LEADING-EDGE ANALYSIS FOR CORRECT SIMULATION OF INTERFACE SEPARATION AND HYDRAULIC FRACTURING

by

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ABSTRACT

An accurate simulation of propagating circular hydraulic fractures is developed. Despite prior efforts, this problem had not previously been properly resolved, even for ideal assumptions of linear, homogeneous, isotropic and elastic material behavior. The reasons for this lie in the highly non-linear and coupled equations associated with the simultaneous solution of mass conservation, fluid flow, and crack-opening deformation relations.

Although the circular geometry is of limited interest, the associated solutions serve two important roles in the context of more general, practical hydraulic-fracture simulations:

- Serving as a precise reference solution for fully three-dimensional (3-D) simulators.
- Providing the details of the near-perimeter behavior, which such general 3-D simulators cannot handle when modelling realistic conditions for propagating hydraulic fractures because of the excessive amount of computation which would be required.

Comparisons of simulator results with laboratory results are provided for two distinct experiments. One involves the injection of viscous fluid into the interface between a cast rubber material and a transparent PMMA (Plexiglas) block, through which the fracture-equivalent interface separation can be observed by direct visual means. In the other experiment, hydraulic fractures are grown in cement blocks, with fracture growth marked using striations created by perturbing the applied stress field in a triaxial test cell.
A concise description of the near-perimeter behavior has been obtained from the results of circular-crack simulations, for inclusion into a general 3-D simulator. The absence of such a description has been a source of error in prior 3-D simulations. The near-tip behavior is referred to here as the Leading-Edge (LE) component of the propagation process, and it dominates the resulting pressure and fracture growth. The LE component also leads to fundamental differences between laboratory and field data-sets; such differences may be explained in terms of dilatant material behavior in deep rock fracturing, which is largely absent in laboratory experiments.

The simulator developed will serve as the basis for generating comprehensive results for LE behavior. The functional basis established for these LE results should facilitate their incorporation into general 3-D hydrafrac simulators for realistic, practical applications.

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NOMENCLATURE

This section defines the variables used in Chapters 1 through 5. In some cases, the nomenclature used in an appendix differs from these definitions; such differences are defined within that appendix as variables are introduced.

\( E \) Young’s modulus of elasticity

\( E(q,k) \) elliptic integral of the second kind

\( \bar{E} \) effective modulus of elasticity for crack opening, \( E/[4(1-v^2)] \)

\( \bar{E}_{\text{eff}} \) effective value of \( \bar{E} \) for a crack embedded in a cylinder of radius \( R_c \)

\( F(q,k) \) elliptic integral of the first kind

\( F_{\text{rheol}} \) rheological flow relation

\( k_i \) permeability of solid to the leakoff fluid

\( k' \) consistency index for power-law fluid

\( K \) bulk modulus of the solid

\( K_I, K_{II}, K_{III} \) stress intensity factors for Modes I, II, and III

\( K_{IC} \) critical value of \( K_I \) at which crack propagation occurs

\( K_{Icyl} \) \( K_I \) value for a crack embedded in a cylinder of radius \( R_c \)

\( K_{\infty} \) \( K_I \) value for a crack in an infinite medium (when comparing with \( K_{Icyl} \))

\( l_i \) depth of leakoff zone adjacent to the fracture face

\( L_n \) global interpolation vectors
two-element local interpolation vector

$\overline{M}_n$  global gradient interpolation vectors

$\overline{M}_n$  two-element local gradient interpolation vector

$n$  node number

$n_{ff}$  node number of node at $r_{ff}$

$n_{wb}$  node number of node at $r_{wb}$

$n'$  flow behavior index for power-law fluid

$N_f$  number of discretization intervals between $r_{wb}$ and $r_{ff}$

$p$  excess pressure, $P - \sigma_c$

$p_{est}$  estimate of excess pressure

$p_{th}$  "top hat" distribution of excess pressure, defined in Equation (3.2.2)

$p_{tip}$  excess pressure within the non-penetrated zone at the crack tip

$p^*$  characteristic excess pressure

$P$  fluid pressure

$P_{est}$  estimate of fluid pressure

$P_{res}$  resultant fluid pressure

$P_{est}$  estimate for the vector of nodal pressure values

$P_{res}$  resultant vector of nodal pressure values
$P_{tip}$ fluid pressure within the non-penetrated zone at the crack tip

$P_{wb}$ fluid pressure at the wellbore

$\Delta P_{ff}$ pressure drop across the fluid interface at the fluid front

$P_{pore}$ fluid pressure in the pores of the solid being fractured

$q$ fluid volume flowrate

$q'$ fluid volume flowrate per unit perimeter, $q'(r) = q(r)/(2\pi r)$

$q'_\in$ global fluid influx vector

$q_i''$ fluid leakoff rate (leakoff volume flowrate through both fracture faces per unit area of a single fracture face)

$Q_{wb}$ volume flowrate of fluid injected into fracture at the wellbore

$r$ radius

$r_{ff}$ radius of fluid front, $R-\omega$

$r_n$ radius to the $n$th node

$r_{wb}$ wellbore radius

$r$ radial distance from crack tip, using polar coordinates centered at crack tip

$r'$ dummy variable of integration, representing radius

$\Delta r_{max}$ maximum spacing allowed between adjacent nodes in the region between the wellbore and tip regions

$\Delta r_{max(ref)}$ reference value of $\Delta r_{max}$
\( R \)  
crack tip radius  

\( R_{jk} \)  
angular functionality of \( \sigma_{jk} \) in the vicinity of the crack tip  

\( \dot{R} \)  
fracture propagation speed, \( dR/dt \)  

\( Re \)  
Reynolds number  

\( s \)  
dummy variable of integration  

\( s_n \)  
normalized position within the \( n \)th discretization interval  

\( t \)  
time  

\( \Delta t \)  
difference in time between successive timesteps  

\( u \)  
dummy variable of integration  

\( u_l \)  
superficial leakoff velocity  

\( w_b \)  
influence (weighting) function for the crack opening integral  

\( X \)  
representation of a general variable having dependence on radial position  

\( \dot{X}_n \)  
two-element local field vector  

\( z \)  
spatial coordinate for distance between crack faces, normal to the crack plane  

\( \beta \)  
exponent used in leading edge pressure gradient relation  

\( \gamma_{nt} \)  
surface tension coefficient  

\( \dot{\gamma} \)  
fluid shear rate
\[ \delta \] crack opening width or aperture  
\[ \delta_{fr} \] crack width at the fluid front  
\[ \dot{\delta} \] crack opening rate, \( \frac{d\delta}{dt} \)  
\[ \varepsilon_{fr}, \varepsilon_{wb} \] coefficients used in determining nodal spacing near the fluid front and wellbore  
\[ \varepsilon_{fr(\text{act})}, \varepsilon_{wb(\text{act})} \] actual values of \( \varepsilon_{fr} \) and \( \varepsilon_{wb} \) used to compute nodal spacing  
\[ \varepsilon_{fr(\text{ref})}, \varepsilon_{wb(\text{ref})} \] reference values of \( \varepsilon_{fr} \) and \( \varepsilon_{wb} \)  
\[ \zeta \] distance back from the crack closure point in the leading-edge module  
\[ \eta \] variation coefficient used to compute \( \varepsilon_{fr(\text{act})} \) and \( \varepsilon_{wb(\text{act})} \) from \( \varepsilon_{fr(\text{ref})} \) and \( \varepsilon_{wb(\text{ref})} \)  
\[ \theta_w \] wall angle, half the angle subtended by the fracture surfaces  
\[ \Theta \] arbitrary function used in the variational development of the finite element scheme  
\[ \Lambda \] \( \omega_{w}/\omega_d \)  
\[ \mu \] dynamic viscosity  
\[ \overline{\mu} \] channel viscosity, \( -(\delta P/\delta r)/(q'/\delta^3) \)  
\[ \mu_l \] viscosity of the leakoff fluid  
\[ \nu \] Poisson ratio  
\[ \xi \] effective distance from the crack tip in the near-tip wedge-flow analysis  
\[ \pi \] \[ 3.1415926535... \]  
\[ \rho \] normalized radius, \( r/R \)
\( \rho_f \)  
fluid density

\( \rho_{ff} \)  
radius of curvature of the fluid interface at the fluid front

\( \sigma_c \)  
confining stress

\( \sigma_{jk} \)  
\((j,k)\) components of the stress tensor

\( \tau \)  
fluid shear stress

\( \tau_l \)  
time of local leakoff initiation

\( \tau^* \)  
characteristic time for crack propagation

\( \phi \)  
angular position for polar coordinates centered at crack tip

\( \phi_s \)  
porosity of the solid

\( \Phi \)  
transformation operating on \( P_{est} \) to produce \( P_{rea} \)

\( \psi \)  
contact angle between the liquid interface and the solid face of the fracture, measured through the vapor phase

\( \omega \)  
size of non-penetrated zone at crack tip

\( \omega_d \)  
size of dilational closure zone in leading-edge module

\( \omega_{le} \)  
distance between the edge of the crack mesh and the crack closure point in the leading-edge module (\( \zeta=\omega_{le}^{+} \) and \( \zeta=\omega_{le}^{-} \) represent values of \( \zeta \) infinitesimally larger or smaller than \( \omega_{le} \))
1.1 Introduction

Hydraulic fracturing of rock at depths of thousands of feet below the ground surface is an operation which is used extensively in the petroleum industry to increase the flow of fluids from oil and gas reservoirs. In this process, fluids are pumped into a petroleum (oil or gas) well at pressures high enough to cause the surrounding rock to fracture and pre-existing natural fractures to open; as pumping continues, sand-like "proppant" material is mixed with the fluid and remains in the opened fractures after fluid pressure declines, producing a narrow but highly conductive path through which the reservoir fluids can be extracted. About 35% to 40% of all recently drilled petroleum wells have been hydraulically fractured, and about 25% to 30% of total U.S. oil reserves have been made producible by this process [Veatch, 1983]. The percentage of wells subjected to hydraulic fracturing has increased as lower permeability formations account for more and more of the new fields tapped for production.

Hydraulic fractures (or "hydrafracs") are expensive: a typical fracture treatment costs from $50,000 to $500,000, a large fraction of the overall cost of the well. If the hydrafrac operation is not done properly, the well may not produce petroleum at a rate which justifies the expense of fracturing. Even worse, the well
can be damaged, and even rendered unusable, a possibility which puts the cost of both drilling and fracturing at risk during the hydrafrac.

Although hydraulic fracturing of conventional oil and gas wells accounts for most of the hydrafrac expenditures by industry, there are several other uses of hydraulic fracturing which would also benefit from advances in hydrafrac technology. Hydrafracs are used to produce methane from coal seams, both to increase mine safety and as a primary objective in seams for which mining is not feasible [Brandenburg, 1983-88]. Hydrafracs allow circulation of water in high-temperature, competent rock for the extraction of heat in so-called Hot Dry Rock geothermal energy recovery schemes [Armstead and Tester, 1987]. Hydrafracs are used in conjunction with water- and steam-flooding operations in petroleum reservoirs to increase the sweep efficiency [Settari and Raisbeck, 1981]. Grouting operations employ hydrafracing for subterranean applications (mine safety, slope/footing stabilization, etc.) as well as structural applications [Cambefort, 1964]. Recently, hydraulic fracturing has been considered as a possible means for isolation of hazardous waste spills in groundwater aquifers [Brunsing, 1985]. In addition, a number of other processes have essential components which involve the separation of solid surfaces by fluid motion between them. These vary from applications such as elasto-hydrodynamic lubrication to deformation of rollers in mechanical expulsion of fluid from slurries in papermaking.
1.2 Background

Although hydraulic fracturing of petroleum wells is a technique which has been in use for over forty years [Veatch et al., 1989], general methods for predicting the details of fracture growth have been developed only in the past decade. Early models made the simplifying assumption that the fractures have a constant height (Figures 1.1, 1.2), providing useful estimates for the special cases of fractures completely contained by bounding barriers. In the context of oil and gas reservoirs, these barriers would be horizontal geologic strata. One such set of models, referred to as the PKN-type models [Perkins and Kern, 1961; Nordgren, 1972], assumed crack-opening distributions which depend only on the vertical position within the fracture (Figure 1.1). This analysis provides useful estimates of fractures having large length-to-height ratios and no slippage at the bounding barriers.

A complementary analysis was provided by the CGD-type models [Christianovitch and Zheltov, 1955; Geerstma and deKlerk, 1969; Daneshy, 1973] which assumed crack widths independent of vertical position (Figure 1.2), a situation achievable with total slippage at the barriers.

These constant-height models are useful for special cases, but are inadequate for most conditions of fracture growth, both because of the limited geometries which they approximate and also because they require constant rheology and uniform conditions for the portion of the reservoir covered by the fracture.
Another idealization which can be useful in certain situations is the assumption that the fracture shape is circular [Wong and Cleary, 1985]. This is adequate for early stages of fracture growth and for uniform reservoir conditions. Conditions in petroleum reservoirs are never uniform, however; at the very least, the applied stress field varies with depth. Nevertheless, we shall find the circular assumption extremely useful as the basis for checking more general models which handle non-circular fractures and for analyzing the important near-tip conditions.

An attempt to analyze non-constant-height and non-axisymmetric fractures more completely with minimum computational effort led to the "pseudo three-dimensional hydrafrac" (P3DH) models (Figure 1.3) [Cleary, 1980; Settari and Cleary, 1986]. The three-dimensionality referred to here indicates that the same two-dimensional crack shape is not required for every cross section along the fracture length. In these models, the distance along the fracture's main direction of growth is discretized and fluid flow is treated as though it were one-dimensional in both this direction and the normal direction, with the latter being computed separately for each discretized "slice". While individual determination of the height of each slice allows this formulation to account for variation in fracture height, the accuracy is poor unless the flow rate in the main direction of growth is much greater than the flow rate in the normal direction, i.e. unless the fracture is well-contained by (horizontal) barriers. In addition, the region near the crack tip is inadequately represented, as discussed below.
More recently, a "lumped three-dimensional" model has been developed [Crocket et al., 1986a] which solves separately for the growth rate of fracture tips in the upper, lower, and lateral directions without attempting to determine the detailed crack shape along the entire fracture perimeter. This formulation makes use of coefficients from more detailed ("three-dimensional" or "3-D") analyses to give the correct values for the spatially integrated expressions used (e.g. volume = coefficient x width x length²). An additional set of coefficients is used to account for the effects of variations in applied stress fields, elastic modulus, and fluid leakoff permeability. The lumped 3-D model has been used with numerous real-time data sets from actual fracture treatments of gas wells, and the time history of excess pressure (fracture fluid pressure minus relevant confining stress) has been successfully reproduced in most cases, giving some credence to the fracture geometry histories predicted [Cleary, 1988].

A number of large computer programs have been developed [Clifton, 1989] for more detailed analysis of fracture growth in hydrafracs, calculating the distributions of relevant parameters (e.g. fracture width, fluid pressure, fluid velocity, etc.) as functions of position on the fracture surface, as well as the position of the crack tip at all points around the fracture perimeter. Although these programs typically analyze planar (two-dimensional) fractures, they are referred to as 3-D models because their analyses account for the full three-dimensional nature of the stress field due to a localized displacement event at the crack surface, so that fracture geometries are not limited to those in which all cross sections are identical,
with two-dimensional, plane strain conditions, such as those of the PKN- and CGD-type models.

While some of these 3-D models use a finite element formulation which explicitly discretizes the entire volume of rock containing the fracture [e.g. Morita et al., 1988], it is more efficient to discretize only the fracture surface and account for poroelastic response in the surrounding rock by surface or boundary integral schemes using appropriate influence functions [Cleary et al., 1983]. If heterogeneities in the surrounding rock are not adequately accounted for by the available influence functions, a tractable hybrid scheme can be used in which the main character of the response to the fracture presence is obtained through influence functions, and any needed corrections are provided by a coarse discretization of the surrounding volume [Keat, 1989].

1.3 Problem Statement

Adequate accuracy for the 3-D analyses requires discretization of sufficient resolution to allow the piece-by-piece interpolated representation of the primary variable (pressure, in this case) to be sufficiently close to the actual spatial distribution. Because the pressure gradient in fluid flowing within the fracture is inversely proportional to the cube of the crack opening, a large pressure drop occurs near the crack tip, where crack opening is small. As a result, the majority of the pressure drop occurring across the fracture face typically occurs within a region near the fracture tip which is a small fraction (e.g. 1%) of the fracture
radius. This is illustrated by Figure 1.4, where the radial pressure distribution is plotted for a 300-foot hydraulic fracture propagating under conditions representative of an oil or gas reservoir. Because the pressure at the fracture tip is the only known reference pressure within the fracture, it is essential to adequately determine the pressure drop across the near-tip region in order to properly predict the average pressure level in the fracture. Since this average pressure level determines the average amount of crack opening, and consequently the pressure gradient throughout the fracture, both the levels and shapes of the pressure and crack-opening distributions within the fracture are uncertain if the near-tip region is not adequately modelled. Furthermore, since the pressure level and crack aperture in the vicinity of the tip determine the extent to which the crack propagates in each direction against the confining stress field, even the fracture positions predicted are in question when the near-tip conditions are uncertain.

The conditions in the near-tip region of a propagating hydraulic fracture are shown schematically in Figure 1.5. A small region adjacent to the crack tip is not penetrated by the fracturing fluid and has a pressure of zero in the absence of significant flow of pore fluid from the surrounding (porous) solid into the crack opening. Behind this non-penetrated zone the pressure gradient is very high, because of the narrowness of the crack. As the crack widens farther from the tip, the gradient decreases.

The size \( \omega \) of the non-penetrated region is determined by a balance between the factors working to open the crack and squeeze it shut. Those portions of the
crack which have pressures greater than the confining stress make positive contributions to crack openings and to the stress intensity factor at the tip, while those portions which have pressures less than the confining stress make negative contributions. The stress intensity factor $K_I$ is a measure of the stress level in the immediate vicinity of the crack tip and when a critical value for $K_I$ is exceeded [Thiercelin et al., 1989], the material at the crack tip fails and the crack tip propagates. If the fluid front were to move closer to the crack tip, the narrower widths there would lead to higher pressure gradients and the positive contributions to $K_I$ would increase while the negative contributions from a smaller non-penetrated zone would decrease. As $K_I$ climbed above the critical value, the tip would propagate until the original value of $w$ was restored.

Because the conditions in the region of high pressure gradient determine the overall pressure level in the fracture, the interaction between the various near-tip mechanisms must be adequately modelled to properly predict the behavior of propagating hydraulic fractures.

At present, the 3-D hydraulic fracturing analyses available do not accurately determine the pressure and crack-opening distributions near the crack tips. In this current work, an analysis technique is developed and implemented in a computer program (A3DH) which gives an accurate description of the crucial near-tip region. Simulations of hydraulic fracture growth using A3DH allow correlations describing the near-tip behavior to be incorporated into a computer module (a "leading-edge module" - see Figure 1.6) which is efficient enough for inclusion into a general 3-D
hydraulic-fracturing analysis program such as the R3DH fracture simulator [Cleary et al., 1983; Lam et al., 1986; Cleary et al., 1988].
Figure 1.1  Schematic diagram of fracture geometry having crack-opening distribution independent of horizontal position ("PKN Model").
Figure 1.2  Schematic diagram of fracture geometry having crack-opening distribution independent of vertical position ("CGD Model").
Figure 1.3  Concept of the P3DH model formulation.
Figure 1.4  Pressure distribution in a hydraulic fracture propagating under conditions representative of underground hydrocarbon reservoirs.
Figure 1.5  Schematic representation of crack-opening and pressure distributions near the tip of a propagating hydraulic fracture.
Figure 1.5  Schematic representation of crack-opening and pressure distributions near the tip of a propagating hydraulic fracture.
Figure 1.6 Fracture discretization for the fully three-dimensional hydraulic fracture model (R3DH) coupled with a "leading-edge module" which succinctly describes the near-tip conditions.
CHAPTER 2
DEVELOPMENT OF AN AXISYMMETRIC
HYDRAULIC FRACTURE SIMULATOR

2.1 Introduction

To gain a better understanding of the conditions in the near-tip region of a hydraulic fracture, it is useful to develop a numerical simulation of axisymmetric (circular) fractures. Since an axisymmetric fracture needs to be discretized only along a single radius rather than over the entire fracture area, it is possible to achieve extremely high resolution in the region just behind the crack tip with a reasonable amount of computing effort. Although the general results obtained from such simulations apply strictly only to axisymmetric fractures, the conditions which are found to hold in the immediate vicinity of the crack tip are expected to be universally applicable, since the size of this region is too small for the shape of the fracture perimeter to be evident and since parameter variations along the perimeter are unimportant on the characteristic scale of the near-tip region.

An axisymmetric hydraulic-fracture simulator also allows comparison of results with a general 3-D simulator for axisymmetric test cases. Because of the known (circular) geometry of an axisymmetric fracture, some calculations used in the axisymmetric analysis can be carried out exactly, using analytical expressions. A computer program which calculates axisymmetric fracture growth is therefore less dependent on the approximations inherent in a numerical implementation.
Several prior analyses of propagating axisymmetric hydraulic fractures have been made by Cleary and Wong. Their initial investigations were restricted to cracks growing in a self-similar fashion [Cleary and Wong, 1983], so that a fully transient response of the crack was not determined. A further analysis of unsteady cracks [Cleary et al., 1978-81; Cleary and Wong, 1985] treated the crack growth process as one of alternately extending the crack tip with no fluid motion and propagating the fluid front with a fixed position for the crack tip. This formulation would be expected to produce results which are not fully self-consistent, since the final pressures and crack openings at any time are computed using intermediate distributions which do not correspond to the current physical conditions. In addition, the computational scheme exhibited rather strong instabilities, producing oscillatory results in both time and position.

The analytical capability developed here for simulating the growth of axisymmetric hydraulic fractures is fully transient and self-consistent, producing stable results. This chapter describes the relevant physical phenomena modelled by the axisymmetric simulator and the general procedure for computing a solution to the resulting equations. The details of the numerical implementation are provided in the Appendices.

2.2 Conditions Satisfied by the Simulator

The procedure developed here to calculate details of the growth of axisymmetric hydraulic fractures satisfies the following conditions:
(1) The crack-opening width at every point in the fracture is consistent with the pressure distribution throughout the fracture and the existing level of confining stress.

(2) The fluid flowrate at every point in the fracture satisfies mass conservation, the radial variation of flowrate being determined by the rates of crack opening ("storage") and leakoff into the surrounding solid at the point in question.

(3) The pressure gradient at every point in the fracture satisfies the governing constitutive equation which relates pressure gradient, local flowrate, rheology and crack-opening width.

(4) The Mode I stress intensity factor $K_I$ at the crack tip, resulting from the distribution of pressure and the level of confining stress over the fracture surface, must equal the value $K_{IC}$ of the fracture toughness of the solid restraining the fracture growth. This condition is consistent with fractures opening against the minimum principal stress so that no sheer stresses are relieved on the fracture surface.

(5) The pressure at the fluid front is consistent with the pressure in the non-penetrated zone and the pressure difference existing across the fluid interface due to fluid surface-tension effects.

The solution procedure must determine the radial distributions of the fluid pressure, crack-opening width, fluid velocity, and fluid leakoff rates as well as the crack propagation rate and size of the non-penetrated region behind the crack tip. The procedure uses the specified rate of fluid injection into the fracture, plus information on fluid rheology, elastic properties (elastic modulus and Poisson's ratio) of the surrounding solid, and the level of the confining stress. The procedure could be modified easily to alternatively accept a specified fluid injection pressure.
The equations resulting from each of the above conditions will be developed in the following sections.

2.3 Crack-Opening Distribution

The shape of the crack-opening profile is determined by the response of the surrounding material to the distribution of pressure within the fracture and the pre-existing stress state. This material response can be complicated by such effects as the presence of microcracking around the crack tip [Kobayashi and Fourney, 1978] or nonlinear stress/strain behavior, such as dilatancy effects [Warpinski and Smith, 1989]. To permit the inclusion of these and other effects, the computational procedure developed here is not inherently tied to any specific model of material response, but employs an interchangeable module to compute crack deformations resulting from the pressure distribution. As the most generally applicable representation of the solid behavior, the response is modelled as being linearly elastic for development purposes. To the extent that adequate information is available on alternative material descriptions, an appropriate module could be used in its place.

In order to preserve the axisymmetric nature of the fracture, the initial state of stress in the solid (before the introduction of the fracture) must be axisymmetric as well. While any such stress distribution can be accommodated by the simulator described here, a uniform stress field is sufficient for the study of near-tip phenomena, since the near-tip behavior is not expected to be affected significantly by variations in the initial stress state, unless these variations occur over a length
scale approaching the size of the near-tip region, a situation not often encountered in practice.

The time scale for change in hydraulic fracturing operations is set by the resistance of the fluid to movement within the fracture and is much longer than the time scales required for stresses in the surrounding solid to respond to changes in the applied loads. As such, the hydraulic fracturing operation can be considered to occur quasi-statically, from the point of view of the elastic response of the solid. This response to the pressure distribution within the fracture satisfies the constraints of stress equilibrium, strain compatibility, and stress-strain constitutive relations.

The linear-elastic problem of solving for the crack-opening distribution resulting from a fluid pressure distribution within the crack is equivalent to the superposition of the two problems A and B shown in Figure 2.1. Problem A represents the body with the same far-field stresses as applied in the complete problem, but no fracture is present; the resulting equilibrium represents the stresses and displacements present in the body before the fracture is introduced. Problem B is the solid containing the fracture but with no stresses applied in the far-field and with tractions acting on the fracture surface which are equal to the actual fracture tractions minus the initial stress state at those points. The boundary conditions, i.e. the applied far-field stresses and the fracture-face tractions, are the same for the complete problem as for the superposition of problems A and B, and therefore the linear stress state and displacements resulting from the superposition is the same as for the complete problem.
The crack-opening distribution represents the displacements on the fracture surface with respect to the initial state of problem A and, as such, is determined solely by the traction distribution of problem B. The traction acting at radius $r$ on the crack face in problem B is the pressure $P$ in the fracture minus the initial confining stress $\sigma_c$ acting normal to the plane of the fracture. This traction is referred to as the excess pressure $p$,

$$ p(r) = P(r) - \sigma_c(r) \tag{2.3.1} $$

For a linearly elastic medium of infinite extent, the crack-opening width $\delta$ at the normalized radius $\rho = r/R$, where $R$ is the crack tip radius, can be expressed as a weighted integral of the excess pressure over the fracture surface [Sneddon and Lowengrub, 1969]:

$$ \delta(\rho) = \frac{2R}{\pi E} \int_0^1 \frac{du}{u^2 - \rho^2} \int_0^u \frac{ds \, s \, p(s)}{\sqrt{u^2 - s^2}} \tag{2.3.2} $$

with $E = E/[4(1-v^2)]$ being the effective elasticity modulus for crack opening. For the general (non-axisymmetric) three-dimensional fracture, the inverse of this relation must be used, expressing the local excess pressure as a weighted integral of the crack opening distribution relative to the local crack opening [Lam et al., 1986]. The formulation used here can be satisfactorily numerically integrated for arbitrary positioning of nodal points, whereas the integration of the general relation is most effectively performed by constraining nodal points to lie at zero points of Chebyshev polynomials [Cleary and Wong, 1985]. This restriction would require
the highest density of nodal points to be positioned in the nonpenetrated zone and in the wellbore region, a very inefficient placement.

When the order of integration in Equation (2.3.2) is reversed, the double integration can be replaced by two single integrations, which are computationally more efficient, and the crack opening can be expressed as:

$$\delta(\rho) = \frac{2R}{\pi E} \left[ \int_0^\rho ds \ s P(s) \left\{ \frac{1}{\rho} F\left(\sin^{-1}\left(\frac{1-\rho^2}{1-s^2}\right), \frac{s}{\rho}\right) \right\} 
+ \int_1^\rho ds \ s P(s) \left\{ \frac{1}{s} F\left(\sin^{-1}\left(\frac{1-s^2}{1-\rho^2}\right), \frac{\rho}{s}\right) \right\} \right]$$

(2.3.3)

where the function F is the elliptic integral of the first kind.

The derivation of this equation, as well as its numerical implementation, are presented in Appendix A.

2.4 Fluid Mass Conservation

The fluid injection which creates and propagates the hydraulic fracture generally occurs within a restricted region near the center of the circular crack. For specificity of analysis, it is assumed that fluid is injected into the fracture at an effective wellbore radius $r_{wb}$ and that the fluid pressure within the fracture is constant for radii less than $r_{wb}$. This assumption will not have a significant effect if $r_{wb}$ is much smaller than the crack-tip radius. It is not feasible or realistic to use a point source for the fluid injection because of the singular pressure distribution that would result.
If the material surrounding the hydraulic fracture is permeable, some of the injected fluid volume will leak out of the fracture into the adjacent pore space in the solid, displacing the original pore fluid in the process. The remaining fluid goes into increased fracture volume, causing both crack-tip propagation and increased fracture width. For compressible flow, of course, crack volume will also be influenced by changes in the density of the fluid within the fracture. Only incompressible flows are considered in this analysis, however.

The flowrate per unit perimeter $q'$ at a given radius $r$ therefore satisfies the condition

$$\frac{1}{r} \frac{d}{dr}(rq'(r)) = -\left(q''(r) + \dot{\delta}(r)\right)$$

(2.4.1)

where $q''$ is the leakoff rate and $\dot{\delta}$ is the crack opening rate. To be specific, $q''$ is the leakoff flowrate through both fracture faces, at radius $r$, per unit area of a single fracture face and $q'$ is the integral of the radial fluid velocity across the fracture width at radius $r$.

In addition, the flowrate at $r_{wb}$ must also be consistent with any specified injection rate $Q_{wb}$

$$2\pi r_{wb} \cdot q'(r_{wb}) = Q_{wb}$$

(2.4.2)

and the flowrate at the fluid front, at a distance $\omega$ behind the crack tip, must be consistent with the rate at which the fluid front is propagating

$$q'(R-\omega) = \delta(R-\omega) \frac{d}{dt} (R-\omega) .$$

(2.4.3)
2.5 Constitutive Flow Relation

Knowing the rheology of the fluid and the position and motion of the fracture surfaces allow a relationship to be determined for the pressure gradient as a function of the flowrate $q'(r)$ and the crack opening width $\delta(r)$,

$$\frac{\partial P}{\partial r} = F_{\text{rheol}} (q'(r), \Delta(r)). \quad (2.5.1)$$

The simulator which has been developed can readily incorporate any appropriate rheological behavior. Several representative fluid behaviors are explored in this section.

The fluid flow within a hydraulic fracture is dominated by viscous forces. The crack-opening widths are small and the Reynolds numbers based on these apertures are in most cases very small, typically being several orders of magnitude less than unity. Variations in the flow field with radius and time are small enough at most points within the fracture that the flow can be taken to be steady and fully developed. The variation of crack width, in both time and radius, is also generally small enough to allow the flow to be considered as one-dimensional, having a radial component only. Under these conditions, the flow field is equivalent to steady flow between parallel plates.

For a Newtonian fluid experiencing steady radial flow between parallel plates, with no variation of flow with respect to angle and no flow in the $z$-direction normal to the plate faces, the radial momentum equation is
\[ \frac{dP}{dr} = -\rho_f v \frac{\partial v}{\partial r} + \mu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right). \quad (2.5.2) \]

The first three terms in the parentheses, being of order \( v/r^2 \), are much smaller than the last term, which is of order \( v/\delta^2 \). Similarly, the ratio of \( \rho_f \nu \partial v / \partial r \) to \( \mu \partial^2 v / \partial z^2 \) is of order \( Re(\delta) \cdot \delta/r \), where \( Re(\delta) = \rho_f \nu \delta / \mu << 1 \) is the Reynolds number based on the opening width. The remaining terms,

\[ \frac{dP}{dr} = \mu \frac{\partial^2 v}{\partial z^2} \quad (2.5.3) \]

are the same as in the fully-developed, linear flow (non-radial) case, and gives a pressure gradient of

\[ \frac{\partial P}{\partial r} = -\frac{12\mu}{\delta^3} q' = -\frac{\mu}{\delta^3} q' \quad (2.5.4) \]

where \( q' \) is the flow per unit perimeter, \( \mu \) is the fluid viscosity, and \( \bar{\mu} \) is referred to here as the channel viscosity.

For "power law" fluids, exhibiting shear stresses \( \tau \) proportional to the \( (n') \)th power of the shear rates \( \dot{\gamma} \)

\[ \tau = k' \dot{\gamma}^{n'} \quad (2.5.5) \]

the pressure gradient associated with a given flow rate is more generally

\[ \frac{\partial P}{\partial r} = -\frac{2k'}{\delta^{2n'+1}} \left[ 4 + \frac{2}{n'} \right] q' \quad (2.5.6) \]
For Newtonian fluids, \( n' = 1 \) and \( k' \) represents the fluid viscosity. In this case, Equation (2.5.6) reduces to Equation (2.5.4). The derivation of Equation (2.5.6) is given in Appendix B.

### 2.5.1 General deviation from fully developed, 1-D flow

Although the assumption of one-dimensional, fully-developed, steady flow is entirely sufficient for most of the fracture, there are some cases where a more complete description of the flow is needed. The one-dimensional condition does not necessarily apply in the region near the fracture tip where a significant component of fluid flow moving toward the fracture face results from the crack-opening rate being comparable to the magnitude of the radial fluid velocity in this region. In addition, the flow may not be fully-developed for very small radii, where the flow is strongly divergent. In these regions, an additional component is present in the pressure gradient, referred to here as the "deviational" pressure gradient \( \frac{\partial P}{\partial r_{\text{dev}}} \). An approximate expression for this component is developed in Appendix C, for a Newtonian fluid, as

\[
\frac{\partial P}{\partial r_{\text{dev}}} = -\mu \left[ \frac{\partial}{\partial r} \left( \frac{\delta + q''_i}{\delta} \right) \right] - \rho_f \left[ \frac{\partial v}{\partial t} + \frac{s}{\delta} \frac{\partial v}{\partial r} - \frac{v}{\delta} \left( \delta + q''_i \right) \right].
\] (2.5.1.1)

The first term results from the two-dimensionality of the flow and is more important near the crack tip. The second term is due to the inertia of the fluid and is only important when there is a strong divergence of the fluid, e.g. near a small-radius injection zone with a very high volume flowrate. Both of these terms, however, are generally negligible.
2.5.2 Wedge flow at crack tip

In the immediate vicinity of the crack tip, the fluid flow can be strongly two-dimensional if the crack-opening rates are comparable to the radial velocity of the fluid. This condition is more likely to occur when fracturing takes place in soft material at a high relative propagation rate $\dot{R}/R$, such as occurs in the DISLASH laboratory hydrafrac experiments described in Chapter 3, where fracture growth is studied using rubber samples. For conditions representative of these laboratory experiments, performed at MIT’s Resource Extraction Laboratory, Figure 2.2 shows that the crack-opening rate is a sizeable fraction (e.g. from 10-40%) of the radial fluid velocity for points within a distance of $0.001R$ from the crack tip. In this region, therefore, pressure gradients normal to the crack surface might be expected to be comparable to the radial pressure gradients. Although this is a very small region, the pressure gradients in it are very large and Figure 2.3 shows that, under the conditions of the previous figure, most of the pressure difference between the tip and the wellbore occurs within this region, so an accurate description of the pressure gradient here is critical.

A detailed analysis of the flow field in the near-tip region for a Newtonian fluid is described in Appendix D. The large radius of curvature of the fracture surface (being several orders of magnitude greater than the crack width even in this region) allows the flow to be computed for the case of a crack which has a linear opening profile.
The ratio of the lateral pressure gradient across the crack width to the radial pressure gradient is found to be

\[
\frac{\text{lateral pressure gradient}}{\text{radial pressure gradient}} = \frac{1 - \cos \theta_w}{\theta_w \cos \theta_w} = \frac{1}{2} \theta_w + \frac{5}{24} \theta_w^3 + \frac{61}{720} \theta_w^5 \tag{2.5.2.1}
\]

where \( \theta_w \) is half the angle subtended by the fracture surfaces.

The difference in pressure between the fracture wall and the center of the flow channel is found to be

\[
P_{\text{fracture wall}} - P_{\text{channel center}} = \frac{2 \mu \dot{R}}{\dot{r}} \tan \theta_w \left( 1 - \cos \theta_w \right) \left( \theta_w - \frac{1}{2} \sin 2 \theta_w \right) \tag{2.5.2.2}
\]

where \( \dot{r} = \delta/2 \sin \theta_w \) is the (effective) distance from the crack tip.

The pressure gradient along the wall of the fracture surface, projected onto the plane of the fracture, is found to be

\[
\left. \frac{\partial P}{\partial r} \right|_{\text{fracture wall}} = \frac{-12 \mu \dot{R}}{\delta^2} \left( 1 + \frac{1}{5} \delta^2 \right), \tag{2.5.2.3}
\]

which is nearly the same as the gradient for the parallel plates case. Consequently, even in the near-tip region where the flow field is strongly two-dimensional, the parallel plate relation is generally appropriate.

2.6 Stress Intensity Factor

The stress intensity factors \( K_i \) measure the strength of the (theoretical) singularity in the stress field at the tip of a fracture in a linearly elastic material. The stresses and displacements near the crack tip can be obtained by the
superposition of three "modes" of crack opening. In Mode I the crack experiences only opening displacements of the surface points (no sliding parallel to the fracture surface). Mode II and Mode III conditions are associated with lateral relative displacements (sliding) of the fracture surfaces, due to shear tractions on the fracture surface. Specifically, for a crack occupying the half plane $\xi=0, \eta<0$ of a Cartesian coordinate system $(\hat{x}, \hat{y}, \hat{z})$, the Mode I, II or III situations are characterized by crack face displacements in only the $\hat{x}, \hat{y}$ or $\hat{z}$ directions, respectively. For the mode $i$ case, the near-tip stresses are $\sigma_{jk} = K_i R_{jk}(\phi)/\sqrt{2\pi R}$, where $R_{jk}$ are well-known functions of the angular position $\phi$ and $R$ is the radial distance from the crack tip [Sneddon and Lowengrub, 1969]. While nonlinear effects will prevent the actual stress from being infinite, the above expression for stresses is still valid for values of $R$ much larger than the size of the region of nonlinearity and much smaller than other dimensions of the problem.

Since axisymmetric fluid flow conditions do not produce lateral fracture displacements, only the Mode I stress intensity factor is considered here. Under circumstances where previously existing fractures are being reopened, such that the fracture is not necessarily aligned with the current principal stresses, shear stresses may be relieved on the fracture surfaces, requiring consideration of $K_{II}$ and $K_{III}$ contributions. The computation of these contributions could be trivially added to the current formulation, using integrals analogous to those presented below, with the shear stresses replacing the excess pressure.
For a circular fracture, the Mode I stress intensity factor $K_I$ is expressible as a weighted integral of the excess pressure in the fracture [Sneddon and Lowengrub, 1969]:

$$K_I = \frac{2}{\sqrt{\pi R}} \int_0^R \frac{drp(r)r}{\sqrt{R^2 - r^2}}. \quad (2.6.1)$$

It is useful to split this integral to highlight the separate contributions from the fluid-filled region and the "non-penetrated zone" of length $\omega$ between the fluid front and the crack tip

$$K_I = \frac{2}{\sqrt{\pi R}} \int_0^{R-\omega} \frac{drp(r)r}{\sqrt{R^2 - r^2}} + \sqrt{\frac{8\omega}{\pi}} p_{\text{ip}} \sqrt{1 - \frac{\omega}{2R}}. \quad (2.6.2)$$

For fractures growing in impermeable solids, the excess pressure within the non-penetrated zone, $p_{\text{ip}}$, is approximately $-\sigma_e$, since the absolute fluid pressure in this region is virtually zero. It may be elevated slightly by the presence of any vapor of the fluid within the fracture, but this would generally be a negligible effect.

If the solid is permeable, then $p_{\text{ip}}$ is dependent on the extent to which pore fluid flows into the non-penetrated zone. The excess pressure in this region is then between $-\sigma_e$ and $P_{\text{pore}} - \sigma_e$, with higher values resulting from more flow of pore fluid into the non-penetrated zone. The volume of pore fluid which flows into this region increases with high "leak-in" rates (caused by large permeability, high porosity, low pore fluid viscosity, and high pore fluid pressure) as well as with slow fracture propagation which allows more time for fluid transport. If the pore
fluid is a gas, its low viscosity will lead to little pressure variation across the non-penetrated zone, and $p_{tip}$ may be essentially uniform. It is conceivable that the gas could move with the crack front and accumulate to the point where the gas pressure eventually reached a steady state value of $P_{pore}$. If the pore fluid is a liquid, the region ahead of the fracturing fluid will be composed of two zones. The first, just behind the tip, is a region in which the pore fluid has leaked into the crack but does not fill the crack, so that the pressure acting on the fracture faces is the pressure in any vapor phase which is present in this region. Another zone, behind the first, is one in which the pore fluid does fill the crack width. This fluid-filled zone can be treated as part of the region containing fracturing fluid, with perhaps different fluid properties from the actual fracturing fluid.

Therefore, for the case of either gas or liquid pore fluid, there is, in general, a region ahead of the fluid front in which the pressure acting on the fracture faces is controlled by vapor pressure and therefore is uniform.

Because the weighting factor for pressure in the integral for $K_r$ becomes very large near the crack tip (being singular at the tip itself), and because the magnitude of the excess pressures near the crack tip is often much larger than the magnitude of the excess pressures within the rest of the fracture, the contribution to $K_r$ from pressure in a small near-tip region is comparable to the contribution from the pressures in rest of the fracture. In fact, the near-tip pressure distribution can be viewed as setting the level of the (nearly uniform) pressure in the rest of the fracture such as to achieve the proper balance in $K_r$ at the tip. This highlights the
importance of properly determining the size of the non-penetrated zone at the crack tip and of accurately computing the pressure distribution in the near-tip region. Computationally, the non-penetrated zone sets the length scale for the numerical refinement needed to adequately analyze the near-tip region.

In the current work, it is assumed that $P_{ap}$ is uniform and equal to $-\sigma_c$ (i.e. $P_{ap} = 0$). The discretized expressions used to implement Equation (2.6.1) for computations are given in Appendix E.

### 2.7 Fluid Interface Pressure

There is a difference in pressure across the fluid interface separating the fracturing fluid from the non-penetrated zone, due to the effect of surface tension. This jump in pressure is expressed as

$$\Delta P_{ff} = \frac{\gamma_{st}}{\rho_{ff}} = \gamma_{st} \cdot \frac{\cos(\psi)}{(\delta_f/2)}$$

(2.7.1)

where $\gamma_{st}$ is the surface tension, $\rho_{ff}$ is the radius of curvature of the interface, $\delta_f$ is the crack width at the fluid front, and $\psi$ is the contact angle the liquid interface makes with the solid face of the fracture, measured through the vapor phase.

While this surface-tension effect is generally small for a propagating fracture, it becomes significant as the fracture tip slows down and the fluid penetrates closer to the tip into a narrower fracture width. It is possible that the surface-tension effect allows for a smooth transition from the rapidly propagating
regime to one in which the fracture propagation has nearly stopped. In the latter case, the low flowrates in the fracture result in small pressure variation between the wellbore and the fluid front, in which case the pressure drop across the interface would be approximately equal to $P_{wb} - P_{tip}$.

2.8 Fluid Leakoff

The flow of fluid from the fracture into the surrounding pore space (or vice versa in the near-tip region where fracture fluid pressures are below the pore pressure) can be adequately analyzed as a localized phenomenon in geotechnical applications because the slow diffusion of the fluid through the pore matrix limits the extent to which disturbances propagate. Leakoff rates are currently computed in A3DH for conditions in which the pore pressure $P_{pore}$ outside the leakoff zone is not affected by the leakoff, such as in an underground natural gas reservoir. In this case the superficial velocity (equal to the leakoff volume flowrate per unit area of fracture surface) of the fluid leaking into the porous solid is

$$v_i = \frac{k_i}{\mu_i} \cdot \frac{(P - P_{pore})}{l_i}$$

(2.8.1)

where $k_i$ is the permeability of the solid to the leakoff fluid, $\mu_i$ is the viscosity of the fluid being leaked off, and $l_i$ is the depth of the leakoff zone adjacent to the fracture face. The time rate of change of $l_i$ is

$$\frac{dl_i}{dt} = \frac{v_i}{\phi_s}$$

(2.8.2)

where $\phi_s$ is the porosity of the solid. Combining Equations (2.8.1, 2.8.2) and denoting the leakoff rate through both fracture surfaces as $q_i'' = 2v_i$ yields
where \( \tau_l \) is the time at which leakoff began at the point on the fracture under consideration.

Alternative representations of fluid leakoff (e.g. [Vogeler et al., 1985] or [Crockett et al., 1986b]) can readily be substituted in A3DH by a simple module replacement.

Two-dimensional leakoff effects near the crack tip would become important for very slow crack growth or very high solid permeability, with flow of fluid through the porous solid toward the crack tip, parallel to the fracture surface, becoming comparable to the near-tip flow within the fracture. This flow outside the fracture serves as a source of "leak-in" fluid, increasing the pressure in the region immediately behind the crack tip and reducing the role required of other mechanisms, such as surface tension effects, in providing a large pressure drop at the fluid front for a slow-moving fracture. A criterion to determine whether this porous flow parallel to the crack surface is significant is the condition of the superficial velocity of the flow through the porous solid being at least equal to the crack propagation speed,

\[
q_l'' = \frac{2k_i \phi_s}{\mu_i (t - \tau_l)} (P - P_{pore})
\]  

(2.8.3)
An equivalent criterion would be for the superficial velocity through the porous medium to be at least equal to the fluid velocity in the crack channel, at a common pressure gradient:

\[ \frac{v_s}{v} = \frac{-k_i \frac{dP}{dr}}{-\frac{\delta^2}{\mu} \frac{dP}{dr}} = \frac{\mu k_i}{\mu_i \delta^2} \geq 1. \quad (2.8.5) \]

### 2.9 Iterated Values

Since many of the conditions which must be satisfied in determining the state of a propagating circular hydraulic fracture rely on parameters and distributions which cannot be specified \textit{a priori}, but which must ultimately have values consistent with each other, the process of computing a consistent state is an iterative one.

Nonlinearities in the relationships among these parameters preclude the direct solution of a consistent set of values. One example of these extremely nonlinear inter-relationships serves to illustrate the situation: the time derivative of the crack opening distribution is computed from the current and previous crack width distributions, the crack width is a weighted integral of the pressure distribution, the pressure is an integral of the pressure gradient, the pressure gradient is proportional to the flowrate (divided by the cube of the crack width), and the flowrate is the integral of the crack opening rate distribution which was computed to begin the sequence.
In particular, iteration is necessary to achieve consistent values for:

1. pressure distribution and the resulting crack opening distribution
2. fluid leakoff rates
3. crack propagation speed
4. size of the non-penetrated zone
5. interface pressure at the fluid front
6. crack-opening rates consistent with crack-opening values at the current and previous timesteps.

It is this last requirement, achieving a consistent crack opening rate distribution, which proves to be the most difficult. This is largely because both the crack opening rates and the pressure gradients are very high near the crack tip. The extreme sensitivity of the pressure gradient to the crack width means that changes made in the crack opening rates (and hence in the crack width values) while iterating toward a converged solution result in large relative changes in the distributions of flowrate and pressure near the crack tip and throughout the fracture.

2.10 Discretization

Development of a procedure which computes the fracture and fluid conditions consistent with the foregoing equations requires that the integrals and derivatives be approximated by algebraic equations which can be solved simultaneously or iteratively. The resulting formulation is implemented in a computer program. This program has been named A3DH, indicating that it is the axisymmetric counterpart to the fully 3-D hydrafac simulator called R3DH.
The first step in this numerical implementation is the discretization of the physical domain. Although this could entail discretizing the entire solid in which the fracture is propagating, the influence function for the crack opening, $w_6(p,s)$ in Equation (2.3.3), allows the computational domain to be limited to the fracture surface itself, because $w_6(p,s)$ accounts for the response of the entire solid to the tractions on the crack faces.

Since the problem is axisymmetric, it is sufficient to discretize the crack surface along a single representative radial cut of the fracture. The distribution of each variable as a function of radius is characterized by a set of discrete values at a finite number of "nodes" at different radii $r_n$, with values of the variable in the region between adjacent nodal points interpolated from the nodal values.

The region between the wellbore radius $r_{wb}$ and the fluid-front radius $r_{ff}$ is divided into $N_f$ intervals $[r_n, r_{n+1}]$ with $r_n < r_{n+1}$ for $n = n_{wb}, \ldots, n_{ff}-1$ and $r_{n_{wb}} = r_{wb}, r_{n_{ff}} = r_{ff}, N_f = n_{ff} - n_{wb}$.

The assignment of the nodal positions makes use of three regions (Figure 2.4), with nodal spacings in each region chosen to avoid drastic changes between adjacent nodes in the variable which tends to change most rapidly there.

In the region near the wellbore where the flow divergence can give rise to a large radial variation in fluid flowrate, the nodal radii $r_n$ are constrained by the relation

$$r_n/r_{n-1} \leq 1 + \epsilon_{wb} \quad (2.10.1)$$
where \( r_n \) is the radial position of node \( n \) and \( \varepsilon_{ub} \) is a specified limiting value above unity for the ratio. This constraint limits the amount of change in flow that is allowed between adjacent nodes.

In the region near the fluid front, the pressure gradients are very high because of the small crack-opening values \( \delta(r) \) (cf. Equation (2.5.4)), requiring very small nodal spacings immediately behind the fluid front and gradually larger spacings farther from the front. In this region, the nodal radii are constrained by the relation

\[
\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \leq 1 + \varepsilon_{ff}
\]

where \( \varepsilon_{ff} \) is a specified limiting value above unity for the ratio. This region typically requires about half of the total number of nodes.

Finally, in the region near the zero excess-pressure value, \( \frac{\partial^2 P}{\partial r^2} \) is large, and the rapidly varying pressure gradient would be expected to lead to errors when integrating to obtain the pressure distribution if nodal spacings are too large. Therefore, in the region between the wellbore and tip regions, a maximum relative nodal spacing \( \Delta r_{\text{max}}/R \) is enforced:

\[
\frac{r_n - r_{n-1}}{R} \leq \frac{\Delta r_{\text{max}}}{R}.
\]

Typically about one-third of the nodes fall in this region.

The first nodal spacing behind the fluid front is assigned to be a small fraction (typically 1%) of the size of the non-penetrated zone. Successive nodal
spacings moving away from the fluid front are increased by the factor \((1+\varepsilon_f)\), until the maximum spacing size \(\Delta r_{\text{max}}\) is reached.

The first node beyond the node at the wellbore radius \(r_{wb}\) is at a radius \((1+\varepsilon_{wb})\) times \(r_{wb}\) and successive nodal radii increase by the same factor, again until the maximum spacing size \(\Delta r_{\text{max}}\) is reached.

The region between the wellbore and tip regions is filled with nodes spaced at the maximum value.

Values for the spacing parameters which have been found to give good results are \(\varepsilon_{wb}=0.05, \varepsilon_f=0.1\). The value of \(\Delta r_{\text{max}}\) should be as small as can be accommodated by the number of nodes used. Using \(\Delta r_{\text{max}}=2R/N_f\) has been found to work well. Some flexibility is needed in setting the values of these parameters, since a consistent set of values for a given crack geometry can only be obtained by iteration. Denoting the suggested (reference) values given above as \(\varepsilon_{wb(\text{ref})}, \varepsilon_f(\text{ref})\), and \(\Delta r_{\text{max}(\text{ref})}\), a variation coefficient \(\eta\) is defined and evaluated by iterating to obtain a set of "actual" spacing parameters \(\varepsilon_{wb(\text{act})}, \varepsilon_f(\text{act}), \Delta r_{\text{max}(\text{act})}\) which are self-consistent under the current conditions. The expressions

\[
\begin{align*}
\varepsilon_{wb(\text{act})} &= \eta \varepsilon_{wb(\text{ref})} \\
1 + \varepsilon_f(\text{act}) &= \eta (1 + \varepsilon_f(\text{ref}) ) \\
\Delta r_{\text{max}(\text{act})} &= \Delta r_{\text{max}(\text{ref})}
\end{align*}
\]  

(2.10.4)

are used to relate the two sets of parameters. If the \(\varepsilon_{wb(\text{act})}\) or \(\varepsilon_f(\text{act})\) values exceed limiting prescribed maximums (e.g. \(\varepsilon_{wb(\text{act})} > 1.0, \varepsilon_f(\text{act}) > 3.0\)), \(\Delta r_{\text{max}(\text{act})}\) is increased.
to the extent needed to keep $\varepsilon_{wb(\text{act})}$ and $\varepsilon_{\text{eff}(\text{act})}$ within the allowable limits, although this is an indication that more nodal points are probably needed.

Just as the physical region needs to be discretized to permit numerical solution of the governing equations, discretization in time is also required. The continuum of time is divided into successive "timesteps" of varying length and the instantaneous state of the fracture is solved for at the end of each timestep. The position and condition of the fracture obtained for the previous timestep are used with assumptions regarding the time variation of specific variables to determine any needed time derivatives. The size of each timestep is determined by the fracture propagation speed $\dot{R}$. The increase in the crack tip radius $\Delta R$ over the timestep is specified to be a certain fraction (typically 1 - 5%) of the value at the previous timestep. The timestep size $\Delta t$ is then $\Delta t = \Delta R/\dot{R}$. As the propagation speed is adjusted to satisfy the volume conservation conditions, the timestep size is adjusted accordingly.

2.11 Crack-Opening Rates

Obtaining a consistent distribution of crack opening rates $\dot{\delta}(r)$ has required considerable effort. Previous attempts at obtaining axisymmetric hydraulic fracture simulators with general transient behavior were unsuccessful and either ultimately relied on self-similar expressions for calculating crack opening rates [Cleary and Wong, 1983] or produced unacceptable (oscillatory) results [Cleary et al., 1978-1981; Cleary and Wong, 1985].
The first simulator versions developed in this current work were implicit methods which incorporated into the governing equations expressions for \( \delta(r) \) in terms of the current and previous crack opening distributions. This is a generally desirable scheme since it reduces the number of variables to be solved for and automatically ensures that \( \dot{\delta}(r) \) and \( \delta(r) \) are consistent. However, once a consistent \( \ddot{\delta}(r) \) distribution is obtained, additional iterations are still needed to solve for consistent values for the remaining variables, and since there is no direct control on the extent to which the \( \dot{\delta}(r) \) values change during those iterations, it was found that these \( \dot{\delta}(r) \) changes, especially near the crack tip, led to large fluctuations in the other variables which prevented convergence.

Consequently, an explicit scheme was implemented in which an estimate for \( \dot{\delta}(r) \) is initially specified and a self-consistent pressure distribution is found which agrees with the prescribed \( \delta(r) \). Only after this has been done is the consistency of \( \dot{\delta}(r) \) itself checked and the distribution revised, if needed. The iteration process by which a consistent \( \dot{\delta}(r) \) distribution is obtained is described in Section 2.15.

Three different methods were used to compute \( \dot{\delta}(r) \) from the \( \delta(r) \) values at the current and previous timesteps, in an effort to reduce the errors involved as much as possible. These methods are described in Appendices F and G.
2.12 Consistent Pressure Distribution

Two different formulations have been used in this work to obtain a pressure distribution \( P(r) \) which is consistent with the specified crack opening rate distribution \( \delta(r) \) and the resulting flowrate distribution \( q'(r) \). In this context, \( P(r) \) is considered to be consistent when its gradient \( \partial P/\partial r \), at every \( r \), is equal to the value given by the rheology relation

\[
\frac{\partial P}{\partial r} = F_{\text{rheol}}(q'(r), \Delta(r)). \tag{2.12.1}
\]

The crack opening values \( \delta(r) \) used in this equation must, of course, be the values which result from \( P(r) \), and are given by Equation (2.3.3).

Each formulation can be considered to be a transformation \( \Phi \) which uses a current estimate for the vector of nodal pressure values \( P_{\text{est}} \) to produce a vector of resultant nodal pressure values \( P_{\text{res}} \):

\[
P_{\text{res}} = \Phi(P_{\text{est}}). \tag{2.12.2}
\]

The pressure distribution is consistent when the two pressure vectors are equal,

\[
P_{\text{est}} = P_{\text{res}} = \Phi(P_{\text{est}}). \tag{2.12.3}
\]

Since the transformations \( \Phi \) are nonlinear, each method forms the basis for an iterative process to find \( P_{\text{est}} \) which satisfies Equation (2.12.3). The methods used to revise the estimated pressure distribution, in a manner which allows smooth and consistent convergence, is described in Appendix H.
As described in Section 2.15, the pressure iteration must be combined with other iterations to satisfy all of the required conditions.

The first method for obtaining a consistent $P(r)$ is based on successive integrations of the pressure distribution. The second uses a variational approach and results in a finite element scheme. The integration scheme is generally faster than the finite element method, because of fewer calculations required, and was the first method tried. Difficulties in achieving a consistent crack-opening-rate distribution with this scheme prompted the development of the finite element method, which was not expected to have the same problem since the various conditions are satisfied simultaneously, rather than being invoked sequentially, as in the integration scheme. The finite element method was found to exhibit the same behavior, however. A convergent algorithm was eventually developed for obtaining consistent crack-opening rates with the finite element method, at which point the algorithm was also applied to the integration scheme and found to work just as well. The details of the two formulations are presented in the following two sections.

2.13 Successive Integration Scheme

Beginning with the current estimate of the pressure distribution in the fracture $P_{est}(r)$, it is possible to apply a series of transformations in succession and eventually recover the pressure distribution again. Since more than half of the
transformations involved are integrals, the process is being referred to here as one of "successive integrations".

The scheme starts by computing the crack-opening values resulting from the estimated excess pressure distribution \( p_{\text{est}}(r) = p_{\text{est}}(r) - \sigma_c \), using Equation (2.3.3)

\[
\delta(r) = \int_0^1 ds \ p_{\text{est}}(s) \ w_\delta(r/R, s).
\]  

(2.13.1)

If the crack-opening-rate distribution \( \dot{\delta}(r) \) were not specified, the \( \delta(r) \) values at the current time would be used with the \( \delta(r) \) values at an earlier time to determine \( \dot{\delta}(r) \). In the present work, however, a distribution \( P(r) \) is being sought which is consistent with a currently assumed distribution for \( \dot{\delta}(r) \), so this step is not needed.

The crack-opening rates \( \delta \), along with the leakoff rates \( q'' \), are then integrated to determine the fluid flowrate \( q(r) \) as indicated by Equation (2.4.1):

\[
q(r) = q(r_{\text{wb}}) - \int_{r_{\text{wb}}}^r 2\pi r' \left\{ \dot{\delta}(r') + q''(r') \right\} dr'.
\]  

(2.13.2)

Using the flowrate per unit perimeter \( q'(r) = q(r)/(2\pi r) \) and the crack openings \( \delta(r) \) in the appropriate rheological flow relation provides the pressure gradient

\[
\left. \frac{\partial P}{\partial r} \right|_r = F_{\text{rheol}}(q'(r), \delta(r)).
\]  

(2.13.3)

Finally, integrating the pressure gradient yields the resulting pressure distribution,

\[
P_{\text{res}}(r) = P(r_{\text{ff}}) - \int_{r_{\text{ff}}}^r \frac{\partial P}{\partial r'} dr'.
\]  

(2.13.4)
2.14 Finite Element Scheme

The finite element implementation of the governing equations developed here is based on the volume conservation equation, Equation (2.4.1):

\[ \frac{1}{r} \frac{d}{dr} (rq'(r)) + q_i''(r) + \delta(r) = 0. \]  \hspace{1cm} (2.14.1)

Requiring that this equation be satisfied for all radii \( r_{wb} \leq r \leq r_{ff} \) is equivalent to requiring that the equation

\[ \int_{r_{wb}}^{r_{ff}} \Theta(r) \left[ \frac{1}{r} \frac{d}{dr} (rq') + q_i'' + \delta \right] dr = 0 \]  \hspace{1cm} (2.14.2)

be satisfied when \( \Theta(r) \) is an arbitrary function of \( r \). Using integration by parts eliminates the derivative of the flowrate

\[ \int_{r_{wb}}^{r_{ff}} \left[ \Theta \left( \frac{q'}{r} + q_i'' + \delta \right) - q' \frac{d\Theta}{dr} \right] dr = -\left[ \Theta q' \right]_{r_{wb}}^{r_{ff}}. \]  \hspace{1cm} (2.14.3)

Applying the integration separately to each of the discretization intervals introduced in section 2.10 gives

\[ \sum_{n=n_{wb}}^{n_{ff}+1} \int_{r_n}^{r_{n+1}} \left[ \Theta \left( \frac{q'}{r} + q_i'' + \delta \right) - q' \frac{d\Theta}{dr} \right] dr = -\left[ \Theta q' \right]_{r_{wb}}^{r_{ff}}. \]  \hspace{1cm} (2.14.4)

The value of a general field variable \( X(r) \) (defined as any parameter which varies with radius, e.g. pressure, crack opening, flowrate, etc.) at a point within the \( n^{th} \) interval \( r_n \leq r \leq r_{n+1} \) is computed by linear interpolation within the interval. Defining a two-element local interpolation vector \( \tilde{L} \), and using a normalized position \( s_n = (r-r_n) / (r_{n+1}-r_n) \), \( 0 \leq s_n \leq 1 \), at point \( r \) within the \( n^{th} \) interval, this
interpolation is expressed as the dot product between $\vec{L}$ and a two-element local field vector $\vec{X}_n$ for the $n$\textsuperscript{th} interval,

$$X(s_n) = \vec{L} \cdot \vec{X}_n = \vec{L}^T \vec{X}_n$$

where

$$\vec{L} = \begin{bmatrix} (1-s_n) \\ s_n \end{bmatrix}, \quad \vec{X}_n = \begin{bmatrix} \dot{X}_n(1) \\ \dot{X}_n(2) \end{bmatrix}$$

$\dot{X}_n(1) = X(r_n), \quad \dot{X}_n(2) = X(r_{n+1})$.

$\vec{X}_n$ represents any of two-element local vectors $\vec{P}_n, \vec{\delta}_n, \vec{\delta}_n'', \vec{\theta}_n$ for the $n$\textsuperscript{th} interval. The derivative of a field variable with respect to $r$ can similarly be expressed as a product of the local field vector and a local gradient interpolation vector $\vec{M}_n$

$$\frac{dX(r)}{dr} = \frac{d}{ds} \left( \vec{L}^T \vec{X}_n \right)$$

$$= \frac{d}{ds} \left( \vec{L}^T \vec{X}_n \right) \frac{ds_n}{dr}$$

$$= \left( \frac{d}{ds} \vec{L}^T \frac{ds_n}{dr} \right) \vec{X}_n$$

$$= \frac{1}{\Delta r_n} \begin{bmatrix} -1 & 1 \end{bmatrix} \vec{X}_n$$

$$= \vec{M}_n^T \cdot \vec{X}_n$$

where $\Delta r_n = r_{n+1} - r_n$ and $\vec{M}_n = \frac{1}{\Delta r_n} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Using these definitions and Equation (2.5.4) in Equation (2.14.4) gives
Then, since $\hat{\Theta}_n^T \hat{\Theta}_n = \hat{\Theta}_n^T \hat{\Theta}_n$ and $\hat{\Theta}_n^T \hat{\Theta}_n = \hat{\Theta}_n^T \hat{\Theta}_n$ and since $\hat{\Theta}_n^T$ is independent of $s_n$, Equation (2.14.8) becomes

$$\sum_{n=n_w}^{n_r-1} \Delta r_n \int_0^1 ds_n \left\{ -\frac{(L^T \delta_n)^3}{\mu r} \hat{M}_n^T \hat{P}_n + L^T \hat{q}_n'' + L^T \hat{\delta}_n \right\} + \frac{(L^T \delta_n)^3}{\mu} \hat{M}_n^T \hat{M}_n^T \hat{P}_n = -[\Theta q']_{r_{wb}}.$$  

(2.14.9)

Global vectors $X$ are now defined, corresponding to the local vectors $X_n$, $n = n_w, \ldots, n_f$ with $X(n)$, the $n_{th}$ element of $X$, being equal to the value of $X$ at $r=r_n$, i.e.

$$X(n) = X(r_n) = \hat{X}_n (1) = \hat{X}_{n-1} (2).$$  

(2.14.10)

Introducing global interpolation vectors $L_n$ allows determination of values of a field variable $X$ at points within interval $n$, using the global vector $X$. The global interpolation vectors therefore satisfy the relation

$$L_n (s_n) \cdot X = \hat{L}(s_n) \cdot \hat{X}_n$$  

(2.14.11)
for \( s_n = (r - r_n) / (r_{n+1} - r_n) \) within the region \( r_n \leq r \leq r_{n+1} \).

Similarly, global gradient interpolation vectors \( \mathbf{M}_n \) are defined such that

\[
\mathbf{M}_n(s_n) \cdot \mathbf{X} = \mathbf{M}_n(s_n) \cdot \mathbf{X}_n. \tag{2.14.12}
\]

Also defining

\[
q'_\text{in} = \begin{bmatrix}
q'_{\text{in}(n_{wb})} \\
\vdots \\
q'_{\text{in}(n_{ff})}
\end{bmatrix}
= \begin{bmatrix}
q'(r_{wb}) \\
0 \\
\vdots \\
0 \\
-q'(r_{ff})
\end{bmatrix}, \tag{2.14.13}
\]

where \( q'(r_{wb}) \) is the flow into the fracture at the wellbore and \( q'(r_{ff}) \) is the flow toward the fracture tip at the fluid front radius \( r_{ff} \), allows the substitution

\[
\Theta q'_\text{in} = \Theta q'_\text{in}(r_{wb}) - \Theta q'_\text{in}(r_{ff}) = \Theta \cdot q'_\text{in} = \Theta q'_\text{in}. \tag{2.14.14}
\]

In terms of these global vectors, the governing equation then becomes

\[
\sum_{n=1}^{n_{wb}} \Theta^T \Delta r_n \int_{s_n}^1 ds_n \left\{ \frac{(L_n^T \delta)^3}{\mu} \left( M_n - \frac{1}{r} L_n \right) M_n^T P + L_n L_n^T (q'' \delta) \right\} - \Theta^T q'_\text{in} = 0. \tag{2.14.15}
\]
Now, since $\Theta^T$ can be taken outside the summation, Equation (2.14.15) has the form

$$\Theta^T \left( \sum_{n=n_{\text{ab}}}^{n_{\tau}-1} Y_n - q'_n \right) = 0 \quad (2.14.16)$$

and, since $\hat{\Theta}^T$ is arbitrary, this is equivalent to

$$\sum_{n=n_{\text{ab}}}^{n_{\tau}-1} Y_n - q'_n = 0 \quad (2.14.17)$$

so that the governing relation becomes

$$\sum_{n=n_{\text{ab}}}^{n_{\tau}-1} \Delta r_n \int_0^1 ds_n \left\{ \frac{(L_n^T \delta)^3}{\mu} \left( M_n - \frac{1}{r} L_n \right) M_n^T \right\} - L_n L_n^T (q'' + \hat{\delta}) - q'_n = 0. \quad (2.14.18)$$

This can be expressed as an explicit equation for $P$ if the flowrates and crack opening rates are known:

$$[B] P = Q \quad (2.14.19)$$

where

$$[B] = \sum_{n=n_{\text{ab}}}^{n_{\tau}-1} \Delta r_n \int_0^1 ds \left( \frac{L_n^T \delta}{\mu} \right)^3 \left( M_n - \frac{1}{r} L_n \right) M_n^T \quad (2.14.20)$$

$$Q = q'_n - \sum_{n=n_{\text{ab}}}^{n_{\tau}-1} \Delta r_n \int_0^1 ds \ L_n L_n^T (q'' + \hat{\delta}). \quad (2.14.21)$$
Equation (2.14.19) is the pressure transformation for the finite element scheme, since both \([B]\) and \(Q\) are dependent on the pressure distribution. To make this readily apparent, Equation (2.14.19) can be written as

\[
\Phi(P_{\text{est}}) = P_{\text{res}} \tag{2.14.22}
\]

where \(\Phi\) transforms the estimated pressure distribution \(P_{\text{est}}\) to the resulting pressure distribution \(P_{\text{res}}\), with

\[
\Phi(P) = [B(P)]^{-1} Q(P). \tag{2.14.23}
\]

A consistent distribution has been obtained when \(P_{\text{est}} = P_{\text{res}}\).

### 2.15 Overall Iteration Procedure

The highly nonlinear inter-relationships among the governing equations means that their solution is an iterative process, as was discussed briefly in Section 2.9. Because it is not feasible to converge simultaneously on all the different conditions that must be satisfied, the iteration procedure used necessarily involves several levels of iteration loops.

The innermost iteration loop (Level 1) provides a consistent pressure distribution \(P(r)\) for the current estimate of the crack opening rate distribution \(\dot{\delta}_{\text{est}}(r)\). The resulting \(P(r)\) satisfies the rheology relation

\[
\left. \frac{\partial P}{\partial r} \right|_r = F_{\text{rheol}} (q'(r), \delta(r)) \tag{2.15.1}
\]
where \( \delta(r) \) are the crack openings consistent with \( P(r) \) and \( q'(r) \) are the flowrates consistent with \( \delta_{est}(r) \) and \( q''(r) \).

Once a consistent pressure distribution has been obtained, the resulting \( \delta(r) \) for the current timestep is used, along with \( \delta(r) \) from the previous timestep, to compute the resulting crack opening rates \( \delta_{res}(r) \), using one of the methods described in Appendix F.

The second level of iteration (Level 2) is the process of getting \( \delta_{res}(r) \) to agree with the estimated values \( \delta_{est}(r) \).

This task is complicated by the non-linear manner in which the different parts of the fracture interact, causing revisions in \( \delta(r) \) at any radius \( r \) to result in changes in \( \delta(r) \) at all other radii and in the crack propagation rate \( \dot{R} \), in ways that are difficult to predict without excessive computation time. These resulting fluctuations in \( \delta(r) \) are typically at least as large as the errors which are to be eliminated, making convergence difficult.

Convergence to a consistent \( \delta(r) \) can be accomplished in spite of these complications by separating the \( \delta(r) \) distribution into two terms, one which is predictable and which accommodates the bulk of the changes in crack opening rates needed to maintain volume conservation and a second term which remains relatively steady in spite of larger fluctuations in the total crack opening rate.
Toward this end, the estimated distribution is expressed as a self-similar distribution plus a deviation term:

\[ \delta(r)_{\text{est}} = \delta(r)_{\text{as}} + \delta(r)_{\text{dev}} \]  \hspace{1cm} (2.15.2)

The self-similar term represents the crack opening rates under conditions for which the relative shape of the crack opening distribution remains constant as the opening changes with time. The derivation of this term is given in Appendix G and results in the estimated crack opening rate becoming expressed as

\[ \delta(r)_{\text{est}} = \delta(r) \cdot \frac{\delta(0)}{\delta(0)} - r \frac{\delta\delta}{\delta r} \frac{\dot{R}}{R} + \delta(r)_{\text{dev}}. \]  \hspace{1cm} (2.15.3)

The deviation term can contribute a significant fraction (e.g. 25%) of the total opening rate.

Because of the difficulties in obtaining a consistent \( \delta(r) \) distribution, revisions to \( \delta(r) \) are carried out in a three-step procedure:

(a) Revise the estimated crack opening rate at the wellbore \( \delta(r_{wb})_{\text{est}} \) until it agrees with the resulting value \( \delta(r_{wb})_{\text{res}} \). In order to achieve convergence during this process, the distributions for \( \delta_{\text{dev}}(r) \) and for the estimated crack opening distribution \( \delta_{\text{est}}(r) \) which is used in computing \( \delta_{\text{as}}(r) \) had to be held fixed.

(b) Revise the estimated crack opening distribution \( \delta_{\text{est}}(r) \) used in the computation of \( \delta_{\text{as}}(r) \). While this is done, \( \delta_{\text{dev}}(r) \) is held fixed.

(c) Revise \( \delta_{\text{dev}}(r) \) until \( \delta_{\text{res}}(r) \) agrees with \( \delta_{\text{est}}(r) \).
After each revision, the previous revisions, including the revisions of \( P(r) \), must be performed again to regain the consistency which is lost when the change is made.

Once a consistent \( \delta(r) \) has been obtained, outer levels of iteration loops are required to satisfy the remaining conditions:

Level 3: Consistent surface tension pressure drop \( \Delta P_f \) across the fluid interface at the fluid front.

Level 4: Consistent size for the non-penetrated zone, to give a stress intensity factor which is equal to the critical value for the material, \( K_i = K_{ic} \).

Level 5: Check for overall volume conservation, to obtain a crack volume change over the current timestep which is equal to the time integral of the injection rate minus the total leakoff rate.

Each time a revision is made at any of these levels, the preceding conditions need to be checked again for continued consistency. In general, new revisions and iterations will be needed at each preceding level.

The typical number of iterations required for each of the iteration levels to converge is shown in Tables 2.1 and 2.2.
Figure 2.1 Equivalency of an elastic fracture problem to the superposition of two problems: A, with the far field boundary conditions and no fracture, and B, with no far field loads and a fracture present.
Figure 2.2  Ratio of crack opening rate to fluid velocity as a function of normalized distance from the crack tip, for an A3DH simulation of representative DISLASH conditions.
Figure 2.4 Representative crack pressure distribution showing the three different criteria used to determine nodal positions.
TABLE 2.1

Iterations required for convergence at each iteration level, for a typical timestep.

<table>
<thead>
<tr>
<th>Iteration Level</th>
<th>Total number of Iterations</th>
<th>Number of Iterations per Cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Pressure</td>
<td>1152</td>
<td>6</td>
</tr>
<tr>
<td>2a. $\delta(r_{wb})$</td>
<td>192</td>
<td>4</td>
</tr>
<tr>
<td>2b. $\delta(r)_{est}$</td>
<td>48</td>
<td>4</td>
</tr>
<tr>
<td>2c. $\delta(r)_{dev}$</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>3. $\Delta P_{ff}$</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>4. $K_I$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5. Volume</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE 2.2

Average number of iterations required for convergence at each iteration level, for a typical simulation, averaging the iterations from all timesteps.

<table>
<thead>
<tr>
<th>Iteration Level</th>
<th>Average number of Iterations needed for convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Pressure</td>
<td>6.3</td>
</tr>
<tr>
<td>2a. $\delta(r_{wb})$</td>
<td>3.5</td>
</tr>
<tr>
<td>2b. $\delta(r)_{est}$</td>
<td>4.2</td>
</tr>
<tr>
<td>2c. $\delta(r)_{dev}$</td>
<td>3.3</td>
</tr>
<tr>
<td>3. $\Delta P_{ff}$</td>
<td>1.9</td>
</tr>
<tr>
<td>4. $K_I$</td>
<td>1.2</td>
</tr>
<tr>
<td>5. Volume</td>
<td>1.4</td>
</tr>
</tbody>
</table>
CHAPTER 3
TESTING AND RESULTS

3.1 Introduction

In this chapter the performance of the axisymmetric hydraulic fracture simulator A3DH is tested for accuracy and validity. In the next four sections, the different functional segments of the program are checked to ensure that they are correctly and accurately performing the algorithms that they embody. In the last two sections, the predictions of the simulator are compared with the results of lab experiments to verify the adequacy of the physical modelling that has been translated into the numerical procedures of the program.

3.2 Testing Crack Opening

Tests to verify the accuracy of the calculations used in A3DH to obtain the crack-opening distribution \( \delta(r) \) resulting from a given excess pressure distribution \( p(r) \) are developed in this section. They are used to validate the numerical computations and to determine the extent to which the nodal spacing affects the error in the crack-opening values.

Using Equation (2.3.2), the crack opening \( \delta(\rho) \) at the normalized position \( \rho=r/R \) can be expressed as
\[
\delta(\rho) = \frac{2R}{\pi E} \int_0^1 \frac{du}{\sqrt{u^2 - \rho^2}} \int_0^u \frac{p(s)ds}{\sqrt{u^2 - s^2}}.
\]  

(3.2.1)

For a given set of nodal positions, the most extreme test for algorithm accuracy is the series of "top-hat" distributions shown in Figure 3.1 and defined as

\[
p_{\text{th}}(s;\rho_1,\rho_2) = \begin{cases} 
0, & 0 \leq s < \rho_1 \\
1, & \rho_1 \leq s \leq \rho_2 \\
0, & \rho_2 < s \leq 1
\end{cases}
\]  

(3.2.2)

for each \((\rho_1 = r_n/R, \rho_2 = r_{n+1}/R)\) corresponding to the discretization regions \([r_n, r_{n+1}]\).

This series of distributions can be used to test the accuracy of the computed contributions from each region \([r_n, r_{n+1}]\) to \(\delta(r_n)\) at all nodal \(r_n\). Another obvious possible set of test functions which was also tried is the set of Heaviside functions \((p_H(s;\rho_1) = 0 \text{ for } s < \rho_1, = 1 \text{ for } s > \rho_1)\). The set of distributions \(p_{\text{th}}(s;\rho_1,\rho_2)\) chosen for these tests has an advantage over \(p_H(s;\rho_1)\), however, since the Heaviside pressures involve contributions to each \(\delta(r_n)\) from many nodal regions at the same time and large contributions from one region may dominate the total crack-opening value and mask large relative errors in the smaller contribution to \(\delta(r_n)\) from another region.

Using Equation (3.2.2) in Equation (3.2.1), and referring to \(p_{\text{th}}(s;\rho_1,\rho_2)\) as simply \(p_{\text{th}}(\rho_1,\rho_2)\), gives the expressions
\[
\delta(\rho, p_{sh}(\rho_1, \rho_2)) = \frac{2R}{\pi E} \left\{ \int_{\rho_1}^{\rho_2} \frac{du}{\sqrt{u^2 - \rho^2}} \int \frac{udx}{sds} \right\} \text{ for } \rho \leq \rho_1
\]
\[
= \frac{2R}{\pi E} \left\{ \int_{\rho}^{\rho_1} \frac{du}{\sqrt{u^2 - \rho^2}} \int \frac{udx}{sds} \right\} \text{ for } \rho_1 \leq \rho \leq \rho_2
\]
\[
= \frac{2R}{\pi E} \left\{ \int_{1}^{\rho_2} \frac{du}{\sqrt{u^2 - \rho^2}} \int \frac{udx}{sds} \right\} \text{ for } \rho_2 \leq \rho
\]

Carrying out some of the indicated integrations and recognizing that other integrals can be replaced by the elliptic integral of the first kind

\[
F(\varphi, k) = \int_{0}^{\varphi} \frac{d\varphi'}{\sqrt{1 - k^2 \sin^2 \varphi'}}
\]

and the elliptic integral of the second kind

\[
E(\varphi, k) = \int_{0}^{\varphi} \sqrt{1 - k^2 \sin^2 \varphi'} \, d\varphi'
\]

allows the resulting crack-opening distributions to be written as

\[
\delta(\rho, p_{sh}(\rho_1, \rho_2)) = \frac{2R}{\pi E} \left\{ \sqrt{(1-\rho_1^2)/(1-\rho^2)} - \rho_1 E\left(\varphi(\rho_1, \rho), \frac{\rho}{\rho_1}\right) \right\}
\]
\[
- \sqrt{(1-\rho_2^2)/(1-\rho^2)} + \rho_2 E\left(\varphi(\rho_2, \rho), \frac{\rho}{\rho_2}\right) \right\} \text{ for } \rho \leq \rho_1
\]
\[\delta(\rho_1, P_{\alpha}(\rho_1, \rho_2)) = \frac{2R}{\pi E} \left\{ 1 - \rho_1 - \sqrt{\frac{1-\rho_1^2}{1-\rho_2^2}} \right\} + \rho_2 E \left( \frac{\varphi(\rho_2, \rho_1), \rho_1}{\rho_2} \right) \text{ for } \rho_1 \neq 1 \]
\[
\delta(\rho_2, P_a(\rho_1, \rho_2)) = \frac{2R}{\pi E} \left[ \left( \frac{\rho_2^2 - \rho_1^2}{\rho_2} \right) F\left( \varphi(\rho_2, \rho_1), \frac{\rho_1}{\rho_2} \right) \right.

- \left(1 - \rho_2^2/(1 - \rho_1^2) \right) \left(1 - \rho_2^2 \right) \right] \text{ for } \rho_2 \neq 1
\]

(3.2.8b)

\[
\delta(1, P_a(\rho_1, \rho_2)) = 0.
\]

(3.2.8c)

A series of tests were performed, comparing the values of \( \delta(r) \) from Equations (3.2.6, 3.2.8) with the values computed by the algorithms used in A3DH. For these tests, the wellbore radius was held at 1\% of the crack-tip radius, the nodal-spacing parameters used were those recommended in Section 2.10 (viz. \( \varepsilon_{wb} = 0.05, \varepsilon_{fr} = 0.1, \) and \( \Delta r_{\text{max}} = 2R/N_f \)). Tests were made with 25, 50, 100, and 150 nodes used to discretize the fracture, and three relative sizes of the non-penetrated zone were used, \( \omega/R = 10^{-2}, 10^{-4}, 10^{-6} \). For all these tests, the relative error in \( \delta(r) \) was less than \( 10^{-4} \) except at the origin where the relative errors were \( 10^{-3} \). The low error levels demonstrate that the calculations used for \( \delta(r) \) are sufficiently accurate, even for a coarse discretization of the fracture.

### 3.3 Testing Stress Intensity Factor

In this section, tests are developed to verify the accuracy of the stress intensity factor calculated in A3DH resulting from a given excess-pressure distribution \( p(r) \). Again, these tests are used to validate the numerical
computations and to determine the extent to which the nodal spacing affects the error in the value of $K_I$.

Using Equation (2.6.1), the stress intensity factor $K_I$ due to a distribution of excess pressure $p(s=r/R)$ is

$$K_I = 2\sqrt{R/\pi} \int_0^1 \frac{p(s)s \, ds}{\sqrt{1-s^2}}. \quad (3.3.1)$$

Now the same series of "top-hat" pressure distributions adopted in Section 3.2

$$p_{th}(s; \rho_1, \rho_2) = \begin{cases} 0, & 0 \leq s < \rho_1 \\ 1, & \rho_1 \leq s \leq \rho_2 \\ 0, & \rho_2 < s \leq 1 \end{cases} \quad (3.3.2)$$

for each pair of $(\rho_1 = r_n/R, \rho_2 = r_{n+1}/R)$ corresponding to the discretization regions $[r_n, r_{n+1}]$ are also used here as test cases. The stress intensity factor due to one of these test cases is readily seen to be

$$K_I(p_{th}(\rho_1, \rho_2)) = 2\sqrt{R/\pi} \int_{\rho_1}^{\rho_2} \frac{sd\rho}{\sqrt{1-\rho^2}} = 2\sqrt{R/\pi} \left(\sqrt{1-\rho_2^2} - \sqrt{1-\rho_1^2}\right). \quad (3.3.3)$$

A series of tests were performed which were similar to those used to check the crack opening relations, comparing the values of $K_I$ from Equation (3.3.3) with the values computed by the algorithms used in A3DH. The same nodal spacing parameters were used as were employed in those tests. Again tests were made with 25, 50, 100, and 150 nodes used to discretize the fracture, and three relative sizes of the non-penetrated zone were used, $\omega/R = 10^2, 10^4, 10^6$. For all these tests, the relative error in $K_I$ was less than $5 \times 10^{-8}$, even though the values of $K_I$ ranged over...
many orders of magnitude. The low error levels again demonstrate that the calculations used for $K_f$ are sufficiently accurate, even for a coarse discretization of the fracture.

3.4 Testing Fluid Flow

To test the accuracy of the portion of A3DH which solves the fluid flow relations for the pressure distribution, the results of the simulator for two test cases were compared against known analytical results. The first case is the steady flow of a Newtonian fluid between parallel plates having a uniform spacing of $\delta_0$, with a specified flowrate $Q_{wb}$ and a specified pressure $P(R)$ at the crack tip. The pressure distribution for this case is

$$P(r) = P(R) + \frac{12 \mu Q_{wb}}{2 \pi \delta_0^3} \ln(R/r) \quad (3.4.1)$$

The second test case is the same except that the crack opening varies linearly with radius

$$\delta(r) = \delta_{wb} \cdot \left( \frac{R-r}{R-r_{wb}} \right) \quad (3.4.2)$$

and the pressure is specified as $P_{ff}$ at some intermediate radius $r_{ff}$. The pressure distribution in this case is

$$P(r) = P_{ff} + \frac{12 \mu Q_{wb}}{2 \pi \delta_{wb}^3} \left( 1 - \frac{r_{wb}}{R} \right)^3 \left[ \frac{1}{2} \left( \frac{2R-r_{ff}}{R} \right)^2 - \left( \frac{2R-r_{ff}}{2R-r} \right)^2 \right] + \ln \left( \frac{r_{ff}}{r} \frac{R-r}{R-r_{ff}} \right) \quad (3.4.3)$$
Simulations of these two test cases were performed with A3DH using both the Finite Element and Successive Integration formulations. Simulations were run using 10, 20, 50, 100, and 150 nodes within the flow region and the nodal spacing parameter values suggested in Section 2.10 were used. The errors between the computed pressures and the expected pressures, normalized by the overall pressure difference for the appropriate test, are shown in Figures 3.2-5. The maximum errors are less than 1% when 50 or more nodes are used, with most errors being much less. Figure 3.6 summarizes the results for the maximum errors in each case.

3.5 Solution Efficiency

The number of computations which must be performed to obtain a solution obviously increases as the number of nodes increases. The rate at which this increase in computation occurs is an important consideration when running simulations with large numbers of nodes. A direct solution of an \( N \times N \) matrix requires on the order of \( N^3 \) computations, so that a problem with 150 nodes would be expected to take 27 times longer to run than a problem with 50 nodes. The average time required for the innermost iteration loop in A3DH (the pressure convergence loop), which is where the simulation spends most of its time, is shown in Figure 3.7, as a function of the number of nodes used. The times shown are specific for a Silicon Graphics 4D25 workstation which can perform 1.5 Megaflops (1.5 million floating point operations per second), but the scaling with number of nodes would be the same on any other machine. The schemes used in A3DH are
seen to be of order less than 2, indicating an efficient solution methodology. The low order is mainly due to the use of an iterative solver (typically an order\(N^2\) algorithm) to obtain solutions of the resulting matrix equations when the error between the estimated solution and resulting solution becomes small. Approximately ten iterations are generally required to get the matrices solved, a number which seems to be rather insensitive to the number of nodes used. This would indicate that the overall scheme should be approximately second order in \(N\), which is consistent with results shown in Figure 3.7.

3.6 Comparison with DISLASH Experiments

To verify the physical validity of the modelling used in A3DH, its results have been compared against two sets of data from laboratory experiments being carried out at the Resource Extraction Laboratory at MIT. The first experiment compared against is the Desktop Interface Separation and Laboratory Apparatus for Simulation of Hydrafracs (DISLASH) [Johnson, 1990; Johnson and Cleary, 1991]. The apparatus, shown in Figure 3.8, consists of a rubber cylinder (14 cm diameter x 17 cm height) which is contained by a flexible jacket inside a metal sleeve. A rigid polymethylmethacrylate (PMMA or Plexiglas) block is clamped to the top of the metal sleeve and the rubber sample is pressed against the PMMA by air pressure introduced into the region between the containment jacket and the metal sleeve. A highly viscous silicon fluid (polydimethylsiloxane with viscosity between 1,000 and 300,00 centipoise) is injected into the planar interface between the rubber
and the PMMA through a wellbore in the center of the PMMA block. A circular "puddle" of liquid grows in the interface in the same manner as a fluid-driven crack would grow in a homogeneous medium which is twice as stiff as the rubber (to give the same crack opening widths for a given pressure distribution) and for which $K_{lc}=0$. Pressure taps in the PMMA measure the pressure in the growing fracture at radii of 1.0 cm, 2.0 cm, 3.0 cm and 4.0 cm, as well as at the injection point. Growth of the "fracture" is monitored visually through the transparent PMMA.

The parameters and conditions relevant to the growth of the fracture in these experiments are given in Table 3.1. A3DH simulations were run for these conditions and the results compared with the experimental data. Figure 3.9 is a plot of crack tip radius as a function of time, both for an A3DH simulation and for a series of DISLASH experiments run under the same conditions. The difference between the A3DH prediction and the experimental results is greatest for early times, as can be seen more clearly in Figure 3.10, where the crack growth rates are plotted. Figure 3.11 shows a comparison between the A3DH prediction and the DISLASH data for the excess pressure history at the wellbore. The experimental pressures are considerably higher. The pressure distributions for three crack tip radii are shown in Figure 3.12. Consistent with the previous comparison, the experimental values show higher pressure levels and gradients than the A3DH predictions. This discrepancy is apparently due to a difference between the elastic modulus used in the A3DH simulations (150 psi, as reported in Table 3.1) and the effective elastic modulus which properly reflects the response of the rubber.
The silicon rubber which deforms in the DISLASH experiments has a non-linear stress-strain response, as the results of an unconfined compression test show in Figure 3.13a, where the strain displayed is \( \frac{\text{deflection}}{\text{initial sample length}} \) and the stress displayed is \( \frac{\text{force}}{\text{initial surface area}} \) [Johnson, 1990]. Much of this apparent non-linearity, however, is due to the unconfined nature of the compression test, which is not representative of DISLASH conditions. When the large changes in sample dimensions are accounted for, as in Figure 3.13b, where the adjusted values of \( \frac{\text{incremental deflection}}{\text{current sample length}} + (\text{previous strain}) \) and \( \frac{\text{force}}{\text{current surface area}} \) are plotted, the response is more nearly linear. The remaining non-linearities which remain in the true stress vs. true strain curve are characteristic of the response of rubber, both at high and low strains. The elastic modulus obtained from the slope of the engineering stress vs. engineering strain curve in Figure 3.13a is the relevant one to use in the linear elastic relations currently employed in A3DH, however, and this curve is nearly linear for small (e.g. less than 25%) strains. Although some strains in the immediate vicinity of the fracture tip may be large enough to bring the nonlinearities of the elastic response into play, the strains over nearly the entire fracture are small enough to be within the linear region of the stress-strain curve.

There is, however, a Poisson-ratio effect in the DISLASH apparatus which would be expected to produce a higher effective modulus than indicated by the stress-strain curve. For an infinite medium having a Poisson ratio of exactly \( \nu=0.5 \), subjected to normal (pressure) loading over a specified region of a plane
through the material, there is no in-plane displacement on the loading plane (Figure 3.14a). Therefore, a half-infinite body also having \( v = 0.5 \), in contact with a rigid solid and subjected to the same loading at the interface plane with no slippage allowed at the interface, will have the same response as the infinite region (Figure 3.14b). This equivalence is the reason that rubber with \( v = 0.5 \) was originally chosen as the material to be used in the DISLASH experiment, allowing deformation analyses to be carried out using the readily available infinite medium influence functions. In order for the crack apertures to be the same in both cases, the normal deformations over the region of the applied loading must be twice as large for the material occupying the half-space and so this material must have half the elastic modulus of the infinite medium.

However, if \( v = 0.5 \) then there will in general be in-plane displacements along the plane of loading in the infinite-region case (Figure 3.14c) and there will be shear stresses generated on the no-slip interface in the half-space case (Figure 3.14d). The displacements on the loading plane in the infinite region case are of the order \( (1-2v)P/E \), where \( P \) is the applied pressure. Removing the unwanted displacements from the infinite region case, to properly model the no-slip condition of the half-space case, would induce additional stresses. Although the determination of the exact displacements and stresses is very involved due to the presence of the fracture, the magnitude of the stresses involved would be on the order of \( K(1-2v)P/E \), where \( K \) is the bulk modulus of the material. For rubber with \( K \approx 5 \times 10^5 \) psi and \( E \approx 150 \) psi, this gives additional stresses which are of the same
order as the applied excess pressures in DISLASH when \((1-2v)=0.001\), as is the case for the rubber in DISLASH. These additional stresses of the same magnitude as the applied loads suggest that the half-region case is behaving as a material which is several times stiffer than the elastic modulus would indicate.

Further evidence of the need for a higher modulus is provided when the crack volumes associated with the measured DISLASH pressure distributions are computed using A3DH algorithms. In this analysis, the pressures measured at the DISLASH pressure taps for different crack-tip radii are used to assign nodal pressures in A3DH, linearly interpolating for nodes between the pressure tap locations. The solid deformation components of A3DH can then be used to compute the resulting crack-opening distribution and crack volume. When this volume is divided by the flowrate, a total required injection time is obtained which can be compared with the observed injection times from the DISLASH experiments. The times obtained in this manner for crack-tip radii of 2.5, 3.5, and 4.5 cm, using \(\bar{E} = 100\) psi, are all too large by a factor of approximately 2, as seen in Table 3.2. The computed injection times could be reduced by reducing the excess pressures or by increasing the modulus. A reduction in the pressures would be in order if there is an error in the measurement of the confining stress. As Table 3.2 shows, though, an increase in the confining stress does not reconcile the times. However, increasing in the modulus to \(\bar{E} = 225\) psi gives good agreement between the computed times and the measured times. Figures 3.15-18 show the
comparison with experimental data when an elastic modulus of $\bar{E}=225$ psi is used in A3DH simulations. With this value, the excess pressure history and distributions match well, as do the growth rate predictions.

There are a number of additional factors which are potentially important in the context of the DISLASH experiments. These include experimental uncertainties as well as factors which may be important for the conditions under which the DISLASH experiments are performed, but not generally significant under normal field conditions (i.e. for the hydraulic fracturing of underground hydrocarbon reservoirs). These factors include:

(a) Free-surface effects
(b) Surface tension
(c) Pressure transducer response-time
(d) Confining stress level and uniformity
(e) Two-dimensional flow effects near the crack tip
(f) Shear-thinning of the fluid
(g) Fluid shear stress acting on fracture surface

The effect of each of these is discussed below.

**Free-Surface Effects**

The rubber sample in the DISLASH experiments is unconstrained on its sides and bottom, whereas the fracture being analyzed in A3DH is contained in a solid of infinite extent in all directions. As the radius of the fracture-equivalent separation
zone in DISLASH becomes comparable to the radius of the sample, the presence of the free surfaces will produce crack widths which are larger than the infinite region case, similar to what would be experienced in an infinite solid which is softer than the sample. An indication of the extent of this free surface effect can be inferred from Figure 3.19, which shows the relative value of the Mode I stress intensity factor $K_I$ for a uniformly loaded circular crack in an infinite cylinder compared to $K_I$ for the same crack embedded in an infinite solid [Kassir and Sih, 1975].

Since the near-tip crack opening values are proportional to $K_I$, the crack openings near the tip would show the same relative increase due to the free surfaces as the $K_I$ value. Also, simulations of the free-surface case using a Surface Integral Finite Element Hybrid (SIFEH) computer program, [Keat, 1989] show that the effect on deflections is uniform over the entire fracture surface for fractures of 4.5 cm radius within a cylinder having a radius of 7.0 cm [Keat, 1991]. (Direct simulation of the free-surface effect cannot be performed with SIFEH for $\nu=0.5$ because of numerical limitations at the incompressible limit.) The same uniformity would also be expected for cracks smaller than 4.5 cm. Consequently, the effect of the free surfaces on the crack opening values, for DISLASH cracks less than 4.5 cm radius, can be determined by the relative change in $K_I$ from Figure 3.19. The progressive increase in crack opening values as the crack grows, compared to the infinite region values, can be modelled using an effective elastic modulus $\overline{E}_{eff}$ that
decreases as the crack grows, such that, for a crack tip radius \( R \) and a cylinder radius \( R_c \),

\[
\bar{E}_{\text{eff}}(R) = \frac{\bar{E}}{K_{Icyl}/K_{I\infty}}
\]  

(3.6.1)

where \( K_{Icyl}/K_{I\infty} \) is the value of the \( K_I \) ratio from Figure 3.19 for the appropriate \( R/R_c \) value.

Modifying the modulus used in A3DH as a function of crack tip radius, in accordance with Equation (3.6.1), to account for the free surfaces in DISLASH, gives the results shown in Figures 3.20-21. The maximum revision of \( \bar{E} \) is about 20% at the larger crack radii, resulting in changes in excess pressure of about 15% and in crack radius of only a few percent.

Although this effect is small for the DISLASH experiments, it is very important in the CIA experiments described in Section 3.7 because of larger \( R/R_c \) ratios, and the variable \( \bar{E} \) is used in computing the results presented there.

**Surface Tension**

The effects of surface tension at the fluid front are shown in Figures 3.22-24 for A3DH simulations of DISLASH conditions, using \( \bar{E} = 225 \) psi. The two extreme cases shown are for a totally non-wetting fluid, having a contact angle between the fluid surface and the solid surface of 0° and a surface tension of 21.5 dyne/cm, and a neutrally wetting fluid which has a contact angle of 90°. Figures
3.22-23 show the distribution of excess pressure as a function of normalized distance back from the crack tip. Although the interface pressure is 5 psi higher in the fully non-wetting case, this has an effect only very near the fluid front and, as Figure 3.24 demonstrates, has no effect on the crack propagation rate, the curves for both cases coinciding.

**Pressure Transducer Response Time**

The pressure taps at 1-cm, 2-cm, 3-cm, and 4-cm positions have very small-bore passages connecting the pressure transducers to the interface. This is to prevent significant "siphoning off" of the fluid in the fracture. However, the narrow passages also result in a slow response of each transducer to changes in the pressure being measured, as fluid motion to accommodate compressibility in these passages and in the transducer assembly is impeded by viscous drag. Because transducers near the tip of the crack are responding to smaller changes in pressure and since these readings have had less time to equilibrate, the lagging of the measured pressure values would have a tendency to underpredict near-tip pressures more than near-wellbore pressures, increasing the measured apparent pressure gradient. Figure 3.25 shows the pressure history for the five probes in a typical DISLASH experiment. Although it is not possible to separate pressure-tap lag from the actual time-variation of the pressure at the probe locations, the response time for the pressure measurements appears to be no more than ten seconds. This lag time could possibly account for some of the difference shown in Figure 3.18
between the A3DH prediction and the measured pressure value at the 4-cm pressure
tap for the 4.5-cm crack length. Some further factors are apparently involved,
since there would be no pressure lag affecting the 3-cm pressure reading at the
same crack length.

Confining Stress Level and Uniformity

An underestimate of the confining stress $\sigma_c$ which is applied to the sample
would lead to an overprediction of the excess pressure $p(r) = P(r) - \sigma_c$ at any point
in the fracture. In addition, any variation in confining stress over the
PMMA/rubber interface would lead to errors in comparing the experimental excess
pressure gradients with the A3DH simulations.

To verify the level and uniformity of the confining stress, independent
measurement of the confining stress at the pressure monitoring ports was attempted,
by injecting air into the PMMA/rubber interface through the pressure ports and
observing the pressures required to maintain localized interface separation [Johnson,
1991]. These tests were difficult to perform and interpret because an initial
pressure greater than the expected confining stress was needed to initiate separation
and then a rapid reduction of the air pressure was required to maintain a stationary
size for the air bubble in the interface. In many cases a stationary bubble could not
be maintained. In different trials the motion of the bubble occurred in different
directions, indicating inconsistent variations in confining stress. The level of these
variations was not quantified; due to the low viscosity of air and the slow bubble
motion the variation could have been very small. With the hydrostatic pressure surrounding the sample, significant variations in confining stress would seem to be difficult to maintain.

**Two-Dimensional Flow Effects Near the Crack Tip**

The crack opening rates very near the fluid front in the DISLASH experiments are large enough that in addition to the flow radially outward within the flow channel there is also a significant transverse flow component toward the fracture walls. This could result in a pressure variation across the flow channel, so that the pressure distribution acting on the fracture walls to cause opening displacement differs from the distribution for the average fluid pressure (across the channel width) used to compute the fluid flow rates. The magnitude of this pressure difference can be computed using the analysis of the two-dimensional flow field presented in Appendix D. For the conditions of the DISLASH experiment, the ratio of the lateral pressure gradient to the radial pressure gradient, as a function of normalized distance from the crack tip, is shown in Figure 3.26. The resulting pressure difference between the fracture face and the centerline of the flow channel is shown in Figure 3.27. Except for the immediate vicinity of the fluid front, the pressure difference is very small and is not expected to influence the predicted results. The pressure difference at the fluid front is less than that seen for the surface tension effect and should therefore not be a factor either.
**Shear-Thinning of the Fluid**

The polydimethylsiloxane fluid used in the DISLASH experiments is shear-thinning at high shear rates. For shear rates \( \dot{\gamma} \) above a threshold value \( \dot{\gamma}_0 \), the fluid behaves as a power-law fluid, such that

\[
\tau = \mu_0 \dot{\gamma}, \quad \dot{\gamma} < \dot{\gamma}_0 \\
\tau = k' \dot{\gamma}^{n'} = \mu_{\text{eff}} \dot{\gamma}, \quad \dot{\gamma} > \dot{\gamma}_0
\]

where

\[
\mu_{\text{eff}} = k' \dot{\gamma}^{n'-1} = \mu_0 \left( \frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^{n'-1}
\]

with \( n' = 0.37 \) and \( \dot{\gamma}_0 = 1000 \text{ sec}^{-1} \). The shear rates in DISLASH experiments approach the threshold value only in the region directly behind the fluid front. Using the rheological description of Equations (3.6.2-3) in A3DH simulations of DISLASH showed no significant effect.

**Shear Stress Acting on Fracture Surface**

The fluid pressure is not the only traction component acting on the fracture surface. The shear stress due to the fluid motion, though negligible compared to the fluid pressure for fracturing in stiff materials like rock and cement, can be significant for the soft rubber used in DISLASH. The shear stresses acting on the fracture surface are shown in Figure 3.28, for early (1-cm tip radius) and late (4.5-cm tip radius) stages of DISLASH experiments. The shear stresses in the outer portions of the fracture approach the level of the excess pressure and may
affect the fracture shape. The effect of fluid shear stress on the fracture walls has not been explicitly included in this analysis. The effect of these shear stresses differs from the effect of pre-existing shear stresses in the solid. In the latter case, the negative of the relieved shear stresses are applied to the modelled fracture faces, with the tractions on the two faces being in opposite directions. These shear stresses contribute to the Mode II and Mode III stress intensity factors at the crack tip. The fluid shear stresses, on the other hand, have the same orientation on both surfaces and do not contribute to stress singularities at the crack tip.

3.7 Comparison with CIA Experiments

A second set of hydraulic fracturing experiments is being carried out at MIT's Resource Extraction Laboratory, called the Crack Interaction Apparatus (CIA) experiments [Motamed, 1986]. The "Apparatus" referred to is a pressure vessel (shown in Figure 3.29) large enough to hold 45 cm x 30 cm x 20 cm cement samples and apply both a hydrostatic confining pressure to the fluid surrounding the sample as well as an independent uniaxial auxiliary stress to the sample by means of a hydraulic ram. A wellbore and small initiation crack are cast into each sample and the same high-viscosity silicon fluid used in the DISLASH experiments (polydimethylsiloxane) is injected into the sample to generate a hydraulic fracture. The "Crack Interaction" part of the name refers to the fact that more than one crack can be cast into the samples and grown simultaneously, to permit the study of the interaction among multiple growing cracks. By casting the cracks at an oblique
angle to the direction of the auxiliary stress and then momentarily changing the level of the auxiliary stress, the direction of crack tip growth changes momentarily, creating a small bump in the crack surface, the position of which can later be measured to determine the crack tip position at the time the stress level was changed. Perturbing the stress level at regular time intervals allows the tip position as a function of time to be determined. For the CIA experiments used for comparison in this work, the auxiliary stress was set equal to the hydrostatic confining pressure except when the auxiliary stress was momentarily perturbed to mark the tip position.

Figures 3.30-31 show the results of A3DH simulations of a CIA experiment carried out under the conditions shown in Table 3.3. The predicted crack growth from A3DH is faster than the experimental data. There are indications that fluid compliance and flow control difficulties led to actual injected flowrates at early times which were lower than the specified flowrate. To eliminate this effect, the comparisons are replotted in Figures 3.32-33, with the experimental curves having been shifted in time to give a consistent time for which the crack tip reaches 1.75 inch. When this is done, the predicted values for both propagation and excess pressure are in good agreement with the experimental data.

As mentioned in the Free-Surface Effects discussion in Section 3.6, Equation (3.6.1) has been used to modify the Young’s modulus at each timestep of the A3DH simulations of CIA experiments, to account for the effect of the finite sample radius.
Figure 3.34 shows A3DH predictions of crack growth as a function of time for three different values of $K_{IC}$: 100, 200, and 400 psi$\sqrt{\text{in}}$. The material toughness is seen to have little effect on the crack growth. This is consistent with the fact that the contributions to the stress intensity factor from the regions of positive and negative excess pressures are both large relative to $K_{IC}$ and nearly balance each other, so the level of $K_{IC}$ which they sum to is not very important.

The positive contribution is of the order $p\sqrt{R}$ so the negative contribution is of the order $K_{IC} - p\sqrt{R}$. That both the positive and negative contributions are much larger than $K_{IC}$ is seen to be true for CIA conditions where

$$\frac{p\sqrt{R} - K_{IC}}{K_{IC}} = \frac{(1200\text{psi})(4\text{inch})^{1/2} - 200\text{psi}}{200\text{psi}\sqrt{\text{in}}} = 11.$$ \hspace{1cm} (3.7.1)

For typical hydrafracturing conditions in hydrocarbon reservoirs, the ratio is even larger:

$$\frac{p\sqrt{R} - K_{IC}}{K_{IC}} = \frac{(500\text{psi})(500\text{ft})^{1/2} - 1000\text{psi}}{1000\text{psi}\sqrt{\text{in}}} = 40.$$ \hspace{1cm} (3.7.2)

Another measure of the small role which $K_{IC}$ typically plays is the fact that most of the work required to propagate the fracture goes into opening the fracture against the confining stress and a relatively small fraction is required to fracture the material at the crack perimeter. The energy per unit of fracture area required for the opening is of the order $P \cdot \delta = P \cdot \frac{p R}{E}$. The energy per unit area required for
fracturing the perimeter is of the order \( \frac{K_{IC}^2}{E} \). The ratio of these two is \( \frac{PPR}{K_{IC}^2} \), which for the CIA experiment is approximately

\[
\frac{PPR}{K_{IC}^2} = \frac{(3000 \text{ psi})(1200 \text{ psi})(4 \text{ in.})}{(200 \text{ psi/\text{in}.})^2} = 350.
\]

(3.7.3)

For hydrocarbon reservoir conditions the ratio is considerably larger, being

\[
\frac{PPR}{K_{IC}^2} = \frac{(5000 \text{ psi})(500 \text{ psi})(500 \text{ ft})}{(1000 \text{ psi/\text{in}.})^2} = 15,000.
\]

(3.7.4)

### 3.8 Normalized Comparisons

When the parameters describing fracture growth are properly normalized, the results for disparate conditions should collapse onto a single set of curves. The appropriate normalizations for crack growth and excess pressure level as functions of time are determined from the expected scaling for circular hydraulic fractures [Crocket et al., 1986a]:

\[
R = \left( \frac{EQ^{3/4}t^4}{\mu} \right)^{1/9}
\]

(3.8.1)

\[
p = \left( \frac{\mu \bar{E}^2}{t} \right)^{1/3}
\]

(3.8.2)

These relations can be rearranged to the normalized forms

\[
\frac{R}{r_{ub}} = f(t/\tau^*)
\]

(3.8.3)

\[
\frac{p}{p^*} = f(t/\tau^*)
\]

(3.8.4)

where
\[ \tau^* = \left[ \frac{r_{wb}^3}{Q} \cdot \frac{\mu}{E} \right]^{1/4} \]

\[ p^* = \left[ \frac{E}{r_{wb}^3} \cdot \frac{\bar{\mu}Q}{E} \right]^{1/4} \]

represent the characteristic time and excess pressure.

Figures 3.35-36 show the normalized results of A3DH simulations for fracture growth under the conditions of the DISLASH experiments (with three different fluid viscosities) and the CIA experiments as well as conditions representative of hydraulic fracture treatments in underground hydrocarbon reservoirs ("field conditions"). As expected, the different sets of results all fall on a single curve in each plot, providing further evidence that the simulator is properly representing the hydraulic fracturing process.
"Top-hat" pressure distribution used to test the algorithms for crack opening and stress intensity factor.
Figure 3.2  Relative errors for flow tests with uniform crack opening, using the Successive Integration formulation.
Figure 3.3 Relative errors for flow tests with uniform crack opening, using the Finite Element formulation.
Figure 3.4    Relative errors for flow tests with linear crack opening, using the Successive Integration formulation.
Relative errors for flow tests with linear crack opening, using the Finite Element formulation.
Figure 3.6 Comparison of maximum relative errors for the various flow tests.
Figure 3.7  Computation time required for a single pressure-convergence iteration, as a function of the total number of nodes used for the solution.
Figure 3.8  Schematic representation of the DISLASH experimental apparatus.
Figure 3.9  Comparison of A3DH prediction with DISLASH experimental results for crack tip radius vs. time.
Figure 3.10  Comparison of A3DH prediction with DISLASH experimental results for crack propagation speed vs. time.
Figure 3.11 Comparison of A3DH prediction with DISLASH experimental results for wellbore excess pressure vs. time.
Figure 3.12 Comparison of A3DH prediction with DISLASH experimental results for excess pressure vs. radius at three crack sizes.
Figure 3.13a  Engineering stress vs. engineering strain for rubber used in DISLASH experiments.
Figure 3.13b  True stress vs. true strain for rubber used in DISLASH experiments.
(a) Infinite medium with $\nu = 0.5$. No in-plane displacement or shear stress.

(b) Semi-infinite medium with $\nu = 0.5$, having a no-slip planar contact with a rigid solid. No in-plane displacement or shear stress.

Figure 3.14 Infinite and semi-infinite bodies subjected to pressure loading over a specified planar region.
(c) Infinite medium with $\nu \neq 0.5$. In-plane displacement but no in-plane shear stress.

(d) Semi-infinite medium with $\nu \neq 0.5$, having a no-slip planar contact with a rigid solid. No in-plane displacement but in-plane shear stress.

Figure 3.14 Infinite and semi-infinite bodies subjected to pressure loading over a specified planar region.
Figure 3.15  Comparison of A3DH simulation (using adjusted $\bar{E}$) with DISLASH experimental results for crack tip radius vs. time.
Figure 3.17
Comparison of A3DH simulation (using adjusted $\bar{E}$) with DISLASH experimental results for wellbore excess pressure vs. time.
Figure 3.18  Comparison of A3DH simulation (using adjusted $E'$) with DISLASH experimental results for excess pressure vs. radius at three crack sizes.
Figure 3.19  Effect of free surface on stress intensity factor for circular crack in a cylinder [Kassir and Sih, 1975].
Figure 3.21  A3DH predictions for crack tip radius vs. time under DISLASH conditions, with and without a variable elastic modulus to account for the presence of the free surface.
Figure 3.21  A3DH predictions for crack tip radius vs. time under DISLASH conditions, with and without a variable elastic modulus to account for the presence of the free surface.
Figure 3.22  A3DH predictions for excess pressure vs. normalized radial distance from the crack tip under DISLASH conditions, for two extreme values of the fluid interface contact angle.
Figure 3.23  A3DH predictions for excess pressure vs. normalized radial distance from the crack tip under DISLASH conditions, for two extreme values of the fluid interface contact angle.
Figure 3.24  A3DH predictions for crack tip radius vs. time under DISLASH conditions, for two extreme values of the fluid interface contact angle.
Figure 3.25  Pressure response as a function of time at the five pressure taps monitored during a representative DISLASH experiment.
Figure 3.26  Radial distributions of the ratio of lateral pressure gradient to radial pressure gradient as computed by A3DH for DISLASH conditions with crack tip radii of 1.0 and 4.5 cm.
Figure 3.27  Radial distributions of pressure difference across the flow channel computed by A3DH for DISLASH conditions with crack tip radii of 1.0 and 4.5 cm.
Figure 3.28  Radial distributions of wall shear stress computed by A3DH for DISLASH conditions with crack tip radii of 1.0 and 4.5 cm.
Figure 3.29  Schematic representation of the Crack Interaction Apparatus (CIA) experimental setup.
Figure 3.30  Comparison of A3DH prediction with CIA experimental results for crack tip radius vs. time.
Figure 3.31  Comparison of A3DH prediction with CIA experimental results for excess wellbore pressure vs. time.
Figure 3.32 Comparison of A3DH prediction with CIA experimental results for crack tip radius vs. time when experimental curves are shifted to coincide at $R=1.75$ inch.
Figure 3.33 Comparison of A3DH prediction with CIA experimental results for excess wellbore pressure vs. time when experimental curves are shifted to coincide at $R=1.75$ inch.
Figure 3.34  A3DH predictions of crack tip radius vs. time for CIA experimental conditions, using $K_{IC} = 100$, 200, and 400 psi$\sqrt{\text{in}}$. 
Figure 3.35  Normalized excess pressure vs. normalized time for A3DH simulations of DISLASH and CIA experiments and field conditions.
Figure 3.36  Normalized crack-tip radius vs. normalized time for A3DH simulations of DISLASH and CIA experiments and field conditions.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>150 psi</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>0.4995</td>
</tr>
<tr>
<td>Confining stress</td>
<td>50.84 psi</td>
</tr>
<tr>
<td>Flow rate</td>
<td>0.042 cm³/sec</td>
</tr>
<tr>
<td>Fluid viscosity</td>
<td>9800 cp</td>
</tr>
<tr>
<td>Fracture toughness</td>
<td>0.0 psi√in</td>
</tr>
<tr>
<td>Surface tension</td>
<td>21.5 dyne/cm</td>
</tr>
<tr>
<td>Specimen size</td>
<td>7 cm radius x 17 cm height</td>
</tr>
<tr>
<td>Wellbore radius</td>
<td>0.25 cm</td>
</tr>
</tbody>
</table>
TABLE 3.2
Comparison of measured DISLASH injection times (in seconds) with injection times required to attain fracture volumes computed by A3DH using measured DISLASH pressures with the indicated values of $E$ and corrections $\Delta \sigma_c$ in the confining stress.

<table>
<thead>
<tr>
<th>Crack Tip Radius</th>
<th>2.5 cm</th>
<th>3.5 cm</th>
<th>4.5 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured times for DISLASH</td>
<td>15</td>
<td>31</td>
<td>52</td>
</tr>
<tr>
<td>$E = 100\text{psi}$</td>
<td>$\Delta \sigma_c = 0$</td>
<td>30.1</td>
<td>69.6</td>
</tr>
<tr>
<td></td>
<td>$\Delta \sigma_c = 2 \text{ psi}$</td>
<td>20.4</td>
<td>42.4</td>
</tr>
<tr>
<td></td>
<td>$\Delta \sigma_c = 3 \text{ psi}$</td>
<td>15.8</td>
<td>28.8</td>
</tr>
<tr>
<td>$E = 225\text{psi}$</td>
<td>$\Delta \sigma_c = 0$</td>
<td>13.4</td>
<td>30.9</td>
</tr>
<tr>
<td><strong>TABLE 3.3</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CIA experimental conditions</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| **Young’s modulus:**   | $1.0 \times 10^6$ psi  
| **Poisson ratio:**     | 0.2  
| **confining stress:**  | 1800 psi  
| **flow rate:**         | 0.002 cm$^3$/sec  
| **fluid viscosity:**   | 940,000 cp  
| **fracture toughness:**| 200 psiv/in  
| **surface tension:**   | 21.5 dyne/cm  
| **specimen size:**     | 12.4 cm radius x 35 cm height  
| **wellbore radius:**   | 0.28 cm  

CHAPTER 4
LEADING EDGE

4.1 Leading Edge Concept

If a general three-dimensional hydraulic-fracture simulator were to account for the conditions in the near-tip region by explicitly solving the governing equations there, the highly refined discretization required by the small length scales in this region would necessitate a large part of the simulator’s computation time being devoted to determining the behavior in this region. A more reasonable approach is to have a concise description of the conditions in the near-tip region of a propagating hydraulic fracture which can be included as a module in a general 3-D simulator, and allow the effect of the near-tip region on the rest of the fracture to be adequately accounted for without having to explicitly solve the governing equations there. Such a module is developed in this chapter. Prior to this development, a discussion of the role of dilatancy on crack opening values is presented.

4.2 Role of Dilatancy

A solid exhibits the characteristic called dilatancy if it experiences non-linear expansion in the directions normal to an applied compressive stress. Such dilatant behavior is characteristic of sandstone in triaxial compression, as Figures 4.1a-c indicate [Warpinski and Smith, 1989]. Figures 4.1a and 4.1b show a general
tendency for nonlinear (increased) lateral expansion under triaxial compression, manifested as an increase in lateral strain and a decrease in volumetric strain with increasing applied stress and as a Poisson’s ratio which increases with stress. These curves are for conditions of 2900-psi confining stress, but Figure 4.1c shows that the tendency for dilatancy is even more pronounced at lower stress levels. These characteristics of sandstone are of interest in the fracturing of oil and gas reservoirs because it is sandstone formations which typically contain the hydrocarbons being extracted.

The effect of dilatancy on propagating hydraulic fractures is most evident near the crack tip, where absolute pressures are at or near zero. The stress normal to the fracture surface there (the stress component in the y-direction in Figure 4.2), being equal to the pressure, is also near zero, indicating a stress relief of \(-\sigma_{c2}\). Because of the two-dimensional nature of the region very near the crack tip, the stress relief in the x-direction (parallel to the crack face, and normal to the line of the crack tip) is also \(-\sigma_{c2}\) and the stress relief in the third direction is \(-2\nu\sigma_{c2}\). The stress state therefore has a single dominant component (in the z-direction in Figure 4.2) with zero or near-zero stresses in the other two orthogonal directions. Under these conditions, the dilatant behavior of sandstone is fully evident. The zero-stress curves of Figure 4.1c show that nonlinear lateral expansion becomes significant for applied stress levels above 6000 psi, levels which are routinely encountered in fracturing of hydrocarbon reservoirs.
The significance of this dilatant behavior is that the rock will expand into the crack-opening region predicted from a linearly elastic analysis, giving a narrower crack opening (see Figure 4.2). Because the crack widths near the tip are so small, the relative reduction in crack width in this region is large for even a small amount of dilation. In fact, it is possible that dilational effects may effectively close a portion of the crack near the tip.

The effect of such a narrowing of the crack near the tip is shown in Figure 4.3, where pressure distributions are given for cracks which have had the crack opening artificially constricted over a region approximately ten times the size of the non-penetrated zone. The constriction applied is a maximum at the tip and reduces to zero as the region is traversed. The maximum relative amount of constriction applied is referred to in the figure as the percent of Leading Edge Dilatancy (% LED). The crack constriction in this small region is seen to have a dominant effect on the overall pressure distribution.

4.3 Leading-Edge Formulation and Implementation in R3DH

Figure 4.4 shows the fluid pressure gradient as a function of the normalized distance back from the crack tip for simulations of the DISLASH and CIA experiments, as well as a representative field case. In each case, the log-log curve is quite linear, for a large region near the tip, indicating that
\[ \frac{dP}{dr} \propto (R-r)^{-\beta} \]  

(4.3.1)

for some constant \( \beta \).

This relation has been used as the basis for a concise representation of the conditions in the near-tip region, and incorporated as an independent module within a general hydraulic fracture simulator. This module, referred to here as the "leading-edge module" (LEM), allows the effect of the conditions in the near-tip region to be taken into account by the general simulator without having to solve the detailed governing equations in that region. Such a detailed solution would require an inordinate number of the degrees of freedom to be devoted to the near-tip region, whereas, with the LEM implemented, very few degrees of freedom are required.

The general fracture simulator, called R3DH, is a result of research efforts at MIT's Resource Extraction Laboratory and of further work sponsored by the Gas Research Institute [Cleary et al., 1983; Lam et al., 1986]. R3DH employs a finite element formulation to solve the fluid flow relations and a surface integral formulation to determine the elastic deformation of the solid. With the use of appropriate influence functions in the surface integral method, it is possible to precisely represent the elastic behavior of homogeneous materials as well as materials having one or two planes of discontinuity in the elastic properties [Keat and Cleary, 1989]. The use of the surface integral scheme also avoids the need for
performing a volume discretization of the solid containing the fracture, requiring an explicit solution of the solid response only on the fracture surface.

R3DH therefore uses a finite element/surface integral mesh of elements which discretize only the fracture surface in solving for the fracture growth and conditions within the fracture. The leading-edge module is introduced into R3DH, positioned around the perimeter of the mesh, with one LEM element adjacent to each of the exterior mesh elements, as shown in Figure 4.5. The LEM elements describe the detailed behavior of the near-tip regions, based on the local conditions, and a matching of the appropriate parameters must be enforced at the interface between each LEM element and the adjacent mesh element.

A schematic representation of the near-tip region represented by the LEM is shown in Figure 4.6. The module is assumed to have a length of $\omega_\text{le}$ and contains a non-penetrated zone of size $\omega$. A dilational zone of size $\omega_d$ can also be included. The sizes of $\omega_\text{le}$ and $\omega$ are determined by adjusting them until the conditions

$$K_I = K_{IC} \quad (4.3.2)$$

$$v_{\text{fluid}}(\zeta = \omega_\text{le}) = \dot{R}\quad (4.3.3)$$

are satisfied, using the constraints

$$P(\zeta = \omega_\text{le}^-) = P(\zeta = \omega_\text{le}^+) \quad (4.3.4)$$

$$\frac{\partial P(\zeta = \omega_\text{le}^-)}{\partial r} = \frac{\partial P(\zeta = \omega_\text{le}^+)}{\partial r} . \quad (4.3.5)$$
Using the relation in Equation (4.3.1), leads to the expressions

\[ P(\zeta = \omega_{le}) = P(\omega) - \frac{dP}{dr} \bigg|_{\zeta = \omega_{le}} \frac{\omega_{le}}{(\beta - 1)} \left[ \left( \frac{\omega_{le}}{\omega} \right)^{\beta - 1} - \left( \frac{\omega_{le}}{\zeta} \right)^{\beta - 1} \right] \]  
(4.3.6)

\[ \frac{dP}{dr} (\zeta = \omega_{le}) = -(1 - \beta) \left[ \frac{P(\omega_{le}) - P(\omega)}{\omega_{le}} \right] \cdot \left[ \frac{\left( \frac{\zeta}{\omega_{le}} \right)^{-\beta}}{1 - (\omega/\omega_{le})^{1-\beta}} \right] \]  
(4.3.7)

and, specifically,

\[ P(\zeta = \omega_{le}) = P(\zeta = \omega) - \frac{dP}{dr} \bigg|_{\zeta = \omega_{le}} \left( \frac{\Lambda^{\beta-1} - 1}{\beta - 1} \right) \omega_{le} \]  
(4.3.8)

\[ \frac{dP}{dr} (\zeta = \omega_{le}) = - \left( \frac{P(\omega_{le}) - P(\omega)}{\omega_{le}} \right) \left( \frac{\beta - 1}{\Lambda^{\beta-1} - 1} \right) \]  
(4.3.9)

where \( \Lambda = \omega_{le}/\omega \).

The stress intensity factor, with the LEM in place, is computed using contributions separately from the mesh region and from the leading-edge region

\[ K_I = K_{Imr} + K_{Ile} \]  
(4.3.10)

where

\[ K_{Ile} = \frac{\sqrt{2/\pi} \omega_{le}}{R} \int_{-\omega_d}^{\omega_d} \frac{d\zeta P(\zeta)(R - \omega_d - \zeta)}{\sqrt{\zeta + \omega_d}} \]  
(4.3.11)

and the contribution from the mesh region has two components

\[ K_{Imr} = K_{Imr(comp)} + K_{Imr(corr)} \]  
(4.3.12)

The \( K_{Imr(comp)} \) term accounts for the computed effect of the irregular geometry of the fracture and the distribution of excess pressure over the fracture surface. Since
the crack opening values calculated by the surface integral formulation account for 
these factors, \( K_{Imr(\text{comp})} \) is computed based on these crack-opening values at the 
edge of the finite element mesh:

\[
K_{Imr(\text{comp})} = \delta(\omega_e) E \sqrt{\frac{\pi}{2(\omega_e + \omega_d)}} .
\]  

(4.3.13)

The crack-opening distribution in the leading-edge region, for this 
computation only, is assumed to be that which would be obtained if the pressure 
in this region were uniform, i.e. an opening distribution which varies as the square-
root of the distance from the point at which \( z=0 \). The uniform pressure assumed 
in this region for this calculation is the current pressure value at the edge of the 
mesh, avoiding the introduction of a discontinuity in the pressure distribution which 
would make the assumed square-root crack opening in this region inadequate.

Since Equation (4.3.11) computes the contribution to \( K_r \) from the actual 
excess-pressure distribution in the leading-edge region, the correction term \( K_{Imr(\text{corr})} \) 
removes the contribution to \( K_r \) due to the uniform excess pressure \( P(\zeta=\omega_e) - \sigma_c \) 
which is used in the computation of \( K_{Imr(\text{comp})} \):

\[
K_{Imr(\text{corr})} = -\frac{8}{\pi} \left( \frac{\omega_e + \omega_d}{\omega_e - \omega_d} \right) \left[ 1 - \frac{1}{2} \frac{\omega_e + \omega_d}{R} \right] \cdot \left[ P(\zeta=\omega_e) - \sigma_c \right].
\]  

(4.3.14)

Because the equations being solved by R3DH are as non-linear as those in 
A3DH, the process of determining values for the leading edge parameters which
satisfy all of the above relations is an iterative one. The procedure used is the following:

1. Choose initial values for \( \omega, \omega_{te}, \beta, \) and \( \dot{R} \) at each element on the crack perimeter, based on experience and local conditions.

2. Each time the pressure distribution in the mesh is revised, during iterations to achieve a distribution consistent with the governing equations, revise \( \omega \) to satisfy Equation (4.3.2) and revise \( \omega_{te} \) to satisfy Equation (4.3.3).

3. If the local value of \( \dot{R} \) needs to be revised, as is sometimes necessary to satisfy overall volume conservation, revise the local values of \( \omega_{te} \) and \( \beta \) to initially maintain the current local value of \( \partial P/\partial r \) at the edge of the mesh. This step has been found to aid in convergence of the subsequent iterations for a revised pressure distribution.

Any dilational zone \( \omega_d \) used in this implementation is scaled to the size of the crack opening at edge of the LEM, \( \delta(\zeta=\omega_{te}) \). The size could also be scaled to other near-tip length scales, such as \( \omega \) or \( \delta(\zeta=\omega) \), but a more accurate representation of the dilation effects, for a given dilational behavior, must be based on a non-linear analysis of the near-tip region, the results of which could readily be incorporated into the LEM.

This LEM formulation has been found to be a stable, convergent and efficient means of incorporating the near-tip behavior into the general 3-D fracture simulator.
Figures 4.7-8 show a comparison of simulations run with A3DH and R3DH for a circular fracture under conditions representative of hydraulic fracture treatments in underground hydrocarbon reservoirs ($E=2.3 \times 10^6$ psi, $\sigma_c=5350$ psi, $\mu=500$ cp, flowrate=50 barrel/min=2100 gal/min). The good agreement between the two sets of results demonstrates that the relations used in the LEM adequately represent the conditions near the crack tip.
Figures 4.1a-c Evidence of dilatant behavior in sandstone [Warpinski and Smith, 1989].
Figure 4.2  Near-tip region showing initial stress state, crack-surface stress state in the non-penetrated zone, and the narrowing of the crack aperture due to dilatant effects.
Figure 4.3   Radial distributions of pressure, computed by A3DH for various amounts of Leading Edge Dilatancy (LED). Fracture conditions are: $E=1.5 \times 10^6$ psi, $\sigma_c=6000$ psi, $\mu=40$ cp, $Q=20$ bpm.
Figure 4.4  Pressure gradient vs. normalized distance from the crack tip, computed by A3DH for conditions representative of the DISLASH and CIA experiments as well as "field" conditions of underground hydrocarbon reservoirs.
Fracture discretization for the fully three-dimensional hydraulic fracture model (R3DH) coupled with a "leading-edge module" (LEM) which succinctly describes the near-tip conditions.
Schematic representation of the near-tip region represented by the "leading-edge module" (LEM) in R3DH.
Figure 4.7  Comparison of A3DH and R3DH simulator results for excess wellbore pressure vs. time under conditions representative of hydraulic fracture treatments in underground hydrocarbon reservoirs.
Figure 4.8 Comparison of A3DH and R3DH simulator results for excess pressure vs. radius at three crack sizes under conditions representative of hydraulic fracture treatments in underground hydrocarbon reservoirs.
CHAPTER 5
CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

The capability has been developed in this thesis to accurately analyze hydraulic fractures propagating axisymmetrically, accounting fully for the time-dependence of the crack-opening distributions which govern fluid flow and fracture growth. The axisymmetric geometry employed permits detailed analysis of the important region near the crack tip as well as providing a means of generating results with which more general hydraulic fracture analyses can be validated. The strong non-linear coupling between the flow of fluid within the fracture and the deformation of the surrounding solid requires several levels of iteration and specialized convergence schemes to solve the resulting computational algorithms. In spite of these complications, which exist even for linearly elastic solids and Newtonian fluids, the procedure developed can accommodate any appropriate pressure-deformation or fluid rheology relations.

The validity of the resulting computer program, called A3DH, has been verified through comparison with measurements made in two separate laboratory experiments. The agreement with the lab data is very good, when actual experimental conditions are properly taken into account. For the DISLASH experiments, the main factor to be accounted for was the effective stiffening of the rubber sample, while uncertain initial flowrates for the various CIA tests required adjustments of their initial starting times to provide a self-consistent data set for comparison.
The program has been employed to show that simulated dilatant material behavior in a very small region near the crack tip can dramatically increase the fracture pressure, with associated changes in the fracture growth behavior. This effect demonstrates the importance of adequately modelling the conditions in the small near-tip region.

Generalization of the thesis results may now be achieved with a more general 3-D hydraulic fracture simulator, called R3DH, which has been developed prior to and concurrent with this work. The essential missing component in R3DH as well as other industry simulators, has been a sufficiently detailed description of the conditions in the critical region near the crack tip. The work of this thesis now provides a basis for determining this near-tip behavior. Leading-edge results from A3HD have been reduced to a general functional form and implemented as a "leading edge module" (LEM), providing near-tip boundary conditions for R3DH. Using the appropriate LEM allows accurate simulation of hydraulic fracture growth for realistic reservoir conditions.

There are many possibilities for further work using the capabilities developed here. Much can be done using A3DH to perform additional detailed analysis of the near-tip region, examining the influence which different conditions have on fracture response and near-tip behavior. Further investigation of the effects of fluid rheology (especially non-Newtonian effects), solid elasticity and toughness, stress conditions, fracture size and growth rate is warranted.
In addition, the role of specific mechanisms not currently implemented in the description of the fracturing process can be investigated. This would include a more detailed study of the effects of dilatancy, which was examined to a limited extent in the current work. That analysis, using a rather arbitrary description of crack narrowing in the near-tip region due to dilatancy, showed that such partial closure over a very small region can totally alter the levels and distributions of the pressure and crack opening over the entire fracture. Coupling A3DH with a module capable of incorporating a dilatant material description into calculations for the deformation of the solid would provide a more definitive evaluation of the consequences of dilatant behavior. The effects of other causes of nonlinear solid response, such as microfracturing near the crack tip, could be studied if the appropriate solid deformation relations are available.

A study of two-dimensional leakoff effects near the crack tip, as discussed in Section 2.8, is also a suitable topic for further research. In extreme cases of high permeability or slow crack propagation, the near-tip leakoff zone would extend beyond the normal fluid front, providing an alternative flow path through the porous solid into the near-tip region, supplementing the flow within the fracture itself. In these cases, the near-tip pressure gradient would no longer have the $-\mu \overline{H}/\delta^2$ character, but would tend to be more uniform, controlled by the gradient behind the moving front of the leakoff zone which is travelling with the crack tip.

Another mechanism of interest is the tortuosity of the fracture channel resulting from successive incremental propagations of the crack tip in directions
which deviate from the plane of the fracture. Even though the extent of deviation from the average plane of propagation would generally be small, the longer fluid flow path resulting from such a "zigzag" tip growth could be significant near the tip where the flow channel is very narrow, leading to higher pressure drop across the near-tip region. This increased pressure drop could equivalently, and perhaps more conveniently, be modelled by modifying another parameter on which the apparent pressure gradient is linearly dependent, for instance by increasing the fluid viscosity in that portion of the crack where tortuosity is a factor.

Additional investigation of the near-tip behavior will undoubtedly lead to modifications in the leading edge module (LEM) which is used in the fully three-dimensional hydraulic fracture simulator R3DH, relative to that which was proposed in Chapter 4. In fact, the author has already undertaken revisions to the LEM to improve the coupling between the leading edge and the rest of the fracture under dilatant conditions and to facilitate more direct comparisons between A3DH and R3DH in these cases.

Finally, because the fracture simulation capability developed here accounts for the dynamic response of the solid/fluid system, it will allow analysis of the transient behavior of hydraulic fractures under fracturing conditions which change with time. Of particular interest in this regard would be investigation of the fracture diagnostic procedure called "surge" which has been proposed by Cleary [Cleary, 1991]. In a surge test the rate at which fluid is being injected into the fracture is changed rapidly and the time-dependent response of the injection
pressure is used to infer the size of the fracture. A3DH is well suited to a study of this problem and work has already begun on it.
REFERENCES


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APPENDIX A

INTEGRAL EXPRESSION FOR CRACK OPENING

The crack opening $\delta$ at the normalized radius $\rho=r/R$ due to an excess pressure distribution $p(r/R)$ in a linearly elastic material of infinite extent can be expressed as [Sneddon and Lowengrub, 1969]:

$$\delta(\rho) = \frac{2R}{\pi E} \int_0^1 \frac{dt}{\rho \sqrt{t^2 - \rho^2}} \int_0^t \frac{ds}{\sqrt{t^2 - s^2}} s p(s).$$  \hspace{1cm} (A.1)

This is an inefficient form for computation, since it requires two integrations, one carried out within the other. An equivalent form having a single integral can be obtained by expressing Equation (A.1) as

$$\delta(\rho) = \frac{2R}{\pi E} \int_0^1 \frac{dt}{\rho} \int_0^t ds \ g(s,t)$$  \hspace{1cm} (A.2)

where

$$g(s,t) = \frac{s p(s)}{\sqrt{(t^2 - s^2)(t^2 - \rho^2)}}$$  \hspace{1cm} (A.3)

and changing the order of integration:

$$\delta(\rho) = \frac{2R}{\pi E} \left\{ \int_0^\rho ds \int_0^1 dt \ g(s,t) + \int_0^1 ds \int_0^1 dt \ g(s,t) \right\}$$

$$= \frac{2R}{\pi E} \left\{ \int_0^\rho ds \cdot s p(s) \int_0^1 \frac{dt}{\rho \sqrt{(t^2 - s^2)(t^2 - \rho^2)}} \right. \hspace{1cm} (A.4)$$

$$\hspace{2cm} + \left( \int_0^1 \frac{dt}{s \sqrt{(t^2 - s^2)(t^2 - \rho^2)}} \right) \}.$$
Now for $t_0 > t_1$,

\[
\int_{t_0}^{1} \frac{dt}{\sqrt{(t^2-t_0^2)(t^2-t_1^2)}} = \frac{1}{t_0} F(\varphi, k) \tag{A.5}
\]

where $F(\varphi, k)$ is the elliptic integral of the first kind [Abramowitz and Stegun, 1964; Press et al., 1986].

\[
F(\varphi, k) = \int_{0}^{\varphi} \frac{d\varphi'}{\sqrt{1-k^2 \sin^2 \varphi'}} = \int_{0}^{\sin \varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} \tag{A.6}
\]

and

\[
\varphi = \sin^{-1} \left( \frac{1-t_0^2}{1-t_1^2} \right) \tag{A.7}
\]

\[
k = \frac{t_1}{t_0}. \tag{A.8}
\]

Therefore,

\[
\int_{\rho}^{1} \frac{dt}{\sqrt{(t^2-\rho^2)(t^2-s^2)}} = \frac{1}{\rho} F \left( \sin^{-1} \left( \frac{1-\rho^2}{1-s^2} \right), \frac{s}{\rho} \right) \quad \text{when } \rho > s \tag{A.9}
\]

\[
\int_{s}^{1} \frac{dt}{\sqrt{(t^2-\rho^2)(t^2-s^2)}} = \frac{1}{s} F \left( \sin^{-1} \left( \frac{1-s^2}{1-\rho^2} \right), \frac{\rho}{s} \right) \quad \text{when } s > \rho \tag{A.10}
\]

So the crack opening at $\rho=r/R$ can be expressed in terms of a single integral involving the elliptic integral which can be efficiently computed using readily available algorithms [Press et al., 1986]:

\[
\delta(\rho) = \frac{2R}{\pi E} \left\{ \int_{0}^{\rho} ds \cdot s \cdot p \left[ \frac{1}{\rho} F \left( \sin^{-1} \left( \frac{1-\rho^2}{1-s^2} \right), \frac{s}{\rho} \right) \right] \right. \\
\left. + \int_{\rho}^{1} ds \cdot s \cdot p \left[ \frac{1}{s} F \left( \sin^{-1} \left( \frac{1-s^2}{1-\rho^2} \right), \frac{\rho}{s} \right) \right] \right\}. \tag{A.11}
\]
Discretization of Crack-Opening Integral

If the pressure distribution used in Equation (A.11) is discretized as

\[ p(s) = \sum_{i=0}^{N-1} \left\{ \left( \frac{s_{i+1} - s_i}{s_{i+1} - s_i} \right) p(s_i) + \left( \frac{s - s_i}{s_{i+1} - s_i} \right) p(s_{i+1}) \right\} \]

(A.12)

\[ = \sum_{i=0}^{N-1} \left\{ (a_i + b_i s_i) p_i + (c_i + d_i s_i) p_{i+1} \right\} \]

where

\[ a_i = \frac{s_{i+1}}{s_{i+1} - s_i} \]

(A.13a)

\[ b_i = \frac{-1}{s_{i+1} - s_i} \]

(A.13b)

\[ c_i = \frac{-s_i}{s_{i+1} - s_i} \]

(A.13c)

\[ d_i = \frac{1}{s_{i+1} - s_i} \]

(A.13d)

\[ p_i = p(s_i) \]

(A.13e)

\[ s_0 = 0 \]

(A.13f)

\[ s_N = 1 \]

(A.13g)

Using Equations (A.12) and (A.13) in Equation (A.11) then gives

\[ \delta(p) = \frac{2R}{\pi E} \left[ \frac{1}{\rho} \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} ds \left[ (a_i s + b_i s^2) p_i + (c_i s + d_i s^2) p_{i+1} \right] F \left( \sin^{-1} \sqrt{(1 - p^2)/(1 - s^2)} , \frac{s}{\rho} \right) \right] \]

\[ + \sum_{i=m}^{N-1} \int_{s_i}^{s_{i+1}} ds \left[ (a_i + b_i s) p_i + (c_i + d_i s) p_{i+1} \right] F \left( \sin^{-1} \sqrt{(1 - s^2)/(1 - \rho^2)} , \frac{\rho}{s} \right) \]

(A.14)

where \( s_n = \rho \).
Using the notation

\[
F_1[s, \rho] = F\left(\sin^{-1}\sqrt{(1-\rho^2)/(1-s^2)}, \frac{s}{\rho}\right) \tag{A.15a}
\]

\[
F_2[s, \rho] = F\left(\sin^{-1}\sqrt{(1-s^2)/(1-\rho^2)}, \frac{\rho}{s}\right), \tag{A.15b}
\]

each region \([s_i, s_{i+1}]\) is divided into enough subregions \([s_{i,j}, s_{i,j+1}]\), \(j = 0, M_i - 1\), such that \(F_k[s, \rho]\) is approximately linear within each subregion, for \(k=1,2\):

\[
F_k[s, \rho] = \left(\frac{s_{i,j+1} - s}{s_{i,j+1} - s_{i,j}}\right) F_k[s_{i,j}, \rho] + \left(\frac{s - s_{i,j}}{s_{i,j+1} - s_{i,j}}\right) F_k[s_{i,j+1}, \rho]
\]

\[
= (A_{ij} + B_{ij}s) F_{k,i,j} + (C_{ij} + D_{ij}s) F_{k,i,j+1}
\]

for

\[
s_i \leq s_{i,j} \leq s \leq s_{i,j+1} \leq s_{i+1} \tag{A.17}
\]

where

\[
A_{ij} \equiv \frac{s_{i,j+1}}{s_{i,j+1} - s_{i,j}} \tag{A.18a}
\]

\[
B_{ij} \equiv \frac{-1}{s_{i,j+1} - s_{i,j}} \tag{A.18b}
\]

\[
C_{ij} \equiv \frac{-s_{i,j}}{s_{i,j+1} - s_{i,j}} \tag{A.18c}
\]

\[
D_{ij} \equiv \frac{1}{s_{i,j+1} - s_{i,j}} \tag{A.18d}
\]

\[
F_{k,i,j} = F_k[s_{i,j}, \rho] \tag{A.18e}
\]

\[
s_{i,0} = s_i \tag{A.18f}
\]

\[
s_{i,M_i} = s_{i+1} \tag{A.18g}
\]
Using these relations in Equation (A.14) results in

\[ \delta(p) = \frac{2R}{\pi E} \left\{ \frac{1}{\rho} \sum_{i=0}^{n-1} \sum_{j=0}^{M-1} \int_{S_{i,j}}^{S_{i,j+1}} ds \left[ (a_i s + h_i s^2)p_i + (c_i s + d_i s^2)p_{i+1} \right] \right. \\
+ \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \int_{S_{i,j}}^{S_{i,j+1}} ds \left[ (a_i s + b_i s)p_i + (c_i + d_i s)p_{i+1} \right] \right. \\
\cdot \left[ (A_{ij} + B_{ij} s)F_{1,i,j} + (C_{ij} + D_{ij} s)F_{1,i,j+1} \right] \\
\left. + \sum_{i=m}^{N-1} \sum_{j=0}^{M-1} \int_{S_{i,j}}^{S_{i,j+1}} ds \left[ (a_i s + b_i s)p_i + (c_i + d_i s)p_{i+1} \right] \right. \\
\cdot \left[ (A_{ij} + B_{ij} s)F_{2,i,j} + (C_{ij} + D_{ij} s)F_{2,i,j+1} \right] \} \\
\]

\[ = \frac{2R}{\pi E} \left\{ \frac{1}{\rho} \sum_{i=0}^{n-1} \sum_{j=0}^{M-1} \int_{S_{i,j}}^{S_{i,j+1}} ds \left[ [(a_i s + c_i s)p_{i+1}](A_{ij} F_{1,i,j} + C_{ij} F_{1,i,j+1}) \right] s \\
+ [(a_i s + c_i s)p_{i+1}](B_{ij} F_{1,i,j} + D_{ij} F_{1,i,j+1}) \right. \\
+ [(b_i s + d_i s)p_{i+1}](A_{ij} F_{1,i,j} + C_{ij} F_{1,i,j+1}) s^2 \\
+ [(b_i s + d_i s)p_{i+1}](B_{ij} F_{1,i,j} + D_{ij} F_{1,i,j+1}) s^3 \left. \right] \right. \\
\right. \\
\left. + \sum_{i=m}^{N-1} \sum_{j=0}^{M-1} \int_{S_{i,j}}^{S_{i,j+1}} ds \left[ [(a_i s + c_i s)p_{i+1}](A_{ij} F_{2,i,j} + C_{ij} F_{2,i,j+1}) \right] s \\
+ [(a_i s + c_i s)p_{i+1}](B_{ij} F_{2,i,j} + D_{ij} F_{2,i,j+1}) \right. \\
+ [(b_i s + d_i s)p_{i+1}](A_{ij} F_{2,i,j} + C_{ij} F_{2,i,j+1}) s^2 \\
+ [(b_i s + d_i s)p_{i+1}](B_{ij} F_{2,i,j} + D_{ij} F_{2,i,j+1}) s^3 \left. \right] \right. \\
\left. \right\} \\
\]

(A.19)

Carrying out the integrations, gives a final expression for the crack opening value:
\[ \delta(\rho) = \frac{2R}{\pi E} \left\{ \frac{1}{\rho} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left[ \left[ a_i \psi_{11ij} \phi_{2ij} + (a_i \psi_{21ij} + b_i \psi_{11ij}) \phi_{3ij} + b_i \psi_{21ij} \phi_{4ij} \right] p_i \right. \\
+ \left. \left[ c_i \psi_{11ij} \phi_{2ij} + (c_i \psi_{21ij} + d_i \psi_{11ij}) \phi_{3ij} + d_i \psi_{21ij} \phi_{4ij} \right] p_{i-1} \right] \right. \\
+ \left. \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left[ a_i \psi_{12ij} \phi_{1ij} + (a_i \psi_{22ij} + b_i \psi_{12ij}) \phi_{2ij} + b_i \psi_{22ij} \phi_{3ij} \right] p_i \right. \\
+ \left. \left[ c_i \psi_{12ij} \phi_{1ij} + (c_i \psi_{22ij} + d_i \psi_{12ij}) \phi_{2ij} + d_i \psi_{22ij} \phi_{3ij} \right] p_{i-1} \right\} \]

(A.20)

where

\[ \psi_{1kij} = A_{ij} F_{k,i,j} + C_{ij} F_{k,i,j+1} \]  

(A.21a)

\[ \psi_{2kij} = B_{ij} F_{k,i,j} + D_{ij} F_{k,i,j+1} \]  

(A.21b)

\[ \phi_{lij} = \frac{1}{l} (s_{ij} - s_{ij}). \]  

(A.21c)
APPENDIX B

CONSTITUTIVE FLOW RELATION
FOR POWER-LAW FLUID

In this appendix, expressions are developed for the velocity distribution and pressure gradient due to the flow of a "power-law" fluid between parallel plates. A power-law fluid is one for which the shear stress \( \tau \) is a power \( n' \) of the strain rate \( \dot{\gamma} \):

\[
\tau = k' \dot{\gamma}^{n'}
\]  

(B.1)

For laminar flow, the conservation of linear momentum is expressed as [White, 1974]:

\[
\rho g - \nabla P + \nabla \cdot \tau = \rho \frac{Du}{Dt}
\]  

(B.2)

where the stress tensor \( \tau \) is composed of a normal pressure component \( P \) and a shear stress component \( \tau \):

\[
\sigma = -P I + \tau = \begin{bmatrix} -P + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -P + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -P + \tau_{zz} \end{bmatrix}.
\]  

(B.3)

If the body force \( g \) is conservative, a potential \( \phi \) can be defined such that

\[
g = -\nabla \phi,
\]  

(B.4)

and the pressure and body force potential can be combined into a piezometric pressure.
\[ \dot{P} = P + \rho \phi. \quad (B.5) \]

Using this definition, the components of the momentum conservation equation can be expressed, for uniform density, as

\[
\begin{align*}
-\frac{\partial \dot{P}}{\partial x} + \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) &= \rho \left( \frac{\partial u}{\partial t} + \frac{u \partial u}{\partial x} + \frac{v \partial u}{\partial y} + \frac{w \partial u}{\partial z} \right) \\
-\frac{\partial \dot{P}}{\partial y} + \left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) &= \rho \left( \frac{\partial v}{\partial t} + \frac{u \partial v}{\partial x} + \frac{v \partial v}{\partial y} + \frac{w \partial v}{\partial z} \right) \\
-\frac{\partial \dot{P}}{\partial z} + \left( \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) &= \rho \left( \frac{\partial w}{\partial t} + \frac{u \partial w}{\partial x} + \frac{v \partial w}{\partial y} + \frac{w \partial w}{\partial z} \right).
\end{align*} \tag{B.6}
\]

For two-dimensional flow in the xy-plane, between parallel bounding surfaces at \( y = \pm y_{\text{wall}} \), there is no flow in the z-direction

\[ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0. \quad (B.7) \]

and no variation of velocity with respect to \( z \)

\[ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0. \quad (B.8) \]

In this case, Equation (B.6) becomes

\[
\begin{align*}
-\frac{\partial \dot{P}}{\partial x} + \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) &= \rho \left( \frac{\partial u}{\partial t} + \frac{u \partial u}{\partial x} + \frac{v \partial u}{\partial y} \right) \\
-\frac{\partial \dot{P}}{\partial y} + \left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) &= \rho \left( \frac{\partial v}{\partial t} + \frac{u \partial v}{\partial x} + \frac{v \partial v}{\partial y} \right) \\
-\frac{\partial \dot{P}}{\partial z} + \left( \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) &= 0.
\end{align*} \tag{B.9}
\]
The power-law assumption of Equation (B.1) is written for specific components of the shear rates and shear stresses as

\[ \tau_{ij} = k' |u_{i,j} + u_{j,i}|^{n'} s_{ij} \]  \hspace{1cm} (B.10)

where \( s_{ij} = \frac{u_{i,j} - u_{j,i}}{|u_{i,j} - u_{j,i}|} = \pm 1 \) \hspace{1cm} (B.11a)

\[ u_{i,j} = \frac{\partial u_i}{\partial x_j}. \]  \hspace{1cm} (B.11b)

For two-dimensional flow in the xy-plane, Equations (B.7, B.8) in Equation (B.10) implies that

\[ \tau_{xx} = \tau_{xx} = \tau_{yx} = \tau_{xy} = \tau_{zz} = 0 \]  \hspace{1cm} (B.12)

and the momentum equation becomes

\[ \frac{-\partial P}{\partial x} + \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \right) = \rho \left( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]

\[ \frac{-\partial P}{\partial y} + \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \right) = \rho \left( \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]  \hspace{1cm} (B.13)

\[-\frac{\partial P}{\partial z} = 0.\]

If the flow is fully-developed in the x-direction, then

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0. \]  \hspace{1cm} (B.14)

Using this with Equation (B.7) in the volume conservation equation \( \nabla \cdot \mathbf{v} = 0 \) results in

\[ \frac{\partial v}{\partial y} = 0. \]  \hspace{1cm} (B.15)
For stationary channel walls at $y = \pm y_{wall} = \pm \delta/2$, $v = 0$ at the walls and Equation (B.15) implies that $v = 0$ everywhere. Using Equations (B.14) and (B.15) in Equation (B.10) gives

$$
\tau_{xx} = \tau_{yy} = 0.
$$

(B.16)

If the flow is also steady, so that $\frac{\partial u}{\partial t} = 0$, then Equation (B.13) simplifies to

$$
-\frac{\partial P}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0
$$

$$
-\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0
$$

(B.17)

Using Equations (B.10, B.11) gives

$$
\tau_{xy} = \tau_{yx} = k' \left| \frac{\partial u}{\partial y} \right|^n s_{xy}.
$$

(B.18)

Therefore

$$
\frac{\partial \tau_{xy}}{\partial x} = k' n' \left| \frac{\partial u}{\partial y} \right|^{n'-1} \frac{\partial}{\partial x} \left| \frac{\partial u}{\partial y} \right| s_{xy}
$$

$$
= k' n' \left| \frac{\partial u}{\partial y} \right|^{n'-1} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right)
$$

$$
= k' n' \left| \frac{\partial u}{\partial y} \right|^{n'-1} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = 0,
$$

(B.19)

since $s_{xy} = \pm 1$, $\left| \frac{\partial u}{\partial y} \right| s_{xy} = \frac{\partial u}{\partial y}$, and using Equation (B.14). Similarly
\[
\frac{\partial \tau_{xy}}{\partial y} = k \cdot n' \left| \frac{\partial u}{\partial y} \right|^{n'-1} \frac{\partial u}{\partial y} s_{xy} \\
= k \cdot n' \left| \frac{\partial u}{\partial y} \right|^{n'-1} \frac{\partial^2 u}{\partial y^2} .
\]

Using Equations (B.19, B.20) in Equation (B.17) yields

\[
\frac{\partial \hat{P}}{\partial x} = k \cdot n' \left| \frac{\partial u}{\partial y} \right|^{n'-1} \frac{\partial^2 u}{\partial y^2} \quad (B.21a)
\]
\[
\frac{\partial \hat{P}}{\partial y} = 0 \quad (B.21b)
\]
\[
\frac{\partial \hat{P}}{\partial z} = 0 . \quad (B.21c)
\]

Since Equations (B.21b) and (B.21c) show \( \hat{P} \) to depend only on \( x \), the left-hand side of Equation (B.21a) is a function of \( x \) while the right-hand side is a function of \( y \), indicating that both sides must be constant. Solving Equation (B.21a) for \( u(y) \), subject to the boundary conditions

\[
u(y = \pm y_{wall}) = 0
\]
\[
\frac{\partial u}{\partial y}(y = 0) = 0 ,
\]

\( B.22 \)
gives

\[
u(y) = \left( -\frac{y_{wall}}{k'} \frac{d\hat{P}}{dx} \right)^{\frac{1}{n'}} \left( \frac{n' y_{wall}}{n' + 1} \right) \left\{ 1 - \frac{y}{y_{wall}} \right\}^{\frac{n'+1}{n'}} . \quad (B.23)
\]

Integrating this velocity distribution across the channel provides an expression for the flowrate per unit channel depth.

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\[ q' = \int_{-y_{wall}}^{y_{wall}} u(y)dy = \left( -\frac{y_{wall}}{k'} \frac{dP}{dx} \right)^{\frac{1}{n'}} \left( 2n'y_{wall}^2 \right) \quad \text{(B.24)} \]

The pressure gradient as a function of the flowrate is therefore

\[ \frac{dP}{dx} = -k' \left( \frac{2n' + 1}{2n'y_{wall}^2} \right)^{n'} \quad \text{(B.25)} \]

and, since the channel width \( \delta = 2y_{wall} \),

\[ \frac{dP}{dx} = -k' \left( \frac{2}{\delta} \right)^{2n'+1} \left[ \frac{(2n' + 1)}{2n'} q' \right]^{n'} \quad \text{(B.26)} \]

This is the constitutive relationship for the pressure gradient of steady, fully-developed, one-dimensional flow of power law fluid given in Equation (2.5.6).
APPENDIX C
"DEVIATIONAL" PRESSURE GRADIENT

In this appendix an approximate relationship between the radial flowrate and fluid pressure gradient in the fracture is developed for cases in which the flow does not satisfy the conditions of being steady, fully-developed and one-dimensional, as required in the derivation of Equation (2.5.4). An "integral-method" of analysis [e.g. see Rohsenow and Choi, 1961] has been used, in which the momentum and continuity equations are satisfied in integral form over the entire width of the flow channel, \(-\delta/2 \leq z \leq \delta/2\), rather than in differential form at each point in the flow. The resulting solution satisfies the overall momentum and continuity conditions.

For radial flow of Newtonian fluid between parallel plates, with no angular variation of pressure or velocity and negligible body forces, the equation governing the radial linear momentum is

\[
\rho_f \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left[ \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right]
\]

(C.1)

Assumptions are made for the variations of radial and transverse flows across the channel which satisfy the boundary conditions of

\[
v_r(z = \pm \delta/2) = 0
\]

(C.2)
where \( v_z(wall) \) is the fluid velocity in the \( z \)-direction at the upper wall \((z = \delta/2)\). The distributions used here are

\[
\frac{\partial v_r}{\partial z} (z=0) = 0 \tag{C.3}
\]

\[
v_z(z = \pm \delta/2) = \pm v_z(wall) \tag{C.4}
\]

\[
v_z(z = 0) = 0 \tag{C.5}
\]

Using Equation (C.6),

\[
\frac{\partial v_r}{\partial z} = -12\bar{v}_r \frac{z}{\delta^2} \tag{C.10}
\]

and

\[
\bar{v}_r = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} v_r(z)dz \tag{C.8}
\]

\[
v_z(wall) = v_z(z = \delta/2) = \frac{1}{2} (\dot{\delta} + q_l'' ) \tag{C.9}
\]

with \( \dot{\delta} = \frac{\partial \delta}{\partial t} \) and \( q_l'' \) being the leakoff rate through both channel walls.
\[ \frac{\partial^2 v_r}{\partial z^2} = -12 \frac{\bar{\nu}_r}{\delta^2}. \]  

(C.11)

Using Equations (C.6-C.11) to help integrate the terms in Equation (C.1) from \( z=-\delta/2 \) to \( z=\delta/2 \) gives

\[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \nu_r \frac{\partial v_r}{\partial r} \, dz = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \left[ \frac{3}{2} \bar{\nu}_r \left( 1 - \frac{z^2}{(\delta/2)^2} \right) \right] \cdot \left[ \frac{3}{2} \frac{\partial \bar{\nu}_r}{\partial r} \left( 1 - \frac{z^2}{(\delta/2)^2} \right) \right] \, dz \]

\[ = \frac{\delta}{\delta} \frac{\partial \bar{\nu}_r}{\partial r} \]

(C.12)

\[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} v_z \frac{\partial v_r}{\partial z} \, dz = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \left( \frac{2z}{\delta} \cdot v_z(wall) \right) \cdot \left( -12 \bar{\nu}_r \frac{z}{\delta^2} \right) \, dz \]

\[ = -2 \bar{\nu}_r \frac{v_z(wall)}{\delta} \]

\[ = -\bar{\nu}_r \left( \delta + q_i'' \right) \]

(C.13)

\[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial^2 v_r}{\partial z^2} \, dz = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \left( -12 \bar{\nu}_r \right) \, dz = -12 \bar{\nu}_r \]

(C.14)

\[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial v_r}{\partial r} \, dz = \frac{\partial \bar{\nu}_r}{\partial t} \]

(C.15)

\[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) \, dz = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r \bar{v}_r) \right). \]

(C.16)

Equation (C.1) can then be expressed as

\[ \frac{\partial P}{\partial r} = \rho_f \left[ \frac{-\partial \bar{\nu}_r}{\partial t} - \frac{\delta}{\delta} \frac{\partial \bar{\nu}_r}{\partial r} + \bar{\nu}_r \frac{\delta + q_i''}{\delta} \right] \]

\[ + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r \bar{v}_r) \right) - 12 \bar{v}_r \right]. \]

(C.17)
This relation can be simplified using the continuity condition for incompressible flow in polar co-ordinates with no angular variation

\[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) = -\frac{\partial v_z}{\partial z}. \]  

(C.18)

Integrating the terms of this equation across the channel width gives

\[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} dz \left[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) \right] = \frac{1}{r} \frac{\partial}{\partial r} \left( rv_r \right) \]

\[ = \frac{1}{r} \frac{\partial}{\partial r} \left( rv_r \right) \]

(C.19)

\[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} dz \left( -\frac{\partial v_z}{\partial z} \right) = -\frac{1}{\delta} \left[ v_z(z=\delta/2) - v_z(z=-\delta/2) \right] \]

\[ = -\frac{2v_z(\text{wall})}{\delta} \]

(C.20)

Therefore

\[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) = -(\delta + q_i'' \right)/\delta. \]  

(C.21)

Using this in Equation (C.17) yields

\[ \frac{\partial P}{\partial r} = -12\mu \frac{\bar{v}_r}{\delta^2} - \frac{\partial}{\partial r} \left( \frac{\delta + q_i'' \right)}{\delta} + \rho_f \left[ -\frac{\partial \bar{v}_r}{\partial t} - \frac{6}{5} \bar{v}_r \frac{\partial \bar{v}_r}{\partial r} + \bar{v}_r \frac{\delta + q_i'' \right)}{\delta} \]

\[ = -12\mu \frac{\bar{v}_r}{\delta^2} + \frac{\partial P}{\partial r \text{ dev}}. \]  

(C.22)

The "deviational" pressure gradient of Equation (2.5.1.1) is therefore

\[ \frac{\partial P}{\partial r \text{ dev}} = -\mu \frac{\partial}{\partial r} \left( \frac{\delta + q_i'' \right)}{\delta} + \rho_f \left[ -\frac{\partial \bar{v}_r}{\partial t} - \frac{6}{5} \bar{v}_r \frac{\partial \bar{v}_r}{\partial r} + \bar{v}_r \frac{\delta + q_i'' \right)}{\delta} \]  

(C.23)
APPENDIX D

TWO-DIMENSIONAL (WEDGE) FLOW NEAR THE CRACK TIP

The crack opening rate near the crack tip can be of the same magnitude as the crack propagation rate, indicating that a large fluid velocity component exists in the transverse direction (normal to the crack surface) in this region, possibly resulting in large transverse pressure gradients. If this were the case, the pressure gradient relation for plane Poiseuille flow (Equation (2.5.4)) would be inadequate and, in addition, the radial distribution of pressure on the walls of the fracture would be different than the radial distribution of the average pressure across the flow channel between the walls. The former controls the crack width and the latter controls the fluid flow. An analysis of the two-dimensional flow field in this near-tip region is developed in this appendix to determine the extent to which such transverse flows and pressure gradients are important.

The crack width is a small fraction (typically less than 1%) of the radius of curvature $(\partial^2 \delta / \partial r^2)^{-1}$, even near the crack tip, so it is reasonable to consider a geometry in which the crack surfaces are flat but not parallel -- a wedge-shaped region such as shown in Figure D.1. Although the slope of the actual crack surface varies with distance from the crack tip, the actual flow, being dominated by viscous effects, with negligible inertia, will locally have the same behavior as the flow in a wedge with the same wall slope as the local crack surface, at the point in the wedge where the channel width equals the local crack width.
Considered in a stationary coordinate system, as in Figure D.1, the walls are taken to be moving in the \(-x\)-direction with speed \(\dot{R}\) and the width at a fixed \(x\)-position is increasing at the rate \(d\delta(x)/dt = \dot{R}\tan\theta_w\), where \(\theta_w\) is the channel half-angle. The total flow in the \(x\)-direction is therefore \(Q(x) = -2x\dot{R}\tan\theta_w\) and the average \(x\)-velocity at any cross section \(x\) is
\[
\bar{u}(x) = \frac{Q(x)}{2h(x)} = \frac{-2x\dot{R}\tan\theta_w}{2x\tan\theta_w} = -\dot{R}.
\]  
(D.1)
The constant average \(x\)-velocity at all values of \(x\) is consistent with the near-tip results for simulations of propagating circular fractures, a representative example of which is shown in Figure D.2.

When a frame of reference is adopted which moves with the crack tip, as shown in Figure D.3, the solid walls now move away from the tip of the wedge with the speed \(v_w = \dot{R}/\cos\theta_w\). The fluid next to the wall would have the same velocity \(v_w\) as the wall and the velocity distribution across the channel would be of the form shown in Figure D.3, having an integrated value of zero at any cross section. In this analysis, the \(x\)-independence of the average \(x\)-velocity is taken to indicate a velocity field which is \(r\)-independent in the polar co-ordinates of Figure D.3.

For incompressible, Newtonian fluid the flow is governed by the Navier-Stokes equation
\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla P + \mu \nabla^2 \mathbf{v} + \mathbf{F}_{\text{body}}
\]  
(D.2)
and continuity

\[ \nabla \cdot \mathbf{v} = 0. \]  
(D.3)

For steady-state conditions, small Reynolds number \((\rho R \dot{\theta} / \mu << 1)\), negligible body forces \( \left( \frac{h}{R} \frac{\rho R \dot{\theta}}{\mu} \frac{g}{R} << 1 \right) \), and velocities independent of \( \dot{\theta} \), these relations reduce to

\[ \begin{align*}
\dot{r} - \text{momentum:} & \quad \frac{\partial P}{\partial \dot{r}} = \mu \left[ \frac{1}{\dot{r}^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{\dot{r}^2} - \frac{2}{\dot{r}^2} \frac{\partial v_\theta}{\partial \theta} \right] \\
\theta - \text{momentum:} & \quad \frac{1}{\dot{r}} \frac{\partial P}{\partial \theta} = \mu \left[ \frac{1}{\dot{r}^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{\dot{r}^2} \frac{\partial v_r}{\partial \theta} \frac{v_\theta}{\dot{r}^2} \right] \\
\text{continuity:} & \quad \frac{\partial v_\theta}{\partial \theta} = -v_r
\end{align*} \]

(D.4)  
(D.5)  
(D.6)

Using Equation (D.6) in Equations (D.4) and (D.5) gives

\[ \begin{align*}
\dot{r} - \text{momentum:} & \quad \frac{\partial P}{\partial \dot{r}} = \frac{\mu}{\dot{r}^2} \left[ \frac{\partial^2 v_r}{\partial \theta^2} + v_r \right] \\
\theta - \text{momentum:} & \quad \frac{1}{\dot{r}} \frac{\partial P}{\partial \theta} = \frac{\mu}{\dot{r}^2} \left[ \frac{\partial v_r}{\partial \theta} - v_\theta \right]
\end{align*} \]

(D.7)  
(D.8)

The \(-\mu v_\theta / \dot{r}^2\) term in Equation (D.8) results in a pressure which drops in the direction of the fluid flow, the intuitive situation for viscous-dominated laminar flow. However, as will be seen below, this term will be completely cancelled out by part of the \(\frac{\mu}{\dot{r}^2} \frac{\partial v_r}{\partial \theta}\) term, the remainder of which will determine the angular pressure variation.

Now if the angular component of velocity is expressed as an unknown function of the angle

\[ v_\theta(\theta) = f(\theta) \]  
(D.9)
then Equation (D.6) implies that

$$v_\theta(\theta) = -f''(\theta).$$  \hspace{1cm} (D.10)

The momentum equations then can be written as

$$\frac{\partial P}{\partial \hat{r}} = -\frac{\mu}{\hat{r}^2} (f''' + f') = -\frac{F(\theta)}{\hat{r}^2},$$  \hspace{1cm} (D.11)

$$\frac{1}{\hat{r}} \frac{\partial P}{\partial \theta} = -\frac{\mu}{\hat{r}^2} (f'' + f).$$  \hspace{1cm} (D.12)

Equation (D.12) implies that

$$\frac{1}{\hat{r}} \frac{\partial^2 P}{\partial \theta^2} = -\frac{\mu}{\hat{r}^2} (f'' + f') = -\frac{F(\theta)}{\hat{r}^2}.$$  \hspace{1cm} (D.13)

Integrating Equation (D.11) with respect to $\hat{r}$ yields

$$P = \frac{F(\theta)}{\hat{r}} + G(\theta)$$  \hspace{1cm} (D.14)

and using Equation (D.13) in Equation (D.14) gives

$$P + \frac{\partial^2 P}{\partial \theta^2} = G(\theta).$$  \hspace{1cm} (D.15)

Now Equations (D.14) and (D.15) together imply that

$$\left[ \frac{F(\theta)}{\hat{r}} + G(\theta) \right] + \left[ \frac{F''(\theta)}{\hat{r}} + G''(\theta) \right] = G(\theta),$$  \hspace{1cm} (D.16)

or

$$\frac{1}{\hat{r}} \left[ F(\theta) + F''(\theta) \right] = -G''(\theta).$$  \hspace{1cm} (D.17)

Since the left-hand side of Equation (D.17) depends on $\hat{r}$ while the right-hand side doesn't, each side must equal zero:
\[ F(\theta) + F''(\theta) = 0 \quad \text{(D.18)} \]
\[ G''(\theta) = 0. \quad \text{(D.19)} \]

The solutions to these differential equations are
\[ F(\theta) = c_1 \sin \theta + c_2 \cos \theta \quad \text{(D.20)} \]
\[ G(\theta) = c_3 \theta + c_4. \quad \text{(D.21)} \]

The symmetry condition on the centerline \( \frac{\partial P}{\partial \theta} \bigg|_{\theta=0} \) forces \( c_4 = c_3 = 0 \), so that

\[ P(\hat{r},\theta) = \frac{F(\theta)}{\hat{r}} + G(\theta) \]
\[ = c_2 \frac{\cos \theta}{\hat{r}} + P(\hat{r} = \infty). \quad \text{(D.22)} \]

Using Equation (D.22) in Equation (D.12) results in

\[ f''(\theta) + f(\theta) = \frac{c_2}{\mu} \sin \theta \quad \text{(D.23)} \]

which has the solution

\[ f(\theta) = -\frac{c_2}{\mu} \frac{\theta}{2} \cos \theta + c_5 \sin \theta + c_6 \cos \theta. \quad \text{(D.24)} \]

The velocities can now be expressed as

\[ v_\theta = f(\theta) = -\frac{c_2}{\mu} \frac{\theta}{2} \cos \theta + c_5 \sin \theta + c_6 \cos \theta \quad \text{(D.25)} \]
\[ v_\rho = -\frac{\partial v_\theta}{\partial \theta} = \frac{c_2}{2\mu} (\cos \theta - \theta \sin \theta) - c_5 \cos \theta + c_6 \sin \theta. \quad \text{(D.26)} \]
Using the boundary conditions

\[ v_\theta(\theta = 0) = 0 \]  \hspace{1cm} (D.27)

\[ v_\theta(\theta = \theta_w) = 0 \]  \hspace{1cm} (D.28)

\[ v_r(\theta = \theta_w) = v_w \]  \hspace{1cm} (D.29)

allows the constants to be evaluated. The resulting velocities are

\[ v_r = v_w \frac{[\theta_w \cos \theta_w \cos \theta + \sin \theta_w (\theta \sin \theta - \cos \theta)]}{[\theta_w - \frac{1}{2} \sin 2\theta_w]} \]  \hspace{1cm} (D.30)

\[ v_\theta = v_w \frac{[-\theta_w \cos \theta_w \sin \theta + \theta \sin \theta_w \cos \theta]}{[\theta_w - \frac{1}{2} \sin 2\theta_w]} . \]  \hspace{1cm} (D.31)

These distributions are plotted in Figure D.4, for several values of \( \theta_w \). The variation in \( v_r \) with \( \theta_w \) is too small to be distinguishable on the plot.

The corresponding pressure distribution is

\[ P(\hat{r}, \theta) = P(\hat{r} = \infty) - \frac{2\mu v_w}{\hat{r}} \frac{\sin \theta_w \cos \theta}{(\theta_w - \frac{1}{2} \sin 2\theta_w)} . \]  \hspace{1cm} (D.32)

The spatial derivatives of pressure are

\[ \frac{\partial P}{\partial \hat{r}} = \frac{2\mu v_w}{\hat{r}^2} \frac{\sin \theta_w \cos \theta}{\theta_w - \frac{1}{2} \sin 2\theta_w} \]  \hspace{1cm} (D.33)

\[ \frac{\partial P}{\partial \theta} = \frac{2\mu v_w}{\hat{r}} \frac{\sin \theta_w \sin \theta}{\theta_w - \frac{1}{2} \sin 2\theta_w} , \]  \hspace{1cm} (D.34)

indicating that pressure increases with \( \hat{r} \) and with angle away from the centerline.

Making the substitution \( v_w = \hat{R}/\cos \theta_w \) in Equation (D.32) gives the pressure difference between the centerline and the wall, at a given radius, as

\[ P(\theta = \theta_w) - P(\theta = 0) = \frac{2\mu \hat{R}}{\hat{r}} \frac{\tan \theta_w (1 - \cos \theta_w)}{(\theta_w - \frac{1}{2} \sin 2\theta_w)} . \]  \hspace{1cm} (D.35)
The relative magnitudes of the angular and radial pressure gradients can be compared by the following ratio:

\[
\frac{\left[ P(\theta = \theta_w) - P(\theta = 0) \right]}{\theta_w \cos \theta_w} = \frac{1 - \cos \theta_w}{\theta_w \cos \theta_w}
\]  

(D.36)

which is plotted as a function of \( \theta_w \) in Figure D.5.

Figure D.6 shows \( \theta_w \) near the crack tip for an A3DH simulation of the DISLASH experiment. Figure D.7 shows the pressure difference \( P(\theta = \theta_w) - P(\theta = 0) \) as computed from Equation (D.35), also for DISLASH conditions. The pressure difference is on the order of 1 psi near the fluid front, but quickly drops off to much smaller values. Fractures propagating in rock have much smaller values of \( \theta_w \) (less than 1° for the CIA experiments) so the two-dimensionality of the flow field is not a factor in those cases.

The pressure gradient at the wall, projected into the \( x \)-direction, is

\[
\frac{\partial P}{\partial x} \bigg|_{\theta_w} = \left[ \frac{\partial P}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} \right]_{\theta_w}
\]

\[
= \left[ \frac{\mu \nu_w}{\rho^2 \sin \theta_w} \cdot \frac{2 \cos \theta_w \sin \theta_w}{\theta_w - \frac{1}{2} \sin 2 \theta_w} \right] \cdot \left[ \frac{1}{\cos \theta_w} \right].
\]  

(D.37)

Using \( \nu_w = \dot{R} / \cos \theta_w \) and \( h(x) = \delta(x)/2 = \rho(x) \sin \theta_w \) gives
\[
\left. \frac{\partial P}{\partial x} \right|_{\theta_w} = \frac{8\mu R}{\delta^2(x)} \frac{\sin^3 \theta_w}{\cos \theta_w \left( \theta_w - \frac{1}{2} \sin 2\theta_w \right)}
\]  
(D.38)

which can be expanded for small \( \theta_w \) as

\[
\left. \frac{\partial P}{\partial x} \right|_{\theta_w} = \frac{12\mu R}{\delta^3(x)} \left( 1 + \frac{1}{5} \theta_w^2 + \ldots \right).
\]  
(D.39)

The correction to the plane Poiseuille flow relation is negligible except very near the tip.
Figure D.1  Wedge flow geometry, viewed in a stationary co-ordinate system.
Figure D.2  Average fluid velocity vs. normalized distance from crack tip. A3DH simulation of DISLASH conditions.
Figure D.3  Wedge flow geometry, showing radial fluid velocity, for a co-ordinate system moving with the crack tip.

\[ v_f(\theta = \theta_w) = v_w = \frac{\dot{R}}{\cos \theta_w} \]
Figure D.4  Normalized radial and angular velocity components vs. normalized angular position within the near-tip wedge for $\theta_w = 5^\circ, 10^\circ, 15^\circ, 20^\circ$. 
Figure D.5  Ratio of average angular pressure gradient to radial pressure gradient vs. wall angle.
Figure D.6  Wall angle $\theta_w$ vs. normalized distance from crack tip. A3DH simulation of DISLASH conditions.
Figure D.7 \( P(\theta=\theta_{\text{ep}}) - P(\theta=0) \) vs. normalized distance from crack tip. A3DH simulation of DISLASH conditions.
APPENDIX E

CALCULATION OF STRESS INTENSITY FACTOR

The Mode I stress intensity factor $K_I$ measures the strength of the singularity in the stress field at the tip of a fracture whose surfaces experience only opening displacements (no sliding parallel to the fracture surface). The scaling of the near-tip stress is

$$
\sigma_{j,k}(r,\phi) = \frac{K_I}{\sqrt{2\pi r}} R_{jk}(\phi) + O(r^{\frac{1}{2}})
$$

(E.1)

in polar co-ordinates centered at the crack-tip, where $r$ is the distance from the crack tip and $\phi$ represents angular position. The second term on the right-hand side indicates additional terms, the largest of which is of order $r^{\frac{1}{2}}$.

For a circular fracture subjected to a specified distribution of excess pressure $p(r)$ as a function of position $r$ from the crack center, the Mode I stress intensity factor is computed as

$$
K_I = \frac{2}{\sqrt{\pi R}} \int_0^R \frac{p(r)rdr}{\sqrt{R^2-r^2}}
= 2\sqrt{\frac{R}{\pi}} \int_0^1 \frac{p(s)ds}{\sqrt{1-s^2}}
$$

(E.2)

where $s=r/R$ [Sneddon and Lowengrub, 1969].
Now if the pressure distribution is assumed to be well approximated by the piecewise linear distribution

\[ p(s) = \left( \frac{s_{n+1} - s}{s_{n+1} - s_n} \right) \cdot p(s_n) + \left( \frac{s - s_n}{s_{n+1} - s_n} \right) \cdot p(s_{n+1}) \]  

(E.3)

for \( s_n \leq s \leq s_{n+1}; \ n=0,\ldots, N-1; \ s_0=0; \ s_N=1 \), then Equation (E.2) can be expressed as

\[ K_I = 2 \left[ \frac{R}{\pi} \sum_{n=0}^{N-1} \int_{S_n}^{S_{n+1}} \frac{s}{\sqrt{1-s^2}} \left[ \left( \frac{s_{n+1} - s}{s_{n+1} - s_n} \right) p(s_n) + \left( \frac{s - s_n}{s_{n+1} - s_n} \right) p(s_{n+1}) \right] ds \right]. \]  

(E.4)

Evaluating the integrals gives

\[ K_I = 2 \left[ \frac{R}{\pi} \sum_{n=0}^{N-1} \left\{ \frac{s_{n+1}p_n - s_np_{n+1}}{s_{n+1} - s_n} \cdot \left[ \sqrt{1-s_n^2} - \sqrt{1-s_{n+1}^2} \right] \right. \right. \]

\[ + \ \left. \frac{p_{n+1} - p_n}{2(s_{n+1} - s_n)} \cdot \left[ \sin^{-1}\left( s_{n+1} \sqrt{1-s_n^2} \right) - s_n \sqrt{1-s_{n+1}^2} \right] \right) \]

\[ + \ s_n \sqrt{1-s_n^2} - s_{n+1} \sqrt{1-s_{n+1}^2} \]  

(E.5)

with \( p_n = p(s_n = r_n/R) \).

This expression is adequate for computing \( K_I \) if none of the points \( s_n \) are so near to adjacent points or to 1 that near-cancellation of terms affects the accuracy of the calculation for the precision of the computer being used to evaluate the expression. For the nodal positions required to simulate some fracture conditions, this near-cancellation will occur and an alternate expression must be developed to achieve an accurate calculation of \( K_I \).
To do this, use is made of the series representations [Abromowitz and Stegun, 1964]

\[
\sqrt{1+x} = \sum_{k=0}^{\infty} a_k x^k \quad \text{for } |x| < 1
\]

where \( a_0 = 1 \)

\[
a_k = -\frac{(2k-3)}{2k} a_{k-1}
\]

\[
\sin^{-1}x = \sum_{k=0}^{\infty} b_k \frac{x^{2k+1}}{2k+1} \quad \text{where } b_0 = 1
\]

\[
b_k = \frac{(2k-1)}{2k} b_{k-1}
\]

For \( |x|=1 \), it is useful to define two functions which separate out the leading behavior of the functions in Equations (E.6) and (E.7), to eliminate the tendency for near-cancellation:

\[
S_1(x) = \sqrt{1-x} - \left(1 + \frac{x}{2}\right)
\]

\[
= \sum_{k=2}^{\infty} a_k x^k, \quad |x| < 1
\]  

(E.8)

\[
S_2(x) = \sin^{-1}x - x
\]

\[
= \sum_{k=1}^{\infty} b_k \frac{x^{2k+1}}{2k+1}.
\]  

(E.9)

Then if three parameters are defined to measure the relative closeness of adjacent points to each other and of a single point to 1,

\[
\varepsilon_n = s_{n+1} - s_n 
\]

(E.10)

\[
\Delta_{1n} = 1 - s_n^2
\]

(E.11)

\[
\Delta_{2n} = s_{n+1}^2 - s_n^2 \leq \Delta_{1n}
\]

(E.12)
it is possible to rewrite the radicals used in Equation (E.5) as

\[
\sqrt{1 - s_{n+1}^2} = \sqrt{\Delta_{1n} - \Delta_{2n}} = \sqrt{\Delta_{1n} (1 - \Gamma_n)} \quad (E.13)
\]

\[
\sqrt{1 - s_n^2} = \sqrt{\Delta_{1n}} \quad (E.14)
\]

where

\[
\Gamma_n = \frac{1}{2} \frac{\Delta_{2n}}{\Delta_{1n}} - s_1 \left( \cos \left( \frac{\Delta_{2n}}{\Delta_{1n}} \right) \right) \quad (E.15)
\]

Using these expressions, the argument of the arcsin term in Equation (E.5) can be expressed in a manner which similarly avoids near-cancellation:

\[
s_{n+1} \sqrt{1 - s_{n+1}^2} - s_n \sqrt{1 - s_n^2} = \sqrt{\Delta_{1n} \left[ \varepsilon_n (1 - \Gamma_n) + s_{n+1} \Gamma_n \right]} = \Psi_n \quad (E.16)
\]

Using Equation (E.9) with Equation (E.16) gives

\[
\sin^{-1} \left( s_{n+1} \sqrt{1 - s_{n+1}^2} - s_n \sqrt{1 - s_n^2} \right) = \Psi_n + S_2(\Psi_n) \quad (E.17)
\]

Putting these expressions in Equation (E.5) and grouping terms for each pressure together gives a final expression which avoids the near-cancellation pitfall:

\[
K_I = \sqrt{\frac{R}{\pi}} \sum_{n=0}^{N-1} \left\{ p_n \left[ \sqrt{\Delta_{1n} \Gamma_n} - \frac{1}{\varepsilon_n} S_2(\Psi_n) \right] + p_{n+1} \left[ \sqrt{\Delta_{1n} \Gamma_n} + \frac{1}{\varepsilon_n} S_2(\Psi_n) \right] \right\} \quad (E.18)
\]
APPENDIX F

CRACK OPENING RATE

Three methods have been implemented to compute the crack opening rate distribution \( \delta(r) \), given the crack opening distribution at the current timestep \( \delta(r,t) \) and at the previous timestep \( \delta(r,t-\Delta t) \). Since the region between the previous and current crack tip positions has only the current crack opening distribution available, all three methods use a self-similar crack opening rate in this region. The expression for the self-similar rate is given in Appendix G. Over the remaining region of the fracture, the methods used to calculate \( \delta(r) \) are:

- Simple difference ratio
- Localized self-similarity
- Simultaneous solution

Simple Difference Ratio

The crack opening rate at radius \( r \) is computed as the ratio of the change in crack opening between timesteps to the change in time

\[
\dot{\delta}(r,t) = \frac{\delta(r,t) - \delta(r,t - \Delta t)}{\Delta t}.
\]  

(F.1)

This expression would be accurate if the crack opening rate at a given radius remained unchanged during the timestep. This is a reasonable assumption near the wellbore where \( \delta(r) \) changes slowly. Near the crack-tip, however, the opening rates
are initially very high and then drop rapidly as the crack tip moves away. Thus, the use of Equation (F.1) would generally overestimate the $\delta(r)$ values.

**Localized Self-Similarity**

As discussed in Appendix G, for a self-similar crack opening distribution the time dependence and the spatial dependence are separable, so that

$$\delta(r,t) = \delta(r = 0, t) \cdot f(r/R).$$

(F.2)

If a less restrictive condition is imposed, assuming only that

$$\delta(r,t) = \delta(r/R,t),$$

(F.3)

then the time derivative at a fixed radius is

$$\frac{\partial \delta(r/R,t)}{\partial t} \bigg|_r = \frac{\partial \delta}{\partial t} \bigg|_{r/R} + \frac{\partial \delta}{\partial r/R} \bigg|_{t} \cdot \frac{\partial r/R}{\partial t} \bigg|_r$$

$$= \frac{\partial \delta}{\partial t} \bigg|_{r/R} - r \frac{\partial \delta}{\partial r} \bigg|_{t} \frac{R}{R}.$$  

(F.4)

In this formulation, the time variation at a fixed value of $r/R$ is not tied to the variation at $r=0$, but can vary with the local conditions. The expression used here for this variation is

$$\frac{\partial \delta}{\partial t} \bigg|_{r/R} = \frac{\delta(r/R,t) - \delta(r/R, t-\Delta t)}{\Delta t}$$

(F.5)

so that the crack opening rate at a fixed radius $r$ is
\[ \frac{\delta \delta}{\delta t} \bigg|_r = \frac{\delta(r/R, t) - \delta(r/R, t-\Delta t)}{\Delta t} - r \frac{\delta \delta}{\delta r} \frac{\dot{R}}{R}. \] (F.6)

**Simultaneous Solution**

In this formulation, the current values for \( \delta(r) \) at all nodal positions are solved for simultaneously, taking into account the variation of \( \delta(r) \) with time at each location. The resulting crack opening rates satisfy the condition

\[ \int_{t-\Delta t}^{t} dt' \dot{\delta}(r', t') = \delta(r', t) - \delta(r', t-\Delta t) \] (F.7)

at each nodal radius \( r' \). The time dependence of the crack opening rate at a given radius \( r' \), over the time interval \( t_1 = t-\Delta t \leq t' \leq t = t_2 \), is assumed to be

\[ \dot{\delta}(r', t') = \left( \frac{t_2 - t'}{t_2 - t_1} \right) \dot{\delta}_1 \left( \frac{r'}{R(t')} \right) + \left( \frac{t' - t_1}{t_2 - t_1} \right) \dot{\delta}_2 \left( \frac{r'}{R(t')} \right) \] (F.8)

where

\[ \dot{\delta}_1 \left( \frac{r'}{R(t')} \right) = \frac{\delta \delta}{\delta t} \left( r = R(t_1) \cdot \frac{r'}{R(t')}, t_1 \right) \] (F.9)

\[ \dot{\delta}_2 \left( \frac{r'}{R(t')} \right) = \frac{\delta \delta}{\delta t} \left( r = R(t_2) \cdot \frac{r'}{R(t')}, t_2 \right). \] (F.10)

This assumption sets the crack opening rate at a normalized position \( r/R \), at any time within the timestep, equal to a value interpolated between the rates for the bounding times for the timestep, at the same normalized position, with the time interpolation being linear.
The difference among the three methods, for the conditions of the DISLASH experiments described in Section 3.6, are shown in Figure F.1. As expected, the simple difference ratio gives the highest values, and the differences among the methods is greatest near the crack-tip where the opening rates change the most during the timestep.
Figure F.1  Comparison of calculated crack opening rates using three different computation methods, for DISLASH conditions.
APPENDIX G

SELF-SIMILAR CRACK OPENING RATE

To assume that the time-dependence of the crack opening distribution \( \delta(r) \) is self-similar is to assume that the relative shape of the distribution remains constant with time. In other words, a plot of \( \delta(r, t)/\delta(r=0, t) \) versus \( r/R \) would at all times be represented by the same curve. This condition can be represented by the relation

\[
\frac{\delta(r, t)}{\delta(r=0, t)} = f(r/R) \tag{G.1}
\]

where the function \( f(r/R) \) is independent of time. Under these conditions

\[
\frac{\partial \delta(r, t)}{\partial t} \bigg|_r = f(r/R) \frac{\partial \delta(0, t)}{\partial t} + \delta(0, t) \frac{\partial f}{\partial (r/R)} \cdot \frac{\partial (r/R)}{\partial t} \bigg|_r . \tag{G.2}
\]

But

\[
\delta(r = 0, t) \frac{\partial f}{\partial (r/R)} = \frac{\partial \delta(r, t)}{\partial (r/R)} \bigg|_t = \frac{\partial \delta(r, t)}{\partial r} \bigg|_t \frac{\partial r}{\partial (r/R)} \bigg|_t = R \frac{\partial \delta(r, t)}{\partial r} \bigg|_t . \tag{G.3}
\]

and
\[
\frac{\partial (r/R)}{\partial t} \bigg|_r = \frac{\partial (r/R)}{\partial R} \bigg|_r \frac{dR}{dt} \\
= -\frac{r}{R^2} \dot{R},
\]  
\text{(G.4)}

where the dot over a symbol represents the time derivative.

Using Equations (G.1), (G.3), and (G.4) in Equation (G.2) gives

\[
\frac{\partial \delta}{\partial t} \bigg|_r = \delta(r,t) \frac{\dot{\delta}(0,t)}{\delta(0,t)} - r \frac{\partial \delta(r,t)}{\partial r} \bigg|_t \frac{\dot{R}}{R}.
\]  
\text{(G.5)}

This is the crack opening rate that results from the self-similar assumption.
APPENDIX H
PRESSURE - REVISION ALGORITHMS

This appendix describes the procedures used to revise the current estimate for the pressure distribution to obtain one which is more consistent with the requisite conditions, as described in Section 2.12. Although the procedure used for revision of the pressure in the Successive Integration scheme (Section 2.13) is conceptually the same as the procedure used for the Finite Element scheme (Section 2.14), the details of implementation are different enough that it is convenient to describe them separately.

Finite Element Formulation

The finite element formulation results in Equation (2.14.19)

\[ [B(P)] \cdot P = Q(P) \] (H.1)

where both the matrix \([B]\) and the right-hand-side vector \(Q\) depend on the vector of nodal pressure values \(P\). When Equation (H.1) is solved after a particular estimate \(P\text{est}\) has been chosen as a trial vector for the correct pressure distribution \(P\), a resultant vector \(P\text{res}\) is obtained, such that

\[ [B(P\text{est})] \cdot P\text{res} = Q(P\text{est}). \] (H.2)

A vector for the difference between \(P\) and \(P\text{est}\) is introduced,

\[ \Delta P = P - P\text{est} \] (H.3)
along with the notation $\Delta P_n$, $P_n$, $P_{est}$, and $P_{res}$ for the $n^{th}$ elements of $\Delta P$, $P$, $P_{est}$, and $P_{res}$, respectively, and $B_{jk}$ as the element in row $j$ and column $k$ of $[B]$.

The desired matrix $[B(P)]$ and right-hand-side vector $Q(P)$ can be approximated as first-order Taylor series expansions centered around the current estimates:

\[
[B(P)] = [B(P_{est})] + \sum_n \frac{\partial[B]}{\partial P_n} \bigg|_{P_{est}} \Delta P_n \quad (H.4)
\]

\[
Q(P) = Q(P_{est}) + \sum_n \frac{\partial Q}{\partial P_n} \bigg|_{P_{est}} \Delta P_n \cdot (H.5)
\]

Substituting Equations (H.3, H.4 and H.5) into Equation (H.1) gives

\[
[B(P_{est})] \cdot P_{est} + \left\{ \sum_n \frac{\partial[B]}{\partial P_n} \bigg|_{P_{est}} \Delta P_n \right\} \cdot P_{est} + [B(P_{est})] \cdot \Delta P
\]

\[
= Q(P_{est}) + \sum_n \frac{\partial Q}{\partial P_n} \bigg|_{P_{est}} \Delta P_n \quad (H.6)
\]

when terms of second order in $\Delta P_n$ are ignored. Subtracting Equation (H.2) from Equation (H.6) yields

\[
[B(P_{est})] \cdot (P_{est} - P_{res}) + \left\{ \sum_n \frac{\partial[B]}{\partial P_n} \bigg|_{P_{est}} \Delta P_n \right\} \cdot P_{est}
\]

\[
+ [B(P_{est})] \cdot \Delta P = \sum_n \frac{\partial Q}{\partial P_n} \bigg|_{P_{est}} \Delta P_n \quad (H.7)
\]
This can be expressed as a matrix equation for \( \Delta P \) by expanding and rearranging the second term as

\[
\sum_k \left( \sum_n \frac{\partial B_{jk}}{\partial P_n} \mid_{P_{est}} \Delta P_n \right) P_{est_k} = \sum_k \sum_n \frac{\partial B_{jk}}{\partial P_n} \mid_{P_{est}} \Delta P_n \ P_{est_k}
\]

\[
= \sum_n \sum_k \frac{\partial B_{jk}}{\partial P_n} \mid_{P_{est}} \Delta P_{est_k} \ P_n
\]

\[
= \sum_n \left( \sum_k \frac{\partial B_{jk}}{\partial P_n} \mid_{P_{est}} \ P_{est_k} \right) \Delta P_n
\]

(H.8)

for each value of \( j \).

Then Equation (H.7) can be written as

\[
\sum_n \left( \sum_k \frac{\partial B_{jk}}{\partial P_n} \mid_{P_{est}} + B_{jn}(P_{est}) - \frac{\partial Q_j}{\partial P_n} \mid_{P_{est}} \right) \Delta P_n = -\sum_n B_{jn}(P_{est}) \cdot (P_{est_n} - P_{res_n})
\]

(H.9)

for each value of \( j \).

Returning to vector form, the matrix equation to be solved for \( \Delta P \) is

\[
[B^*] \cdot \Delta P = \left[ B(P_{est}) \right] \cdot (P_{res} - P_{est})
\]

(H.10)

where the \((j,n)\) element of \([B^*]\) is given by
Once Equation (H.10) has been solved for $\Delta P$ the revised estimate for $P$ is

$$\text{P}_{est}^{(\text{revised})} = \text{P}_{est} + \Delta \text{P}.$$  \hspace{1cm} (H.12)

Because $[B(P)]$ and $Q(P)$ are nonlinear functions of $P$, this revised estimate will not exactly satisfy Equation (H.1) and it will be necessary to repeat this process until the difference between $P_{est}$ and $P_{res}$ is small enough to be neglected.

The derivatives $\frac{\partial [B]}{\partial P_n} \bigg|_{P_{est}}$ and $\frac{\partial Q}{\partial P_n} \bigg|_{P_{est}}$ are computed by perturbing the $n^{th}$ element of $P_{est}$ by an amount $P_{pert}^{n}$, calculating the perturbed $[B]_{pert}$ and $Q_{pert}$ which result, and using the estimates

$$\frac{\partial [B]}{\partial P_n} \bigg|_{P_{est}} = \frac{[B(P_{est} + P_{pert}^{n})] - [B(P_{est})]}{P_{pert}^{n}}.$$ \hspace{1cm} (H.13)

$$\frac{\partial Q}{\partial P_n} \bigg|_{P_{est}} = \frac{Q(P_{est} + P_{pert}^{n}) - Q(P_{est})}{P_{pert}^{n}}.$$ \hspace{1cm} (H.14)

Because only one element of $P_{est}$ is being changed, efficient methods of calculating the differences in $[B]$ and $Q$ can be computed without computing the full perturbed matrix and vector.
Successive Integration Formulation

Using the terminology of Section 2.12, the pressure estimate $P_{est}$ is in error when

$$P_{res} = \Phi(P_{est}) \neq P_{est}. \quad (H.15)$$

A correction vector to $P_{est}$ is sought such that

$$\Phi(P_{est} + \Delta P) = P_{est} + \Delta P. \quad (H.16)$$

Using first-order Taylor series expansion to represent the revised transformation

$$\Phi(P_{est} + \Delta P) = \Phi(P_{est}) + \sum_n \frac{\partial \Phi(P_{est})}{\partial P_n} \Delta P_n \quad (H.17)$$

allows the difference between Equations (H.16 and H.15) to be written as

$$\sum_n \frac{\partial \Phi(P_{est})}{\partial P_n} \Delta P_n - \Delta P = P_{est} - P_{res} \quad (H.18)$$

or

$$[A] \cdot \Delta P = P_{est} - P_{res} \quad (H.19)$$

where

$$A_{jk} = \frac{\partial \Phi_j(P_{est})}{\partial P_k} - \delta_{jk} \quad (H.20)$$

$$\delta_{jk} = \begin{cases} 0 & \text{for } j \neq k \\ 1 & \text{for } j = k \end{cases} \quad (H.21)$$

with $\Phi_j(P_{est})$ being the $j^{th}$ element of the vector $\Phi(P_{est})$.

Equation (H.19) is solved for $\Delta P$ and

$$P_{est}^{\text{(revised)}} = P_{est} + \Delta P \quad (H.22)$$

is the revised pressure estimate.