final frame shows that the anticyclonic outer pools are larger and stronger than the cyclones. The anticyclones are still circular in shape, in contrast to the cyclones that
Figure 2-8: $Ro = 1.0$ and $Fr = 0.1$: asymmetry in the shape of vortices is apparent. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 

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have a circular core but have a wedge shape on the left hand side of the outer most contour. This wedge structure will be shown to be characteristic of the cyclonic fluid for intermediate and large Rossby numbers. This particular choice of parameters lies in a regime where the anticyclones are the stronger vortices that form. The cyclones and anticyclones injected are relatively stronger than in those Figure 2-4 which thereby generates stronger dipoles.

![Figure 2-9: a) amplitude and b) position of the vorticity extrema for $Ro = 1.0$ and $Fr = 0.1$. The anticyclones are stronger than the cyclones.](image)

Figure 2-9 illustrates how the anticyclones are stronger than the cyclones because the street has experienced a net shift towards deeper waters. This shift moves the anticyclones (cyclones) into a region where the gradient of the free surface is weaker (stronger). Therefore, even though both sets of vortices oscillate in the across channel direction at nearly the same amplitude, the cyclones experience stronger vortex tube stretching and therefore more oscillations in Figure 2-9(a).
Ro=1.0 and Fr=0.5  Figure 2-10 is the first simulation where the amplitude deformations are no longer small. Consequently, strongly non-QG instabilities arise. The anticyclonic fluid develops outer pools and relatively weak inner pools, as can be seen in the final frame. These inner pools do not pinch off as in Figures 2-4 and 2-8, nor do they merge as in Figure 2-6. Instead, they remain between the cyclonic outer pools on the edge of the front a fixed distance from the center of the jet. They cannot be injected into the shallow waters since the cyclonic fluid does not leave any space for them to pass through, thereby impeding the fluid transport across the jet. This is a result of the stabilizing effects of baroclinicity.

The evolution of the cyclonic fluid resembles Figure 2-6 in that the inner pools are the structures that emerge early on. Unlike Figure 2-6, the outer pools never form: the only cyclones generated are the inner pools. However, they are unusual vortices since they are boomerang shaped. The end result is a highly asymmetric vortex street where the cyclones and anticyclones possess completely different size, strengths and shapes. By plotting the extrema we determined that by the final frame the cyclones and anticyclones have vortices larger and smaller than 10 and –6 units, respectively.

Ro=1.6 and Fr=0.5  In the simulation presented in Figure 2-11 the anticyclonic fluid behaves in nearly the same manner as in Figure 2-10. However, the cyclonic fluid is more boomerang in shape since the back of these vortices are indented even more than in Figure 2-10. We speculate that these structures are due to gravity wave instabilities that arise because of Ripa’s second criterion being violated. The asymmetries in the along channel direction develop by frame two where the top portions of the cyclones extend further back. These extensions grow and eventually form feet like structures. In frame four we see the origins of an anticyclonic pool between these feet. This fluid grows because of anticyclonic fluid that seeps through the street into shallower waters: this transport of fluid was observed by plotting the evolution of PV; which is advected following the motion of the fluid. The end result is rows of anticyclones and cyclones that lie on the shallow side of the street. We speculate that the formation of these feet-like structures are due to gravity waves.

We observe that the trend with increasing nonlinearities is for cyclones to be less circular and more elongated. Arai and Yamagata (1994) studied the evolution of two dimen-
Figure 2-10: $Ro = 1.0$ and $Fr = 0.5$: in this non-QG regime the cyclones are stronger and boomerang-shaped. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 
Figure 2-11: \( Ro = 1.6 \) and \( Fr = 0.5 \): asymmetries of the cyclones in the along channel direction have developed. The across channel and along channel coordinates correspond to the \( x \) and \( y \) axis, respectively. The free surface decreases monotonically with increasing \( x \). These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to \( Ro \).
sional turbulence in the SW model and found that the cyclones are more elongated and, in the highly nonlinear regimes, they can split in two. The cyclones in Figure 2-11 are indeed filament-like and appear to be near the point of splitting but this does not occur.

Figure 2-12: a) amplitude and b) position of the vorticity extrema for $Ro = 1.6$ and $Fr = 0.5$. The strength of the cyclones is far beyond that of the anticyclones.

The vorticity extrema plots for Figures 2-10 and 2-11 are very similar hence we only present the later, Figure 2-12. The plot of the position shows that there is a net shift of the system towards deeper water which, through vortex tube stretching, generates asymmetrically stronger cyclones. The large difference in free surface height causes the vortices movement into deeper waters. The system could not have moved in the other direction since then the anticyclones would be contracted as much as these cyclones are stretched. But, the gradient wind balance indicates that arbitrarily strong anticyclones cannot exist while no such restriction exists on cyclones.
c. Large Rossby Number

*Ro=5.0 and Fr=0.1* In Figure 2-13 the anticyclonic fluid forms both inner and outer pools where the inner pools pinch off and are injected into the shallow region. The cyclonic fluid generates inner pools which are first elongated and then settle into a triangular vortex with feet-like extensions. These triangular cyclones have been previously observed in both Flierl et al. (1987) and Arai and Yamagata (1994) and are the barotropic versions of the boomerang shaped objects observed in Figure 2-10. At the head of the cyclones, small portions of fluid are peeled off by the anticyclones and injected into the deep waters.

The feet shaped protrusions and anticyclonic inner pools are parallel and drift downstream towards shallow water. These wave patterns indicate that Rossby waves are being radiated from the jet; note that they travel with the shallow fluid on the right. The feet pinch off after frame four to join the anticyclonic inner pools to form dipoles. The length of the feet in Figure 2-13 enables them to detach from the vortex street, unlike Figures 2-10 and 2-11 where protrusions appear but they do not elongate sufficiently to detach.

*Ro=8.0 and Fr=0.1* The larger Rossby number in Figure 2-14 adds some additional ageostrophic effects that did not arise in the previous runs. Since the maximum of the vorticity contours become larger than twice the magnitude of the minimum we chose the positive contour increment to be twice that of the negative increments. As the thin-elongated cyclones travel into deeper fluid they are injected across the street, thereby transporting shallow fluid into the deep waters. The larger deformation amplitude causes the vortex tubes to be stretched more than in Figure 2-13 which is what enables this injection to occur. The injected filaments roll up to form circular vortices. These strong cyclones peel off some anticyclonic fluid that initially forms rings around the cyclones but then they settle to form dipoles. The cyclonic fluid left behind to form the street are triangular in shape, as was previously seen in Figure 2-11.
Figure 2-13: $Ro = 5.0$ and $Fr = 0.1$: in the case of large free surface deformations and small Froude number, the cyclones become triangular in shape. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$; deep water on the left and shallow water on the right. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 

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Figure 2-14: $Ro = 8.0$ and $Fr = 0.1$: an example of a highly non-QG jet that injects a strong cyclones into the deep waters. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$; deep water on the left and shallow water on the right. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. The snapshots are taken at times $t \times Ro = 8.5, 12.5, 17.4, 19.0, 21.5$ and 26.0.
In order to consider more realistic perturbations we present the results from two sets of polychromatic perturbations. We perturb the basic state with the first six along channel modes, each mode is of the same amplitude. The parameters chosen correspond to those in Figures 2-10 and 2-13, respectively.

**Ro=1.0 and Fr=0.5** Since there are six vortices in Figure 2-15 during the early stages of growth from which, we deduce that \( k = 3\pi \) is the most unstable mode. The evolution of the instability is as in 2-10 except that the crests and troughs are no longer of equal strength because of the polychromatic nature of the solution. The growth of sub-harmonic instabilities causes the merger that takes place (see Saffman and Schatzman (1981)). The merger occurs because the mode three vortex street is unstable to perturbations in mode two. The end state is a vortex street with two sets of cyclones and anticyclones with vortices that are asymmetric in size, strength and shape.

**Ro=5.0 and Fr=0.1** Our last simulation is presented in Figure 2-16. The most unstable mode is mode two. It gives rise to dipole type instabilities (see Flierl et al. (1987)); a vortex street never materializes. This mode is more unstable than mode three, and manages to inject a strong cyclone into the deep waters; something that did not occur in Figure 2-13. This demonstrates how the instability of the vortex street to subharmonics can also aide to transport more fluid across the jet. This transport generates very strong cyclones because the vortex tube is stretched a great deal because of the large free surface deformations.

Both of these polychromatic runs show that the anticyclonic eddies that form tend to be circular in shape. The cyclones that form in Figure 2-15 tended to be boomerang-shaped as in some of the previous simulations. As well, the cyclones of Figure 2-16 were filamented as in associated monochromatic perturbations. These two runs demonstrate that the structures that arise in monochromatic runs also arise in polychromatic runs.
Figure 2-15: $Ro = 1.0$ and $Fr = 0.5$: a vortex street emerges and it is highly asymmetric in form. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$; deep water on the left and shallow water on the right. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 
Figure 2-16: $Ro = 5.0$ and $Fr = 0.1$: strong cyclones are injected into the deep waters. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$; deep water on the left and shallow water on the right. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. The snapshots are taken at times $t \times Ro = 9.0, 11.0, 15.0, 17.5, 22.0$ and $27.0$. 
4. Growth Rates

If we substitute (2.15) into the integral for the total kinetic energy and exclude the kinetic energy of the geostrophic state, we find that the perturbation kinetic energy is the sum of the following three integrals,

\[
\int_{-1}^{+1} \int_{-1}^{+1} \bar{h} \bar{v}' + \frac{1}{2} \bar{v}^2 \bar{h}' \, dx \, dy
\]  
(2.21)

\[
\int_{-1}^{+1} \int_{-1}^{+1} \bar{v} \bar{v}' \bar{h}' + \frac{1}{2} \bar{h}(u'^2 + v'^2) + \frac{1}{2} \bar{h}'(u'^2 + v'^2) \, dx \, dy
\]  
(2.22)

\[
\int_{-1}^{+1} \int_{-1}^{+1} \frac{1}{2} \bar{h}'(u'^2 + v'^2) \, dx \, dy.
\]  
(2.23)

If the perturbations are small the motion is governed by the linear dynamics and the solution can be decomposed into a sum of normal modes (2.19). This decomposition implies that the integral of any field, linear or cubic in the perturbation, is identically zero. When the linear theory breaks down this result will no longer be true since the decomposition of modes is no longer possible. Therefore, we chose the quadratic term (2.22) as the measure of the growth rate since it contains terms that do not vanish in the linear regime (see Hayashi and Young (1987)). The calculations of the logarithm of the perturbation kinetic energy terms are shown in Figure 2-17.

From these plots it is straightforward to calculate the lines that best fit the curves of exponential growth. The slopes of these lines give the growth rates of the nonlinear simulations, the results of which are presented in table 2.1. The values obtained from the numerical simulations are within twenty percent of the theoretical predictions except for the cases where the Rossby number is 0.1.

In all four cases of Figure 2-17 the initial curve is slightly oscillatory but, soon after, becomes a smooth curve which is nearly linear. The straightness of these lines support the hypothesis that there is exponential growth. The closeness of the growth rate calculations, combined with the linearity of the curve, strongly suggests that linear growth is indeed the mechanism that dominates the initial stage of the instability.

One source of discrepancy in the two growth rates is because of diffusion. First, our
Figure 2-17: The logarithm of the Growth Rates for \( Fr = 0.1 \) and various Rossby Numbers. Each curve initially levels out to a linear curve, the slope of which yields the growth rate.

diffusion terms are slightly different in the linear calculation and the nonlinear scheme. Second, there is implicit diffusion in the numerical scheme that is not accounted for in the instability calculations. This implies that the growth rates for the nonlinear experiments should be less than the predicted values which is indeed the case for all but one simulation. Also, the small Rossby number simulations which have the largest time scales should have the greatest discrepancies, which they do.

Another source of error is that the system is not truly linear. The small nonlinear terms can affect the growth rate by introducing other waves into the system. The presence of multiple modes alters the growth rate since the additional modes grow at a different rate and hence alter the overall growth of the system. This conjecture is supported by our polychromatic perturbations that consistently yield growth rates smaller than their monochromatic equivalents.

It is of interest to note that RUN11 has a growth rate larger than that predicted by linear theory. The difference is not very large but this is qualitatively different from the other runs which undershoot the theoretical values. This suggests that perhaps linear instability is not the strongest mechanism at work in this simulation and that nonlinear instability may be important.
### Table 2.1: A summary of the results from the numerical simulations of the instability of a geostrophic jet over a flat bottom.

<table>
<thead>
<tr>
<th>RUN</th>
<th>Fr</th>
<th>Ro</th>
<th>$\Delta \eta$</th>
<th>$\sigma_{\text{num}}$</th>
<th>$\sigma_{\text{lin}}$</th>
<th>type</th>
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<td>0.1</td>
<td>0.1</td>
<td>0.01</td>
<td>1.04</td>
<td>1.43</td>
<td>Mono</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.2</td>
<td>0.02</td>
<td>1.22</td>
<td>1.41</td>
<td>Mono</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
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<td>0.1</td>
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5. Conclusions

In chapter two we studied the formation of vortices through the barotropic instability of a geostrophic jet in the context of the SW model. It was determined that the linear theory predicts the initial development instability in the nonlinear simulations. The increase of Rossby number decreases the relative growth rates signifying that QG overestimates the instability of jets in non-QG regimes. An increase of Froude number stabilized the flow. As anticipated, the results of the nonlinear simulations for small Rossby number compare very well to QG theory. We ventured into the non-QG regime to observe the cyclone-anticyclone asymmetries in eddy generation due to finite deformations in the free surface. The anticyclones were always circular in shape and were larger than the cyclones. For small Froude number and moderate Rossby number the anticyclones were stronger than the cyclones. The case of large free-surface deformations, the cyclones were clearly stronger and were triangular and boomerang in shape. Therefore, there is a transition from moderate to large Rossby numbers in that the strongest vortices change from anticyclones to cyclones; the parameter choice is important in determining which vortices are stronger.
Linear instability theory predicts the initial development in all of the simulations. The increase of Rossby number decreases the relative growth rates signifying that QG overestimates the instability of jets in non-QG regimes. An increase of Froude number stabilized the flow.

In a following chapter we study the instability of a jet overlying a smooth shelf. There, we will investigate how the instability is affected by changes in magnitude, direction and width of the topography. This will have implications to coastal problems where the flow is along the shelf.
Chapter 3

The Effect of Topography on the Stability of Jets

Section 3.1 explains the model that we use to study the instability of a geostrophic jet overlying step-like topography; this consists of the governing equations and the particular jet and topographic profiles. Then in §3.2 we present the results of the liner stability analysis to learn how the instabilities are affected by the nondimensional parameters. Finally, §3.3 explains the differences in the numerical method compared to Chapter 2 and shows the results from the nonlinear evolution of the instability that verifies the dependency predicted from the linear theory.

1. The Model

In this chapter we aim to further our understanding of the stability of jets that flow along topographic ridges near coastlines. Instead of dealing with the many intricacies involved in realistic models, we instead consider an idealized setting which aids us to better focus on the physical processes at work. In particular, we assume the geometry of a periodic channel with smooth, one-dimensional topography of the form,

\[ h_B = H \left( 1 - |\beta| + \beta \tanh \left( \frac{x}{\alpha L} \right) \right). \]  

(3.1)
The coordinates $x$ and $y$ are directed in the across shelf and along shelf directions, respectively. The parameters $L$ and $H$ denote the length and depth scales of the system, which we will take to be one. Therefore, $\alpha$ and $\beta$ specify the relative width and amplitude of the topography and are nondimensional parameters.

To describe the instability of a jet, it would be advantageous to assume that the motion is Quasi-Geostrophic (QG). However, this restricts the fluid to small amplitude deformations, where we must allow for finite amplitude topography and free-surface variations. Hence, QG is an inadequate model for our motions of interest. Instead, we assume the motion is governed by the fully nonlinear SW equations (see Pedlosky (1987)). The inviscid, reduced gravity, SW model in dimensional form is governed by the same set of equations as in Chapter 2,

\begin{align}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f \hat{k} \times \mathbf{u} &= -g' \nabla (h - h_B) \\
\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) &= 0.
\end{align}

The two parameters, $g'$ and $f$, are the reduced gravity, because of the infinitely deep upper layer, and the Coriolis parameter, both of which we set to be constant. The variables $\mathbf{u} = (u, v)$, $h$ and $h_B$ denote the horizontal velocity field, total depth and bottom topography respectively. The five nondimensional parameters are the Rossby number $Ro = U/(fL)$, Froude number $Fr = (fL)^2/(g' H)$, numerical Reynolds number $Re = UL/\nu$ for some friction coefficient $\nu$, topographic width $\alpha$ and height $\beta$. Chapter 2 used this model to study the effects of finite amplitude variations in the free surface in the absence of any topography. This chapter is a direct extension of chapter two, focusing on topographic, rather then free-surface, variations. It is seen that there are some similarities in the structures that develop in these two cases.

The alongshore flow is a geostrophic Bickley Jet which only varies on the across shelf coordinate, $x$;
\[ \tilde{\eta} = -\Delta \eta \tanh \left( \frac{x}{\ell} \right), \]  
\[ \bar{u} = 0, \]  
\[ \bar{v} = -\frac{g^\prime \Delta \eta}{f \ell} \operatorname{sech}^2 \left( \frac{x}{\ell} \right). \]

The parameter $\Delta \eta$ denotes the maximum amplitude of the surface deformations from the mean which we take to be positive, $\ell$ is the length scale of the width of the jet. Note that equations (3.5) and (3.6) imply the motion is strictly along shore. If $\beta$ is positive or negative the motion is prograde or retrograde, respectively. It will be seen that this parameter is instrumental in determining the stability of the jet. Figure 3-1 illustrates the two possible topographic configurations.

![Figure 3-1](image-url)

**Figure 3-1:** These profiles show slices through the channel in the $x$ direction. Since the flow is always in the negative $x$ direction, out of the page, the two configurations above denote prograde and retrograde topography, respectively.

# 2. Linear Stability Analysis

Before we delve into the nonlinear dynamics we investigate the linear regime since it reveals how the stability of the jet depends on the given parameters. The governing equation
of the linear perturbation dynamics is the same as in chapter two except that $H(x) = h_B(x) + \tilde{\eta}(x)$

$$
\begin{bmatrix}
(\bar{v} - i k \nu + \frac{u}{k} \frac{d^2}{dx^2}) & \left( \frac{dH}{dx} + H \frac{d}{dx} \right) & \frac{H}{k^2} \\
-\frac{g'}{k^2} \frac{d}{dx} & (\bar{v} - i k \nu + \frac{w}{k} \frac{d^2}{dx^2}) & \frac{f}{k^2} \\
g' & \left( \frac{d\tilde{v}}{dx} + f \right) & (\bar{v} - i k \nu + \frac{w}{k} \frac{d^2}{dx^2})
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta} \\
\tilde{u} \\
\tilde{v}
\end{bmatrix}
= c
\begin{bmatrix}
\tilde{\eta} \\
\tilde{u} \\
\tilde{v}
\end{bmatrix}.
$$

(3.7)

This is the Rayleigh equation for the viscous SW model for a one-dimensional basic state in geostrophic balance overlying one-dimensional topography. The parameter $k$ is the wavenumber and $\tilde{\eta}$, $\tilde{u}$ and $\tilde{v}$ are the perturbation variables. Our numerical code solves this problem using a spectral scheme (see Trefethen (2000)).

The solid lines in Figure 3-2 represent contours of the growth rates normalized by the Rossby number at $Fr = 0.1$ and $\alpha = 0.2$ for a wide range of Rossby numbers and topographic heights; the small Froude number is chosen to simulate a flow that is nearly barotropic but not exactly so. The two dashed lines denote the curves, beneath which the PV of the basic state do not change sign anywhere throughout the domain. This is sufficient to guarantee that the basic state is linearly stable. We see that that these curves are close to where the zero contours should appear. Therefore, the Rayleigh’s criteria for instability is close to being a sufficient condition for instability. Note that we do not plot the zero contour because numerical error makes it difficult to calculate it precisely.

Figure 3-2 can be interpreted to be split into four sectors. The two outer most sectors are the regions beyond the dashed lines; this is the region in parameter space where the topography completely stabilizes the flow. The two inner sectors of the unstable region are separated by a curve stemming out from the positive $\beta$ axis with positive slope which defines the line of maximum relative growth. We have not calculated this line but instead we have drawn it in a dashed-dotted pattern where it should approximately be situated. Observe that this line should slice the 1.8 contour loop in two even parts and then extend outwards in the same direction. On both sides of this ray the growth rates are monotonically decreasing in moving away from the maximal ray. By calculating stability plots for other topographic widths, we have observed that this division of Figure 3-2 into these four different sectors is
Figure 3-2: Growth Rate of $\beta$ versus $Ro$ for $Fr = 0.1$, $\alpha = 0.2$ and $\nu = 0$. Observe that retrograde topography ($\beta < 0$) is strictly stabilizing in contrast to prograde topography ($\beta > 0$) that can be destabilizing or stabilizing. For growth rates between 0.1 to 1.5, the contour interval is 0.2, whereas above 1.5 the interval is 0.05.

characteristic of this type of topography, with the maximal ray having a positive slope.

The division of the plot into these various sectors has several implications on how topography affects the stability of jets. First, retrograde topography ($\beta < 0$) always stabilizes the jet and, for small enough values of the Rossby number, the jet can be completely stabilized. This is what is predicted in the barotropic limit, i.e. $Fr = 0$, of Li and McClimans (2000). Second, prograde topography ($\beta > 0$) can either stabilize or destabilize the jet depending on its height and the Rossby number. There are three cases. For the first, large values of the Rossby number, the flow is always destabilized. However for the second, intermediate values of $Ro$, the flow is destabilized up to a critical value of $\beta$ beyond which the growth rate decreases monotonically and hence the flow is said to be stabilized. For the third, small Rossby number, there are regions of destabilization, stabilization and even a region where the flow is entirely stabilized.

This complicated dependence of stability on topography indicates that the magnitude of the topography is crucial in determining the stability of the flow. This is in contrast to the barotropic limit in Li and McClimans (2000). They determined that both types of
topography stabilized the flow. They did this by calculating the neutral curves between stability and instability, which are symmetric about $\beta = 0$. However, they did not calculate the growth rates within the unstable regions and it is conceivable that the same sort of behavior arises there as we see here. Indeed, the fact that they found that the prograde topography had more unstable modes suggests that there is an asymmetry in the stability of the two flows.

Figure 3-3 focuses on the barotropic region of the previous figure and can be seen as the QG reduction of our problem. Clearly, in QG a positive $\beta$ stabilizes and negative $\beta$ destabilizes or stabilizes for small or large values, respectively. However, it need be remarked that the QG approximation only holds for small amplitude topography, so that the prediction for large topographic slopes is extrapolating beyond its range of validity. In contrast, the SW model can describe these regimes without any difficulty.

Figure 3-3: This figure is precisely the same as Figure 3-2 except that it focuses in on the barotropic region, $0 < Ro < 0.1$.

The reason why retrograde topography stabilizes the jet is due to the fact that Rossby waves have a preferred direction of travel; topographic Rossby waves travel with the shallow water on the right (see Pedlosky (1987)). The speed of the waves increases as the
height of the topography increases. The instability process can be interpreted as a resonance between the mean flow and a Rossby wave. In the case of retrograde topography, the waves travel in the opposite direction as the jet which is why it is more difficult for the resonance to arise. Very steep retrograde topography makes the phase speed of the waves too fast to resonantly interact with the mean flow, which is why these particular mean flows are stable (see Schmidt and Johnson (1997)).

Results from a series of linear stability calculations for various topographic heights and slopes are presented in a contour plot in Figure 3-4 for two different Rossby numbers. In the first plot, for a given slope, the growth rate is an increasing function of the parameter except in the regions of large and small slopes. Therefore, decreasing below zero always stabilizes whereas, increasing above zero usually destabilizes. The case of behaves in the same fashion but it has the added feature that the stabilizing region of prograde topography is larger. It is not shown here but, if the plot is extended to larger slopes we would discover a region where the flow is stabilized entirely. The QG limit is similar to the second plot of Figure 3-4 for the case of small slopes and topographic variations. As is shown in the plot, in this regime there can be no stabilizing effects of
prograde topography.

Figure 3-5: a) The growth rates predicted by solving the linear stability problem with $Fr = 0.1$ and $\alpha = 0.2$, for various topographic heights and Rossby numbers. b) A slice through $\beta = 0.0$ and c) a slice through $\beta = 0.4$. The flat bottom case is symmetric in contrast to the case of uneven topography.

To conclude this section we investigate what implications the stability analysis has on the tidally forced problem that is addressed in Chapter 5. The jets that arise because of the tides and Coriolis force are complicated since they are time-dependent and they are not unidirectional. However, if we consider for the moment the simpler case of oscillatory unidirectional flow along a step, we observe that as the shear reverses it changes from prograde to retrograde flow and then back again. For a fixed topographic height the oscillations are equivalent to oscillations in the Rossby number: positive and negative Rossby numbers signify prograde and retrograde flow respectively. Figure 3-5a) presents the linear growth rates of $\beta$ versus $Ro$. The oscillatory flow we have just described corresponds to moving back and forth along a horizontal slice of this plot. The two lower plots, b) and c), show the growth rates along a slice for two different values of $\beta$. As is to be expected the case of $\beta = 0$, corresponding to flat topography, has symmetric growth rates. The other case,
β = 0.4, illustrates how the topography does create an asymmetry between prograde and retrograde motion. We anticipate that this asymmetry caused by uneven topography will aide in the development of instabilities. This is something that will be discussed further in the follow chapter.

3. Numerical Simulations

We proceed by exploring the nonlinear dynamics to verify the predictions of the linear stability theory. Moreover, these experiments will illustrate how various topographic heights alter the structure of the finite amplitude waves and eddies. The numerical model is precisely the same as in Chapter 2 except that the $x$ momentum transport equation has an additional term because of the topography beneath,

$$
\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\mathcal{U}^2}{h} + \frac{g' h^2}{2} \right) + \frac{\partial}{\partial y} \left( \frac{\mathcal{U} \mathcal{V}}{h} \right) = f \mathcal{V} + g' h \frac{d h_B}{d x} \tag{3.8}
$$

$$
\frac{\partial \mathcal{V}}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\mathcal{U} \mathcal{V}}{h} \right) + \frac{\partial}{\partial y} \left( \frac{\mathcal{V}^2}{h} + \frac{g' h^2}{2} \right) = -f \mathcal{U} \tag{3.9}
$$

$$
\frac{\partial h}{\partial t} + \frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{V}}{\partial y} = 0. \tag{3.10}
$$

The $x$ and $y$ coordinates are the same as before. The free-surface and bottom topography are located at $z = 1 + \eta$ and $z = 1 - h_B$, respectively. Therefore, the total depth is $h = h_B + \eta$. In the absence of any free-surface deformation and for the case of flat topography the depth of the fluid is uniform, i.e. $h = 1$.

The numerical experiments were aimed at capturing the dependence of growth rates on topographic height as predicted by the linear theory and to discover what asymmetries develop between cyclones and anticyclones because of topography. In all simulations we set $\alpha = 0.2$ and $Fr = 0.1$. In the first set of experiments $Ro = 1.0$ and $\beta$ increases from $-0.4$ to $0.4$ in increments of $0.1$. This range is sufficient to illustrate how prograde and retrograde topography destabilize and stabilize the jet, respectively. The second set of simulations is exactly the same except that $Ro = 0.25$. What is different about this selection...
is that it confirms the prediction that, for small enough Rossby number, sufficiently large prograde topography stabilizes the jet. The first frame in each series of plots denotes which side of the channel has deep and shallow topography.

A synopsis of the growth rates for both series of runs is presented in the two plots of Figure 3-6. These graphs compare the growth rates predicted from the linear theory with those calculated in the nonlinear simulations. The qualitative behavior of the dependence of the growth rates on topographic height is certainly in agreement. The root-mean-square error in the cases of $Ro = 0.25$ and $Ro = 1.0$ are 0.020 and 0.018, respectively. The reason that the error is larger in the first case is that all simulations have the same frictional parameter but the first set of runs have a longer time scale. This produced more net diffusion, both implicit and explicit, and hence a greater discrepancy between the linear and nonlinear calculation than in the runs with $Ro = 1.0$.

![Figure 3-6](image.png)

Figure 3-6: A comparison of the growth rates predicted from the linear theory and those calculated from the nonlinear simulations.

a. Simulations with $Ro = 1.0$

For the case of $Ro = 1.0$ we present results from the two retrograde simulations with $\beta = -0.1$ and $\beta = -0.3$. The evolution of the total vorticity of an unstable jet overlying
Figure 3-7: $Ro = 1.0$ and $\beta = -0.1$: resembles the simulation of the case with $\beta = 0$. The two uppermost plots show the topographic profile. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$ overlying uneven topography. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 
small retrograde topography with $\beta = -0.1$ is illustrated in Figure 3-7; the perturbation is of wavenumber three which is the most unstable mode. The results from this small amplitude topography is qualitatively similar to those instabilities studied in Chapter 2 with a flat bottom and small free-surface deformations. The dramatic differences that occur are as follows. First, the vorticity plots suggest that there is no cyclonic fluid ejected by the street. Indeed, it has been verified by viewing the evolution of the PV that anticyclonic fluid, but no cyclonic fluid, is transported past the street. The vortex street itself is translated to the left, which compensates for the fluid injected into the shallow water. Second, the anticyclonic fluid that is injected into the shallow water is contracted more than in the flat-bottom cases, which creates larger anticyclones than in the $\beta = 0$ situation. This anticyclone thereby peels off a larger portion of cyclonic fluid forming a rather large dipole pair. The implications of these differences is that the slightly retrograde topography stabilizes the jet by preventing any transport into the deep waters, but still allows for injection into the shallow water.

Figure 3-8 presents contour plots of the vorticity field for the case where the topographic height is $\beta = -0.3$. The vortices that form are more triangular in shape than in the flat bottom simulations. These triangular eddies are reminiscent of the stabilizing effect of the $\beta$-plane in (see Figure 2-6). The PV profiles indicate that there is no significant portion of fluid transported across the jet; there might be a small portion injected down the shelf but if so it is very weak. The stabilizing effects of topography create a barrier for across shelf transport. This barrier will not only impede the transport of fluid but of salinity, temperature, biology and chemistry. If this barrier is not established then fluid is interchanged between coastal and deep waters. This can act to either inject nutrients into desolate waters that can save a dying species or it may inject biology into nutrient rich water so that they may flourish. Clearly, this transport may have a significant impact on the life cycles of the coastal and near coastal water biology.

The linear stability analysis stated that prograde topography destabilizes jets. Results from the prograde simulations confirm this and also illustrate how this destabilization is manifested. In particular, Figure 3-9 with $\beta = 0.3$, is the first of the prograde simulations that is significantly different from the flat bottom scenarios of Chapter 2. In this simulation,
Figure 3-8: $Ro = 1.0$ and $\beta = -0.3$: the retrograde topography is larger and it stabilizes instability more dramatically then before. The two uppermost plots show the topographic profile. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$ overlying uneven topography. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 

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Figure 3-9: $Ro = 1.0$ and $\beta = 0.3$: this prograde simulation is destabilized by the topography. The two uppermost plots show the topographic profile. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$ overlying uneven topography. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 

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the arms of the cyclonic (anticyclonic) fluid swing across the center of the jet and the fluid is contracted (stretched) as it moves into shallower (deeper) waters. Moreover, it advects neutral fluid, i.e. fluid of zero relative vorticity, from behind the arm, which is also contracted (stretched) to generate anticyclonic (cyclonic) anomalies, as is apparent in frame $b$). The transportation of this neutral fluid implies that, in this parameter regime, the prograde topography increases the transport across the jet in both directions. The injected fluid do not however form dipoles. After this initial injection, the PV profiles indicate that there is a secondary injection of anticyclonic fluid which is what causes the cyclones to be preferentially stronger.

The extreme case of $\beta = 0.4$ is presented in Figure 3-10. The increase in the height of the prograde topography increases the transport across the jet. The vorticity, and PV, profiles reveal that inner pools of both types of fluid are transported across the jet. Afterwards, a significant portion of the anticyclonic outer pools, which form the vortex street, is injected down the shelf. This fluid then splits in two, half of which forms a dipole and travels away from the jet. The other half, as well as the inner pool that proceeded it, is advected around the cyclone and injected back up the slope. The cyclonic fluid that is injected does not form dipoles but instead forms a long filament that extends across the entire length of the channel. The resulting vortex street has the cyclones being much stronger and larger than the anticyclones. This instability is rather interesting since, after fluid is injected across the jet in both directions, the anticyclonic fluid manages to get re-injected where it then forms the base of the vortex street.

In summary, the simulations with $R_o = 1.0$ revealed that prograde and retrograde topography destabilized and stabilized the jet respectively. This behavior is characteristic of Rossby numbers beyond a critical value. As the Rossby number decreases below this critical value there is a bifurcation where large prograde topography then acts to stabilize the jet. In order to demonstrate this behavior we have performed a series of numerical experiments for the case of $R_o = 0.25$ in the nonlinear regime. For this choice of parameters, the linear stability calculations predict that both very large prograde and retrograde topography act to stabilize the unstable jet.
Figure 3-10: $Ro = 1.0$ and $\beta = 0.4$: A more dramatic example of destabilization by prograde topography. The two uppermost plots show the topographic profile. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$ overlying uneven topography. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 

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b. Simulations with $Ro = 0.25$

Figure 3-11 depicts the instability of a jet over retrograde topography with $\beta = -0.1$. The simulation is qualitatively similar to that of Figure 3-8 because of the triangular structure of the vortices and the reduction in across shelf transport. This verifies our intuition that the stabilizing effect of retrograde topography acts similar for various Rossby numbers. The only other retrograde topography that generated instability is that with $\beta = -0.2$. The vortex street that is produced is a tight cats eye pattern with much stronger cyclones. In addition, their cores oscillated in position which consequently, produced oscillations in strength. No fluid was transported in either direction.

Of the prograde simulations with $Ro = 0.25$, Figure 3-6 indicates that the most unstable is that with $\beta = 0.2$. Figure 3-12 shows the vorticity plots for this case as the jet destabilizes. The inner pools are injected across the jet but are then advected around the outer pools of opposite sign to merge with the outer pools of the same sign. This is similar to what happened in the prograde simulations with $Ro = 1.0$ except that in those simulations the fluid split to then generated dipoles in the deep water. There is no evidence of any splitting and therefore no dipoles form. Presumably there is an equal amount of fluid injected but, since the anticyclones are shallower features, the injection alters them more dramatically. The result in a vortex street that has stronger cyclones. The simulation with $\beta = 0.3$ is very similar except that the cyclones that form the vortex street are more oval in shape.

The final simulation we present is with $\beta = 0.4$ in Figure 3-14. This jet, overlying large prograde topography, produces a weaker instability than the case of $\beta = 0$. The most unstable mode has wavenumber four. This indicates that the stabilizing effects of large prograde topography transfers the instabilities to small wavelengths. Both of the simulations with monochromatic perturbations of wavenumbers three and four are very similar. We choose to present the wavenumber four because the growth rates are larger and therefore, the characteristics are more clearly exemplified then in the other case. The vortex-street that is generated is composed of a cats-eye pattern where the vortices are all elongated. This is unlike those observed in the simulations of Flierl et al. (1987) since the
Figure 3-11: $Ro = 0.25$ and $\beta = -0.1$: a near QG jet is stabilized by retrograde topography. The two uppermost plots show the topographic profile. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$ overlying uneven topography. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 
Figure 3-12: $Ro = 0.25$ and $\beta = 0.2$: a near QG jet is destabilized by prograde topography. The two uppermost plots show the topographic profile. The across channel and along channel coordinates correspond to the $x$ and $y$ axis, respectively. The free surface decreases monotonically with increasing $x$ overlying uneven topography. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 

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Figure 3-13: $Ro = 0.25$ and $\beta = 0.4$: a near QG jet is stabilized by prograde topography. The two uppermost plots show the topographic profile. The across channel and along channel coordinates correspond to the $x$ and $y$ axes, respectively. The free surface decreases monotonically with increasing $x$ overlying uneven topography. These are contour plots of the vorticity at six different instants in time during the destabilization of the jet. The solid and dashed lines denote positive and negative vorticity, respectively, and the contour spacing is equal to $Ro$. 
Figure 3-14: a) amplitude and b) position of the vorticity extrema for $Ro = 0.25$ and $\beta = 0.4$. The oscillations in extrema are accompanied by lateral motions across the shelf because of vortex tube stretching.
strength of the cyclones and anticyclones both oscillate due to lateral motions across the shelf. This is well depicted in Figure 3-14. When the anticyclones are strongest and the cyclones are weakest, the anticyclones are half-moon shaped; otherwise, all the vortices are oval. The PV indicates these oscillations by the meanders growing and then decaying in amplitude. Through time, viscosity causes the half-moon shapes to become increasingly more oval in form. This is remarkable since it clearly demonstrates that topography can generate oscillatory motions in the overlying vortices.

4. Conclusion

In this chapter, we have extended the linear analysis of Li and Mcclimans (2000) in the SW model to include the effects of a free-surface. We also went beyond the linear theory and studied the nonlinear evolution of the instability through the use of numerical simulations. The growth rates calculated from the linear analysis indicate that retrograde topography is stabilizing, as in the barotropic limit. However, the case of small but non-zero Froude numbers yields some dramatic differences in regards to prograde topography. For small Rossby numbers it can be either destabilizing or stabilizing depending on whether the topographic height is small or large, respectively. Moreover, there is a critical Rossby number beyond which prograde topography is strictly destabilizing.

The results from the linear theory that describe the instability of a geostrophic jet force by topography, have been verified by performing a series of numerical simulations of the nonlinear SW equations with a free-surface. These numerical simulations confirm the qualitative and quantitative results of the numerical analysis, thereby, proving the importance of knowing both the orientation and magnitude of topography in order to determine how it alters the stability properties of a jet.

The fact that retrograde topography preferentially stabilizes shear flows more than prograde flows creates an asymmetry. This asymmetry may help to induce instability for shear flows that oscillate between prograde and retrograde; as in the study of the tidally forced problem addressed in chapter five. Presumably, the instabilities will grow more during the prograde phase of the cycle.
Chapter 4

Parametric Instability in Oscillatory Shear Flow

1. Introduction

In §4.1 we derive the equations that govern the linear stability of an oscillatory shear flow. This is used in §4.2 to study the stability properties of pure oscillatory shear flow. Subsequently in §4.3, we examine the interaction of a steady background flow of constant shear with an oscillatory shear with two jumps in vorticity. We derive criteria to calculate the transition wavenumbers and then perform a multiple scale analysis to determine the growth rate in the first subharmonic tongue. In §4.4, we show the results from the Floquet analysis and the structure of the pure parametric mode. In §4.5, we study the mixed barotropic-parametric instability to discern what effect oscillations have on a barotropically unstable shear flow. Then in §4.6, we present a four-contour example in order to determine how this higher order system compares to the simple two-contour model.

2. Linear Stability Problem

Consider the homogeneous two-dimensional Euler equations for a fluid of uniform depth. The incompressibility of the system permits the horizontal velocity field to be written in terms of a streamfunction, \( \mathbf{u} = \hat{k} \times \nabla \psi \). This implies that the vorticity \( q = \hat{k} \cdot \nabla \times \mathbf{u} \) and
streamfunction are related by

\[ q = \nabla^2 \psi. \tag{4.1} \]

The two-dimensional vorticity equation is

\[ \frac{\partial q}{\partial t} + J(\psi, q) = F(y, t), \tag{4.2} \]

where \( J(A, B) = A_x B_y - A_y B_x \) is the Jacobian operator and the vorticity is given by (4.1). The ranges of the across flow coordinate, \( y \), and along flow coordinate, \( x \), are infinite and periodic, respectively. The function \( F(y, t) \) denotes the time dependent forcing which maintains the oscillatory shear flow and defines the vorticity of the basic state \( \tilde{q} \) by the equation

\[ \frac{\partial \tilde{q}}{\partial t} = F(y, t). \tag{4.3} \]

To determine the stability of an oscillatory basic state, we perturb the vorticity and streamfunction in the standard fashion:

\[ q(x, y, t) = \tilde{q}(y, t) + q'(x, y, t), \tag{4.4} \]
\[ \psi(x, y, t) = \tilde{\psi}(y, t) + \psi'(x, y, t). \tag{4.5} \]

The substitution of this into (4.1) implies that the perturbation vorticity is equal to the Laplacian of the perturbation streamfunction. To derive the equation that governs the linear perturbation dynamics we substitute (4.4) and (4.5) into (4.2) and assume the perturbations are infinitesimal. This allows us to linearize the equation in the primed variables, and so obtain

\[ \frac{\partial q'}{\partial t} + \tilde{u} \frac{\partial q'}{\partial x} + \tilde{q} \frac{\partial \psi'}{\partial x} = 0. \tag{4.6} \]

The fields \( \tilde{u}(y, t) \) and \( \tilde{q}(y, t) \) are the velocity and vorticity gradients of the basic state.

The fact that our geometry is periodic in the \( x \) direction enables us to decompose the
perturbation fields as normal modes in $x$ such that both $q(x, y, t)$ and $\psi'(x, y, t)$ are proportional to $\exp(ikx)$. This simplifies the inversion of equation (4.1) and yields the following equation for the streamfunction in terms of the vorticity:

$$\psi'(x, y, t) = -\frac{1}{2k} \int_{-\infty}^{+\infty} \exp(-k|y' - y|) q(x, y', t) \, dy'.$$

When this, along with the modal decomposition, is substituted into (4.6) we obtain

$$\frac{\partial q}{\partial t} + ik \, \bar{u} \, q = \frac{i}{\bar{q}_y} \int_{-\infty}^{+\infty} \exp(-k|y' - y|) q(x, y', t) \, dy'.$$

Rather than study this integral equation in all generality, let us narrow our focus to forcing functions of the form

$$F(y, t) = \frac{dS}{dt} \sum_{n=1}^{N} \Delta_n \mathcal{H}(y - y_n),$$

where $\mathcal{H}(y)$ is the Heaviside step function and $S(t) \Delta_n$ is the vorticity jump across the line $y_n$. By substituting (4.9) into (4.3) and integrating, we solve for the vorticity field of the basic state and its spatial gradient:

$$\bar{q}(y, t) = S(t) \sum_{n=1}^{N} \Delta_n \mathcal{H}(y - y_n) + q_0(y),$$

and

$$\frac{\partial \bar{q}}{\partial y}(y, t) = S(t) \sum_{n=1}^{N} \Delta_n \delta(y - y_n) + \frac{d q_0}{dy}(y).$$

A non-zero function of integration, $q_0(y)$, introduces a non-oscillatory component in the flow; it is a control parameter for stability. Throughout, we restrict our attention to the case where $q_0$ is constant. The integration of equation (4.10) yields the velocity of the basic state,

$$\bar{u}_m = -q_0 \, y_m - S(t) \sum_{n=1}^{N} \Delta_n \max\{(y_m - y_n), 0\} \, dy.$$

The forcing specified in (4.9) yields a system composed of $N + 1$ strips of uniform periodic
strips of vorticity with \( N \) contour interfaces. The system of dynamical equations is equivalent to what would arise if we calculated the linearized equations from a contour dynamical perspective with \( N \) interfaces (see Dritschel (1989)). The case of two contour steps with \( q_0 = 0 \) is illustrated in Figure 4-1.

<table>
<thead>
<tr>
<th>Shear Flow Profile</th>
<th>Velocity</th>
<th>Vorticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = a )</td>
<td>( U_3 = S(t)a )</td>
<td>( Q_3 = 0 )</td>
</tr>
<tr>
<td>( y = -a )</td>
<td>( U_2 = S(t)y )</td>
<td>( Q_2 = -S(t) )</td>
</tr>
<tr>
<td>( y = a )</td>
<td>( U_1 = -S(t)a )</td>
<td>( Q_1 = 0 )</td>
</tr>
</tbody>
</table>

Figure 4-1: A example of a pure oscillatory shear flow profile for the case of two contours.

Since the forcing is a piecewise constant function, the vorticity equation may be discretized: we replace \( \tilde{\omega}_y \) by the differences in the vorticity between two regions divided by the width of a band; we assume all bands are of the same width. The integral is discretized and the vorticity is evaluated at the discrete levels between regions of constant vorticity. The relation that describes the evolution of the perturbation vorticity at the interface \( y_m \) is given by

\[
\frac{d\eta}{dt} = -i k \tilde{\omega}_m \eta_m + \frac{i}{2} S(t) \Delta m \sum_{n=1}^{N} \exp(-k|y_m - y_n|) \eta_n. \tag{4.13}
\]

This system of equations describes the linearized motion of the \( N \) contour interfaces in terms of the velocity and vorticity of the oscillatory flow. In the derivation we have allowed for a linear non-oscillatory background shear. In the next section we set this mean shear
to be zero. In this case all fields are directly proportional to $S(t)$, which is taken to be a periodic function, and hence we call these basic states pure oscillatory flows. Note that the time average of these flows need not be zero, but all the fields are dependent on the same function of time. In the subsequent section, we study the more general case where the oscillation is superimposed upon a nonzero mean shear: a mixed oscillatory flow. We acknowledge that there is no physical reason for the function $q_0$ to be constant except for those cases when the oscillatory shear has a smaller length scale than the mean shear and so locally, its velocity field appears linear. However, this choice yields the simplest setting in which to study the interaction of steady and oscillatory flows. After this foundational work has been completed it would then be appropriate to consider more complicated background flows.

Equation (4.13) indicates that the perturbation field is discrete. This signifies that we are restricting our attention to instabilities that arise in the discrete spectrum. The instabilities that originate from the continuous spectrum are not accounted for in our analysis. If we are looking for examples of instability, then finding an instability in the discrete spectrum is sufficient to conclude the system is unstable. However, if we determine that the solution is stable, it is possible that there are still instabilities in the continuous spectrum. In the analysis that is to follow we will find basic states that are stable, but it should be kept in mind that this result says nothing about the continuous spectrum.

3. Pure Oscillatory Flows

Before we delve into the study of oscillatory flows, let us first analyze the stability of steady flows, i.e. $S$ is constant. Since the basic state only varies in the across flow coordinate $y$, it is natural to decompose the perturbation in normal modes and then treat each mode separately:

$$q(x, y_m, t) = \text{Re} \left\{ \hat{q}_m \exp(ikx + i\sigma t) \right\}, \quad (4.14)$$

where $k$ and $\sigma$ are the wavenumber and frequency of the perturbation. For notational
simplicity we define an $N$-vector $\vec{q}$ which has $\hat{q}_m$ as the $m$th component. If we substitute (4.14) into (4.13), we obtain the algebraic eigenvalue problem

$$\sigma \vec{q} = M \vec{q},$$

(4.15)

where the $(m,n)$-component of the $N \times N$ matrix $M$ is

$$-k\delta_{mn}\vec{u}_m + \frac{S}{2}\Delta_m \exp(-k|y_m - y_n|)$$

(4.16)

and $\delta_{mn}$ is the Kronecker delta function. From (4.14) it is obvious that if the imaginary component of $\sigma$ is negative then the system is linearly unstable: the growth rate is the imaginary part of $\sigma$. To determine whether a wavenumber is unstable one must solve for the eigenvalues of (4.16) for the particular basic state in question. The mechanism by which a steady flow destabilizes is referred to as shear instability or, in geophysical fluid dynamics, as barotropic instability.

The instability of steady shear flows has been studied quite extensively (see Pedlosky (1987), Drazin and Reid (1995) and Dritschel (1989)). However, the stability of oscillatory shear flows has not received much attention so far. What effects does oscillating a velocity profile periodically have on the stability of the modes? Are precisely the same wavenumbers unstable as in the steady case or are instabilities generated at new wavenumbers? These are two questions we address in this chapter.

To analyze time dependent basic flows, we rewrite (4.13) as a system of ordinary differential equations

$$\frac{d\vec{q}}{dt} = iS(t)\vec{M}\vec{q}.$$  

(4.17)

Here, $\vec{q}$ is the vector composed of $\hat{q}$ and the coordinates of $\vec{M}$ are given by (4.16) with the $S(t)$ factored out from the $u_m$ and the second term as well. In general, if $\vec{M}$ is a function of time this system cannot in general be solved analytically. However, in this case the matrix is constant which enables us to solve the problem easily.
\[ \tilde{q} = \exp \left( i \tilde{M} \int_0^t S(s) ds \right) \tilde{q}_0. \]  

(4.18)

Note that the right hand side of this equation is the matrix exponential of the quantity in parenthesis and \( \tilde{q}_0 \) are the initial conditions. If we assume that \( \tilde{M} \) has an eigenvalue decomposition with no degenerate eigenvalues, we can rewrite it as \( \tilde{M} = XDX^{-1} \) where \( D \) is the matrix with eigenvalues in the diagonal and \( X \) is the matrix of column eigenvectors placed in the appropriate order. By substituting the eigenvalue decomposition into (4.18) and setting \( S(t) = \delta + \epsilon \cos(\omega t) \), the solution becomes

\[ \tilde{q} = X \exp \left( it \left[ \delta + \epsilon \frac{\sin(\omega t)}{\omega t} \right] D \right) X^{-1} \tilde{q}_0. \]  

(4.19)

Since \( D \) is diagonal the exponential matrix above is simply a diagonal matrix with each entry of the form

\[ \exp \left( it \left[ \delta + \epsilon \frac{\sin(\omega t)}{\omega t} \right] c_n \right) \]  

(4.20)

\( n \) ranges between 1 and \( N \) and \( c_n \) is the \( n \)th eigenvalue. For exponential growth it is necessary that at least one element in this matrix has a phase speed with a positive imaginary part. Note that if \( \epsilon = 0 \) equation (4.19) reduces to the steady state solution.

If the mean of the oscillatory flow is zero, \( \delta = 0 \), the oscillatory flow is always stable. To see this, consider a mode \( k_c \) that is unstable (asymptotically stable) in the steady regime. When the sine function is positive, the perturbations grow (decay) exponentially; when sine is negative, they decay (grow) exponentially. Even though there are intervals of both growth and decay, after an entire period there is no net growth. Therefore all the modes, even those that are unstable and asymptotically stable in the steady case, are stable in pure oscillatory flows. The reason for this stabilization is that at every instant the growth rate is simply a scalar multiple of the growth rate of the steady profile. Since the multiplying function, \( \epsilon \sin(\omega t)/\omega \), is periodic and zero at \( 2\pi/\omega \), the net growth is zero. This result applies to any \( S(t) \) that has an integral of zero over one period. Clearly, if the steady profile is stable, the growth rate will be zero at every instant in time and therefore no new instabilities can develop.
Care should be taken in interpreting these results from the linear stability analysis which results state that all pure oscillatory shear flows are stable. The modes that are unstable or asymptotically stable according to the steady theory, when oscillated, experience intervals of growth where the small perturbations grow exponentially. If the interval of growth is long enough, the perturbations may grow sufficiently so that linear theory is no longer applicable and they enter in the fully nonlinear regime. If this occurs there is no reason to believe that the perturbations will ever decay to their initial state by the end of the period, and presumably they will not. Therefore, even though the linear theory predicts stability, the existence of intervals of growth might still lead to instability. In order to determine whether the growth is sufficient to escape the linear regime one would need to take into account the size of the perturbation and the amount of growth that occurs; something that depends on both the growth rate and the length of the period.

In the general case where \( \delta \) and \( \epsilon \) are both nonzero we see a superposition of the previous two cases. If \( k_c \) is unstable (asymptotically stable) the sine term in (4.19) produces instants of growth and decay but it will, according to linear theory, have no net growth after one period. If \( \delta \) is nonzero, without loss of generality we take it to be positive, the mode will continually grow as in the steady state theory and hence the solution is unstable. Since for every growing mode there is a corresponding decaying mode, whatever shear flows are unstable in the steady case will be precisely those that are unstable in the oscillatory case.

The linear stability results we have just proved do not depend whatsoever on the number of contour interfaces chosen in the forcing function (4.9). The fact that these results hold with increasing \( N \) suggests that they should also hold for continuous profiles. This is the next issue we will investigate, but first, we study the stability of continuous steady flows. The perturbation is decomposed as

\[
q(x, y, t) = \text{Re} \{ \hat{q}(y) \exp(ikx + i\sigma t) \}. \tag{4.21}
\]

If we substitute this into (4.8) we recover the following eigenvalue problem

\[
\sigma \hat{q}(y) = S(t)M(\hat{q}(y)) \tag{4.22}
\]
in terms of the integral operator $\mathcal{M}$ defined as

$$\mathcal{M}(\hat{q}(y)) = -k\hat{u}\hat{q} + \frac{\hat{q}_y}{2} \int_{-\infty}^{+\infty} \exp(-k|y-y'|)\hat{q}(y')dy'. \quad (4.23)$$

For time-dependent basic states we must decompose the perturbation as

$$q'(x,y,t) = \text{Re} \left\{ \hat{q}(y,t) \exp(ikx) \right\}. \quad (4.24)$$

The linear stability problem is then governed by the integral equation

$$\frac{\partial \hat{q}}{\partial t}(y,t) = iS(t)\mathcal{M}(\hat{q}(y,t)), \quad (4.25)$$

in terms of the operator (4.23); $\hat{q}_y$ is now interpreted as a partial derivative. Assuming that $\hat{q}$ is an eigenfunction enables us to replace $\mathcal{M}$ by the eigenvalue $\sigma$, and get

$$\frac{\partial \hat{q}}{\partial t} = iS(t)\sigma\hat{q}, \quad (4.26)$$

which is solved to give

$$\hat{q} = q_0 \exp \left[ i\sigma \int_{t}^{t} S(s) \, ds \right]. \quad (4.27)$$

This solution is entirely analogous to that of (4.18). If $S(t)$ is constant we obtain the steady state case. If $S(t)$ is an oscillatory function with zero mean, all basic states are linearly stable; there are intervals of growth and decay but after one period the solution has no net growth. In addition, if $\delta$ and $\epsilon$ are both nonzero we get instabilities at precisely the same wavenumbers that are unstable in the steady case.

If the integral operator $\mathcal{M}$ is complete then this analysis is sufficient to imply that the oscillatory continuous case is only unstable if the snapshot of the extreme position is unstable. Also, any discretization of the continuous system yields the same results. However, we can not make the same conclusions in general.

One type of instability that occurs in systems with continuous systems but that does not exist in the discrete contour dynamical systems are those due to critical layers. We remark that it would be interesting to study how critical layer instabilities develop in oscil-
latory systems since then the critical layer would change throughout the period. To address this problem would require analyzing the continuous system, but this is not something we address.

We conclude that a single profile multiplied by $S(t)$ does not have a rich enough structure to generate new types of instabilities for any discrete system. This is why we shall next study a superposition of two profiles: one steady and the other oscillating. The theory for such a mixture is more complex than the case of pure oscillatory flow and consequently we have not been able to solve the linear stability problem in general. By choosing appropriate profiles we will discover that interesting and new phenomena arise.

4. Analysis of Mixed Oscillatory Shear

Our analysis of the previous section explains the direct connection between the linear stability of pure oscillatory shear flows and their steady analogues. In this section we delve into the more complex phenomenon of mixed oscillatory shears. The non-oscillatory component is a flow with a uniform vorticity which, in the context of (4.10), means that $q_0$ is a nonzero constant. Superimposed upon this steady flow is an oscillatory flow that contains two jumps in vorticity. The two jumps are located at $y_1 = -a$ and $y_2 = +a$ and the magnitude of these jumps are such that $\Delta_1 = -S(t)$ and $\Delta_2 = S(t)$. The velocities at the two interfaces are denoted by $\tilde{u}_m = (-1)^m(S(t) - q_0)a$. The governing linear equation is obtained by specializing (4.13) to this particular shear flow and rewriting $K = 2ak$:

$$\frac{dq}{dt} = \frac{i}{2}S \begin{bmatrix} K(1 - q_0/S) - 1 & -\exp(-K) \\ \exp(-K) & -K(1 - q_0/S) + 1 \end{bmatrix} \tilde{q}. \quad (4.28)$$

To derive the equation for the steady state we decompose the above equation with

$$q'_m = \exp(i\sigma t)\tilde{q}_m \quad (4.29)$$

to yield the following eigenvalue problem
Figure 4-2: The oscillatory shear flow profile to be investigated. It is a superposition of a mean steady flow of uniform vorticity and an oscillatory profile with two jumps in its vorticity field.

We calculate the eigenvalues of the matrix in order to derive an expression for the frequency:

$$\sigma \bar{q} = \frac{1}{2} S \left[ \begin{array}{cc} K(1 - q_0/S) - 1 & -\exp(-K) \\ \exp(-K) & -K(1 - q_0/S) + 1 \end{array} \right] \bar{q}. \quad (4.30)$$

We calculate the eigenvalues of the matrix in order to derive an expression for the frequency:

$$\frac{4\sigma^2}{S^2} = (K(1 - q_0/S) - 1)^2 - \exp(-2K). \quad (4.31)$$

A criteria that is necessary and sufficient for the imaginary part of the frequency to be positive, and therefore for linear stability, is

$$|K(1 - q_0/S) - 1| \geq \exp(-K). \quad (4.32)$$

Observe that the right hand side of (4.32) is a positive function that is equal to 1 at $K = 0$, and then exponentially decays to zero as $K \to \infty$. The function on the left hand side
is a line that, at \( K = 0 \), has a value of 1, and a slope of \( |1 - q_0/S| \). If \( q_0/S \geq 1 \), the slope is non-negative and the line is monotonically non-decreasing, it never intersects with \( \exp(-K) \), and the basic state is stable. Alternatively, if \( q_0/S < 1 \) the line is decreasing for small wavenumbers and must necessarily intersect the \( K \) axis, thereby violating (4.32) and yielding instability. In summary, a necessary and sufficient condition for the steady state to be unstable is that

\[
\frac{q_0}{S} < 1. \tag{4.33}
\]

If \( q_0 = 1 \) the steady state is only stable for \( 0 \leq S \leq 1 \). Figure 4-3 illustrates the the two critical profiles at \( S = 0 \) and \( S = 1 \) and two examples of unstable flows.

Figure 4-3: The flow is stable for the values of \( 0 \leq S \leq 1 \) and unstable beyond this interval.

The mathematical framework we use to analyze the stability of periodic shear flows is Floquet Theory, a review of which is in Appendix A. The particular oscillatory state we will study is that with \( q_0 = 1, \delta = 1/2, \epsilon = 1/2 \) and \( \omega = 1 \). It produces the oscillations of the largest radius possible for this value of \( q_0 \) such that the snapshots at every instant in time are linearly stable according to steady theory. The absence of any steady shear instability implies that if instabilities exist, they must be due solely to the oscillatory behavior of the basic state. It is for this reason that we entitle them parametric resonances or instabilities. This example of oscillatory shear flow is simple enough to allow for some analytical results to be obtained. First we determine the location of the transition points and then use a
multiple scale analysis to determine the growth rates of the first subharmonic; it is the most unstable region and hence the most important.

We customize equation (4.28) to our specific choice of parameters and then rewrite it as a scalar second order equation,

$$\frac{d^2 q}{dt^2} - \frac{1}{S} \frac{dS}{dt} \frac{dq}{dt} + \left( \frac{S^2}{4} \left[ (K - \frac{q_0}{S}) - 1 \right]^2 - \exp(-2K) \right) \frac{1}{2} + \frac{i}{2} K q_0 \frac{1}{S} \frac{dS}{dt} \right) q = 0. \quad (4.34)$$

This equation is satisfied at both interfaces and hence the subscript and hat are dropped. The Wronskian of this equation is

$$W(t) = W(0) \exp \left( - \int_0^t \frac{1}{S} \frac{dS}{dt} \right) = W(0) \frac{\delta + \epsilon}{\delta + \epsilon \cos(\omega t)}. \quad (4.35)$$

$W(t)$ is a periodic function of period, $T = 2\pi/\omega$. This is different from Mathieu’s equation, which has a constant Wronskian. However, the periodic Wronskian is sufficient to conclude, as in Mathieu’s equation, that the transition solutions between stability and instability must be of period $T$ or $2T$.

Consider the case of a nearly steady basic state; i.e. the asymptotic limit of $\epsilon \ll 1$ and $S \sim \delta$. The leading order equation (4.34) reduces to the simple harmonic oscillator where the frequency $\sigma$ is defined by (4.31). Since the transition solutions must be of periods $T$ or $2T$, it is necessary that the frequency of the motion must satisfy the relationship

$$\sigma^2 = \frac{n^2 \omega^2}{4}. \quad (4.36)$$

where $n$ are non-negative integers. The case of $n = 0$ does not allow for temporal instabilities and hence we neglect it. This situation is similar to the case of the Mathieu equation. This relationship designates where the transition points must be in the case of $\epsilon = 0$ and hence where the regions of instabilities develop for small $\epsilon$; the odd and even $n$ correspond to subharmonics and harmonics, respectively. In particular, Figure 4-4 illustrates the dispersion relationship for the waves in this two-contour problem. The vertical lines indicate the position of the subharmonic and harmonic transition points, respectively.
Figure 4-4: The curved lines are the dispersion relations of the waves in the system with $S = 1/2$, $q_0 = 1$ and $\omega = 1$. The solid and dashed vertical lines indicate where the subharmonic and harmonic instabilities are located, respectively. The vertical bars are drawn to show the location of the transition wavenumbers, i.e. where a resonance occurs. The length of each vertical bar is a multiple of the frequency, in this case one.

Our next aim is to analytically calculate the growth rate by using the method of multiple scales for any value of the two parameters, $\delta/\omega$ and $q_0/\delta$. To begin, if we define the matrix

$$M_0 = \frac{i}{2\omega} \begin{bmatrix} K(1 - q_0/\delta) - 1 & -\exp(-K) \\ \exp(-K) & -K(1 - q_0/\delta) + 1 \end{bmatrix},$$

(4.37)

and

$$M_1 = \frac{i}{2\omega} \begin{bmatrix} K - 1 & -\exp(-K) \\ \exp(-K) & -K + 1 \end{bmatrix},$$

(4.38)

we can rewrite the governing linear system (4.28) as

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\[ \frac{\partial \vec{q}}{\partial t} = M_0 \vec{q} + \epsilon \cos t M_1 \vec{q}, \quad (4.39) \]

where \( \vec{q} \) is a time-dependent 2-vector. The fact that for Mathieu’s equation the first subharmonic region grows linearly with the amplitude of the oscillations suggests that we should choose the long-time variable to be \( T = \epsilon t \); the analysis for the higher order harmonics require different powers of \( \epsilon \).

If we expand \( q \) as a perturbation series,

\[ \vec{q} = \vec{q}^{(0)} + \epsilon \vec{q}^{(1)} + O(\epsilon^2), \quad (4.40) \]

and take the long time variable to be \( T = \epsilon t \), (4.39) becomes

\[ \left( \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T} \right) \left( \vec{q}^{(0)} + \epsilon \vec{q}^{(1)} + \ldots \right) = (M_0 + \epsilon \cos t M_1)(\vec{q}^{(0)} + \epsilon \vec{q}^{(1)} + \ldots). \quad (4.41) \]

The leading order equation is

\[ \frac{\partial \vec{q}^{(0)}(T)}{\partial t} = M_0 \vec{q}^{(0)}. \quad (4.42) \]

Since the vorticity is symmetric about \( y = 0 \), the characteristic equation has no linear term and therefore the solution to our system is simply

\[ \vec{q}^{(0)}(T) = S_+(T) \exp(i \sigma t) + S_-(T) \exp(-i \sigma t), \quad (4.43) \]

where \( \pm i \sigma \) are the eigenvalues of \( M_0 \) and \( S_\pm \) are right eigenvectors. That is,

\[ M_0 S_\pm = \pm i \sigma S_\pm. \quad (4.44) \]

The functions \( S_\pm \) are complex conjugates of each other and therefore \( \vec{q}^{(0)} \) is a real quantity. Moreover, it is useful to define the adjoint problem, using dagger superscripts, that is independent of \( T \):
\[ s_\pm^4 M_0 = \pm i \sigma s_\pm^4. \] (4.45)

The second order equation is, after having substituted in for \( \bar{q}^{(0)} \) and rewritten \( \cos(\omega t) \),

\[
\frac{\partial \bar{q}^{(1)}}{\partial t} - M_0 \bar{q}^{(1)} + \frac{dS_+}{dT} \exp(i\sigma t) + \frac{dS_-}{dT} \exp(-i\sigma t) = \\
\frac{1}{2} \left( \exp(i\omega t) + \exp(-i\omega t) \right) M_1 \left( S_+ \exp(i\sigma t) + S_- \exp(-i\sigma t) \right).
\]

To remove the secular terms we multiply the equation by \( \exp(\mp i\sigma) \), integrate \( \frac{1}{\tau} \int_0^\tau dt \), take \( t \ll \tau \ll T \) in order that the average of \( \bar{q}^{(1)} \) is zero on this intermediate time scale, and then use the eigenvalue problem (4.44) to eliminate \( \bar{q}^{(1)} \):

\[
\frac{dS_\pm}{dT} = \frac{1}{2} M_1 S_\mp.
\] (4.46)

In the derivation of this equation we set \( \sigma = \omega/2 \) which keeps the terms that have a non-zero projection onto this solution space. This choice of \( \sigma \) is equivalent to restricting our attention to the first subharmonic instabilities.

Let us decompose the right eigenvectors into a function of \( T \) multiplied by a constant vector, i.e. \( S_\pm = S_\pm(T) \hat{s}_\pm \). We multiply (4.46) on the left by \( S_\mp^\dagger \) to recover

\[
(\hat{s}_\mp^\dagger \hat{s}_\pm) \frac{dS_\pm}{dT} = \frac{1}{2} (\hat{s}_\mp^\dagger M_1 \hat{s}_\pm) S_\mp.
\] (4.47)

These two equations can then be combined to yield two identical scalar second order equations in \( T \),

\[
\frac{d^2 S_\pm}{dT^2} = \frac{1}{4} \left( \frac{(\hat{s}_\mp^\dagger M_1 \hat{s}_-)(\hat{s}_\mp^\dagger M_1 \hat{s}_+)}{(\hat{s}_\pm^\dagger \hat{s}_-)(\hat{s}_\pm^\dagger \hat{s}_+)} \right) S_\pm.
\] (4.48)

If the quantity in front of \( S_\pm \) is positive then this signifies that the base state is unstable and the square root of this quantity gives the slope at which the growth rate increases in terms of \( \epsilon \). The left and right eigenvectors are
\[
\hat{s}_\pm = \begin{bmatrix}
-\delta \exp(-K) \\
\pm \omega - K(\delta - q_0) + \delta
\end{bmatrix} \tag{4.49}
\]

and

\[
\hat{s}_\pm^\dagger = \begin{bmatrix}
\delta \exp(-K) \\
\pm \omega - K(\delta - q_0) + \delta
\end{bmatrix}^T. \tag{4.50}
\]

The vector products are

\[
\hat{s}_\pm^\dagger \hat{s}_\pm = 2\omega^2 \pm 2\omega(-K(\delta - q_0) + \delta), \tag{4.51}
\]

\[
\hat{s}_\pm^\dagger M_1 \hat{s}_\pm = -i\frac{\delta q_0}{\omega} K \exp(-2K). \tag{4.52}
\]

When we substitute these identities into (4.48), we conclude that

\[
\Sigma = \frac{|q_0| K^* \exp(-K^*)}{4\omega^2}, \tag{4.53}
\]

with

\[
(K^*(1 - q_0/\delta) - 1)^2 - \exp(-2K^*) = \frac{\omega^2}{\delta^2}. \tag{4.54}
\]

This equation is equivalent to (4.31) with \(\sigma = \omega/2\), since this is criteria for the first subharmonic. The growth rate of the first subharmonic is equal to \(\epsilon \Sigma\).

It appears in equation (4.53) that the growth rate varies linearly with the slope of the background shear. This is somewhat deceptive since the wavenumber \(K^*\) is implicitly defined by (4.54) in terms of \(q_0\) and \(\delta\), which makes the dependence of growth rate on these two parameters highly nonlinear. If we assume that the wavenumber is fixed then we deduce the general tendencies of the growth rate to increase (decrease) with respect to the background vorticity shear (forcing frequency). The fact that the growth rate is exponentially decaying with respect to the wavenumber indicates that the strength of the parametric resonance is highly dependent on the closeness of the contours.

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Throughout our work, we focus on the simple case in which the basic state oscillates with only one frequency. This idealization rarely, if ever, occurs in the real world since there are always a multitude of frequencies forcing the motion. Our analysis is essentially treating each of these frequencies individually. For a particular resonance, if there is only one frequency that generates this instability, the inclusion of the other frequencies would have a negligible contribution to the instability process. Therefore, this approach of considering a solitary frequency still yields meaningful results, even in the case of multiple forcing frequencies. One can imagine a case of multiple contours where two forcing frequencies happen to generate instabilities at the same wavenumbers; this scenario is more complicated and is beyond the scope of this thesis.

5. Growth Rates for Pure Parametric Instability

The analysis of the previous section is limited to small amplitude oscillations. To explore beyond this asymptotic regime we use Floquet Theory and solve the problem numerically. First, the column vectors of the identity matrix are taken to be initial conditions for the dynamical system (4.28), which is integrated numerically by one period. The resulting matrix, called the Floquet matrix, determines the stability of the system. If the Floquet matrix has an eigenvalue that has a positive real component the system is unstable. The real part of this eigenvalue gives the growth rate over one period; we divide this by the period to determine the growth rate.

Figure 4-5 presents the growth rates of the simple mixed shear in the steady case. When oscillations are introduced the parameter $q_0$ remains fixed, it is the $\delta$ parameter that oscillates. The oscillations are equivalent to moving back and forth along a horizontal slice of Figure 4-5a). The time-dependent stability calculations determines what the net effect is of having moved along one slice and back again. The case of no background shear, $q_0 = 0$ as in Figure 4-5b), is perfectly symmetric about $\delta = 0$. However, as $q_0$ is increases we see that asymmetries develop; Figure 4-5c) focuses on $q_0 = 1$, something that we will study further very shortly. This figures is the analogue of Figure 3-5. The mean background shear is different then the topography studied in Chapter three yet it produces a similar asymmetry.
in the growth rates.

Figure 4-5: a) The contour plot of the growth rates in the steady case. b) Shows the growth rates for \( q_0 = 0 \). c) Shows the growth rates for \( q_0 = 1 \). These plots together illustrate how a mean background shear produces an asymmetry in the steady growth rates, as was apparent in Figure 3-5 because of topography.

Figure 4-6 shows the results from a series of calculations for various wavenumbers and forcing amplitudes. The solid lines are the numerical approximations to the transition curves for the first subharmonic, first harmonic and second subharmonic; these curves should extend down to the \( K \) axis but do not because of the limited resolution in \( K \) and small numerical error that arises in each integration.

If we tailor equation (4.31) to the specific parameters under consideration, we obtain

\[
4n^2 = (K + 1)^2 - \exp(-2K). \tag{4.55}
\]

The roots of this equation for \( n = 1, 2, 3 \), are \( K = 1.0315, K = 3.0003 \) and \( K = 5,0000 \). For larger \( n \) the solutions are, to a very good approximation, \( K = 2n - 1 \). These transition wavenumbers are where the instability tongues touch the wavenumber axis. The growth rate for the first subharmonic region, calculated from equation (4.53), is 0.091925\( \varepsilon \). The one calculated from the direct linear stability calculation is accurate to this many significant digits; the next digit is different and we attribute this difference to numerical error.

There are certain similarities between Figure 4-6 and the equivalent result for the Math-
Figure 4-6: The upper plot depicts the stability boundaries of the first three unstable regions; first subharmonic, first harmonic and second subharmonic. The lower three focus on each of these modes and show the stability boundaries along with equally spaced growth rate contours. The dashed lines are contours of the growth rates in one period, with increments of $5 \times 10^{-2}$, $5 \times 10^{-3}$ and $5 \times 10^{-4}$ for the lower three figures respectively.
ieu equation. First, the unstable regions originate from the transition wavenumbers predicted from the leading order theory. Second, the transition curves for the first subharmonic are linear and then increase in concavity with the higher order harmonics. Third, the growth rate increases most slowly in moving to higher harmonics; this is shown more clearly in Figure 4-7 which is a slice of the previous figure at $\epsilon = 1/2$. The lower plot in Figure 4-7 is a logarithm plot of the growth rate showing a linear decrease. This is not surprising given the analytical expression of the growth rate of the first subharmonic. This decay, approximately $-1.03$, illustrates how parametric resonance requires that the contour interfaces in the basic state be in close proximity to each other; otherwise, the required coupling is very weak. This is because the instability is inherently due to a coupling of two interfaces. If the interfaces are far away we approach the limit where they are each alone, and hence the growth rates tend to zero.

![Graph](image.png)

Figure 4-7: The growth rate of our example of pure parametric instability. Observe that the unstable modes are evenly spaced for large enough $K$ and that the envelop decays exponential.
Figure 4-8: First subharmonic. The two plots in the left column show that the modulus of each component of the growing (decaying) mode have the same functional dependence on time. The phase plots show that the change in phase after one period is $\pi$ which indicates that this is the first subharmonic.

In Figure 4-8, we plot the structure of the wavenumber $K = 1.02$ at $\epsilon = 1/2$; we choose this particular wave since it the most unstable of the parametric modes plotted in Figure 4-7. In the top and bottom rows we plot the modulus and phase of the growing and decaying modes, respectively. All unstable positions in parameter space give rise to a growing and a decaying mode. The solutions are obtained by numerically integrating the governing system forward one period using the eigenvectors of the Floquet matrix as the initial conditions. Observe that the plots of the modulus have only one curve, as is true in general, since the perturbation vorticity at two interfaces grow equally at all times. The majority of the growth (decay) occurs during the first and final quarters of the forcing period; this coincides with the intervals where $\cos(\omega t)$ is positive. We speculate that it is during these times that the resonant triad, between the two waves and the basic state, can most effectively extract energy from the mean flow. The phases of the interfaces $y = -a$ and $y = a$ are decreasing and increasing respectively. After one period they have a net change of $\pi$ radians; i.e. it is the first subharmonic.

Figures 4-9 and 4-10 are the plots of the growing and decaying modes for $K = 3$ and
Figure 4-9: First harmonic. The two plots in the left column show that the modulus of each component of the growing (decaying) mode have the same functional dependence on time. The phase plots show that the change in phase after one period is $2\pi$ which indicates that this is the first harmonic.

Figure 4-10: Second subharmonic. The two plots in the left column show that the modulus of each component of the growing (decaying) mode have the same functional dependence on time. The phase plots show that the change in phase after one period is $3\pi$ which indicates that this is the second subharmonic.
\( K = 5 \). What are qualitatively different from the previous figures are the frequencies of the modes; they are \( 1, 3/2 \) times that of the frequency of the forcing. Therefore, the numerical calculations verify that the frequencies are precisely what is predicted from the leading order theory for the case of small \( \epsilon \).

There is a qualitative difference between pure and mixed oscillatory flows since our simple example of the latter produces parametric instability but the entire class of the former possess none. The reason is that in mixed oscillatory flows there is the possibility of a resonant wave triad formed between the oscillatory basic state and the two waves at certain intervals of the period. In contrast to pure oscillatory flows, if a resonant triad does occur, it persists throughout the entire period: sometimes growing, other times decaying, but having no net growth. The mixed oscillatory flows avoid this since their dispersion relations change shape throughout the period, which is what allows for times when the resonance is excited and other times of little or no growth.

In order to derive a necessary condition for parametric resonance, consider (4.39) for the general case where the matrices \( M_0 \) and \( M_1 \) are arbitrary but constant. If they commute, as is true for pure oscillatory flow \( M_0 = M_1 \), the exact solution is

\[
\tilde{q} = \exp \left( M_0 t + \epsilon M_1 \frac{\sin(\omega t)}{\omega} \right) \tilde{q}_0.
\]  

This solution does not give rise to parametric instability since the only instabilities are those that exist when \( \epsilon \neq 0 \). Therefore, a necessary condition for parametric resonance is that \( M_0 \) and \( M_1 \) do not commute, as is the case with Mathieu’s equation and with equation (4.28). In general, these matrices do not commute; this is why we believe that parametric instability is a common phenomena in periodic shear flows.

Our example demonstrates that time dependence may be crucial in correctly determining the stability of a flow. Even though every snapshot in time is stable, and hence the average is certainly stable, the oscillatory flow may be unstable. This demonstrates the importance of incorporating time dependence into the stability calculations if one wishes to correctly determine the stability of the flow and subsequently its growth rate. In modeling geophysical phenomena one often uses the average flow as representative of the time
dependent flow. This may seem like a good approximation but it may not account for instabilities that are inherent to the time dependent nature of the flow. Clearly, care should be taken in order to correctly model the stability of time dependent shear flows.

6. Mixed Barotropic-Parametric Instability

Mixed barotropic-parametric instability is of fundamental interest, but is also important because nearly all shear flows in the real world tend to be modeled as steady, even though they are actually time dependent. Our analysis, which incorporates time dependence as an oscillation, should yield next order corrections of barotropic instability of realistic shear flows. In this work we will not be so bold as to model a particular physical phenomenon but instead we will investigate what qualitative differences arise between the steady and time dependent theories.

The previous section demonstrated that parametric instability can develop in a mixed oscillatory shear that is barotropically stable when considered as a snapshot at every instant in time. The next question we address is: how are barotropically unstable modes affected by oscillatory shear? To explore this scenario we consider the case where \( q_0 = -1 \), \( \delta = 1/2 \) and \( \epsilon \) is equal to 0, 1/4 and 1/2. This is qualitatively different from the previous example in that at every instant in time the flow is barotropically unstable. The calculations of the growth rates within a period for a range of wavenumbers are presented in Figure 4-11. The top plot shows the growth rates for the three different \( \epsilon \) and the bottom two focus in on the two separate modes, the barotropic and the parametric. As \( \epsilon \) increases from 0 to 1/2 the barotropic modes are stabilized: the maximum growth rate decreases and the range of the barotropic modes decreases. Moreover, the parametric modes are introduced and their strength increases with \( \epsilon \), as the range of wavenumbers increases. This one example depicts how oscillations can either stabilize or destabilize waves, in comparison to their stability in the limit of a steady state.

To predict the position of these transition points we specialize (4.31) to the case where \( q_0 = -1 \):
Figure 4-11: We see the transition from the purely barotropic instability to a mixed barotropic-parametric instability: as $\epsilon$ increases the growth rate of the barotropic mode gradually decreases and parametric modes are introduced and become more unstable.

\[ 4n^2 = (3K - 1)^2 - \exp(-2K). \]  

(4.57)

We solve this nonlinear algebraic equation for $n = 1, 2, 3$ and find the solutions are $K = 1.0109$, $K = 1.6681$ and $K = 2.3336$, respectively. For larger $n$ the solutions are, to a very good approximation, $K = (2n + 1)/3$. The order of accuracy of this solution increases exponentially with the wavenumber. We can apply the multiple scale analysis to the first subharmonic and deduce that the growth rate is $0.091965\epsilon$. The growth rate calculated from the numerical method is accurate to within $10^{-6}$. It is important to note that the first subharmonic wavenumber in this example differs by $10^{-2}$ from that deduced in the previous section. This causes a difference of $10^{-3}$ in the growth rates, which suggests that the strength of the parametric instability is independent of whether the oscillatory flow is unstable.

In Figure 4-12 we present the growing and decaying modes for $\epsilon = 1/2$ at the wavenum-
ber $K = 1$. Observe that the modulus and phase are very similar to that found in Figure 4-8. Therefore, not only do the first subharmonics in these two examples have the same growth rates, but they have the same modal structure.

Figure 4-13 has the same parameter values as before except for the fact that $\epsilon = 1/10$. The oscillation is only slightly unstable. The phase plots are very similar to before but the growth rates are quite different. Both the unstable and stable modes grow for the first half of the period and then decay for the second half. However, the amount of growth does not equal the amount of decay, which is why there is a net growth or decay at the end of the period, albeit small.

Figure 4-14 depicts the modulus and phase for the barotropic mode and $\epsilon = 1/10$. The increasing and decreasing moduli are both nearly exponential, as is predicted by the steady theory. The phase at each interface is nearly uniform throughout the period as predicted by the steady theory.

![Growing and Decaying Modes](image)

Figure 4-12: The first subharmonic mode for parametric instability. It behaves qualitatively in the same fashion as the example of pure parametric instability.

The final set of structural plots, is illustrated in figure 4-15 and shows how the barotropic mode is altered by strong oscillations, in particular $\epsilon = 1/2$. In contrast to the previous case, the phase relationships are time-dependent and the growth rates are no longer exponential.
Figure 4-13: The first subharmonic mode for parametric instability with smaller forcing amplitude than in Figure 4-12.

Figure 4-14: The barotropic mode slightly modified by an oscillatory basic state. The growth and decay are exponential and the phase shift for growth or decay is nearly fixed.
Figure 4-15: The barotropic mode altered by parametric resonance.

The evolution of the moduli have become similar to those of the parametric modes but the phase relationship still distinguishes it from the parametric modes.

Finally, Figure 4-16 depicts the functional dependence of growth rate on wavenumber and $\delta$ for $q_0 = 1$ and various values of $\epsilon$. The case of no or weak oscillation consists of the two distinct regions of barotropic instability. The areas on the right and left correspond to $S > q_0$ and $S < q_0$ and are distinguishable in that instability sets in at large and small wavenumbers, respectively. As $\epsilon$ increases these modes do not change dramatically but by comparing the first and last frame we observe that increasing $\epsilon$ reduces the size and strength of the unstable barotropic regions. In addition, there is a new mode that is introduced at $\epsilon = 0.3$ near $K = 1$, the parametric mode. As $\epsilon$ increases the strength of the parametric mode increases, as does its support.

To summarize this section, we have studied an oscillatory shear flow that is barotropically unstable at every instant in time. Parametric modes were introduced that have the same phase shifts after one period as in the pure parametric mode. The first subharmonic occurred at nearly the same wavenumber as in the first example we considered. Consequently, the growth rates in the two cases are very close. We thus conclude that whether or not snapshots of an oscillatory flow are barotropically unstable, does not appear to have any bearing on the growth rates of the parametric modes.
Figure 4-16: Plots that illustrate the functional dependence of the growth rate on wavenumber, $\delta$ and $\epsilon$. As $\epsilon$ increases the parametric mode is introduced and the barotropic modes are modified by being reduced in strength and size.
7. Four-Contour Example

The two-contour problem demonstrates the existence of parametric resonance in oscillatory shear flows. In a step towards understanding this instability in a continuous profile we study an oscillatory flow with four-contours. The vorticity jumps across the four-contours, \( y = -3a, y = -a, y = a \) and \( y = 3a \) are \( -R, -(S - R), S - R \) and \( R \), respectively, with \( 0 \leq R \leq S \). We recover the two-contour model by either taking \( R = 1 \) and replacing \( a \) by \( a/3 \) or \( R = 0 \). We dedicate special attention to the case where \( R = S/2 \) since that yields the shallowest shear profile, and yields the weakest growth rates. A snapshot of the velocity profile is presented in Figure 4-17 along with the velocity and vorticities of the basic state; as before \( q_0 \) denotes the vorticity of the mean background shear. If we set \( K = 2ak \), the governing equations are

\[
\frac{dq_m}{dt} = \frac{iS}{2} \left( K\alpha_m q_m - \beta_m \sum_{n=1}^{4} \exp(-K(y_m - y_n))q_n \right),
\]

with

\[
\begin{array}{ll}
\text{Shear Flow Profile} & \text{Velocity} & \text{Vorticity} \\
\hline
y = 3a & U_5 = a[S(t) + 2R(t)] - q_0 y & Q_5 = q_0 \\
y = a & U_4 = a[S(t) - R(t)] + [R(t) - q_0] y & Q_4 = q_0 - R(t) \\
y = -a & U_3 = Sy - q_0 y & Q_3 = q_0 - S(t) \\
y = -3a & U_2 = -a[S(t) - R(t)] + [R(t) - q_0]y & Q_2 = q_0 - R(t) \\
\end{array}
\]
In the special case of $S$ and $R$ both constant, the system reduces to the steady state problem. Unfortunately, computing the eigenvalues of the resulting four by four matrix symbolically is not very insightful. We choose to solve the problem numerically. By solving the eigenvalue problem for a range of wavenumbers, we determine the dispersion relations for the four waves in the system. For $q_0 = 1$ and $K \geq 0$ the calculations indicate that the shear flow is stable for $0 \leq S \leq 1$, as in the two-contour analogue. The criteria for parametric resonance is that, for a given wavenumber, the difference between the frequencies of two distinct waves is a multiple of the forcing frequency. This means $\sigma_p - \sigma_q = n\omega$ where $1 \leq p, q \leq 4$, $n$ is a positive integer and $\omega$ is the forcing frequency which we take to be unity. Moreover, we require that the vorticity changes sign at the two interfaces $p$ and $q$. Each pair of waves interacts with the basic state to give rise to a resonant triad interaction. In the two-contour case there were only two dispersion curves and they were negatives of one another, hence the resonance criteria reduced to the statement that $\sigma = n/2$. The four-contour problem allows for the possibility of four distinct resonances, two of which occur at the same wavenumber and are mirror images of each other about the $x$-axis. Figure 4-18 plots the positions of three qualitatively different types of these resonances for the first subharmonics and harmonics; the fourth redundant resonance is ignored for the sake of clarity. Clearly, as the number of contours increases there are more possibilities for pairs, i.e. resonant wave triads, and therefore more potential for parametrically unstable wavenumbers.

$$\vec{a} = \begin{bmatrix} 1 + 2R/S - 3q_0/S \\ 1 - q_0/S \\ -(1 - q_0/S) \\ -(1 + 2R/S - 3q_0/S) \end{bmatrix}, \quad \text{and} \quad \vec{\beta} = \begin{bmatrix} R/S \\ 1 - R/S \\ -(1 - R/S) \\ -R/S \end{bmatrix}. \quad (4.59)$$

$$\vec{q}^{(0)} = S_+^1(T) \exp(i \sigma_1 t) + S_-^1(T) \exp(-i \sigma_1 t) + S_+^2(T) \exp(i \sigma_2 t) + S_-^2(T) \exp(-i \sigma_2 t); \quad (4.60)$$
Figure 4-18: The dispersion relations for \( S = 1/2, \ R = 1/4, \ q_0 = 1, \ \omega = 1 \) and the transition points where the first subharmonic and harmonic resonant triads are located. The solid and dashed vertical lines indicate where the subharmonic and harmonic instabilities are located, respectively. The vertical bars are drawn to show the location of the transition wavenumbers, i.e. where a resonance occurs. The length of each vertical bar is a multiple of the frequency, in this case one.

as before \( S_{\pm}^{1/2} \) and \( S_{\mp}^{1/2} \) are complex conjugates of each other, respectively. To write this solution, we have exploited the symmetry in the dispersion relation by expressing the four frequencies in terms of \( \sigma_1 \) and \( \sigma_2 \), the two distinct eigenvalues, with positive real parts, of the steady system. The previous analysis follows through analogously except for the fact that now there are different waves that may interact. The three different values of the growth rates are given by

\[
\frac{1}{2} \left( \frac{(\hat{s}_+^{1,2} M_1 \hat{s}_-^{1,2})(\hat{s}_-^{1,2} M_1 \hat{s}_+^{1,2})}{(\hat{s}_+^{1,2} M_1 \hat{s}_+^{1,2})(\hat{s}_-^{1,2} M_1 \hat{s}_-^{1,2})} \right)^{1/2} \epsilon,
\]

where the components of the matrix \( M_1 \) are defined component-wise as
\[
\frac{i}{2} \left( K \alpha_m - \beta_m \sum_{n=1}^{4} \exp(-K(y_m - y_n)) \right) \tag{4.62}
\]

and the two constant vectors are

\[
\vec{\alpha} = \begin{bmatrix} 2, & 1, & -1, & -2 \end{bmatrix}^T \quad \text{and} \quad \vec{\beta} = \begin{bmatrix} 1/2, & 1/2, & -1/2, & -1/2 \end{bmatrix}^T.
\tag{4.63}
\]

All the adjoint vectors must have a common superscript, 1 or 2, as must all of the non-adjointed vectors, but these two superscripts need not be the same. This allows for four different cases within this one equation.

It is possible to compare the growth rates from weakly nonlinear analysis with those computed numerically. Since the steady four-contour profile is stable for \(0 \leq S \leq 1\), we pick as before \(S(t) = \delta + \epsilon \cos(t)\) with \(\delta = 1/2\) and \(\epsilon \leq 1/2\). A series of calculations were done for a wide range of wavenumbers and amplitude oscillations in order to determine the stability boundaries for this dynamical system. Figure 4-19 depicts the first several unstable regions. As in the two-contour example, numerical errors prevent the transition curves from extending down to the wavenumber axis.

By solving the steady state problem numerically, we have determined that there are three transition wavenumbers for the first subharmonic, \(K = 0.3610\), \(K = 0.6083\) and \(K = 1.5058\). The first two generate instability tongues stemming from the \(K\) axis. However, the third is separated from the transition point and it appears to have bifurcated into two distinct tongues. The growth rates of the first two regions are \(0.0605\epsilon\) and \(0.0406\epsilon\), as computed by solving the stability problem numerically. These values are in good agreement with the growth rates predicted from the multiple scales analysis and are accurate to three significant digits. The boundaries of the tongues are linear, as is true for the Mathieu equation and our two-contour example of pure parametric resonance. The dominant tongue in the lower plot of Figure 4-19 stems from \(K = 1.4008\), which is one of the first harmonics. In particular, it arises from an asymmetric resonance. The transition wavenumber at \(K = 0.8714\) has not yet generated a visible tongue and we have decided to not plot the third wavenumber at

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Figure 4-19: The stability regions and growth rates for $S = 1/2$, $R = 1/4$, $q_0 = 1$ and
$\omega = 1$.

$K = 3.5001$ since it is far away and weak. The quadratic nature of the transition curves for
the third tongue appears for moderate values of $\varepsilon$. Note that the transition curves appear a
bit jagged because of numerical error.

The maximum growth rate during one period, at $\varepsilon = 1/2$ is 0.19184. It lies between
the growth rates of the two-contour problem at the same $\varepsilon$ for vorticities of $S = 1/2$ and
$S = 1/4$, which are 0.2886 and 0.1443, respectively. This suggests that it may be possible
to find upper and lower bounds of the parametric growth rates of an $N$-contour problem by
evaluating the two-contour problems at the shallowest and steepest slopes.

Figure 4-20 depicts the perturbation vorticity of the growing eigenmodes after one pe-
period. The specific magnitude is not important since this is a linear system. What is im-
portant is the relative amplitude: the interfaces of large amplitudes are the ones that are
instrumental in forming that particular type of parametric resonance. For the first sub-
harmonics, the stability tongue that is the weakest is the symmetric resonance involving
the two inner interfaces. For the first harmonic, the strongest instabilities are due to the
asymmetric resonance.
Figure 4-20: $S = 1/2$, $R = 1/4$, $q_0 = 1$ and $\omega = 1$ (first subharmonic resonance). These figures are obtained by first calculating the four eigenvectors of the Floquet matrix, then evolving each of these vectors forward by one period to see how the eigenvectors have grown in one period. This indicates which contour interfaces are involved in the instability process.
8. Conclusions

Previously, the study of parametric instability in geophysical systems has been most limited to motions involving gravity waves. In chapter four we have studied parametric shear instability in the context of two-dimensional vorticity equation. The uninteresting case of pure oscillatory flows demonstrated that if \( \delta = 0 \) then all the flows are linearly stable. If the snapshot of the extreme profile is unstable then in the oscillatory case there are intervals of growth and decay but after one period there is no net growth. The circumstances where \( \delta \) is nonzero yield oscillatory flows that are unstable for exactly the same wavenumbers that are unstable according to the steady theory. Therefore, no new instability processes were discovered for these flows.

Our simple example where a steady flow of uniform vorticity interacts with an adverse oscillatory shear illustrated a simple example of parametric instability in oscillatory shear flow; the background shear creates an asymmetry analogous to the topography in the case of a tidally forced oscillating jet. The snapshot of the profile at every instant in time is stable to shear instability, but the oscillating state itself is linearly unstable, giving rise to a series of unstable wavenumbers. The modes generated were subharmonics and harmonics in alternating order, as in Mathieu’s equation. This simple example and the Mathieu equation share similar stability boundaries. The multiple scales method gave us an analytical solution to the growth rate of the first subharmonic which agreed with the predictions from numerical simulations. In general, it appears that the oscillating instability of shear flow increase with increasing background shear, but decrease with increasing wavenumbers and frequencies.

We studied an oscillatory flow whose snapshot profile is, at least sometimes, barotropically unstable. The introduction of oscillations of increasing amplitude demonstrated that the barotropic mode is stabilized by the oscillations. Moreover, there are higher wavenumbers that are excited that are harmonics and subharmonics. These modes have the same qualitative behavior as in the purely parametric instability case and hence we call them parametric modes. This example illustrated the point that the strength of the parametric instabilities had negligible dependence on whether snapshots of the oscillating state are
unstable to steady barotropic instability; they depend most dramatically on the size of the oscillations.

We finished by studying an example of a four-contour problem. For each type of harmonic there are generally three possible transition wavenumbers, which give rise to more possibilities of parametric resonance, hence more instability tongues. However, these regions tend to be of weaker strength than the two-contour analogue. Certainly, it is of great interest to extend this study to a higher number of contours in order to understand the importance of parametric modes in smoother profiles.
Chapter 5

The Instability of Tidally and Topographically Forced Jets

Section 5.1 develops the model that we study and explains how it resembles the oscillatory jet in the Cape Cod Bay. Then, in §5.2 we explain the method that is used to solve the linear stability problem and then present our results from these calculations. Finally, in §5.3 we show our results from the nonlinear simulations, compare them with the results from linear theory and describe the instabilities and turbulence that we have observed.

1. The Model

The tidally and topographically forced SW equations are,

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f \hat{k} \times \mathbf{u} = -g \nabla (\eta - \eta_e) \]  \hspace{1cm} (5.1)

\[ \frac{\partial \eta}{\partial t} + \nabla \cdot ([\eta + h_B] \mathbf{u}) = 0. \]  \hspace{1cm} (5.2)

The two parameters, $g$ and $f$, are the -gravity and the Coriolis parameter, both of which we assume are constant. The variables $\mathbf{u} = (u, v)$, $\eta$ and $h_B$ denote the horizontal velocity field, free-surface depth and topographic height, respectively. The topography is the same as in Chapter 3, see (3.1). In addition to appearing explicitly in the equation, the tidal forcing $\eta_e$ also appears in the inflow and outflow boundary conditions at the offshore open
boundary. We assume that the tides are well approximated by the barotropic tide; it has a single frequency and a much longer horizontal length scale than that of the topography and free surface variations. Therefore, we take

$$\eta_e = \eta_0 \cos(\omega t).$$  \hspace{1cm} (5.3)

where $\eta_0$ is the amplitude of the tide and $\omega$ the tidal frequency.

![Diagram](image)

Figure 5-1: The geometry of the model which is forced by step-like topography and the tides along with all of the geometric length scales.

The ambient geometry of the model is illustrated in Figure 5-1 along with the many geometric length scales. Since the tidal frequency is nearly half of the inertial frequency, the Coriolis force is likely to be important. Therefore, the physics imposes an additional length scale; the Rossby radius of deformation,

$$L_R = \frac{\sqrt{gH_0}}{f}. \hspace{1cm} (5.4)$$

We collect the various length scales and form the following five nondimensional parameters. First, a tidal Froude number, $F_\omega$, that is a ratio of the period of the tides to the period of the relatively fast surface gravity waves.
We include the factor of \( \pi \) to simplify our expressions. Second, \( \delta \) is a ratio of the tidal and inertial frequencies,

\[
\delta = \frac{\omega}{f}.
\]  

(5.6)

Third, \( \epsilon \) is a ratio of the free-surface depth to that of the total depth

\[
\epsilon = \frac{\eta_0}{H_0}.
\]

(5.7)

Fourth, \( \alpha \) and \( \beta \) are the nondimensionalized half widths and heights of the topography,

\[
\alpha = \frac{L_B}{2L_1} \quad \text{and} \quad \beta = \frac{H_B}{2H_2}.
\]

(5.8)

With all of these nondimensional parameters there is a vast parameter space to be explored. Since our interests lie in studying motions that are most relevant to those in the Cape Cod Bay, we use the parameters in this region as our guide. Observations dictate that the parameters for this region are

\[
L_1 \sim 10^3 \text{ m} \quad H_1 \sim 5 \text{ m} \quad g \sim 7.3 \times 10^{10} \text{ m/day}^2
\]

\[
L_2 \sim 10^4 \text{ m} \quad H_2 \sim 50 \text{ m} \quad \omega \sim 3.9\pi \text{ rad/day}
\]

\[
L_B \sim 10^2 \text{ m} \quad H_B \sim 45 \text{ m} \quad f \sim 2.7\pi \text{ rad/day}
\]

\[
\eta \sim 1 \text{ m}
\]

(5.9)

The appropriate nondimensional groups that describe this system are \( F_\omega = 10^3, \delta = 1.44, \epsilon = 10^{-2}, \alpha = 5 \times 10^{-2} \) and \( \beta = 5 \times 10^{-1} \). In our numerical model we take the similar values of \( \delta = 1.44, \epsilon = 0.1, \alpha = 0.1, \beta = 0.4 \) unless otherwise specified. We vary the tidal Froude number to learn how the instabilities are affected. The hope is that we
will be able to extrapolate from this work into the regime of physical interest, to learn of the type of instabilities that may occur in the Cape Cod Bay. In the domain of large tidal Froude numbers, as is appropriate to the Cape Cod Bay, asymptotic solutions can be found that are accurate to $O(F_{\omega}^{-1})$. In this limit the leading order solution to the linearized SW equations is, given (5.3),

$$\eta \sim \eta_e = \eta_0 \cos(\omega t)$$

(5.10)

$$u \sim -\frac{(x + 1)}{H} \frac{\partial \eta_e}{\partial t} = \frac{(x + 1)}{H} \omega \eta_0 \sin(\omega t)$$

(5.11)

$$v \sim \frac{(x + 1)}{H} f \eta_e = \frac{(x + 1)}{H} f \eta_0 \cos(\omega t)$$

(5.12)

It is clear that in this asymptotic regime the velocity field decreases as the tidal frequency decreases, i.e. the tidal Froude number increases. Moreover, the topography directly determines the location and strength of the shear in the velocity fields. Presumably, this qualitative dependence will carry over even beyond this asymptotic limit.

The inflow (outflow) open boundary conditions for the nonlinear dynamics are determined in the following systematic fashion. First, we assume that the equilibrium tide is defined by (5.10). With this we solve the linear equations for the case of a flat bottom in the limit of $F_{\omega} \gg 1$. The solution is precisely (5.10), (5.11) and (5.12). Second, we evaluate this solution at $x = 1.2$ and then impose these boundary condition on the linear dynamics with the actual topography. Due to the complex topography this equation is solved numerically. The resulting linear solution is what we use as the initial conditions for the nonlinear simulations. This linear solution also specifies the inflow (outflow) boundary conditions and is the state that the nonlinear solution is forced to relax to in the sponge layer.

Even though this method uses the limit of large tidal Froude number to obtain the first linear solution, we still apply it to situations when this parameter is order one. The reason being that we do not feel the specific structure of the of the inflow and outflow boundary conditions should alter the results significantly. What is important is that they are
oscillatory, and indeed, our linear solution always satisfies this properly.

To determine the origin of (5.10) consider the following. Assume for the moment that we wanted to model the tides as a wave that travels perpendicular to the coast in a large ocean basin. The appropriate representation would be that of a traveling wave, \( \eta_e = \eta_0 \cos(kx - \omega t) \). If we take the limit as \( x \ll L \), we recover (5.10). This limit is equivalent to narrowing our solution to near the solid boundary or, in other words, narrowing our focus to the coastal region. Similarly, if we assume the tides travel parallel to the coast the equilibrium tide would be the following, \( \eta_e = \eta_0 \cos(ly - \omega t) \). When we restrict this to the small length scales, we recover (5.10). This means that to this order of approximation it does not matter whether the tides come in across the shelf or along the shelf. This is due to the fact that the length scale of the tides is much longer then the length scales of the coastal region.

To obtain the solution outside for general tidal Froude numbers we linearize the governing equations and impose the periodic boundary conditions at the open boundary. The system of three equations can be expressed as a solitary second order differential equation in terms of the velocity. This allows us to express the linear fields, \( \tilde{\eta}_L, \tilde{u}_L \) and \( \tilde{v}_L \), in terms of one function \( A(x) \),

\[
\begin{align}
\tilde{\eta}_L &= \frac{dA(x)}{dx} \cos(\omega t) \\
\tilde{u}_L &= \frac{\omega A(x)}{H(x)} \sin(\omega t) \\
\tilde{v}_L &= \frac{f A(x)}{H(x)} \cos(\omega t).
\end{align}
\]

The equation that defines the unknown function is,

\[
\frac{d^2 A}{dx^2} + \frac{\omega^2 - f^2}{gH} A = 0.
\]

If \( H \) is constant, it is trivial to solve this equation. However, for general topographic profiles this equation must be solved numerically, which we do using a simple finite difference scheme. The first boundary condition we impose is no normal flow at the wall; the second
is the open boundary inflow (outflow) condition explained above.

Figure 5-2 compares the linear solutions for several different Tidal Froude numbers with the asymptotic limits. Clearly, for $F_\omega = 100$ or larger, the asymptotic limit is a very close approximation to the linear solution. Since the Cape Cod Bay has $F_\omega \sim 10^3$, the asymptotic limit is an appropriate approximation to that particular linear solution. Observe that all the jets are located on the shelf and therefore we should expect the instabilities to develop in the same region.

2. Linear Stability Analysis

Before exploring the nonlinear regime we analyze the linear stability problem. This has the advantage of not only determining whether a given basic state is unstable, but if so, it gives a means to calculate the growth rates and structures of the instability. For steady shear flows in SW dynamics Ripa’s Theorem (see Ripa (1983)) gives necessary conditions for two distinct types of instabilities, one due to the slow vortical motion and the other fast gravity waves. There exists no such analogue for oscillatory shear flow even in the simple QG model. In SW theory three types of instabilities that may arise, inertial (see Drazin and Reid (1995)), vortical (see Pedlosky (1987)) and parametric (see Rosenblat (1968)).

The simple normal mode analysis that is used for steady basic flows, decomposing the perturbations in space and time, can not be applied to time-dependent ones. A method that is traditionally employed to determine the stability of periodic states is Floquet Theory (see Coddington and Carlson (1997)). The first step is to derive the system of linear perturbation equations; the coefficients of this system are periodic functions of time with period $T$. Second, for a particular wavenumber and set of nondimensional parameters, integrate the perturbation equations forward one period with the identity matrix as the initial conditions; the resulting matrix is the Floquet matrix. Third, determine the eigenvalues of the Floquet matrix. The eigenvalues with real parts of magnitude greater than one represent unstable modes and their corresponding eigenvectors reveal the structure of these instabilities. An alternative approach to Floquet analysis is to solve the linear perturbation equations numerically for many periods while renormalizing the numerical solution after every period. This
Figure 5-2: The four different sets of three plots for $F_\omega = 100, 10, 3$ and 2, respectively. They compare the linear solutions with the asymptotic solutions in the case of $F_\omega \to 0$. 
is the method that has produced meaningful results and therefore the one that we present; the details will follow shortly.

We remark that the linear solution is not an exact solution to the nonlinear dynamics. In the linearization, the following terms are neglected from the continuity and $x$- and $y$-momentum equations are, respectively,

$$\frac{\partial (\tilde{\eta}_L \tilde{U}_L)}{\partial x}, \quad \tilde{U}_L \frac{\partial \tilde{U}_L}{\partial x}, \quad \text{and} \quad \tilde{U}_L \frac{\partial \tilde{v}_L}{\partial x}. \quad (5.17)$$

It is justifiable to neglect these terms if the basic state is small amplitude, but this need not be the case. To obtain an exact periodic solution to the nonlinear equations we must calculate it numerically. In particular, we denote this nonlinear basic state as $\tilde{\eta}(x, t)$, $\tilde{u}(x, t)$ and $\tilde{v}(x, t)$. This solution is found by numerically integrating the SW equations with the linear solution as the initial condition. After several periods it equilibrates to a fully nonlinear solution, which is the nonlinear periodic basic state we analyze.

We perturb the basic state and derive the linear perturbation equations for the basic state,

$$\eta = \tilde{\eta} + \text{Real} \{ \tilde{n}(x, t) \exp (iky) \} \quad (5.18)$$
$$u = \tilde{u} + \text{Real} \{ \tilde{u}(x, t) \exp (iky) \} \quad (5.19)$$
$$v = \tilde{v} + \text{Real} \{ \tilde{v}(x, t) \exp (iky) \}, \quad (5.20)$$

and assume that the quantities $\tilde{\eta}$, $\tilde{u}$ and $\tilde{v}$ are infinitesimal. We substitute these decompositions into the nonlinear equations and then neglect the quadratically primed quantities since they are of smaller order than the linear terms

$$\frac{\partial \tilde{\eta}}{\partial t} + \frac{\partial}{\partial x} (\tilde{h} \tilde{u} + \tilde{a} \tilde{\eta}) + ik(\tilde{h} \tilde{v} + \tilde{v} \tilde{\eta}) = 0 \quad (5.21)$$
$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial x} (\tilde{u} \tilde{u} + g \tilde{\eta}) + ik \tilde{v} \tilde{u} - f \tilde{v} = 0 \quad (5.22)$$
$$\frac{\partial \tilde{v}}{\partial t} + \frac{\partial}{\partial x} (\tilde{v} \tilde{v} + f) \tilde{u} + (ik \tilde{v} - \tilde{u}_x) \tilde{v} + ik g \tilde{\eta} = 0. \quad (5.23)$$
The subscript $x$ denotes the partial differentiation with respect to coordinate $x$. The linear
system governs the perturbations for any basic state that is a function of $x$ and $t$ and exactly
solves the nonlinear tidally and topographically forced SW equations, (5.1) and (5.2).

The processes through which we compute the linear stability of a basic state involves
several steps. First, we initialize the nonlinear numerical model with the linear solution,
with no perturbations, and integrate it forward in time for ten periods so that it can equi-
librate to a periodic state. Second, we integrate the nonlinear dynamics for an additional
period, while storing the profile of the basic state at each instant; this represents the non-
linear solution $\tilde{\eta}(x, t)$, $\tilde{u}(x, t)$ and $\tilde{v}(x, t)$. Third, to determine the linear stability of this
basic state, we choose a wavenumber in the $y$ direction and an abundance of sine modes in
the $x$ direction and integrate the linear perturbation equations for a hundred periods, while
renormalizing the solution after every period by dividing all fields by the ratio of the peak
value of the vector with $h$, $u$ and $v$ as the three components, to its peak value in the initial
state. The purpose of the many modes in the $x$ direction is that we require a rich enough
spectrum that the most unstable mode, or a reasonably good approximation to it, can be
expressed as a sum of this basis. As the system evolves forward the most unstable mode
grows fastest and dominates over all other modes. After sufficient time has passed, the sys-
tem equilibrates with the scaling factor being $\exp(\sigma T)$ where $\sigma$ is the growth rate and $T$ the
period. The fields take on the structure of the most rapidly growing mode. The particular
method that we use is a direct extension of the nonlinear method.

Figure 5-3 presents results from the linear stability calculations of the nonlinear peri-
odic solutions. Each plot varies one of the nondimensional parameters to determine what
effect this has on the instabilities. The results are summarized in Table 5.1. This series of
calculations are by no means complete but they do indicate the qualitative dependence of
growth rate on the governing parameters for the parameter space examined.

As $F_\omega$ increases the growth rates decreases and the most unstable wavelength becomes
larger. This is not surprising since as this parameter increases the scale of the velocity
decreases, as is suggested by (5.11). This suggests that the growth rates for typical $F_\omega$, as
may be found in the Cape Cod Bay, are even smaller still.
Figure 5-3: The growth rates produced from the linear stability calculations for the oscillatory jet. The plots indicates that the growth rate decreases as $F_\omega$ increases, $\delta$ increases, $\epsilon$ decreases, $\alpha$ increases and $\beta$ decreases.
Table 5.1: A summary of the growth rate dependence on the nondimensional parameters, as predicted from the linear stability calculations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Growth Rate</th>
<th>Most Unstable Wavenumber</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_\omega$</td>
<td>Decrease</td>
<td>Decrease</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Decrease</td>
<td>Increase</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Increase</td>
<td>Increases</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Decrease</td>
<td>Decrease</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Increase</td>
<td>Increase</td>
</tr>
</tbody>
</table>

a. Linear Analysis of $F_\omega = 2$

We will now present the structures of the instabilities that are predicted from the linear theory. The particular along shelf wavenumber we use is the one observed to be the most unstable wavenumber in the corresponding nonlinear simulation. Figure 5-4 presents the perturbation vorticity field at four equally spaced times in one period for the case of $F_\omega = 2$. There are a couple of observations to be made. First, the structure consists of three rows of alternating vortices, the center of which is much stronger than the other two. Second, the entire street travels in a tidal orbit which induces stretching and contraction; this causes oscillations in the strength and size of the eddies. Frames two and four are taken when the orbit is furthest and closest to shore; therefore, the vortices appear smallest and largest, respectively. This linear mode moves fluid in orbits on the shallow side of the step but does not transport any fluid between the deep and shallow waters. If any across shelf transport occurs in the numerical simulations it must be attributed to nonlinear effects. This steps from the fact that stable jets do not change very much and therefore naturally act as barriers for transport between the shallow and deep waters.

b. Linear Analysis of $F_\omega = 3$

The structure of the most unstable mode for the case of $F_\omega = 3$ is illustrated in Figure 5-5. The snapshots are all taken during the same period. The modal structure is very similar
Figure 5-4: Snapshots in one period of the most unstable mode calculated from the linear stability analysis for $F_w = 2$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 4$. 
Figure 5-5: Snapshots throughout one period of the most unstable mode calculated from the linear stability analyze for $F_o = 3$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 6$. 
to that in Figure 5-4 except for the fact that the row of vortices closest to the deep waters is relatively weak. The comparison between this figure and the previous one suggests that this structure is not exceptional but may indeed be common for a variety of parameters.

c. Linear Analysis of $F_\omega = 10$

The third model structure we present, in Figure 5-6, is the most unstable mode for $F_\omega = 10$; this tidal Froude number is close to that appropriate in the Cape Cod Bay. The row of vortices nearer to the deep water is like that of the previous figure. However, the one on the shallow side is quite different in that the eddies have tails moving downstream and away from the jet. This series of plots does appear to be noisier than the previous two which we attribute to numerical dispersion.

3. Numerical Solutions

If the tidally and topographically forced viscous SW equations are written in terms of the momentum transport functions, $U = uh$ and $V = vh$, the governing equations are as follows:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left( \frac{U^2}{h} \right) + \frac{\partial}{\partial y} \left( \frac{UV}{h} \right) = fV + gh \frac{\partial h_B}{\partial x} + \nu \nabla^2 U - r(x)(U - U_L) \quad (5.24)$$

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x} \left( \frac{UV}{h} \right) + \frac{\partial}{\partial y} \left( \frac{V^2}{h} + \frac{gh^2}{2} \right) = -fU + \nu \nabla^2 V - r(x)(V - V_L) \quad (5.25)$$

$$\frac{\partial h}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \nu \nabla^2 h - r(x)(h - h_L). \quad (5.26)$$

The parameters $\nu$ and $r(x)$ represent the Laplacian and Rayleigh friction coefficients. The first is used for numerical stability whereas the second is important in creating the sponge layer. The across-shelf and along-shelf coordinates $x$ and $y$ range from $-1$ to $1.2$ and $-1$
Figure 5-6: Snapshots of the most unstable mode calculated from the linear stability analysis for $F_w = 10$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 20$. 
to 1, respectively. The reason for the elongated geometry is that we have a sponge layer between $x = 1$ and $x = 1.2$ to damp out the waves that are reflected at the right boundary, in an attempt to simulate an open boundary. The manner in which this is done is that the function $r(x)$ is zero on the interval $[-1, 1]$ and then increases quadratically to a specified value. The functions $U_L, V_L$ and $h_L$ denote the linear solution that the nonlinear solution is forced to relax to in the sponge layer. The free-surface and bottom topography are located at $z = 1 + \eta$ and $z = 1 - h_B$, respectively. Therefore, the total depth is $h = h_B + \eta$.

An alternative method to applying a sponge layer is to apply a radiation boundary condition, as was originally done in Oranski (1976). This ensures that there are only outward traveling waves that reach the open boundary, thereby eliminating the inward traveling waves. This is well documented for the wave equation and other similar unforced systems. Our problem has external forcing which complicates matters, and hence why we have instead decided to use the sponge layer, since it is more straightforward to implement.

The physics and geometry of the model defines five nondimensional parameters of the inviscid dynamics. Our objective is not to fully map out the different regions of parameter space but instead to focus on a few examples that depict some interesting phenomena. The gravity waves in the SW model dictate that the Courant-Friedrichs-Lewy condition is satisfied, (2.9). Unfortunately, this numerical constraint prevents us from exploring the parameter space that is relevant to the oscillating current in the Cape Cod Bay, because of the very long time integrations required. The tidal Froude number appropriate for this physical system is $F_\omega \leq 10^3$. Our limited computer resources have constrained us to pick $F_\omega = 10$. This means that the frequencies we are studying are faster by a factor of 100. Therefore, we can only extrapolate from our numerical experiments to learn about parameter regimes relevant to the oscillatory current in the Cape Cod Bay.

a. **Nonlinear Simulations for $F_\omega = 10$**

The first numerical experiment is for $F_\omega = 10$ and is initialized with the appropriate linear solution, as was done with all of our nonlinear simulations. The perturbations are polychromatic and demonstrated that the most unstable mode has a wavenumber of $k = 2\pi$ in the
Figure 5-7: Snapshots of the total vorticity for $F_\omega = 10$ at $t= 590, 670, 950, 955, 960, 965$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 20$. 
along shelf direction, as predicted in the linear theory. Throughout the sixty periods that the
code is run there are no subharmonic instabilities that cause the resulting oscillating vor-
tices in the vortex street to split or merge. Figure 5-7 shows snapshots of the total vorticity
field at six instances. The contour plots of the vorticity only extend as far as \( x = 1 \) since
the sponge layer is a numerical artifact and what transpires in this region is not physically
meaningful. The first three frames are at different times but at the same phase of the period.
The fact that the third frame occurs after nearly fifty periods indicates that these instabili-
ties is slow in relation to the time-scale of the oscillation. The final four frames are evenly
spaced snapshots of the same period. As in the linear modes, the oscillating vortex street
travels in a tidal period in an anticyclonic sense in the shallow portion of the domain. The
eddies are largest in frame 5 where the street is closest to the solid boundary. Throughout
the majority of the period there is an asymmetry where the cyclones are stronger than the
anticyclones.

To compare the nonlinear evolution of the instability with the linear theory, we per-
formed two other numerical simulations for the same choice of parameters. One has a
single perturbation in the along shelf direction, that being the most unstable mode, and
the other has no perturbations. The difference between these two vorticity fields yields the
perturbation field for the monochromatic run and it is plotted in Figure 5-8. The first four
frames show snapshots within one period where the most unstable mode is growing expo-
nentially. First, we observe that the perturbation field is much cleaner than that in Figure
5-6. This indicates that the linear stability calculation is more sensitive and generates more
numerical noise then the nonlinear code. Second, the row of eddies closest to the shore
does have a similar slanted behavior as seen in the linear calculation. Frames five and six
are snapshots at the same phase as frame one and indicate how the nonlinearities alter the
perturbation field. We will see why more vortices are generated in the next example.

The growth rates of the monochromatic experiment are presented in Figure 5-9. The
oscillatory behavior of the growth is obvious. The envelopes increase linearly, thus signifying
exponential growth. There are slight differences between the growth rates determined
by the four different envelopes but they are all very similar. We choose the maximum of
these four growth rates, that being 0.012, as being representative of the growth of the sys-
Figure 5-8: Snapshots of the perturbation vorticity for $F_\omega = 10$ at $t= 400, 405, 410, 415, 740, 1160$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 20$. 
Figure 5-9: The plots of the logs of the growth rates for $F_\omega = 10$. The legend states what the growth rate is for each set of curves. Even though the growth rates oscillate the envelopes grow exponentially.
tem. That predicted from linear theory is 0.010. The closeness of these two numbers is remarkable especially considering the differences in the structures that develop.

\[ b. \ Nonlinear \ Simulations \ for \ F_\omega = 3 \]

The next set of numerical experiments is for \( F_\omega = 3 \). The snapshots of the total vorticity field throughout one period with polychromatic perturbations are presented in Figure 5-10. By polychromatic we mean that there are multiple wavenumber perturbing the basic state. The first frame has many ripples in the shallow region which is due to numerical dispersion that is generated by a shock traveling away from the solid boundary. The vortical structures are not dissimilar to those in Figure 5-7. Two obvious differences are first, that the most unstable wavenumber is \( k = 3\pi \) and second, that there are two sets of streets. One street is located near the topographic jump and the other in the middle of the shallow zone. The first street behaves in a similar fashion to the previous simulation in that the strength and size of the vortices oscillate as they travel in tidal orbits. The cyclones, which tend to be strongest, are boomerang shaped (see chapter two), which indicates there are strong non-QG effects present because of the large topographic difference across the step. The second street is much weaker but also oscillates in tidal orbits.

The perturbation vorticity of the monochromatic simulation is presented in Figure 5-11. As in the previous case, the first four frames are part of the same period during an interval of linear growth. These frames indicate that the growing structures that develop in the nonlinear evolution are nearly identical to those predicted from the purely linear theory (Figure 5-4). The comparison is indeed remarkable. The last two frames have the same phase as frame one but later in the evolution of the simulation. They give a glimpse of the structure of the second street superimposed with the first.

Figure 5-12 shows the perturbation field of the same monochromatic experiment early on. In the center, above the steep topography, there is a vortex street that is the most unstable mode before it has become dominant. However, there are also two other vortex streets. The street located in the interior of the shallow fluid grows but not as quickly as the most unstable mode; thus it does not appear in the early frames in Figure 5-11. Frames
Figure 5-10: Snapshots of the total vorticity for $F_\omega = 3$ at $t=240, 241, 242, 243, 244, 245$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 6$. 
Figure 5-11: Snapshots of the perturbation vorticity for $F_\omega = 3$ at $t= 90, 91.5, 93, 94.5, 120, 144$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 6$. 

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Figure 5-12: Snapshots of the perturbation vorticity for the monochromatic simulation of $F_\omega = 3$. This depicts the three different structures that grow. The street closest to the center of the channel is the most unstable mode. The street just to the left is another unstable mode that grows more slowly. The street adjacent to the boundary is caused by numerical error and therefore is unphysical.
five and six are taken after this secondary structure has grown and been equilibrated by
the nonlinear effects. The end result is a superposition of the two vortex streets, altered by
nonlinearities. There is a third vortex street that emerges adjacent to the solid boundary. If
they traveled with the coastline on the right we would have reason to believe that they are
Kelvin-waves. However, they are stationary which suggests that they are due to numerical
errors. Indeed, when doing a similar computation for a coarser grid resolution we find that
these eddies adjacent to the solid boundary span the same number of grid points and are of
larger amplitude, thereby signifying that they are numerical artifacts. This is in contrast to
the other two vortex streets whose length scales do not depend on the resolution of the grid.
We attribute the existence of these boundary layer vortices to the fact that the higher order
schemes we are using typically have difficulties at the boundary. Even though we have not
been able to eliminate this error, we have chosen to present numerical simulations that stop
well before these boundary layers become significant.

Figure 5-13 depicts the growth rates for the particular monochromatic simulation in
question. The nonlinear simulation yields growth rates that the maximum growth rate is
0.057. That computed from the linear theory is 0.058. The close agreements between
the growth rates and the structures of the most unstable modes confirms that the linear
theory is indeed capturing the right behavior. It should be remarked that the linear theory
does have its defects however. It predicts that the most unstable mode is $k = 4\pi$ whereas
the polychromatic perturbation indicated that $k = 3\pi$ was the most unstable mode. To
further investigate this discrepancy we did a numerical experiment with a monochromatic
perturbation of $k = 4\pi$ to determine its growth rate. The nonlinear theory indicates that the
growth rate is 0.042 and not 0.064, as predicted from the linear theory. As of yet, we have
been unable to account for this error.

c. Nonlinear Simulations for $F_\omega = 2$

The final choice of tidal Froude number that we consider is $F_\omega = 2$. Figure 5-14 shows the
total vorticity of a polychromatic perturbation at six different times, none of which are in
the same period. The most unstable mode is $k = 5\pi$. The instabilities that grow are initially
similar to the previous plots, but the flow is more unstable which gives rise to turbulence.
Figure 5-13: The plots of the logs of the growth rates for $F_w = 3$. The legend states what the growth rate is for each set of curves. Even though the growth rates oscillate the envelopes grow exponentially.
Figure 5-14: Snapshots of the total vorticity of the polychromatic simulation for $F_\omega = 2$ at $t=56, 71, 91, 107, 134, 156$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 4$. 

This example proves that simple one dimensional topography and monochromatic tides are indeed a mechanism through which turbulence can be generated. Presumably, this is not a solitary example but occurs with other parameter as well. The unstable oscillatory jet generates coherent circular cyclones that remain above the topographic step.

The perturbation vorticity for this monochromatic run with \( k = 5\pi \) is shown in Figure 5-15. The first four frames are from the same period during a time of linear growth and the final two are the perturbation fields after the nonlinearities have equilibrated the growth. Even though this mode is more unstable than the previous two we have considered, the structure is not qualitatively different. The appearance of essentially the same structures in these four different examples suggests that this particular structure is not an abnormality.

For the sake of completeness, the growth rate curves are presented in Figure 5-16. The maximum of the various measures of the growth rates from the nonlinear simulation is 0.091 which is close to that predicted from linear theory, which is 0.098.

The final plot in Figure 5-17 depicts the total vorticity of a polychromatic run that has parameters that are identical to the previous case except that \( \delta = 1.1 \). This simulation is very similar to that of Figure 5-14. The first difference that is noticeable in frame two is that some of the growing cyclones are asymmetric since the peak in vorticity is located off center of the vortex. This elongated structure is reminiscent of the very non-QG simulations in chapter two that yielded strong instabilities. As the instability develops in Figure 5-17, the cyclones travel upstream but remain adjacent to an anticyclone. The cyclones are strong enough to roll up and form a dipole pair with an anticyclone. This dipole is injected into the deep water, thereby causing a large amount of vortex tube stretching which results in asymmetrically stronger cyclones. There is a second dipole that forms along the current that presumably will fall down the topography as well, if given enough time. This simulation contains more turbulence than before as the dipoles form and are injected into deep waters. This instability process creates a means by which nutrients, biology and chemistry are transported from shallow into deep waters through tidal sloshing. The injection of a shallow fluid down topography occurred quite frequently for this particular topographic profile in the case of a steady geostrophic jet (see chapter three). These oscillatory jets are weaker which is presumably why the injection does not happen so readily.
Figure 5-15: Snapshots of the perturbation vorticity for $F_w = 2$ at $t = 64, 65, 66, 67, 100, 152$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals change at every time step. The period for one tidal oscillation is $T = 4$. 
Figure 5-16: The plots of the logs of the growth rates for $F_w = 2$. The legend states what the growth rate is for each set of curves. Even though the growth rates oscillate the envelopes grow exponentially.
Figure 5-17: Snapshots of $F_\omega = 2$, $\delta = 1.1$ at $t=118, 180, 205, 236, 247, 284$. The water is deep on the left and shallow on the right. The topography varies in the $x$-direction and is periodic in the $y$-direction. The contour intervals changes at every time step. The period for one tidal oscillation is $T = 4$. 

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4. Conclusions

This chapter addressed the main objective of this thesis, that is to investigate an idealized jet that is forced by both steep one-dimensional topography and monochromatic tides in the SW model. The time-dependent nature of the jet makes it more complicated to analyze than steady flows. Moreover, since the linear solutions are not exact solutions we must integrate the one-dimensional SW equations to find a periodic nonlinear solution.

The linear stability analysis revealed the structures of the most unstable modes as well as their growth rates. It was determined that the growth rate increases with increasing tidal Froude number and that the most unstable mode moved to larger wavelengths. The numerical experiments of the nonlinear evolution of the instability determined that the linear theory was good at predicting both the growth rate and structure of the unstable modes. There are significant discrepancies in the structures for the case of very small growth rates, which we attribute to the fact that the numerical errors grow relatively rapidly.

By studying the particular case of $F_\omega = 3$ we observed, as is true in the other cases, that there are three different sets of unstable modes. The one located directly above the topography was the most unstable but the second one in the shallow region, although lagging behind, did manage to grow significantly. The result was a final state filled by a nonlinear superposition of both modal structures. We specifically chose examples and cut off times such that the third mode did not appear to be important since this mode is due to numerical errors and not any physical mechanism.

The two simulations where $F_\omega = 2$ proved that simple topography and tidal forcing can produce turbulent instabilities. These instabilities transported water from the shallow region down the shelf into deeper waters. If such an instability occurs in a coastal region, this transport is certain to have a significant impact on the chemical and biological distributions in that domain.

The eddies that were generated preferentially had stronger cyclones where the vortices oscillated at the tidal frequency. This suggests that parametric resonance is not a significant factor and that these are simply oscillatory modified shear instabilities. Perhaps, in the case of larger tidal Froude numbers, where the shear instabilities become weaker the
parametric instabilities will dominate and take form. This is something that requires further investigation.
Chapter 6

Conclusions

Chapter 2 focused on studying the barotropic instability of a geostrophic Bickley Jet in the SW model in order to determine what asymmetries can develop between cyclones and anticyclones. First, we solved the linear stability problem to determine how the growth rates depend on the two nondimensional parameters; the Rossby and rotational Froude numbers. It was determined that the growth rate decreases with increasing Froude numbers. Also, the relative growth rate was nearly constant for small Rossby numbers and then decreased with respect to Froude number for larger Rossby numbers. These growth rates predicted from linear theory were verified, to within experimental error, by numerically integrating the fully nonlinear SW equations. The case of small Rossby number, in the QG regime, produced symmetric vortices: small Froude numbers produce structures similar to the simulations of Flierl et al. (1987): $Fr = 0.5$ generated a street similar to the case of $\beta > 0$ in the same paper. Clearly, the stabilizing effect of the free surface slope is similar to that of a positive ambient PV gradient.

Exploration of the non-QG regime produced a variety of asymmetries between vortices of opposite polarity because of the finite amplitude deformations in the free surface. For small Froude number and order one Rossby number, the anticyclones were stronger than the cyclones. In the case of large free-surface deformations, the cyclones were clearly stronger and were triangular or boomerang in shape. This indicates that there is a transition from order one to $Ro \leq 5$ where the strongest vortices change from anticyclones to cyclone. The reason for the development of an asymmetry between the vortices is because these eddies
are maintained due to gradient-wind balance (see Olson (1991)). As previously explained, this does not permit anticyclones to become arbitrarily strong. Thus, the vortices translate large distances into deeper waters but not in the other direction; the latter would lead to violations of the gradient wind-balance.

This work has explored the barotropic instability of a jet in the non-QG realm, which had not previously been exampled. This analysis predicts the asymmetries which arise between the two types of vortices. This regime is relevant for the world’s western boundary currents such as the Gulf Stream since they have Rossby numbers of 0.5 or possibly higher. This instability process is important since it generates transport between the deep and shallow waters. This transport has implications on the ambient biology, chemistry and physical properties. A next step that should be taken is to quantify how the transport itself is altered with the changes in parameters. We have observed qualitatively how this is affected, but have not made any attempt to quantify it. This could be done by injecting tracer fields into the fluid and tracking their trajectories to determine where they settle down.

Chapter 3 explored how the barotropic instability of a jet running along a shelf is altered by the topography beneath. The most thorough analysis done to date had been the linear analysis of Li and McClimans (2000), that found that both prograde and retrograde topography were stabilizing. They did this by plotting the neutral stability curves and observed that they are symmetric. They did not however, calculate the growth rates in these unstable regions to verify that the growth rates are indeed symmetric. We have considered only a slightly different jet and topography then Li and McCLimans (2000) and we have found qualitatively different results.

Our linear stability analysis calculated the growth rates for various Rossby numbers, topographic heights and orientations. They indicated that retrograde topography is always stabilizing. However, even the case of small but non-zero Froude numbers yields some dramatic differences in regards to prograde topography. For small Rossby numbers it can be either destabilizing or stabilizing depending on whether the topographic height is small or large, respectively. Moreover, there is a critical Rossby number beyond which prograde topography is strictly destabilizing.

The linear theory was verified by a series of numerical simulations of the fully nonlin-
ear dynamics. These numerical simulations confirm the qualitative and quantitative results of the numerical analysis, thereby, proving the importance of knowing both the orientation and magnitude of topography in order to determine how it alters the stability properties of a jet. The asymmetry between prograde and retrograde topography has implications on the tidally forced problem addressed in Chapter 5. An oscillatory jet over a flat bottom would have no asymmetries in the growth rates, and would consequently not generate any instability. In contrast, topography produces asymmetries in the two phases of flow which can generate instabilities as the jet oscillates. The instabilities should develop mainly during the prograde phase of the cycle.

A problem that is related to the one studied in Chapter 3 is the instability of a jet flowing along ridges. The analysis in Schmidt and Johnson (1997) illustrates that ridges can destabilize a jet. It would be straightforward and insightful to apply our methods to that problem. This way we could gain a more accurate picture as to how ridges alter the stability in various dimensional regimes. Then, the numerical simulations could validate these calculations but also discover what interesting structures arise due to this interesting topography.

Studies of parametric instability in geophysical fluid dynamics have focused on their relevance to gravity waves. Very little has been investigated as to how they can arise in oscillatory shear flow (see Greenspan and Benney (1963), Kelly (1965), Kelly (1965) and Rosenblat (1968)). Chapter 4 of this thesis studied horizontal shear flow in more detail than previous work. Examination of pure oscillatory shear flows (fixed horizontal profiles with amplitudes oscillating around a mean) demonstrated that all flows with zero mean are linearly stable. When the mean is non-zero, the oscillatory flows are unstable for exactly the same wavenumbers that are unstable according to the steady theory. There is no evidence of a new instability process in the case of pure oscillatory shear flow.

Next we studied the interaction of a steady flow of uniform vorticity with an adverse oscillatory shear flow that can act to stabilize the flow. This mean background shear creates an asymmetry from one phase of the oscillation to the other, as did the shelf like topography considered in Chapters 3 and 5. We found an example whose profile at every instant in time is stable to shear instability, but the oscillating state itself is linearly unstable, giving
rise to a series of unstable wavenumbers. The instabilities generated were subharmonics and harmonics in alternating order, as in Mathieu’s equation. Then we applied a multiple scales method to calculate the growth rates in the first subharmonic tongue. We focused on this region of instability since it is of the greatest physical interest. This determined that the oscillating instability of shear flow increases with increasing background shear, but decreases with increasing wavenumbers and frequencies.

Subsequently, we studied an oscillatory shear flow that was barotropically unstable, according to the steady theory, at every instant in time. This determined that as the amplitude of the oscillations increases from zero, the barotropic mode is stabilized and parametric modes are introduced. We also studied other scenarios that were sometimes unstable and other times stable according to the steady theory. This yielded similar behaviors to before. We surmised that the strength of the parametric instabilities had negligible dependence on whether snapshots of the oscillating state are unstable to steady barotropic instability; they depend most dramatically on the size of the oscillations.

The final example addressed is that of a four-contour problem. The object was to learn how a more continuous profile would compare with the simple two-contour example. Each harmonic produces three instability tongues that have smaller growth rates than the two-contour analogue. This indicates that the instabilities of smoother oscillatory profiles will be weaker; the existence of more modes seems to act against the resonance and instability.

An issue that should be addressed in future work is the linear instability of more continuous profiles to determine which profiles give rise to parametric instability. This can be done by extending the linear contour dynamical model of this chapter to $N$ interfaces and then increasing this parameter. With this it would be possible to find an example of a continuous flow that is stable by the steady criteria, at every instant in time, but yields parametric instabilities. The role of critical layers and dissipation could be significant and could also be difficult to analyze. It would be of interest to study the nonlinear evolution of this problem to observe the instability process, in order to learn about the structure of the resulting eddies.

Chapter 5 is the focus point of the thesis. It presents an idealized model of the oscillatory jet in the Cape Cod Bay that is generated by steep topography and the tides. The
time-dependent and non-unidirectional nature of the jet makes the analysis more difficult than what is done in steady problems. By initializing the one-dimensional code with the linear solution, and evolving it for many periods, it equilibrated to a periodic solution. This is the solution that we analyze the stability of.

Instead of using Floquet theory, we solved the linear perturbation equations directly and renormalized the solution every period. This yielded the growth rates and structures of the most unstable modes, which are the modes of greatest interest. We also calculated these by perturbing the nonlinear equations with only a single wavenumber and then solving the problem numerically. For the most part, there was very good agreement between the structures predicted from the linear and nonlinear dynamics. There were discrepancies for large tidal Froude numbers which we account for because these integrations are done over long time scales which creates more numerical error than in the other instances.

In general there were three different sets of structures that appeared early on in the nonlinear simulations. The one which grew fastest is the most unstable mode and is the one predicted by the linear theory. The other two modes did not arise in the linear theory which indicates that they grow more slowly. The one that arose near the solid boundary was observed to change width and height with the grid scale. This indicates that this mode is due to numerical error. In the case of small tidal Froude numbers, this was the fastest growing mode. We only considered parameters where these erroneous structures did not grow to contaminate the solution. The end results in our plots were typically a superposition of the two unstable physical modes.

The CFL condition constrained the time step dramatically which prevented us from exploring tidal Froude numbers similar to those observed in the Cape Cod Bay. This, coupled with the spurious numerical mode mentioned above, gave a small window in parameter space that we could explore. Regardless, we explored the parameter range in this window to determine what interesting dynamics can arise, in hopes that it might lead to extrapolations as to what may arise in the Cape Cod Bay.

All of the simulations presented yielded instabilities. They created vortices that traveled in tidal orbits on the shallow side of the shelf; there was no evidence of fluid transport between the deep and shallow water. There are only two simulations in this chapter, those
with \( F_\omega = 2 \), that were turbulent. These managed to transport fluid across the jet and inject strong dipoles into the deep waters. This transport is certain to have a significant impact on the chemical and biological distributions in coastal waters.

The vortices that formed typically had cyclones that were stronger. All the eddies traveled in tidal orbits which caused them to oscillate in strength. The fact that the structure of the instabilities resembled those that occur in steady cases and the frequency of the oscillations is the same as the forcing, indicates that parametric resonance is not the dominant instability mechanism. These instabilities are due to barotropic instability. Perhaps, in the case of larger tidal Froude numbers, where the shear instabilities become weaker, the parametric resonance will dominate. This is something that requires further investigation.

We have only touched the surface of this idealized model and consequently, there are numerous directions in which this work can be extended. First, as mentioned previously, we are interested in the effect of these more irregular motions on the biology. Second, we have been limited to a parameter space that is far removed from those appropriate to the Cape Cod Bay. Ideally, the numerical method should be modified in order that more realistic situations can be investigated. Third, the observations of this current are relatively coarse in comparison to the length scales of the motion. Readings were taken every 50 m where the width of the jet is on the order of 100 m. A finer sampling would allow us to learn more about the structure of this oscillatory jet and thereby construct a more appropriate model. Fourth, it is of interest to explore the effects other physical processes such as bottom friction and stratification will have on these instabilities in a more complete and realistic model.
Appendix A

Floquet Theory

To determine whether an oscillatory solution is unstable we use Floquet theory (Coddington and Carlson (1997)). Consider the general case where the system of ordinary differential equations is of the form

\[
\frac{d\vec{y}}{dt} = A(t)\vec{y}.
\]  

(A.1)

Note that \(\vec{y}\) is the \(N\)-vector solution and \(A(t)\) is an \(N \times N\) matrix that is a periodic function of time of period \(T\). The solution may be periodic, with a period that is a rational multiple of \(T\), or it may even be aperiodic; it is the nature of \(A(t)\) that will determine which of these will arise.

Our system of equations that describes the linear motion of an oscillatory flow in a periodic domain is in the standard form of (A.1). If the forcing has \(N\) jumps then we obtain a system of \(N\) differential equations. Therefore, the general solution can be written as a sum of \(N\) linearly independent vectors. We pick our basis to be \(\vec{y}_m(t)\) such that \(\vec{y}_m(0)\) is the \(m\)th column of the \(N \times N\) identity matrix. Note that the collection of \(N\) functions \(\vec{y}_m(t + T)\) must still be a solution since the Wronskian is non-zero initially, and therefore non-zero for all time. Hence, we can rewrite these functions in terms of our basis solution

\[
\vec{y}_m(t + T) = \sum_{n=1}^{N} b_{mn}\vec{y}_n(t) \quad \text{or} \quad \mathcal{Y}(t + T) = B\mathcal{Y}(t) \quad \text{(A.2)}
\]

where \(b_{mn}\) are the components of \(B\) and \(\mathcal{Y}\) is a matrix that has \(\vec{y}_m\) in the \(n\)th column. \(B\) is
referred to as the Floquet matrix.

Since $A(t)$ is periodic, Floquet theory implies (Stoker (1950)) that there exist “normal” solutions $\bar{Y}(t)$ to (A.1) such that

$$\bar{Y}(t + T) = \sigma \bar{Y}(t).$$

(A.3)

The variable $\sigma$ is the Floquet multiplier that can be rewritten as $\sigma = \exp(\lambda T)$ where $\lambda$ is the Floquet exponent. Since $\bar{Y}(t)$ is a solution we can decompose it in terms of our basis solution:

$$\bar{Y}(t) = \sum_{m=1}^{N} \alpha_m \bar{y}_m(t).$$

(A.4)

If we substitute (A.4) into (A.3) we find, with the help of (A.2), that

$$\sum_{m=1}^{N} \alpha_m \bar{y}_m(t + T) = \sum_{m=1}^{N} \sum_{n=1}^{N} \alpha_m b_{mn} \bar{y}_n(t) = \sigma \sum_{m=1}^{N} \alpha_m \bar{y}_m(t).$$

(A.5)

Since the $\bar{y}_n(t)$ are linearly independent we can project this equation on each basis function to obtain the following $N$ equations

$$\sum_{m=1}^{N} \alpha_m b_{mn} = \sigma \alpha_n \quad \text{or} \quad B\tilde{\alpha} = \sigma \tilde{\alpha}$$

(A.6)

where $\tilde{\alpha}$ is the vector composed of $\alpha_n$ as the components.

This equation illustrates that $\sigma$ is the eigenvalue of the Floquet matrix. Therefore in order to determine the growth of the oscillatory system we must evolve the identity matrix forward one period. The eigenvalues of the resulting matrix, the Floquet matrix, determine whether the modes are decaying or growing. If there exists at least one value of $\sigma$ whose absolute value is larger than one, then the system is unstable. If we use the eigenvectors as initial conditions for our differential equations the resulting solution determines the structure of these modes.
Bibliography


