SHAPE AND VIBRATION CONTROL OF DISTRIBUTED PARAMETER SYSTEMS – EXTENSION OF MULTIVARIABLE CONCEPTS USING SPATIAL TRANSFORMS

by

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ABSTRACT

A new method of analyzing linear distributed parameter control systems is presented, based
upon their input/output representation in a spatially- and temporally-transformed frequency
space. The plant model is first augmented with discrete and/or distributed sensors and
actuators. The resulting input/output relation is then analyzed in the transformed space using
the singular value decomposition. Performance is quantified over spatial and temporal
bandwidths in terms of command following, disturbance rejection, sensor noise rejection,
controllability, observability, and stability/robustness in the presence of spatial and temporal
modelling errors. Suitable design measures are presented. The analysis is applicable to
stationary distributed plants described by spatially-symmetric or isoplanatic Green’s
functions, or self-adjoint differential operators.

The analysis is applied to the shape control problem by introducing discrete (Fourier series)
spatial transforms, which both quantify performance over a discrete spatial bandwidth and
provide a convenient set of basis functions for specifying the shapes of the control task. The
discrete transform technique yields plant models amenable to extant control analysis and
design software. A closed-loop dynamic shape control experiment is presented that
incorporates all elements of the new analysis. A 40in pinned-pinned steel beam’s shape is
controlled, in the presence of quasi-static and resonant disturbances, over a band-limited set
of discrete sinusoidal basis functions at a closed-loop temporal bandwidth of 2Hz, using
distributed piezoelectric actuators. Temporal compensation is provided by
digitally-implemented LQG/LTR compensators. A novel inner-loop damping formulation,
based upon the second method of Lyapunov, is developed and implemented to damp the
beam resonant response beyond the LQG/LTR control bandwidth.

An analysis of distributed piezoelectric transducers is included, with emphasis on their
application to the vibration damping of structural elements using a non-model-based control
synthesis strategy. The technique can be applied to control all vibrational modes of thin
beams and plates with nearly arbitrary boundary conditions. An experiment is presented
demonstrating the efficacy of spatially-varying control distributions for the control of beams
having both symmetric and asymmetric modal responses.

Thesis Supervisor: Dr. James E. Hubbard, Jr.
Lecturer, Department of Mechanical Engineering
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Shawn Burke

Permission is hereby granted by The Charles Stark Draper Laboratory, Inc. to the Massachusetts Institute of Technology to reproduce any or all of this thesis.
For my Mother, Frances,
and my late Father, Phillip;

"There is a light that shines in my heart."
NOMENCLATURE

Throughout the text scalar mathematical symbols are represented by upper- or lower-case italics characters, vectors by lower-case bold characters, and matrices by upper-case bold characters.

\[a_i\] Fourier coefficient; normalization coefficient

\[A\] amplitude

\[A_b\] beam/film cross-sectional area

\[A, B, C, D\] state matrices

\[b\] lumped boundary element linear viscous damping coefficient

\[B\] beam width

\[c_i\] Fourier coefficient

\[C\] transformed closed-loop response matrix

\[d\] feed-through correction

\[d_{31}\] piezoelectric constitutive constant

\[d\] transformed disturbance vector

\[D\] flexural rigidity

\[D\] spatial domain

\[e\] transformed error vector

\[E\] Young’s modulus; mean-square error

\[E\] transformed plant error matrix; output coupling operator matrix

\[F\] input coupling operator matrix

\[G\] shear coefficient

\[G\] augmented plant response matrix

\[h\] thickness

\[h(x - c)\] Heaviside step function

\[h(x, \xi, t, \tau)\] Green’s function

\[i, j, k, l, m, n, p\] indices

\[I\] moment of inertia

\[I_t\] lumped boundary element rotary inertia

\[I\] identity matrix
$J$  polar moment of inertia
$J(t)$  Lyapunov functional
$k$  spatial transform variable (generically, the "wavenumber" spatial frequency)
$k_i$  lumped boundary element linear spring stiffness
$k$  spatial transform variable (generically, the "wavenumber" spatial frequency) vector
$K,L,M,N,P$  index limits
$K$  transformed compensator response matrix
$L$  beam length
$m$  film actuator gain; mass density
$M$  moment
$M_t$  lumped boundary element mass
$M(\alpha)$  transformed plant/actuator matrix
$M(\beta)$  transformed sensor matrix
$n$  transformed noise vector
$p(x,\omega)$  sensing distribution (temporally transformed)
$p(k,\omega)$  spatial transform or sensing aperture
$p$  sensor spatial transform vector
$q$  distributed load
$q_{ij}$  actuator influence function Fourier coefficient
$q(x,\omega)$  actuator distribution
$q$  actuator spatial transform vector
$r$  transformed reference command vector
$s$  sensor output Fourier coefficient
$s$  sensor output Fourier coefficient vector
$S$  transformed sensitivity response matrix
$t$  time variable
$T$  tension
$T$  transformed forward loop response matrix
$u$  scalar distributed input signal
$u$  exogeneous control signal vector
Matrix of left singular vectors
control voltage
matrix of right singular vectors
scalar spatial coordinate
vector spatial coordinate; state vector
displacement; scalar distributed output signal
vector output; discrete spatial transform output vector

Greek Symbols

Sensor gain; plant/actuator Fourier coefficients
lumped boundary element rotary viscous damping coefficient; Fourier coefficient; discrete spatial transform sensor characterization
slope
degenerate actuator temporal filter characteristic
Dirac delta function
point moment function
Kronecker delta function
computational delay
loss factor
degenerate sensor temporal filter characteristic
Timoshenko shear coefficient; distributed stiffness
lumped boundary element rotary spring stiffness
spatial bandwidth (vector)
structural damping coefficient
eigenvalue; Fourier coefficient
\( \Lambda(x) \) control distribution

\( \mu_i \) eigenvalue

\( \nu \) shear force

\( \xi \) nondimensional boundary location; input spatial argument

\( \xi \) Green's function input spatial variable (vector)

\( \rho(t) \) control time dependence

\( \rho_b \) composite beam/film density

\( \sigma \) singular value

\( \Sigma \) matrix of singular values

\( \tau \) Green's function input time variable; dummy time variable; state propagation rate; time delay

\( \phi(t) \) beam displacement time dependence; orthogonal eigenfunction

\( \Phi(\omega) \) transition matrix

\( \psi \) slope

\( \psi(x) \) beam displacement field; orthogonal shape function

\( \Omega \) temporal bandwidth

**Mathematical Operations**

\( \cdot' \) differentiation with respect to the argument

\( \cdot_x \) partial derivative with respect to \( x \)

\( \cdot_t \) partial derivative with respect to \( t \)

\( L[\cdot] \) linear operator
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There is, at the surface, infinite variety of things; at the center there is simplicity and unity of cause.

—Ralph Waldo Emerson

Adde parvum parvo, magnus acervis erit.

—Ovid
INTRODUCTION

Distributed parameter control – the control of systems described by space and (usually) time variables – possesses many of the characteristics of a multivariable control problem. In practical applications it is often necessary to use a number of discrete and/or distributed sensors and actuators to achieve specified performance goals. By fixing the distribution of these sensors and actuators one constructs a multivariable control system. In the design of multi-input, multi-output (MIMO) controllers, practical design considerations can be realized using the system's input/output relation in the frequency domain, now in matrix form, where "directional" information about the system's response becomes important. Recent developments have extended optimal control methods to meet frequency domain performance specifications as well as stability/robustness measures for MIMO controllers [1,2,3]. However, the machinery of modern control analysis, while incorporating frequency domain information via the Fourier or Laplace transform, says nothing about spatial performance; the essentially distributed nature of the system is neglected, or treated using ad hoc methods.

There are a number of distributed parameter systems whose performance requirements must be posed in spatial terms. The problem of shape control is of particular interest. Large space-based radar platforms have strict shape control requirements which will require the inclusion of active profile maintenance systems [4,5,6]; performance is quantified in spatial rather than temporal terms. In optics, deformable mirrors can be employed as spatial light modulators in optical correlators [7,8], and for real-time optical wavefront correction [9,10]. The control task is stated in terms of correcting for a spatially band-limited set of wavefront aberrations by varying the shape of a mirror in real time. A lumped parameter, time domain analysis can only address these performance measures in a limited fashion, if at all.

A number of other engineering systems require shape control. Recent investigations have revealed the utility of shape control approaches in sheet metal forming [11]. It is also anticipated that advanced manufacturing processes, such as custom crystal growth for optical applications, will require shape control to achieve precise internal structures within the working piece; the concept of "shape" is not restricted to mechanical structures.

The continuum representation of distributed parameter systems plays a vital role in
applications other than shape control. In the aerospace community, stringent vibration requirements on many lightly-damped, highly flexible space structures for precise pointing and tracking have necessitated the incorporation of active control systems [12,13,14,15,16,17,18]; additional passive damping treatments are often not applied because of weight constraints. While the performance requirements for these systems are generally presented in the time domain, the modal response of elastic structures is distributed in space; actuators placed at node points will not contribute favorably to vibration damping performance.

In light of these requirements, can a distributed parameter control methodology be devised that explicitly addresses spatial performance requirements? If so, can it be reduced to practice, not only in simulations, but on real hardware?

Central questions remain open in the area of distributed parameter control theoretic development. The major difficulty results from the state of distributed parameter control design and implementation. Much of the literature on distributed parameter control is of a highly theoretical nature; references [19,20] provide summaries and examples of the state of the art. As such, much of the relevant technology lies buried in a plethora of existence, uniqueness, and invariance theorems. While such an assessment may be overly harsh, it does reflect the fact that, by comparison, there are few papers describing distributed parameter control experiments and implementations. The goal of this thesis is to develop a distributed parameter control analysis method that is analytically accessible, and can be extended to the design of control systems for experimental hardware, thus bridging the gap between analysis and application.

The principal concept of the thesis is that of the "spatial response" of distributed parameter systems, or, more specifically, their "spatial frequency response". Given the success – and accessibility – of temporal frequency domain concepts in classical and multivariable lumped parameter systems control theory, the introduction of spatial frequency transforms to distributed parameter systems and control follows naturally. Wavenumber (spatial frequency) transform methods have been utilized in acoustics for decades. The entire field of Fourier optics is based upon the notion of the spatial frequency response, or "spatial bandwidth", of optical elements [21]. Using these well-developed modelling techniques, it will be shown that one can construct an input/output relation representing a distributed
parameter system in a temporal and spatial frequency domain. This representation is then joined with the concepts of modern multivariable control theory to synthesize analysis and design approaches for distributed parameter control systems. The extension to the spatial domain is natural, and intuitive.

The thesis can logically be divided into three parts. The first portion, Chapters I and II (and Appendices A through C), is concerned with the analytical and experimental application of distributed transducers to the vibration control of structural elements [22,23,24]. This introduces a particular class of distributed parameter systems, and reveals the advantages of fully distributed actuators and sensors, challenging the reader to "think distributed". The control synthesis scheme is based upon Lyapunov's direct method [25,26], and, because of the a priori introduction of distributed transducers, has the advantage of not being model based. This is in sharp contrast to most model-based control synthesis schemes, which truncate the system representation to a finite number of discrete modes [14,15], and often suffer from control/observation spillover where unmodelled (and hence uncontrolled) modes are destabilized [15,16].

The spatial frequency transform concept is introduced in Chapter III. Distributed parameter systems amenable to this modelling approach are classified based upon the form of their Green's functions; numerous examples are included. The method is applicable to plants with convolution kernels, as well as systems described by modal expansions. In Chapter IV, the multidimensional transform representation is combined with elements of modern multivariable control analysis to construct performance measures for distributed parameter control systems, over both temporal and spatial bandwidths [27,28]. These Chapters comprise the second portion of the thesis.

In the third portion, commencing with Chapter V, the spatial transform paradigm is extended to the shape control problem. Shape control requires both a prescribed spatial bandwidth and a set of shapes that characterize the control task. Consequently, the concepts of Chapter III are extended to more general spatial transforms, namely expansions in orthogonal functions, to provide supplementary performance measures specific to shape control. A flexible beam, closed-loop dynamic shape control experiment is then presented in Chapter VI. This experiment utilizes the insight and analysis methods of Chapters I through V to design an actuator and sensor distribution, along with a digital control computer, to
successfully control a beam's shape over a prescribed spatial bandwidth, in real time. This system incorporates all of the experimental and analytical aspects of the thesis in a single proof-of-concept experiment. The experiment is also significant because no other closed-loop, dynamic shape control experiment of this kind has been reported in the open literature; the literature on structural shape control is largely devoted to static, open-loop systems [29,30,31], or the problem of actuator placement [32,33,34].

In Chapter VII the spatial transform method is used to extend and interpret the Lyapunov-based controllers of the first two Chapters. This analysis serves to unite the seemingly disparate portions of the thesis, providing further insight into the so-called "Lyapunov dampers", and sensor and actuator distributions (e.g. the transducer placement problem) in general.

The analysis presupposes some familiarity with multivariable control concepts, and some further applied mathematics, including transform methods, orthogonal functions, and Fredholm integral equations, although relevant concepts from these fields are developed where necessary. The general reader with this or similar background will have no difficulty forging through the thesis.
I. DISTRIBUTED ACTUATOR VIBRATION CONTROL OF THIN BEAMS

1.1 INTRODUCTION

AN ACTIVE DAMPER DESIGN that circumvents many of the problems associated with modal truncation was recently developed at MIT [1.1]. The actuator is spatially distributed, and made of the piezoelectric polymer film polyvinylidene fluoride (PVF$_2$). In their prototype experiment, Bailey and Hubbard [1.2] used a uniform layer of the material bonded to one face of a cantilevered beam. They implemented a control scheme based upon Lyapunov's direct method, and demonstrated that the film actuator could control all vibrational modes of the beam; the feedback scheme only required a measurement of the beam's angular tip velocity. The first mode loss factor was increased by up to a factor of forty in these experiments. The technique has been applied to the vibration control of larger cantilevered beams [1.3,1.4].

Most significantly, the distributed film actuator facilitated a control design where all modes of the distributed plant could be controlled. The resultant controller was simple, and, unlike other control synthesis methods, required no detailed knowledge of the system's natural frequencies, or other physical parameters. This is a very desirable property, as many distributed parameter systems may have imprecisely-measured (or even largely unknown) constitutive properties. It is of interest to extend this research to the control of more general distributed elastic systems.

An analysis of the beam/film active damper is presented in the balance of this chapter. Arbitrary combinations of clamped, free, pinned, and sliding boundary conditions are considered. Control strategies are derived using Lyapunov's direct method. It will be shown that the \textit{a priori} assumption of a \textit{non-uniform} spatial distribution of the film actuator can greatly simplify the control formulation. Additionally, a few beam configurations will be uncontrollable using a spatially-uniform control distribution. Candidate spatially-varying control distributions are presented. A set of design guidelines is deduced for the application of piezoelectric film actuators for controlling all vibrational modes of flexible beams.
1.2 THE DISTRIBUTED ACTUATOR

The active element used as a candidate actuator is the piezoelectric film polyvinylidene fluoride, or PVF$_2$. Polyvinylidene fluoride is a polymer that can be polarized, or made piezoelectrically active, by appropriate processing during manufacture. In its polarized form PVF$_2$ is essentially a tough, flexible piezoelectric crystal. It is commercially available as a thin polymeric film. The film generally has a layer of nickel or aluminum deposited on each face to conduct a voltage or electric field.

For uniaxially polarized film, a voltage or electric field applied across its faces results in a longitudinal strain in one direction. The strain occurs over the entire plated area of the film, making it a distributed parameter actuator. If the field is varied spatially, the strain will also vary spatially; this gives the added possibility of varying the control spatially as well as temporally.

Note that the present analysis is not limited to the description of systems that use PVF$_2$ as their distributed actuator. Rather, the analysis assumes only that the film actuator responds to control inputs via longitudinal strain, and is bonded to one face of the beam. Consequently, the analysis can be applied to the study of other candidate materials as well; the analysis itself can also accommodate other discrete or distributed actuators.
I.3 DERIVATION OF BEAM GOVERNING EQUATIONS AND CONTROL CONSTRAINTS

The flexural vibrations of a Bernoulli-Euler beam having a film actuator bonded to one face [1.1], including in-plane tension, are described by the dimensional governing equation

\[
(EI \ddot{\hat{y}} - \hat{T} \dot{\hat{y}} - m \ddot{\hat{V}})_{\hat{x} \hat{x}} + \rho_{b}A_{b}\dddot{\hat{y}}_{tt} = 0; \quad 0 < \hat{x} < L,
\]

where the subscripts \((\cdot)_{\hat{x}}\) and \((\cdot)_{\hat{t}}\) denote partial differentiation. The displacement of the beam \(\hat{y}\) and the control voltage across the film \(\hat{V}\) are functions of space and time. The in-plane tension \(\hat{T}\) is assumed to be constant. The beam's flexural rigidity \(EI\) is defined in terms of beam \((\cdot)_{1}\) and film \((\cdot)_{2}\) parameters as

\[
EI = E_{1}I_{1} + E_{2}I_{2},
\]

and

\[
m = -d_{31}(h_{1} + h_{2}) \frac{E_{1}h_{1}E_{2}B}{2(E_{1}h_{1} + E_{2}h_{2})}.
\]

Here, \(h\) is thickness, \(B\) is the beam's width, and \(d_{31}\) is the piezoelectric constitutive constant describing the strain response of the film to an electric field across its faces. Equation (1.1) is a linear inhomogeneous partial differential equation. The inhomogeneous term represents space and time variations of the control distribution; the control input can be regarded as a distributed moment. Since equation (1.1) describes a Bernoulli-Euler beam, it will be valid for long, thin beams at "low" temporal frequencies [1.5].

Introducing the nondimensionalizations

\[
x = \frac{\hat{x}}{L}, \quad y = \frac{\hat{y}}{L}, \quad T = \frac{EI\hat{T}}{L^{2}},
\]

\[t = \hat{t} \left(\frac{EI}{\rho_{b}A_{b}L^{4}}\right)^{\frac{1}{2}}, \quad V = \hat{V} \left(\frac{mL}{EI}\right),\]

\[1.4\]
permits equation (1.1) to be expressed as

$$y_{xxxx} - y_{xx} + y_{tt} = V_{xx} ; \quad 0 < x < 1.$$ \hspace{1cm} 1.5

The boundary conditions to be considered here are

$$y(\xi,t) = y_x(\xi,t) = 0 \quad \text{(clamped)} \hspace{1cm} 1.6a$$

$$y(\xi,t) = 0, \quad y_{xx}(\xi,t) = V(\xi,t) - I_0 y_{xt}(\xi,t) - \kappa_0 y_x(\xi,t) - \beta_0 y_{xt}(\xi,t) \quad \text{(pinned)} \hspace{1cm} 1.6b$$

$$y_{xx}(\xi,t) = V(\xi,t) - I_1 y_{xt}(\xi,t) - \kappa_1 y_x(\xi,t) - \beta_1 y_{xt}(\xi,t), \quad y_{xxx}(\xi,t) = V_x(\xi,t) + M_t y_{tt}(\xi,t) + k_t y(\xi,t) + b y_t(\xi,t) \quad \text{(free)} \hspace{1cm} 1.6c$$

$$y_x(\xi,t) = 0, \quad y_{xxx}(\xi,t) = V_x(\xi,t) + M_t y_{tt}(\xi,t) + k_t y(\xi,t) + b y_t(\xi,t) \quad \text{(sliding)} \hspace{1cm} 1.6d$$

where $\xi$ is the nondimensional boundary point $x = 0$ or $x = 1$. These describe clamped, pinned, free, and sliding boundary conditions, respectively. The boundary conditions include linear lumped elements as well. Exogenous inputs at the boundaries can be incorporated in the analysis, but are neglected here for simplicity.

The second, or direct method of Lyapunov is used to derive the system's control constraint equations [1.6,1.7]. Choosing the energy functional

$$J(t) = \frac{1}{2} \int_0^1 \left[ (y_{xx})^2 + (y_x)^2 + (y_t)^2 \right] dx + \frac{1}{2} M_t [y_t(\xi,t)]^2$$

$$+ \frac{1}{2} I_t [y_{xt}(\xi,t)]^2 + \frac{1}{2} \kappa_t [y_x(\xi,t)]^2 + \frac{1}{2} k_t [y(\xi,t)]^2$$ \hspace{1cm} 1.7

for the beam is particularly advantageous, for it represents the total system energy. The
terms outside the integral denote lumped energy storage elements. As before, $\xi$ represents the boundary points; naturally, the discrete elements can appear at either or both of the boundaries. Vibration damping can then be approached based upon total energy considerations. This approach has the advantage over conventional methods by dealing with the distributed system model without resorting to approximations (save any approximations due to limitations of the Bernoulli-Euler model). As a result, conditions sufficient for asymptotic stability can be derived [1.8].

The time derivative of the functional (1.7) is combined with the governing equation (1.5), and then integrated by parts. Upon combining with any pair of the boundary conditions (1.6a) - (1.6d), one obtains an expression for the power flow from the system,

$$\dot{J}(t) = \int_0^l y_{x't}(x,t) V(x,t) \, dx - \beta [y_x(\xi,t)]^2 - b[y_t(\xi,t)]^2.$$  \hspace{1cm} 1.8

Note that the control input $V(x,t)$ appears only in the spatial integral. The boundary lumped damper terms, which are decoupled from the control input, always extract energy from the system (by convention, negative power means power flow out of the system), hence they are stabilizing. They will henceforth be dropped from equation (1.8). [As an aside, the same result (1.8) can be derived by assuming that the boundary conditions (1.6) are homogeneous with respect to the control voltage $V$.]

The control design now involves choosing $V(x,t)$ to make (1.8) as negative as possible, so as to extract the most energy. An example of how equation (1.8) can be used to synthesize a spatially-uniform control can be found in [1.2]. Bailey and Hubbard found that, for a cantilevered beam with a tip mass and a spatially-uniform film actuator distribution, equation (1.8) takes the form

$$\dot{J}(t) = V(1,t) y_{xt}(1,t),$$  \hspace{1cm} 1.9

where $x = 1$ is the free end. To maximize $\dot{J}(t)$, and hence extract the most energy, they designed a controller where
\[ V(1,t) = -\text{sgn}[y_{xt}(1,t)] V_{\text{max}}. \]

Equation (1.8) in no way precludes the formulation of boundary control problems defined by spatially-uniform control distributions. However, it does permit the examination of more general spatial control distributions. It will be shown that a spatially-uniform film actuator distribution is ineffective for certain beam boundary condition combinations, and that equation (1.8) can be used to gain insight into the design of appropriate *spatially-varying* controls.
I.4 LIMITATIONS OF SPATIALLY-UNIFORM CONTROL

The results of Bailey and Hubbard will now be extended to more clearly understand the implications of equation (1.8). For the ensuing discussion, equation (1.8) will be written

\[ \dot{J}(t) = V(x,t) y_{xt}(x,t)|_0^1 - V_x(x,t) y_t(x,t)|_0^1 + \int_0^1 y_t(x,t) V_{xx}(x,t) \, dx. \]  

A spatially-uniform control distribution is defined in terms of step functions \( h(x - c) \) as

\[ V(x,t) = V_{\text{max}} [ h(x) - h(x - 1) ] \rho(t), \]  

where \( \rho(t) \) is the time portion of the control. Then,

\[ V_{xx}(x,t) = V_{\text{max}} [ \delta'(x) - \delta'(x - 1) ] \rho(t). \]

A sketch of this distribution and its effective loading interpretation is shown in Figs.1.1a and 1.1b. The distribution will only produce boundary control terms in equation (1.11) in terms

![Figure 1.1a: Spatially uniform control distribution.](image)

of linear and angular velocities at \( x = 0 \) and \( x = 1 \), since the uniform distribution provides a pair of point moments at the boundaries. Thus, a beam with clamped-clamped boundary
conditions is not controllable using a spatially-uniform film control. Beams with clamped ends, where equations (1.6a) holds, will not display boundary control at those ends in terms of linear and/or angular velocities, because \( y_x \) and \( y \) must vanish there. For example, a beam with clamped-sliding boundary conditions will not be controllable with a uniform distribution; the sliding end of the beam, like the root, must have \( y_x = 0 \). One can show, however, that a beam clamped at one end, with a pinned boundary condition at the other end is controllable with such a spatial distribution. In this case equation (1.11) takes the form

\[
\dot{J}(t) = V_{\text{max}} \rho(t) y_x(1,t),
\]

where \( x = 1 \) is the pinned end.

![Figure 1.1b: Effective loading interpretation for the spatially-uniform control distribution.](image)

As a further example, consider a pinned-pinned beam with the uniform control distribution, equation (1.12). For this configuration, equation (1.8) becomes

\[
\dot{J}(t) = V_{\text{max}} \rho(t) \left[ y_x(1,t) - y_x(0,t) \right].
\]

The beam is controllable for motions where the slopes at its ends are unequal. Physically, this corresponds to odd-numbered (symmetric about mid-span) modes of the pinned-pinned beam. However, even-numbered (asymmetric about mid-span) modes have slopes at the boundaries that are equal, hence \( \dot{J} = 0 \) for these modes. Since the time derivative of the energy functional corresponds to instantaneous power flow, equation (1.15) shows no energy can be removed from or added to the system for asymmetric vibrational modes. Parenthetically, ones notes that, for even-numbered modes and the uniform control
distribution, the integrand in equation (1.8) is the product of odd and even (about mid-span) functions; the integral must vanish over the length of the beam for the pinned-pinned beam.

The spatially uniform film actuator distribution, essentially a portion of a square wave, can be expressed mathematically in a Fourier sine series containing only odd harmonics. When this series is substituted into the spatial integral in equation (1.8), one obtains a restatement of modal orthogonality for the pinned-pinned beam, for odd-numbered modes. As a result, only symmetric modes lead to non-zero contributions to the beam's vibration control. The distribution demonstrates this property because it can be decomposed into the same set of orthogonal functions as the Laplacian of the system eigenfunctions.

Similarly, a free-free beam with uniform spatial control yields the same results as equations (1.15); its asymmetric modes are not controllable using counter-opposed moments at the boundaries. Thus, in spite of the utility and simplicity of the uniform film spatial control distribution for most boundary condition combinations, there exist configurations for which it is inappropriate. Non-uniform spatial control distributions will be considered in the next Section for the vibration damping of systems having both symmetric and asymmetric eigenfunctions.
1.5 APPLICATION OF SPATIALLY-VARYING CONTROL DISTRIBUTIONS

The Pinned-Pinned Beam

The pinned-pinned beam will be discussed in detail as a model problem for spatially non-uniform film control distributions. We begin by expressing the beam's displacement \( y(x,t) \) as a product of spatial and temporal functions, viz

\[
y(x,t) = A \psi(x) \varphi(t).
\]

This is an admissible solution form for equation (1.5). The control input is similarly defined,

\[
V(x,t) = V_{\text{max}} \Lambda(x) \rho(t),
\]

where one requires \(|\Lambda(x)| < 1, |\rho| < 1\). Substituting into equation (1.8), and neglecting the boundary damping terms, gives

\[
\dot{J}(t) = AV_{\text{max}} \rho(t) \varphi'(t) \int_0^1 \psi''(x) \Lambda(x) \, dx.
\]

Fig. 1.2a: Periodic spatially-discontinuous control distribution.
The notation \((\cdot)\)' denotes differentiation with respect to the argument. Owing to the separation in equation (1.18), the control design can now divided into a spatial part and a temporal part. The integral in equation (1.18) could be maximized with a constant amplitude spatial control distribution that "switched" in space at zeroes of the beam curvature. Such a spatial control distribution and its loading interpretation are sketched in Figs 1.2a and 1.2b. This "spatial bang-bang" controller, however, would require an inordinate amount of associated instrumentation to implement, and would necessitate segmenting the film over the beam's span at zeroes of the beam's curvature for some number of modes \(n\), thus imposing a bandwidth limit on the effectiveness of the control. The scheme would require an infinite number of segments to control all modes.

![Diagram](image-1.2b.png)

Fig. 1.2b: Effective loading for distribution of Fig. 1.2a.

However, noting that the pinned-pinned beam's even-numbered modes have odd symmetry about \(x = \frac{1}{2}\), and odd-numbered modes have even symmetry, one is led to choose a spatially-varying film control distribution having both even and odd spatial symmetry of the form

\[
\lambda(x) = (1-x)[h(x) - h(x-1)].
\]

![Diagram](image-1.3a.png)

Fig. 1.3a: Linearly-varying ("ramp") spatial distribution.
This distribution is shown in Fig. 1.3a. The control can be implemented in practice, for example, by varying the electrode plating on the film layer over the beam's length, or by varying the film's thickness (subject to the constraint, in the present analysis, that such a variation does not affect the homogeneity assumption). The film distribution's loading on the beam, given by its Laplacian [see equation (1.1)], can be deduced using the properties of generalized functions \([1.10, 1.11]\) as

\[
\Lambda''(x) = \delta'(x) - \delta(x) + \delta(x - 1).
\]

1.20

This loading distribution is sketched in Fig. 1.3b. Substituting the expression for the varying

\[
\Lambda''(x)
\]

\[
\delta'(x)
\]

\[
\delta(x - 1)
\]

\[0\]

\[-\delta(x)\]

\[1 \quad x\]

Fig. 1.3b: Effective loading for "ramp" spatial control distribution.

spatial distribution, equation (1.19), into (1.18) gives an expression for the power flow from the beam/film composite,

\[
\dot{J}(t) = -AV_{\text{max}} \rho(t) \varphi'(t) \psi'(0)
\]

1.21

\[= -V_{\text{max}} \rho(t) y_\alpha(0,t).
\]

In order to extract the most energy from the pinned-pinned beam system for damping (for the given spatial distribution), the time portion of the control \(\rho(t)\) should be
\[ \rho(t) = sgn \left[ y_{xt}(0,t) \right]. \] \hspace{1cm} (1.22)

Naturally, other stabilizing time dependencies can be chosen, such as proportional control,

\[ \rho(t) = y_{xt}(0,t). \] \hspace{1cm} (1.23)

The physical significance of the control distribution described by equation (1.19) is most easily seen by examining the inhomogeneous governing equation (1.5). The forcing term in equation (1.5) due to the chosen distribution will involve point forces and a point moment, as shown in equation (1.20), and illustrated in Fig. 1.3b. Since translational motion is prohibited by the pinned boundary conditions, terms involving the delta functions will vanish in the time derivative of the energy functional, as seen in equation (1.11). Thus, the spatially-varying film control appears to the system as a boundary controller at one end. Forcing the distribution to zero "smoothly" (e.g. without a discontinuity in amplitude) at \( x = 1 \) does not give rise to a point moment there, precluding the result for the uniform control distribution, equation (1.25). Discontinuities in amplitude give rise to point moments, while discontinuities in slope give rise to point forces. This is analogous to the interpretation of static loading distributions on beams in elementary structural mechanics texts; the film distribution \( \Lambda(x) \) is precisely the moment distribution for the pinned-pinned beam corresponding to the loading given by equation (1.20).

One might alternatively choose a somewhat more general linearly-varying distribution of the form

\[ \Lambda(x) = (1 - bx) \left[ h(x) - h(x - 1) \right], \] \hspace{1cm} (1.24)

where \( 0 < b < 2 \). This permits recovery of the uniform control distribution result, equation (1.15), by choosing \( b = 0 \). However, setting \( b = 2 \) gives

\[ J(t) = -V_{max} \rho(t) \left[ y_{xt}(0,t) + y_{xt}(1,t) \right]. \] \hspace{1cm} (1.25)

In this case one can now control asymmetric modes, but not symmetric modes; for \( b = 2 \) the distribution of equation (1.24) has odd symmetry about \( x = \frac{1}{2} \), since \( \Lambda(x) \) changes sign.
there. Both even- and odd-numbered modes are weighted equally when the modal weighting factor $b$ equals 1. By choosing a modal weighting factor for the distribution $0 < b < 1$, one weights the symmetric modes more than the asymmetric modes, with the opposite true for $1 < b < 2$. This can be exploited in the design of systems requiring the control of only certain sets of modes.

Finally, note that the Fourier sine series representation of the "ramp" distribution, equation (1.19), is expressible as a superposition of sinusoids with both odd and even harmonic components. Thus, this actuator distribution has Fourier components coincident with the all modal components of the beam's curvature. Using this insight, one is tempted to design actuator distributions that are expressed as superpositions of the beam's curvature functions for only certain modes, so as to control only those modes; this will be exploited in Chapters VI and VII.

The Clamped-Sliding Beam

A beam with clamped-sliding boundary conditions exhibits the same limitations of the clamped-clamped beam, as the zero-slope requirement at the boundaries precludes the use of point moments there for control. The spatially-uniform control distribution, which was seen to exert moment controls at the boundaries, is therefore inappropriate.

However, the linearly-varying film spatial distribution designed for the pinned-pinned beam can also be applied to the clamped-sliding beam. As seen in equation (1.20) and Fig. 1.3b, this control exerts both a point moment and a point force at $x = 0$, and a point force at $x = 1$, owing to its discontinuities at the boundaries. Substituting the distribution (1.19) into the Lyapunov functional's time derivative (1.8), and imposing the boundary conditions (1.6d) gives

$$\dot{J}(t) = V_{\text{max}} \rho(t) y_t(1,t).$$

where the sliding end is $x = 1$. An appropriate control is then

$$\rho(t) = - \text{sgn} \ [y_t(1,t)].$$
The "ramp" control is effective because of its effective loading characteristics. The point force at $x = 1$ works against the rectilinear motion of the sliding end. The energy removed by the control can be doubled by using the generalized linearly-varying distribution, equation (1.24), with a modal weighting factor $b = 2$ to obtain

$$\dot{J}(t) = 2V_{\text{max}} \rho(t) y_{x}(1,t).$$  \hspace{1cm}  \text{(1.28)}$$

Unlike the pinned-pinned beam result, the augmented linear distribution (1.24) controls all modes here rather than just a subset. This is due to the asymmetry of the boundary conditions of the clamped-sliding beam configuration.

The Clamped-Clamped Beam

One might naturally ask whether the "ramp" spatial distribution will work for the clamped-clamped beam, which was shown to be unaffected by the uniform control distribution. Since the boundary conditions for this system require vanishing slope and displacement at $x = 0$ and $x = 1$, the spatial distribution $\Lambda(x)$ must contain delta functions (e.g. point actuation) to affect this system with boundary control. This is not physically realizable with the film. As a result, alternative spatial distributions must be considered.

A clamped-clamped beam will have modes with either a vanishing displacement or slope at the center, but never both. A distribution $\Lambda(x)$ with a discontinuous amplitude at $x = 1/2$ (giving a point moment, and hence an angular velocity term) and a discontinuous slope there as well (giving a point force, resulting in a linear velocity term) of the form

$$\Lambda(x) = \left(\frac{3}{2} - x\right) \left[h \left(x - \frac{1}{2}\right) - h \left(x - 1\right)\right]$$  \hspace{1cm}  \text{(1.29)}$$

will then give the result

$$\dot{J}(t) = -V_{\text{max}} \rho(t) \left[y_{x\left(\frac{1}{2},t\right)} + y_{x\left(\frac{1}{2},t\right)}\right].$$  \hspace{1cm}  \text{(1.30)}$$
The distribution of equation (1.29) and its effective loading (1.30) are sketched in Figs 1.4a and 1.4b. This distribution places a point moment (for even-numbered, asymmetric modes) and a point force (for odd-numbered, symmetric modes) at mid span.

To extract the most energy from the system to damp vibrations, choose

\[ \rho(t) = \text{sgn} \left[ y_{x_1} \left( \frac{1}{2}, t \right) + y_{x_2} \left( \frac{1}{2}, t \right) \right]. \]  \hspace{1cm} (1.31)

Naturally, other stabilizing control time dependences (such as proportional control) can be chosen. The energy removal can be made more "severe" by choosing
\[ \Lambda(x) = \left( \frac{2}{3} - 1 \right) \left[ h(x) + h(x - 1) - 2h \left( x - \frac{1}{2} \right) \right]. \]  

This distribution is depicted in Fig. 5; using it, equation (1.30) becomes

\[ J(t) = \frac{4}{3} V_{\text{max}} \rho(t) \left[ y_{\frac{1}{2}}(t) + y_{\frac{1}{2}}(t) \right]. \]

For the clamped-clamped beam, this introduces the added complexity of changing the control distribution's sign in space. Damping capability may have to be traded for ease of implementation. These basic design considerations can be readily used to synthesize spatial control distributions that weight the linear velocity measurement more than the angular velocity, and hence control odd-numbered modes more than even-numbered modes, by removing the amplitude discontinuity in \( \Lambda(x) \). The converse holds true for even-numbered modes. The important result, however, is that a control design for the clamped-clamped beam was constructed using spatially discontinuous film control distributions, and for other systems as well using identical arguments.
1.6 SPATIALLY-VARYING CONTROL DISTRIBUTIONS AND ENERGY EXTRACTION

To estimate the effectiveness of the control distribution described by equation (1.19) for the pinned-pinned beam, an analysis of its energy extraction compared to any inherent passive structural damping is now presented. This beam configuration was chosen for its analytical simplicity, and because all Bernoulli-Euler beam configurations have resonant characteristics like the pinned-pinned beam after their first several modes [1.5]. The $n$-th mode of the pinned-pinned beam is assumed to have the form

$$y(x,t) = A \sin(n \pi x) \cos(n^2 \pi^2 t).$$

This describes a single mode having an initial displacement and zero initial velocity. It is further assumed for the purpose of illustration that the modes remain uncoupled. (This is, of course, not rigorously correct, although present film materials used as distributed actuators do provide low authority control.) The frequency $n^2 \pi^2$ is chosen to satisfy the beam's dispersion relation. The beam's initial strain energy is

$$E_{strain} = \frac{1}{2} \int_0^1 \left[ y_{xx}(x, 0) \right]^2 dx = \frac{n^4 \pi^4 A^2}{4}.$$  

The energy that the switching control, equation (1.22), can extract over a half-period of oscillation [using the form of $A(x)$ found in equation (1.19)] is found by integrating the expression for the instantaneous power flow, equation (1.21), over a half period:

$$E_{control, T/2} = -2n \pi AV_{max}.$$  

If the passive internal damping is modelled as a viscous loss proportional to the beam's transverse velocity, over a half period of oscillation it will remove energy of the amount

$$E_{viscous, T/2} = \frac{-\eta n^2 \pi^2 A^2}{4}.$$
where $\eta$ is the associated modal loss factor. Thus, over a half period of oscillation, the controller will remove more energy from the beam than the passive viscous damping if

$$\frac{8V_{\text{max}}}{\eta n \pi A} > 1.$$  

Consequently, the controller is most effective at low frequencies (where structural damping is generally insufficient to damp modes) and low amplitudes. This result has been observed experimentally [1.3] and in simulations [1.4] of cantilevered beams. The two damping mechanisms, then, complement each other. Essentially, one can regard equation (1.38) as a bandwidth limit on the control effectiveness compared to the passive damping.
I.7 DESIGN GUIDELINES

The results of the previous sections can be summarized in a set of design guidelines, as follows:

(1) Uniform control distributions are effective for the active damping of many beam configurations. This control might be chosen for the sake of simplicity, unless other considerations preclude it.

(2) The film actuator distributions described herein, having discontinuous slopes and/or amplitudes (e.g. superpositions of step and ramp functions), will only yield control laws, using the Lyapunov design method, in terms of linear and angular velocity feedback. This may affect the choice of sensors and sensor placement for a given beam configuration.

(3) Discontinuities in amplitude will produce point moments, and will result in controls in terms of angular velocity at the point of discontinuity. The amplitude of these terms in the control law is proportional to the magnitude of the amplitude discontinuity.

(4) Discontinuities in slope will produce point forces, and will result in control laws in terms of linear velocity at the point of discontinuity. The amplitude of these terms in the control law is proportional to the magnitude of the slope discontinuity.

(5) The film actuator will be most effective at "low" frequencies and "low" amplitudes. The controller is most effective in the "energy regime" where the system is most flexible and lightly damped.

(6) The film distribution function Λ(x) is the moment distribution corresponding to the loading given by Λ''(x). This makes physical sense, since the film actuator is seen to exert a distributed moment on the beam. As a result, one can synthesize a film actuator distribution function by prescribing a desired loading Λ''(x), and then integrating. However, the film actuator will only provide loading distributions that are self-reacting.

(7) The location of the film distribution discontinuities can be varied so as to implement a control which will affect only certain modes.
1.8 SUMMARY AND OBSERVATIONS

For a number of boundary condition combinations a Bernoulli-Euler beam can be controlled effectively using a spatially-uniform film actuator distribution. The technique has been demonstrated experimentally for cantilevered beams. However, the spatially-uniform distribution was shown to be inappropriate for other configurations of practical interest. Also, it might be desirable to formulate a control solely in terms of certain sets of vibrational modes. As a result, nonuniform spatial film actuator distributions were studied, and potential implementations were investigated for pinned-pinned, clamped-sliding, and clamped-clamped beams. Through these analyses insights were gained into the design of spatially-varying film controllers, and a set of design guidelines was deduced.

The advantage of the spatially-varying control distributions is that they are simple spatial configurations that can be implemented on real systems. Further, they provide insight about controllability and observability that appeal to ones intuition. The requisite states that need to be measured for implementing these controls are realizable. Additionally, at no time was it necessary to model the system in terms of its component modes. In principle, the control will not suffer due to model truncation. The approach provides insight into a design methodology for applying distributed film controllers to (nearly) arbitrary beam configurations, to control all modes. One can also design systems to selectively control certain modes for specialized applications.

While the actuator distributions discussed here have lumped actuator counterparts, one can readily synthesize film distributions that have no discrete equivalent [1.12]. Given an arbitrary specification of the requisite spatially-varying loading on the beam, two integrations will give the corresponding moment distribution [1.13]; this precisely equals the necessary film distribution. For applications other than vibration damping, this may lead to film distributions that do not extend over the entire length of the structure, or multiple actuator segments.

The analysis heretofore has examined the application of distributed film vibration dampers to the Bernoulli-Euler beam model. However, the method can also be applied to Rayleigh and Timoshenko beam models, as shown in Appendix A. The method can also be extended to thin plates for biaxial film layers; the Lyapunov analysis for plates is presented
in Appendix B.

In the next Chapter, experimental results will be presented for the vibration control of a pinned-pinned beam incorporating a linearly-varying film distribution. These experiments demonstrate the necessity of spatially-varying control distributions to ensure controllability for both odd- and even-order modes, and hence validate the theoretical assertions of this Chapter.
II. ACTIVE VIBRATION CONTROL OF A PINNED-PINNED BEAM USING A SPATIALLY DISTRIBUTED ACTUATOR

II.1 EXPERIMENTAL CONFIGURATION

A BENCH-TOP BEAM EXPERIMENT was designed and constructed to demonstrate that a spatially-varying film actuator distribution is needed to control both odd- and even-order modes of certain beam configurations. A pinned-pinned beam was chosen as the candidate plant. This system was analyzed in detail in Chapter I, where a linearly-varying film distribution was synthesized to control all modes of the beam.

A photo of the experimental setup is presented in Fig. 2.1, while an experimental schematic is shown in Fig. 2.2. The beam's static properties are summarized in Table 2.1.

The experimental

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>0.2762 m</td>
</tr>
<tr>
<td>Width</td>
<td>1.27 x 10^{-2} m</td>
</tr>
<tr>
<td>Beam Thickness</td>
<td>3.81 x 10^{-4} m</td>
</tr>
<tr>
<td>Film Thickness (including bond)</td>
<td>6.35 x 10^{-5} m</td>
</tr>
<tr>
<td>Beam Elastic Modulus</td>
<td>2.10 x 10^{11} Pa</td>
</tr>
<tr>
<td>Film Elastic Modulus</td>
<td>1.50 x 10^{9} Pa</td>
</tr>
<tr>
<td>Beam Density</td>
<td>780 kg/m^3</td>
</tr>
<tr>
<td>Film Density</td>
<td>180 kg/m^3</td>
</tr>
<tr>
<td>Film Piezoelectric Coefficient d_{31}</td>
<td>1.20 x 10^{-11} m/V</td>
</tr>
</tbody>
</table>
structure consisted of a beam made of stainless-steel feeler gauge stock, with knife edges machined into the ends via EDM (Electrical Discharge Machining) to provide the pinned supports. A modal study of the configuration demonstrated that the beam indeed displayed pinned-pinned sinusoidal mode shapes. Also, the inherent passive structural damping of the
modes was found to be much less than one percent of critical. Values of the damping coefficient $\zeta$ are given in Table 2.2. Having a beam with low structural damping was important, as it permitted the effects of the active damper to be easily differentiated from passive damping. Note, however, that the passive structural damping differed when the beam had a spatially-uniform layer of film bonded to it compared to the linearly-varying distribution. This reflects the larger bond area for the uniform distribution, which is twice that of the tapered film layer; the loss factor differed by a factor of approximately two as well.

Table 2.2: Beam/Film Dynamic Properties

<table>
<thead>
<tr>
<th>Mode</th>
<th>Freq. (Hz)</th>
<th>$\zeta$, Uniform Dist. (Controlled)</th>
<th>$\zeta$, Ramp Dist. (Controlled)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>27.5</td>
<td>.0044 (.019)</td>
<td>.0018 (.0080)</td>
</tr>
<tr>
<td>2</td>
<td>63.4</td>
<td>.0057 (.006)</td>
<td>.0025 (.0071)</td>
</tr>
<tr>
<td>3</td>
<td>123.7</td>
<td>.0034 (.011)</td>
<td>.0023 (.0064)</td>
</tr>
</tbody>
</table>

The control time function $\rho(t)$ was chosen to be $-\text{sgn}(w_x(0,t))$. This near-optimal control provides for greater energy extraction from the system than proportional or other controllers, and is simple to construct. The control was implemented digitally, using an assembly language routine on an IBM PC/AT. The control computer was outfitted with a MetraByte Dash-16 A/D-D/A board.

The control also required the measurement of angular velocity at $x = 0$. Since the beam was so small, no commercially available (e.g. large) angular accelerometer could be mounted at the pinned support without severely altering the beam's response. Instead, an Endevco EGA-125-5D 0.5g linear accelerometer was mounted near (but not at) the boundary, as shown in Fig. 2.2. A Vishay 2110 power supply/2120 strain gauge amplifier provided the necessary excitation voltage and signal conditioning. The switching controller
only required the sign of the angular velocity at $x = 0$. As a result of this placement, the linear accelerometer had the same sense as an angular (acceleration) measurement, for modes low enough that no node exists between the accelerometer location and the support. This nodal requirement, combined with a 150Hz bandwidth limit on the accelerometer, provided the bandwidth limit on the control implementation.

The acceleration output was integrated in one of two ways. The first method used a simple analog integrating circuit. The second used a pair of fourth-order active low-pass filters, where the cut-off frequency was tuned until the phase response of the filters at the frequency of a particular mode was set to $+90$ or $-270$ degrees. This latter option was used extensively during experimentation because of the accelerometer's bandwidth limitation. Since the phase response of the accelerometer begins to shift beyond its bandwidth limit, higher-frequency modes could be excited. The filters not only adjusted the phase of the input signal to provide the necessary sensed parameter, but they also attenuated the higher-frequency modes in the input, preventing their excitation by the controller (observation spillover). Since the goals of the experiment were to demonstrate the necessity of incorporating spatially-varying control distributions and to determine the effectiveness of the postulated "ramp" distribution, these compromises were deemed acceptable. Future work will attempt to verify the contention that "all" modes can be controlled simultaneously with this actuator, as suggested by the analysis, using more sophisticated sensing and signal conditioning.

A spatially-uniform layer of PVF$_2$ was bonded to the beam for the first series of tests using 3M Super-77 spray adhesive; a linearly-varying layer was later bonded for the second test series. The film was driven by a Kepco BOP-1000M high-voltage power supply. The power supply was configured for a ±250V operating range. Since the film has a very high electrical input impedance (>30MΩ), very little current is required by the controller, and, hence, little electrical power. After each film layer was mounted, an electrical lead was bonded to the outer metalized face using a thin layer of Amicon conductive ink. The beam itself provided the other electrode.

The entire test structure was mounted to a 1000lb floating optical bench to isolate the experiment from external vibrations. The beam was inserted into its supports, and tension
was applied to the beam to tune the vibrational modes to those given in Table 2.2. This mounting procedure was found to give surprisingly repeatable results for the beam dynamic response. Since the voltage limits were modest, no special cabling was employed. However, the entire setup was connected to earth ground with a grounding strap.

Once the beam was in place, the high-voltage amplifier was connected to the film and turned on. The amplifier DC bias voltage was slowly swept over the entire ±250V range to "burn in" the film layer. This step is needed because, in the process of cutting the film using single-edged razor blades, its plated faces can be pressed together along the cut edges by the blade, causing small contact shorts. This was largely avoided by using each razor blade for only one cut (e.g. while it is sharpest), then throwing it away. Finally, in the course of testing, the film's outer metalization broke at a few spots, electrically isolating segments of the actuator. These breaks were painted over with small amounts of the conductive ink to restore the electrode. The ink was easy to apply, flexible, and took less than one hour to dry.
II.2 EXPERIMENTAL PROCEDURE AND RESULTS

The goal of the experiments was to demonstrate the necessity of spatially-varying film control distributions for controlling modes with both even and odd symmetry. To this end two series of tests were defined. The first involved free-decay tests of the first three beam modes, controlled and uncontrolled, with a uniform film actuator layer. The second series consisted of the same tests using a linearly-varying film actuator distribution.

To provide an initial condition to the system, the controller was driven using positive feedback. This proved to be much more repeatable than providing an initial displacement mechanically, but assumes that the actuator distribution is capable of exciting each mode of interest (if only slightly). Then, either the controller was turned off (to measure the free-decay response) or switched to control the beam.

Superimposed plots of the acceleration decay envelopes for the first three modes, using the uniform film layer, are shown in Figs 2.3-2.5. Note that the controller is quite effective for modes 1 and 3, increasing the effective damping ratio $\xi$ by up to a factor of five compared to passive damping. However, mode 2 was essentially impervious to active control with the uniform distribution; the initial amplitude for this mode is much smaller as
well. Thus, even though the uniform distribution seems to weakly excite mode 2 due to inherent system asymmetry, its effectiveness as a controller for this mode is small. This upholds the contention of the analysis — a spatially-uniform control distribution will not control even-order modes of the beam.

![Graph showing controlled and uncontrolled displacement](image1)

**Fig. 2.4:** Mode 2 with spatially-uniform distribution (acceleration response envelope).

![Graph showing controlled and uncontrolled displacement](image2)

**Fig. 2.5:** Mode 3 with spatially-uniform distribution (acceleration response envelope).
Next, a linearly-varying layer of film was adhered to the beam. One could vary the control distribution in space by varying the film thickness over the length of the beam, or by similarly shaping its nickel-aluminum plating. However, standard commercially-available PVF$_2$ was merely cut into a wedge shape to provide the requisite distribution, as shown in Figs 2.1 and 2.2. Free- and controlled-decay tests were then conducted using this configuration. Plots of the corresponding decay envelopes are shown in Figs 2.6-2.8 for the first three modes. Again, modes 1 and 3 are effectively controlled by the film. Now, however, mode 2 is controlled as well, demonstrating the utility of this spatial control distribution for the pinned-pinned beam. The modal loss factors are increased by up to a factor of 4.5 by the controller. Also note that the controlled decay envelopes are more linear than exponential; the switching control affects the beam like a Coulomb damper.

Even though the linearly-varying distribution has solved the controllability problem for mode 2, the control merit for modes 1 and 3 has been reduced because of the smaller actuator area, as compared to the uniform distribution. Also, the control's damping augmentation

![Graph](image)

**Fig. 2.6:** Mode 1 with linearly-varying spatial distribution (acceleration response envelope).
Fig. 2.7: Mode 2 with linearly-varying spatial distribution (acceleration response envelope).

Fig. 2.8: Mode 3 with linearly-varying spatial distribution (acceleration response envelope).

decreases with increasing frequency; the film controller will extract more energy from the beam than inherent passive damping when the (nondimensional) condition (1.38) holds. As a result, the distributed actuator is most effective at low frequencies (where structural damping is generally insufficient to damp modes) and low amplitudes. The two damping mechanisms complement each other.
A simplified computer simulation was written that incorporated expressions for the strain energy and dissipated energy lost to the controller, and averaged them over a half-period of oscillation. Internal damping was included using a viscous model [2.1], where the damping coefficient was determined from the free-decay experiments. Using the initial amplitudes set by the experiments, and the beam and film properties found in Table 2.1 (except as noted below), the simulation determined the strain energy in the beam. For each successive time step, the energy dissipated by the control and by internal damping was subtracted from the beam’s total strain energy, whereupon a new value for the strain energy (and hence amplitude) was calculated. This "averaged energy subtraction" continued until the system’s energy went to zero, or a user-defined time limit was met.

A comparison of the settling times predicted by the simulations and those measured with the linearly-varying distribution is presented in Table 2.3. Since the experiments were constrained to control only one mode at a time by use of the cascaded filters, it is not surprising that these results corroborate the model. It was found, however, that the film’s $d_{31}$ piezoelectric coefficient had to be increased by a factor of two compared to its published value [2.2] to fit the data. This no doubt reflects the imperfect mechanical coupling between the film and the beam due to the adhesive bond. Such an effect has been observed in cantilever beam experiments as well [2.3,2.4,2.5].

Table 2.3: "Ramp" Distribution Controlled Settling Times

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experimental $T_s$ (sec)</th>
<th>Simulation $T_s$ (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.45</td>
<td>1.68</td>
</tr>
<tr>
<td>2</td>
<td>0.68</td>
<td>0.82</td>
</tr>
<tr>
<td>3</td>
<td>0.36</td>
<td>0.44</td>
</tr>
</tbody>
</table>
II.3 SUMMARY AND RELATED RESEARCH

The experimental application of a distributed piezoelectric film actuator to a pinned-pinned beam has been explored in this Chapter. The experiments demonstrated that a spatially-varying film layer was necessary to control modes with both odd and even symmetry. In addition, the film actuator facilitated the design and hardware implementation of a simple, lightweight, low-power active vibration control for this beam configuration.

The experiments validated the controllability issues of the analysis. Nonetheless, it is of interest to enhance the controller, and to apply the technique to other beam boundary conditions as well. In related research, Procopio and Hubbard [2.6] have applied spatially-varying film actuator distributions to the control of a clamped-sliding beam experiment [a simple model of a Remote Center Compliance (RCC)], utilizing the analytical results of the previous Chapter. They also applied both a uniform and a "ramp" film actuator distribution simultaneously to a cantilevered beam. This configuration proved especially useful for the lowest vibrational modes, where the linear displacement at the cantilever beam's tip is larger than the angular motion there. The ramp distribution provided a point force control at the tip that used linear tip velocity feedback, while the uniform distribution provided a point moment control at the tip utilizing angular tip velocity feedback. Consequently, these two independent controllers complemented each other over a wide frequency range.

Miller and Hubbard [2.5] have studied the use of piezoelectric film as both an actuator and as a sensor for beam vibration control. They have found, using analysis motivated by the results of the previous Chapter, that the film can be used as a sensor to measure the feedback parameter required by the Lyapunov control design. They bonded a sensing film layer to one side of a cantilevered beam, and an actuation layer to the other side, then interconnected the two through a differentiating compensator. The resulting "smart structural component" provided vibration control for several beam modes, both individually and simultaneously. They then successfully extended the concept to a system of interconnected beam elements, where each component has its own independent sensing and actuating film layers. In Appendix C, it is shown that any colocated film sensing and actuation distributions applied to an arbitrary beam configuration can provide a stabilizing controller, even though they may not control all the beam modes.
III. SPATIAL FILTERING CONCEPTS IN DISTRIBUTED PARAMETER SYSTEMS

III.1 INTRODUCTION

THE RESULTS OF THE TWO PREVIOUS CHAPTERS, and Appendices A-C, have demonstrated the elegance and utility of the Lyapunov method as a design technique for structural element active vibration dampers. However, it has some notable limitations as well. The method requires that the system to be controlled be represented by a governing partial differential equation. This precludes the large class of systems described by integral equations, modal analyses, and finite element analyses. Also, the method does not incorporate performance measures to quantify tracking, disturbance rejection, and robustness; it merely indicates whether or not a given controller will stabilize the system. It will be inappropriate for applications such as shape control, except to indicate whether or not the system will be stable about its non-zero distributed set point [3.1]; the method says nothing about spatial performance.

Are there existing methods for explicitly describing the spatial response of distributed parameter systems? If so, can they be incorporated into modern control analysis and design techniques, and (most importantly) subsequently reduced to practice on real hardware?

The notion of "spatial bandwidth" found in acoustics and Fourier optics is especially apt. There are systems in acoustics that are routinely described in terms of their spatial frequency response. Maidanik et al [3.2] have demonstrated the utility of spatial Fourier transforms in quantifying the performance of sonar arrays. The sensing apertures of hydrophones (i.e. underwater microphones) are transformed in space to reveal their spatial filtering performance, both singly and in "steered" arrays. For example, a fictitious one-dimensional pressure sensor having a sensing aperture of length \( L \), defined in space as

\[
p(x) = h(x) - h(x - L),
\]

has a corresponding spatial frequency response
derived by Fourier transforming equation (3.1). The variable $k$ is called a wavenumber, or spatial frequency. It is inversely proportional to spatial wavelength. A sketch of the "boxcar" aperture (3.1) appears in Fig. 3.1. The sensor's wavenumber

$$p(k) = \frac{i}{k} (1 - e^{ikL}),$$

**Fig. 3.1: Boxcar aperture.**

**Fig. 3.2: Wavenumber acceptance for boxcar aperture.**

response (3.2) is presented in Fig. 3.2. For long wavelengths (low wavenumbers) the
sensor has a constant amplitude pass band. When the wavelength of an incident acoustic field equals the length of the sensing aperture, the average pressure across the aperture is zero. This corresponds to the first null of the wavenumber response (3.2). Nulls at higher wavenumber correspond to acoustic pressures having spatial periodicities (wavelengths) equal to integral multiples of the aperture length. The response rolls off above the first wavenumber null because of pressure averaging across the sensor. Consequently, the sensor acts as a spatial filter to the incident acoustic excitation. This analysis assumes that any temporal resonances of the sensor are much higher than the highest temporal frequency component of the incident sound field.

Aupperle and Lambert [3.3], Martin [3.4], Jameson [3.5], and Martin and Leehey [3.6] have extended these spatial filtering concepts to structural elements such as beams, plates, and membranes, including spatial and temporal response characteristics of the system. They recognized that the forced resonant response of a distributed parameter system is a combination of temporal and spatial coincidence. A common example is the response of a beam to point excitation, where the exciter is driven at a resonance frequency of the beam, but placed at a spatial node of the resonance. There is a temporal coincidence, but no spatial coincidence,

![Envelope of Peaks for Ideal Clamped Beam (9,1) Mode](image)

Fig. 3.3: Wavenumber acceptance of S-C-S-C plate's (9,1) mode (after Martin [3.4]).
hence the beam will not respond. For each temporal resonance of a beam or plate, there is a corresponding spatial resonance wavenumber (or characteristic spatial periodicity), as well as off-resonant wavenumber response. An experimentally-measured plot of a simply-supported/clamped rectangular plate's (9,1) mode spatial frequency response appears in Fig. 3.3. The "spatial resonance" is due to the spatial decomposition of the corresponding mode shape; each mode has a preferred spatial wavelength. Martin et al exploited this spatial filtering characteristic of mechanical structures to measure the low-wavenumber content of a turbulent boundary layer's wall pressure field.

This successful utilization of wavenumber transforms in describing the spatial response of certain distributed parameter systems motivates their application to the distributed parameter control problem. In the next Section, a new modelling technique is presented wherein a distributed system's input/output relation is transformed into a temporal and spatial frequency domain. The plant is then augmented with sensor and actuator distributions to construct a multiplicative MIMO input/output relation. Consequently, the well-developed tools of MIMO controls analysis can be applied directly. The classes of distributed parameter systems amenable to the modelling paradigm are defined in the ensuing Section. Examples are presented; the multidimensional transform technique is shown to be very general. In the following Chapter, the modelling technique will be combined with elements of modern MIMO controls analysis to construct performance measures for distributed parameter control systems, thereby addressing the shortcomings of the Lyapunov analysis.
III.2 THE DISTRIBUTED PLANT

The mathematical theory of distributed parameter systems is largely based upon the theory of integral equations [3.7]. A distributed parameter system described by partial differential equations requires initial and boundary conditions, while an integral equation already contains all information about the plant, including these conditions. The integral equation formulation is more general. In practice it is often more convenient to use integral equations to analyze a problem described by differential equations. (The inverse is not always possible.) The integral equation approach will be adopted here to characterize distributed plants by their Green's functions.

Consider, then, a distributed parameter system with input \( u(x,t) \) and output \( y(x,t) \). These functions are scalar distributed signals, defined over a suitable domain, that describe the excitation and corresponding system response over the plant's entire spatial domain as a function of time. Most generally, these are related via a composition integral [3.7] of the form

\[
y(x,t) = \int_{\xi}^{x} \int_{0}^{t} h(x,\xi,t,\tau) u(\xi,\tau) d\xi \, d\tau. \tag{3.3}
\]

The function \( h(x,\xi,t,\tau) \) is given several different names. In linear systems theory, as an analogy to lumped parameter systems, it is called the plant's impulse response function. In the theory of integral equations it is called the kernel of the integral operator (3.3). In mathematical physics it is variously dubbed the point source function, influence function, or Green's function; the last name will be used hereafter. The Green's function is the response of the plant at \((x,t \geq 0)\) to an impulsive input at \( \xi \) at time \( \tau \geq 0 \). Consequently, the integral (3.3) can be regarded as a superposition of impulsive inputs over the domain of the plant weighted by \( u(x,t) \). If the distributed parameter system is an interconnected structure of individual distributed components, then \( h \) can have a matrix form [3.7].

It is assumed in the present analysis that the composition integral is linear, so that if the input consists of a summation of independent inputs the principle of superposition can be used. Since the system is linear, the response to a given input is unique. (Note, however,
that while the output is unique, the Green's function is not! The integral equation (3.3) is also homogeneous with respect to any boundary or initial conditions; if need be, this is achieved through the introduction of suitable standardizing functions [3.7]. We further assume that the distributed parameter system is stationary, hence (3.3) takes the form

\[ y(x,t) = \int_0^t \int_{\mathbf{D}} h(x,\xi, t-\tau) u(\xi, \tau) \, d\xi \, d\tau. \]  

This is the distributed parameter extension of the usual time-domain convolution integral for stationary systems. The temporal portion of (3.4) is a convolution because of the stationarity assumption.

For a large number of distributed plants (the extent of which is detailed in the next Section), the Green's function can be manipulated so that the space/time Fourier transform of equation (3.4) taking the form

\[ y(k, \omega) = h(k, \omega) u(k, \omega), \]  

for the spatial transform variable \( k \), temporal frequency \( \omega \), transformed output \( y(k, \omega) \), transformed input \( u(k, \omega) \), and space/time response function \( h(k, \omega) \). The distributed parameter plant is seen to act as a space/time filter which filters the distributed input \( u \). If the spatial transform is a Fourier integral, then the elements of \( k \) are wavenumbers. The spatial transform (and hence the spatial control task) can be posed in terms of other bases; this will be exploited in the sequel for the shape control problem.

As an example, Fig. 3.4 presents the wavenumber/frequency response function for a nondimensional simply-supported beam. This "multidimensional Bode plot" shows that the forced response of a distributed parameter system is a combination of both temporal and spatial phenomena. Each mode has a preferred spatial frequency corresponding to the temporal resonances of the plant. The locus of resonance peaks follows the plant's dispersion relation [3.8], or space/time characteristic equation. The lobing in \( k \) reflects the plant's spatial aperture (e.g. its length). Note that the wavenumber response above the
resonant wavenumber rolls off with increasing \( k \); this is controlled by the boundary conditions [3.4] and by the strain/curvature relation (more "wiggles" in the beam's shape require more strain energy).

![Graph showing wavenumber/frequency response function for nondimensional simply-supported beam.](image)

Fig. 3.4: Wavenumber/frequency response function for nondimensional simply-supported beam.

For a scalar Green's function the multiplicative input/output relation (3.5) in the transformed space is also scalar. No distinct sensors or actuators have been introduced. Let the input be represented as a superposition of \( N \) inputs, each of which is filtered in space and time by an actuator distribution that has temporal dynamics, vis

\[
u(k, \omega) = \sum_{i=1}^{N} q_i(k, \omega) u_i(k, \omega).
\]

\( q_i(k, \omega) \) is the space/time transform of each actuator distribution and its associated dynamics. Equation (3.6) facilitates the description of actuators that reconfigure themselves in space over time (i.e. moving inputs, "fully distributed" actuators). If the actuator distribution has temporal dynamics that do not vary over its spatial aperture, then the actuator distribution can be represented as a product of space and time functions, and is called degenerate [3.7].
For a degenerate actuator, equation (3.6) takes the special form

\[ q_i(k, \omega) = q_i(k) \gamma(\omega). \]  \hspace{1cm} \text{(3.7)}

This will be the form for most common actuators (i.e. piezoceramic stacks, heaters, shakers), but need not be assumed for the ensuing analysis. If \( u_i = u_i(\omega) \) alone in equation (3.6), this corresponds to a command input with point spatial extent, i.e. a time-varying electrical signal connected to the physical actuator.

Each sensor affixed to a distributed parameter plant convolves its output. Consider a point displacement measurement by a sensor that has infinite temporal bandwidth;

\[ y_s(x, \omega) = \int_D \delta(x - x_s) y(x, \omega) \, dx. \]  \hspace{1cm} \text{(3.8)}

\( x_s \) is the location of the sensor. This is a convolution integral, which admits the spatial Fourier integral transform

\[ y_s(k, \omega) = e^{ik \cdot x_s} y(k, \omega). \]  \hspace{1cm} \text{(3.9)}

This point sensor has infinite spatial bandwidth (the complex exponential in the spatial transform variable has unit magnitude for all wavenumbers); this is the spatial analog of a time domain impulse, which has infinite \textit{temporal} bandwidth.

Similarly, a sensor with the "boxcar" aperture (3.1) also convolves the distributed output signal; the convolution integral can be broken up into two separate integrals, with the transformed kernel (3.2) representing the wavenumber response function of this sensor. One can construct such spatial filter functions for other sensing distributions, and for other sensor types as well. This is done most commonly in acoustics, especially in sonar applications.

Consequently, for the \( j \)-th sensor, including temporal dynamics, the transformed sensor output takes the form
\[ y_j(k,\omega) = p_j(k,\omega) y(k,\omega). \]  \hspace{1cm} 3.10

\( p_j(k,\omega) \) is the space/time transform of the sensor distribution and its associated dynamics. Most commonly, \( p_j(k,\omega) \) is degenerate (e.g. separable into a product of space and time functions), and will assume the special form

\[ p_j(k,\omega) = p_j(k) \theta(\omega), \]  \hspace{1cm} 3.11

e.g. the sensor extends over a fixed subset of the output domain, with associated temporal dynamics uniform over the entire sensing aperture. However, this is also not a prerequisite of the analysis; the more general form includes sensors whose apertures can vary with time.

For a distributed system with \( N \) actuators and \( M \) sensors, each sensor's output in the transformed wavenumber/frequency space becomes

\[ y_1(k,\omega) = p_1(k,\omega) h(k,\omega) q_1(k,\omega) u_1(k,\omega) + \]
\[ \cdots + p_1(k,\omega) h(k,\omega) q_N(k,\omega) u_N(k,\omega), \]
\[ \vdots \hspace{1cm} \vdots \]
\[ y_M(k,\omega) = p_M(k,\omega) h(k,\omega) q_1(k,\omega) u_1(k,\omega) + \]
\[ \cdots + p_M(k,\omega) h(k,\omega) q_N(k,\omega) u_N(k,\omega). \]  \hspace{1cm} 3.12

or, in matrix form, one constructs the multiplicative MIMO relation

\[
\begin{bmatrix}
  y_1(k,\omega) \\
  y_2(k,\omega) \\
  \vdots \\
  y_M(k,\omega)
\end{bmatrix} =
\begin{bmatrix}
  p_1(k,\omega) h(k,\omega) q_1(k,\omega) & \cdots & p_1(k,\omega) h(k,\omega) q_N(k,\omega) \\
  p_2(k,\omega) h(k,\omega) q_1(k,\omega) & \cdots & p_2(k,\omega) h(k,\omega) q_N(k,\omega) \\
  \vdots & \cdots & \vdots \\
  p_M(k,\omega) h(k,\omega) q_1(k,\omega) & \cdots & p_M(k,\omega) h(k,\omega) q_N(k,\omega)
\end{bmatrix}
\begin{bmatrix}
  u_1(k,\omega) \\
  u_2(k,\omega) \\
  \vdots \\
  u_N(k,\omega)
\end{bmatrix}. \]  \hspace{1cm} 3.13

which can be expressed more compactly as
\[ y(k, \omega) = G(k, \omega) u(k, \omega). \]

Hereafter, \( G(k, \omega) \) will be referred to as the augmented plant response matrix. If the plant, sensors, and actuators were not distributed in space (a lumped parameter system), e.g. if they were independent of \( k \), then equation (3.14) reduces to the familiar time domain matrix response function relation,

\[ y(\omega) = G(\omega) u(\omega). \]

Note that one can always write equation (3.14) in the form

\[ y(k, \omega) = \underbrace{h(k, \omega)}_{\text{scalar}} G'(k, \omega) \underbrace{u(k, \omega)}_{\text{matrix}}, \]

and equation (3.15) can always be written as

\[ y(k, \omega) = \underbrace{1}_{\text{scalar}} G'(\omega) u(\omega), \]

where \( \Delta(\omega) \) is the determinant relation defining the characteristic equation for a lumped-parameter system. \( \Delta(\omega) \) contains information about the transfer matrix poles, an invariant property of the system. \( G'(\omega) \) contains information about the interaction of the inputs and outputs with the system.

Similarly, \( h(k, \omega) \) contains information about temporal and spatial poles, while \( G'(k, \omega) \) reflects the interaction of the actuators and sensors with the system, in time and space. The invariant scalar factorable part of (3.16) will be the inverse of the distributed plant dispersion relation, which relates its temporal and spatial poles; this too reflects the combination of temporal and spatial coincidences defining resonance in a distributed parameter system.
To better illustrate the augmented plant relations, consider the following simple example: an infinite string on an elastic foundation. A sketch of this system appears in Fig. 3.5. The system has two degenerate point displacement sensors at \( x_1 \) and \( x_2 \), and two degenerate point force actuators at \( x_3 \) and \( x_4 \). The displacement of the string is given by \( y(x,t) \). The corresponding dimensional governing equation is

\[
T \frac{\partial^2 y}{\partial x^2} - m \frac{\partial^2 y}{\partial t^2} - \kappa y = u(x,t),
\]

where \( T \) is the tension in the string (assumed to be spatially uniform), \( m \) is its mass density, \( \kappa \) is the stiffness of the elastic foundation, and \( u(x,t) \) is the input. Introducing the Fourier integral space/time transform

\[
y(k,\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(kx + \omega t)} y(x,t) \, dx \, dt
\]

permits equation (3.18) to be space/time transformed;

\[
(-Tk^2 + m\omega^2 - \kappa) y(k,\omega) = u(k,\omega).
\]

The plant's space/time response function \( h \) is then
\[ h(k, \omega) = \frac{1}{-Tk^2 + m \omega^2 - \kappa}. \]  

3.21

The actuator filter functions \( q_i \) for degenerate point actuation take the form

\[
q_1(k, \omega) = e^{ikx_1} \gamma_1(\omega), \\
q_2(k, \omega) = e^{ikx_2} \gamma_2(\omega).
\]  

3.22

Consequently,

\[
u(k, \omega) = e^{ikx_1} \gamma_1(\omega) u_1(\omega) + e^{ikx_2} \gamma_2(\omega) u_2(\omega).
\]  

3.23

The sensor filter functions \( p_j \) for degenerate point displacement measurement are

\[
p_1(k, \omega) = e^{ikx_1} \theta_1(\omega), \\
p_2(k, \omega) = e^{ikx_2} \theta_2(\omega).
\]  

3.24

Combining equations (3.21) - (3.24) gives the MIMO input/output response relation:

\[
\begin{bmatrix} y_1(\omega) \\ y_2(\omega) \end{bmatrix} = \begin{bmatrix} e^{ikx_1} \theta_1(\omega) e^{ikx_3} \gamma_1(\omega) \\ -Tk^2 + m \omega^2 - \kappa \\ e^{ikx_2} \theta_2(\omega) e^{ikx_3} \gamma_1(\omega) \\ -Tk^2 + m \omega^2 - \kappa \\ e^{ikx_1} \theta_1(\omega) e^{ikx_4} \gamma_2(\omega) \\ -Tk^2 + m \omega^2 - \kappa \\ e^{ikx_2} \theta_2(\omega) e^{ikx_4} \gamma_2(\omega) \\ -Tk^2 + m \omega^2 - \kappa \end{bmatrix} \begin{bmatrix} u_1(\omega) \\ u_2(\omega) \end{bmatrix}.
\]  

3.25

\[ \equiv G(k, \omega) \]

Note that, since the sensed output and command signal vectors are temporal functions only, the wavenumber dependence of \( G(k, \omega) \) is an internal representation of the plant's spatial dependence. One can obviously factor out the plant's dispersion relation \(-Tk^2 + m \omega^2 - \kappa\), which defines a continuous locus of resonances along two \( k \) branches. This relation also defines the phase velocity of wave propagation within the system; since the phase and group velocities are equal here, the system is not dispersive [3.8].
The MIMO relation (3.25) changes considerably if the wavelength of the string's spatial response becomes infinite (i.e. $T \to 0$, or the wavelength becomes much larger than the largest spacing between any of the sensors and/or actuators). Since the wavenumber $k$ is $2\pi$ divided by the wavelength, the matrix $G(k, \omega)$ loses its wavenumber dependence; $k \to 0$. At $k = 0$,

$$
\begin{bmatrix}
\gamma_1(\omega) \\
\gamma_2(\omega)
\end{bmatrix} = \frac{1}{m \omega^2 - \kappa} \begin{bmatrix}
\rho_1(\omega) \gamma_1(\omega) & \rho_1(\omega) \gamma_2(\omega) \\
\rho_2(\omega) \gamma_1(\omega) & \rho_2(\omega) \gamma_2(\omega)
\end{bmatrix} \begin{bmatrix}
u_1(\omega) \\
u_2(\omega)
\end{bmatrix}.
\tag{3.26}
$$

This corresponds to two actuators (with associated dynamics) forcing an infinite spring/mass oscillator! There is now no "spatial phase" between the sensors and actuators, as the wavelength is much larger than the greatest distance between and sensor/actuator pair. All that remains is temporal dynamics.

Essentially, this simple example shows the similarity of form between the wavenumber/frequency formalism and the usual MIMO temporal formalism. Most practical distributed parameter systems will require a more involved analysis. The general concepts presented in the previous example, however, will still apply.

In the next Section, the classes of Green's functions to which the space/time transform modelling technique can be applied will be studied. The section is replete with examples of each form. The presentation is restricted to continuous (integral) Fourier spatial transforms; the application of discrete spatial transforms is left for the discussion of shape control in Chapter V.
III.3 MULTIDIMENSIONAL TRANSFORMS: CLASSES OF APPLICATION

In this Section, the classes of distributed parameter plants to which the multidimensional transform model can be applied will be defined. In some few cases it will be possible to transform a governing partial differential equation directly. The plants are classified based upon the form of their Green's functions. Throughout, it is assumed that the temporal properties of the distributed parameter systems are stationary. This permits the associated composition integral to assume a temporal convolution form, and hence a product form in \( \omega \) when Fourier transformed in time. As before, the input/output composition is assumed to be linear. The Green's functions are also assumed to be Fourier-transformable, which requires that they be square integrable [3.9,3.10]. The input and output distributed signals \( y \) and \( u \) are also assumed to be square integrable. We consider here only continuous (integral) Fourier spatial transforms; discrete (Fourier series) spatial transforms will be studied in Chapter V.

The Green's functions can be static, and hence have no temporal frequency dependence. The input \( u(x,t) \) and output \( y(x,t) \) appearing in the ensuing composition integrals can be vectors, with matrix kernels. Each element of such a Green's function could assume one of the forms illustrated in Cases 1-4. Consider then the following forms for \( h(x,\xi,t - \tau) \):

**Case 1: "Right" Convolution Form**

In this case the Green's function assumes the form

\[
h(x,\xi,t - \tau) = h(x - \xi,t - \tau).
\]

The associated composition integral becomes a convolution in space and time;

\[
y(x,t) = \int_{-\infty}^{\infty} \int_{D} h(x - \xi,t - \tau) u(\xi,\tau) d\xi d\tau.
\]
Since (3.28) is a convolution integral, when Fourier integral transformed in space and time it assumes a product form in wavenumber/frequency space,

\[ y(k, \omega) = h(k, \omega)u(k, \omega), \tag{3.29} \]

where \( h(k, \omega) \) is the transform of the Green's function \( h(y, \alpha) \) for \( y = x - \xi, \alpha = t - \tau \). Since the kernel of equation (3.28) is only a function of the difference between the spatial variables \( x - \xi \), rather than their particular locations, the system displays a type of "spatial stationarity". A system whose Green's function depends spatially on only the sum and/or difference of the input and output spatial arguments is called isoplanatic, a term borrowed from the optics literature. The spatial convolution form (3.27) is found commonly in the theory of Fourier optics where it is often posed as time-independent [3.11]. It is also referred to as a "causal" spatial convolution form as an analogy to time domain convolutions.

Note that \( h \) could represent a superposition of right convolution form Green's functions. Since the system is linear, the transform would be applied to each term of \( h \), then summed to obtain \( h(k, \omega) \).

Case 2: "Left" and "Right" Convolution Form

The Green's function for some plants will be a sum of "right" and "left" convolution forms,

\[ h(x, \xi, t - \tau) = h_0(x - \xi, t - \tau) \pm h_0(x + \xi, t - \tau). \tag{3.30} \]

Linearity also permits any superposition of functions having the form (3.30). The corresponding composition integral becomes
\[ y(x,t) = \int_{-\infty}^{\infty} \int_{D} \left[ h_0(x - \xi, t - \tau) \pm h_0(x + \xi, t - \tau) \right] u(\xi, \tau) d\xi d\tau. \quad 3.31 \]

From the multidimensional convolution theorem, the first product component of the integral in (3.31) becomes a product in the transformed wavenumber/frequency space;

\[ F\left\{ \int_{-\infty}^{\infty} \int_{D} h_0(x - \xi, t - \tau) u(\xi, \tau) d\xi d\tau \right\} = h_0(k, \omega) u(k, \omega). \quad 3.32 \]

The second product component of the integral in (3.31) assumes a similar multiplicative form when space/time transformed, using a related convolution theorem;

\[ F\left\{ \int_{-\infty}^{\infty} \int_{D} h_0(x + \xi, t - \tau) u(\xi, \tau) d\xi d\tau \right\} = h_0(k, \omega) u^*(k, \omega). \quad 3.33 \]

The asterisk \((\cdot)^*\) denotes the complex conjugate. The input/output relation in the transformed space becomes

\[ y(k, \omega) = h_0(k, \omega) \left[u(k, \omega) \pm u^*(k, \omega)\right], \quad 3.34 \]

which can be re-written more compactly in the product form

\[ y(k, \omega) = h_0(k, \omega) u_{\text{effective}}(k, \omega). \quad 3.35 \]

Once again, \(h_0(k, \omega)\) is the space/time transform of \(h(y, \alpha)\), for \(y \equiv x - \xi, \alpha \equiv t - \tau\).

One can think of the Green's function form (3.30) as follows: A time-domain lumped-parameter system has information that propagates with increasing time, hence its Green's function is \textit{causal}, and of the form \(h(t - \tau)\). In a spatially-bounded (e.g. finite) distributed parameter system, the response at any point in space is effected by information
(e.g. excitation and boundary conditions) on both sides of it. The portion of the Green's function that is a function of \( x + \xi \) is sometimes called the 'anti-causal' part as an analogy with lumped systems. Information moves in "two" directions [actually, in several directions for a system with \( x = (x,y,z) \)], hence the corresponding Green's function takes the form (3.30).

As an example, the Green's function for a string of length \( L \) takes the form [3.12]

\[
h(x,\xi,\omega) = \sum_{n=1}^{\infty} \frac{a_n \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi \xi}{L} \right)}{\omega^2 + c^2 \left( \frac{n\pi}{L} \right)^2}. \tag{3.36}
\]

where \( c \) is associated wave speed. Each term in the summation (3.36) can be written as

\[
a_n \frac{\sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi \xi}{L} \right)}{\omega^2 + c^2 \left( \frac{n\pi}{L} \right)^2} = \frac{a_n}{2} \frac{\cos \left( \frac{n\pi}{L}(x-\xi) \right) - \cos \left( \frac{n\pi}{L}(x+\xi) \right)}{\omega^2 + c^2 \left( \frac{n\pi}{L} \right)^2}, \tag{3.37}
\]

which has the form (3.30). The corresponding transformed response function has components

\[
F\left( \frac{a_n}{2} \frac{\cos \left( \frac{n\pi}{L}y \right)}{\omega^2 + c^2 \left( \frac{n\pi}{L} \right)^2} \right) = \frac{a_n}{4} \frac{\left[ (-1)^n \sin(kL) + i \left( \frac{1}{k - \frac{n\pi}{L}} + \frac{1}{k + \frac{n\pi}{L}} \right) + i \left( -\cos(kL)(-1)^n - \cos(kL)(-1)^n \right) \right]}{\omega^2 + c^2 \left( \frac{n\pi}{L} \right)^2}. \tag{3.38}
\]

Then, \( h(k,\omega) \) (as per (3.35)) is the superposition of these components. This technique was used to synthesize the bandlimited representation of the wavenumber/frequency response function of a simply-supported beam depicted in Fig. 3.4. Many distributed plants have Green's functions expressed as expansions in orthogonal functions; this will be explored
further in Case 4, and in Chapter V.

Case 3: "Damped" Convolution Form

Here, the Green's function takes another summation form;

\[ h(x, \xi, t - \tau) = h_0(x - i\xi, t - \tau) \pm h_0(x + i\xi, t - \tau). \]  \hspace{1cm} 3.39

The associated composition integral becomes

\[ y(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ h_0(x - i\xi, t - \tau) \pm h_0(x + i\xi, t - \tau) \right] \mu(\xi, \tau) d\xi d\tau. \]  \hspace{1cm} 3.40

The Fourier space/time transform of the first part of equation (3.40) is defined by a modified convolution theorem, so that

\[ \mathcal{F} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_0(x - i\xi, t - \tau) \mu(\xi, \tau) d\xi d\tau \right\} = i h_0(k, \omega) u(ik, \omega). \]  \hspace{1cm} 3.41

Similarly, the second term of equation (3.40) has the transform

\[ \mathcal{F} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_0(x + i\xi, t - \tau) \mu(\xi, \tau) d\xi d\tau \right\} = i h_0(k, \omega) u^*(ik, \omega). \]  \hspace{1cm} 3.42

Combining these gives the desired multiplicative input/output in the wavenumber/frequency space,
\[ y(k, \omega) = h_0(k, \omega)[i u(i k, \omega) \pm i u^*(i k, \omega)] \]
\[ = h_0(k, \omega) u_{\text{effective}}(k, \omega). \] \hspace{1cm} (3.43)

As an example, the Green's function for a free-free Bernoulli-Euler beam will contain terms of the form [3.12]

\[ \frac{a_n \sin(\mu_n x) \cosh(\mu_n \xi)}{\omega^2 + c^2 \mu_n^4}. \] \hspace{1cm} (3.44)

Using the trigonometric identity

\[ \cosh(y) = \cos(iy), \] \hspace{1cm} (3.45)

expression (3.44) can be re-written as

\[ \frac{a_n \sin(\mu_n x) \cos(i \mu_n \xi)}{\omega^2 + c^2 \mu_n^4}, \] \hspace{1cm} (3.46)

and thence

\[ \frac{a_n \sin(\mu_n x) \cosh(\mu_n \xi)}{\omega^2 + c^2 \mu_n^4} = \frac{a_n}{2} \left[ \frac{\sin[\mu_n(x - i \xi)] + \sin[\mu_n(x + i \xi)]}{\omega^2 + c^2 \mu_n^4} \right]. \] \hspace{1cm} (3.47)

The transform follows immediately from elementary manipulations.

Thin beams and plates with any combination of clamped, pinned, free, or sliding boundary will have Green's functions containing terms defined in Cases 1, 2 and/or 3, as will many problems arising in heat transfer and acoustics. Consequently, the transform method can be applied to a wide range of distributed plants whose Green's functions can be derived readily.
Case 4: Symmetric Green's Function

In this case, the Green's function assumes the elegant form

\[ h(x, \xi, t - \tau) = h(\xi, x, t - \tau); \]  

that is, the input and output spatial arguments are interchangeable. Such a Green's function is called symmetric. Mechanical systems with symmetric Green's functions display the property of reciprocity; this property is found in other distributed parameter plants as well. Systems described by self-adjoint partial differential operators (i.e. Sturm-Liouville problems) also have symmetric Green's functions \([3.13,3.14]\). Symmetric Green's functions can always be expressed in a bilinear Hilbert-Schmidt expansion (hereafter called a bilinear expansion), viz

\[ h(x, \xi, \omega) = \sum_{i=1}^{N} \frac{\varphi_i(x) \varphi_i(\xi)}{\lambda_i(\omega)}, \] 

where, for simplicity, the temporal transform has already been taken. The functions \( \varphi_i \) are the eigenfunctions of the kernel \( h \), with \( \lambda_i(\omega) \) the corresponding eigenvalue relation. For self-adjoint distributed parameter systems, these are the same as the systems eigenfunctions and eigenvalues. The summation in equation \((3.49)\) will often have \( N \) going to infinity. Since the bilinear form only requires that the Green's function be symmetric, it is applicable to an large number of distributed plants; Butkovskiy \([3.12]\) has published a tabulation of over 500 bilinear expansions for various distributed parameter systems.

The composition integral corresponding to the bilinear expansion \((3.49)\) (using the temporal transform) becomes
\[ y(x, \omega) = \int \sum_{i=1}^{N} \frac{\varphi_i(x) \varphi_i(\xi)}{\lambda_i(\omega)} u(\xi, \omega) \, d\xi \]
\[ = \sum_{i=1}^{N} \frac{\varphi_i(x)}{\lambda_i(\omega)} \int_{D} \varphi_i(\xi) u(\xi, \omega) \, d\xi. \tag{3.50} \]

One next assumes that the input control signal \( u \) is degenerate, and takes the special separable form
\[ u(\xi, \omega) = q(\xi, \omega) u(\omega). \tag{3.51} \]

That is, the input is a product of an actuator spatial distribution (with associated dynamics) and an exogenous control signal. The input could also be a superposition of inputs such as (3.51), representing several actuators. Substituting (3.51) into (3.50) gives
\[ y(x, \omega) = u(\omega) \sum_{i=1}^{N} \frac{\varphi_i(x)}{\lambda_i(\omega)} \int_{D} \varphi_i(\xi) q(\xi, \omega) \, d\xi. \tag{3.52} \]

Thus,
\[ y(x, \omega) = u(\omega) \sum_{i=1}^{N} a_i \frac{\varphi_i(x)}{\lambda_i(\omega)}. \tag{3.53} \]

Performing a spatial Fourier integral transform of equation (3.53) yields the input/output relation
\[ y(k, \omega) = \left[ \sum_{i=1}^{N} a_i \frac{\varphi_i(k)}{\lambda_i(\omega)} \right] u(\omega) \]
\[ = h'(k, \omega) u(\omega). \tag{3.54} \]
This has the desired multiplicative form in the transformed space. However, the exogenous control input $u(\omega)$ cannot have its own factorable spatial dependence owing to the spatial integral in (3.50). Consequently, the bilinear expansion is restricted to actuator distributions that interact with their associated compensator via a temporal control signal only, i.e. an electrical connection. This too is another kind of degenerate compensation. The "spatial compensation" must reside solely in the actuator distribution, rather than in a fully distributed parameter compensator (e.g. using a second distributed parameter system to compensate a first). However, a large number of distributed parameter control problems will have this characteristic, hence it is not a severe restriction.

III.4 SUMMARY

This Chapter has introduced the space/time transform technique as a modelling tool for distributed parameter systems. The method offers the possibility of quantifying the spatial performance of distributed parameter controllers, as the spatial transform readily lends itself to "spatial bandwidth" interpretations pertinent to many control tasks. The plant is represented as a space/time filter. Actuator and sensor distributions, themselves space/time filters, then augment the plant model to construct a multiplicative MIMO response matrix.

The applicability of the paradigm was defined in terms of plant Green's functions. For a wide class of stationary, isoplanatic systems, space/time convolution theorems were presented that convert integral input/output relations to products in a wavenumber/frequency space. Failing all else, if the Green's function is at least symmetric a bilinear expansion can be introduced to construct the necessary input/output relation.

The multiplicative MIMO input/output representation presents the tantalizing possibility of applying modern control analysis methods directly to distributed parameter systems. In the next Chapter, this potential will be realized. Since the analysis tools only assume a multiplicative matrix input/output relation, where the matrix is a function of one (or more) independent transform variable(s), the extension is a straightforward application of linear operator theory.
IV. PERFORMANCE AND ROBUSTNESS MEASURES FOR DISTRIBUTED PARAMETER CONTROL SYSTEMS

IV.1 INTRODUCTION

An analysis method is developed in this Chapter wherein a distributed plant's input/output relation is studied in a transformed "wavenumber/frequency" space. The presentation reveals the essential similarities to and extensions from research in MIMO control analysis. Particular emphasis is placed upon utilizing a well-developed tool of modern control theory, the singular value decomposition, to analyze compensation and sensor/actuator placement in distributed systems. Criteria for achieving spatial performance goals such as command following and disturbance rejection over "spatial bandwidths" are presented that appeal to physical insight, and are generalized to include temporal performance as well. Controllability and observability measures are derived. A stability-robustness measure is developed. The method is shown to be applicable to both discrete and distributed sensors and actuators, using continuous (integral) or discrete (series) spatial transforms.

IV.2 DISTRIBUTED PARAMETER CONTROLLER RELATIONS

In order to motivate the ensuing performance measures, the loop and closed-loop constitutive relations for a distributed parameter controller will now be derived. The exposition parallels similar developments for lumped-parameter MIMO controllers [4.1,4.2]. Consider the control system topology depicted in Fig. 4.1. The forward loop consists of a distributed MIMO compensator matrix $K(k,\omega)$ followed by the distributed plant. If the compensator is purely temporal, it is called degenerate; the spatial compensation is provided by the sensor and actuator distributions. A transformed space-time disturbance $d(k,\omega)$ is
included at the physical plant output, as is additive "sensor noise" $n(k, \omega)$ at the filtered outputs. Disturbances at the plant input are assumed to be reflected through the plant and included in the output disturbance. The loop transfer matrix from the error signal $e(k, \omega)$ to the output $y(k, \omega)$, in the absence of disturbances, is given by

$$T(k, \omega) = h(k, \omega) p(k, \omega) q(k, \omega) K(k, \omega)$$

$$= G(k, \omega) K(k, \omega). \tag{4.1}$$

The closed-loop input/output relation for the distributed controller is given by

$$y(k, \omega) = S(k, \omega) d(k, \omega) + C(k, \omega) [r(k, \omega) - n(k, \omega)], \tag{4.2}$$

where the sensitivity transfer matrix $S(k, \omega)$, or inverse return difference, is defined by

$$S(k, \omega) = [I + T(k, \omega)]^{-1}. \tag{4.3}$$

and the closed-loop transfer matrix $C(k, \omega)$ is defined by

$$C(k, \omega) = S(k, \omega) T(k, \omega). \tag{4.4}$$

Defining the output error as the difference between the reference command and outputs,
\[ e_o(k,\omega) = r(k,\omega) - y(k,\omega), \] (4.5) 

leads to the important relation

\[ e_o(k,\omega) = S(k,\omega)[r(k,\omega) - d(k,\omega)] + C(k,\omega)n(k,\omega). \] (4.6)

This definition of the error differs from the error signal \( e(k,\omega) \) represented in Fig. 4.1. However, equation (4.6) denotes the impact of the output noise \( n(k,\omega) \) on how well the controller accomplishes command following, which is much more meaningful in quantifying performance [4.1].

For good command following, the control designer wants

\[ y(k,\omega) \equiv r(k,\omega) ; \ \omega \in \Omega_r, \ k \in K_r, \] (4.7)

where \( \Omega_r \) and \( K_r \) denote the temporal and spatial frequency domains of interest, as commanded by \( r(k,\omega) \). The frequency and wavenumber bands \( \Omega_r \) and \( K_r \) are those where \( r(k,\omega) \) has significant energy content; note that the present formulation makes the extension to the spatial domain natural. Also, \( K_r \) can be a continuous or discrete bandwidth, depending on whether the spatial transform is continuous or discrete. From equation (4.2), the command following requirement is met, in the absence of disturbances and noise, if

\[ C(k,\omega) \equiv I ; \ \omega \in \Omega_r, \ k \in K_r. \] (4.8)

Good tracking also requires the expression (4.6) for the output error be "small", hence

\[ S(k,\omega) \equiv [0] ; \ \omega \in \Omega_r, \ k \in K_r. \] (4.9)

If the loop transfer matrix \( T(k,\omega) \), defined in equation (4.1), is "large", then \( S(k,\omega) \) will be "small", and thence satisfy (4.9). Also, a "large" loop transfer matrix will lead to
\[ S(k, \omega) \equiv T^{-1}(k, \omega), \quad \text{(4.10)} \]

consequently the requirement (4.8) will be satisfied. Naturally, the qualifiers "large" and "small" have yet to be rigorously defined.

Disturbance rejection, from equation (4.2), requires

\[ S(k, \omega) \equiv [0] \ ; \ \omega \in \Omega_d, \ k \in K_d, \quad \text{(4.11)} \]

where \( \Omega_d \) and \( K_d \) denote the frequency and wavenumber bands where the disturbance has significant energy. This reinforces the requirement (4.8), and the role of \( S(k, \omega) \) in the output error (4.6).

Finally, immunity to "sensor noise" \( n(k, \omega) \) requires

\[ C(k, \omega) \equiv [0] \ ; \ \omega \in \Omega_n, \ k \in K_n, \quad \text{(4.12)} \]

where \( \Omega_n \) and \( K_n \) denote the frequency and wavenumber regions where the "sensor noise" has significant energy. This requirement must be balanced by the command following requirement (4.8); the command input cannot intrude into high wavenumber-frequency bands, while the closed-loop transfer matrix must be vanishingly small within the bands \( \Omega_n \) and \( K_n \). The "sensor noise" can also include high-wavenumber modelling errors associated with quantifying the sensor spatial distributions. Similarly, the output disturbance \( d(k, \omega) \) can denote plant modelling errors; these observations will be exploited in the stability-robustness development.
IV.3 PERFORMANCE MEASURES FOR DISTRIBUTED PARAMETER CONTROL

The multidimensionally-transformed distributed plant model provides a multiplicative input/output relation. Treating actuator and sensor distributions as spatial filters then leads to the development of a matrix input/output relation, and matrix loop and closed-loop relations. Given this formulation, the singular value decomposition can now be applied to precisely quantify the performance requirements of the previous Section. Distributed parameter control performance measures will now be derived that incorporate temporal as well as spatial information. Each performance measure will be considered separately, in the absence of other contributions, as the system is assumed to be linear (and hence superposition holds).

First, consider the command following requirement. In the absence of noise and disturbances, the tracking error (4.6) is given by

\[ e_o(k, \omega) = S(k, \omega) r(k, \omega). \]  

Taking the conjugate transpose of this equation, multiplying by equation (4.13), and taking the square root of both sides gives an expression for the Euclidean norm of the error,

\[ \| e_o(k, \omega) \|_2 = \| S(k, \omega) r(k, \omega) \|_2. \]  

Using the Cauchy-Schwarz inequality,

\[ \| e_o(k, \omega) \|_2 \leq \| S(k, \omega) \|_2 \| r(k, \omega) \|_2. \]  

However, since the spectral norm of a matrix is defined to be its maximum singular value, this can be re-written as

\[ \| e_o(k, \omega) \|_2 \leq \sigma_{\max}[S(k, \omega)] \| r(k, \omega) \|_2. \]  

Thus, good command following in the presence of a general, non-zero reference input \( r \) requires
$\sigma_{\text{max}}[S(k,\omega)] < 1$ ; $\omega \in \Omega_r, k \in k_r$. \hfill (4.17)

A matrix is then "small" when its maximum singular value is small. The maximum singular value of a matrix is its maximum multiplicative gain.

The requirement (4.17) extends the usual frequency domain specification for lumped-parameter systems to include the distributed plant's performance in the spatial domain. The worst tracking error at any fixed $(k,\omega)$ occurs when $r(k,\omega)$ points along the right singular vector of $S(k,\omega)$ associated with $\sigma_{\text{max}}[S(k,\omega)]$. A similar result holds for the best tracking error. The inequality (4.17) essentially provides a measure of the "size" of $S(k,\omega)$ to meet the performance requirement (4.9), in both a temporal and spatial sense.

Using the definition of $S(k,\omega)$, and other matrix singular value properties [4.3], the performance specification (4.17) can be re-written as

$$\sigma_{\text{max}}[S(k,\omega)] = \frac{1}{\sigma_{\text{min}}[I + T(k,\omega)]} < 1 ; \omega \in \Omega_r, k \in k_r. \hfill (4.18)$$

Thus, make $\sigma_{\text{min}}[I + T(k,\omega)] >> 1$ over the specified bandwidth. However,

$$\sigma_{\text{min}}[I + T(k,\omega)] < \sigma_{\text{min}}[T(k,\omega)] + 1, \hfill (4.19)$$

hence the command following performance requirement becomes

$$\sigma_{\text{min}}[T(k,\omega)] >> 1 ; \omega \in \Omega_r, k \in k_r. \hfill (4.20)$$

$T(k,\omega)$ will be "large" if its minimum singular value is large. $T(k,\omega)$ is made large, as seen in equation (4.1), by a judicious choice of the distributed compensator $K(k,\omega)$ and the actuator and sensor filter functions $p(k,\omega)$ and $q(k,\omega)$. If the compensator is degenerate,
then the sensor and actuator spatial distributions provide the only spatial compensation. This demonstrates the heretofore intuitive result that sensor and actuator placement will influence a distributed parameter controller's spatial bandwidth response. The performance measure (4.20), then, becomes a means of screening candidate compensators and sensor/actuator locations (and types) for achieving the desired temporal and spatial command following for distributed parameter systems.

The disturbance rejection performance measure is similarly derived. The tracking error, in the absence of noise or command inputs, is

$$e_d(k, \omega) = -S(k, \omega) d(k, \omega).$$  \hspace{1cm} 4.21

This can be manipulated to yield

$$\| e_d(k, \omega) \|_2 \leq \| S(k, \omega) \|_2 \| d(k, \omega) \|_2,$$ \hspace{1cm} 4.22

hence

$$\| e_d(k, \omega) \|_2 \leq \sigma_{max}[S(k, \omega)] \| d(k, \omega) \|_2.$$ \hspace{1cm} 4.23

Rejecting disturbances and providing insensitivity to plant modelling errors then requires

$$\sigma_{max}[S(k, \omega)] \ll 1 \ ; \ \omega \in \Omega_d, \ k \in K_d.$$ \hspace{1cm} 4.24

Equation (4.24) quantifies the requirement that $S(k, \omega)$ be "small" over the disturbance temporal and spatial bandwidths. The same directional interpretation as for equation (4.17) holds as well concerning directions of maximum and minimum disturbance sensitivity. As with the command following result, equation (4.24) can be re-written as

$$\sigma_{min}[T(k, \omega)] \gg 1 \ ; \ \omega \in \Omega_d, \ k \in K_d.$$ \hspace{1cm} 4.25

Thus, high loop gain in both the frequency and wavenumber bands $\Omega_d$ and $K_d$ will provide
for good disturbance rejection, and insensitivity to plant modelling errors.

Finally, to address the "sensor noise" performance measure, one must consider the tracking error in the absence of disturbances and command inputs,

$$e_o(k, \omega) = C(k, \omega)n(k, \omega).$$  \hspace{2cm} 4.26

As before,

$$||e_o(k, \omega)||_2 \leq \sigma_{\max}[C(k, \omega)]||n(k, \omega)||_2,$$  \hspace{1cm} 4.27

so that insensitivity to "sensor noise" requires

$$\sigma_{\max}[C(k, \omega)] \ll 1, \quad \omega \in \Omega_n, \ k \in K_n.$$  \hspace{1cm} 4.28

This requirement, which quantifies the relation (4.12), leads to high disturbance sensitivity and low loop gain in the temporal and spatial bands $\Omega_n$ and $K_n$, and potentially conflicts with

Fig. 4.2: Visualization of $(k, \omega)$ tracking performance specification.
the tracking requirement (4.8), unless the bandwidths $\Omega_r$, $\Omega_d$ and $K_r$, $K_d$ are well separated from $\Omega_n$ and $K_n$. Consequently, the command input cannot have significant energy content within $\Omega_n$ and $K_n$. If $n(k, \omega)$ has a noise bandwidth $K_r$ associated with high-wavenumber sensor spatial modelling errors, the requirement (4.28) indicates that one should not design a response bandwidth outside the spatial and temporal bandwidth of the sensors.

The performance requirement that the loop transfer function singular values be large for good command following are visualized in Fig. 4.2 for a continuous (integral) spatial transform. For a distributed plant having one spatial dimension, the range of desired spatial and temporal response appears as a rectangular box in $(k, \omega)$. To achieve good performance the surface of minimum singular values of the loop transfer function matrix must not penetrate this box; the higher the amplitude of the singular values in this region, the smaller the tracking error.
IV.4 CONTROLLABILITY AND OBSERVABILITY MEASURES

To assess the controllability and observability of a distributed parameter system in terms of its actuator and sensor distributions, consider the matrix input/output relation describing the augmented plant. Let the input $u(k, \omega)$ be wavenumber and frequency white, so that all responses of the system can be excited, hence

$$y(k, \omega) = p(k, \omega) h(k, \omega) q(k, \omega).$$

To determine the impact of the actuator distribution $q(k, \omega)$ on the system achieving a non-zero output at every $(k, \omega)$, let the sensor's wavenumber/frequency filter vector be wavenumber and frequency white; consider $p(k, \omega)$ to consist of a single scalar filter function with infinite spatial and temporal bandwidth and unit magnitude. Equation (4.29) then becomes

$$y(k, \omega) = h(k, \omega) q(k, \omega).$$

If there exists some $(k_c, \omega_c)$ where $y(k, \omega)$ vanishes, then the system is not controllable at that wavenumber and frequency combination given the actuator distribution $q(k, \omega)$. Also, the spectral norm

$$||y(k, \omega)||_2 = ||h(k, \omega) q(k, \omega)||_2.$$  

will vanish as well. Equation (4.31) can be re-written as

$$||y(k, \omega)||_2 = \sigma [h(k, \omega) q(k, \omega)].$$

where the fact that $h(k, \omega) q(k, \omega)$ is a vector requires there be only one singular value $\sigma$. The
criteria for loss of controllability at some \((k_c, \omega_c)\) becomes

\[
\sigma[h(k_c, \omega)q(k_c, \omega)] = 0. \tag{4.33}
\]

This bears a strong resemblance to the common modal controllability matrix measure for lumped parameter systems. If \(k_c\) and \(\omega_c\) are eigenvalues and natural frequencies of the distributed plant that satisfy (4.33), and the spatial transform is a modal expansion, then that mode is uncontrollable. Practically, it is difficult to determine numerically when the singular value (4.33) vanishes; one might instead use the positive semi-definite left-hand side of equation (4.33) as a measure of controllability for all \((k, \omega)\) given the actuator distribution \(q(k, \omega)\).

To assess the impact of the sensor distribution \(p(k, \omega)\) on observability, let the actuator filter vector \(q(k, \omega)\) be wavenumber and frequency white; consider \(q(k, \omega)\) to consist of a single scalar filter function with infinite spatial and temporal bandwidth and unit magnitude. Equation (4.29) then becomes

\[
y(k, \omega) = p(k, \omega)h(k, \omega). \tag{4.34}
\]

If there exists some \((k_o, \omega_o)\) where \(y(k, \omega)\) vanishes, where the unfiltered output \(y(k, \omega)\) does not vanish, then the system's output is not observable at that wavenumber/frequency; given the sensor distribution \(p(k, \omega)\). Also, the corresponding spectral norm of the output will vanish, and hence

\[
\sigma[p(k_o, \omega)h(k_o, \omega)] = 0. \tag{4.35}
\]

becomes the criteria for loss of observability. Note the resemblance of equation (4.35) to the common modal observability matrix norm for lumped parameter systems. As with the
controllability norm, if \( k_o \) and \( \omega_o \) correspond to any of the eigenvalues and natural frequencies of the distributed plant that satisfy (4.35), and if the spatial transform is a modal expansion, then the mode is unobservable. The positive semi-definite left-hand side of equation (4.35) becomes a measure of the observability of the plant output for all wavenumbers and frequencies given the sensor distribution \( p(k, \omega) \).
IV.5 STABILITY–ROBUSTNESS MEASURES FOR DISTRIBUTED PARAMETER SYSTEMS

In addition to performance and controllability/observability measures, the control designer must also consider the robustness of a control system's stability in the presence of modelling uncertainties. The plant model may be imperfectly specified. The sensor and actuator distributions also may be idealizations; for example, force actuators may be assumed to make point contact with a structure, when in truth their contact pad is of finite spatial extent.

Lehtomaki [4.4,4.5] has developed stability–robustness measures for lumped-parameter MIMO control systems using various forms for the modelling error. The measures were posed in terms of the boundedness of the closed-loop input/output characterization, with the interpretation for destabilization utilizing the MIMO Nyquist criteria. He developed a sufficient condition for stability in terms of a maximum bound on the worst error in the system model. The criteria itself reduces to an application of the small gain theorem [4.6]. Consequently, the robustness measures, which only use magnitude information about the closed-loop system and its modelling errors, is conservative.

The stability–robustness measure developed here follows a derivation of Lehtomaki's result [4.4], where the generalization for distributed parameter systems arises from the multidimensionality of the independent transform variables. The measures are developed in terms of the boundedness of the closed-loop distributed system's response in the presence of modelling uncertainty. The transformed forward loop response matrix and the closed-loop system are assumed to be nominally stable. Naturally, the technique can be extended to open-loop unstable plants through the introduction of a stabilizing inner loop. The interpretation for destabilization utilizes the notion of parametric loci [4.7].

Consider the closed-loop input/output relation for a distributed parameter control system in the transformed $(k, \omega)$ space, in the absence of disturbances or sensor noise;

\[ y(k, \omega) = [I + T(k, \omega)]^{-1} T(k, \omega) r(k, \omega). \] \hspace{1cm} 4.36

$r$ is the command input vector, $y$ is the filtered output vector, and $T$ is the actual forward
loop transfer matrix (e.g., the product of the physical, implemented compensator matrix $K$ and the true, physical plant transfer matrix). This input/output relation is bounded input/bounded output (BIBO) stable if $(I + T)$ is non-singular, for

$$y(k, \omega) = \frac{adj(I + T(k, \omega))}{det(I + T(k, \omega))} T(k, \omega) r(k, \omega),$$  \hspace{1cm} 4.37

and singularity implies $det(I + T) = 0$. (Parenthetically, this also implies $\sigma_{\text{min}}(I + T) = 0$.)

Note that the system's compensation (e.g. the compensator $K$ and the sensor and actuator filter functions $p$ and $q$) is always designed for a nominal plant model. The forward loop transfer matrix for this nominal model is denoted by $T_N$. Various relationships between the nominal and "true" forward loops will now be studied to assess the impact of modelling error on the destabilization of the closed-loop plant.

**Case 1: Additive Error**

Let the difference between the nominal and "true" augmented plant models be represented by an additive error matrix $E_a$,

$$T(k, \omega) = T_N(k, \omega) + E_a(k, \omega).$$  \hspace{1cm} 4.38

This is a common form for the modelling error in distributed structural systems, which are often represented using finite element or other modal models that are truncated to some finite subset of the plant's modes. Now, if $(I + T)$ is singular, $(I + T_N + E_a)$ is singular as well. Thus, there exists a vector $x$, $\|x\|_2 > 0$, in the null space of $(I + T_N + E_a)$ (denoted by $x \in N$), such that

$$(I + T_N + E_a)x = 0,$$  \hspace{1cm} 4.39

thus,
\[(I + T_N)x = -E_a x.\]  \hspace{1cm} 4.40

Taking the Euclidean norm of each side,
\[
\| (I + T_N)x \|_2 = \| E_a x \|_2,
\]

hence
\[
\frac{\| (I + T_N)x \|_2}{\| x \|_2} = \frac{\| E_a x \|_2}{\| x \|_2}.
\]

\hspace{1cm} 4.42

Now, for some arbitrary \( z \), \( \| z \|_2 > 0 \),
\[
\min_{z \neq 0} \frac{\| (I + T_N)z \|_2}{\| z \|_2} \leq \min_{x \in N} \frac{\| (I + T_N)x \|_2}{\| x \|_2};
\]

that is, an \textit{unconstrained} minimization must be smaller than, or at most as large as, a \textit{constrained} minimization, since \( x \) is not arbitrary but rather restricted to lie in the null space of \((I + T_N)\). Thus, using the definition of the singular value,
\[
\sigma_{\min}(I + T_N) \leq \min_{x \in N} \frac{\| (I + T_N)x \|_2}{\| x \|_2}.
\]

\hspace{1cm} 4.44

Further,
\[
\min_{x \in N} \frac{\| E_a x \|_2}{\| x \|_2} \leq \max_{z \neq 0} \frac{\| E_a z \|_2}{\| z \|_2};
\]

that is, a constrained minimization is always bounded by an unconstrained maximization. Using the definition of the singular value,
\[
\min_{x \in \mathbb{N}} \frac{||E_a x||_2}{||x||_2} \leq \sigma_{\max}[E_a].
\]

Consequently, if \((I + T)\) is singular, then

\[
\sigma_{\min}[I + T_N(k, \omega)] \leq \sigma_{\max}[E_a(k, \omega)].
\]

Thus, if

\[
\sigma_{\max}[E_a(k, \omega)] < \sigma_{\min}[I + T_N(k, \omega)], \quad \text{all } (k, \omega),
\]

then \((I + T)\) is non-singular, and the input/output relation (4.36) is BIBO stable with respect to the additive modelling error matrix \(E_a\). Note that this is a sufficient rather than a necessary condition for the robustness of stability.

**Case 2: Multiplicative Error**

Let the difference between the nominal and "true" augmented plant models be represented by a multiplicative error matrix \(E_m\) at the plant output,

\[
T(k, \omega) = [I + E_m(k, \omega)]T_N(k, \omega)
\]

\[
= T_N(k, \omega) + E_m(k, \omega)T_N(k, \omega).
\]

Then, if \((I + T)\) is singular, \((I + T_N + T_N E_m)\) is singular as well. Thus, there exists a vector \(x, ||x||_2 > 0\), in the null space of \((I + T_N + T_N E_m)\), such that

\[
(I + T_N + E_m T_N)x = 0.
\]

This can be re-written as

\[
(I + T_N^{-1} + E_m)T_N x = 0.
\]
If $T_N$ is non-singular (e.g. the forward loop is open-loop stable), then

$$(I + T_N^{-1} + E_m)x = 0,$$  \hspace{1cm} 4.52

or

$$(I + T_N^{-1})x = -E_mx.$$  \hspace{1cm} 4.53

One can now proceed as before, and obtain the result that, if $(I + T)$ is singular, then

$$\sigma_{\text{min}}[I + T_N^{-1}(k, \omega)] \leq \sigma_{\text{max}}[E_m(k, \omega)].$$  \hspace{1cm} 4.54

So,

$$\sigma_{\text{max}}[E_m(k, \omega)] < \sigma_{\text{min}}[I + T_N^{-1}(k, \omega)], \text{ all } (k, \omega),$$  \hspace{1cm} 4.55

is a sufficient condition for $(I + T)$ to be non-singular, and hence ensure that the closed-loop system is robustly stable with respect to the multiplicative modelling error. Using various singular value identities, and the relationships between the various system matrices, equation (4.55) can be re-written in a more easily-applied and physically-meaningful form;

$$\sigma_{\text{max}}[C(k, \omega)] < \frac{1}{\sigma_{\text{max}}[E_m(k, \omega)]}, \text{ all } (k, \omega).$$  \hspace{1cm} 4.56

Equation (4.56) signifies that the closed-loop system must have small gain in the regions of $(k, \omega)$ where the modelling error is large. For example, if the modelling errors are primarily high-frequency (e.g. poorly-quantified high-frequency resonances) and high-wavenumber (e.g. small errors associated with the location of sensors and actuators), then (4.56) codifies the well-known result that one should not specify a closed-loop bandwidth greater than the model bandwidth.

Since the test (4.56) is often the most easy to apply, it may be convenient to redefine
the additive modeling error as an equivalent multiplicative error. One can readily show that

\[ E_m(k, \omega) = E_a(k, \omega)G^{-1}(k, \omega). \]

This transformation requires that the augmented plant be open-loop stable for \( G^{-1} \) to exist.

In the present development, stability-robustness has been based upon perturbations in the plant model that cause the determinant of the return difference to vanish. If \( T \) is wavenumber white (e.g. independent of \( k \)), then the condition \( \text{det}(I + T) = 0 \) corresponds directly to the lumped-parameter MIMO Nyquist contour touching the critical point (the point of neutral stability) [4.4]. Consider a distributed parameter system that can only be destabilized via temporal instabilities; for example, the mean-square amplitude of a distributed parameter control system might grow without bound over time, but not over space, as the system is of finite size. Then the \( k \)-dependent generalization \( \text{det}[I + T(k, \omega)] = 0 \) follows immediately – the spatial transform variable is a parameter, so that one can construct a Nyquist contour for each value of \( k \). For an integral spatial transform, this leads to a Nyquist surface, while for a discrete spatial transform it yields a series of Nyquist contours.

This notion of "parametric loci" has been utilized by Postlethwaite and MacFarlane [4.7] to study the role of plant parameter variations on system stability. If, however, one permits the distributed parameter control system to be destabilized spatially (as seen, for example, in the Euler buckling criteria in statics, where a change in the eigenvalues of the governing equation denotes instability) as well as temporally, then a multidimensional Nyquist test must be developed. Such tests have been developed for two-frequency control systems and multidimensional digital filters used in image processing [4.8]; these would provide a more general interpretation for the stability/robustness criteria (4.56).
IV.6 EXAMPLE PROBLEM: THE PINNED-PINNED BEAM

As a simple example problem, consider the vibration control of a pinned-pinned Bernoulli-Euler beam using a distributed piezoelectric actuator. This problem was investigated in depth in Chapters I and II. The goal here is to discern the merits of the Lyapunov-designed controller within the framework of the spatial/temporal performance measures. While the problem is single input, single output (SISO), it nonetheless contains all elements of the analysis. The analysis utilizes a discrete spatial transform as a prelude to the shape control analysis presented in the next chapter.

The flexural vibrations of a thin beam, including a structural damping model [4.10], are described by the nondimensional governing equation

$$\frac{\partial^4 y}{\partial x^4} + \lambda \frac{\partial^3 y}{\partial x^2 \partial t} + \frac{\partial^2 y}{\partial t^2} = u \quad ; \quad 0 < x < 1.$$  \hspace{1cm} 4.58

The input $u$ and output $y$ are both functions of the position $x$ and time $t$. The form of the input $u$ will later be specialized to model a piezo film actuator distribution. The boundary conditions are assumed to be homogeneous, and take the form

$$y(0,t) = y(1,t) = \frac{\partial^2 y(0,t)}{\partial x^2} = \frac{\partial^2 y(1,t)}{\partial x^2} = 0.$$  \hspace{1cm} 4.59

A nondimensional finite sine spatial transform/continuous time Fourier integral transform, defined by

$$\hat{f}(n, \omega) \equiv F_s[f(x,t)] = \int_0^1 f(x,t) \sin(k_n x) e^{i \omega t} dt \quad , \quad k_n = n\pi, \quad n = 1, 2, \ldots ,$$  \hspace{1cm} 4.60

with inverse
\[ f(x,t) \equiv F_s^{-1}\{\tilde{f}(k_n,\omega)\} = 2 \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_n,\omega) \sin(k_n x) e^{-i\omega \cdot d\omega} \]

is applied to the governing equation (4.58) and the boundary conditions. This yields the transformed space/time response function of the plant,

\[
\frac{\tilde{y}(k_n,\omega)}{\tilde{u}(k_n,\omega)} = h(k_n,\omega) = \frac{1}{k_n^4 + i\omega \lambda k_n^2 - \omega^2}.
\]

This function is plotted in Fig. 4.3 over a portion of its space and time bandwidths; note that

the spatial transform is discrete, and that the damping parameter \( \lambda = 0.002 \) in this and all succeeding plots. The peaks correspond to the resonances of the plant. The plant rolls off like \( \omega^2 \) in temporal frequency beyond resonances, and like \( k^4 \) in spatial frequency.

The introduction of the actuator, sensor, and temporal compensator to the distributed plant will be studied separately. First, for the film actuator, consider a spatially-uniform
control distribution over the length of the beam, such that

\[ u(x,t) = V_{\text{max}}[h(x) - h(x - 1)]u(t). \]  \hspace{1cm} (4.63)

\( V_{\text{max}} \) is the nondimensional maximum gain associated with the actuator, and \( u(t) \) is the temporal control modulation; the actuator is degenerate. This film distribution is sketched in Fig. 1.1a. The spatial/time transform of the actuator distribution is

\[ q(k_n, \omega) = V_{\text{max}}k_n[(-1)^n - 1]\tilde{u}(\omega), \]  \hspace{1cm} (4.64)

hence

\[ \tilde{y}(k_n, \omega) = \frac{V_{\text{max}}k_n[(-1)^n - 1]}{k_n^2 + i\omega\lambda k_n^2 - \omega^2} \tilde{u}(\omega). \]  \hspace{1cm} (4.65)

From equation (4.33), the system will not be controllable by this uniform actuator distribution for even values of \( n \). All odd-numbered sinusoidal modes of the beam, however, will be controllable. This is because the spatially-uniform distribution appears to the beam as two counter-opposed moments at the boundary. In fact, the uniform film actuator has the same spatial transform as a discrete moment actuator pair,

\[ F_{\delta}\{\delta'(x) - \delta'(x - 1)\} = k_n[(-1)^n - 1]. \]  \hspace{1cm} (4.66)

As determined in Chapter I, an alternative film distribution that will control both even- and odd-numbered modes is a "ramp" distribution,

\[ u(x,t) = V_{\text{max}}(1 - x)[h(x) - h(x - 1)]u(t). \]  \hspace{1cm} (4.67)

This distribution is sketched in Fig. 1.3a. Its space/time transform is

\[ q(k_n, \omega) = -V_{\text{max}}k_n \tilde{u}(\omega), \]  \hspace{1cm} (4.68)

hence
\[ \bar{y}(k_n, \omega) = \frac{-V_{\text{max}}k_n}{k_n^4 + i\omega\lambda k_n^2 - \omega^2} \bar{u}(\omega). \]  

From equation (4.33), the system will be controllable using this ramp distribution, for all values of \( n \). Thus, the system is controllable for all \((k_n, \omega)\). Interestingly, this actuator distribution has the same spatial transform as that of a single control moment at the end \( x = 0 \); a single actuator distributed over the entire spatial domain of the plant can have the same influence on the system as a single discrete actuator located at one point.

A band-limited portion of the space/time transform (4.69) is plotted in Fig. 4.4 for \( V_{\text{max}} = 1 \). This plot shows the "spatial compensation" provided by the "ramp" actuator distribution on the plant. The system now rolls off as \( k^{-3} \), while the temporal behavior, including the resonance characteristics, is unchanged.

![Image of frequency response](image)

**Fig. 4.4:** Actuator-augmented plant response function.

A stabilizing compensator for this actuator-augmented system takes the form
\[ u(t) = -\alpha \frac{\partial^2 y(0,t)}{\partial x \partial t} \]  

4.70

e.g. angular velocity feedback with nondimensional positive-definite gain \( \alpha \). The spatial filter function associated with this discrete angular velocity sensor (assumed to have infinite temporal bandwidth) is

\[ p(k_n) = \alpha k_n, \]  

4.71

while the temporal derivative compensator function is

\[ K(\omega) = -i \omega. \]  

4.72

The distributed space/time forward loop response function of the compensated plant is then

\[ T(k_n, \omega) = \frac{i\alpha V_{max} \omega k_n^2}{k_n^4 + i\omega k_n^2 - \omega^2}. \]  

4.73

The magnitude of this function (since it is scalar, it has but one singular value, equal to its magnitude) is plotted in Fig. 4.5 for \( \alpha = V_{max} = 1 \). The angular velocity sensor has added further spatial compensation, as the augmented plant now rolls off as \( k^-2 \) at large \( k_n \). The temporal compensation provided by the angular velocity feedback control law now provides more forward loop gain at higher \( \omega \), while the gain parameters \( \alpha \) and \( V_{max} \) determine how large the forward loop gain is for disturbance rejection (e.g. vibration damping) performance. This is emphasized by the plots of the sensitivity transfer function \( S(k_n, \omega) \) in Fig.s 4.6 and 4.7 for the product \( \alpha V_{max} = 1 \) and 100, respectively. In Fig. 4.6, the smaller loop gain means the controller rejects vibrations best only at the resonances, which leads to poor tracking performance; this is corrected in Fig. 4.7 with the larger gain parameters. Plots of the closed-loop transfer function magnitude, Fig.s 4.8 and 4.9 (for the product \( \alpha V_{max} = 1 \) and 100, respectively), reinforce these conclusions. Nonetheless, the control was formulated to damp vibrations, rather than to provide tracking, which would require
integral control. The system does have an "infinite" spatial bandwidth, as the control affects the system response at all $k_r$, hence the actuator distribution may be an ideal candidate for shape control. However, since the closed-loop response function does not roll off quickly at high $(k_r, \omega)$, the control design will be sensitive to high temporal and spatial frequency modelling errors, as contended in the stability/robustness development. This has been observed experimentally as high-frequency spillover owing to the finite bandwidth of the control electronics in providing derivative control.

Fig. 4.5: Forward loop response function ($\alpha = V_{max} = 1$).
Fig. 4.6: Sensitivity response function ($\alpha = V_{max} = 1$).

Fig. 4.7: Sensitivity response function ($\alpha = V_{max} = 10$).
Fig. 4.8: Closed-loop response function ($\alpha = V_{\text{max}} = 1$).

Fig. 4.9: Closed-loop response function ($\alpha = V_{\text{max}} = 10$).
IV.7 SUMMARY AND OBSERVATIONS: A PRELUDE TO SHAPE CONTROL

A new method of assessing the performance of linear distributed parameter control systems has been presented in this Chapter. The method is based upon the system's multiplicative input/output representation in a transformed wavenumber/frequency space, including spatially-transformed representations of sensor and actuator distributions. The analysis has presented performance measures for distributed parameter control, in both a temporal and spatial sense, for command following, disturbance rejection, noise immunity, and stability-robustness in the presence of modelling errors. Additionally, observability and controllability norms were derived. The formulation readily lends itself to the quantification of spatial performance, such as achievable spatial bandwidth.

The performance measures provide a means for evaluating distributed parameter control designs. The distributed parameter control problem was shown to reduce to designing an appropriate space/time compensator. If the compensation is degenerate, then spatial compensation is provided solely by the sensor and actuator distributions. An example problem demonstrated the application of the transform formalism, and reinforced conclusions about a system designed and analyzed in Chapter I by Lyapunov methods.

This analysis technique must now be extended into a distributed control design methodology. The example problem was seen to have "infinite" spatial bandwidth using the ramp control distribution. In the next Chapter discrete spatial transforms will be studied in the context of shape control. The aforementioned example problem will be shown to be inappropriate for shape control applications. Additional performance measures specific to the shape control problem will be introduced. These additional conditions will show that the discrete spatial transform is more appropriate than the continuous (integral) spatial transform for shape control analysis. Further, discrete spatial transforms can be folded into extant computer-aided design methods directly.
V. SHAPE CONTROL: DISCRETE TRANSFORMS, MODELLING, AND PERFORMANCE MEASURES

V.1 INTRODUCTION

SHAPE CONTROL IS THE MANIPULATION of a distributed plant's output to achieve a distributed, non-zero set-point. One can think of it as a distributed servo problem. In this sense shape control is distinguished from damping the resonant response of a plant, e.g. state regulation.

The open literature holds few examples of shape control analysis, design, or experiment. Hardy [5.1] presented a summary of adaptive optical systems through 1978, including deformable mirrors. At best, these mirrors are operated in the open loop, with most tests being dynamic (temporal) performance evaluations [5.2,5.3]. Weeks [5.4] presented an accessible analytical study of structural shape control, using an integral equation formulation. The analysis is restricted to the static, open-loop problem, using discrete point displacement sensors and force actuators. Spatial performance is posed in terms of determining actuator sites to best achieve (in a mean square sense) a single shape. Schaechter [5.5,5.6,5.7] presented a series of papers describing an experimental application of Week's analysis. The experiment consisted of driving a flexible beam, suspended vertically in a gravity field, to two shapes. The experiment was static, and quasi open-loop. Others have presented similar analyses of segmented mirrors for astronomical applications [5.8]. Much of the balance of literature on "shape control" is actually concerned with figure control in the presence of resonant response, i.e. vibration damping, or shape maintenance.

The spatial transform paradigm developed in the two previous Chapters, with its notion of "spatial bandwidth", will be applied to the shape control problem. It is best to first understand how one may define a "shape", and then pose a meaningful performance specification for shape control. Consider the shape profile depicted in Fig. 5.1. It is tempting
Fig. 5.1: Shape profile.

to quantify this shape in terms of its spatial Fourier integral transform: this spatial transform is presented in Fig. 5.2. The profile is seen to have a spatial passband in wavenumber $k$,

Fig. 5.2: Wavenumber transform of shape profile.
with a structure suggestive of spatial resonances, or preferred spatial wavelengths. (Note that the structure of the transform differs somewhat from the usual time series analysis form because of the bounded spatial domain of the shape.) It is reasonable to require, then, that a shape control system possess a specified spatial bandwidth. This extends the usual temporal performance specification in an intuitive way. More importantly, it folds directly into the analysis of the two previous Chapters.

The Fourier integral transform of the shape depicted in Fig. 5.1 is merely a decomposition of the profile on a basis of complex exponentials with a continuous spectrum. Interestingly, the shape shown in Fig. 5.1 can also be represented in a decomposition on a basis of functions with a discrete spectrum, e.g. a Fourier series. In fact, this shape can be exactly decomposed as

\[ y(x) = 0.1 \sin(\pi x) + 0.5 \sin(2\pi x) + 0.3 \sin(3\pi x) + 0.1 \sin(4\pi x). \]

The discrete transform spectrum, equation (5.1), is plotted in Fig. 5.3. The shape has a finite discrete spatial bandwidth.

![Fig. 5.3: Discrete spatial transform of shape profile.](image)
Although this example may seem a bit contrived, it demonstrates two important concepts. First, the shape can be described by either a continuous, integral Fourier transform, or by a discrete spatial transform (Fourier series) in a set of orthogonal functions. The shape itself can be reconstructed by a corresponding inverse transform. The reader may even find it more intuitive to think of a shape profile as a superposition of distinct functions than in terms of an integral transform. Second, the discrete transform offers the possibility of posing the shape control performance requirement not only in terms of spatial bandwidth, but in achieving specific, distinct shapes.

To illustrate this important distinction, we shall re-examine the model problem presented at the end of Chapter IV. Consider a simply-supported Bernoulli-Euler beam with a linearly-shaped piezoelectric film actuator bonded to one face. The distributed space/time response function of the augmented plant took the form

\[
h(k_n, \omega) = \frac{-\alpha V_{\text{max}} k_n^2}{k_n^4 + i\omega \lambda k_n^2 - \omega^2}. \tag{5.2}
\]

This response function was derived using a discrete sine spatial transform with discrete wavenumber spectrum \( k_n = n\pi \), and a continuous Fourier time transform. Since the response function is non-zero for all \((k_n, \omega)\), it appears that this system might be amenable to shape control: it has infinite discrete spatial bandwidth.

However, equation (5.2) is misleading. Recall that the beam has a single actuator. Since the actuator was appropriately chosen, the system has infinite discrete spatial bandwidth. Nonetheless, the lone actuator can only drive the output profile of the beam to a single shape, with various amplitudes. The output shape will always have a fixed ratio among its spatial Fourier components, i.e. one shape; this is the true implication of equation (5.2). To achieve more than one independent shape requires more than one actuator.

The shape control performance specification must then impose not only a spatial bandwidth, but a set of independent shapes to be achieved. Orthogonal functions are particularly appropriate for the latter specification, as they possess the property of spatial independence due to their orthogonality. We would then require that the shape controller be
capable of attaining each of these functions independently, and thence any superposition. Any desired shape would then be represented by

\[ y(x) = c_1 \psi_1(x) + c_2 \psi_2(x) + \ldots, \quad 5.3 \]

where \( \{ \psi_i(x) \} \) are the orthogonal shape functions that represent the requisite component profiles, and hence the shape control task.

The wavefront control task can easily be posed in terms of controlling a set of orthogonal shape functions. Zernike polynomials [5.9], a complete set of orthogonal functions, are often used to describe wavefront aberrations [5.10]. Interestingly, Creedon and Lindgren posed the shape control task for a model deformable rectangular mirror in terms of plate eigenfunctions [5.11]. These also form a complete set of orthogonal functions.

The rest of this Chapter is devoted to an extension of the spatial transform representation and analysis presented in the previous two Chapters to discrete spatial transforms. In the next Section, conditions under which a shape has a discrete expansion are elucidated. An expression for the associated expansion coefficients (i.e. the spatial transform) is derived. The discrete transform is used to derive an input/output representation of the actuator-augmented distributed plant for two important and general forms of the system's Green's function. Performance measures for shape control are derived. An example problem is considered. In order to extend the range of application to truncated plant representations obtained from finite element or modal analyses, a quasi-static correction to the system's input/output relation is derived. Finally, the shape estimation problem for the discrete shape expansion is studied, with associated performance measures, including an example.
V.2 DISCRETE SPATIAL TRANSFORM: ORTHONORMAL EXPANSIONS

Consider a shape \( y(x, \omega) \) defined on \( x \in D \). The task of representing the shape as an expansion in orthogonal functions, as per equation (5.3), depends upon the meaning of "represented". One could require

\[
\lim_{n \to \infty} \left[ y(x, \omega) - \sum_{i=1}^{n} c_i(\omega) \psi_i(x) \right] = 0, \tag{5.4}
\]

e.g. that the expansion precisely equals the shape at every point in \( D \). Such a point-by-point convergence is usually difficult to achieve in practice. However, a slightly weaker requirement that can be satisfied under rather general conditions is

\[
\lim_{n \to \infty} \int_{D} \left[ y(x, \omega) - \sum_{i=1}^{n} c_i(\omega) \psi_i(x) \right]^2 dx = 0. \tag{5.5}
\]

Equation (5.5) requires that the integrated mean square error in representing the shape be minimized over \( D \); this is often referred to as a limit in the mean [5.4,5.5,5.6]. The representation problem reduces to finding the coefficients \( c_i \) that minimize this error.

For the analysis, we assume that the shape functions \( \psi_i(x) \) are orthonormal. That is, they are orthogonal on \( D \) and of unit 2-norm, satisfying

\[
\int_{D} \psi_i(x) \psi_j(x) dx = \delta_{ij}, \tag{5.6}
\]

where \( \delta_{ij} \) is the Kronecker delta function; \( \delta_{ij} = 1 \) for \( i = j \), \( \delta_{ij} = 0 \) otherwise. If the system \( \{ \psi_i(x) \} \) is orthogonal, it can be normalized using suitable normalizing coefficients [5.12]. Finally, we assume that the shape is square integrable over \( D \),

\[ \int_{b} y^2(x, \omega) \, dx < \infty. \quad 5.7 \]

The band-limited mean square error \( E_n \) between the shape and its truncated expansion is defined as

\[
E_n(\omega) = \left[ \int_{b} \left[ y(x, \omega) - \sum_{i=1}^{n} c_i(\omega) \psi_i(x) \right]^2 \, dx \right. \\
= \left[ \int_{b} y(x, \omega)^2 \, dx - 2 \sum_{i=1}^{n} c_i(\omega) \int_{b} y(x, \omega) \psi_i(x) \, dx + \sum_{i=1}^{n} c_i^2(\omega) \right]. \quad 5.8
\]

Define \( a_i \) as the Fourier coefficient of \( y(x, \omega) \) with respect to \( \psi_i(x) \),

\[
a_i(\omega) \equiv \int_{b} y(x, \omega) \psi_i(x) \, dx. \quad 5.9
\]

Then,

\[
E_n(\omega) = \int_{b} y^2(x, \omega) \, dx - \sum_{i=1}^{n} a_i^2(\omega) + \sum_{i=1}^{n} [a_i(\omega) - c_i(\omega)]^2. \quad 5.10
\]

Note that the first two terms defining the error \( E_n \) in equation (5.10) are independent of the undetermined coefficients \( c_i \). Since the sum

\[
\sum_{i=1}^{n} [a_i(\omega) - c_i(\omega)]^2 \quad 5.11
\]

is never negative, for the set of functions \( \{ \psi_i(x) \} \) and shape \( y(x, \omega) \), the mean square error \( E_n \) is minimized if and only if the coefficients \( c_i(\omega) \) are equal to the Fourier coefficients...
$a_i(\omega)$:

\[ c(\omega) = a(\omega). \quad 5.12 \]

Define this minimum error as $E_n^*$:

\[ E_n^*(\omega) \equiv \int_\Omega y^2(x, \omega) \, dx - \sum_{i=1}^n c_i^2(\omega) \geq 0. \quad 5.13 \]

So, for any square-integrable shape $y(x, \omega)$ and any $n$, Bessel's Inequality

\[ \sum_{i=1}^n c_i^2(\omega) \leq \int_\Omega y^2(x, \omega) \, dx \quad 5.14 \]

holds. Since the integral of $y^2$ over $\Omega$ is bounded, the series is always convergent, for any $n$, subject to the assumptions of the analysis.

Equation (5.13) can be re-written as

\[ E_n^*(\omega) = \left[ \int_\Omega y^2(x, \omega) \, dx - \sum_{i=1}^\infty c_i^2(\omega) \right] + \sum_{i=n+1}^\infty c_i^2(\omega). \quad 5.15 \]

As $n \to \infty$, the second summation in (5.15) will vanish. (It can also be proved that $c_i(\omega) \to 0$ as $i \to \infty$ [5.12].) Thus, a necessary and sufficient condition for the shape $y(x, \omega)$ to be approximated in the mean by an expansion in the orthonormal shape functions $\psi_i(x)$ is that Parseval's Equation,

\[ \sum_{i=1}^\infty c_i^2(\omega) = \int_\Omega y^2(x, \omega) \, dx \quad 5.16 \]
be satisfied, where \( c_\omega(\omega) \) are the Fourier coefficients defined by equations (5.12) and (5.9).

An orthonormal system of functions \( \{ \psi_i(x) \} \) is called \textit{complete} with respect to the shape \( y(x,\omega) \) if the corresponding Fourier coefficients satisfy (5.16). \textit{However, even if the system is not complete, choosing the expansion coefficients} \( c_\omega(\omega) \) \textit{to equal the Fourier coefficients of} \( y(x,\omega) \) \textit{with respect to} \( \{ \psi_i(x) \} \) \textit{leads to a representation minimizing the mean square shape error over} \( D \).

One can represent the required shape in terms of orthonormal component functions using a discrete spatial Fourier transform, where the expansion coefficients are given by equations (5.12) and (5.9). The shape control problem, then, evolves into deriving an input/output representation of the distributed plant in terms of this expansion, which is undertaken in the next Section.
V.3 AUGMENTED PLANT REPRESENTATION USING DISCRETE SPATIAL TRANSFORMS

Representing a distributed plant in terms of shape functions consists of equating the system's output with the desired shape expansion, as per equation (5.3), and then expressing the expansion coefficients \( c_i(\omega) \) in terms of the plant's Green's function. Two general forms of the Green's function will be considered.

Case 1: "Generic" Green's Function

In this case the only assumptions are that the plant is linear and time invariant. The Green's function is given by

\[
h(x,\xi,t,\tau) = h(x,\xi,t - \tau). \tag{5.17}
\]

The associated composition integral can be Fourier transformed in time directly, taking the form

\[
y(x,\omega) = \int_D h(x,\xi,\omega) u(\xi,\omega) d\xi. \tag{5.18}
\]

The output \( y(x,\omega) \), assumed to be square integrable over \( D \), admits a discrete expansion in the orthonormal shape functions \( \{\psi_i(x)\} \):

\[
y(x,\omega) = c_1(\omega) \psi_1(x) + c_2(\omega) \psi_2(x) + ... \tag{5.19}
\]

The expansion coefficients are found using equations (5.12), (5.9), and (5.18):
\[ c_i(\omega) = \int_D \psi_i(x) \left[ \int_D h(x, \xi, \omega) u(\xi, \omega) \, d\xi \right] \, dx \]

\[ = \int_D \int_D \psi_i(x) h(x, \xi, \omega) u(\xi, \omega) \, d\xi \, dx. \]

The entire spectrum of Fourier coefficients \( c_i \), equation (5.20), is the discrete shape/integral time transform of the forced system response.

The input \( u(x, \omega) \) naturally can be expressed as a superposition of inputs from \( N \) actuators, with associated spatial and temporal dynamics. The exogenous command signals are assumed to be degenerate. Thus,

\[ u(x, \omega) = \sum_{j=1}^{N} q_j(x, \omega) u_j(\omega). \]

Consequently,

\[ c_i(\omega) = \sum_{j=1}^{N} \left[ \int_D \int_D \psi_i(x) h(x, \xi, \omega) q_j(\xi, \omega) \, d\xi \, dx \right] u_j(\omega). \]

This provides a matrix input/output relation between a vector of inputs and the transformed response. Note that the actuator spatial dynamics are absorbed into the integral. The command signals \( u_j \) can have no shape function transform dependence; the form of the kernel (5.17) does not provide a "discrete spatial convolution form" analogous to certain plants considered in Chapter 3.
Case 2: Symmetric Green's Function

The generic Green's function form (5.17), while very general, is nonetheless quite abstract. A more practical and physically meaningful form describes a stationary plant with a spatially-symmetric kernel,

\[ h(x, \xi, t - \tau) = h(\xi, x, t - \tau). \quad \text{(5.23)} \]

As discussed in Chapter III, a symmetric Green's function can always be expressed in a bilinear expansion,

\[ h(x, \xi, \omega) = \sum_{k=1}^{\infty} \frac{\varphi_k(x) \varphi_k(\xi)}{\lambda_k(\omega)}, \quad \text{(5.24)} \]

where, for simplicity, the temporal transform has already been applied to \( h \). The functions \( \varphi_k(x) \) are the system eigenfunctions, with \( \lambda_k(\omega) \) the corresponding eigenvalue relation. The eigenfunctions are orthogonal in \( D \). We further assume that the distributed plant is self-adjoint, hence the eigenfunctions are complete as well. The bilinear expansion accommodates the use of finite element or experimental modal analyses to provide the requisite eigenfunctions and eigenvalues. Thus, the form (5.24) is applicable to a wide range of engineering systems, including those without explicit analytical representations. The application of (5.24) will be studied in detail.

Substituting the bilinear expansion (5.24) into equation (5.20) gives an expression for the discrete output spectrum in the shape function basis:

\[ c_i(\omega) = \sum_{k=1}^{\infty} \left[ \int_D \psi_i(x) \varphi_k(x) \, dx \right] \int_D \frac{\varphi_k(\xi)}{\lambda_k(\omega)} u(\xi, \omega) \, d\xi. \quad \text{(5.25)} \]

The first integral in equation (5.25) is a decomposition of the shape functions in the eigenfunctions — a transformation from "shape" space to "modal" space. The second integral
shows how the forced response to the distributed input is weighted by the characteristic spatial scales of the plant (the mode shapes $\varphi_k$) and its temporal dynamics ($\lambda_k$). Since this integral is independent of $x$, it can be symbolized as a frequency-dependent constant:

$$f_k(\omega) = \int_B \frac{\varphi_k(\xi)}{\lambda_k(\omega)} u(\xi, \omega) d\xi.$$  \hspace{1cm} 5.26

Equation (5.25) can then be re-written as

$$c_i(\omega) = \sum_{k=1}^{\infty} f_k(\omega) \int_B \psi_i(x) \varphi_k(x) dx.$$ \hspace{1cm} 5.27

The implications of truncating this plant representation will be studied in Section V.5.

It can readily be shown that (5.27) is convergent under the assumptions of the analysis. To prove this, use the "second form" of Parseval's Theorem [5.13],

$$\int_B f(x, \omega) g(x, \omega) dx = \sum_{k=1}^{\infty} a_k(\omega) b_k(\omega).$$ \hspace{1cm} 5.28

The expansion (5.28) is convergent, and can be interpreted as a discrete spatial transform cross-spectrum. Equation (5.28) is valid for any two square integrable functions $f$ and $g$, each of which has an expansion in a set of complete, orthogonal functions $\{\zeta_k(x)\}$, vis

$$a_k(\omega) = \int_B f(x, \omega) \zeta_k(x) dx,$$ \hspace{1cm} 5.29

and

$$b_k(\omega) = \int_B g(x, \omega) \zeta_k(x) dx.$$ \hspace{1cm} 5.30
Then, Parseval's Theorem (5.28) can then be re-written, using (5.30), in the convenient form

$$\int_{\mathcal{D}} f(x, \omega) g(x, \omega) dx = \sum_{k=1}^{\infty} a_k(\omega) \int_{\mathcal{D}} g(x, \omega) \zeta_k(x) dx . \quad 5.31$$

Each of the shape functions $\psi_i$ is orthonormal, and hence square integrable. The distributed plant eigenfunctions $\varphi_k$ are both orthogonal and complete [5.13,5.14]. Further, by definition,

$$c_i(\omega) = \int_{\mathcal{D}} y(x, \omega) \psi_i(x) dx , \quad 5.32$$

hence equation (5.27) can be re-written as

$$\int_{\mathcal{D}} y(x, \omega) \psi_i(x) dx = \sum_{k=1}^{\infty} f_k(\omega) \int_{\mathcal{D}} \psi_i(x) \varphi_k(x) dx . \quad 5.33$$

From the composition integral associated with the bilinear expansion, plus equation (5.26),

$$y(x, \omega) = \sum_{k=1}^{\infty} \varphi_k(x) \int_{\mathcal{D}} \varphi_k(\xi) \frac{u(\xi, \omega)}{\lambda_k(\omega)} d\xi$$

$$= \sum_{k=1}^{\infty} f_k(\omega) \varphi_k(x) , \quad 5.34$$

i.e. $f_k$ are the expansion coefficients of the forced response with respect to the eigenfunctions. The output $y(x, \omega)$ was assumed to be square integrable. Thus, one sees the one-to-one correspondence between equation (5.33) and the second form of Parseval's
Theorem, equation (5.31). \( y(x, \omega) \) can be associated with \( f(x, \omega) \), \( \psi_j(x) \) with \( g(x, \omega) \) (albeit frequency independent), \( f_k(\omega) \) with \( a_k(\omega) \), and \( \phi_k(x) \) with \( \zeta_k(x) \); each of the functions in (5.33) satisfies the restrictions of its corresponding function in (5.31). Consequently, the discrete transform expansion is convergent.

The input \( u(x, \omega) \) can be expressed as a superposition of inputs, as per equation (5.21). Thus,

\[
c_i(\omega) = \sum_{j=1}^{N} \left[ \sum_{k=1}^{\infty} f_{jk}(\omega) h_{ik}(\omega) \right] u_j(\omega), \tag{5.35}
\]

where

\[
f_{jk}(\omega) = \int_{\Omega} \frac{\phi_k(\xi)}{\lambda_k(\omega)} \psi_j(\xi, \omega) d\xi, \tag{5.36}
\]

and

\[
h_{ik}(\omega) = \int_{\Omega} \psi_i(x) \phi_k(x) dx. \tag{5.37}
\]

Equation (5.35) provides the requisite input/output relation between a vector of command inputs and the transformed response.

As before, the actuator spatial and temporal dynamics are absorbed into the integral (5.36). However, each actuator's space/time filter function \( q_j \) can have an expansion in the system eigenfunctions (assuming the eigenfunctions are complete with respect to the distribution),

\[
q_j(x, \omega) = \sum_{l=1}^{\infty} a_{jl}(\omega) \phi_l(x), \tag{5.38}
\]
where

\[ a_{ij}(\omega) = \int_D q_j(x, \omega) \varphi_I(x) \, dx. \]  \hspace{1cm} 5.39

Then, using the orthogonality property of the eigenfunctions,

\[ f_{jk}(\omega) = \frac{a_{jk}(\omega)}{\lambda_k(\omega)} \int_D \varphi_k^2(\xi) \, d\xi. \]  \hspace{1cm} 5.40

This separates the computation of the actuator "transform" from the other components of the discrete spectrum input/output relation (5.35), which will be of use when screening various actuator distributions.

A further simplification arises if the shape functions \( \psi_i \) and the eigenfunctions \( \varphi_k \) are equal. \( h_{ik} \) (equation (5.37)) then reduces to unity, and the input/output relation (5.35) decouples, taking the simple product form

\[ c_i(\omega) = \sum_{j=1}^{N} f_{ji}(\omega) u_j(\omega). \]  \hspace{1cm} 5.41
V.4 PERFORMANCE MEASURES FOR SHAPE CONTROL

The transformed actuator-augmented input/output relations (5.22) and (5.35) can be expressed generically in the form

$$c_i(\omega) = \sum_{j=1}^{N} \alpha_{ij}(\omega) u_j(\omega),$$

where

$$\alpha_{ij}(\omega) = \int_{D} \int_{D} \psi_i(x) h(x, \xi, \omega) q_j(\xi, \omega) d\xi \, dx$$

for the "generic" Green's function form, and

$$\alpha_{ij}(\omega) = \sum_{k=1}^{\infty} f_{jk}(\omega) h_{ik}(\omega)$$

for the bilinear expansion form (see equations (5.35)-(5.37)).

In the shape control problem, one wishes to drive the distributed system output to each of the orthonormal shape functions $\psi_i(x)$ up to a discrete band limit $i = L$, both independently and in combination. That is,

$$y_{desired}(x, \omega) = \sum_{i=1}^{L} \beta_i(\omega) \psi_i(x).$$

This requirement can be compactly stated as a matrix relationship in terms of the plant and actuator characteristics by performing a "harmonic balance" in these expansion functions:
The matrix of coefficients $\alpha_{ij}$ is fixed by the actuator distributions on the distributed plant, as well as the spatial and temporal response characteristics of the plant itself. The partitioned submatrix in (5.46) must be invertible – it must have full rank – in order for a given actuator distribution to achieve the requisite output. Thus, the number of actuators $N$ must equal or exceed the number of "degrees of freedom" $L$ of the commanded output as decomposed in the shape function $\psi_i$, $i = 1, \ldots, L$. This is the first new performance requirement for shape control. Since the control vector $u = [u_1 \ldots u_N]^T$ is degenerate and spatially-independent, all spatial compensation must be provided by the actuator distribution.

The left-hand side of equation (5.46) – the coefficients of the desired output shape expansion – is spanned by a set of orthonormal vectors (the unit vectors $e_i$). Consequently, the output space of the partitioned plant/actuator submatrix must span this $L$-dimensional vector space as well to independently drive the system output to an arbitrary combination of the shape functions. Also, the residual components of the output at higher spatial frequencies ($i > L$) must be minimized. These are the remaining performance requirements for shape control.

The singular value decomposition provides the means of quantifying these performance requirements. Denote the partitioned submatrix in (5.46) as $M(\alpha)$. This matrix has the singular value decomposition

$$M(\alpha) = U \Sigma V^H.$$  \hspace{1cm} (5.47)

$U$ is the matrix of left singular vectors, which span the output (shape) space of $M$. $\Sigma$ is the matrix whose main diagonal consists of the singular values of $M$, and all other entries set to zero. $V$ is the matrix of right singular values, which span the input (control signal) space of
M. The distributed system will precisely achieve the required spatial bandwidth of shapes independently and in combination if the left singular vectors of M are equal to the unit vectors $e_i$, e.g. if $U$ equal the identity matrix $I$. This will serve to decouple the output space, and ease the control task. Also, each of the "directions" (e.g. shapes) in $\psi_i$ space will be equally attainable if the ratio of the maximum and minimum singular values of $M$ is as near to 1 as possible, since these singular values denote gains in the various output directions. The ratio of a matrix's maximum to minimum singular values is called its condition number.

Naturally, the system will be "well" controlled with respect to $\{\psi_i\}$ if $\sigma_{\text{min}}(M)$ is large. Since $M(\alpha)$ is fixed by the actuator types and locations, these requirements become criteria for screening candidate actuator distributions for shape control.

The computation of $M(\alpha)$'s rank and condition number can be undertaken directly with extant numerical analysis packages. The associated singular vector matrices can also be calculated readily. However, screening the structure of these matrices for systems with even a modest number of inputs and outputs quickly becomes tedious. A particularly appropriate measure of actuator participation and output decoupling can be realized by adapting Minto and Knack's notion of "coupling operators" [5.15]. These operators quantify norms associated with the singular value structure, including the roles of the singular matrices.

The method involves expressing the singular value decomposition of the matrix input/output operator in terms of the factorization

$$M(\alpha) = EF.$$ \hspace{1cm} 5.48

The corresponding spatially band-limited form of the input/output relation (5.42) then becomes

$$y = EFu,$$ \hspace{1cm} 5.49

for
\[ y \equiv [c_1(\omega) \ldots c_N(\omega)]^T, \quad 5.50 \]
\[ u \equiv [u_1(\omega) \ldots u_N(\omega)]^T, \quad 5.51 \]

where the requirement \( L = N \) is assumed met.

In terms of an intermediate variable \( z \),

\[ z = F u, \quad 5.52 \]

and

\[ y = E z. \quad 5.53 \]

One obvious choice for \( E \) and \( F \) that satisfies (5.48), using the singular value decomposition (5.47), is

\[ F = \Sigma^{\frac{1}{2}} \Psi^H, \quad 5.54 \]
\[ E = U \Sigma^{\frac{1}{2}}. \quad 5.55 \]

In the present application \( F \) can be interpreted as the matrix transforming the input \( u \) to the "gain space" \( z \) of the singular values. \( E \) applies the transformation from the "gain space" to the output shape space of \( y \).

Consider the impact of each actuator on driving the system to any shape (or, to all the desired shapes) by setting

\[ u = \sum_{i=1}^{N} \eta_i e_i, \quad 5.56 \]

where \( \{e_i\} \) are the standard unit vectors. Then,
\[ z = \sum_{i=1}^{N} \eta_{i} \mathbf{F}_i \mathbf{e}_i = \sum_{i=1}^{N} \eta_{i} \mathbf{F}_i. \]

Without loss of generality, set \( \eta_i = 1 \). Thus, each \( i \)-th column of \( \mathbf{F} \) is the contribution of the \( i \)-th actuator, from the input (control) space to the gain space. Further,

\[ \| z_i \|_2 = \| \mathbf{F}_i \|_2 \]

is a measure of the \( i \)-th actuator's contribution to controlling all the \( N \) shapes.

The application of the aforementioned shape control performance measures will be demonstrated by an example of "design by analysis". We consider the shape control of a nondimensional pinned-pinned beam. The desired shape functions will be the first four sinusoids:

\[ \psi_i(x) = \sin(i\pi x) ; \quad i = 1 \ldots 4, \quad 0 < x < 1. \]

The beam will be driven by abutting distributed piezoelectric film actuators. Four actuators are required to meet the bandwidth specification. Each actuator has been arbitrarily chosen to have a rectangular aperture, and is assumed for simplicity to have infinite temporal bandwidth, hence

\[ q_j(x,x) = \delta'(x - c_j) - \delta'(x - c_j - \Delta_j), \]

where \( 0 < c_j < 1 \), and \( \Delta_j \) is the aperture width. The elements of the plant/actuator matrix \( \mathbf{M}(\alpha) \) are then

\[ \alpha_{ij}(\omega) = \frac{2 \sin \left( \frac{i\pi}{2} (2c_j + \Delta_j) \right) \sin \left( \frac{i\pi}{2} \Delta_j \right)}{\omega^2 + (i\pi)^2}. \]
The matrix $M(\alpha)$ will be studied at zero frequency to find actuator distributions of the form (5.60) that minimize its condition number, display balanced participation among the actuators via the input coupling operators, and decouples the output shape space.

Four candidate film actuator distributions are sketched in Fig.s 5.4a - 5.7a, and shall hereafter be referred to as distributions I through IV. The corresponding input coupling operators appear in Fig.s 5.4b - 5.7b; these plots are annotated with the associated condition number and minimum singular value of $M(\alpha)$.

![Diagram](image)

Fig. 5.4a: Film actuator distribution I.

![Diagram](image)

Fig. 5.4b: Input coupling operators for actuator distribution I.
Distribution I has a large condition number (~80), and the corresponding input coupling operator plot shows that actuators 2 and 3 are doing most of the work. Conversely, this implies that the control voltages sent to actuators 1 and 4 will be larger than those to actuators 2 and 3 for a given output. Distribution II is asymmetric about mid-span, and yields an even larger condition number (~95). Each actuator contributes to the shape control task in proportion to its size, as shown in Fig. 5.5b. This suggests that a symmetric distribution (note that the sinusoids possess symmetry about mid-span) having actuators with equal-length apertures may be best.

![Fig. 5.5a: Film actuator distribution II.](image)

![Fig. 5.5b: Input coupling operators for actuator distribution II.](image)
The symmetric distribution III, shown in Fig. 5.6a, minimizes the condition number of $M(\alpha)$ (~17; subject to the constraint that the actuators have abutting rectangular apertures). Its input coupling operators, Fig. 5.6b, show a more balanced participation of the actuators over the entire bandwidth of the shape control task. In addition, it was found that the matrix of left singular vectors $U$ associated with distribution III is equal to the identity matrix; the output is *decoupled* by this "spatial compensation".

\[
\begin{align*}
\sigma_{\text{min}} &= .002 & \text{cond} &= 17.3183 \\
\Lambda(x) &= [.25 \quad .25 \quad .25 \quad .25]
\end{align*}
\]

![Diagram of distribution III](image)

**Fig. 5.6a**: Film actuator distribution III.

![Diagram of coupling operators](image)

**Fig. 5.6b**: Input coupling operators for actuator distribution III.

Since distribution III did not have equi-valued input coupling operators, distribution IV
was synthesized. The input coupling operators associated with distribution IV are all equal, hence all actuators participate equally over the entire bandwidth of the shape control task.

However, it has a larger condition number for $M(\alpha)$. Further, its left singular vector matrix is *not* decoupled, as for distribution III. Thus, input coupling operators cannot be used alone in screening actuator distributions for shape control. Rather, they are used as figures of merit in the analysis, with minimum condition number and a decoupled output space having priority in the final design selection. In the present system they reflect the trade-offs

\[
\begin{align*}
\sigma_{\text{min}} &= .0016 \\
\text{cond} &= 20.7449
\end{align*}
\]

![Graph showing the distribution of actuator forces with values .316, .184, .184, .316](image)

Fig. 5.7a: Film actuator distribution IV.

![Graph showing the magnitude of input coupling operators](image)

Fig. 5.7b: Input coupling operators for actuator distribution IV.
in actuator participation over the entire spatial bandwidth due to the beam's strain/curvature relation and boundary conditions.

The shape control task could have been posed at any frequency \( \omega > 0 \) equally as well. However, in the present formulation the spatial compensation (e.g. actuator distribution) is designed \textit{first}. After the actuator distribution is set, suitable MIMO control methods can be employed to develop a \textit{temporal} compensator to achieve dynamic performance.
V.5 SHAPE CONTROL AND MODAL REPRESENTATIONS: MODELLING CORRECTION FOR TRUNCATED PLANTS

Many engineering systems of practical interest will be modelled using finite element or modal analyses. Of necessity, these representations are often truncations of the true plant's characteristics; one cannot practically compute the infinity of modes. Some have even suggested that the idea of an infinite number of modes is inappropriate for any physical system [5.16].

This presents a dilemma. The eigenfunctions are complete with respect to the static and dynamic response of linear self-adjoint distributed parameter systems. If the eigenfunction expansion is truncated, the predicted response as a function of \( x \) will be in error. Still, using a model with an inordinate amount of modal dynamics, especially if the required control temporal bandwidth requirement is "low", presents computational problems.

These issues can be addressed by appending a quasi-static correction to the truncated plant model. First, consider the static response of the system, which is conveniently expressed in terms of the plant static Green's function (static influence function) as

\[
y_s(x,0) = \int_{\Omega} h_s(x,\xi) u(\xi,0) d\xi,
\]

where \( h_s(x,\xi) \) is the static Green's function. The input can be expressed as a superposition of inputs, as per equation (5.21), so that equation (5.62) becomes

\[
y_s(x,0) = \sum_{j=1}^{N} \left[ \int_{\Omega} h_s(x,\xi) q_j(\xi,0) d\xi \right] u_j(0).
\]

The dynamic response of a self-adjoint distributed parameter system can be represented by a bilinear expansion in its eigenfunctions, as in equation (5.34). If the eigenfunction expansion is truncated, the modally band-limited response is given by
\[ y_d(x, \omega) \equiv \sum_{k=1}^{K} \varphi_k(x) \int_{\Omega} \frac{\varphi_k(\xi)}{\lambda_k(\omega)} q_j(\xi, \omega) d\xi. \]  

5.64

The input is most often expressed as a superposition of inputs. As in (5.21), hence equation (5.64) becomes

\[ y_d(x, \omega) \equiv \sum_{j=1}^{N} \left[ \sum_{k=1}^{K} \varphi_k(x) \int_{\Omega} \frac{\varphi_k(\xi)}{\lambda_k(\omega)} q_j(\xi, \omega) d\xi \right] u_j(\omega) \]

5.65

\[ = \sum_{j=1}^{N} \left[ \sum_{k=1}^{K} \frac{\varphi_k(x)}{\lambda_k(\omega)} f'_{jk}(\omega) \right] u_j(\omega), \]

where

\[ f'_{jk}(\omega) \equiv \int_{\Omega} \varphi_k(\xi) q_j(\xi, \omega) d\xi. \]  

5.66

At zero frequency, the static and dynamic representations of the output \( y \) must be equal. Due to the modal truncation of the dynamic Green's function, the responses predicted by (5.63) and (5.65) differ by the amount

\[ y_s(x, 0) - y_d(x, 0) = \sum_{j=1}^{N} \left[ \int_{\Omega} h_s(x, \xi) q_j(\xi, 0) d\xi - \sum_{k=1}^{K} \frac{\varphi_k(x)}{\lambda_k(0)} f'_{jk}(0) \right] u_j(0) \]

5.67

\[ = \sum_{j=1}^{N} \hat{a}_j(x, 0) u_j(0), \]

for

\[ \hat{a}_j(x, 0) \equiv \int_{\Omega} h_s(x, \xi) q_j(\xi, 0) d\xi - \sum_{k=1}^{K} \frac{\varphi_k(x)}{\lambda_k(0)} f'_{jk}(0). \]  

5.68
So, the true static response, as well as the *dynamic* response for \( \omega << \omega_{N+1} \) (the resonant response of the \( N+1 \)st mode) can be better approximated by summing the band limited approximation (5.65) and the correction (5.67):

\[
y(x,\omega) = \sum_{j=1}^{N} \left[ \sum_{k=1}^{K} \frac{\varphi_k(x)}{\lambda_k(\omega)} f_{jk}^{'}(\omega) \right] u_j(\omega) + \sum_{j=1}^{N} \tilde{a}_j(x,\omega) u_j(\omega). \tag{5.69}
\]

The quasi-static correction leads to a *feed-through* term in (5.69).

The shape control task is posed in terms of an expansion of the system response in the shape functions \( \{ \psi_i(x) \} \). The Fourier coefficients of this expansion -- the discrete space/continuous time transform of the output -- are given for the quasi-statically corrected response (5.69) using (5.32):

\[
c_i(\omega) = \sum_{j=1}^{N} \left[ \sum_{k=1}^{K} \frac{f_{jk}^{'}(\omega)}{\lambda_k(\omega)} \int_{d} \varphi_k(x) \psi_i(x) \, dx \right] u_j(\omega) + \sum_{j=1}^{N} \left[ \int_{d} \tilde{a}_j(x,\omega) \psi_i(x) \, dx \right] u_j(\omega). \tag{5.70}
\]

Utilizing the definition

\[
\tilde{a}_{ij}(\omega) \equiv \int_{d} \tilde{a}_j(x,\omega) \psi_i(x) \, dx, \tag{5.71}
\]

and the definition of the coefficients \( h_{ik} \) specifying the transformation from modal to shape space coordinates (5.37), equation (5.70) can be expressed more compactly as

\[
c_i(\omega) = \sum_{j=1}^{N} \left[ \sum_{k=1}^{K} \frac{h_{ik}}{\lambda_k(\omega)} f_{jk}^{'}(\omega) \right] u_j(\omega) + \sum_{j=1}^{N} \tilde{a}_{ij}(\omega) u_j(\omega). \tag{5.72}
\]

The definition (5.50) for an output vector of Fourier coefficients, and (5.51) for a
control vector, permit equation (5.72) to take the familiar form

\[ y = [C \Phi(\omega) B + D] u, \]  

where

\[ C \equiv \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1K} \\ h_{21} & h_{22} & \cdots & h_{2K} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NK} \end{bmatrix}, \]

\[ \Phi(\omega) \equiv \begin{bmatrix} \frac{1}{\lambda_1(\omega)} & 0 & \cdots & 0 \\ & \frac{1}{\lambda_2(\omega)} & 0 & \vdots \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_K(\omega)} \end{bmatrix}, \]

\[ B \equiv \begin{bmatrix} f'_{11} & f'_{21} & \cdots & f'_{N1} \\ f'_{12} & f'_{22} & \cdots & f'_{N2} \\ \vdots & \vdots & & \vdots \\ f'_{1K} & f'_{2K} & \cdots & f'_{NK} \end{bmatrix}, \]

\[ D \equiv \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1N} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2N} \\ \vdots & \vdots & & \vdots \\ \tilde{a}_{N1} & \tilde{a}_{N2} & \cdots & \tilde{a}_{NN} \end{bmatrix}. \]

The \( C \) matrix defines a transformation from the modal space to the shape space. The
matrix $\Phi$ is the usual modal transition matrix, while $B$ is a matrix of modal coefficients from an expansion of the actuator distributions in the eigenfunctions; it includes actuator temporal dynamics which, if separable, can be absorbed into $\Phi$. The $D$ matrix provides a feed-through quasi-static correction to the modal truncation. The output vector $y$ consists of discrete \textit{shape} transform coefficients, while the internals (e.g. $\Phi(\omega)$) are \textit{modal}. The form of $C$ permits the shape control formalism to be applied directly to extant modal state space models of distributed parameter systems.

Two observations are immediately forthcoming. First, the presence of any mode in the shape-transformed output depends on whether

$$\int \psi_i(x) \varphi_k(x) \, dx = 0.$$  \hspace{1cm} 5.78

However, this integral can never vanish because of the assumptions of the analysis. To prove this, assume (5.78) vanishes. This implies $\psi_i$ and $\varphi_k$ are orthogonal. Consequently, $\psi_i$ must be an eigenfunction, or the set of eigenfunctions $\{ \varphi_k \}$ is incomplete. Since $\{ \varphi_k \}$ is complete, and $\psi_i(x) \neq 0$ [from (5.6) [5.12]], either $\psi_i$ is an eigenfunction, or (5.78) \textit{cannot} vanish.

This result is fortuitous when the shape functions are \textit{not} equal to the system eigenfunctions. However, in many cases (such as the example in the previous Section) they \textit{are} equal. This leads to the second observation. When $\psi_i = \varphi_k$, the orthogonality properties of the eigenfunctions reduce (5.73) to the particularly simple form

$$y = C \Phi B u.$$  \hspace{1cm} 5.79

The feed-through term vanishes, since
\[ \int_{D} h_{\gamma}(x,\xi) u(\xi,0) \, d\xi = \sum_{k=1}^{K} \frac{\varphi_{k}(x)}{\lambda_{k}(0)} \int_{D} \varphi_{k}(\xi) u(\xi,0) \, d\xi \]
\[ = \sum_{k=K+1}^{\infty} \frac{\varphi_{k}(x)}{\lambda_{k}(0)} \int_{D} \varphi_{k}(\xi) u(\xi,0) \, d\xi, \]

e.g. the quasi-static correction equals the balance of the infinite eigenfunction expansion for \( k > N \). The correction vanishes because the eigenfunctions defining the shape control task \( \varphi_{k}, k = 1,\ldots,N, \) are orthogonal to those for \( k = N+1,\ldots,\infty \) that lie beyond the required discrete spatial bandwidth.

As a simple example of the feedthrough correction strategy, consider a string of unit length with fixed ends. The nondimensional governing equation is

\[ \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = u(x,t), \quad 0 < x < 1. \]

Upon Fourier transforming the governing equation (5.81) in time, the corresponding dynamic Green's function has the bilinear expansion

\[ h(x,\xi,\omega) = -\frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin(k\pi x) \sin(k\pi \xi)}{(k\pi)^2 + \omega^2}. \]

Statically, the string is described by the governing equation

\[ \frac{\partial^2 y}{\partial x^2} = u(x), \]

which has a static Green's function of the form

\[ h_{\delta}(x,\xi) = \begin{cases} 
(1 - \xi)x & \text{for } x \leq \xi, \\
(1 - x)\xi & \text{for } \xi \leq x.
\end{cases} \]

5.80
5.81
5.82
5.83
5.84
Assume that the string is driven by a point load of unit magnitude at $\xi = .75$. The corresponding static displacement of the string, using the static Green's function (5.84), is plotted in Fig. 5.8. Since the string has no flexural rigidity, the slope at the excitation point is discontinuous, and the displacement is a linear function of position; the reader can readily verify this result by experiment.

![Graph showing nondimensional amplitude versus nondimensional position](image)

Fig. 5.8: String static Green's function response: $\xi = .75$.

The static response of the string to the same excitation, using the dynamic Green's function (5.82), is plotted in Fig. 5.9 for 4, 6, 8 and 10 terms in the bilinear expansion. The error in the predicted response is due to the truncation. However, if a quasi-static correction is appended to the truncated dynamic Green's function, the static response prediction becomes exact, as shown in Fig. 5.10. The feedthrough correction will be valid for frequencies below the resonance of the last mode retained in the truncated plant representation.
Fig. 5.9: Band-limited approximation of string static response: $\xi = .75$. 

Fig. 5.10: String static response with feedthrough correction: $\xi = .75$. 
V.6 SHAPE RECONSTRUCTION

In order to construct a shape control system, not only must actuator sites be chosen to satisfy performance requirements, but sensors must be utilized to estimate the shape output. The shape function expansion approach developed in this Chapter provides a convenient framework for shape estimation, or more properly, shape reconstruction. Given a number of sensors, it is possible to estimate the Fourier coefficients of the shape as decomposed on an orthonormal basis of functions, e.g. the shape functions \( \{ \psi_i(x) \} \). These coefficients can then be used directly by the shape controller to represent the system outputs. Or, they can be used to calculate a spatially band-limited reconstruction of the shape.

A derivation of the method follows. The sensors are assumed to be linear devices. They are also assumed to convolve the system output, with associated spatial apertures ("windows") and temporal dynamics. Various types of sensors can be combined to achieve the measurement. While the presentation may seem rather abstract, it is concluded by a demonstrative example problem.

Consistent with these assumptions, the output of the \( l \)-th sensor is given by

\[
  s_l(\omega) = \int_B p_l(x,\omega)L[y(x,\omega)]dx.
\]

5.85

\( p_l(x,\omega) \) is the sensor windowing function, or spatial/temporal filter. \( L[\cdot] \) is a linear spatial operator that models the operation of sensor. For example, a strain gauge would have \( L = (\cdot)_{xx} \), while a point displacement measurement would have \( p_l = \delta(x-x_l) \) and \( L = 1 \). The sensed output \( s_l \) is a time-varying signal. All spatial information about the transducer is buried within the integral.

The output \( y(x,\omega) \) is assumed to have an expansion of the form (5.19) in the shape functions \( \{ \psi_i(x) \} \), consistent with the analysis, up to a band limit \( N \). Thus,
\[ s(\omega) = \int_D p(x, \omega) L \left[ \sum_{i=1}^{N} c_i(\omega) \psi_i(x) \right] dx. \]

Since \( L \) is a linear spatial operator,

\[ s(\omega) = \sum_{i=1}^{N} c_i(\omega) \int_D p(x, \omega) L[\psi_i(x)] dx. \]

\[ = \sum_{i=1}^{N} c_i(\omega) \beta_i(\omega), \]

for

\[ \beta_i(\omega) = \int_D p(x, \omega) L[\psi_i(x)] dx. \]

For \( P \) sensors, one can construct a matrix input/output relation between the shape coefficients \( c_i(\omega) \) and the sensor outputs \( s_k(\omega) \):

\[
\begin{bmatrix}
    s_1(\omega) \\
    s_2(\omega) \\
    \vdots \\
    s_P(\omega)
\end{bmatrix} =
\begin{bmatrix}
    \beta_{11}(\omega) & \beta_{21}(\omega) & \cdots & \beta_{N1}(\omega) \\
    \beta_{12}(\omega) & \beta_{22}(\omega) & \cdots & \beta_{N2}(\omega) \\
    \vdots & \vdots & \ddots & \vdots \\
    \beta_{1P}(\omega) & \beta_{2P}(\omega) & \cdots & \beta_{NP}(\omega)
\end{bmatrix}
\begin{bmatrix}
    c_1(\omega) \\
    c_2(\omega) \\
    \vdots \\
    c_N(\omega)
\end{bmatrix},
\]

which is expressed more compactly as

\[ s = M(\beta) c. \]

In order to estimate the shape function expansion coefficients given the measurement vector \( s \), and thence reconstruct the shape,

\[ c = [M(\beta)]^+ s, \]
where $(\cdot)^\dagger$ denotes the pseudo inverse. To effect this inverse the matrix $M(\beta)$ must have full rank. This implies $P \geq N$, e.g. there must be at least as many sensors as shape functions, and more if one wishes to over-determine the problem to prevent contamination from shapes $\psi_i$ with index $i > N$ (e.g. aliasing).

The condition $P \geq N$ can be interpreted as a generalized sampling theorem requirement \cite{5.17,5.18}. Choosing $P \geq N$ pushes spatial aliasing to shape functions $i = P + 1$. Since the true system output will (in general) have shape components above $i = N$ (the control spatial bandwidth), and the matrix $M(\beta)$ will in truth have more than $N$ columns in its decomposition of the windowing functions $p_t$ (e.g. the acceptance of the sensing array will extend beyond the discrete spatial bandlimit $N$, unless the apertures are suitably shaped), this is a desirable choice. In mechanical structures the system itself acts as a spatial low-pass filter because of its strain/curvature relation; this further knowledge of the system permits an intelligent, minimal choice of $P$ for structural shape estimation and control.

The matrix of coefficients $\beta$ is fixed by the sensor distributions and types, e.g. by the form of (5.85) for all sensors. The singular value decomposition of $M(\beta)$ will quantify the further requirements for shape estimation. The matrix $M(\beta)$ has the singular value decomposition

$$M(\beta) = U \Sigma V^H,$$

where $U$ is the matrix of left singular vectors which span the sensor signal space, $\Sigma$ is the matrix of singular values, and $V$ is the matrix of right singular vectors which span the shape space. All "directions" (e.g. shapes) in $\psi_i$ space will be equally measurable if the ratio of the maximum and minimum singular values of $M$ is as near to 1 as possible, since these singular values denote gains in the various signal/shape directions; this ratio is the matrix condition number. The shapes will also be "best" estimated if $\sigma_{\min}(M)$ is large, implying more inherent
gain in the sensing system. Since $M(\beta)$ is fixed by the sensor types and locations/apertures, these requirements become criteria for screening candidate sensor distributions for shape estimation and reconstruction.

The application of the shape reconstruction performance measures will now be demonstrated by an example problem of "design by analysis". Consider the shape measurement of a pinned-pinned beam of length 40in., where the desired shape functions are the first four sinusoids;

$$\psi(x) = \sin\left(\frac{i \pi x}{40}\right); \; i = 1..4, \; 0 < x < 40.$$  \hspace{1cm} 5.93

The output will be sensed by ten point displacement transducers having infinite (with respect to the control task) temporal bandwidth. Thus,

$$p_f(x) = \delta(x - x_i), \; 0 < x, x_i < 40,$$  \hspace{1cm} 5.94

and

$$L = 1.$$  \hspace{1cm} 5.95

The elements of $M(\beta)$ are then

$$\beta_u = \sin\left(\frac{i \pi x_i}{40}\right).$$  \hspace{1cm} 5.96

Since $P = 10$, and $N = 4$, $P > N$; the output is oversampled. The matrix $M(\beta)$ can then be calculated, and various sensor positions studied to determine placements that minimize its condition number.

Three candidate sensor distributions are sketched in Fig.s 5.11 – 5.13, and shall hereafter be designated distributions I, II, and III. The condition number and minimum singular value for each distribution appears in the corresponding Figure.
\[ \sigma_{\text{min}} = 1.5767 \quad \text{cond} = 1.6027 \]

Fig. 5.11: Displacement transducer distribution I.

\[ \sigma_{\text{min}} = 2.3294 \quad \text{cond} = 1.0890 \]

Fig. 5.12: Displacement transducer distribution II.

\[ \sigma_{\text{min}} = 2.3452 \quad \text{cond} = 1.0000 \]

Fig. 5.13: Displacement transducer distribution III.

Distribution I was randomly chosen, and displays the largest condition number (~1.6) of the three distributions; the fact that the condition number is still close to 1 no doubt reflects the oversampling, even for this distribution. Distribution II is symmetric (as are the shape functions!), and has a better condition number as well as a larger minimum singular value.
This makes it "better" than distribution I. Distribution III uses an equi-spaced placement, and achieves an ideal condition number of 1. Further, its minimum singular value is the largest of all the distributions. Consequently, it is the best distribution of ten point displacement transducers to estimate the first four sinusoidal shapes.

Naturally, this result does not preclude the use of other sensors for the shape reconstruction task. Rather, this is an example of how the formalism can be applied to screen candidate sensor distributions. The matrix $M(\beta)$ can be calculated and analyzed for various types of sensors, not only to screen placements for each type of transducer, but to compare various types of sensors as well using the aforementioned performance measures.
V.7 SUMMARY AND OBSERVATIONS

A new method of assessing the performance of stationary, linear distributed parameter systems for shape control has been presented in this Chapter. The method is based upon modelling the multiplicative plant input/output representation using discrete spatial transforms in sets of orthogonal functions. The discrete spatial transform is posed in terms of a Fourier expansion in these functions, which gives a minimum integrated mean square error over the domain of the system output. If the set of functions is complete over the output space, then this error goes to zero in a limit in the mean.

New performance measures were derived that are specific to shape control. These performance requirements are extensions of those derived in Chapter IV. The shape control problem was seen to require not only a specified spatial bandwidth, but a set of functions that the control system must be able to achieve within that bandwidth. The use of orthogonal functions to define the spatial transform fits immediately into this requirement. The shape control performance specifications, then, drive the form of the analysis directly.

A set of performance measures was derived for shape estimation/reconstruction consistent with the discrete spatial transform analysis derived for actuator placement. In fact, this sensor placement problem is the dual of the actuator placement problem. Conditions for sampling and optimum placement were derived that are useful not only within the context of shape control, but for shape estimation in general. Example problems for actuator and sensor placement were presented. These were undertaken using extant numerical analysis packages.

These analyses will now be combined to design a shape control proof-of-concept experiment. The discrete shape transform provides a system representation that can be posed in a state space form, with the outputs represented as a vector of Fourier shape components; these are calculated using the methods of the shape reconstruction analysis. Consequently, given sensor and actuator distributions that meet spatial performance goals, the design methods of modern MIMO control can be applied to synthesize temporal compensation to meet closed-loop performance requirements. The design and realization of a dynamic, closed-loop shape control system for a flexible beam is presented in the next Chapter.
VI. A DYNAMIC SHAPE CONTROL EXPERIMENT

VI.1. INTRODUCTION

A PROOF-OF-CONCEPT EXPERIMENT was designed and constructed to assess the utility of the previous analyses, and to determine whether the spatial transform paradigm can be reduced to practice. The dynamic, closed-loop shape control of a flexible beam was chosen, since shape control was one of the aforesaid goals of the thesis. Additionally, the open literature holds no examples of dynamic, closed-loop shape control. A beam experiment was chosen because it facilitates the demonstration of all elements of the analysis without obscuring these results with excessive experimental complication. The experiment was dubbed the DYSC experiment, an acronym for DYnamic Shape Control.

The system's performance is quantified in both spatial and temporal terms. The shape functions are chosen to be sinusoids, with a spatial bandwidth of the first four sinusoidal shapes. The temporal bandwidth is specified as the highest achievable consistent with stability-robustness and computational constraints. It is required that the system achieve the four shapes, and any superposition, with zero steady-state error. This necessitates the addition of integrators at the plant inputs. Further, the system must demonstrate static and quasi-static spatial and temporal disturbance rejection, as well as damped resonant response.

A description of the beam system used in the experiments is given in the next Section. This includes a summary of actuator and sensor selection and placement, the associated electronics, and the design and integration of a digital control system. The system model is presented in the ensuing Section. Open-loop system identification tests are then described. A family of LQG/LTR (Linear Quadratic Gaussian with Loop Transfer Recovery) compensators is designed for the shape control experiments. Closed-loop experimental tests using these compensators are presented that demonstrate the efficacy of the modelling formalism and its experimental realization. A novel inner-loop damping procedure is formulated and implemented to damp resonant responses above the bandwidth of the LQG/LTR controller. Tests of the controller's static and quasi-static disturbance rejection are summarized.
VI.2 EXPERIMENTAL HARDWARE

The Test Structure

The experimental beam was designed to have pinned boundary conditions. This simplified the system modelling and analysis, as the plant mode shapes are sinusoids (e.g. the same as the shape performance functions), and the Green's function can be written down in closed form. Further, the constrained boundaries made it representative of certain mirror systems to which shape control will be applied. The beam configuration is shown in Fig. 6.1. The beam was machined out of .004in 904 stainless steel flat shim stock. This thickness was chosen because it was the thinnest practical; it provided as flexible a beam as possible, but was not prone to warping when the adhesive bonding the film actuators cured. The beam was designed to be 2in wide and 40in long between supports, with an additional 1.5in of stock at each end for mounting. The length was constrained by available space within the laboratory, and the width by experience in bonding the actuators as described below.
The pinned boundary conditions were facilitated by .0025in grooves milled using EDM (Electrical Discharge Machining) into the front face of the beam at each boundary location. These grooves provide elastic restraints that perform like pinned supports when their depth is greater than about 60% of the total beam thickness [6.1]. This was deemed superior to the knife edge supports machined into the test beam investigated in Chapter II; such a support would be extremely difficult to machine accurately in .004in stock.

To avoid the effect of deflecting the beam in a gravity field, the beam was mounted on its side horizontally, as shown in Fig. 6.1. The test fixture consists of a 6in x 6in x 56.5in aluminum I-beam, whose faces were milled to be flat and parallel. One side of the I-beam's web was reinforced with phenolic plastic braces glued into place using white epoxy. This provided additionally stability and rigidity. The experimental beam was attached to the I-beam with aluminum brackets. The beam was clamped to these supports with an aluminum face plate, bolted to each bracket using 1/4–20 cap screws. Each bracket was mounted to the base I-beam using similar cap screws.

The I-beam itself was bolted to a seismic base. Initially, the base was a large steel test bed that sat upon the laboratory subfloor on stacked planks. However, residual vibration excitation from nearby computer disk drives and the lab air handling system was deemed unacceptable, as well as the ambient acoustic environment, so the experiment mounted atop a large floating (e.g. air-cushioned) Newport optical bench. The optical bench provided good vertical vibration isolation, but comparatively poor horizontal vibration isolation. As a further precaution, the experimental beam/I-beam system was surrounded with a Lexan acoustical enclosure with a hinged lid. This "fish tank" also isolated the beam from air swirling within the lab from overhead air conditioning vents.

The mounting procedure for the test beam was determined by trial and error using various test beams, and consisted of the following steps. The technique is shown schematically in Fig. 6.2. The beam was clamped to one face of piece of 38in aluminum angle stock using binder clips, with the bottom edge of the beam flush with the corner of the angle, and the ends of the beam extending beyond its length. This kept the beam from sagging during mounting, or developing an uneven tension top to bottom. The angle stock had been machined flat and parallel. The face that touched the beam was covered with layers of drafting tape; this protected the test beams which had layers of PVF₂ bonded to them.
The beam/support was then set upon milled spacers that kept it parallel to the I-beam, and the beam was clamped into both mounting brackets. The cap screws attaching the left mounting bracket to the I-beam were tightened. Then, the aluminum angle support and the spacers were removed, and the right mounting bracket was slid to the right, adding tension to the beam. The tension was set heuristically to eliminate any potential of sagging in the test beam. The beam often had to be mounted and remounted several times before a "satisfactory" mount was achieved.

![Diagram of beam mounting system]

Fig. 6.2: Exploded view of beam mounting system.

The test bed could be modified to incorporate a second, fixed bracket on the right-hand side to which the movable beam mounting bracket could be attached by threaded rods. Then, if these rods were instrumented with strain gauges, the tension, and hence the mounting, could be more easily repeated. This experimental modification was not deemed necessary for the current investigation.
The Distributed Actuators

Distributed PVF$_2$ film actuators were selected to drive the system. Distributed actuators were chosen because of their desirable spatial filtering characteristics, as described in the previous Chapter. PVF$_2$ film was chosen for several reasons: there is a significant body of experience in applying these actuators within the research group, the material is easily available, and it provides a self-reacting actuator that does not significantly alter the structure. Since the spatial bandwidth was chosen to be the first four sinusoids, a minimum of four actuators are required. The actuators are numbered 1 to 4, from left to right in Fig. 6.1.

Based upon the actuator placement design example of Chapter V, the four actuators were cut to equal lengths. Rather than butting the pieces and inviting inter-segment arcing, the actuator apertures were cut to $9.75\text{in}$ length, and spaced $0.25\text{in}$ apart on the beam; the configuration is shown in Fig. 6.1. This shortening of the actuators did not significantly alter the performance figures of merit quantified in Chapter V. However, to ensure that the beam/film composite flexural rigidity was not discontinuous along its length, unplated interstitial film segments were glued in the gaps. These pieces were deplated using a ferric chloride solution.

The film chosen for the experiments was $28\mu\text{m}$ uniaxially-polarized PVF$_2$. This thickness was chosen for its performance and ease of handling. The film was manufactured by Pennwalt Corporation. Four $15\text{cm} \times 20\text{cm}$ sheets were ordered, from production run TO28NA 10101, order 85121. These sheets were plated on both faces with a nickel-aluminum conductor. The film actuator segments were cut with extreme care using single-edged razor blades, and each blade was disposed of after a single cut to guarantee clean film segment edges. If the blade is not extremely sharp, the plated faces of the film can be squeezed together at the cut, resulting in an electrical short. Also, the film was handled with disposable plastic gloves at all times to prevent contamination of its faces with dirt, skin oils, etc. While these measures may seem excessive, they guaranteed an excellent bond to the beam and minimal "burn-in", as described below.

The film segments were cut to the required length, and to a $2.5\text{in}$ width, i.e. wider than
the beam. The beam was then cleaned with acetone to remove any contaminants, and wiped with Kimwipes. This cleaning was repeated until no residue appeared on the wipes. Armstrong C-7 epoxy was used to adhere the film to the beams. This epoxy provides approximately 30 minutes handling time, and it could be thinned and the working time extended by spraying with cellosolve acetate. The epoxy was applied to the beam over the area of a single actuator film segment, and spread to a thin, uniform layer with a single-edged razor blade. A cut film segment was then rolled loosely about a thick, clean dowel, and rolled onto the adhesive-coated beam. The positively-poled side of the film segment faced away from the beam. Any bubbles were squeezed out with the fingers, wiping from the center outwards as the film was laid down. Once the layer was in place, slight positioning adjustments were made by sliding the piece about. When the segment was in its final position it was rolled using a 4in wide rubber printer’s brayer to squeeze out any extra epoxy. This procedure was repeated until all four actuator segments were in place. The beam/film system was left to dry overnight on a special drying jig.

After the epoxy had cured, the beam was laid upon a paper-covered lab bench with the film side down. Since the film segments were cut wider than the beam, they were trimmed using single-edged razor blades and a steel straight edge. The composite beam/film system was then mounted on the support structure, as described above. In order to electrically connect the exposed film faces, four .5in x .5in brass contact pads were cut from .015in shim stock. These pads were carefully filed, and their edges were rounded. Each pad was then buffed with 400 grit emery cloth to remove any sharp edges and contaminants, then cleaned in acetone. A length of Belden 30 AWG 7 x 38 stranded hookup wire was soldered to one face of each pad. These stranded tie wires were flexible enough not to load the beam. Each pad then was glued to the center of a film segment using Tra-Con Tra-Duct 2902 conductive epoxy; the configuration can be seen in Fig. 6.1. The entire setup was left to dry overnight.

Electrical Connections

The 30 AWG hookup wire from each actuator was soldered to the center conductor of a length of Belden 22 AWG shielded microphone cable. These in turn were connected to one side of a screw terminal strip mounted atop the I-beam; one strip handled the first two
actuators, and was mounted on the left side in front of the left beam mounting bracket, while another handled the last two actuators, and was mounted on the right side in front of the right beam mounting bracket. These connector strips, along with the electrical interconnections, can be seen in Fig. 6.3.

![Experimental configuration](image)

Four Kepco BOP 1000M high-voltage power supplies, shown in Fig. 6.3, drove the four film segments. These power supplies are capable of sourcing ±1000V maximum, and have a practical bandwidth of about 1kHz. Since the beam itself served as a common negative conductor across all the film segments, with the outer plated film faces serving as the positive conductors, the four power amplifiers were run with fully differential outputs in voltage mode. The electrical connections from the power supplies to the screw terminal strips were made with lengths of Belden 22 AWG 2-conductor shielded PVC microphone cable. This cable was chosen because it is capable of handling high voltages without shielding breakdown. The connections to the terminal strips were made through lugs soldered to the cable center and outer leads. The common leads were joined together on the "beam side" of the terminal strips to short segments of the microphone cable, which were then bolted to the beam supports to make an electrical connection to the beam. The test bed was then tied to earth ground through another segment of microphone cable; this was the sole ground point of the structure/power supply system, thus eliminating any floating
ground potentials.

Once the entire system was wired, all the supply lines to the experiment were checked for continuity. The film segments were checked for electrical short circuits using a Fluke hand-held digital multimeter. The power supplies, still disconnected from the test beam, were then turned on and allowed to warm up. Once ready, they were connected to the actuators. The voltage offset of each power supply in turn was gradually increased to the ±600V voltage limits. This was done to eliminate any residual electrical shorts around the edges of the film segments; the process is euphemistically called "burning in" an actuator. Shorts will appear during this process as small, quick sparks along the film edges. Few shorts appeared, indicating that the careful cutting process had been successful.

The connections to the power supplies were made through paired banana plug connectors with 1kΩ shunting resistors. These resistors were included to protect the power supplies in case the film segments shorted. Their resistance value was chosen heuristically, in concert with a ±10mA power supply current limits. The current limits of each power supply were adjusted while a square wave of ±100V peak amplitude was sent to the corresponding film segment, and the voltage across the film was monitored on a Tektronix 2230 two-channel digital storage oscilloscope, so as to eliminate overshoot and "ringing" of this voltage. This eliminated any problems associated with control voltages sent to the film that slew quickly to a high voltage level and clip the power supply; subsequent arcing can damage or even ruin the film plating.

During the course of these shake down tests small cracks in the film plating were found in actuators 3 and 4. These cracks electrically isolated portions of the conductive faces, which meant that those portions were not being driven by the power supplies. These cracks were easily fixed using Amicon conductive ink, which could be brushed over the cracks, and left to dry for an hour or so. Drying time was sped up by thinning the ink with cellosolve acetate. Once dry, electrical continuity was restored over the plating. The ink is flexible enough to not stiffen the beam, and made for quick, easy repairs.

Displacement Transducers
The control implementation required estimation of the beam's shape. Various types of sensors were reviewed. Scanning laser profilometers, used in various robotics and manufacturing applications, were both too slow and too expensive. Various discrete displacement transducers were investigated instead. Fotonic sensors were chosen because they require no special target characteristics, are not-contacting, offer excellent sensitivity, superb stability, high bandwidth, and are readily available.

Ten MTI KD300 fotonic sensors were acquired, along with "hemispherical" sensor bundles. These transducers operate by shining light through a fiber optic bundle, reflecting the light from the target, and sensing the reflected light through another fiber optic bundle; both the sending and receiving fiber optic elements are packaged in a single probe. Their associated signal conditioning provides a voltage proportional to the gap spacing between the probe and the target. The calibration curves for each sensor have two linear ranges, a "near" range and a "far" range. The near range was used for all testing since it is more sensitive, typically 55mV/mil.

The sensors were positioned along the test beam midline, on the side opposite the film actuators. The sensor spatial distribution was that designed in Chapter V, with equal sensor spacings. The sensor mounting brackets were machined from Delrin plastic to electrically isolate them from the test structure. The sensor distribution is shown in Fig. 6.4,

Fig. 6.4: Fotonic sensor distribution (beam top view).
with a detail of the brackets in Fig. 6.5. The sensor tips, while encased in metallic sheaths, were nonetheless electrically isolated from the conditioning electronics; this was verified in a call to MTI, and by checking electrical continuity using a multimeter. The probes were positioned so that their nominal gap spacings were in the center of the near sensitivity linear range. This positioning was facilitated by the digital computer system used to acquire data from and control the experiment, described in the sequel. The calibration program is SETUP.SYS.

Fig. 6.5: Fotonic sensor mounting bracket detail.

A schematic of the experimental setup appears in Fig. 6.6. Except for the cables between the power supplies and the film actuators, all connections shown were made with Pomona BNC cables. The outputs from the fotonic sensors were fed into a custom-built set of signal conditioning amplifiers. These amplifiers were adjusted to subtract the sensor voltage offsets, and provide a gain of 10 to the signals. The offsets were subtracted so as to best utilize the dynamic range of the computer's A/D converters, which were bipolar. Since the displacements of the beam were expected to be small (on the order of a few mils), a motion display device was designed and built. The unit is shown in Fig. 6.7. The device
incorporated ten 21-segment bargraph displays, one per fotonic sensor channel. The inputs were buffered using high-impedance op-amp circuits, then conditioned through gain and offset amplifiers. The motion display was driven by the control computer, which made calibration and scaling adjustments of the displacement data. All analog electronics were left powered up throughout the course of the experimental program to assure stability.

Fig. 6.6: Experimental block diagram.
Fig. 6.7: Cilibtron motion display.
VI.3 CONTROL COMPUTER SYSTEM

A digital control system was purchased and configured to implement the necessary temporal compensation for the shape control experiments. The computer was a Digital Equipment Corporation MicroVAX II. The MicroVAX II was chosen for its performance, and because of extensive in-house experience using the these computers in real-time applications. The system is shown in Fig. 6.3. The computer was housed in a BA-123 enclosure, and was outfitted with a VT240 console terminal, RQDX3 controller, RD50 52 MB hard disk, two RX50 floppy disk drives, DEQNA network interface, KA-630-A processor, 1MB base memory, 8MB AST/Caminion expansion memory board, and a Codar M-Timer 15-channel trigger/timer board. The computer was dubbed "MACS", an acronym for Multi-channel Acquisition and Control System.

The system included a Data Translation DT3382-G-32DI 12-bit 32-channel bipolar A/D board with DMA. The board provides a 250kHz aggregate conversion rate, with a maximum sample and hold aperture delay of 100ns. While the board does not include simultaneous sample and hold, this is not important for the present application; the board was always used in a multi-channel DMA mode at maximum throughput, so the inter-channel phase lag is negligible (about 1.44° between channels at 1kHz). The control computer also included a Data Translation DT3366 12-bit 8-channel bipolar D/A board with DMA, plus two DT3376 8-channel D/A expansion boards, bringing the total number of D/A channels to 24. The D/A boards provide a 500kHz aggregate conversion rate. Most importantly, the A/D and D/A boards have nearly identical interface registers, and similar operating configurations. This sped up device driver development. Custom BNC connector boxes were constructed for the analog I/O boards.

The control software was developed for the DEC VAXELN real-time operating system, Version 3.1. The VAXELN toolkit permits one to write real-time application programs including interrupt handlers, multi-tasking, and multi-programming in high level languages such as Pascal, C, and FORTRAN. A separate MicroVAX II running VMS and the VAXELN development toolkit was used to edit, compile, link, and build VAXELN applications. One drawback to using VAXELN was the lack of software support for the analog I/O boards and the trigger/timer board. A library of functions had already been written in-house for the trigger board in EPascal, the extended Pascal compiler DEC
supplies for VAXELN application development. However, Data Translation could only provide device drivers for the analog boards that ran under VMS. Consequently, a library of device drivers and utility functions were developed by the author for the I/O boards in EPascal, totalling 23,808 lines of code. These procedures and functions were "shaken down" on another Draper project [6.2]. A linkable version of the library is found in the file ANALOG.OLB. The EPascal source code can be found in the files ANALOG.PAS, ANALOGDEFS.PAS, ANALOGDECLS.PAS, ANALOGBODY.PAS, and ANALOGDEFSR.PAS. An NCARMake-compatible MAKE file, ANALOG.MAKE, provides the necessary compilation and library construction commands.

After development of a real-time executive program, the application is compiled and built, and then booted on MACS in one of several ways. The first method is to down-line load the application over DECNET, and remotely debug it from the host computer. This technique proved invaluable during the development of the device drivers. A second method is to down-line load an application built without the remote debugger. The application will then boot on MACS, and can be built with a local debugger. The final method is to copy a bootable image to MACS' hard disk, then boot the application locally from the console. This latter technique was used extensively during the open-loop and closed-loop tests.

During the course of developing and testing the device drivers an omission in the analog I/O board manuals was discovered. If a control program was doing consecutive A/D and D/A conversions at even modest closed-loop conversion rates (around 100Hz), the DMA transfer would become muddled, and data intended for a certain D/A channel would appear on the next channel in the D/A board channel list. This channel shifting was caused by the File Access Listener process built into the bootable applications that provided DECNET support. The process had higher priority than the executing program, and would often respond to random network traffic and momentarily take over the Q-bus. If this happened during a DMA transfer the data could be – and often was – corrupted. Consequently, all closed-loop or fast open-loop application programs were built without network support, since VAXELN (as of this writing) does not support turning on and off the File Access Listener from within an application program. This required that all application program be brought down, and a rudimentary operating system brought up, to effect file transfers over the network. Thanks must be extended to Tom Bailey and Alex Gruzen for their contributions to solving this problem.
A flow chart of a typical application program appears in Fig. 6.8. The program reads in data files describing the A/D, D/A, and timer board configurations, and then sets up these boards for operation. The requisite files can be created off-line using the VMS-compatible application CONFIG.EXE. Upon configuration, the program will prompt the user for the names of any data files containing state matrices for compensator(s) or other dynamic filters to be implemented, as well as any application-specific information. Then, the real-time
acquisition and/or control sequence commences. When complete, the user typically can review the data graphically, and has the option of saving it to various disk files for later analysis and plotting.

Since MACS was designed to implement state matrix-based compensators, a set of Ctrl-C macro functions was written to facilitate the creation of formatted compensator matrix data files that could be copied to MACS' hard disk. A description of the use of these functions, as well as their listings, appears in Appendix D. The data files were read by a control application program at run time, and the discretized matrices were used in state propagation. This feature permitted the design and testing of compensators "on the fly", greatly speeding application development and compensator tuning.

![Diagram](image)

**Fig. 6.9: State propagation and I/O board event time line.**

Real-time state propagation as implemented on MACS is shown on a time line in Fig. 6.9. The A/D, upon receiving a trigger from the M-Timer's channel 1, effects a conversion on the desired channels and does a DMA transfer of this data to a mapped array. The
M-Timer's channel 2 is hard wired as a one-shot fixed-delay trigger that is started by the channel 1 trigger. The executive program, after doing some "house keeping", polls the A/D board's DMA Control Status Register to check for the DMA transfer's completion. Polling was used in place of interrupts; even though VAXELN on the MicroVAX II has a 30\(\mu\)s interrupt handling latency, the minimum time from an interrupt occurring to the return of the program context from an interrupt service routine is 300\(\mu\)s, which is too slow. After the main program is assured that the DMA data is valid, the compensator states are propagated, and the resulting output states are written to another mapped DMA array for the D/A. The D/A then receives its trigger, and the output DMA transfer and D/A conversions to the required channels occurs. The main program then does more house keeping, such as storing data to appropriate arrays for post-processing, and finally polls the D/A's DMA Control Status Register to ensure that the DMA transfer is completed successfully. This whole sequence repeats a user-specified number of times. Once done, the timer board is immediately disarmed. Note that all time-critical procedures and functions that appear within the control portion of the time line were written as inline code for maximum speed.

The real-time executive and the I/O hardware are only synchronized by the polling operations. Further, the computation delay \(\Delta\) defined in Fig. 6.9, corresponding to the time elapsed in the A/D DMA transfer, state propagation, and writing the compensator outputs to the D/A DMA array, had to be determined heuristically. The (albeit crude) method used was to set a state propagation rate \(\tau\) (as defined in Fig. 6.9), and shrink \(\Delta\) until an I/O board error was detected or the compensator failed; naturally, these tests were done without connecting the computer to the structure. A more elegant method would have the main program toggle lines on a parallel digital I/O board at various I/O and computation events, and monitor these signals on an oscilloscope to determine the smallest delay for a fixed state propagation rate.

The typical compensator structure implemented a 12- or 8-state Model-Based Compensator (MBC) and four augmenting integrators; the minutiae of compensator design is discussed later in this Chapter. Table 6.1 summarizes the maximum state propagation rates and minimum computation delay \(\Delta\) for these compensators, as found by experiment.
Slightly reducing $\Delta$ or the difference $\tau - \Delta$ further for a fixed compensator size was possible, but led to unreliable performance. The maximum state propagation rates of 150Hz (for the 12-state MBC) and 200Hz (for the reduced-order 8-state MBC) provided a temporal bandwidth limit on the control implementation, as well as a stability-robustness constraint.

**Table 6.1: Characteristic Computation Times for the DYSC Experiment**

<table>
<thead>
<tr>
<th>MBC/Integrator Compensator Order</th>
<th>State Propagation rate (ms)</th>
<th>Computation Delay (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>5.0</td>
<td>4.2</td>
</tr>
<tr>
<td>16</td>
<td>6.667</td>
<td>5.65</td>
</tr>
</tbody>
</table>
VI.4 SYSTEM MODEL

The plant is modelled as a Bernoulli-Euler beam with in-plane tension and structural damping [6.3]. The governing equation, including control inputs from four distributed piezoelectric film actuators, is

\[
EI \frac{\partial^4 y(x,t)}{\partial x^4} - T \frac{\partial^2 y(x,t)}{\partial x^2} - \lambda \frac{\partial^3 y(x,t)}{\partial x^2 \partial t} + \rho A \frac{\partial^2 y(x,t)}{\partial t^2} = m \sum_{i=1}^{4} \frac{\partial^2 V_i(x,t)}{\partial x^2}, \quad 0 < x < L. \quad 6.1
\]

\( EI \) is the beam/film composite flexural rigidity, \( T \) is the in-plane tension, \( \lambda \) is the structural damping coefficient, \( \rho \) is the beam/film composite's mass per unit length, \( A \) is its cross-sectional area, \( L \) is the length of the beam, and \( m \) is the actuator gain constant, expressed in equation (1.3) in terms of beam and film parameters. The displacement \( y \) is assumed to be expandable in a set of sinusoidal shape functions,

\[
y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)y_n(t), \quad 6.2
\]

where the Fourier coefficients \( y_n(t) \) are defined by

\[
y_n(t) = \frac{2}{L} \int_{0}^{L} y(x,t) \sin\left(\frac{n\pi x}{L}\right) dx. \quad 6.3
\]

The film actuators are degenerate, e.g. they can be represented analytically in a product form,

\[
V(x,t) = \sum_{i=1}^{4} \Lambda_i(x)u_i(t). \quad 6.4
\]

Since the actuators have rectangular apertures,

\[
\Lambda_i(x) = h(x - c_i) - h[x - (c_i + \Delta_i)], \quad 6.5
\]
where \( c_i \) is the start of the aperture, and \( \Delta_i \) is its width. Consequently,

\[
\Lambda_i''(x) = \delta'(x - c_i) - \delta'[x - (c_i + \Delta_i)].
\]  

6.6

The control distribution is assumed to be expandable in the shape functions,

\[
\frac{\partial^2 V(x,t)}{\partial x^2} = \sum_{i=1}^{4} \sum_{n=1}^{\infty} q_{in} \sin\left(\frac{n\pi x}{L}\right) \mu_i(t),
\]  

6.7

where the Fourier coefficients \( q_{in} \) of the actuator distributions \( \Lambda_i''(x) \) are given by

\[
q_{in} = \frac{2}{L} \int_{0}^{L} \left( \delta'(x - c_i) - \delta'[x - (c_i + \Delta_i)] \right) \sin\left(\frac{n\pi x}{L}\right) dx
\]  

6.8

\[
q_{in} = -\frac{4n\pi}{L^2} \sin\left(\frac{n\pi}{2L}(2c_i + \Delta_i)\right) \sin\left(\frac{n\pi \Delta_i}{2L}\right).
\]

Define \( \nu_n(t) \) as the time derivative of the \( n \)-th Fourier coefficient,

\[
\nu_n(t) \equiv y_n(t).
\]  

6.9

For the \( n \)-th shape, the governing equation then assumes the form

\[
\left[ EI\left(\frac{n\pi}{L}\right)^2 + T\left(\frac{n\pi}{L}\right)^2 \right] y_n + \lambda_n \left(\frac{n\pi}{L}\right)^2 \nu_n + \rho A \nu_n = m \sum_{i=1}^{4} q_{in} \mu_i,
\]  

6.10

or, in state space form,
\[
\begin{bmatrix}
y_n \\ v_n
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{EI}{\rho A L} (n \pi)^4 & -\frac{T}{\rho A L} \left(\frac{n \pi}{2}\right)^2 & -\frac{\lambda_n (n \pi)^2}{\rho A L}
\end{bmatrix} \begin{bmatrix}
y_n \\ v_n
\end{bmatrix} + \frac{m}{\rho A} \begin{bmatrix}
0 & 0 & 0 & 0 \\
q_{1n} & q_{2n} & q_{3n} & q_{4n}
\end{bmatrix} \begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4
\end{bmatrix}.
\]

A modal damping factor \( \lambda_n \) was introduced in (6.10) to accommodate modes with different damping coefficients, so as to fit experimental data.

Since the shape control task requires a spatial bandwidth of four shapes, equation (6.11) is truncated at \( n = 4 \). Then, a state vector \( x_p \) is defined as

\[
x_p \equiv [y_1 \ y_1 \cdots y_4 \ v_4]^T,
\]

and a control vector \( u_p \) is defined as

\[
u_p \equiv [u_1 \cdots u_4]^T.
\]

The first four shapes become the system outputs, hence one can define an output vector \( y \) as

\[
y \equiv [y_1 \cdots y_4]^T.
\]

Thus, the governing equation assumes a state matrix form in the shape space,

\[
x_p = A_p x_p + B_p u_p,
\]

\[
y = C_p x_p,
\]

for
\[ A_p = \text{diag} \left[ \begin{array}{ccc} 0 & & 1 \\ \frac{EI(n\pi)^4}{\rho A \frac{L}{4}} - T \frac{(n\pi)^2}{\rho A \frac{L}{2}} & \lambda_n \frac{(n\pi)^2}{\rho A \frac{L}{2}} & \end{array} \right], \quad n = 1 \ldots 4, \quad 6.17 \]

\[ B_p = \frac{m}{\rho A} \begin{bmatrix} q_{11} & q_{21} & q_{31} & q_{41} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix}, \quad 6.18 \]

\[ C_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad 6.19 \]

Note that the plant inputs are control voltages, and the outputs are Fourier coefficients. This state space form facilitates the use of control analysis and design software packages to design a temporal compensator for the shape control system. This will be exploited in the sequel.

The known static beam/film parameters are summarized in Table 6.2. The unknown parameters are the damping coefficients \( \lambda_n \) and the in-plane tension \( T \). For each shape an equivalent viscous damping ratio \( \zeta_n \) can be derived,

\[ \zeta_n = \frac{\lambda_n \frac{(n\pi)^2}{L}}{2\sqrt{\rho A [EI \frac{(n\pi)^2}{L} + T]}}. \quad 6.20 \]

The tension \( T \) and the damping coefficients \( \lambda_n \) will be adjusted to fit experimentally-determined resonant frequencies and dampings; this will be explored in the next Section. Further, the plant's zero-frequency static gain matrix \( G_p(0) \) is given [using
\( \mathbf{G}_p(0) = -\mathbf{C}_p \mathbf{A}_p^{-1} \mathbf{B}_p \)

\[
= m \text{diag} \left[ \frac{1}{-E_f(nL)^4 - T(nL)^2} \right] \begin{bmatrix}
q_{11} & q_{21} & q_{31} & q_{41} \\
\vdots & \vdots & \vdots & \vdots \\
q_{14} & q_{24} & q_{34} & q_{44}
\end{bmatrix}
\]

This permits an experimental determination of the actuator filter functions \(q_{im}\), and thence the system \(\mathbf{B}_p\) matrix, from experimentally-measured static shape influence functions. This too will be exploited in the next Section.

**Table 6.2: Beam/film Static Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BEAM:</strong></td>
<td></td>
</tr>
<tr>
<td>material</td>
<td>904 stainless steel</td>
</tr>
<tr>
<td>length</td>
<td>40in</td>
</tr>
<tr>
<td>width</td>
<td>2in</td>
</tr>
<tr>
<td>thickness</td>
<td>.004in</td>
</tr>
<tr>
<td>density</td>
<td>.2815 lb/in^3</td>
</tr>
<tr>
<td>modulus</td>
<td>3.047x10^7 psi</td>
</tr>
<tr>
<td><strong>FILM:</strong></td>
<td></td>
</tr>
<tr>
<td>material</td>
<td>PVF2</td>
</tr>
<tr>
<td>length</td>
<td>9.75in/active segment</td>
</tr>
<tr>
<td></td>
<td>0.25in/interstitial segment</td>
</tr>
<tr>
<td>width</td>
<td>2in</td>
</tr>
<tr>
<td>thickness</td>
<td>0.0011in (film alone)</td>
</tr>
<tr>
<td></td>
<td>0.002in [film+bond (nominal)]</td>
</tr>
<tr>
<td>density</td>
<td>0.065 lb/in^3</td>
</tr>
<tr>
<td>modulus</td>
<td>2.9x10^5 psi</td>
</tr>
<tr>
<td>static piezoelectric constant</td>
<td>8.66x10^-10 in/V</td>
</tr>
</tbody>
</table>
VI.5 SYSTEM IDENTIFICATION

Unlike the so-called Lyapunov dampers studied in Chapter I, the compensators designed for the DYSC experiment require a more detailed knowledge of the plant characteristics. First, a shape control system must drive the system output to a specific setpoint, rather than just remove energy in a global sense. Further, the compensator design is model based. Consequently, an open-loop test program was undertaken to identify the unknown system parameters.

Fig. 6.10: Transfer function measurement schematic.
The first series of tests were dynamic transfer function measurements on the experimental beam. Transfer functions were measured between each of the four actuators and the ten sensors, leading to a total of forty measurements. The transfer functions identified the system's resonant frequencies. From these measurements damping ratios were calculated, as per equation (6.20). Also, a value of the in-plane tension $T$ could be estimated; the tension parameter was adjusted until the eigenvalues of the system A matrix, equation (6.17), matched the measured natural frequencies. These computations are straightforward because the shape functions and the plant eigenfunctions are equal for the DYSC experiment.

![Graph](image.png)

Fig. 6.11: Typical DYSC transfer function measurement (9/1).

The transfer function measurement setup is shown in Fig. 6.10. A Zonic 6088 four-channel spectrum analyzer was used for the tests. The analyzer provided band-limited random noise to one of the Kepco power supplies, which in turn drove one of the actuator segments. The noise source was also connected to the first channel of the analyzer as the reference input. Then, the remaining channels of the analyzer were connected to three of the displacement transducer outputs; the Zonic can measure three transfer functions simultaneously. Sensor probe calibrations were entered into the analyzer, and the system was configured for continuous processing of transfer function measurements with 50% overlap, over the bandwidth 0–62.5 Hz, using Hanning windowing functions. The channel
gains were adjusted in concert with the signal source level to use as much of each A/D's dynamic range as possible, and to provide for the best coherence between channels. Tests were performed with fifty averages. A typical transfer function appears in Fig. 6.11, showing the resonances of the plant. The test procedure was repeated until all forty transfer functions were measured.

The Zonic spectrum analyzer was then employed to pick out the natural frequencies and associated damping coefficients for the first four modes, using the analyzer's own modal analysis software to provide multiple degree of freedom (MDOF) curve fits to the transfer functions. All forty transfer function measurements were used by the software to best estimate these parameters by averaging. The additional off-resonance structure noted in Fig. 6.11 appeared in frequency bands with poor coherence, and did not appear in all the transfer function data, hence the MDOF fit didn't regard these as modes. The MDOF analysis results are tabulated in Table 6.3. The tension $T$ and damping parameters $\lambda_n$ were adjusted until the poles of the system $A$ matrix best fit the experimental values. This process focused first on fitting mode 1, as it is the closest to the control bandwidth; the fit for the rest of the modes is remarkably good as well. The final tension value of $T = 30.937N$ seems a bit high, but the experimental configuration offered no direct tension measurement. This high tension value may in part be attributed to the milled boundary conditions (which provide elastically-restrained rather than purely hinged boundary conditions) or to residual stresses created by the film/beam bond during curing.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency ($Hz$) [Experiment (Model)]</th>
<th>Damping Factor ($Hz$) [Experiment (Model)]</th>
<th>$\lambda_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.918 (12.92)</td>
<td>.064 (.064)</td>
<td>.00379</td>
</tr>
<tr>
<td>2</td>
<td>26.065 (25.85)</td>
<td>.139 (.137)</td>
<td>.00203</td>
</tr>
<tr>
<td>3</td>
<td>39.242 (38.80)</td>
<td>.191 (.189)</td>
<td>.00124</td>
</tr>
<tr>
<td>4</td>
<td>52.128 (51.79)</td>
<td>.234 (.232)</td>
<td>.00086</td>
</tr>
</tbody>
</table>

The second series of system identification tests were static shape influence function measurements. Given the tension and damping parameters, the actuator spatial influence functions $q_{in}$ can be determined from a static response measurement, as shown in equation
(6.21). While the first series of tests supplied all parameters needed to model the "nominal" system, these further tests served both to validate elements of the nominal model and provide a "truth" model for compensator design.

The influence function measurements consisted of quasi-statically driving the beam to an output profile with a single actuator. The tests were conducted for various steady-state voltage levels, from -450V to +450V in increments of 25V for each actuator. The beam was brought to its shape by passing the actuator voltage command through a second-order low pass filter with a 1Hz corner frequency; the filter was implemented digitally on the control computer at a 100Hz state propagation/acquisition rate. The tests lasted for 60 seconds, with the data acquisition program recording the outputs from the photonic sensors. These data points were stored in disk files for later processing.

The displacement files were analyzed off-line to determine the Fourier shape coefficients of the time-averaged steady-state beam profiles. The coefficients were calculated for each time step, and averaged over the acquisition time interval from 2 to 60 seconds to average out oscillations in the beam due to exogenous disturbances. From equation (6.21), choosing \( u_p = V_{max} e_i \), where \( e_i \) is one of the unit vectors, gives an output \( y_i \) defined by

\[
y_i = V_{max} m \text{diag} \left[ \frac{1}{-EI(nL)^4 - T(nL)^2} \right] \begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \\ q_{i4} \end{bmatrix}.
\]

6.22

The actuator shape influence function matrix is then calculated from

\[
\begin{bmatrix}
q_{11} & q_{21} & q_{31} & q_{41} \\
\vdots & \vdots & \vdots & \vdots \\
q_{14} & q_{24} & q_{34} & q_{44}
\end{bmatrix} = \text{diag} \left[ \frac{-EI(nL)^4 - T(nL)^2}{m V_{max}} \right] \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}.
\]

6.23

The normalized Fourier coefficient estimates \( 100y/V_{max} \) for the influence function measurements as a function of the command voltage are plotted in Fig.s 6.12 through 6.15. for each actuator; the Fourier coefficient computation method is described in the next Section. Actuator 1 shows a slight nonlinear stiffness which may be attributed to an asymmetry in the grooves machined into the beam's face that provide the pinned boundary condition. The other actuators exhibit a linear behavior with actuator voltage. Actuators 2
and 3 show similar influence function Fourier coefficients, with actuator 3 appearing to be a bit weaker; shapes 2 and 4 are flipped in sign because of symmetry. Actuators 1 and 4 show some of this symmetry, but some notable differences that may be attributed to variations in the film properties between the two segments, differences in bonding, and, perhaps most

![Fourier Decomposition](image)

**Fig. 6.12:** Fourier decomposition of beam static response to actuator 1.

![Fourier Decomposition](image)

**Fig. 6.13:** Fourier decomposition of beam static response to actuator 2.
Fig. 6.14: Fourier decomposition of beam static response to actuator 3.

Fig. 6.15: Fourier decomposition of beam static response to actuator 4.

importantly, variations in the EDM-machined grooves at the left and right boundaries. Since these two actuators extend to their respective ends of the beam, their response will be most affected by such a mechanical difference in the beam. The matrix of Fourier coefficients in equation (6.23) was calculated by averaging the data appearing in these plots.
Two system models were derived based upon the system identification results. The first, hereafter called the "nominal" plant model, used the static parameters from Table 6.2 and the experimentally-determined tension and damping coefficients to calculate the system state matrices. The second, hereafter called the "experimental" plant model, used these values as well, but substituted the experimental values of the actuator shape influence coefficients $q_{in}$ to form the system state matrices. Preliminary calculations led to a system scaling of 10-mils as outputs, and V/100 as inputs for the compensator design and implementation.

The uncompensated forward loop transfer matrix singular values for the nominal and experimental plant models appear in Fig.s 6.16 and 6.17, respectively. The plant matrices appear in Appendix F. The plots are envelopes of the minimum and maximum singular values over frequency. Both plant models have identical resonant characteristics. However, the experimental plant has more directionality with respect to the shapes at zero frequency than the nominal plant; the static condition numbers are 1.9852 and 1.2315, respectively. The reasons for this directionality are seen in plots of the two plants' input coupling operators, Fig.s 6.18 and 6.19. The estimates of the control influence functions show that actuators 3 and 4 are weaker than actuators 1 and 2, as seen in Fig. 6.19. Actuators 1 and 2
Fig. 6.17: Uncompensated experimental plant model singular value response.

Fig. 6.18: Input coupling operators for nominal plant model.
were cut from the same piece of PVF$_2$, while actuators 3 and 4 were cut from another piece. Given the variability in electromechanical properties between samples of PVF$_2$, these deviations are not surprising. Actuators 3 and 4 also had the most problems with cracks in the outer metallization. Nonetheless, the singular value plots 6.16 and 6.17 show the systems are not highly directional, and that the scaling is appropriate. This simplifies the compensator design and implementation.

The experimental plant model's static gain matrix was used to predict the actuator voltages required to drive the beam to each of the four sinusoidal shape functions. A series of open-loop control experiments was undertaken to test these predictions on the DYSC system. As in the shape influence function measurements, the actuator voltage commands were passed through a 1Hz digital low pass filter. This drove the beam to its steady-state output without exciting any resonances; the goal of these tests was to validate the experimental plant model at zero frequency. The beam was sent to each shape, with tests done at various voltage levels. Sample test time traces are shown in Fig.s 6.20 through 6.23. These plots are, essentially, real-time shape decompositions of the beam's profile, where the amplitude of each shape component appears as a separate curve. If the plant were perfectly identified, and there were no external disturbances, the output of the shape 1 test (Fig. 6.20), for example, would have the solid line (shape 1) achieve a steady-state amplitude of 4, with the other shape components identically zero. The time-averaged
steady-state value of shape 1 is instead 3.79, and the other shape coefficients are non-zero, but small.

Fig. 6.20: Quasi-static shape $1 \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^T$ open loop response (experiment).

Fig. 6.21: Quasi-static shape $2 \begin{bmatrix} 0 & 2 & 0 & 0 \end{bmatrix}^T$ open-loop response (experiment).
Fig. 6.22: Quasi-static shape $3 \begin{pmatrix} 0 & 0 & 2 \end{pmatrix}^T$ open-loop response (experiment).

Fig. 6.23: Quasi-static shape $4 \begin{pmatrix} 0 & 0 & 0 & 2 \end{pmatrix}^T$ open-loop response (experiment).
Based upon the open-loop control tests, the experimental plant appears to quantify the system static response quite well. However, feedback is still required, because the plant is not perfectly identified, e.g. one desires excellent tracking performance in the presence of modelling errors. Also, an open-loop design would not be robust to exogenous disturbances. A feedback control design and implementation strategy for the DYSC experiment is presented in the next Section.

A stability-robustness bound for the experimental plant model was derived by plotting the singular value magnitude of the multiplicative error associated with truncating the plant model. The worst model error was assumed to be bounded by the contribution to the plant response of the fifth mode. This error, while additive, was reflected through the experimental plant matrix to deduce an equivalent multiplicative error. The fifth mode has a resonant frequency of 75.3Hz, and damping factor of .291Hz; this was estimated based on a further transfer function measurement. This stability-robustness bound is plotted in Fig. 6.24. This

![Graph showing stability-robustness bound](image)

Fig. 6.24: Stability-robustness bound for experimental plant model.

shows that the corner frequency of a closed-loop control system design based upon the experimental plant model must not exceed 3Hz, or the controller cannot be guaranteed to be
stable. And finally, a stability-robustness bound for the nominal plant model was derived from the difference between the experimental and nominal plant response matrices; the experimental plant model was assumed to represent a "truth" model. The bound is plotted in Fig. 6.25, which shows that the corner frequency of a closed-loop control system based upon the nominal model must not exceed $1.6\,\text{Hz}$. This is less than half the bound placed upon the experimental plant model, due to the errors inherent in the nominal model at low frequencies.

Fig. 6.25: Stability-robustness bound for nominal plant model.
VI.6 COMPENSATOR DESIGN

Overview

Compensators for the DYSC experiment were designed using the Loon Transfer Recovery (LTR) method. Numerous journal articles have been written about LTR; the reader is referred to [6.4, 6.5, 6.6] for derivations and illustrations of the technique. Essentially, LTR is a structured method for designing output-feedback LQG controllers. The MIMO design is implemented using a Model Based Compensator (MBC).

The LTR design begins by first constructing a suitable "target" closed-loop MIMO controller. This target loop must meet all the performance specifications and stability-robustness constraints of the final design. The target loop is often designed using Kalman filtering techniques. The target loop is not intended to function as an optimal stochastic estimator. Rather, Kalman filter design techniques are employed because Kalman filters have desirable characteristics: their closed-loop dynamics are always stable, their multivariable transmission zeroes are always minimum phase, their sensitivity transfer function matrices have maximum singular value less than 0dB, and they have infinite upward gain margin, -6dB gain reduction margin, and ±60° phase margin at each output feedback channel independently and simultaneously.

Once a suitable target loop has been designed, its characteristics are "recovered" by the MBC/design plant by calculating a gain matrix $G_R$ for a special form of the "cheap control" LQR problem. Then, pointwise in frequency, the recovered forward loop transfer matrix equals the target forward loop transfer matrix. This result is constrained by the fact that the design plant must be minimum phase.

An overview of LTR as applied to the DYSC experiment follows. The target and recovered loop dynamics are summarized, including issues of integral control and loop shaping. A compensator order reduction technique is introduced, and the structure of the shape estimator implementation is derived. What follows is not a rigorous derivation of LTR, but rather a summary of the mathematical formulation relevant to the computer-aided compensator design.
The LTR Design Method

The plant, defined by equations (6.12) through (6.19), is first augmented by integrators at the plant inputs to construct a Design Plant Model (DPM);

\[ \dot{x} = Ax + Bu, \quad 6.24 \]

\[ y = Cx, \quad 6.25 \]

where

\[ x \equiv \begin{bmatrix} u_p \\ x_p \end{bmatrix}, \quad 6.26 \]

\[ u \equiv u_p, \quad 6.27 \]

\[ A \equiv \begin{bmatrix} 0 & 0 \\ B_p A_p \end{bmatrix}, \quad 6.28 \]

\[ B \equiv \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad 6.29 \]

\[ C \equiv \begin{bmatrix} 0 & C_p \end{bmatrix}. \quad 6.30 \]

The integrators are added to meet the zero steady-state error performance specification. Note that \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, \) and \( y \in \mathbb{R}^m, \) with \( n = 12 \) and \( m = 4 \) for the present system.

A special Kalman filtering problem is formulated for the target filter loop, having state equation dynamics

\[ \dot{x} = Ax + L\xi, \quad 6.31 \]

\[ y = Cx + \theta. \quad 6.32 \]
\( \xi \) is fictitious Gaussian zero-mean white noise with an identity intensity matrix. \( \theta \) is fictitious Gaussian zero-mean white noise with constant intensity matrix \( \mu I, \mu > 0 \), independent of the process noise. We assume that \([A, L]\) is stabilizable, and that \([A, C]\) is detectable. The \( n \times m \) matrix \( L \) and the scalar gain \( \mu \) are design parameters in the filter problem. These parameters are not given a priori physical significance. Rather, they can be adjusted for the purposes of design.

![Block diagram](image)

**Fig. 6.26: Target filter loop topology.**

The target filter loop is depicted in Fig. 6.26. The filter gain matrix \( H \) is computed from the filter algebraic Riccati equation,

\[
0 = A\Sigma + \Sigma A^T + LL^T - \frac{1}{\mu} \Sigma C^T C \Sigma, \tag{6.33}
\]

where \( H \) is defined in terms of the covariance matrix \( \Sigma \) as

\[
H = \frac{1}{\mu} \Sigma C^T. \tag{6.34}
\]

Consequently, the target filter closed-loop dynamics are described by the state equations

\[
x = (A - HC)x + Hr, \tag{6.35}
\]

\[
y = Cx. \tag{6.36}
\]

The design matrix \( L \) is chosen to match the singular values of the target filter's forward
loop transfer matrix at low and high frequencies. The technique, based upon the structure of the Kalman frequency domain equality [6.4], first partitions \( L \) as

\[
L = \begin{bmatrix} L_L \\ L_H \end{bmatrix}.
\]  

(6.37)

\( L_L \) is an \( m \times m \) matrix, while \( L_H \) is \( n \times m \). To match the singular values of the target forward loop transfer matrix at low frequencies, so as to eliminate "directional" response and make all shapes equally achievable, one chooses

\[
L_L = -[C_p A_p^{-1} B_p]^{-1} = G_p^{-1}(0).
\]  

(6.38)

The low frequency singular value matching criteria (6.38) implies, as \( \omega \rightarrow 0 \),

\[
\sigma[C(i\omega I - A)^{-1}H] \rightarrow \frac{1}{\omega \sqrt{\mu}}.
\]  

(6.39)

Thus, the design parameter \( \mu \) sets the bandwidth of the target filter. The target filter forward loop crossover frequency is then approximately

\[
\omega_c \equiv \frac{1}{\sqrt{\mu}}.
\]  

(6.40)

To match singular values at high frequencies, choose

\[
L_H = C_p^T(C_p C_p^T)^{-1}.
\]  

(6.41)

This is one of several possible pseudo-inverses that will give the desired high frequency result.

Given the design parameters \( L \) and \( \mu \) one calculates the filter gain matrix \( H \) using (6.33) and (6.34), and then simulates the target loop response using (6.35) and (6.36). Further,
the design's singular value loop shapes are plotted to ensure it meets performance and stability-robustness constraints. Several compensator designs (e.g. values of \( \mu \)) were constructed for the DYSC experiment; these are summarized later in this Chapter.

![Diagram of LTR loop topology](image)

Fig. 6.27: LTR loop topology.

Upon completing the target design, one next constructs a Model Based Compensator which, when cascaded with the design plant, recovers the target filter characteristics. The topology of the LTR compensator loop is shown in Fig. 6.27. The filter gain matrix \( \mathbf{H} \) is that synthesized for the target design, while the \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \) matrices are those defined in equations (6.28) through (6.30) for the design plant. The MBC has the time domain description

\[
\mathbf{z} = (\mathbf{A} - \mathbf{B} \mathbf{G}_\rho - \mathbf{H} \mathbf{C}) \mathbf{z} - \mathbf{H} \mathbf{e},
\]

\[
\mathbf{u} = -\mathbf{G}_\rho \mathbf{z}.
\]

\( \mathbf{z} \) is the compensator state vector; \( \mathbf{z} \in \mathbb{R}^n \), \( \mathbf{e} \in \mathbb{R}^m \) is the output error signal, and \( \mathbf{u} \) is the input to the augmenting integrators defined in equation (6.27). The governing equation for the MBC/DPM combination are found by a series combination of the MBC state equations (6.42),(6.43) and the DPM state equations (6.24),(6.25). The gain matrix \( \mathbf{G}_\rho \) is calculated from the control algebraic Ricatti equation,

\[
0 = -\kappa_{\rho} \mathbf{A} - \mathbf{A}^T \kappa_{\rho} - \mathbf{C}^T \mathbf{C} + \frac{1}{\rho} \kappa_{\rho} \mathbf{B} \mathbf{B}^T \kappa_{\rho},
\]
\[ G_{\rho} = \frac{1}{\rho} B^T K_{\rho}. \]

\( \rho \) is a scalar design parameter used in a special form of the "cheap control" LQR problem.

The loop transfer recovery result is that, pointwise in frequency,

\[ \lim_{\rho \to 0} G(\omega) K_{\rho}(\omega) \to G_F(\omega), \]

for the MBC structure described above. \( G_F \) is the target filter forward loop response matrix. It follows that, for minimum phase design plants (such as the DYSC system), the zeroes of \( K_{\rho} \) and \( G_F \) are equal. Therefore, some poles of the LTR compensator cancel the zeroes of the design plant, while the rest of the poles go to \( -\infty \) along stable Butterworth filter loci as \( \rho \to 0 \). Thus, the LTR compensator stably inverts the design plant, and substitutes the desirable dynamics of the target filter. For each of the DYSC compensator designs \( \rho \) was chosen so as to recover the target loop singular values up to one decade beyond crossover. Further decreasing \( \rho \) to extend the frequency range of recovery did not provide any significant performance enhancement, but did make some of the compensator poles undesirably fast.

Shape Estimation and Compensator Computational Structure

The shape control compensator requires an estimate of the Fourier coefficients of the beam's displacement, rather than the displacement at the individual fotonic sensor locations. The integral defining these coefficients, equation (6.3), cannot be calculated exactly because the displacement \( y(x,t) \) is only measured at ten discrete locations. However, the integral can be approximated as
\[ y_n(t) \equiv \frac{2}{L} \sum_{i=0}^{11} y(x_i, t) \sin\left(\frac{n \pi x_i}{L}\right) \Delta x, \]

where \( x_i \) is the location of the \( i \)-th displacement. The locations \( x_0 \) and \( x_{11} \) are the beam's fixed boundaries, hence \( y = 0 \) there. \( \Delta x = 3.6364\text{in} \), the inter-sensor spacing. The matrix of coefficients defined by (6.47), for \( y \) defined in equation (6.14), is calculated off-line, and stored for use in calculating the requisite Fourier coefficients at each state propagation cycle.

Equation (6.47), being a rectangle rule integration, is exact for the oscillatory integrand as long as the integrand is oversampled. Given the eleven independent displacement measurements (ten photonic sensors plus one fixed end), the shape estimator will not alias until the twelfth sinusoid; the discrete spectrum "folds" about the eleventh shape. While this bears a resemblance to the aliasing phenomena encountered in calculating discrete Fourier transforms, the present computation is instead a "discrete Fourier series".

**Fig. 6.28: Computational structure.**
The computational structure of the digital control implementation is sketched in Fig. 6.28. The user commands the beam to assume one or more of its four sinusoidal shapes, with a prescribed amplitude. These commands can be specified as step inputs, or as a time sequence of shape commands. For each time step the shape estimates are calculated by an in-line procedure, and then subtracted from the current command input to form an error signal. This error signal is then passed to the MBC state propagation routine (also written as in-line code), and the compensator states are propagated. Each output from this calculation is passed to an integrator state propagation routine, and the integrator states are propagated. The routine for the MBC and integrator state propagation is the same; the state matrices specific to each, however, are different. The computation was broken up into separate integrator and compensator calculations to reduce the total number of arithmetic computations, and hence speed the state propagation rate. The hardware and software details of the conversion/computation cycle has been previously described.
VI.7 CLOSED-LOOP TESTS

Four LTR compensator designs were experimentally evaluated on the DYSC beam. The designs, listed in Table 6.4, all satisfied the singular value frequency matching criteria, equations (6.38) and (6.41). They differ by their forward loop crossover frequencies and by their order, e.g. the number of compensator states. Since the uncompensated plant had eight states and four inputs, the integrator-augmented design plant model had twelve states. By design, the MBC must have twelve states as well. One compensator design, MBC1, used a residue expansion method [6.7] to reduce its order to eight states, eliminating two resonant poles. Order reduction was investigated because it offered the possibility of faster state propagation, as shown in Table 6.1.

<table>
<thead>
<tr>
<th>Design</th>
<th>$\mu$</th>
<th>$\rho$</th>
<th>$\omega_c$ (Hz)</th>
<th>Order</th>
</tr>
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<td>.0001</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>MBC1a</td>
<td>.025</td>
<td>.0001</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>MBC1</td>
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<td>.0001</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>MBC2a</td>
<td>.006</td>
<td>.00002</td>
<td>2</td>
<td>12</td>
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</table>

Given the target loop design parameter $\mu$, each design's target filter was constructed and simulated using Ctrl-C. The recovery parameter $\rho$ was chosen to effect recovery of the target loop singular value response approximately one decade beyond crossover. The singular value forward loop, sensitivity, and closed-loop response functions for the target and recovered loops were plotted. The step response to each shape's reference command input was simulated for the target and recovered loops, as well as the step response to a sequence of all four shapes. Then, the compensator was discretized using the Ctrl-C function FILTRBUILDST (see Appendix D) for a time step of $1/150 \, sec$ for the full-order compensator designs, and for a time step of $1/200 \, sec$ for the reduced-order design, and tested on the DYSC system using the EPascal application DYSC_CL2. Experimental data was taken for step responses to each shape's command input, as well as a sequence of all four shapes, for various amplitudes. Representative simulation and experimental data for
Compensator MBC1er was designed using the nominal plant model. The target loop was constructed using the nominal plant, and the recovered loop was simulated using this nominal compensator design cascaded with the experimental plant model. The singular values of the forward loop, sensitivity, and closed-loop response functions are plotted in Figs 6.29 through 6.31, respectively, for the target and recovered loops. The augmenting integrators have provided sufficient loop gain at low frequencies to satisfy the tracking requirement (4.20) for all the shapes. Further, this provides for low frequency disturbance rejection, as seen in the sensitivity response function, Fig. 6.30, for all four spatial shapes. However, the recovered loop is more directional than the target loop, especially at zero frequency and the resonance frequencies: note the spread in the singular value curves. This difference can be attributed to the nominal plant’s errors in quantifying the actuator spatial influence functions, as discussed in the system identification Section. The closed-loop bandwidth of approximately 1Hz borders on the stability-robustness constraint plotted in Fig. 6.25; the design is nonetheless stable.

Fig. 6.29: Forward loop singular value response for the target and recovered loops - design MBC1er (nominal plant model).
Fig. 6.30: Sensitivity function singular value response for the target and recovered loops - design MBC1er (nominal plant model).

Fig. 6.31: Closed-loop singular value response for the target and recovered loops - design MBC1er (nominal plant model).
The poles and zeroes of MBC1er's model-based compensator are listed in Table 6.5. Aside from the integrator poles, the compensator has some "fast" poles; fast in that they are approaching the Nyquist frequency of the 150Hz state propagation rate. These poles become "faster" as the recovery parameter $\rho$ is decreased, going to $-\infty$ along stable Butterworth patterns as $\rho \to 0$ [6.6]. The zeroes of the MBC correspond to the poles of the nominal plant model. The compensator is trying to invert the plant in a stable fashion, cancelling the poles of the plant (both magnitudes and directions, as this is a matrix plant model) with the compensator zeroes, and substituting its own dynamics. This pole/zero structure is seen in the singular values of the compensator, plotted in Fig. 6.32 versus frequency.

<table>
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<th>Poles (Hz)</th>
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<td>- 0.1894 ± 38.8022i</td>
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<td>-15.63 + 00.0i</td>
<td>- 0.2324 ± 51.7884i</td>
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<tr>
<td>- 7.3211 ± 53.4533i</td>
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</table>

The simulated and experimental MBC1er [-4 0 0 0]^T step responses for shape 1 are plotted in Fig.s 6.33 and 6.34, respectively. The experiment consisted of commanding the beam to assume the first sinusoidal shape, and measuring the response for 20 seconds, using a 150Hz state propagation rate. The integrator and compensator states, as well as the command inputs (e.g. the D/A output voltages), were zeroed before this run, and all ensuing runs. The control program saved the shape-transformed displacement data at every time step. The plots are essentially real-time spectral decompositions of the beam's profile in the shape functions, where each curve corresponds to the amplitude of one of the Fourier components.
The corresponding control voltages appear in Figs. 6.35 and 6.36. The control voltage to actuators 3 and 4 reach their steady-state value at a slower rate than the controls to actuators 1 and 2. This reflects once again the errors in quantifying the actuator shape influence functions in the nominal plant model. Actuators 3 and 4 were the most poorly quantified, as seen in the plots of the input coupling operators, Figs. 6.18 and 6.19.

The "wiggles" in the experimental Fourier coefficient curves are due to exogenous vibration disturbances from the laboratory air handling system; the compensator is not causing this response, as the control voltages (Fig. 6.36) do not show the fast variations that would be required to drive these modes. These inputs, which lie outside the frequency bandwidth where the forward loop has high gain (see Fig. 6.30), slightly excite the beam's resonant response. The experimental data was obtained late at night (between 1AM and 5AM), when the disturbances were smallest; the lab air handling system was at its lowest power level during this time, and the nearby subway line was shut down for the night. This forced resonant response was often five to ten times as large during the daytime. The periodicity of each curve's resonant response corresponds to the associated shape's natural frequency; the shape functions and the beam eigenfunctions are equal. An inner-loop damping formulation will be developed in the sequel to alleviate the resonant response.
Fig. 6.33: MBC1er shape 1 [-4 0 0 0]^T step response: simulation.

Fig. 6.34: MBC1er shape 1 [-4 0 0 0]^T step response: experiment.
Fig. 6.35: MBC1er shape 1 [-4 0 0 0]^T step response control voltages: simulation.

Fig. 6.36: MBC1er shape 1 [-4 0 0 0]^T step response control voltages: experiment.
Note how the second shape component is briefly excited during the shape 1 step response test, but is essentially nulled out after about 1 second, the settling time of the controller. This coupling between the shapes is also seen in the shape 2 $[0 \ 3 \ 0 \ 0]^T$ step response simulation and experiment, plotted in Fig.s 6.37 and 6.38. While the second shape component achieves its set point amplitude in about 1 second, the first shape is also excited briefly, but is later nulled out. This coupling between the shapes during the beam's transient response limits the beam's tracking ability. However, while the coupling is present, it is nonetheless small, with the system achieving its setpoint in the 1 second settling time; many designers of deformable mirrors, for example, would be satisfied with this level of decoupling performance.

The simulated and experimental control voltages for the shape 2 step response test are plotted in Fig.s 6.39 and 6.40, respectively. Note once again the slower time response of actuators 3 and 4, attributable to the deficiencies of the nominal plant model.

The shape 3 $[0 \ 0 \ -2 \ 0]^T$ step response simulation and experimental data are plotted in Fig.s 6.41 and 6.42, with the corresponding control voltages plotted in Fig.s 6.43 and 6.44. Once again, the first shape couples briefly to the shape transient response, as well as a bit of the second sinusoidal shape, with the set point being achieved in about 1 second. The shape 4 $[0 \ 0 \ 0 \ 3]^T$ step response simulation and experimental data are plotted in Fig.s 6.45 and 6.46, respectively, with the corresponding control voltages plotted in Fig.s 6.47 and 6.48. These plots show less shape coupling during the shape 4 step response than for the other step responses, along with the usual exogeneously-excited resonant response. Further, one again notes the slower time response characteristic of actuators 3 and 4 in Fig. 6.48.
Fig. 6.37: MBC1er shape 2 $[0 \ 3 \ 0 \ 0]^T$ step response: simulation.

Fig. 6.38: MBC1er shape 2 $[0 \ 3 \ 0 \ 0]^T$ step response: experiment.
Fig. 6.39: MBC1er shape 2 $[0 \ 3 \ 0 \ 0]^T$ step response control voltages: simulation.

Fig. 6.40: MBC1er shape 2 $[0 \ 3 \ 0 \ 0]^T$ step response control voltages: experiment.
Fig. 6.41: MBC1er shape 3 \([0 \ 0 \ -2 \ 0]^T\) step response: simulation.

Fig. 6.42: MBC1er shape 3 \([0 \ 0 \ -2 \ 0]^T\) step response: experiment.
Fig. 6.43: MBC1er shape 3 $[0 \ 0 \ -2 \ 0]^T$ step response control voltages: simulation.

Fig. 6.44: MBC1er shape 3 $[0 \ 0 \ -2 \ 0]^T$ step response control voltages: experiment.
Fig. 6.45: MBC1er shape 4 $[0 \ 0 \ 0 \ 3]^T$ step response: simulation.

Fig. 6.46: MBC1er shape 4 $[0 \ 0 \ 0 \ 3]^T$ step response: experiment.
Fig. 6.47: MBC1er shape 4 $[0 \ 0 \ 0 \ 3]^T$ step response control voltages: simulation.

Fig. 6.48: MBC1er shape 4 $[0 \ 0 \ 0 \ 3]^T$ step response control voltages: experiment.
The final test of the nominal plant compensator design was a sequenced step response to each of the four sinusoidal shapes. The simulated and experimental responses for this test are plotted in Figs. 6.49 and 6.50, respectively, with the corresponding control voltage time histories appearing in Figs. 6.51 and 6.52. The system is first driven to a shape 1 set point, then to a shape 2 set point, a shape 3 set point, and finally a shape 4 set point. This test is representative of the tasks required of a mirror shape control system, especially one employed in an optical correlation system. Note that the time scale is five times as long as for the single shape step response plots, hence the superimposed resonances are "squeezed" a bit. Since the system has achieved its design goal of attaining any of the four sinusoidal shapes, it can attain any superposition. Such a control design is ideal for applications such as optical wavefront correction.

The shape sequence step response test reveals the coupling among the shapes during transient response. Nonetheless, the system is performing admirably. Also, it shows a high degree of fidelity between the simulation model of the experimental plant and the true experimental hardware. This suggests that, using a better plant model, a compensator can be designed to eliminate the shape coupling phenomena; this will be examined in the next compensator design. Note that coupling in the present context is coupling between the various shape components, rather than the spatial coupling often associated with mirror figure control systems, where one divides the distributed plant into various spatial regions, or "pixels", and desires to decouple these regions. Shape decoupling is a more natural application for distributed parameter systems; the idea (for example) of a mirror having pixels is merely an imposition of lumped parameter concepts onto a distributed plant.
Fig. 6.49: MBC1er shape sequence step response: simulation.

Fig. 6.50: MBC1er shape sequence step response: experiment.
Fig. 6.51: MBC1er shape sequence step response control voltages: simulation.

Fig. 6.52: MBC1er shape sequence step response control voltages: experiment.
Compensator MBC1a was designed for the experimental plant model, using the LTR design parameters listed in Table 6.4. The singular values of the forward loop, sensitivity, and closed-loop response matrices are plotted in Figs. 6.53 through 6.55, respectively, for the target and recovered loops. As in design MBC1er, the augmenting integrators have provided sufficient gain at low frequencies to satisfy the tracking requirement (4.20) for all four shapes. Unlike the previous design (MBC1er), the recovered and loop has the same non-directional response as the target filter up to the recovery bandwidth limit. The present compensator is cascaded with the plant for which it was designed, leading to a more desirable frequency response characteristic. The elimination of the directional singular value characteristics that degraded the nominal plant compensator design will lead to a more decoupled shape control response, as well as a more balanced control voltage transient response. The closed-loop bandwidth of approximately 1Hz easily satisfies the stability/robustness constraint plotted in Fig. 6.24 for the experimental plant model, guaranteeing a robustly stable (albeit conservative) design.

Fig. 6.53: Forward loop singular value response for the target and recovered loops – design MBC1a (experimental plant model).
Fig. 6.54: Sensitivity function singular value response for the target and recovered loops – design MBC1a (experimental plant model).

Fig. 6.55: Closed-loop singular value response function for the target and recovered loops – design MBC1a (experimental plant model).
The poles and zeroes of design MBC1a's model-based compensator are listed in Table 6.6. This pole/zero structure is seen in the singular values of the compensator, plotted in Fig. 6.56 versus frequency. The compensator is essentially similar to the nominal plant design, save for the difference in the compensator's directionality and gains at very low frequencies. This variation is due to the difference in the actuator static influence functions for the two plant models. The transmission zeroes are identical, as the poles of the two plant models are identical.

<table>
<thead>
<tr>
<th>Table 6.6: Compensator MBC1a Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poles (Hz)</td>
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<tr>
<td>------------</td>
</tr>
<tr>
<td>-9.1275 + 00.0i</td>
</tr>
<tr>
<td>-10.9530 + 00.0i</td>
</tr>
<tr>
<td>-12.9202 + 00.0i</td>
</tr>
<tr>
<td>-13.7574 + 00.0i</td>
</tr>
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</tr>
<tr>
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</tr>
<tr>
<td>-5.3603 ± 40.0566i</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

Fig. 6.56: Compensator MBC1a singular value response function.
The simulated and experimental MBC1a shape 1 \([3 \ 0 \ 0 \ 0]^T\) step responses are plotted in Figs. 6.57 and 6.58, respectively. As for the previous compensator tests, the experiment consisted of driving the beam to its first sinusoidal shape, and measuring the response for 20 seconds, using a 150Hz state propagation rate. The integrator and compensator states, as well as the control inputs, were zeroed before the run, and all succeeding runs. The shape is achieved in about 1 second, with a superimposed resonant response driven by external vibration sources as before. Note, however, that the shapes are not coupled during the transient response. The improved model of the actuator influence functions has led to a design that decouples the component shapes. This result has significant implications for shape control applications such as optical wavefront correction.

The corresponding control voltages appear in Figs. 6.59 and 6.60. The control signals to actuators 3 and 4 now achieve their steady-state value at the same rate as the other controls, reflecting the improved modelling of the actuators in this design.

The simulated and experimental MBC1a step responses for shapes 2 through 3, along with their corresponding control voltages, are shown in Figures 6.61 through 72. These tests all reinforce the shape decoupling property of the compensator design; the shapes are not coupled during the transient response, and the control signals are more "balanced" in their transient behavior. This can be best appreciated by comparing these results with the corresponding transients for design MBC1er. The combination of spatial and temporal compensation has satisfied the tracking response performance goals of the experiment.

The final test of this experimental plant compensator design was a sequenced step response to each of the four sinusoids. The simulated and experimental responses for this test are plotted in Figs. 6.73 and 6.74, respectively, with the corresponding control voltage time histories appearing in Figs. 6.75 and 6.76. Note that the time scale in these plots is five times as long as for the single shape step response plots, hence the forced resonant response is "squeezed". This sequenced step response test again shows that the experimental plant compensator design decouples the shapes during the beam's transient response. The only desirable improvement would be a faster transient response, and elimination of the forced resonant vibrations; both of these issues will be addressed in subsequent designs.
Fig. 6.57: MBC1a shape 1 $[3 \ 0 \ 0 \ 0]^T$ step response: simulation.

Fig. 6.58: MBC1a shape 1 $[3 \ 0 \ 0 \ 0]^T$ step response: experiment.
Fig. 6.59: MBC1a shape 1 $[3 \ 0 \ 0 \ 0]^T$ step response control voltages: simulation.

Fig. 6.60: MBC1a shape 1 $[3 \ 0 \ 0 \ 0]^T$ step response control voltages: experiment.
Fig. 6.61: MBC1a shape 2 \([0 \, 3 \, 0 \, 0]^T\) step response: simulation.

Fig. 6.62: MBC1a shape 2 \([0 \, 3 \, 0 \, 0]^T\) step response: experiment.
Fig. 6.63: MBC1a shape 2 \([0 \ 3 \ 0 \ 0]^T\) step response control voltages: simulation.

Fig. 6.64: MBC1a shape 2 \([0 \ 3 \ 0 \ 0]^T\) step response control voltages: experiment.
Fig. 6.65: MBC1a shape 3 \([0 \ 0 \ -2 \ 0]^T\) step response: simulation.

Fig. 6.66: MBC1a shape 3 \([0 \ 0 \ -2 \ 0]^T\) step response: experiment.
Fig. 6.67: MBC1a shape 3 $[0 \ 0 \ -2 \ 0]^T$ step response control voltages: simulation.

Fig. 6.68: MBC1a shape 3 $[0 \ 0 \ -2 \ 0]^T$ step response control voltages: experiment.
Fig. 6.69: MBC1a shape 4 \([0 \ 0 \ 0 \ -2]^T\) step response: simulation.

Fig. 6.70: MBC1a shape 4 \([0 \ 0 \ 0 \ -2]^T\) step response: experiment.
Fig. 6.71: MBC1a shape 4 $[0 \ 0 \ 0 \ -2]^T$ step response control voltages: simulation.

Fig. 6.72: MBC1a shape 4 $[0 \ 0 \ 0 \ -2]^T$ step response control voltages: experiment.
Fig. 6.73: MBC1a shape sequence step response: simulation.

Fig. 6.74: MBC1a shape sequence step response: experiment.
Fig. 6.75: MBC1a shape sequence step response control voltages: simulation.

Fig. 6.76: MBC1a shape sequence step response control voltages: experiment.
Design MBC1

Compensator MBC1 was constructed to evaluate compensator order reduction. A reduced-order design is required to accommodate the faster state propagation rate of 200Hz, as shown in Table 6.1. This compensator is a reduced-order, eight-state version of design MBC1a; simulations showed that reducing the order further tended to destabilize the closed-loop system. The order reduction utilized a residue expansion method [6.7]. Other compensator order reduction schemes were investigated [6.8], but all tended to yield unstable closed-loop designs when the order was reduced by more than two states (one resonant mode). Design MBC1 was constructed to determine the maximum bandwidth of the control computer, and as a first step in implementing the inner-loop damping scheme developed in the sequel. The singular values of the forward loop, sensitivity, and closed-loop response functions are plotted in Figs. 6.77 through 6.79, respectively, for the target and recovered loops. The recovered loop is the reduced-order MBC cascaded with the experimental plant model, which shows excellent loop transfer recovery up to the second plant resonance. The succeeding resonances are not cancelled by the lower-order compensator, and hence are

![Graph](image_url)

Fig. 6.77: Forward loop singular value response for the target and recovered loops – design MBC1.
Fig. 6.78: Sensitivity function singular value response for the target and recovered loops – design MBC1.

Fig. 6.79: Closed-loop singular value response for the target and recovered loops – design MBC1.
uncontrolled. The closed-loop system is nevertheless stable, and the transient response simulations showed no noticeable degradation. Doubtless this is attributable to the comparatively slow 1Hz forward loop crossover frequency.

The poles and zeroes of MBC1's model-based compensator are listed in Table 6.7. This pole/zero structure is seen in the singular values of the compensator matrix, plotted in

**Table 6.7: Compensator MBC1 Dynamics**

<table>
<thead>
<tr>
<th>Poles (Hz)</th>
<th>Zeros (Hz)</th>
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</thead>
<tbody>
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<td>-0.0637 ± 12.9192i</td>
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<td>-10.9530 + 00.0i</td>
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<td>-12.9202 + 00.0i</td>
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<td>-13.7574 + 00.0i</td>
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<tr>
<td>-5.5975 ± 27.7965i</td>
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</table>

![Graph](image)

**Fig. 6.80: Compensator MBC1 singular value response function.**
Fig. 6.80 versus frequency. The compensator poles are identical to the experimental plant design, except there are four fewer. The transmission zeroes are also identical, except once again there are four fewer due to the model reduction.

The simulated and experimental MBC1 step responses were essentially identical to those of the full-order design MBC1a. As an example, the simulated and experimental MBC1 shape 1 \([4 \ 0 \ 0 \ 0]^T\) step responses are plotted in Fig.s 6.81 and 6.82, respectively. Once again, the experiment consisted of commanding the beam to assume the first sinusoidal shape, and measuring the response for 20 seconds, but now using a 200\(Hz\) state propagation rate. The integrator and compensator states, as well as the command inputs, were zeroed before this run, and all ensuing runs. The beam achieves its first shape in the about 1 second, with a small amount of externally-excited resonant vibration superimposed. The control voltages, plotted in Fig.s 6.83 and 6.84 show the balanced transient response among the channels found in the full-order compensator tests. Essentially, the reduced-order compensator is performing as well as the full-order version, except with four fewer states and a faster state propagation rate. This is not surprising, since the poles eliminated in the order reduction are an order of magnitude faster than the control bandwidth.

As a final example of the reduced-order compensator's transient performance, the step responses to a sequence of shape command step inputs are plotted in Fig.s 6.85 and 6.86; these correspond to the simulated and experimental data, respectively. The corresponding control voltages are plotted in Fig.s 6.87 and 6.88. Note that the time scale is five times longer than for the single shape step response experiments, hence the superimposed resonant response appears to be denser. The design shows no coupling between the shapes during the test. Also, the simulation demonstrates a high degree of fidelity with the experimental data. Consequently, these tests show that one can safely reduce the compensator order significantly, and suffer no perceptible performance degradation. This result will be exploited in the sequel.
Fig. 6.81: MBC1 shape 1 $[4 \ 0 \ 0 \ 0]^T$ step response: simulation.

Fig. 6.82: MBC1 shape 1 $[4 \ 0 \ 0 \ 0]^T$ step response: experiment.
Fig. 6.83: MBC1 shape 1 $[4 \ 0 \ 0 \ 0]^T$ step response control voltages: simulation.

Fig. 6.84: MBC1 shape 1 $[4 \ 0 \ 0 \ 0]^T$ step response control voltages: experiment.
Fig. 6.85: MBC1 shape sequence step response: simulation.

Fig. 6.86: MBC1 shape sequence step response: experiment.
Fig. 6.87: MBC1 shape sequence step response control voltages: simulation.

Fig. 6.88: MBC1 shape sequence step response control voltages: experiment.
Design MBC2a

Compensator MBC2a was designed for the experimental plant model using a "faster" target loop design parameter, \( \mu = .006 \), to develop a faster compensator consistent with the stability-robustness constraint. The system has a 2Hz forward loop crossover frequency. The singular values of the forward loop, sensitivity, and closed-loop response functions are plotted in Fig.s 6.89 through 6.91, respectively, for the target and recovered loops. While the augmenting integrators provide ample forward loop gain below the crossover frequency, the return difference plot 6.90 shows that the design is nonetheless incapable of rejecting disturbances in the frequency range of the plant resonances. The recovered loop matches the target loop's response quite well up to a decade beyond crossover using the design recovery parameter \( \rho = .00002 \). Since the spread between the minimum and maximum singular values is minimal up to this point, where the closed-loop system has begun to roll off, the design shows no preferred directional response, hence all four shapes are equally achievable. This suggests that, like the other experimental plant-based designs MBC1a and MBC1, the shape step response transients should show no coupling, and the control transients should be well balanced.

Fig. 6.89: Forward loop singular value response for the target and recovered loops – design MBC2a.
Fig. 6.90: Sensitivity function singular value response for the target and recovered loops – design MBC2a.

Fig. 6.91: Closed-loop singular value response for the target and recovered loops – design MBC2a.
The poles and zeroes of MBC2a's model-based compensator are listed in Table 6.8. The compensator poles are faster than for the other LTR designs; this is a faster design. The fastest pole can still be accommodated by a 150Hz state propagation rate, although it is close to being undersampled. The zeroes of the compensator are the same as before: the poles of the beam, as the design is attempting to stably invert the plant. This pole/zero structure is

<table>
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<th>Zeroes (Hz)</th>
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</table>

Fig. 6.92: Compensator MBC2a singular value response function.
seen in the singular values of the compensator, plotted in Fig. 6.92 versus frequency. An order reduction simulation was undertaken for this design. However, the shape 3 and 4 step response simulations showed that the 3rd and 4th modes were excited during these tests. The residues corresponding to these compensator modes were larger than for the $1Hz$ full-order compensator, reflecting their greater importance for the faster design. Consequently, no reduced-order compensators were tested experimentally.

The simulated and experimental MBC2a step responses were essentially identical to those of the other experimental plant model LTR designs, except the settling time was reduced to 0.5 seconds. As an example, the simulated and experimental MBC1 shape 2 $[0 3 0 0]T$ step responses are plotted in Figs. 6.93 and 6.94, respectively. The experiment consisted of commanding the beam to assume the second sinusoidal shape, and measuring the response for 20 seconds, using a $150Hz$ state propagation rate. The integrator and compensator states, as well as the command inputs, were zeroed before this run, and all ensuing runs. The beam achieves its second shape in the about 0.5 seconds, with a small amount of externally-excited resonant vibration superimposed. The control voltages, plotted in Figs. 6.95 and 6.96 show the balanced transient response among the channels found in the MBC1a full-order compensator tests. There is no noticeable coupling among the shapes, hence the faster compensator is a successful shape decoupling design.

As a final example of the compensator's transient performance, the step responses to a sequence of shape command step inputs are plotted in Figs 6.97 and 6.98; these correspond to the simulated and experimental data, respectively. The corresponding control voltages are plotted in Figs 6.99 and 6.100. Note that the time scale is two and a half times longer than for the single shape step response experiments. These tests also show that the compensator has successfully decoupled the shapes, and at a faster bandwidth to boot.

In the next Section, an inner-loop damping scheme will be developed to control the exogenously-excited resonant response, thus augmenting the capabilities of the shape control compensators.
Fig. 6.93: MBC2a shape $2 \begin{bmatrix} 0 & 3 & 0 & 0 \end{bmatrix}^T$ step response: simulation.

Fig. 6.94: MBC2a shape $2 \begin{bmatrix} 0 & 3 & 0 & 0 \end{bmatrix}^T$ step response: experiment.
Fig. 6.95: MBC2a shape 2 \([0 \ 3 \ 0 \ 0]^T\) step response control voltages: simulation.

Fig. 6.96: MBC2a shape 2 \([0 \ 3 \ 0 \ 0]^T\) step response control voltages: experiment.
Fig. 6.97: MBC2a shape sequence step response: simulation.

Fig. 6.98: MBC2a shape sequence step response: experiment.
Fig. 6.99: MBC2a shape sequence step response control voltages: simulation.

Fig. 6.100: MBC2a shape sequence step response control voltages: experiment.
VI.8 INNER-LOOP DAMPING FORMULATION

The forced resonant response seen in the LTR compensator designs is problematic. Although the experimental control implementation has satisfied the performance goals established at the outset of the Chapter, the resonant response detracts from the DYSC experiment's achievement, and, were this were more than a proof of concept experiment, might exceed system vibration requirements as well. The problem is exacerbated by the conflict between model order and achievable state propagation rate. The sensitivity response functions for the four designs show the LTR compensation incapable of rejecting exogenous disturbances beyond the designs' forward loop crossover frequency because of insufficient loop gain. But this bandwidth cannot be pushed higher because of the stability-robustness constraints. The stability-robustness constraint can be extended to higher frequencies by increasing the order of the beam model, e.g. including more shape components (modes). But the resulting LTR compensators would also have to increase in size; The MicroVAX II is constrained by its computational speed of 1 MIP (million instructions per second), so a larger compensator would necessitate slower state propagation rates, and hence these faster compensators could not be implemented. An alternative approach is to further augment the plant inputs to boost the forward loop gain beyond that provided by the integrators, much as Kalman filter designs for band-limited noise processes are augmented. However, this approach would also increase the compensator order by four to eight states, which was deemed unacceptable.

One potential solution is to exploit the results of Chapters I and II, and design in inner-loop "Lyapunov" damper for the beam to control the resonant response. An outer-loop LTR compensator could then be wrapped around this augmented plant to effect the shape control task. Recall how the Lyapunov analysis showed that a linearly-varying film control distribution is capable of damping all modes of a pinned-pinned beam, such as the DYSC system.

The DYSC experimental beam is equipped with four equi-spaced film actuators with rectangular apertures, so the Lyapunov analysis is not immediately applicable. However, one need only recall the physical implications of the Lyapunov analysis, especially the results for colocated sensor and actuator distributions derived in Appendix C, to arrive at an implementable solution. Given an actuator distribution capable of driving the first mode of
the beam, the output from a transducer that senses the beam response with the same spatial distribution can be fed back to damp that mode. Since the DYSC experiment's actuators are capable of driving the four sinusoidal shapes, which for the present system are equal to the first four eigenfunctions, the velocities of these shapes can be fed back to damp the first four vibrational modes. The actuator distribution has a spatial transform decomposition in each of the shape functions, with corresponding Fourier coefficients, or "weights".

A derivation of the necessary feedback relations follows. Consider the state space representation of the plant, equation (6.15), including an inner-loop feedback gain matrix $G_i$:

$$x_p = (A_p - B_p G_i)x_p + B_p u_p.$$  \hspace{1cm} 6.48

An LQR design problem could be formulated to construct the inner-loop gain matrix. However, the resulting compensator would be of eighth order, and would no doubt contain fast dynamics (fast with respect to the state propagation rate) to damp the resonant poles of the plant. An alternative procedure is to exploit the structure of the system matrices. First, construct a truncated $B_p$ matrix,

$$\tilde{B}_p = \frac{m}{\rho A} \begin{bmatrix} q_{11} & q_{21} & q_{31} & q_{41} \\ \vdots & \vdots & \vdots & \vdots \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix}.$$  \hspace{1cm} 6.49

Then, an intermediate matrix is defined by

$$\tilde{G}_i = \tilde{B}_p^{-1}.$$  \hspace{1cm} 6.50

The inner loop gain matrix is then

$$G_i = k_i \begin{bmatrix} 0 & \uparrow & \cdots & 0 & \uparrow \\ 0 & \tilde{G}_i(:,1) & \cdots & 0 & \tilde{G}_i(:,4) \\ 0 & 0 & 0 & 0 & \downarrow \\ 0 & \downarrow & \cdots & 0 & \downarrow \end{bmatrix}.$$  \hspace{1cm} 6.51
where \( k_i \) is a positive semi-definite constant; the nomenclature \( (:,j) \) refers to the \( j \)-th column of the intermediate matrix. The inner-loop gain matrix \( G_i \) has the unique property that

\[
B_p G_i = k_i
\]

Consequently, this state feedback formulation will damp the velocity components of the system response in a decoupled fashion, using a separate velocity feedback channel for each mode.

This feedback gain choice has a simple interpretation in terms of the Lyapunov analysis. One can readily show that

\[
G_i(:,j) \propto G_p^{-1} e_j
\]

that is, the \( 2j \)-th column (a non-zero column) of the inner-loop feedback gain matrix is proportional to the static actuator inputs required to achieve the \( j \)-th shape! The corresponding feedback parameter is the estimate of the \( j \)-th shape's velocity: this is a collocation of actuator and sensor distributions in the shape space.

Fortunately, the Fourier coefficients of the sinusoidal shapes are already calculated in the DYSC control implementation, and the plant \( B_p \) matrix is known. Unfortunately, the feedback scheme requires a differentiator for each mode to be damped. While this is an implementation issue common to the Lyapunov controllers, expediency ruled out the construction of an analog inner-loop damping circuit. Given the availability of the shape coefficient estimates, a digital differentiator was designed instead. The differentiator was implemented with a fast pole, since the state space representation of a plant must be proper, leading to a differentiator response function of the form
\[ d(s) = \frac{\omega_d s}{s + \omega_d}. \]

The MBC2a tests showed that a state propagation rate of 200Hz was feasible for the LTR compensator, hence \( \omega_d \) was chosen to equal 565.4867 \( \text{rad/second} \) (90Hz) to give the best phase characteristic at the first mode frequency. This necessitated the inner-loop damping implementation be reduced in scope to only the first resonant mode of the beam, as the phase characteristic at its resonant frequency was only 78 degrees, rather than the ideal 90 degrees. This was judged sufficient to demonstrate the concept. For the purposes of design, the inner-loop gain matrix for the first mode takes the form

\[
G_i = k_i \begin{bmatrix}
0 & \uparrow & 0 & \ldots & 0 \\
0 & \tilde{G}_i(:,1) & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & \downarrow & 0 & \ldots & 0
\end{bmatrix}
\]

Two compensator designs were undertaken, for values of the inner-loop gain parameter \( k_i = 3 \) and 5. The plant was first augmented with the single-mode inner-loop damper. Then, an outer-loop LTR compensator was designed in the usual fashion, including the augmenting integrators; both designs had a target LTR bandwidth design parameter \( \mu = .025 \), and a recovery parameter \( \rho = .0001 \), leading to a 1Hz forward loop crossover frequency. Finally, the LTR compensators' order was reduced to eight states using the residue expansion method, as per design MBC1r, to accommodate the 200Hz state propagation rate necessitated by the digital differentiator. Simulations demonstrated that this reduced-order compensator did not destabilize the plant, and still satisfied the stability/robustness and performance requirements.

The poles and zeroes of the first design, MBC8Ar, are listed in Table 6.9. This pole/zero structure is seen in the singular values of the compensator, plotted in Fig. 6.101 versus frequency. Note that the compensator zero associated with the first plant mode is less deep than for the four LTR designs, reflecting the action of the inner-loop damper.
### Table 6.8: Compensator MBC8Ar Dynamics

<table>
<thead>
<tr>
<th>Poles (Hz)</th>
<th>Zeroes (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9.1292</td>
<td>-0.3043 ± 12.9192i</td>
</tr>
<tr>
<td>-10.9541</td>
<td>-0.4601 ± 25.8494i</td>
</tr>
<tr>
<td>-12.9198</td>
<td>3.35x10^8 + 00.0i</td>
</tr>
<tr>
<td>-13.7581</td>
<td>-2.18x10^5 + 00.0i</td>
</tr>
<tr>
<td>-4.9613</td>
<td>2.79x10^2 + 00.0i</td>
</tr>
<tr>
<td>-5.5971</td>
<td>1.34x10^2 + 00.0i</td>
</tr>
</tbody>
</table>

---

**Fig. 6.101:** Compensator MBC8Ar singular value response function.
The experimental shape 1 $[4 \ 0 \ 0 \ 0]^T$ step response using this compensator is plotted in Fig.s 6.102 and 6.103; the corresponding control voltages are plotted in Fig. 6.104. The "wiggles" in shape 1's Fourier coefficient due to the exogenous disturbances are significantly reduced in amplitude. The inner-loop damping is actively controlling this mode, as evidenced by the perturbations in the control voltages; compare the control voltages plotted in Fig. 6.104 to those in Fig. 6.36. The other shape component resonant responses are undamped, and of higher amplitude than the first mode component. These tests show the efficacy of the inner-loop Lyapunov damper – LQG/LTR outer shape control loop concept in shape and vibration control.

![Fourier Coefficients](image)

Fig. 6.102: MBC8Ar shape 1 $[4 \ 0 \ 0 \ 0]^T$ experimental step response.
Fig. 6.103: MBR8Ar shape 1(4,0,0)T experimental step response components.
The second compensator design, MBC8Br, with an inner-loop damping gain $k_i = 5$, led to even greater first mode control. The poles and zeroes of MBC8Br are listed in Table 6.10. This pole/zero structure is seen in the singular values of the compensator, plotted in Fig. 6.105. Note that the compensator zero associated with the first plant mode is less deep than for the four LTR designs, as well as the first inner-loop compensator MBC8Ar, reflecting the action of this higher-gain inner-loop damper.

<table>
<thead>
<tr>
<th>Poles (Hz)</th>
<th>Zeroes (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9.1302 + 00.0i</td>
<td>0.4634 ± 12.9110i</td>
</tr>
<tr>
<td>-10.9550 + 00.0i</td>
<td>0.4601 ± 25.8370i</td>
</tr>
<tr>
<td>-12.9202 + 00.0i</td>
<td>3.38×10^8 + 00.0i</td>
</tr>
<tr>
<td>-13.7581 + 00.0i</td>
<td>2.18×10^5 + 00.0i</td>
</tr>
<tr>
<td>-4.9699 ± 15.8020i</td>
<td>2.79×10^2 + 00.0i</td>
</tr>
<tr>
<td>-5.5970 ± 27.7966i</td>
<td>1.34×10^2 + 00.0i</td>
</tr>
</tbody>
</table>
The experimental shape 1 $[-3 \, 0 \, 0 \, 0]^T$ step response using this compensator is plotted in Figs. 6.106 and 6.107; the corresponding control voltages are plotted in Fig. 6.108. The "wiggles" in shape 1's Fourier coefficient due to the exogenous disturbances are further reduced in amplitude due to the higher inner-loop gain, compared to the MBC8Ar tests. The inner-loop damping is successfully controlling the first mode, as evidenced by the larger perturbations in the control voltages. The other shape component resonant responses are undamped, and of higher amplitude than the first mode component. These tests reinforce the efficacy of the inner-loop Lyapunov damper – LQG/LTR outer shape control loop concept in shape and vibration control.

A further inner-loop compensator design was developed using an inner-loop gain $k_i = 10$. However, this compensator drove the beam system unstable during testing; note that, while the outer loop is guaranteed a margin of stability because it satisfies the stability/robustness constraint, the inner-loop damper has no such guarantees. The digital differentiation, combined with the plant $B_p$ matrix, which is assumed to be perfectly known
for the inner-loop feedback, combine to perturb the experimentally-implemented system from the simulated design plant. Nonetheless, the implementation for the DYSC experiment was undertaken as a proof of concept demonstration of a combined LQG/LTR shape control – Lyapunov damping concept: In this light, the experiment was a rousing success. Future implementations will include a higher design fidelity, as well as an analog implementation of the inner loop.

Analytical extensions and implications of the inner-loop damping are studied in detail in Chapter VII.

Fig. 6.106: MBC8Br shape 1 \([-3 \; 0 \; 0 \; 0]\)^T experimental step response.
Fig. 6.107: MBC/Ar shape 1 (300.0) [experimental step response components.]

Shape Coef. 1

Shape Coef. 2

Shape Coef. 3

Shape Coef. 4
Fig. 6.108: MBC8Br shape 1 [-3 0 0 0]^T experimental step response control voltages.
VI.9 QUASI-STATIC DISTURBANCE REJECTION

The inner-loop damping formulation provided the means for successfully dealing with the forced resonant disturbance response of the DYSC beam. The robustness of the nominal plant compensator design MBC1er demonstrated a form of static disturbance rejection by successfully achieving the shape control task in the presence of static model errors. Nonetheless, one further performance requirement had to be demonstrated: quasi-static disturbance rejection. Quasi-static disturbance rejection performance will be most important in applications such as the structural shape control of large-aperture space-based radar, whose nominal profile must be maintained in the presence of quasi-static thermal disturbances due to nonuniform solar heating, which will tend to warp the structure.

Qualitative quasi-static disturbances can be introduced in the DYSC experiment by opening the acoustic enclosure's cover, and exposing the beam to air currents within the lab. The initial tests of compensator design MBC2a (the "2Hz" LTR compensator design) were inadvertently conducted with this cover removed. Fig.s 6.109 and 6.110 show the Fourier coefficients of the beam's shape $1 [-2 0 0 0]^T$ response and the corresponding control voltages for such a test. Save for the resonant response, the compensator is successfully driving the beam to its first shape component. However, the control voltage plot shows that the compensator is working to damp out slow disturbances that are quasi-statically exciting the beam.

A known disturbance excitation experiment was devised to more precisely assess the DYSC compensators' quasi-static disturbance rejection ability. The experiment utilized a magnetic vibration exciter, or solenoid, to drive the beam. Fortunately, the DYSC beam was made from 904 stainless steel, which is ferromagnetic. The solenoid placement was chosen so that it excited as much of each shape component as possible. A final placement of $x_d = 15\sin x$ was chosen based upon a calculation of the solenoid's discrete spatial filter function $q_n$, assuming its spatial distribution can be approximated as a delta function:

$$q_n \propto \int_0^\infty \sin\left(\frac{nx_d}{40}\right) \delta(x - x_d) dx = \begin{cases} 
.9239, & n = 1 \\
.7071, & n = 2 \\
-.3827, & n = 3 \\
-1.000, & n = 4 
\end{cases}$$
Fig. 6.109: MBC2a shape 1 [-2 0 0 0]^T experimental step response with enclosure cover removed.

Fig. 6.110: MBC2a shape 1 [-2 0 0 0]^T experimental step response control voltages with enclosure cover removed.
The solenoid was placed at the beam midline atop an aluminum block, as shown in Fig. 6.111. The exciter was spaced approximately .125\textit{in} from the beam. The solenoid was first driven statically with a Hewlett-Packard power supply to determine the approximate amplitude of the beam's response to a given excitation voltage. Next, an open-loop, uncontrolled baseline experiment was conducted in the presence of a 1\text{rad/sec} disturbance provided by the solenoid, driven by a Wavetek function generator. This frequency was chosen to lie within the bandwidth where the LTR compensator is capable of rejecting disturbances. Then, a series of shape control step response experiments was undertaken in the presence of the same disturbance excitation. All tests were conducted with the compensator MBC8Ar, the inner-loop augmented LTR compensator with an inner-loop gain $k_i = 3$, using the EPascal application DYSC_CL3.

The uncontrolled open-loop response of the beam to the quasi-static disturbance is plotted in Fig. 6.112. Note that the time scale extends for 25 seconds duration, and that the vertical scale is smaller than that shown for the step response tests. The plot shows that the disturbance excitation couples best to the first sinusoidal shape component, with the remaining components being excited at a lower amplitude. This demonstrates that the exciter
Fig. 6.1.2: Open-loop uncontrolled response components to exciter-driven disturbance.
is only crudely modelled as a point excitation source, and that the beam's strain/curvature relation is spatially filtering the beam response (as discussed in Chapter IV). Note also the amplitude of the exogenously-driven resonant response superimposed upon the quasi-static response.

As a representative closed-loop result, the controlled shape 1 \([4 0 0 0]^T\) step response is plotted in Fig. 6.113 over a 10 second time frame. The corresponding control voltages are plotted in Fig. 6.114. The uncontrolled \(\pm 0.5\text{mil}\) shape 1 disturbance component is notably diminished (due to the outer-loop LTR compensator), as well as the amplitude of the first mode response (due to the inner-loop Lyapunov damper). The quasi-static response of the other shape components is damped as well. The control voltages reflect the compensator's performance in rejecting both the quasi-static and resonant response of the beam. These tests, then, combines all aspects of the DYSC experiment's performance goals set forth at the beginning of the Chapter, demonstrating the efficacy of the design and synthesis methods developed in the thesis for achieving the closed-loop dynamic shape control of a distributed parameter system, in the presence of exogenous disturbances.

![Fourier Coefficients vs Time](image)

**Fig. 6.113:** MBC8Ar shape 1 \([4 0 0 0]^T\) experimental step response in the presence of a quasi-static disturbance.
Fig. 6.114: MBC8Ar shape 1 $[4 \ 0 \ 0 \ 0]^T$ experimental step response control voltages in the presence of a quasi-static disturbance.
VI.10 SUMMARY AND DISCUSSION

The DYSC experiment provided a test bed for evaluating the analytical developments of the thesis. The loop transfer recovery method was utilized to construct model-based compensators for the closed-loop, dynamic shape control of a flexible beam. The physical insights of the Lyapunov analysis of Chapter II were extended to formulate an inner-loop damping scheme to supplement the outer-loop LTR controller. The experiment achieved the prescribed goals of a four-sinusoid discrete spatial bandwidth, temporal performance consistent with stability/robustness and implementation constraints, and static, quasi-static, and resonant disturbance rejection. Perhaps just as significantly, the control designs decoupled the output shape response. The experiment is the first such hardware demonstration of its kind in the open literature.

Aside from the aforementioned performance measures, the spatial transform shape control paradigm has other implications. The design implicitly assumes that the control task is completely described by the band-limited set of orthogonal shapes. Further, the spatial compensation (e.g. the actuator distribution) is designed to minimize the excitation of spatial bandwidth components.

The displacements that the fotonic sensors directly measured are plotted in Fig. 6.115. These normalized measurements are time-averaged steady-state shape responses to individual shape step command experiments, taken from the MBC1a compensator test series. The data points also include "error bars" that bound the variances of these displacements over the course of the test, quantifying the zero steady-state error shape requirement. Sinusoidal curves defining the corresponding "ideal" sinusoidal shapes are plotted in the Figure as well. The variances show that the resonant response only slightly detracts from the steady-state error requirement. However, while the averaged transducer measurements of shapes 1, 2, and especially 4 show excellent fidelity with the "ideal" shapes, the shape 3 measurements differ from the ideal significantly around the beam's center.

This deviation is best explained using the discrete spatial transform of these time-averaged shapes, plotted in Fig. 6.116 for the first six spatial frequency components. These plots show that the closed-loop compensators were indeed controlling the first four
Fig. 6.115: Typical time-averaged displacement measurements and variances.
Fig. 6.116: Fourier shape components of time-averaged profiles.
shape components as required. The degradation of the response seen in Fig. 6.115 is attributable to the uncontrolled high spatial frequency components, e.g. those above the first four shapes. This reflects the experimental variation of the actuator shape influence functions from the nominal plant model; recall the variations between the two models’ input coupling operators. The actuator distribution, while designed to best control the prescribed bandwidth of shapes, nonetheless has a spatial decomposition with components above this band limit. Also, the actuator bonds were not precisely uniform, and the film’s piezoelectric characteristics are no doubt slightly nonuniform.

These observations do not detract from the achievement of the DYSC experiment, but serve instead to further define the implications of the discrete spatial transform model as applied to the shape control problem. In a practical design the distributed actuators may have to be further spatially shaped to alleviate this "spatial spillover" problem. Rectangular apertures have significant spatial frequency content. Additionally, the required spatial control bandwidth may have to be increased to sufficiently push the uncontrolled wavenumber components into the spatial regime where the plant naturally acts as a spatial low-pass filter; in structural systems, one can exploit this property of the elastic continua’s strain/curvature relation.
VII. LYAPUNOV DAMPING FORMULATION FOR THIN BEAMS USING DISCRETE SPATIAL TRANSFORMS

VII.1 INTRODUCTION

It is possible, in a surprisingly straightforward fashion, to extend the inner-loop damping formulation developed in Chapter VI to expansions in more general orthogonal functions. This provides a link between the Lyapunov damping formulation for beams, Chapters I and II, and the shape control analysis, serving to unify the various components of the thesis.

A derivation of the Lyapunov damping design approach for beams follows in this short Chapter. The problem is posed in terms of expanding the beam's response in complete sets of orthonormal functions. The analysis shows that a shape control system like that described in Chapter VI can include inner-loop damping directly, for the requisite states and control weighting are "built in" to the controller. The analysis is then specialized to eigenfunction expansions, which permits an interpretation of the Lyapunov controller in terms of completeness of the eigenfunctions with respect to actuator and sensor distributions.

VII.2 DERIVATION OF CONTROL CONSTRAINTS

We begin by reintroducing the nondimensional Lyapunov energy functional for a thin elastic beam presented in Chapter I,

\[ J(t) = \frac{1}{2} \int_0^1 \left[ (y_{xx})^2 + (y_t)^2 \right] dx, \] (7.1)

with time derivative
\[ J(t) = \int_0^t \left( y_{xx}y_{xt} + y_t y_{tt} \right) dt. \] \hspace{1cm} 7.2

The subscripts \((\cdot)_x\) and \((\cdot)_t\) denote partial differentiation with respect to \(x\) and \(t\), respectively. Boundary element energies and energy terms associated with in-plane tension can be included in (7.1), but are neglected for clarity of exposition.

The associated nondimensional governing equation for an undamped Bernoulli-Euler beam is

\[ y_{xxxx} + y_{tt} = u, \quad 0 < x < 1, \] \hspace{1cm} 7.3

where both the displacement \(y\) and the input \(u\) are functions of \(x\) and \(t\). Let the displacement be expanded in a set of orthonormal shape functions \(\{ \psi_i(x) \}\), such that

\[ y(x,t) = \sum_{i=1}^{N} c_i(t) \psi_i(x). \] \hspace{1cm} 7.4

Naturally, \(N\) can go to infinity. Since \(\{ \psi_i(x) \}\) are orthonormal,

\[ \int_0^1 \psi_i(x) \psi_j(x) dx = \delta_{ij}. \] \hspace{1cm} 7.5

Substituting the expansion (7.4) into the expression for the power flow (7.2) gives

\[ J(t) = \int_0^t \left\{ \left[ \sum_{i=1}^{N} \psi_i''(x) c_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i''(x) c_i(t) \right] \right\} dt, \] \hspace{1cm} 7.6

\[ + \left[ \sum_{i=1}^{N} \psi_i(x) \dot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i(x) \dot{c}_i(t) \right] \right\} dx. \]
The associated governing equation has the expansion

\[
\sum_{i=1}^{N} \left[ \psi_i^{(3)}(x) c_i(t) + \psi_i(x) \ddot{c}_i(t) \right] = u. \quad 7.7
\]

Let the control input be degenerate,

\[
u(x,t) = \Lambda(x) \rho(t), \quad 7.8
\]

with a spatial distribution expandable in the orthonormal shape functions,

\[
\Lambda(x) = \sum_{i=1}^{N} \lambda_i \psi_i(x). \quad 7.9
\]

So, for each shape \( \psi_i(x) \),

\[
\psi_i^{(3)}(x) c_i(t) + \psi_i(x) \ddot{c}_i(t) = \lambda_i \psi_i(x) \rho(t). \quad 7.10
\]

Combining the transformed governing equation (7.10) with the time derivative of the energy functional (7.6) gives

\[
j(t) = \int_{0}^{1} \left( \left[ \sum_{i=1}^{N} \psi_i(x) c_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i''(t) \right] + \left[ \sum_{i=1}^{N} \psi_i(x) \ddot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \lambda_i \psi_i(x) \rho(t) \right] \right)
\]

\[
- \left[ \sum_{i=1}^{N} \psi_i(x) \dddot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i^{(3)}(x) c_i(t) \right] \right) \ dx. \quad 7.11
\]

Using the orthonormality property (7.5), the integral of term \( \text{II} \) reduces to
\[
\sum_{i=1}^{N} \lambda_i \dot{c}_i(t) \rho(t). \quad 7.12
\]

Since \( y(x,t) \) and \( \psi_i(x) \) are assumed to be square integrable, consistent with the assumptions of Chapter 5, term I in equation (7.11) can be integrated by parts twice with respect to \( x \) (albeit tediously) to become

\[
\left[ \sum_{i=1}^{N} \psi_i(x) \dot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i''(x) c_i(t) \right] - \left[ \sum_{i=1}^{N} \psi_i(x) \ddot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i'''(x) c_i(t) \right] \bigg|_{0}^{1}
\]

\[+ \int_{0}^{1} \left[ \sum_{i=1}^{N} \psi_i(x) \dot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i'''(x) c_i(t) \right] dx. \quad 7.13
\]

The integral in (7.13) cancels term III in equation (7.11), leaving

\[
\dot{j}(t) = \sum_{i=1}^{N} \lambda_i \dot{c}_i(t) \rho(t) + \left[ \sum_{i=1}^{N} \psi_i'(x) \dot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i''(x) c_i(t) \right] \bigg|_{0}^{1}
\]

\[-\left[ \sum_{i=1}^{N} \psi_i(x) \ddot{c}_i(t) \right] \left[ \sum_{i=1}^{N} \psi_i'''(x) c_i(t) \right] \bigg|_{0}^{1}. \quad 7.14
\]

Since the orthonormal shape functions do not in general satisfy the boundary conditions, the boundary terms in (7.14) must be retained. Specific forms for these functions will now be examined.

**Eigenfunction Expansions**

Consider first the case where the shape functions are equal to the system eigenfunctions; this was true for the DYSC experiment described in Chapter VI. The boundary conditions are assumed to be homogeneous. Consequently, for any combination of clamped, pinned, free, or sliding boundary conditions, the boundary terms in (7.14)
always vanish. These summations are merely eigenfunction expansions of the boundary terms encountered in the Lyapunov analysis of Chapter I. Thus, for $\psi_i(x)$ equal to the system eigenfunctions,

$$\dot{J}(t) = \sum_{i=1}^{N} \lambda_i c_i(t) \rho(t). \tag{7.15}$$

The expansion coefficients $c_i$ are the modal amplitudes, and their time derivatives are the modal velocities. Since, from (7.7) and (7.5),

$$\lambda_i = \int_0^1 A(x) \psi_i(x) \, dx, \tag{7.16}$$

all modes of the beam will be controllable if the eigenfunctions are complete with respect to the actuator distribution $A(x)$, such that $|\lambda_i| > 0$ for all modes $i = 1 \ldots N$. Keep in mind that the eigenfunctions are complete with respect to the beam's displacement as well, hence an identical result will hold for the corresponding sensor problem.

This quantifies the intuitive result developed in Chapters I and II. The Lyapunov control design reduces to choosing a spatial actuator distribution to control all modes of interest by ensuring $|\lambda_i| > 0$ for those modes, then choosing $\rho(t)$ to make the right hand side of (7.15) negative semi-definite, i.e. generalized modal control. In principle, one can synthesize a spatial actuator distribution not only to control, say, odd- or even-order modes (as done in Chapter I), but individual modes, using (7.16).

As an illustration of this analysis, consider the beam/film control system designed by Bailey [7.1] to damp vibrations of a thin cantilever beam. For this system, the film spatial distribution was uniform over the entire length of the beam. Thus,
\[ u(x,t) = \frac{\partial^2}{\partial x^2} \{ mV_{\text{max}} \rho(t) [h(x) - h(x - 1)] \} = mV_{\text{max}} \rho(t) [\delta'(x) - \delta'(x - 1)]. \]

The eigenfunctions for a cantilever beam take the form [7.2]

\[ \psi_i(x) = A[\cos(\mu_i x) - \cosh(\mu_i x)] + B[\sin(\mu_i x) - \sinh(\mu_i x)]. \]

The eigenvalues \( \mu_i \) are defined as the positive definite roots of

\[ \cos(\mu_i) \cosh(\mu_i) = -1. \]

All of the beam's modes will be controllable with respect to the uniform film actuator if none of the Fourier coefficients of the expansion of the the actuator distribution (7.17) in the eigenfunctions (7.18) vanish. These expansion coefficients are defined by

\[ \lambda_i = \int_0^1 [\delta'(x) - \delta'(x - 1)] \psi_i(x) dx. \]

Thus,

\[ \lambda_i = \mu_i A[\sin(\mu_i) + \sinh(\mu_i)] + \mu_i B[\cosh(\mu_i) - \cos(\mu_i)]. \]

Note that the contribution from the point moment at \( x = 0 \) has vanished. This is because the eigenfunctions satisfy the boundary conditions at the clamped end.

Since the eigenvalues \( \mu_i \) must satisfy (7.19), there is no root that also sets \( \lambda_i \) to zero; one can easily see this by testing the first few roots of (7.19), then examining the plots of \( \sin, \cos, \sinh, \) and \( \cosh. \) Thus, the entire set of cantilever beam functions is complete with respect to the film actuator control distribution. This means that the film controller can damp
all modes. A similar result holds for the pinned-pinned beam discussed in Chapters I and II. Consequently, the discrete transform in system eigenfunctions (e.g. a modal expansion) provides a convenient interpretation of the Lyapunov damper results.

General Orthonormal Expansions

Consider the case where the orthonormal shape functions are not equal to the system eigenfunctions. In general, \( \{ \psi_i(x) \} \) will not satisfy the boundary conditions. However, consider the following thought experiment. Let the control \( u(x,t) \) equal zero. Then, all that remains in equation (7.14) are the boundary terms, which are independent of the control. Obviously, if the control is turned off the beam remains open-loop stable in the sense of Lyapunov. Consequently, the boundary terms in (7.14) are, at worst, conservative, e.g. they neither add nor remove energy. Further, since boundary terms are independent of the input \( u(x,t) \), the control design is still based upon

\[
\dot{J}(t) = \sum_{i=1}^{N} \lambda_i \dot{c}_i(t) \rho(t)
\]

alone! So, even if the shape functions are not equal to the system eigenfunctions, the result remains the same: Given an actuator distribution and a colocated sensor distribution, one need only feed back the time derivative of the requisite Fourier coefficients to damp the system's resonant response. However, one is not guaranteed that all modes are controlled, but rather that the Lyapunov-designed controller will always extract energy from the system.
VIII. CONCLUSIONS AND FUTURE DIRECTIONS

An analysis of distributed parameter systems and control has been developed herein, where the essential spatial nature of these systems was explicitly incorporated. So that we may consider the thesis as a whole, let us review the contributions of its parts.

The Lyapunov analysis of Chapter I showed that a beam can be effectively controlled using distributed piezoelectric actuators, for nearly arbitrary boundary conditions. The appeal of the analysis is its lack of reliance on traditional model-based (e.g. modal) plant representations. Consequently, it doesn't suffer from the pitfalls of modal truncation. The distributed actuator's effective loading on the beam can be described using concepts from elementary structural mechanics, making the analysis accessible to the practicing engineer. Further, the aperture shaping derived for the various beam configurations can be readily synthesized. The utility of the approach is best appreciated by the number of experimental realizations it has fostered, including the pinned-pinned beam system described in Chapter II. While the examples given in the development only considered film distributions having lumped actuator interpretations, the film can be shaped spatially to effect control distributions with no lumped counterpart, offering the possibility of more general fully distributed inputs. Further, the sensing and control requirements developed in Appendix B for the plate vibration control problem can only be realized using distributed sensors and actuators.

The space/time transform technique was then introduced as a general modelling tool for distributed parameter systems. This method offers the possibility of quantifying the spatial performance of distributed parameter controllers directly, using the concept of "spatial bandwidth". Distributed parameter plants are merely space/time filters. Actuator and sensor distributions, themselves space/time filters, then augment the plant model to construct a multiplicative MIMO response matrix in the transformed space. For degenerate compensation, the sensor and actuator distributions provide the only spatial compensation.

The method is applicable to stationary distributed plants having convolution-form or symmetric Green's functions. The composition integrals defining their input/output relations were converted to product forms in wavenumber/frequency space. And, since symmetric Green's functions admit bilinear, modal expansions, this extends the method's
range of application to the large number of distributed parameter systems described by finite element or modal analyses.

The modelling technique was combined with the linear algebraic manipulations of modern multivariable control to construct performance measures for distributed parameter systems. These measures quantify command following, disturbance rejection, noise immunity, and stability/robustness for distributed parameter control, in both a temporal and spatial sense. Additionally, observability and controllability measures were derived. The extension to the usual temporal bandwidth concepts is intuitive; there is now an additional bandwidth parameter. The analysis showed that distributed parameter control reduces to the design of an appropriate space/time compensator. The formulation readily lends itself to the quantification of spatial performance, in particular achievable spatial bandwidth.

The shape control problem is simply an extension of the spatial transform methodology to a discrete basis, e.g. an expansion of the system response in orthogonal functions. These functions can be used to describe the shape control task; shape control performance requires not only a spatial bandwidth specification, but a set of independent shapes to achieve. This rectifies the limitations of previous shape control approaches, which designed systems to achieve a single shape. It directly addresses important shape control applications such as wavefront correction. The further developments of Chapter V provide supplemental performance measures specific to shape control. And, perhaps most significantly, the modelling techniques developed therein for reciprocal plants showed the method could be applied to plants having modal descriptions, leading to a state space formulation. This provides a link between the analysis and modern control design and synthesis software.

The acid test of the entire methodology was the dynamic shape control experiment. The experiment embodied the various aspects of the thesis, from analysis to experimental experience with distributed actuators, in constructing a closed-loop, dynamic shape control proof of concept experiment. The experiment qualified the utility of bandlimited orthogonal expansions for the description of a system's shape, demonstrating that the spatial bandlimit, actuator distribution, and plant must be analyzed in concert to satisfy spatial performance goals. This hardware realization is the basis upon which future shape control systems, i.e. wavefront controllers, can be developed.
A significant number of analytical results were developed herein in rather broad strokes. While the examples presented throughout served to demonstrate relevant concepts, they nevertheless utilized simple example distributed systems, such as beams and strings. The method must now be applied to a number of distributed parameter control problems of greater complexity to further assess its utility and range of application. More importantly, the analysis methods must now be extended to become a distributed control design methodology, to uncover techniques for specifying sensor and actuator distributions to achieve space/time performance goals.

In Chapter V, performance measures were derived to quantify actuator placement for shape control. These were based upon the characteristics of an input/output matrix relating input control signals and output Fourier coefficients of the performance shape functions, equation (5.46). Instead of using this as an analysis tool, one can instead specify a design matrix that embodies desirable characteristics, such as small condition number, high gain, balanced participation among actuators, etc; i.e. a specified singular value structure. Denote this design matrix by \( M_d \), evaluated at some design frequency \( \omega_d \). Then,

\[
E_d \equiv ||M_d(\omega_d) - M[\alpha(\omega_d)]||_2
\]  

is a way of defining the error between the realized plant/actuator matrix \( M(\alpha) \) and the desired characteristic. Ideally, one could require

\[
M_d(\omega_d) - M[\alpha(\omega_d)] = 0.
\]  

Then, given the matrix of desired Fourier coefficients, the designer would merely calculate the inverse transform of the required actuator distribution [which is buried within \( M(\alpha) \)] to synthesize the needed actuators.

This technique has obvious shortcomings. First, the requirement (8.2) will often be too severe to meet with any known actuators, discrete or distributed. Instead, one could attempt to extremize the error \( E_d \) defined in (8.1) given certain restrictions on the actuators, such as a set of assumed actuator types with free design parameters such as placement or aperture.
width. However, an even more attractive alternative would be to extremize the $p$-norm

$$J_d = \left( \int_0^\infty \left| M_d(\omega) - M[\alpha(\omega)] \right|^p d\omega \right)^{1/p} ; 1 \leq p < \infty. \quad 8.3$$

This would not only satisfy the spatial performance requirement inherent in the discrete transform model used to derive the plant/actuator matrices, but a temporal requirement specified by the frequency characteristics of the design matrix over a range of frequencies. Fortunately, recent trends in systems and control theory are pointing to input/output based analyses of linear systems, based upon norms such as (8.3). Current developments in $H_\infty$ optimal control might be applied to this formulation to realize the necessary temporal and spatial compensation.

The form of equation (8.3) naturally leads one to pose the design problem for the continuous spectrum spatial transform model as one of extremizing the $p$-norm

$$J_d = \left( \int_{-\infty}^\infty \int_0^\infty \left| M_d(k,\omega) - K(k,\omega) G(k,\omega) \right|^p dk d\omega \right)^{1/p} ; 1 \leq p < \infty. \quad 8.4$$

The design problem is then to determine a space/time compensator matrix $K$ that permits the distributed plant $G$ to have the specified wavenumber/frequency characteristics of the design matrix $M_d$. This is especially significant for the continuous spectrum spatial transform model: the spatial transform technique is quite general, but heretofore only the discrete expansion approach has been amenable to design. Using this as a starting point, it may be possible to specify not only degenerate actuator distributions for meeting spatial performance goals, but no doubt compensators that are themselves distributed parameter systems.

The thesis provides a basis for future research in distributed parameter control theory and applications. The stability-robustness analysis should be extended to address system parametric variations, and further quantification of "spatial robustness". The space/time transform method will be combined with recent developments in performance-robustness
analysis to extend its utility and range of application.

The discrete spatial transform formulation led to representations amenable to Loop Transfer Recovery (LTR) temporal compensator design methods, hence all the robustness guarantees and design tools for LTR are directly applicable to plants having modal representations; the derivation of a formal LTR result for the shape control of more general plants, including complex non-self-adjoint systems in optics and acoustics, is a topic of current research. Recent spectral iteration developments within the optics community offer the possibility of a combined spatial/temporal LTR synthesis method for designing space/time target loops. These design methods will be applicable to sensor fusion applications for distributed parameter system measurement and identification as well.
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CHAPTER VI


CHAPTER VII


APPENDIX A


APPENDIX B


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APPENDIX C

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Appendix A.
DISTRIBUTED ACTUATOR CONTROL DESIGN
FOR THE TIMOSHENKO BEAM MODEL

The analysis of the beam/film damper system presented in Chapter I is valid for "thin" beams, where the effects of rotary inertia and shear forces can be neglected. This is often referred to as a "low-frequency" model. It will be shown in this Appendix that the Lyapunov analysis can be applied to the Timoshenko beam model without modification. This extends the range of application of the film damper.

To start, the governing equation for a Bernoulli-Euler beam having a layer of piezoelectric film bonded to one face will be derived. The technique parallels Graff [A.1]. The derivation differs in method from Bailey's [A.2], but yields the same result. The same derivation method will subsequently be applied to the Timoshenko model. The Lyapunov analysis follows.

Consider a differential beam element of length \( dx \), as shown in Fig. A.1. The beam is of length \( L \). \( v \) is shear force, \( M \) is the bending moment, \( q \) is an external distributed load, and \( mV \) is the moment due to the film actuator. \( m \) is defined by equation (1.3), and \( V \) is the control voltage across the film. The beam has composite flexural rigidity \( EI \). A balance of
vertical forces acting on the beam element gives

\[-v + \left( v + \frac{\partial v}{\partial x} \right) + q \, dx = \rho A \, dx \frac{\partial^2 y}{\partial t^2}, \tag{A.1}\]

or

\[\frac{\partial v}{\partial x} + q = \rho A \frac{\partial^2 y}{\partial t^2}. \tag{A.2}\]

Balancing moments about the differential element, using the convention that positive moments act counter-clockwise, gives

\[-M + \left( M + \frac{\partial M}{\partial x} \right) - mV \left[ \frac{\partial (mV)}{\partial x} \right] - v \frac{dx}{2} - \left( v + \frac{\partial v}{\partial x} \right) \frac{dx}{2} = 0. \tag{A.4}\]

Neglecting nonlinear terms, equation (A.4) reduces to

\[v = \frac{\partial M}{\partial x} + \frac{\partial (mV)}{\partial x}. \tag{A.5}\]

Differentiating (A.5) with respect to \(x\), using the moment-curvature relation

\[EI \frac{\partial^2 y}{\partial x^2} = M, \tag{A.6}\]

and combining the result with (A.3) yields the governing equation for the Bernoulli-Euler beam,

\[\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} + mV \right) + \rho A \frac{\partial^2 y}{\partial t^2} = q. \tag{A.7}\]

This is the usual Bernoulli-Euler beam/film composite governing equation [e.g. equation
(1.1)], which has been studied in depth.

![Diagram of beam element geometry for Timoshenko model](image1)

Fig. A.2a: Beam element geometry for Timoshenko model (after Graff [A.1]).

![Diagram of coordinate system](image2)

Fig. A.2b: Coordinate system (after Graff [A.1]).

For the Timoshenko beam, consider the differential element shown in Fig. A.2a. The
displacement of the centroidal plane is still \( y(x,t) \), and its slope is given by \( \frac{\partial y}{\partial x} \). A new coordinate \( \psi \) is introduced to measure the slope of the cross-section due to bending. In the Bernoulli-Euler model the cross-section is assumed to remain perpendicular to the centroidal plane, hence \( \psi = \frac{\partial y}{\partial x} \) and no new coordinate need be introduced. As seen in Fig. A.2b, the slope of the centroidal axis is still \( \frac{\partial y}{\partial x} \), which now consists of two contributions. The non-zero \( \psi \) coordinate is caused by shearing effects that twist the faces of the cross-section, making them non-perpendicular to the neutral axis. Note, however, that plane sections are still assumed to remain planar.

One next assumes that the relationship between bending moment and curvature still holds. In terms of the new coordinate system,

\[
\frac{M}{EI} = \frac{\partial \psi}{\partial x} \tag{A.8}
\]

To determine the new coordinate \( \gamma_0 \) (see Fig. A.2b), one must invoke the "Timoshenko argument". The shear force over the cross-section in given in terms of the shear stress and strain as

\[
\nu = \int_A \tau dA = G \int_A \gamma dA. \tag{A.9}
\]

If \( \gamma_0 \) is the shear strain at the neutral axis, \( G\gamma_0 A \) denotes the corresponding shear force. However, it will not equal the value obtained by integrating the variable stress distribution, equation (A.9). To bring the value into balance, Timoshenko introduced an adjustment coefficient \( \kappa \) such that

\[
\nu = G \int_A \gamma dA = (G\gamma_0 A) \kappa. \tag{A.10}
\]
The value of $\kappa$ will depend upon the shape of the cross-section, and must be determined for each particular beam configuration. $\kappa$ is called the "Timoshenko shear coefficient". Now, since

$$\frac{\partial y}{\partial x} = \psi + \gamma_0, \quad \text{A.11}$$

the shear force can be defined in terms of the Timoshenko shear coefficient as

$$\nu = AG\kappa \left( \frac{\partial y}{\partial x} - \psi \right). \quad \text{A.12}$$

Given equation (A.12), the derivation proceeds as for the Bernoulli-Euler analysis. Balancing vertical forces on the differential beam element gives

$$\nu + \left( \nu + \frac{\partial \nu}{\partial x} \right) + q \, dx = \rho A \frac{\partial^2 y}{\partial t^2}, \quad \text{A.13}$$

hence

$$\frac{\partial \nu}{\partial x} + q = \rho A \frac{\partial^2 y}{\partial t^2}. \quad \text{A.14}$$

Balancing moments about an axis perpendicular to the $xy$-plane and passing through the center of the differential element gives

$$-M + \left( M + \frac{\partial M}{\partial x} \right) - mV + \left[ mV + \frac{\partial (mV)}{\partial x} \right] - \nu \frac{dx}{2} - \left( \nu + \frac{\partial \nu}{\partial x} \right) \frac{dx}{2} = J \frac{\partial^2 \psi}{\partial t^2}, \quad \text{A.15}$$

where $J$ is the polar moment of inertia of the element. Since
\[ J = \rho I dx, \quad \text{A.16} \]

equation (A.15) reduces to

\[ v = \frac{\partial M}{\partial x} + \frac{\partial (mV)}{\partial x} + \rho l \frac{\partial^2 \psi}{\partial t^2}. \quad \text{A.17} \]

Note that the rotary inertia effect is the correction to the Bernoulli-Euler model introduced by Rayleigh. The Timoshenko model includes it as well as the effects of shear.

Substituting the expression for the bending moment (A.8) into equation (A.17), using (A.12), yields the first governing equation for the Timoshenko beam model,

\[ AG \kappa \left( \frac{\partial y}{\partial x} + \psi \right) + EI \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial (mV)}{\partial x} + \rho l \frac{\partial^2 \psi}{\partial t^2}. \quad \text{A.18} \]

Combining equations (A.12) and (A.14) gives the second governing equation,

\[ AG \kappa \left( \frac{\partial \psi}{\partial x} - \frac{\partial^2 y}{\partial x^2} \right) + \rho A \frac{\partial^2 y}{\partial t^2} = q. \quad \text{A.19} \]

To combine equations (A.18) and (A.19) into a single governing equation, one first solves (A.19) for \( \frac{\partial \psi}{\partial x} \):

\[ \frac{\partial \psi}{\partial x} = \frac{\partial^2 y}{\partial x^2} - \frac{\rho}{G \kappa} \frac{\partial^2 y}{\partial t^2} + \frac{q}{AG \kappa}. \quad \text{A.20} \]

Differentiating (A.18) with respect to \( x \) gives

\[ AG \kappa \left( \frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial x} \right) + EI \frac{\partial^3 \psi}{\partial x^3} = \frac{\partial^2 (mV)}{\partial x^2} + \rho l \frac{\partial^3 \psi}{\partial x \partial t^2}. \quad \text{A.21} \]

Equation (A.20) is then differentiated twice with respect to \( x \),
\[
\frac{\partial^3 \psi}{\partial x^3} = \frac{\partial^4 y}{\partial x^4} - \rho \frac{\partial^4 y}{G \kappa \partial x^2 \partial t^2} + \frac{1}{AG \kappa} \frac{\partial^2 q}{\partial x^2 \partial t^2}.
\]  

A.22

Differentiating (A.20) twice with respect to \( t \) gives

\[
\frac{\partial^3 \psi}{\partial x \partial t^2} = \frac{\partial^4 y}{\partial x^2 \partial t^2} - \rho \frac{\partial^4 y}{G \kappa \partial t^4} + \frac{1}{AG \kappa} \frac{\partial^2 q}{\partial t^2}.
\]  

A.23

Substituting equations (A.22) and (A.23) into equation (A.21) then yields the single governing equation

\[
EI \frac{\partial^4 y}{\partial x^4} - \rho I \left(1 + \frac{EI}{G \kappa}\right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 y}{\partial t^2} + \rho I \frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 (mV)}{\partial x^2} + q - \frac{EI}{AG \kappa} \frac{\partial^2 q}{\partial x^2} + \frac{\rho I}{AG \kappa} \frac{\partial^2 q}{\partial t^2}.
\]  

A.24

If the shear coefficient \( G \to \infty \), and the remaining rotary inertia term is neglected, then equation (A.24) reduces to the familiar Bernoulli-Euler equation, (A.7).

For the ensuing Lyapunov analysis, assume that the distributed load \( q = 0 \). The dimensional energy functional is constructed in the usual fashion –

\[
J(t) = \frac{1}{2} \int_0^L \left[ EI \left( \frac{\partial^2 y}{\partial x^2} \right)^2 + \rho A \left( \frac{\partial y}{\partial t} \right)^2 \right] dx.
\]  

A.25

Energy terms associated with passive lumped boundary elements can be included in (A.25), but are neglected here for simplicity. Further, the energy functional can be generalized to include terms corresponding to the rotary kinetic energy of the differential beam elements, and other contributions contained within the Timoshenko model; these are also neglected for simplicity, as they will not affect the ensuing results.

The time derivative of the energy functional is
\[ j(t) = \int_0^L \left( EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^3 y}{\partial x^2 \partial t} + \rho A \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} \right) dx. \]  
A.26

From equation (A.24),

\[ \rho A \frac{\partial^2 y}{\partial t^2} = -\frac{\partial^2 (mV)}{\partial x^2} - EI \frac{\partial^4 y}{\partial x^4} + \rho l \left( 1 + \frac{E}{G\kappa} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2 I \partial^4 y}{G\kappa \partial t^4}. \]  
A.27

Substituting equation (A.27) into (A.26) gives

\[ j(t) = \int_0^L \left[ EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^3 y}{\partial x^2 \partial t} - \frac{\partial y}{\partial t} \frac{\partial^2 (mV)}{\partial x^2} \right] dx \]

\[ = \int_0^L \left[ \frac{\partial y}{\partial t} \rho l \left( 1 + \frac{E}{G\kappa} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\partial y \rho^2 I \partial^4 y}{G\kappa \partial t^4} \right] dx. \]

A.28

From previous analyses, one notes that the second and third terms in the first integral cancel each other after two integrations by parts, for any combination of free, clamped, pinned, or sliding boundary conditions. Consequently, equation (A.28) reduces to

\[ j(t) = -\int_0^L \frac{\partial y \partial^2 (mV)}{\partial t \partial x^2} dx \]

A.29

\[ + \int_0^L \frac{\partial y}{\partial t} \rho l \left( 1 + \frac{E}{G\kappa} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2 I \partial^4 y}{G\kappa \partial t^4} \right] dx. \]

Note that the control voltage only appears in the first integral. After two integrations by parts
the first integral is identical to the result (1.8). Consequently, the film control distribution
design problem involves only the first integral in (A.29), in the *exact same manner*
presented in Chapter I.

And what of the second integral in (A.29)? Consider the Timoshenko beam in the
absence of any control voltage, \( V(x,t) = 0 \). In the absence of external inputs the Timoshenko
beam is *stable* in the sense of Lyapunov, hence the second integral must be at worst
conservative (e.g. leading to no energy addition or removal), or at best dissipative (e.g.
extracting energy from the system). One can argue that, since there is no odd-order time
derivative in the Timoshenko beam equation, the second integral in (A.29) is conservative
[A.3]. As a result, it can be neglected in the Lyapunov control design. The Lyapunov
control synthesis method, then, remains *unchanged* from the Bernoulli-Euler model.
Appendix B.

DISTRIBUTED ACTUATOR CONTROL DESIGN
FOR THIN PLATES

The distributed piezoelectric film active damper concept developed in Chapter I for Bernoulli-Euler beams can be extended to plates as well. Consider a thin elastic plate ("Bernoulli-Euler plate"), with a layer of biaxial piezoelectric film adhered to one face. A biaxial film layer responds to an electric field across its faces via longitudinal strain in both in-plane dimensions. For small deflections, the governing partial differential equation for the plate/film composite is [B.1,B.2]

\[ D \nabla^4 \tilde{W} + \rho \nabla_{\hat{\xi}}^2 \hat{V} = m \nabla^2 \hat{V}, \quad \hat{\xi} \in D, \quad \text{B.1} \]

where \( \tilde{W} \) is the plate's normal deflection, \( m \) is the coefficient describing the film's electro-mechanical coupling, \( D \) is the plate's composite flexural rigidity, \( \rho \) is its area density, and \( \hat{V} \) is the voltage across the film layer. \( \hat{\xi} \equiv (\hat{x},\hat{y}) \) is the spatial coordinate, while \( \hat{t} \) is time. The subscript \( (\cdot)_{\hat{t}} \) denotes partial differentiation. Both the deflection and the control voltage are functions of space and time. The system is described over a domain \( D \) with boundary \( \Gamma \). The problem considered here is the development of a control law for \( \hat{V}(\hat{x},\hat{t}) \) that will damp vibrations of the plate.

The synthesis technique is Lyapunov's second (or direct) method. First, introducing the nondimensionalizations

\[ x \equiv \frac{\hat{x}}{L}, \quad w \equiv \frac{\tilde{W}}{L}, \quad \text{B.2} \]

\[ t \equiv \frac{\hat{t}}{L^2 \sqrt{\frac{D}{\rho}}}, \quad V \equiv \frac{\hat{V}D}{mL^2}, \]

where \( L \) is a characteristic length of the plate (i.e. the length of one side), permits the governing equation (B.1) to be expressed nondimensionally as
\[ \nabla^4 w + w_{tt} = \nabla^2 V. \] \hspace{1cm} \text{(B.3)}

The corresponding nondimensional Lyapunov energy functional takes the form

\[ J = \frac{1}{2} \int_D \left[ (\nabla^2 w)^2 + (w_{xy})^2 + w_{xx}w_{yy} + (w_t)^2 \right] d\mathbf{x}. \] \hspace{1cm} \text{(B.4)}

The first three terms in the functional describe the system's strain energy, the latter two due to the plate's Gaussian curvature [B.3]. The last term in equation (B.4)'s integrand represents the plate's kinetic energy. Consequently, \( J \) represents the system's total energy within the constraints of the modelling assumptions.

The time derivative of (B.4) is

\[ J = \int_D \left[ \nabla^2 w \nabla^2 (w_t) + w_t w_{xx} + w_{xy}w_{y,t} + \frac{1}{2}(w_{xx}w_{yy} + w_{yy}w_{xx}) \right] d\mathbf{x}. \] \hspace{1cm} \text{(B.5)}

Equation (B.5) is proportional to the power flow into or out of the system. If \( J < 0 \) for all \( t \), then the system is asymptotically stable in the sense of Lyapunov, for power is always removed from it. Using an integral identity [B.4], equation (B.5) can be re-written as

\[ J = \int_\Gamma \frac{\partial (w_t)}{\partial n} \nabla^2 w \, d\Gamma + \int_D w_t \nabla^4 w \, d\mathbf{x} - \int_\Gamma w_t \frac{\partial (\nabla^2 w)}{\partial n} \, d\Gamma \]

\[ \int_D w_t w_{tt} \, d\mathbf{x} + \int_D \left[ w_{xt}w_{xy,t} + \frac{1}{2}(w_{xx}w_{yy,t} + w_{yy}w_{xx,t}) \right] d\mathbf{x}. \] \hspace{1cm} \text{(B.6)}

Combining the governing equation (B.3) and equation (B.6) gives the important result
\[ j = \int_{\Gamma} \frac{\partial (w)}{\partial n} \nabla^2 w \, d\Gamma - \int_{\Gamma} w \frac{\partial}{\partial n} (\nabla^2 w) \, d\Gamma + \int_{\partial} w \nabla^2 \nabla \, dx \]

\[ + \int_{\partial} \left[ w_{xx} w_{yy} + \frac{1}{2} (w_{xx} w_{yy} + w_{yy} w_{xx}) \right] \, dx. \]

The expression for the power flow (B.7) now contains boundary integrals into which appropriate boundary conditions can be substituted for configurations of interest. Additionally, it contains an integral over the plate of the control voltage's Laplacian. Two observations are immediately forthcoming. First, the similarity between this formulation and the beam analysis is striking; the beam analysis is a special case of the plate analysis, obtained by setting \( y = y(x,t) \) only. Second, the expression for \( j \) is valid for any natural coordinate system where \( \nabla^2 \) and \( \nabla^4 \) are defined. This extends the range of application to non-rectangular plates.

The boundary conditions for the beam/film system are assumed to be homogeneous with respect to any (distributed) lumped elements, \textit{and with respect to the control voltage as well}. This approach differs somewhat from the beam development of Chapter I, but yields the desired results without the tedious mathematical manipulations associated with various integral identities. (Note that the beam results can be derived in the same way as well.) One can think of a film distribution over the entire domain of the plate, for example, as extending to, but never reaching (in the limiting mathematical sense) the boundaries. The requisite boundary conditions are then

\[ w(\Gamma, t) = \frac{\partial}{\partial n} w(\Gamma, t) = 0, \]  

\[ w(\Gamma, t) = \nabla^2 w(\Gamma, t) = 0, \]  

\[ \nabla^2 w(\Gamma, t) = \frac{\partial}{\partial n} (\nabla^2 w(\Gamma, t)) = 0, \]
\[ \frac{\partial}{\partial n} w(\Gamma, t) = \frac{\partial}{\partial n} \left( \nabla^2 w(\Gamma, t) \right) = 0. \]  

(B.8d)

These describe clamped, pinned, free, and sliding boundaries, respectively. Substituting any combination of the boundary conditions (B.8a)-(B.8d) into equation (B.7) gives

\[ j = \int_B w_t \nabla^2 V \, dx + \int_B \left[ w_{xx} w_{xy} + \frac{1}{2} (w_{xx} w_{yy} + w_{yy} w_{xx}) \right] dx. \]  

(B.9)

The control design, then, reduces to the selection of an appropriate film actuator voltage space/time distribution that ensures the first integral in (B.9) is negative semi-definite. The second integral in equation (B.9), containing Gaussian curvature terms, is independent of the control voltage. For a rectangular plate with all four edges pinned, it vanishes identically [B.3,B.5]. Since the uncontrolled plate is stable, the second integral is at worst conservative (e.g. not contributing to power flow into or out of the system) for any boundary condition combination, hence it will be neglected in the ensuing analysis.

We consider now the limitations of a spatially-uniform layer of biaxial piezoelectric film for the control of a rectangular plate. A spatially-uniform control distribution is defined here as

\[ V(x, t) \equiv V_{\text{max}} \Lambda(x) \rho(t) \]

\[ = V_{\text{max}}[h(x) - h(x - l_x)][h(y) - h(y - l_y)] \rho(t), \]  

(B.10)

where \( l_x \) and \( l_y \) are the nondimensional length of the sides parallel to the \( x \) and \( y \) axes. Its Laplacian is given by

\[ \nabla^2 V = V_{\text{max}}[\delta'(x) - \delta'(x - l_x)][h(y) - h(y - l_y)] \rho(t) \]

\[ + V_{\text{max}}[h(x) - h(x - l_x)] \left( \delta'(y) - \delta'(y - l_y) \right) \rho(t). \]  

(B.11)

Substituting (B.11) into the controlled portion of equation (B.9) (remembering that the second integral can be neglected) gives
\[ \dot{J}(t) = \int_0^b w_x \nabla^2 V \, dx = -V_{max}(t) \left[ \int_0^b (w_x|_{x=0} - w_x|_{x=b}) \, dy \right] + \int_0^b (w_x|_{y=0} - w_x|_{y=b}) \, dx \]

hence a spatially-uniform control distribution provides boundary control solely in terms of distributed angular moments along the boundaries. If the plate is clamped along all the edges, \( \dot{J} = 0 \), e.g. a spatially-uniform film distribution cannot extract energy from or add energy to a clamped plate, which makes it inappropriate for control.

There are similar restrictions if the plate is simply supported along all sides. If opposing sides of such a plate configuration have identical angular velocity distributions (as opposed to the pinned-pinned beam, where one considers only the value at the boundary points), then modal vibrational components in the direction normal to those sides are not controllable with the uniform film distribution. Additionally, if the integrated angular velocity distributions are equal for two opposing sides, the same result occurs. The response of a simply-supported plate is expressible as

\[ w(x,t) = A \sin(k_x x) \sin(k_y y) e^{-j(\omega t + \phi)}. \]

The eigenvalues,

\[ k_x = \frac{m \pi}{l_x}, \quad k_y = \frac{n \pi}{l_y}, \]

where \( m \) and \( n \) are integers, are chosen to satisfy the boundary conditions. The corresponding angular velocity components are

\[ w_{xt} = -i \omega k_x A \cos(k_x x) \sin(k_y y) e^{-j(\omega t + \phi)}, \]

\[ w_{yt} = -i \omega k_y A \sin(k_x x) \cos(k_y y) e^{-j(\omega t + \phi)}. \]

Consider the case where \( m \) is even. The angular velocities (B.15) and (B.16) evaluated
along the sides \( x = 0 \) and \( x = l_x \) are equal for \( n \) even or odd, hence that portion of the functional (B.12) will vanish. If \( m \) is odd, this term can still vanish if \( n \) is even, for the functional involves the line integral of the angular velocity distribution at the boundaries in the \( y \) direction. When \( m \) is odd and \( n \) is even, the angular velocities at the opposing sides aren't equal, hence the integrand in (B.12) is non-vanishing. But when \( n \) is even the angular velocity distributions along \( x = 0 \) and \( x = l_x \) have odd symmetry about mid-span \( (y = 0.5l_y) \) and the spatial control distribution is uniform (and hence has even symmetry), and the integral will vanish.

A further examination of equation (B.12) for the angular velocity distribution \( w_y \) yields similar results, in that modes with \( n \) even, or \( n \) odd with \( m \) even, are not controllable with the spatially-uniform control voltage distribution. Thus the familiar symmetry arguments for beam mode shapes must be extended; for a plate with a spatially-uniform control distribution, not only are we concerned with the symmetry of mode shapes with respect to linear and/or angular velocities, but also with symmetries of the distributions of these quantities along the boundaries. For the simply-supported example, if \( m \) and/or \( n \) are even, the plate is not controllable. The plate is only controllable with a spatially-uniform distribution if both \( m \) and \( n \) are odd! Hereafter, such a mode will be termed "strictly odd"; those with \( m \) and \( n \) both even will be referred to as "strictly even", while all others will be called "mixed order".

In order to construct a spatial distribution that will control all modes, we shall use results from the pinned-pinned beam analysis. Introduce a film distribution that is a product of "ramp" functions in two dimensions,

\[
\Lambda(x) = \left[ \left( 1 - \frac{x}{l_x} \right) h(x) - \left( 1 - \frac{x}{l_x} \right) h(x - l_x) \right] \left[ \left( 1 - \frac{y}{l_y} \right) h(y) - \left( 1 - \frac{y}{l_y} \right) h(y - l_y) \right].
\]

The loading exerted by this film actuator is given by its Laplacian, which has components

\[
\Lambda_{xx}(x) = \left[ \delta(x) - \frac{1}{l_x} \left[ \delta(x) - \delta(x - l_x) \right] \right] \left[ \left( 1 - \frac{y}{l_y} \right) h(y) - \left( 1 - \frac{y}{l_y} \right) h(y - l_y) \right],
\]

and
\[ A_y(x) = \left[ \left( 1 - \frac{x}{l_x} \right) h(x) - \left( 1 - \frac{x}{l_x} \right) h(x - l_x) \right] \left[ \delta'(y) - \frac{1}{l_y} \delta(y) - \delta(y - l_y) \right]. \]  

This distribution is seen to exert a distributed moment along the sides \( x = 0 \) and \( y = 0 \), as well as distributed point loads along the boundaries. The net loading, of course, provides for static equilibrium, as the film is self-reacting; this was also noted in the beam case. Substituting the Laplacian, equations (B.18) and (B.19), into equation (B.9) yields an expression for the power flow from the system for the new spatial control distribution,

\[
\dot{J}(t) = -V_{\max} \rho(t) \left\{ \int_0^b w_{x|x=0} = 0 \left[ \left( 1 - \frac{x}{l_y} \right) h(y) - \left( 1 - \frac{x}{l_y} \right) h(y - l_y) \right] dy + \int_0^c w_{y|y=0} = 0 \left[ \left( 1 - \frac{x}{l_x} \right) h(x) - \left( 1 - \frac{x}{l_x} \right) h(x - l_x) \right] dx \right\}. \]

Since \( \dot{J} \) only has terms involving the angular velocity distributions along \( x = 0 \) and \( y = 0 \), no cancellation of the component integrands can occur. Further, and perhaps more importantly, the line integrals of the angular velocity distributions are now weighted by ramp functions. Remember that, for the uniform spatial distribution, mixed-order modes were uncontrollable due to the vanishing line integrals, while strictly even modes were uncontrollable both because of asymmetry of the plate's slopes at the boundaries and vanishing line integrals. The double-ramp (or "snow plow") distribution (B.17) can be expressed as the sum of even and odd functions, hence even if the distribution of angular velocity along the boundaries has odd symmetry about mid-span, the integrands of equation (B.20) will have some remaining even component, and thence be non-vanishing. Thus, the distribution (B.17) ensures controllability for the simply supported plate for all modes, as a single input.

The result (B.20), however, requires a measurement of the line integrals of the angular velocity distributions along the boundaries \( x = 0 \) and \( y = 0 \) to implement the control. Needless to say, there is no commercially-available transducer that can measure this. However, extending the results of Miller and Hubbard [B.6], a biaxial film sensing distribution can be synthesized that will yield the requisite measurement. Assume that one has a layer of biaxial piezoelectric film bonded to the plate on the face opposite that with the actuator distribution. The film sensor will provide a voltage output proportional to the
integrated strain on the plate's surface,

\[ V_{out} = \alpha \int_0^b \nabla^2 w \, dx. \]  \hspace{1cm} B.21

The constant \( \alpha \) will be a function of the plate and film geometry, as well as the film's constitutive constants. Introduce a sensor weighting function \( \Lambda_s(x) \), such that

\[ V_{out} = \alpha \int_0^b \Lambda_s(x) \nabla^2 w \, dx. \]  \hspace{1cm} B.22

Using the "second form" of Green's theorem in the plane [B.7],

\[ \int_0^b (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) \, dx = \oint \left( \frac{\partial \psi}{\partial n} - \frac{\partial \varphi}{\partial n} \right) \, d\Gamma, \]  \hspace{1cm} B.23

equation (B.22) becomes

\[ V_{out} = \int_0^b w \nabla^2 \Lambda_s \, dx + \oint \Lambda_s \frac{\partial w}{\partial n} \, d\Gamma - \oint w \frac{\partial \Lambda_s}{\partial n} \, d\Gamma. \]  \hspace{1cm} B.24

For the simply-supported plate, the displacement at the boundaries must vanish, hence the last integral in (B.24) equals zero. Choosing \( \Lambda_s \) equal to the spatial distribution of the actuator distribution, (B.17), transforms the sensed output (B.24) to

\[ V_{out}(t) = \alpha \left\{ \int_0^b w_{x,1} x = 0 \left[ \left( 1 - \frac{y}{l_y} \right) h(y) - \left( 1 - \frac{y}{l_y} \right) h(y - l_y) \right] dy \right. \\
+ \int_0^b w_{y,1} y = 0 \left[ \left( 1 - \frac{x}{l_x} \right) h(x) - \left( 1 - \frac{x}{l_x} \right) h(x - l_x) \right] dx \right\}. \]  \hspace{1cm} B.25
Save for the time dependence, this is the desired sensed parameter for the Lyapunov vibration controller. One need only differentiate the output (B.25) from the shaped sensor distribution. Interestingly, the use of distributed biaxial film actuator and sensor distributions, spatially shaped, can provide for vibration damping of the plate using a single feedback channel. And the sensed parameter needed to implement the control can only be measured using distributed sensing.

The symmetry arguments used in the beam analysis, as well as the above plate analysis, can be applied to the study of thin plates with more general boundary conditions. However, a few important constraints must be considered. First, no closed-form solution to the plate equation exists for most boundary conditions, save for those that have at least two opposing sides that are simply-supported [B.5]. Additionally, even if an appropriate film spatial distribution can be designed, as well as the corresponding distributed sensor, it will be difficult to guarantee that controllability will always be achieved. For a beam, this is rather simple. The beam solution can be written in closed form, including an equation that defines the eigenvalues, hence controllability/observability can be demonstrated by reduction [B.8]. Since these relations cannot in general be derived for the plate, this technique is inappropriate. One possible solution is to use the Rayleigh hypothesis: If the vibration response of a plate is expressed as a doubly-infinite sum of the product of beam modes in the $x$ and $y$ directions, then in the limit the series converges to the solution of the plate equation [B.9]. Using this argument, the actuator and sensor distributions required to control a plate with arbitrary boundary conditions will equal the product of the distributions designed to control beams with the same boundary conditions as the plate's opposing sides. Naturally, the requisite sensing distributions will be the same as the actuator distributions.
Appendix C.

VIBRATION DAMPING OF THIN BEAMS USING
COLOCATED DISTRIBUTED SENSORS AND ACTUATORS

Using the nondimensional Lyapunov formulation of Chapter I, it is a straight-forward exercise to demonstrate that colocated (equivalent) piezoelectric film sensor and actuator distributions can be used to construct stabilizing vibration controllers for Bernoulli-Euler beams. From equation (1.8), the time derivative of the energy functional for the beam/film system is given by

\[ \dot{J}(t) = \int_0^1 V(x,t)y_{xx}(x,t) \, dx. \]  

Contributions from lumped boundary elements have been neglected, as they were shown to be (at worst) conservative. The actuator's spatial distribution \( A_a(x) \) is as yet unspecified. The nondimensional voltage output from a film sensor bonded to the beam on the face opposite the actuator distribution is given by [C.1]

\[ V_{out}(t) = \alpha \int_0^1 A_a(x)y_{xx}(x,t) \, dx, \]  

where the constant \( \alpha \) is a function of the beam and film static parameters, as well as the film's constitutive piezoelectric properties. \( A_a(x) \) is the sensor's spatial distribution. Let the beam's displacement field \( y(x,t) \) be expressed as a product of space and time functions,

\[ y(x,t) = \psi(x) \varphi(t). \]  

Similarly, the control voltage \( V(x,t) \) is specified as a separable function,

\[ V(x,t) = A_a(x) \rho(t). \]
The expression for the power flow, equation (C.1), can then be rewritten as

\[ J(t) = \rho(t) \psi'(t) \int_0^1 \Lambda_o(x) \psi''(x) \, dx \]

\[ = \rho(t) \frac{d}{dt} \int_0^1 \Lambda_o(x) \psi''(x) \psi(t) \, dx. \]  \hspace{1cm} \text{C.5}

The sensor output voltage can be rewritten as

\[ V_{out}(t) = \alpha \int_0^1 \Lambda_s(x) \psi''(x) \psi(t) \, dx. \]  \hspace{1cm} \text{C.6}

Choose the sensor and actuator distributions \( \Lambda_s(x) \) and \( \Lambda_a(x) \) to be equal, with this colocated distribution denoted by \( \Lambda(x) \). If the time portion of the control \( \rho(t) \) is chosen to be proportional to the time derivative of the sensing distribution's voltage output,

\[ \rho(t) = -a^2 \alpha \frac{d}{dt} \int_0^1 \Lambda(x) \psi''(x) \psi(t) \, dx, \]  \hspace{1cm} \text{C.7}

then the power flow (C.5) takes the form

\[ J(t) = -a^2 \alpha \left( \frac{d}{dt} \int_0^1 \Lambda(x) \psi''(x) \psi(t) \, dx \right)^2. \]  \hspace{1cm} \text{C.8}

Note that (C.8) is negative semi-definite. Consequently, no matter what the sensor and actuator distributions are, if they are equal, and the two are interconnected via velocity feedback as per (C.7), then the resulting closed loop system is stabilized. This result is true for any spatial distribution, including those that control/sense only certain modal components. This result occurs because the sensing and actuation design spatial integrals are equal. If a mode is not controllable with the chosen actuator distribution, then it is also not observable using the same sensing distribution. Also, the Bernoulli-Euler beam is open-loop
stable, hence all modes are stabilizable/detectable with respect to any distribution. This is an extension of a well known result [C.2,C.3] (e.g. velocity feedback using colocated discrete sensors and actuators will always stabilize distributed elastic systems, subject to certain robustness constraints) to distributed actuators and sensors. Using this result, one can construct controllers for tasks other than vibration damping, and then utilize the resultant actuator distribution for at least some level of vibration reduction. Interpretations for specific configurations can be deduced using the discrete transform analysis of Chapter VII.
Appendix D.
Ctrl-C – MACS INTERFACE

In order to expedite the testing of various compensator designs, a set of Ctrl-C functions was written that stored discretized compensator matrices in formatted files. These files could then be transferred to MACS' hard disk over DECNet, and read in by a VAXELN real-time control program using the custom EPascal procedure GENFILTER_INIT in ANALOG.OLB. This permitted testing and modifying compensator designs "on the fly".

While in Ctrl-C, after defining the functions FILTRBUILD and FILTRBUILDST in the workspace (which is best done in the user's LOGIN.CTR file) the user constructs a compensator design. If the compensator is defined as a rational fraction in the frequency domain, with numerator coefficient vector NUM and denominator coefficient vector DEN, and the desired time interval for discretizing the compensator is DELTA_T, one need only enter

FILTRBUILD(NUM,DEN,DELTA_T) <Return>

FILTRBUILD.CTR will then prompt the user for the name of the desired output file name; the current implementation uses the DCL file FILTRBUILD.COM to prompt for the file name, although Ctrl-C now has a built-in function to do this. When the file has been created successfully in the correct format FILTRBUILD.CTR returns a "1".

If the compensator is designed instead using a state space representation, with matrices A, B, C, and D in the Ctrl-C workspace and time step for discretization DELTA_T, then one would invoke FILTRBUILDST.CTR as

FILTRBUILDST(A,B,C,D,DELTA_T) <Return>

FILTRBUILDST.CTR will then prompt the user for the name of the desired output file name; the current implementation uses a DCL file to prompt for the name, as did FILTRBUILD.CTR. When the file has been created successfully in the correct format FILTRBUILDST.CTR returns a "1".
Listings of these files follow –

FILTRBUILD.CTR

//[done]=filtrbuild(n,q,dt);
[a,b,c,d]=tf2ss(n,q);
[phi,gam]=c2d(a,b,dt);
save dt phi gam c d >tempsave -f -(1x,e22.16)
$@filtrbuild.com
done=1;
//done

FILTRBUILDST.CTR

//[done]=filtrbuildst(a,b,c,d,dt);
[phi,gam]=c2d(a,b,dt);
save dt phi gam c d >tempsave -f -(1x,e22.16)
$@filtrbuild.com
done=1;
//done

FILTRBUILD.COM

$ inquire file "type in name of output data file"
$ rename tempsave.dat 'file'
$ exit

Credit must be given to Alex Gruzen, who developed the original version of FILTRBUILD, and the original GENFILTER EPascal routines.
Appendix E.
DYSC EXPERIMENT STATE MATRICES

The state matrices for the DYSC system nominal and experimental models follow. The outputs are scaled as 10·mil, and the inputs as V/100. For further definitions, see Chapter VI, especially equations (6.17) through (6.19).

EXPERIMENTAL PLANT MODEL STATE MATRICES

\[
A_p = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6.5893D+3 & -8.0700D-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2.6380D+4 & -1.7290 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.9440D+4 & -2.3763 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0588D+5 & -2.9299
\end{bmatrix}
\]

\[
B_p = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
2.3772D+3 & 5.1260D+3 & 3.6644D+3 & 7.2478D+2 \\
0 & 0 & 0 & 0 & 0 \\
1.2243D+4 & 1.2991D+4 & -1.0057D+4 & -8.2380D+3 \\
0 & 0 & 0 & 0 & 0 \\
3.2501D+4 & -1.6688D+4 & -1.3101D+4 & 2.0480D+4 \\
0 & 0 & 0 & 0 & 0 \\
-4.7057D+4 & -4.4412D+4 & 3.5233D+4 & -3.2088D+4
\end{bmatrix}
\]

\[
C_p = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]
NOMINAL PLANT MODEL STATE MATRICES

\[
A_p = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6.5893 \times 10^3 & -8.0700 \times 10^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B_p = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.8797 \times 10^3 & 4.5381 \times 10^3 & 4.5381 \times 10^3 & 1.8797 \times 10^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.2887 \times 10^4 & 1.2887 \times 10^4 & -1.2887 \times 10^4 & -1.2887 \times 10^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3.3242 \times 10^4 & -1.3769 \times 10^4 & -1.3769 \times 10^4 & 3.3242 \times 10^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5.2551 \times 10^4 & -5.2551 \times 10^4 & 5.2551 \times 10^4 & -5.2551 \times 10^4 \\
\end{bmatrix}
\]

\[
C_p = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
I was born in Waterville, Maine, on October 28th, 1959. I grew up in Fairfield, Maine, and attended the local schools. I was deeply interested in the space program as I grew up. I suppose this is why I was drawn to science at an early age. However, my career path was set when I heard my first stereo. I believe I was twelve. My brother had just bought a $69 Grant's Bradford stereo, and after we first opened it up and modified it I knew I wanted to design loudspeakers.

I graduated from Lawrence High School in 1977. I participated in the usual student activities, did the usual academic things, and ran on the track team. My track coach and geometry teacher, Mr. Dave Martin, encouraged me to apply to Princeton. I had never heard of the place, although I did know that New Jersey was somewhere south of New England.

I enrolled at Princeton, and graduated in 1981. I found out about academic failure early on. However, my career path was "adjusted" by Professor Don Bliss, who was also interested in acoustics. He became my advisor, and helped me define a challenging and rewarding undergraduate thesis project. From then on I was hooked on research. He also taught me that acoustics is much more subtle and interesting than loudspeaker design. Luckily, he also was able to convince the MIT Admissions Office that grades weren't everything.

I found an apartment in Cambridge and was offered a Research Assistantship at MIT on the same day. I received my Masters Degree in Mechanical Engineering in June of 1983, under the supervision of Professor Patrick Leehey. I'm positive no one has read my thesis since. I spent a year as the Departmental Instructor in Mechanical Engineering, and met Jim Hubbard in the process. He taught a lab section of 2.14 across the hall from my office, and we would often chat before his students arrived. He called me up on September 4th, 1985. He said he had a Draper Fellowship, and asked if I was interested. I had no idea what distributed parameter control was. I think I know a little more now.
Simplify, simplify, simplify!

—Henry David Thoreau