Competitive Multi-period Pricing for Perishable Products

by

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Abstract

Pricing of a perishable product over a multi-period time horizon is a challenging problem under an oligopolistic market framework. We propose and study a model for multi-period pricing in an oligopolistic market for a single perishable product. Each participating seller in the market has a fixed inventory of the product at the beginning of the time horizon and additional production is not an available option. Any unsold inventory at the end of the horizon is worthless. The sellers do not have the option of periodically reviewing and replenishing their inventory. Such a model is appropriate for modelling competition in situations where inventory replenishment decisions are made over a longer time horizon and can be considered exogenous to the pricing decision process. This kind of a setup can be used to model pricing of air fares, hotel reservations, bandwidth in communication networks, etc.

In this thesis, we study two issues related to multi-period pricing of a perishable product. First we study the competitive aspect of the problem. Second we study the setup where the demand function for each seller has some associated uncertainty. We assume that the sellers would like to adopt a policy that is robust to adverse uncertain circumstances. We discuss the challenges associated with the analysis for this model.

We study non-cooperative Nash equilibrium policies for the sellers. We discuss why known results from the literature do not extend to this model. We introduce an optimization approach using results from variational inequality theory and robust optimization to establish existence of the pricing equilibrium policy and comment on the uniqueness of the pricing equilibrium policy. We also introduce an iterative learning algorithm for computing the equilibrium policy and analyze its convergence. We study how much is lost in terms of efficiency (in terms of total system profit) due to competition. Finally, we illustrate our results with some numerical examples and discuss some insights.

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Chapter 1

Introduction

1.1 Motivation

> Pricing right is the fastest and most effective way for managers to increase profits. Consider the average income statement of an S&P 500 company: A price rise of 1%, if volumes remained stable, would generate an 8% increase in operating profits–an impact nearly 50% greater than that of a 1% fall in variable costs such as materials and direct labor and more than three times greater than the impact of a 1% increase in volume. Unfortunately, the sword of pricing cuts both ways. A decrease of 1% in average prices has the opposite effect, bringing down operating profits by that same 8% if other factors remain steady.
>
> **The Power Of Pricing**
> *The McKinsey Quarterly*
> *Number 1, 2003*

Pricing has been recognized as a critical lever for revenue management in the industry. As shown by the above excerpt from a McKinsey study, it is a high impact factor determining the overall profitability of a firm. Firms in the retail industry hire pricing experts to help determine optimal pricing mechanisms when introducing new products, entering new markets, scheduling promotions, determining discounting and determining markup/markdown schedules. Firms in the transportation industry often outsource their entire revenue management (including processes like pricing, inventory control and overbooking) to companies specializing in this area such as *Sabre*. Other specialized consultants like *Demandtec*, *Khimetrics*, *Knowledge Support Systems*, *ProfitLogic* and *ProS Revenue Management* offer pricing solutions for firms in retail and transportation.

Pricing has been studied extensively in the academic literature in Economics, Revenue Management and Supply Chain Management. Monopoly pricing problems are generally concerned with sellers finding prices that maximize some revenue-based objective function subject to some resource constraints. In that sense pricing problems are essentially constrained optimization problems. However, there is a fundamental difference between the two when the market is not monopolistic. What makes the problem of revenue maximization through pricing different from other optimization problems is that one seller's pricing is influenced by the pricing policies of other sellers in the same market. This is one of the key factors that affects the demand behavior. Unlike the monopolistic pricing problem, where a single objective function is optimized by a single seller setting all control prices, in the oligopolistic pricing problem there are several sellers trying to maximize their individual objectives by setting their respective control prices.

There are very few products for which the market can be modelled as a monopoly or in which one could assume collaborative pricing between different firms. In fact most developed economies have laws that prohibit monopolistic conduct, price fixing agreements, and other actions that are considered to restrain fair trade. Trusts and monopolies are thought to be injurious to the general consumer. They minimize, if not obliterate normal marketplace competition, and yield undesirable price controls causing markets to stagnate. The United States Antitrust Act is one example of a law that actively seeks to promote competition and competitive pricing. The main source of the antitrust law in the United States is the Sherman Antitrust Act that the US Congress passed in 1890, in order to prevent trusts from creating restraints on fair trade and reduce competition.

A widely used model of such markets is an oligopoly in which the market consists of a finite, but typically small, number of competing firms. The fact that the pricing problem involves considerations about the pricing policies of competitors gives rise to very different dynamics. These interactions, where each agent's payoff depends on not only her individual decisions but is also affected by her competitors' decision variables, cause these models to take the form of a game. Under these circumstances a game-theoretic framework is required to study this problem.

1.2 A practical application

Consider the problem of pricing a one-way flight, say, from Boston to New York the evening before Thanksgiving (between 6 and 10 pm on 26^{th} November, 2003). The competing sellers in this case are airlines operating flights in the same time window. The starting inventory in this case for a particular airline is the total number of seats on the aircraft scheduled for this trip. In theory, an airline should be able to reschedule a bigger plane in case it sees a surge in demand on a particular leg, but this rarely happens in practice because fleet schedules are set well in advance and last minute rescheduling is avoided except under adverse circumstances. Hence for purposes of the pricing problem, the total available inventory is fixed and generation of additional inventory is not an available option.

From the consumer's point of view one has an entire range of options to choose from. One can pick the airline one wants to fly with. One can also pick one out of three airports within a 50 miles radius in the Boston area to fly out of and one of the three airports in a 50 mile radius in the New York Area to fly into. One can decide to take flights that are direct, one-stop or two stops, and pick from a number of flights leaving Boston roughly every 10 minutes. The prices for these options vary widely and the variety of prices for different airport, airlines and route options[1] on the 26^{th} of November, 2003, are shown in Table A.1 and Table A.2.

The prices also vary depending on when and how one books tickets. Booking for tickets usually opens anywhere from 6 to 12 months before the actual flight. Prices are typically lower if one can make travel plans well in advance (and lock in a date and time for the flight) like for a planned vacation trip. Ticket prices rise as the flight date approaches and prices could be substantially higher if the ticket is bought a short time before the flight. This is usually the case for business travellers who have to plan trips at short notice and often need the flexibility to change their itineraries at the last minute. The reason behind this kind of pricing pattern is that the airlines try to maximize their revenues by, among other methods, customer differentiation. The idea is to get each customer to pay as much as her personal budget would allow. Since vacation travellers have limited budgets and higher price sensitivities than business travellers, airlines switch to higher prices for business travellers who typically buy their tickets late. This is done by time-varying pricing and also price differentiation for different booking channels. The price you would be quoted would be different depending on whether you booked through a corporate travel agency like *American Express Travel*, a local travel agent, a college travel advisory like *STA Travels*, an online travel site like *Orbitz*, *Travelocity*, *Hotwire* or *Priceline*, or directly from the Airline toll free call-in number or booking website.

There are a number of additional factors that come into play when pricing products. Airlines price seats on a leg of a flight as part of a larger package consisting of multiple connecting legs or even a round trip ticket. This aspect could play a major role in determining how much discounting the airline would be willing to offer. Such network effects have been ignored for the purposes of this thesis. Consequently, other pricing practices like those involving the creation of differentiated products by artificial restrictions like Saturday-night stay are also ignored.

From the buyer's perspective, not everyone will go for the cheapest option when picking a flight even if the choice was between flights leaving in the same time window. The most important factor that influences preference in choice of flights today is the *frequent flier mile* programs run by airlines. Secondary factors influencing this choice could be the choice of airports, perception of in-flight service and restricted access to some booking channels. It would require a certain price differential between two airlines before a frequent flier member on one airline would switch to another airline. When this assumption is applied to individual buyers in a population, each possessing their own individual price differential thresholds and the demand behavior of the whole population towards one seller tends to show a continuous decrease when that seller's price is raised. This contrasts with a sudden drop when the price crosses that of a competitor as is modelled in the Bertrand models of oligopoly.

[1]Source: QPX airfare pricing and shopping system by ITA Software, Inc.

1.3 Oligopoly pricing models

Oligopolistic competition has been a topic of research in the Economics academic community since the early 19^{th} century. Literature in Economics has studied a variety of models for such oligopolistic markets. In the static (single period) quantity-competition model of oligopoly (for example, Cournot models [21]), the sellers simultaneously decide on the quantity that they will each individually release into the market. There is a market clearing price resulting from a market mechanism that is a function of the total quantities from all the sellers. The market clearing price is a decreasing function of the total quantity. Each seller then earns a revenue equal to the product of the quantity she had released into the market and the market clearing price. This can be extended to a multi-period problem where each seller is endowed with a given inventory of the product for sale and decides what fraction of that to sell in each period. Nevertheless, this model is not suitable for settings like the sale of flight tickets since each seller (in this case, the airlines) declares a price for her own product (tickets for a particular origin-destination leg leaving in a small time window) while deciding on the maximum allowable quantity that she wants to sell in a period (inventory control).

In the single period price-competition model of oligopoly (for example, Bertrand models [6] and Bertrand-Edgeworth models [26]), the sellers declare their respective prices simultaneously. In the Bertrand model, the seller with the lowest price is obliged to fill the entire demand and has positive revenue while all the other sellers earn nothing. In case of a tie, the sellers with the tied prices split the demand according to some rational rule. In the Bertrand-Edgeworth model, the seller with the lowest price fills as much demand as is economical at her production cost level while the remaining demand is filled by other sellers in a cascading fashion. Such models can be extended to the multi period case and are suitable for supply-chain problems which typically have a periodic production-review framework. These are not suitable for settings like airlines selling flight seats since for these situations, the inventory of each seller is fixed and additional production is not an option.

It is important here to distinguish between two different types of multi-period pricing models that are fundamentally different. The first one, which we call the **periodic production-review model**, assumes a framework that is suitable for supply chain problems. In this model each seller start with a given level of inventory at the beginning of the time horizon. In each period she sets a price along with all the other sellers and realizes a certain demand that is a function of all the price levels. She fills the demand realized with her current inventory. If the demand is less than the inventory she had at the beginning of the period, she has some leftover inventory at the end of the period. In that case, she incurs some holding cost that is a function of the amount of inventory left. The rationale behind this is that the holding cost could be representative of the storage costs, interest on capital cost of inventory, etc. On the other hand, if the demand is more than her inventory level at the beginning of the period, she fills the demand with all her current inventory and promises to fill the remaining demand in the next period. This is sometimes represented as negative inventory. She also incurs a backorder cost which is described as a function of the amount of products in backorder. At the beginning of the next period, she reviews her inventory level, and makes a decision whether she would produce more quantity in order to increase her inventory for the subsequent periods. Production costs are typically increasing functions of the quantity produced. It has been shown under certain conditions that the resulting optimal policies consist of a base-stock policy where the optimal amount produced is enough to fill the inventory to a certain level irrespective of the level at the beginning of the period. In case the inventory is already higher than the base stock then no quantity is produced.

In this thesis we introduce a model to study the pricing problem faced when the firm does not have the option to produce additional inventory between periods and the initial inventory is a given. We call this model the **fixed inventory model**. This model is better suited than the periodic production-review model for some situations. For example, such a model is suited for Airlines that are selling seats on a particular flight, or Hotels selling advance room reservations for a particular day or weekend. For these problems there are no holding or backorder costs. There are no holding costs since there is no tangible product that the seller has to hold on to from period to period if unsold. There are no backorder costs since the seller can sell only if she has the product in inventory and loses the sale otherwise. Note that this case is not a trivial extension of the periodic production review model. The challenges that arise for analysis of equilibrium are very different from that of the periodic review model. This thesis focuses on this model.

1.4 Literature review

There are several excellent surveys of the literature in this field. McGill and van Ryzin [47] and the references therein also provide a thorough review of different issues in revenue management, for example, seat inventory control, overbooking, and pricing models. Bitran and Caldentey [9] provide an overview of pricing models for the monopolistic version of the revenue management problem in which a perishable and non-renewable set of resources satisfy stochastic price-sensitive demand processes over a finite period of time. They survey results on deterministic as well as non-deterministic, single as well as multi-product, and static as well as dynamic pricing cases. Elmaghraby and Keskinocak [30] review the literature and current practices in dynamic pricing in industries where capacity or inventory is fixed in the short run and perishable. They classify monopolistic models on the basis of whether inventory can be replenished or not, whether demand is dependent over time or not, and whether customers are myopic or strategic optimizers. Yano and Gilbert [68] review models for joint pricing and production under a monopolistic setup.

On the competitive side, Vives [66] discusses the development of oligopoly pricing models. A survey by Chan et al [15] summarizes research on joint pricing, inventory control and production decisions in a supply chain. They also survey literature on price and quantity competition in supply chain settings. Cachon and Netessine [14] also survey the problem of competition from a supply chain perspective where the problem is characteristically a periodic production-review model. They discuss both non-cooperative and cooperative games in static and dynamic settings.

There is rich literature in Economics on price and quantity competition. For example, the seminal models by Cournot, Bertrand and Edgeworth mentioned in the previous section. Kirman and Sobel [43] develop a multi-period model of oligopoly where a set of competing firms decide in each period the price and the production level in the face of random demand. They show the existence of equilibrium price-quantity strategies for the firm. Rosen [58] proves existence and uniqueness results for general oligopolistic games. The paper shows existence under concavity of the payoff to a seller with respect to it's own strategy space and convexity of the joint strategy space and uniqueness under strict diagonal dominance of the payoff function. Murphy et al [49] analyze equilibrium in a single-period quantity competition model using mathematical programming results. Harker [39] analyze the same model using variational inequalities. Eliashberg and Jeuland [28] model a two stage problem. The market in the first stage is a monopoly and becomes a duopoly in the second stage with the entry of a second seller. The sellers dynamically price their product. The paper analyzes the pricing behavior under the cases that the incumbent seller foresees or does not foresee the entrant. Eliashberg and Steinberg [29] also study a duopoly over a multi-period time horizon. The first firm faces a convex production cost and a linear inventory holding cost. The second firm faces linear production cost and holds no inventory. The paper studies the behavior of the two firms and characterize the conditions under which the second firm's prices are strictly lower than the first firm's prices over the entire time horizon. Tanaka [64] considers a multi-seller game and analyzes the profits of firms when they choose price or quantity as a strategic variable. They show that the quantity strategy is a better strategy for the seller, irrespective of whether all the other sellers choose price or quantity as their strategic variables.

A number of papers have proposed and studied periodic production-review models. One group of models are inventory management models (see Zipkin [72]) where the price for a product is a static single price and is exogenous to the problem. The other group of models are those that allow price to be a decision variable and vary from period to period. There is a single product being sold in a multi-period setting, and demand is not dependent on sales in previous periods. Some examples of such papers are Zabel [70] and Federgruen and Heching [33]. These models assume convex production, holding and ordering costs and unlimited production capacity. Chen and Simchi-Levi [18] extend the model to include fixed ordering costs and Chan, Simchi-Levi and Swann [16] extend the model to include limited production capacity. In all of these models however, the seller is a monopolist and issues that arise with competitive interactions regarding equilibria are not addressed. Petruzzi and Dada [56] develop various extensions to the newsvendor problem. They consider joint pricing and inventory control in the face of uncertain demand. For the single period problem they discuss the cases where the stochastic demand is additive and multiplicative in nature. For the multi-period extension of the problem, they consider shortage cost and holding costs for inventory between periods and discuss the additive case. They do not consider the influence of competition on demand. Cachon and Netessine [14] survey the application of game theory to supply chain analysis using, for example, newsvendor games. They give an overview of techniques used for the analysis of general cooperative and non-cooperative games in static and dynamic settings and discuss the issues related to existence and uniqueness of equilibrium. In particular, they discuss the significance of quasi-concavity and supermodularity requirements for the payoff function for equilibrium.

The literature on competitive pricing for supply chains is surveyed in Chan et al [15]. We refer the reader to their survey for a description of papers which consider the effect of factors like delivery time in market competition models.

One application where the fixed-inventory model has been studied and applied is in retail pricing and clearance sales. The variety of products is high and product life cycles are short. The combination of long lead times (because of supply chains spanning multiple continents and markets) and shorter selling seasons results in a scenario where production/inventory decisions have to be made well in advance with little information about demand, before the actual selling begins. Pricing becomes a vital component in balancing supply and demand since the inventory levels and the length of the selling season is predetermined. Some analytical models have been presented that study pricing for such products. However, these models typically assume conditions of monopolistic markets or imperfect competition and hence do not address issues regarding the existence of pricing equilibria. These include Lazear [45], Bitran and Mondschein [11], Bitran, Caldentey and Mondschein [10], Feng and Gallego [34], Gallego and Van Ryzin [37], Smith and Achabal [62] and Zhao and Zheng [71]. For a detailed comparison of these models, we refer the reader to a survey paper by Elmaghraby and Keskinocak [30].

A key component of the problem we address in this thesis is the aspect of uncertainty. To achieve this we employ ideas from the newly emerging field of robust optimization. Therefore, in what follows we cite some literature on robust solutions for optimization problems. Soyster [63] was the first to propose the idea of robust optimization for a linear optimization model in which the solution would be feasible for all data belonging to a convex set. Ben-Tal and Nemirovski [3] introduced robust convex optimization and showed that for certain types of linear and nonlinear optimization problems, the robust problem could also be efficiently solved exactly, or approximately, using polynomial-time algorithms. Ben-Tal and Nemirovski [4] focussed specifically on robustness in linear programming problems with uncertain data. In particular, they show that under ellipsoidal uncertainty sets, the robust counterpart of a linear programming problem is a conic quadratic program which is solvable in polynomial time. Ben-Tal and Nemirovski [5] demonstrates numerically the robust optimization methodology introduced in [3] and [4] by applying it to some linear programming problems. Papers that study semidefinite robust optimization problems include Ben-Tal, El-Ghaoui and Nemirovski [2] and El-Ghaoui, Oustry and Lebret [27]. Recently, Bertsimas and Sim [7] studied the tradeoff between robustness of a solution to a linear programming problem and the sub-optimality of the solution. In their paper they adjust the level of conservatism (robustness) of the solution in terms of probabilistic bounds of constraint violation and numerically study how optimality is affected when robustness is increased. Bertsimas and Thiele [8] apply robust optimization principles to supply chain management. In this thesis, we use some results presented in the cited literature. Later we will discuss how we use the concept of robustness to find policies that optimize the payoffs for the sellers in the worst case scenario.

We compare the system optimal and the user optimal solutions to competitive pricing problem and we comment on both price competition models and quantity competition models. The system optimal and user optimal solutions are not the same for problems of competition in general. There has been an increasing attention in literature in recent years towards this issue. Some effort has been made to quantify the inefficiency of Nash equilibrium problems in non-cooperative games. The fact that there is not full efficiency in the system is well known both in the economics but also in the transportation literature (see Braess [12], Dubey [25], etc). Wardrop [67] first stated equilibrium principles in the context of transportation. Dafermos and Sparrow [23] coined the terms *user-optimized* and *system-optimized* in order to distinguish between Nash equilibrium where users act unilaterally in their own self interest versus when users are forced to select the routes that optimize the total network efficiency. Smith [61] and Dafermos [22] recognized that this problem can be cast as a variational inequality. Hearn and Yildirim [41], Hearn and Ramana [40], and Cole, Dodis and Roughgarden [19] have also studied the notion of introducing tolls (taxes) in order to make the decentralized problem efficient in a centralized manner. The review paper by Florian and Hearn [35], the book by Nagurney [50], and the references therein summarize the relevant literature in traffic equilibrium problems. Traffic equilibrium problems are typically modelled through variational inequalities. The books by Facchinei and Pang [31] summarize the developments in the area of variational inequalities.

This inefficiency of user-optimization was first quantified by Papadimitriou and Koutsoupias [44] in the context of a load balancing game. They coined the term *the price of anarchy* for characterizing the degree of efficiency loss. Subsequently, Roughgarden and Tardos [60] and Roughgarden [59] applied this idea to the classical network equilibrium problem in transportation with arc cost functions that are separable in terms of the arc flows. They established worst case bounds for measuring this inefficiency for affine separable cost functions and subsequently for special classes of separable nonlinear ones (such as polynomials). It should be noted that Marcotte presented in [46], results on the price of anarchy for a bilevel network design model. Recently, Johari and Tsitsiklis [42] also studied this problem in the context of resource allocation between users sharing a common resource. In their case the problem also reduces to one where each player has a separable payoff function. Correa, Schulz and Stier Moses [20] have also studied the price of anarchy in the context of transportation for capacitated networks. The cost functions they consider are also separable. The paper by Chau and Sim [17] has recently considered the case of nonseparable, symmetric cost functions giving rise to the same bound as Roughgarden and Tardos [60]. We refer the reader to Perakis [53] for an analysis of the difference between costs arising from an user optimal and a system optimal solution for general asymmetric and non-separable cost functions. We would like to bring to the attention of the reader, the fact that the cited literature compares efficiencies for cost minimizing games. In this thesis, we have a profit maximizing game and some of the issues that arise make a direct application of some general results inappropriate.

1.5 Challenges and contributions

Previously published results prove existence and uniqueness for equilibrium strategies for competitive pricing problems under various conditions. We found that none of these conditions hold for our model, hence requiring a new approach for analysis. One possible approach used in the literature requires results for supermodular games. In brief, supermodular games can be described as games with supermodular objective functions and lattice strategy spaces for each player. We refer the reader to Vives [66] for a detailed description of such games. Other approaches use results that require the payoff function to be concave or quasi-concave (See Nash [51]) over a convex strategy space. The model discussed in this thesis does not fall under any of the above categories. For example, the model can be reformulated so that the strategy space is a lattice and nicely convex but the resulting objective function to be maximized is neither concave nor supermodular. Alternatively, it can be formulated to have a concave objective function, but then the resulting strategy space is no longer a convex set. This makes it difficult to prove equilibrium results for this model. These observations prompted us to take a different approach using ideas from variational inequalities.

Our model differs from other models of oligopolies, like the competitive supply chain models, previously studied in the literature, since we have rigid inventory constraints over the entire horizon and the flexibility to replenish inventory between periods through additional production is absent. Under these modelling restrictions, we lose the convenient structure of the problem which would otherwise allow us to analyze equilibrium pricing with the above standard techniques. To the best of our knowledge, there are no general results in the literature that can be directly used to analyze such a model. This is the main challenge behind the fixed-inventory competitive pricing problem.

The main contributions of this thesis are as follows.

1. We formulate a multi-period pricing model for an oligopoly where each seller has a pre-determined starting inventory and additional production is not an option. We show that this problem does not have a structure that falls under the framework of game theoretic models such as quasi-concave games or supermodular games.

2. We first focus on addressing the competitive aspect of the problem. We establish existence of equilibrium pricing policies with deterministic demand and comment on the uniqueness of the solution. As no traditional approach applies to this problem, a key innovation of this thesis is a quasi variational inequality reformulation. This reformulation allows us to study existence of equilibrium prices and does not require the payoff functions to be concave. To the best of our knowledge, no such analysis for multi-period price competition models for perishable products has been done before.

3. We address the issue of uncertainty in demand for the model via robust optimization. We establish existence of robust equilibrium policies under such uncertain demand.

4. We establish equilibrium results for the model when sellers are faced with stochastic linear demand and each seller adopts policies that maximize their expected payoff.

5. We introduce and study an algorithm for computing equilibrium pricing policies and analyze its convergence in the deterministic demand and robust demand settings.

6. We compare the combined payoff from all sellers in the user optimal equilibrium policies and in the system optimal equilibrium policies for both the price competition and quantity competition settings.

7. We illustrate our results through numerical examples and compare the performance of robust policies with non-robust policies.

1.6 Outline of thesis

The thesis is structured as follows. In Chapter 1, we give the motivation behind the model and discuss some applications. We also describe a broad literature review that covers relevant literature in this area. In Chapter 2 we formulate the model for the fixed inventory pricing problem. We introduce the terminology used throughout this thesis and give the general bilevel program formulation for the deterministic demand model, the robust demand model and the stochastic demand model. In Chapter 3 we consider the deterministic demand model. We describe the formulations for the best response problem and the market equilibrium problem under deterministic demand. We provide the proofs for existence and uniqueness of the best response policy and existence of the equilibrium policies. In Chapter 4 we discuss the concepts of uncertainty in demand and robustness of a policy to this uncertainty. We discuss formulations for the best response problem and the market equilibrium problem under robust demand. We provide proofs for existence and uniqueness of the best response policy and existence of equilibrium policies for this model. In Chapter 5 we discuss the stochastic demand model. We discuss two approximations of this model in brief for which we can show equilibrium results using convex games. In Chapter 6 we compare user optimal and system optimal solutions for some price competition games and quantity competition games. In Chapter 7 we start by showing how the best response problem can be formulated as a network flow optimization problem and is convenient to solve. We introduce the iterative learning algorithm and show convergence under both deterministic and robust demand models. In Chapter 8 we use some simple numerical examples to demonstrate some general properties of the model. These include the basic properties of the equilibrium policies, convergence behavior of the algorithm and the performance of robust equilibrium policies. In Chapter 9 we conclude and talk about possible future extensions and research directions that we feel might be worth exploring.

Appendix A contains all tables and data that are referenced in the thesis. Appendix B contains all figures and graphs. This is followed by a bibliography of the literature referenced in this thesis.

Chapter 2

Model formulation

In this chapter we introduce the notation and terminology used throughout this thesis and describe the oligopolistic market models under deterministic demand and robust demand. We break the development and analysis of the model into two steps. The first step considers the problem faced by an individual seller. Let us assume that the seller has either prior knowledge, or an estimate of the pricing policy of her competitors. In such a case, the seller would then adopt a policy that maximizes her revenue from the sale of her inventory over the entire time horizon. We call this step the best response problem and the resulting policy, the best response policy. We formulate the best response problem as a nonlinear optimization problem that has an underlying structure similar to a min-cost network flow problem.

The second step involves addressing the broader issues concerning the market as a whole. Is there an equilibrium state that the market will converge to? Is there a set of Nash equilibrium policies for the sellers? Questions regarding the existence and uniqueness of equilibrium policies can only be answered through the market equilibrium model. The market equilibrium problem is formulated as a quasi-variational inequality problem derived from the sellers' best response problems considered together. We will discuss the formulations and their solution methodology in greater detail in subsequent chapters in this thesis.

2.1 Notation and terminology

We denote the set of sellers by $\mathbf{I}$. A single seller is denoted by $i \in \mathbf{I}$. With slight abuse of notation, we denote the set of all competitors of i by $-i$. The product inventory belonging to seller i at the beginning of the time horizon is denoted by C_i.

The time horizon is divided into a finite number of time periods. A time period is denoted by $t \in \mathbf{T}$. The price set by seller i in period t is denoted by p_i^t. Seller i's pricing policy over the entire horizon consists of the prices $(p_i^1, p_i^2, \ldots, p_i^T)$ and is denoted by $\mathbf{p}_i$. The pricing policy variables for the entire set of sellers consist of the price vectors $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_I)$ and we denote this by $\mathbf{p}$.

The buyers are represented in an aggregate form by a demand function. Seller i's share of demand (observed demand) in period t is denoted by the demand function

$h_i^t(\mathbf{p}^t)$ and is a function of the price levels set by all sellers in that period.

The actual amount of product sold by seller i in period t is denoted by d_i^t. Clearly d_i^t is less than or equal to $h_i^t(\mathbf{p}^t)$ since the sale made cannot be greater than seller i's observed demand. The relation is an inequality since the seller might be restricted by the actual inventory level available. We use the notation $\mathbf{d}_i = (d_i^1, d_i^2, \ldots, d_i^T)$ and $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_I)$ to denote the realized demand.

We denote the strategy of Seller i by $\mathbf{z}_i$ consisting of the prices set and the realized demand $(\mathbf{p}_i, \mathbf{d}_i)$. Consequently, we also denote the pricing policies of all sellers together $(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_I)$ by $\mathbf{z}$. We use the notation $\mathbf{h}_i(\mathbf{p}) = (h_i^1(\mathbf{p}^1), \ldots, h_i^T(\mathbf{p}^T))$ and $\mathbf{h}(\mathbf{p}) = (\mathbf{h}_1(\mathbf{p}), \ldots, \mathbf{h}_I(\mathbf{p}))$. Note that we assume that period t demand for Seller i depends on the prices of all sellers in the market but only on period t.

Given the inventory information $\mathbf{C} = (C_1, \ldots, C_I)$, the total payoff to seller i over the entire time horizon is a function of the sellers' policies $\mathbf{p}$, and is denoted by $J_i(\mathbf{p})$.

We denote the best response policy that maximizes the payoff of seller i over the entire time horizon given that her competitors have adopted policies $\mathbf{z}_{-i}$ by $\mathcal{BR}_i(\mathbf{z}_{-i})$. In Subsections 2.3, 2.4 and 2.5, we formulate this as a bilevel optimization problem for different demand models. We denote the resulting best response policy $\mathcal{BR}_i(\mathbf{z}_{-i})$ for seller i by $\mathbf{z}'_i$.

Next we define the concept of Nash equilibrium policies for the sellers.

Definition 2.1.1 (Nash equilibrium policies). *The pricing policies for each seller are Nash equilibrium pricing policies if no single seller can increase her payoff by unilaterally changing her policy.*

This definition implies that each seller sets her equilibrium pricing policy as the best response to the equilibrium pricing policies of her competitors. This set of policies would then, by definition, be a Nash equilibrium set of policies. See Nash [51] for further details on the notion of a Nash equilibrium in non-cooperative games. We denote the equilibrium price levels by $\mathbf{p}^* = (\mathbf{p}^*{}_i, \mathbf{p}^*{}_{-i})$. Later in the thesis, we provide a quasi-variational inequality formulation of the problem for determining the market equilibrium policies.

2.2 Best response and market equilibrium problems

There are two problems that are of interest to us. The first problem, which we will call the **best response problem**, is faced by an individual seller and solves the following problem: If a seller also has precise information regarding the pricing policy of each of her competitors in addition to the knowledge about the demand function, how should she choose her pricing policy in order to maximize her revenue? The assumption regarding complete prior knowledge of competitors' policies is, of course, unrealistic and makes this problem hypothetical in nature. This step, however, is crucial in solving the second problem that does not require this assumption. The second problem, which we will call the **market equilibrium problem**, answers the following questions: Is there a set of pricing policies, one for each seller, such that no seller can increase her revenue by unilaterally deviating from it? Does such an equilibrium exist in the market? If the market starts from a non-equilibrium state, will it achieve an equilibrium and how fast?

In this section we formulate models that will solve the best response problems for different demand models. First we assume that the policies of the competitors are fixed and known to the seller. We start by formulating the model for the best response problem under three different demand models. In the **deterministic demand model**, the demand function is modelled as a deterministic function of the price levels. In the **robust demand model**, the demand function is modelled as a function of the price levels but involves some parameters whose values are uncertain. In this case, the seller adopts a pricing policy that is robust to this uncertainty. In the **stochastic demand model**, the aforementioned parameters follow a known distribution and the seller adopts a policy that maximizes the expected revenue.

In all the following models, we formulate the best response problem for a seller as a bilevel program. The control variables that the individual seller sets are the prices and protections levels for each time period. These are the variables that are set in the upper level of the bilevel program. The actual amount of inventory sold is determined after prices and protection levels are set by the sellers and the demand parameters assume their actual vales. The amount of inventory sold in all periods is denoted by variables in the lower level of the bilevel program formulation of the best response problem since they are not controlled by the seller. This bilevel program can be viewed as a best response problem formulation where the feasible policy space for a seller does not depend on the policy of her competitors but the payoff does. In later chapters we will consider formulations of the best response problems as single level optimization problems. These are equivalent to the bilevel formulations given in this chapter. Unlike the bilevel formulations, which make more sense from a modelling perspective, the single level optimization formulations of the best response problems involve feasible policy spaces that depend on the competitors' policies. However, they are used since the single level optimization problems are easier to analyze and solve.

2.3 Deterministic demand model

We start with the deterministic demand model. The seller's problem can be formulated as a bilevel optimization problem. The variables in the higher level program, $(\mathbf{p}_i, \mathbf{D}_i)$, are the policy variables set by the seller. p_i^t is the price set by seller i in period t and D_i^t is the amount of inventory protected for sale in period $t+1$ and later. The variables in the lower level program, $(\mathbf{d}_i)$, are the sale variables that assume their values depending on the policy variables. The formulation ensures that the sale in any period is exactly the amount that is the lower of the following two quantities: demand in that period, and the maximum allowed quantity available for sale in that period. In the lower level program, we do this by maximizing the weighted sum of d_i^ts with strictly decreasing weights in t. The exact weights could be chosen arbitrarily as long as they satisfy this property.

$$
\begin{array}{lll}
\max\limits_{(\mathbf{p}_i,\mathbf{D}_i)} & \sum_{t=1}^{T} p_i^t d_i^t & \\
\text{such that} & p_{i\mathtt{min}}^t \le p_i^t \le p_{i\mathtt{max}}^t & \forall\, t \in \mathbf{T} \\
& 0 \le D_i^T \le \cdots \le D_i^1 \le C_i & \\
\text{where} & & \\
\max\limits_{(\mathbf{d}_i)} & \sum_{t=1}^{T} (T-t+1) d_i^t & \\
\text{such that} & d_i^t \le h_i^t(p_i^t, \bar{p}_{-i}^t) & \forall\, t \in \mathbf{T} \\
& \sum_{\tau=1}^{t} d_i^\tau \le C_i - D_i^t & \forall\, t \in \mathbf{T} \\
& d_i^t \ge 0 & \forall\, t \in \mathbf{T}
\end{array}
\tag{2.1}
$$

2.4 Robust demand model

In the robust demand model, the seller's problem can be formulated as a slightly modified bilevel optimization problem. The variables in the higher level program, $(\mathbf{p}_i, \mathbf{D}_i)$, are the policy variables set by the seller without prior knowledge of the actual values of the uncertain parameters. The variables in the lower level program, $(\mathbf{d}_i)$, are the sale variables that assume their values depending on the policy variables and the realization of the uncertain parameters. This formulation ensures that the sale in any period is the amount that is the minimum between the following two quantities: the demand in that period (allowing for adverse values of the uncertain parameters), and the maximum allowed quantity available for sale in that period.

$$
\begin{array}{lll}
\max\limits_{(\mathbf{p}_i,\mathbf{D}_i)} & \sum_{t=1}^{T} p_i^t d_i^t & \\
\text{such that} & p_{i\mathtt{min}}^t \le p_i^t \le p_{i\mathtt{max}}^t & \forall\, t \in \mathbf{T} \\
& 0 \le D_i^T \le \cdots \le D_i^1 \le C_i & \\
\text{where} & & \\
\max\limits_{(\mathbf{d}_i)} & \sum_{t=1}^{T} (T-t+1) d_i^t & \\
\text{such that} & d_i^t \le h_i^t(p_i^t, \bar{p}_{-i}^t, \xi_i^t) & \forall\, \xi_i^t \in \mathcal{U}_i^t, \forall\, t \in \mathbf{T} \\
& \sum_{\tau=1}^{t} d_i^\tau \le C_i - D_i^t & \forall\, t \in \mathbf{T} \\
& d_i^t \ge 0 & \forall\, t \in \mathbf{T}
\end{array}
\tag{2.2}
$$

In the above robust demand model, the demand realized by a seller i in a period t is modelled as a function of the prices set by the seller i in period t, the prices set by the competitors of seller i in period t and an uncertainty factor. We denote the function by $h_i^t(p_i^t, p_{-i}^t, \xi_i^t)$, where ξ_i^t is the uncertainty factor. The uncertainty factor is a parameter that can take any value from a given closed uncertainty set $\mathcal{U}_i^t$. The vector of uncertain parameters for all time periods, for a seller i, is denoted by ξ_i. ξ_i can take any value from the set $\mathcal{U}_i$ where

$$
\mathcal{U}_i \;=\; \mathcal{U}_i^1 \times \mathcal{U}_i^2 \times \cdots \times \mathcal{U}_i^T.
$$

Similarly, the vector of uncertain parameters for all time periods and sellers is denoted by $\xi \in \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_I$.[1]

Since the seller does not have prior information about what values the uncertainty variable will take, the natural question that arises is what objective function should the seller maximize? In the stochastic demand model, an *a priori* distribution is assumed for the uncertainty variable, and the seller adopts a policy that maximizes the expected revenue. This however, involves assuming knowledge about the distribution of the uncertain parameters. In a lot of cases, it is difficult to estimate accurately the mean and variance of such parameters, let alone their distribution. The most basic information that is available is a likely interval or set within which the realized parameter values will fall. A robust policy is a policy that would maximize the objective function for a seller even when under the most adverse instances of the uncertainty factor within such a set. Note that by introducing a budget of uncertainty, Γ_i^t, we try to reduce being very conservative. This allows us to control a tradeoff between optimality and robustness.

It has been seen that the robust policy typically improves the worst case payoff with regards to uncertainty at the loss of optimality of the best case payoff. Such a tradeoff can be beneficial for a number of reasons. One such reason is linked to the fact that the robust policy reduces the variance in the payoff compared to the optimal policy corresponding to the nominal values of the uncertainty parameters. A lot of firms might find it more attractive to adopt a policy that guarantees revenues that are less variable and uncertain even if they are lower on an average basis than to adopt a policy that on average generates higher revenues but also potentially could generate very poor revenues.

2.5 Stochastic demand model

In the stochastic demand model, we model the seller's problem as yet another bilevel optimization problem. The variables in the higher level program, $(\mathbf{p}_i, \mathbf{D}_i)$, are the same as before, i.e. policy variables set by the seller without prior knowledge of the actual values of the uncertain parameters. In the stochastic demand model, the

[1] For example, for a linear demand function in a duopoly setting, the demand function for a time period t is

$$\begin{aligned} h_1^t(p_1^t, p_2^t, \xi_1^t) &= D_{1_{\text{base}}}^t - \beta_1^t p_1^t + \alpha_1^t p_2^t \\ h_2^t(p_2^t, p_1^t, \xi_2^t) &= D_{2_{\text{base}}}^t - \beta_2^t p_2^t + \alpha_2^t p_1^t \end{aligned}$$

where $\xi_i^t = (D_{i_{\text{base}}}^t, \beta_i^t, \alpha_i^t)$. Parameters ξ_i^t could take any value in the uncertainty set $\mathcal{U}_i^t$. An example of such uncertainty sets is:

$$\mathcal{U}_i^t = \left\{ (D_{i_{\text{base}}}^t, \beta_i^t, \alpha_i^t) \;\middle|\; \left| \frac{D_{i_{\text{base}}}^t - \bar{D}_{i_{\text{base}}}^t}{\sigma_{D_{i_{\text{base}}}^t}} \right|^n + \left| \frac{\beta_i^t - \bar{\beta}_i^t}{\sigma_{\beta_i^t}} \right|^n + \left| \frac{\alpha_i^t - \bar{\alpha}_i^t}{\sigma_{\alpha_i^t}} \right|^n \le \Gamma_i^{t\,n} \right\},$$

where n may be 1 or 2 (polyhedral or ellipsoidal sets), $\bar{D}_{i_{\text{base}}}^t$, $\bar{\beta}_i^t$ and $\bar{\alpha}_i^t$ are the nominal values of the uncertainty parameters and $\sigma_{\bar{D}_{i_{\text{base}}}^t}$, $\sigma_{\bar{\beta}_i^t}$ and $\sigma_{\bar{\alpha}_i^t}$ are some measure of the parameters' dispersion around the nominal values.

uncertain parameters take values from a countably finite set $\mathcal{U}$ with a probability distribution (f_ξ) known to the seller. The seller adopts a policy that maximizes the expected revenue. There are several lower level programs (one for each stochastic scenario) resulting in a lot more lower level variables, i.e. $(\mathbf{d}_{\xi i})$. These are sale variables for each realization of the uncertain parameters, and they assume their values depending on the policy variables from the higher level program. This formulation ensures as before, that the sale in any period is the amount that is the minimum between the following two quantities: the demand in that period given the corresponding realization of the uncertain parameters, and the maximum allowed quantity available for sale in that period.

$$
\begin{array}{rll}
\max\limits_{(\mathbf{p}_i,\mathbf{D}_i)} & \sum_{t=1}^{T} \sum_{\xi\in\mathcal{U}} f_\xi p_i^t d_{\xi i}^t & \\
\text{such that} & p_{\mathtt{imin}}^t \le p_i^t \le p_{\mathtt{imax}}^t & \forall\, t \in \mathbf{T} \\
& 0 \le D_i^T \le \cdots \le D_i^1 \le C_i & \\
\text{for each } \xi \in \mathcal{U} & & \\
\max\limits_{(\mathbf{d}_{\xi i})} & \sum_{t=1}^{T} (T-t+1) d_{\xi i}^t & \\
\text{such that} & d_{\xi i}^t \le h_i^t(p_i^t, \bar{p}_{-i}^t, \xi) & \forall\, t \in \mathbf{T} \\
& \sum_{\tau=1}^{t} d_{\xi i}^\tau \le C_i - D_i^t & \forall\, t \in \mathbf{T} \\
& d_{\xi i}^t \ge 0 & \forall\, t \in \mathbf{T}
\end{array}
\tag{2.3}
$$

The bilevel program formulations for the best response problem are hard to solve. The analysis for equilibrium policies is even harder. In this thesis, we consider some settings for which the bilevel program structure can be simplified into a simpler optimization problem. This is then followed by a quasi-variational inequality reformulation that is used to model the market equilibrium problem. In Chapter 3 (See also Perakis and Sood [54]) we examine the deterministic demand case and in Chapter 4 (See also Perakis and Sood [55]) we analyze the robust demand case. We will discuss the stochastic demand case in Chapter 5.

Chapter 3

Deterministic demand model

In this chapter we will assume that the demand is a deterministic function of the prices with known parameters. Uncertainty in demand will be considered in Chapters 4 and 5. The formulation and analysis of the best response problem and the market equilibrium problem under this assumption of deterministic demand follow below.

3.1 Best response problem

The best response pricing policy for seller i is the policy that maximizes seller i's payoff in response to all others sellers' pricing policies.

Definition 3.1.1 (Multi-period Pricing Problem). *Consider a set of sellers* $\mathbf{I}$ *with inventories* $\mathbf{C}$ *and time horizon* $\mathbf{T}$. *The strategy of each seller consists of setting her price levels* $\mathbf{p}_i$ *optimally, i.e., as best response prices arising from formulation (3.1) below. The demand observed by seller* i *in any period is equal to the number of buyers who are willing to buy from her, given the price levels for all sellers. Seller* i *will realize that demand if she has enough inventory.*

The best response policy $\mathbf{p}_i'$ of seller i, given all competitors' policies $\bar{\mathbf{p}}_{-i}$, is the solution of the following optimization problem:

$$\begin{array}{rll} \operatorname{argmax}_{\mathbf{d}_i,\mathbf{p}_i} & \sum_{t=1}^{T} d_i^t p_i^t & \\ \text{such that} & d_i^t \le h_i^t(p_i^t, \bar{p}_{-i}^t), & \forall t \in \mathbf{T} \\ & \sum_{t=1}^{T} d_i^t \le C_i & \\ & p_{\min}^t \le p_i^t \le p_{\max}^t, & \forall t \in \mathbf{T} \\ & d_i^t \ge d_{min}, & \forall t \in \mathbf{T}. \end{array} \tag{3.1}$$

In vector notation, the above can be rewritten as:

$$\begin{array}{rl} \max\limits_{\mathbf{z}_i=(\mathbf{d}_i,\mathbf{p}_i)} & J_i(\mathbf{z}_i) = \frac{1}{2}\mathbf{z}_i'\mathbf{Q}\mathbf{z}_i \\ \text{such that} & \mathbf{d}_i \le \mathbf{h}_i(\mathbf{p}_i, \bar{\mathbf{p}}_{-i}) \end{array}$$

$$\begin{aligned} \mathbf{1} \cdot \mathbf{d}_i &\leq C_i \\ \mathbf{p}_{\min} \leq \mathbf{p}_i &\leq \mathbf{p}_{\max} \\ \mathbf{d}_i &\geq \mathbf{d}_{\min}, \end{aligned}$$

where $\mathbf{Q} = \begin{pmatrix} \mathbf{0} & \mathcal{I} \\ \mathcal{I} & \mathbf{0} \end{pmatrix}$, $\mathcal{I}$ denotes a square identity matrix of suitable dimension.

Note that in this optimization problem, given a $\bar{\mathbf{z}}_{-i}$, seller i selects the vector $\bar{\mathbf{z}}_i$ that maximizes the objective function $J_i(\mathbf{d}_i, \mathbf{p}_i) = \sum_{t=1}^{T} d_i^t p_i^t$ within the feasible space $\mathcal{K}_i(\bar{\mathbf{z}}_{-i})$,

$$\mathcal{K}_i(\bar{\mathbf{z}}_{-i}) \quad = \quad \left\{ (\mathbf{d}_i, \mathbf{p}_i) \left| \begin{array}{cc} d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t) & \forall t \in \mathbf{T} \\ \sum_{t=1}^{T} d_i^t \leq C_i & \\ p_{\min}^t \leq p_i^t \leq p_{\max}^t & \forall t \in \mathbf{T} \\ d_i^t \geq d_{min} & \forall t \in \mathbf{T} \end{array} \right. \right\}.$$

In Subsection 3.3.2, we reformulate the best response problem for seller i, given $\bar{\mathbf{z}}_{-i}$, as variational inequality problem:

$$-\nabla J_i(\mathbf{z}_i') \cdot (\mathbf{z}_i - \mathbf{z}_i') \geq 0, \qquad \forall \mathbf{z}_i \in \mathcal{K}_i(\bar{\mathbf{z}}_{-i}), \tag{3.2}$$

and establish existence of solution.

3.2 Market equilibrium problem

The definition of a Nash equilibrium (Definition 2.1.1) implies that, at equilibrium, each seller would select a pricing policy that optimally solves her own best response problem. Notice that all competitors solve their best response problems simultaneously (and as a result, variational inequality (3.2)). Given equilibrium pricing policies for her competitors, $\mathbf{z}_{-i}^*$, seller i sets her equilibrium pricing policy by solving variational inequality problem (3.2). That is, the following set of variational inequality problems:

$$-\nabla J_i(\mathbf{z}_i^*) \cdot (\mathbf{z}_i - \mathbf{z}_i^*) \geq 0, \quad \forall \mathbf{z}_i \in \mathcal{K}_i(\mathbf{z}_{-i}^*) \quad i \in \mathbf{I}. \tag{3.3}$$

In Subsection 3.3.3, we study this reformulation further, combine it into a single quasi-variational inequality problem, and establish that equilibrium pricing policies indeed exist.

3.3 Analysis

In order to establish that the model in this thesis has a solution, we first need to establish existence of solution for the best response problem when competitors' strategies are given (see Subsection 3.3.2). This allows us to subsequently study existence of solution for the market equilibrium problem through an equivalent quasi-variational inequality reformulation (see Subsection 3.3.3). Furthermore, we study some additional interesting properties of the model such as when it gives rise to unique policies.

3.3.1 Conditions

In this subsection, we describe the conditions we impose on the model of this thesis and discuss the intuition behind them. Although we often use the linear demand case as an example for illustrating these conditions, the results in this thesis hold for a general non-linear demand.

Condition 3.3.1. *Price in any period is allowed to vary between a minimum and maximum allowable price level. We require $p^t_{\min}$ to be strictly positive and $p^t_{\max}$ to be a level at which demand for seller i vanishes irrespective of competitor prices in that period. Mathematically, we require that $p^t_{\min} > 0$ and $\sup_{\bar{p}^t_{-i}} \left(h^t_i(p^t_i, \bar{p}^t_{-i})\right) = 0$ at $p^t_i = p^t_{\max}$ for all $t \in \mathbf{T}$.*

Condition 3.3.2. *The amount of sale made by any seller in any period should be strictly positive. ie. $d^t_i > 0$ for all $i \in \mathbf{I}$, $t \in \mathbf{T}$. This forces each seller to participate in the market in every period. We enforce this with a constraint $d^t_i \geq d_{\min}\ \forall\ i, t$ where $d_{\min}$ is a arbitrarily small strictly positive value.*

Condition 3.3.3. *The demand function $h^t_i(p^t_i, p^t_{-i})$ is a concave function of (p^t_i, p^t_{-i}) over the set of feasible prices for all $i \in \mathbf{I}, t \in \mathbf{T}$.*

Condition 3.3.4. *For any period t, for any fixed $\bar{p}^t_{-i}$, the function $h^t_i(p^t_i, \bar{p}^t_{-i})$ is decreasing with respect to p^t_i over the set of feasible prices. Mathematically,*

$$\left(-h^t_i(\hat{p}^t_i, \bar{p}^t_{-i}) + h^t_i(\check{p}^t_i, \bar{p}^t_{-i})\right) \cdot (\hat{p}^t_i - \check{p}^t_i) \geq 0, \qquad \forall (\hat{p}^t_i, \check{p}^t_i)\ ,\ i \in \mathbf{I}.$$

Condition 3.3.5. *For any period t, for any fixed $\bar{p}^t_{-i}$, the function $h^t_i(p^t_i, \bar{p}^t_{-i})$ is strictly decreasing with respect to p^t_i over the set of feasible prices. Mathematically,*

$$\left(-h^t_i(\hat{p}^t_i, \bar{p}^t_{-i}) + h^t_i(\check{p}^t_i, \bar{p}^t_{-i})\right) \cdot (\hat{p}^t_i - \check{p}^t_i) > 0 \qquad \forall (\hat{p}^t_i, \check{p}^t_i)\ ,\ \hat{p}^t_i \neq \check{p}^t_i\ ,\ i \in \mathbf{I}.$$

Condition 3.3.6. *The function $-\mathbf{h}(\mathbf{p})$ is strictly monotone with respect to $\mathbf{p}$, over the set of feasible pricing policies $\mathcal{K}$. That is,*

$$([-\mathbf{h}(\hat{\mathbf{p}}) + \mathbf{h}(\check{\mathbf{p}})] \cdot (\hat{\mathbf{p}} - \check{\mathbf{p}})) > 0 \qquad \forall \hat{\mathbf{p}}, \check{\mathbf{p}} \in \mathcal{K}\ ,\ \hat{\mathbf{p}} \neq \check{\mathbf{p}}.$$

Condition 3.3.1 ensures that the space of allowed prices is bounded. We achieve this boundedness property by constraining the prices between some *allowable* upper and lower limits. Under this condition, we can eliminate strategies involving infinitely high price levels. Note that the lower limit could be arbitrarily close to the zero price level and the higher limit would be the price level at which the demand function vanishes (demand becomes zero).

Condition 3.3.2 ensures that each seller participates in each period with a strictly positive sale. The implication, if this were not true, would be that a seller with nothing to sell in a period could influence the demand seen by her competitors by setting a price. In other words, setting a price would make sense only if there is a non zero sale in that period.

Condition 3.3.3 ensures that the demand for a seller is concave in the seller's price for each period. This condition ensures that the strategy space in the best response problem is convex. This holds for products where demand decreases faster as price increases. The linear demand model trivially satisfies this condition.

Condition 3.3.4 ensures that the demand in any period for any seller does not increase with an increase in her price. The condition allows us to show existence of solution of the best response problem. Condition 3.3.5 ensures that the demand is strictly decreasing in price. This is required to ensure that the best response policy is unique. For a linear demand case, this implies that the demand function is strictly downward sloping with respect to price as is true for normal goods.

Condition 3.3.6 is used in the uniqueness result for the market equilibrium model. It requires strict monotonicity on the demand function as a whole. For a two seller linear demand case this is equivalent to saying that the sensitivity of seller i's demand to seller i's price is higher than the sensitivity of seller $-i$'s demand to seller i's price and the sensitivity of seller i's demand to seller $-i$'s price. This makes intuitive sense since we expect the decrease in demand seen by seller i when she raises prices to be more than the resulting increase in demand seen by her competitor. This can be interpreted as saying that upon seeing an increase in seller i's price, some of her customers will prefer to switch to her competitor and some will prefer not to buy at all.

3.3.2 Best response problem

In Section 3.1, we discussed that to find such a best response policy, seller i solves optimization problem (3.1). Note that under Condition 3.3.3, best response optimization problem (3.1) has a compact and convex feasible space denoted by $\mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$, where

$$\mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}) = \left\{ (\mathbf{d}_i, \mathbf{p}_i) \;\middle|\; \begin{array}{ll} d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t) & \forall t \in \mathbf{T} \\ \sum_{t=1}^T d_i^t \leq C_i & \\ p_{\min}^t \leq p_i^t \leq p_{\max}^t & \forall t \in \mathbf{T} \\ d_i^t \geq d_{min} & \forall t \in \mathbf{T} \end{array} \right\}$$

The objective function $J_i(\mathbf{d}_i, \mathbf{p}_i) = \sum_{t=1}^T d_i^t p_i^t$ is not concave. We now consider the corresponding variational inequality problem that seeks $(\mathbf{p}_i', \mathbf{d}_i') \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$ such that

$$-\nabla J_i(\mathbf{p}_i', \mathbf{d}_i') \cdot \begin{pmatrix} \mathbf{p}_i - \mathbf{p}_i' \\ \mathbf{d}_i - \mathbf{d}_i' \end{pmatrix} \geq 0, \quad \forall\, (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}). \tag{3.4}$$

It is easy to show that any solution to best response optimization problem (3.1) is a solution to variational inequality problem (3.4). To show the converse, one traditionally requires concavity of the objective function in the optimization problem. However, as noted before, the objective function for optimization problem (3.1) is not concave. Nevertheless, the variational inequality problem structure allows us to establish this result. First, Lemma 3.3.1 below proves that in any solution to the variational inequality problem (3.4), the variables p_i^t and d_i^t must be related through an equality relation.

Lemma 3.3.1. *Given a competitor strategy $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$, the solution $(\mathbf{d}_i', \mathbf{p}_i')$ to variational inequality problem (3.4) satisfies the following relation:*

$$d_i'^t \;=\; h_i^t(p_i'^t, \bar{p}_{-i}^t).$$

Proof. Assume that at some period t, $d_i'^t \neq h_i^t(p_i'^t, \bar{p}_{-i}^t)$. There are two possible cases:

1. For some $t \in \mathbf{T}$, let $d_i'^t < h_i^t(p_i'^t, \bar{p}_{-i}^t)$. The properties of the demand function (see Condition 3.3.4) imply that $p_i'^t$ can be increased by δ^t while maintaining $d_i'^t < h_i^t(p_i'^t + \delta^t, \bar{p}_{-i}^t)$. Note that since $h_i^t(p_i'^t + \delta^t, \bar{p}_{-i}^t) > d_i'^t > 0$, we can find a $\delta^t > 0$ such that $p_i'^t + \delta^t < p_{\max}^t$. Variational inequality problem (3.4) seeks $(\mathbf{p}_i', \mathbf{d}_i') \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$ such that

$$-\nabla J_i(\mathbf{p}_i', \mathbf{d}_i') \cdot \begin{pmatrix} \mathbf{p}_i - \mathbf{p}_i' \\ \mathbf{d}_i - \mathbf{d}_i' \end{pmatrix} \geq 0, \qquad \forall\, (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}).$$

   We will choose a $(\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$ such that the above condition is violated and hence prove that $(\mathbf{p}_i', \mathbf{d}_i')$ could not be a solution to the variational inequality. For any $t \in \mathbf{T}$, choose

$$p_i^t = \begin{cases} p_i'^t & \text{if } d_i'^t = h_i^t(p_i'^t, \bar{p}_{-i}^t) \\ p_i'^t + \delta^t & \text{if } d_i'^t < h_i^t(p_i'^t, \bar{p}_{-i}^t) \end{cases} \tag{3.5}$$

   Choose $\mathbf{d}_i = \mathbf{d}'_i$. Considering the above point $(\mathbf{d}_i, \mathbf{p}_i)$ in variational inequality problem (3.4) we get $\sum_t \delta^t d_i^t \leq 0$. Nevertheless, since $\sum_t \delta^t d_i^t > 0$ this yields a contradiction. Thus, $(\mathbf{d}_i', \mathbf{p}_i')$ could not be a solution to variational inequality problem (3.4).

2. For some $t \in \mathbf{T}$, $d_i'^t > h_i^t(p_i'^t, \bar{p}_{-i}^t)$. Notice that $(\mathbf{d}_i', \mathbf{p}_i')$ is infeasible and could not be a solution to variational inequality problem (3.4).

Both cases lead to a contradiction and the result follows. □

Proposition 3.3.1 below proves the existence of a solution to best response optimization problem (3.1).

Proposition 3.3.1. *For any fixed $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$, there exists a solution $(\mathbf{d}_i', \mathbf{p}_i')$ to best response optimization problem (3.1).*

Proof. It is easy to show that the feasible space is non-empty and compact and the objective function is continuous. Under these conditions the result follows from the well known Weierstrass theorem (See Bazaraa, Sherali and Shetty [1]). □

Having shown that there exists an optimal policy for best response optimization problem (3.1), Proposition 3.3.2 proves that this optimal policy is also a solution to variational inequality problem (3.4).

Proposition 3.3.2. *Fixing competitor policies at $(\bar{\mathbf{d}}_{-i}, \bar{\mathbf{p}}_{-i})$, under Conditions 3.3.3, 3.3.4, best response optimization problem (3.1) and variational inequality problem (3.4) have the same solutions.*

Proof. We first show that a solution to best response problem (3.1) also solves variational inequality problem (3.4). Under Condition (3.3.3), the feasible space for best response optimization problem (3.1) is a convex and closed set. Moreover, the objective function is continuously differentiable. Consider any point $(\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$. The convexity of the feasible space $\mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$ ensures that the point $\mathbf{z}_i' + \theta \cdot (\mathbf{z}_i - \mathbf{z}_i')$ also belongs to the feasible space for all $\theta \in [0, 1]$. We define function $\xi(\theta) = -J_i(\mathbf{z}_i' + \theta \cdot (\mathbf{z}_i - \mathbf{z}_i'))$, for $\theta \in [0, 1]$. Function $\xi(\theta)$ achieves its minimum at $\theta = 0$. It follows that $\xi'(0) = -\nabla J_i(\mathbf{z}_i') \cdot (\mathbf{z}_i - \mathbf{z}_i') \geq 0$. It follows that if the feasible space is a closed and convex set, and the objective function is continuously differentiable, the solution to best response optimization problem (3.1) solves variational inequality problem (3.4).

We now establish the opposite, i.e., that any solution to variational inequality problem (3.4) is also a solution to best response problem (3.1). The policy that solves variational inequality problem (3.4) satisfies the following condition,

$$-\sum_t p_i^{t'}(d_i^t - d_i^{t'}) - \sum_t d_i^{t'}(p_i^t - p_i^{t'}) \geq 0, \quad \forall\ (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}). \tag{3.6}$$

Condition 3.3.4 states that

$$\left(-h_i^t(p_i^{t'}, \bar{p}_{-i}^t) + h_i^t(p_i^t, \bar{p}_{-i}^t)\right) \cdot (p_i^{t'} - p_i^t) \geq 0, \qquad \forall\ (p_i^{t'}, p_i^t),\ p_i^{t'} \neq p_i^t.$$

From Lemma 3.3.1 it follows that for all $d_i^t = h_i^t(p_i^t, \bar{p}_{-i}^t)$,

$$\left(-d_i^{t'} + d_i^t\right) \cdot (p_i^{t'} - p_i^t) \geq 0, \qquad \forall (p_i^{t'}, p_i^t),\ p_i^{t'} \neq p_i^t.$$

Summing over all t

$$\sum_t \left(-d_i^{t'} + d_i^t\right) \cdot (p_i^{t'} - p_i^t) \geq 0, \qquad \forall (p_i^{t'}, p_i^t),\ p_i^{t'} \neq p_i^t,\ d_i^t = h_i^t(p_i^t, \bar{p}_{-i}^t). \tag{3.7}$$

Adding (3.6) and (3.7) we get that $\forall\ (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i})$

$$\sum_t p_i^{t'} d_i^{t'} \geq \sum_t p_i^t h_i^t(p_i^t, \bar{p}_{-i}^t) \geq \sum_t p_i^t d_i^t.$$

The second inequality results from the fact that $d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t)$, for all feasible d_i^t. The variational inequality solution is thus an optimal policy for seller i and solves best response optimization problem (3.1). $\square$

In conclusion, Proposition 3.3.2 establishes that variational inequality problem (3.4) is an equivalent reformulation of best response problem (3.1).

3.3.3 Market equilibrium problem

We now consider the quasi-variational inequality formed by combining the variational inequality formulations (3.4) for each seller $i \in \mathbf{I}$. We define the joint feasible space as

$$\mathcal{K}(\mathbf{z}^*) \quad = \quad \left\{ \mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_I) \mid \mathbf{z}_i \in \mathcal{K}_i(\mathbf{z}^*_{-i}),\ \forall i \in \mathbf{I} \right\}.$$

Furthermore, the joint quasi-variational inequality problem seeks a point $(\mathbf{p}^*, \mathbf{d}^*) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*)$ such that

$$\mathbf{F}(\mathbf{p}^*, \mathbf{d}^*) \cdot \begin{pmatrix} \mathbf{p} - \mathbf{p}^* \\ \mathbf{d} - \mathbf{d}^* \end{pmatrix} \geq 0, \qquad \forall\ (\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*), \tag{3.8}$$

where $F_i(\mathbf{z}^*) = -\nabla J_i(\mathbf{z}_i^*)$, $\forall i \in \mathbf{I}$.

Proposition 3.3.3. *Solving the joint quasi-variational inequality problem (3.8) is equivalent to solving simultaneously for each $i \in \mathbf{I}$, variational inequality problems (3.4) with $\bar{\mathbf{z}}_{-i} = \mathbf{z}^*_{-i}$.*

Proof. It is easy to show that a policy $(\mathbf{p}^*, \mathbf{d}^*)$ that solves variational inequalities problem (3.4), for each $i \in \mathbf{I}$ also solves joint quasi-variational inequality (3.8).
Let us now consider the converse. Joint quasi-variational inequality (3.8) can be rewritten as $(\mathbf{p}^*, \mathbf{d}^*) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*)$ so that

$$-\sum_{i,t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{i,t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0, \ \ \forall\ (\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*).$$

Consider an individual seller i. Let $(\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*)$ be a feasible policy such that $p^t_{-i} = p^{t*}_{-i}$ and $d^t_{-i} = d^{t*}_{-i}$, for all her competitors and any feasible $(p_i^t, d_i^t) \in \mathcal{K}_i(p^{t*}_{-i}, d^{t*}_{-i})$. Then

$$-\sum_{t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0, \ \ \forall\ (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(p^{t*}_{-i}, d^{t*}_{-i}).$$

Notice this coincides with variational inequality problem (3.4). Repeating the argument for every $i \in \mathbf{I}$, it is easy to show that $(\mathbf{p}_i^*, \mathbf{d}_i^*)$ solves variational inequality problem (3.4) for every seller when the competitor's policy is $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}) = (\mathbf{p}^*_{-i}, \mathbf{d}^*_{-i})$. $\square$

Lemma 3.3.2 proves that any solution to the joint quasi-variational inequality (3.8) satisfies a relation similar to the one described in Lemma 3.3.1.

Lemma 3.3.2. *Let $(\mathbf{p}^*, \mathbf{d}^*)$ be a solution to the joint quasi-variational inequality problem (3.8). $(\mathbf{p}^*, \mathbf{d}^*)$ satisfies the following relation:*

$$d_i^{t*} = h_i^t(p_i^{t*}, p^{t*}_{-i}), \ \ \forall\ i \in \mathbf{I},\ t \in \mathbf{T}.$$

Proof. Proposition 3.3.3 shows that for every $i \in \mathbf{I}$, $(\mathbf{p}_i^*, \mathbf{d}_i^*)$ solves variational inequality problem (3.4) for the corresponding seller i. Lemma 3.3.1 shows that $(\mathbf{p}_i^*, \mathbf{d}_i^*)$ follows the relation

$$d_i^{t*} = h_i^t(p_i^{t*}, p_{-i}^{t*}), \quad \forall\, t \in \mathbf{T}.$$

Repeating the argument for every $i \in \mathbf{I}$ implies the result. □

Proposition 3.3.4 shows that the solution to the joint quasi-variational inequality problem (3.8) is an optimal policy for each seller i and hence is a Nash equilibrium policy.

Proposition 3.3.4. *Conditions 3.3.1 - 3.3.4 imply that the policy that solves the joint quasi-variational inequality problem (3.8) is also a Nash equilibrium policy.*

Proof. The policy that solves joint quasi-variational inequality (3.8) satisfies the following condition,

$$-\sum_{i,t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{i,t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0, \quad \forall\, (\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*).$$

Consider an individual seller i. Since the above holds for all feasible $(\mathbf{p}, \mathbf{d})$, consider the feasible policy $(\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*)$: $p_{-i}^t = p_{-i}^{t*}$ and $d_{-i}^t = d_{-i}^{t*}$ for all her competitors and any $(p_i^t, d_i^t) \in \mathcal{K}_i(p_{-i}^{t*}, d_{-i}^{t*})$. For this seller,

$$-\sum_{t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0, \quad \forall (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(p_{-i}^{t*}, d_{-i}^{t*}). \tag{3.9}$$

Under Condition 3.3.4 it follows that

$$\left(-h_i^t(p_i^{t*}, p_{-i}^{t*}) + h_i^t(p_i^t, p_{-i}^{t*})\right) \cdot (p_i^{t*} - p_i^t) \geq 0, \qquad \forall p_i^{t*},\ p_i^t.$$

From Lemma 3.3.1 it follows that for $d_i^{t*} = h_i^t(p_i^{t*}, p_{-i}^{t*})$ and $d_i^t = h_i^t(p_i^t, p_{-i}^{t*})$,

$$\left(-d_i^{t*} + d_i^t\right) \cdot (p_i^{t*} - p_i^t) \geq 0, \qquad \forall p_i^{t*},\ p_i^t.$$

Summing over all t

$$\sum_{t} \left(-d_i^{t*} + d_i^t\right) \cdot (p_i^{t*} - p_i^t) \geq 0, \qquad \forall p_i^{t*},\ p_i^t,\ d_i^t = h_i^t(p_i^t, p_{-i}^{t*}). \tag{3.10}$$

Adding (3.9) and (3.10) implies that $\forall\, (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(p_{-i}^{t*}, d_{-i}^{t*})$

$$\sum_{t} p_i^{t*} d_i^{t*} \geq \sum_{t} p_i^t h_i^t(p_i^t, p_{-i}^{t*}) \geq \sum_{t} p_i^t d_i^t.$$

The second inequality results from the fact that $d_i^t \leq h_i^t(p_i^t, p_{-i}^{t*})$, for all feasible d_i^t. Thus, the joint quasi-variational inequality solution is a Nash equilibrium policy for each individual seller and the result follows. □

We can now establish existence of a market equilibrium policy. First, we give a result due to Pang and Fukushima [52] regarding the existence of solution to quasi-variational inequalities.

Theorem 3.3.1 (Pang and Fukushima [52]). *Let F be a continuous point-to-point map from $\Re^n$ into itself and let K be a point-to-set map from $\Re^n$ into subsets of $\Re^n$. If there exists a compact convex set $T \subset \Re^n$ such that*

(a) for every $x \in T$, $K(x)$ is a nonempty, closed, convex subset of T;

(b) K is continuous at every point in T,

then the $QVI(K, F)$ has a solution.

We use this to show existence of a market equilibrium policy in the following theorem.

Theorem 3.3.2. *Conditions 3.3.1 - 3.3.3 imply that a solution to the joint quasi-variational inequality problem (3.8) exists and as a result, a market equilibrium policy exists.*

Proof. Notice that the feasible region in joint quasi-variational inequality (3.8) is non empty for any feasible $(\mathbf{p}, \mathbf{d})$. Indeed, consider the point $(\dot{\mathbf{p}}, \dot{\mathbf{d}})$ where for all $i \in \mathbf{I}$ and $t \in \mathbf{T}$, $(\dot{p}_i^t, \dot{d}_i^t) = (p_{\min}, d_{\min})$. It is easy to show that $(\dot{\mathbf{p}}, \dot{\mathbf{d}})$ lies in the feasible space $\mathcal{K}(\mathbf{p}, \mathbf{d})$. $\mathcal{K}(\mathbf{p}, \mathbf{d})$ is thus a non-empty set. Condition 3.3.3 implies that the feasible space $\mathcal{K}(\mathbf{p}, \mathbf{d})$ is also a convex and compact set. We construct the set T as follows.

$$T = \left\{ (\mathbf{d}, \mathbf{p}) \left| \begin{array}{cl} d_i^t \leq h_i^t(p_i^t, p_{-i}^t) & \forall t \in \mathbf{T},\ \forall i \in \mathbf{I} \\ \sum_{t=1}^{T} d_i^t \leq C_i & \forall i \in \mathbf{I} \\ p_{\min}^t \leq p_i^t \leq p_{\max}^t & \forall t \in \mathbf{T},\ \forall i \in \mathbf{I} \\ d_i^t \geq d_{min} & \forall t \in \mathbf{T},\ \forall i \in \mathbf{I} \end{array} \right. \right\}$$

This satisfies condition **(a)** of Theorem 3.3.1. It is also easy to show that **(b)** is true. Furthermore, the quasi-variational inequality function $\mathbf{F}$ is continuous. Thus, a solution to the quasi-variational inequality problem exists due to Theorem 3.3.1. Therefore, Proposition 3.3.4 implies that a market equilibrium policy exists. □

Remark: Till now, we have considered undiscounted revenue cash flows. It is easy to incorporate discounting into the model, for example discounting the cash flow in period t by a factor δ^t where $\delta > 0$ is the discounting factor. The best response problem for the discounted case is

$$\begin{array}{rl} \operatorname{argmax}_{\mathbf{d}_i, \mathbf{p}_i} & \sum_{t=1}^{T} \delta^t d_i^t p_i^t \\ \text{such that} & d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t), \quad \forall t \in \mathbf{T} \\ & \sum_{t=1}^{T} d_i^t \leq C_i \\ & p_{\min}^t \leq p_i^t \leq p_{\max}^t, \quad \forall t \in \mathbf{T} \\ & d_i^t \geq d_{min}, \quad \forall t \in \mathbf{T}. \end{array} \tag{3.11}$$

The conditions for existence remain exactly the same as those for the undiscounted case.

3.4 Uniqueness of best response policy

In this subsection we discuss conditions guaranteeing that the best response problem gives rise to a unique policy. In Proposition 3.4.1 we prove that there is a unique solution to variational inequality problem (3.4) (and hence to the best response optimization problem (3.1)).

Proposition 3.4.1. *Under Condition 3.3.5, there is a unique solution to best response optimization problem (3.1).*

Proof. Under Condition 3.3.5, relation (3.7) in the proof of Proposition 3.3.2, becomes a strict inequality:

$$\sum_t \left(-d_i^{t'} + d_i^t\right) \cdot (p_i^{t'} - p_i^t) > 0, \qquad \forall p_i^{t'}, p_i^t,\ p_i^{t'} \neq p_i^t.$$

Adding (3.6) and the previous inequality implies that $\forall\ (\mathbf{p}_i, \mathbf{d}_i) \in \mathcal{K}_i(p_{-i}^{t'}, d_{-i}^{t'})$

$$\sum_t p_i^{t'} d_i^{t'} > \sum_t p_i^t d_i^t.$$

Thus, $(\mathbf{p}_i', \mathbf{d}_i')$ is the unique solution to best response optimization problem (3.1). From Proposition 3.3.2, it follows that the variational inequality also has a unique solution. □

3.5 Remarks on uniqueness of equilibrium

The market equilibrium policy is the policy that solves the joint quasi-variational inequality problem (3.8). To the best of our knowledge, there are currently no uniqueness results for a general quasi-variational inequality problem in the literature. Hence it is hard to prove the uniqueness of a market equilibrium policy. There is however, one characterization that ensures uniqueness. This is presented in Proposition 3.5.1 below.

Proposition 3.5.1. *If there are two distinct equilibrium solutions, say* $(\mathbf{p}^1, \mathbf{d}^1)$ *and* $(\mathbf{p}^2, \mathbf{d}^2)$*, to the joint quasi-variational inequality (3.8), then the following two conditions cannot hold simultaneously:*

1. $(\mathbf{p}^1, \mathbf{d}^1) \in \mathcal{K}(\mathbf{p}^2, \mathbf{d}^2)$
2. $(\mathbf{p}^2, \mathbf{d}^2) \in \mathcal{K}(\mathbf{p}^1, \mathbf{d}^1)$

Proof. Let us assume that there exist two distinct equilibrium solutions $(\mathbf{p}^1, \mathbf{d}^1)$ and $(\mathbf{p}^2, \mathbf{d}^2)$ and that the two conditions hold. Then, substituting $(\mathbf{p}^1, \mathbf{d}^1)$ in (3.8) describing the QVI with solution $(\mathbf{p}^2, \mathbf{d}^2)$ and substituting $(\mathbf{p}^2, \mathbf{d}^2)$ in (3.8) describing the QVI with solution $(\mathbf{p}^1, \mathbf{d}^1)$ and adding the two, we get

$$(\mathbf{p}^2 - \mathbf{p}^1)(\mathbf{d}^2 - \mathbf{d}^1) \geq 0.$$

Using Lemma 3.3.2 leads to

$$(\mathbf{p}^2 - \mathbf{p}^1)(\mathbf{h}(\mathbf{p}^2) - \mathbf{h}(\mathbf{p}^1)) \geq 0.$$

This contradicts Condition 3.3.6. □

Remark: In Section 6.2 we will provide some insight as to when these conditions hold for some special symmetric cases (See Lemma 6.2.2 and Lemma 6.2.1).

Remark: In the absence of Condition 3.3.6, the equilibrium need not be unique. We show an example below that has multiple market equilibrium. Consider a two seller, single period, uncapacitated example. Instead of requiring that the demand function $h_i^t(p_i^t, p_{-i}^t)$ is jointly concave in (p_i^t, p_{-i}^t), we will require it to be concave with respect to only p_i^t for any fixed p_{-i}^t.

Example: For both Seller 1 and Seller 2, let the demand be given by

$$h_i(p_i, p_{-i}) = D - \beta p_i + \alpha p_{-i}^2$$

for $i = 1, 2$. Since both the sellers have no capacity constraints, the best response for Seller i given the price level $\bar{p}_{-i}$ is given by

$$p_i^*(\bar{p}_{-i}) = \frac{D + \alpha \bar{p}_{-i}^2}{2\beta}$$

for $i = 1, 2$. If p^* is an equilibrium policy, then for $i = 1, 2$,

$$p_i^*(\bar{p}_{-i}) = \frac{D + \alpha {p_{-i}^*}^2}{2\beta}.$$

This has two solutions:

$$p_i^* = p_{-i}^* = \frac{\beta \pm \sqrt{\beta^2 - \alpha D}}{\alpha}$$

For example, consider $D = 1$, $\alpha = 1$ and $\beta = 2$. Notice that $-\mathbf{h}(\mathbf{p})$ is a strictly monotone function. Furthermore,

$$p_i^* = p_{-i}^* = 2 + \sqrt{3}$$

$$d_i^* = d_{-i}^* = 4 + 2\sqrt{3}$$

and

$$p_i^* = p_{-i}^* = 2 - \sqrt{3}$$

$$d_i^* = d_{-i}^* = 4 - 2\sqrt{3}.$$

Both of these are equilibrium solutions.

Chapter 4

Robust demand model

4.1 Preliminaries

In this chapter, we discuss a model for pricing in markets with oligopolistic competition and uncertainty in demand.

1. **Competition**: Each seller optimizes her own payoff simultaneously by solving a best response problem.
   (a) The problem that each seller solves is to find a policy that maximizes her payoff given her competitors' policies. We analyze the best response problem and study the existence and uniqueness of the solution to this problem in Section 4.2.
   (b) When viewed from a market perspective, an equilibrium exists when all sellers adopt a policy that simultaneously solves each seller's best response policy. This is the market equilibrium problem which we analyze in Section 4.3.3. We study the existence of solution to this problem.
2. **Demand Uncertainty**: We address the issue of uncertainty in the demand function by studying the model when sellers adopt robust policies described in Section 4.1.2. The uncertain parameters introduced into the demand function pose an additional challenge in the analysis of equilibrium.

In this chapter, we will assume that the demand is a function of the prices with parameters belonging to some uncertainty sets. The sellers adopt policies that are robust to this uncertainty. We start by describing the concepts of uncertain demand and robust policies. The formulation and analysis of the best response problem and the market equilibrium problem under this case follow next.

4.1.1 Robust demand

We model the demand realized by Seller i in period t as a function of her prices in period t, the prices set by her competitors in period t and some uncertainty factors. We denote the function by $h_i^t(p_i^t, p_{-i}^t, \xi_i^t)$, where ξ_i^t is the uncertainty factor. The uncertainty factor is a parameter that can take any value from a given closed uncertainty set $\mathcal{U}_i^t$. The vector of uncertain parameters for all time periods for a seller i is denoted by ξ_i. This vector can take any value from the set $\mathcal{U}_i$, where

$$\mathcal{U}_i = \mathcal{U}_i^1 \times \mathcal{U}_i^2 \times \cdots \times \mathcal{U}_i^T$$

Similarly, the vector of uncertain parameters for all time periods and sellers is denoted by $\xi \in \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_I$. For example, for a linear demand function in a duopoly, the demand function for a time period t can be defined as follows.

$$\begin{aligned} h_1^t(p_1^t, p_2^t, \xi_1^t) &= D_{1_{\text{base}}}^t - \beta_1^t p_1^t + \alpha_1^t p_2^t \\ h_2^t(p_2^t, p_1^t, \xi_2^t) &= D_{2_{\text{base}}}^t - \beta_2^t p_2^t + \alpha_2^t p_1^t, \end{aligned}$$

where $\xi_i^t = (D_{i_{\text{base}}}^t, \beta_i^t, \alpha_i^t)$ and ξ_i^t lying in the uncertainty set $\mathcal{U}_i^t$. Examples of such uncertainty sets are:

$$\mathcal{U}_i^t = \left\{ (D_{i_{\text{base}}}^t, \beta_i^t, \alpha_i^t) \;\middle|\; \left| \frac{D_{i_{\text{base}}}^t - \bar{D}_{i_{\text{base}}}^t}{\sigma_{D_{i_{\text{base}}}^t}} \right| + \left| \frac{\beta_i^t - \bar{\beta}_i^t}{\sigma_{\beta_i^t}} \right| + \left| \frac{\alpha_i^t - \bar{\alpha}_i^t}{\sigma_{\alpha_i^t}} \right| \le \Gamma_i^t \right\}$$

and,

$$\mathcal{U}_i^t = \left\{ (D_{i_{\text{base}}}^t, \beta_i^t, \alpha_i^t) \;\middle|\; \left| \frac{D_{i_{\text{base}}}^t - \bar{D}_{i_{\text{base}}}^t}{\sigma_{D_{i_{\text{base}}}^t}} \right|^2 + \left| \frac{\beta_i^t - \bar{\beta}_i^t}{\sigma_{\beta_i^t}} \right|^2 + \left| \frac{\alpha_i^t - \bar{\alpha}_i^t}{\sigma_{\alpha_i^t}} \right|^2 \le \Gamma_i^{t\,2} \right\}$$

where $\bar{D}_{i_{\text{base}}}^t$, $\bar{\beta}_i^t$ and $\bar{\alpha}_i^t$ are the nominal values of the uncertainty parameters and $\sigma_{\bar{D}_{i_{\text{base}}}^t}$, $\sigma_{\bar{\beta}_i^t}$ and $\sigma_{\bar{\alpha}_i^t}$ are some measure of the parameters' dispersion around the nominal values.

4.1.2 Robust policy

Since information about the values that the uncertainty parameters will take is not available to the seller, the natural question that arises is what objective function should the seller maximize? As is done in certain models, if an *a priori* distribution is assumed for the uncertainty variable, the seller could adopt a policy that maximizes the expected revenue. This however, involves assuming knowledge about the distribution of the uncertain parameters. In a lot of cases, it is difficult to estimate accurately the mean and variance of such parameters, let alone their distribution. The most basic information that is available is an interval or set in which the realized parameter values will fall. A robust policy is a policy that would maximize the objective function for a seller even when under the most adverse instances of the uncertainty factor within such a set.

It has been seen that the robust policy typically improves the worst case payoff with regards to uncertainty at the loss of optimality of the best case payoff. Such a tradeoff can be beneficial for a number of reasons. One such reason is linked to the fact that the robust policy reduces the variance in the payoff compared to the optimal policy corresponding to the nominal values of the uncertainty parameters. A lot of firms might find it more attractive to adopt a policy that guarantees revenues that are less variable and uncertain even if they are lower on an average basis than to adopt a policy that on average generates higher revenues but also potentially could generate very poor revenues.

4.2 Best response problem

In this section we consider the problem of finding the robust best response policy for seller i given the policy of all her competitors $-i$. To find such a robust best response policy, seller i solves the following optimization problem.

$$
\begin{array}{llll}
\max\limits_{\mathbf{p}_i,\mathbf{d}_i,\mathbf{D}_i} & \sum_{t=1}^{T} d_i^t p_i^t & & (4.1)\\
\text{such that} & d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t, \xi_i^t) & \forall \xi_i^t \in \mathcal{U}_i^t, \forall t \in \mathbf{T} & \\
 & \sum_{\tau=1}^{t} d_i^\tau \leq C_i - D_i^t & \forall t \in \mathbf{T} & \\
 & 0 \leq D_i^T \leq \cdots \leq D_i^1 \leq C_i & & \\
 & p_{i,\,\mathtt{min}}^t \leq p_i^t \leq p_{i,\,\mathtt{max}}^t & \forall t \in \mathbf{T} & \\
 & d_i^t \geq d_{i,\,\mathtt{min}}^t & \forall t \in \mathbf{T} &
\end{array}
$$

Note that under Condition 4.3.3 which we introduce in the following section, the best response optimization problem (4.1) has a compact and convex feasible space denoted by $\mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ where

$$
\mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i}) = \left\{ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \;\middle|\; \begin{array}{ll}
d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t, \xi_i^t) & \forall \xi_i^t \in \mathcal{U}_i^t, \forall t \\
\sum_{\tau=1}^{t} d_i^\tau \leq C_i - D_i^t & \forall t \in \mathbf{T} \\
0 \leq D_i^T \leq \cdots \leq D_i^1 \leq C_i & \\
p_{i,\,\mathtt{min}}^t \leq p_i^t \leq p_{i,\,\mathtt{max}}^t & \forall t \in \mathbf{T} \\
d_i^t \geq d_{i,\,\mathtt{min}}^t & \forall t \in \mathbf{T}
\end{array} \right\}
$$

but an objective function $J_i(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) = \sum_{t=1}^{T} d_i^t p_i^t$ that is not concave. The lack of a concave objective function makes the problem hard. For this reason, we consider a variational inequality reformulation. We now consider the corresponding variational inequality: We want to find $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i') \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ such that

$$
-\nabla J_i(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i') \cdot \begin{pmatrix} \mathbf{p}_i - \mathbf{p}_i' \\ \mathbf{d}_i - \mathbf{d}_i' \\ \mathbf{D}_i - \mathbf{D}_i' \end{pmatrix} \geq 0, \quad \forall\; (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i}) \qquad (4.2)
$$

4.3 Analysis

4.3.1 Conditions

The analysis for the robust best response problem and the market equilibrium problem holds under certain conditions. We start by listing these conditions and in order to obtain additional intuition on these conditions, we will use the linear demand case to illustrate. Please note that although we illustrate the intuition behind these conditions via linear demand functions, the results hold for general non-linear demand as well.

Condition 4.3.1. *Prices in any period is allowed to vary between a minimum and maximum allowable level. We require $p^t_{i,\ \mathtt{min}}$ to be strictly positive and $p^t_{i,\ \mathtt{max}}$ to be a level at which demand for seller i vanishes irrespective of competitor prices in that period. Mathematically, we require that $p^t_{i,\ \mathtt{min}} > 0$ and $\sup_{\bar{p}'_{-i},\ \xi^t_i} \left(h^t_i(p^t_i, \bar{p}^t_{-i}, \xi^t_i) \right) = 0$ at $p^t_i = p^t_{i,\ \mathtt{max}}$ for all $t \in \mathbf{T}$.*

Condition 4.3.2. *The amount of sale made by any seller in any period should be strictly positive. ie. $d^t_i > 0$ for all $i \in \mathbf{I}$, $t \in \mathbf{T}$. This forces each seller to participate in the market in every period. We enforce this with a constraint $d^t_i \geq d^t_{i,\ \mathtt{min}}\ \forall\ i, t$ where $d^t_{i,\ \mathtt{min}}$ is a arbitrarily small strictly positive value.*

Condition 4.3.3. *The demand function $h^t_i(p^t_i, p^t_{-i}, \xi^t_i)$ is a concave function of the price variables (p^t_i, p^t_{-i}) over the set of feasible prices, for all $i \in \mathbf{I}, t \in \mathbf{T}$, for all $\xi^t_i \in \mathcal{U}^t_i$.*

Condition 4.3.4. *For any period t, for any fixed $\bar{p}^t_{-i}$ and $\xi^t_i \in \mathcal{U}^t_i$, the function $h^t_i(p^t_i, \bar{p}^t_{-i}, \xi^t_i)$ is decreasing with respect to p^t_i over the set of feasible prices. Mathematically,*

$$\left(-h^t_i(\hat{p}^t_i, \bar{p}^t_{-i}, \xi^t_i) + h^t_i(\check{p}^t_i, \bar{p}^t_{-i}, \xi^t_i)\right) \cdot (\hat{p}^t_i - \check{p}^t_i) \geq 0 \qquad \forall(\hat{p}^t_i, \check{p}^t_i)\ ,\ i \in \mathbf{I}.$$

Condition 4.3.5. *For any period t, for any fixed $\bar{p}^t_{-i}$ and $\xi^t_i \in \mathcal{U}^t_i$, the function $h^t_i(p^t_i, \bar{p}^t_{-i}, \xi^t_i)$ is strictly decreasing with respect to p^t_i over the set of feasible prices. Mathematically,*

$$\left(-h^t_i(\hat{p}^t_i, \bar{p}^t_{-i}, \xi^t_i) + h^t_i(\check{p}^t_i, \bar{p}^t_{-i}, \xi^t_i)\right) \cdot (\hat{p}^t_i - \check{p}^t_i) > 0 \qquad \forall(\hat{p}^t_i, \check{p}^t_i)\ ,\ \hat{p}^t_i \neq \check{p}^t_i\ ,\ i \in \mathbf{I}.$$

Condition 4.3.1 ensures that the space of allowed prices is bounded. We achieve this boundedness property by constraining the prices between some *allowable* upper and lower limits. Under this condition, we can eliminate strategies involving infinitely high price levels. Note that the lower limit could be arbitrarily close to the zero price level and the higher limit would be the price level at which the demand function vanishes (demand becomes zero).

Condition 4.3.2 ensures that each seller participates in each period with a strictly positive sale. The implication, if this were not true, would be that a seller with nothing to sell in a period could influence the demand seen by her competitors by setting a price. In other words, setting a price would make sense only if there is a non zero sale in that period.

Condition 4.3.3 ensures that the demand for a seller is concave in the seller's price for each period. This condition ensures that the strategy space in the best response problem is convex. This holds for products where demand decreases faster as price increases. The linear demand model trivially satisfies this condition.

Condition 4.3.4 ensures that the demand in any period for any seller does not increase with an increase in her price. We will use this condition to show that the solution to the variational inequality for seller i formulated in the next section is the optimal solution for the seller given the pricing policy for all her competitors. Condition 4.3.5 ensures that the demand is strictly decreasing in price. This condition will be useful to establish that the best response policy is unique. For a linear demand case, this implies that the demand function is strictly downward sloping with respect to price as is true for normal goods.

4.3.2 Best response problem

It is easy to show that any solution to the best response optimization problem (4.1) is a solution to the variational inequality problem (4.2). The converse can be shown under conditions of concavity of the objective function in the optimization problem. However, as noted before, the objective function for the optimization problem (4.1) is not concave. It is easy to see that replacing the inequality involving the protection levels in variational inequality problem (4.2) with an equality would not change the resulting optimal policy. Hence we establish that the following equality holds at the solution:

$$D_i^t = C_i - \sum_{\tau=1}^{t} d_i^\tau \tag{4.3}$$

It should be clear by now that the protection level variables D_i^ts do not affect the outcome of the robust best response problem. This is because the optimal protection levels for the robust best response policy can be removed from the problem formulation and later independently determined from the values of the optimal demand variables. However, these variables play a non-trivial role in the actual implementation of the policy in which case the demand parameters take different values and the protection levels act as cushions for absorbing variance in demand across different periods.

By utilizing (4.3) and some properties of the variational inequality formulation, we can show that any solution to the variational inequality problem (4.2) will also be a solution to the best response optimization problem (4.1) even though the objective function in the optimization problem is not concave. Lemma 4.3.1 proves that in any solution to the variational inequality problem (4.2) the variables p_i^t and d_i^t must be related through an equality relation as we describe below.

Lemma 4.3.1. *Given a competitor strategy $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$, the solution to the variational inequality problem (4.2) $(\mathbf{p}'_i, \mathbf{d}'_i, \mathbf{D}'_i)$ satisfies the following relations:*

$$d_i'^t = h_i^t(p_i'^t, \bar{p}_{-i}^t, \xi_i^{t'})$$

for some $\xi_i^{t'} \in \mathcal{U}_i^t$.

Proof. Assume that at some period t, $d_i'^t \neq h_i^t(p_i'^t, \bar{p}_{-i}^t, \xi_i^t)$ for all $\xi_i^t \in \mathcal{U}_i^t$. There are two possible cases:

1. For some $t \in \mathbf{T}$, $d_i'^t < h_i^t(p_i'^t, \bar{p}_{-i}^t, \xi_i^t)$ for all $\xi_i^t \in \mathcal{U}_i^t$. The properties of the demand function (see Condition 4.3.4) imply that $p_i'^t$ can be increased by δ^t while maintaining $d_i'^t < h_i^t(p_i'^t + \delta^t, \bar{p}_{-i}^t, \xi_i^t)$ for all $\xi_i^t \in \mathcal{U}_i^t$. Note that since $h_i^t(p_i'^t + \delta^t, \bar{p}_{-i}^t, \xi_i^t) > d_i'^t > 0$, we can find a $\delta^t > 0$ such that $p_i'^t + \delta^t < p_{\max}^t$. Variational inequality problem (4.2) states that we want to find $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i') \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ such that

$$-\nabla J_i(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i') \cdot \begin{pmatrix} \mathbf{p}_i - \mathbf{p}_i' \\ \mathbf{d}_i - \mathbf{d}_i' \\ \mathbf{D}_i - \mathbf{D}_i' \end{pmatrix} \geq 0 \ \ \forall \ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$$

   We will choose a $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ such that the above condition is violated and hence prove that $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i')$ could not be a solution to the variational inequality. For any $t \in \mathbf{T}$, choose

$$p_i^t = \begin{cases} p_i'^t & \text{if } d_i'^t = h_i^t(p_i'^t, \bar{p}_{-i}^t, \xi_i^t) \text{ for some } \xi_i^t \in \mathcal{U}_i^t \\ p_i'^t + \delta^t & \text{if } d_i'^t < h_i^t(p_i'^t, \bar{p}_{-i}^t, \xi_i^t) \text{ for all } \xi_i^t \in \mathcal{U}_i^t \end{cases} \tag{4.4}$$

   and $\mathbf{d}_i = \mathbf{d}'_i$, $\mathbf{D}_i = \mathbf{D}'_i$. Considering the above point $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i)$ in variational inequality problem (4.2) implies that $\sum \delta^t d_i^t \leq 0$. Since $\sum \delta^t d_i^t > 0$, this is a contradiction. Thus, $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i')$ could not be a solution to variational inequality problem (4.2).

2. For some $t \in \mathbf{T}$, $d_i'^t > h_i^t(p_i'^t, \bar{p}_{-i}^t)$. Notice that $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i')$ is infeasible and could not be a solution to variational inequality problem (4.2).

Both cases lead to a contradiction. Thus, $d_i'^t = h_i^t(p_i'^t, \bar{p}_{-i}^t, \xi_i^t)$, for some $\xi_i^t \in \mathcal{U}_i^t$. □

Proposition 4.3.1 proves the existence of a solution to the best response optimization problem (4.1). This shows that given the competitors policies, there is a policy that maximizes the revenue for seller i.

Proposition 4.3.1. *For any fixed* $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$, *there exists a solution* $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i')$ *to best response optimization problem (4.1).*

Proof. It is easy to show that the feasible space is non-empty and compact and the objective function is continuous. Under these conditions the result follows from the well known Weierstrass theorem (See Bazaraa, Sherali and Shetty [1]). □

Having shown that there exists an optimal policy for the best response optimization problem (4.1), Proposition 4.3.2 proves that this optimal policy is also a solution to variational inequality problem (4.2).

Proposition 4.3.2. *Given the competitor policies* $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$, *let* $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i')$ *be a solution to best response optimization problem (4.1). Under Condition 4.3.3,* $(\mathbf{p}_i', \mathbf{d}_i', \mathbf{D}_i')$ *also solves variational inequality problem (4.2).*

Proof. Under Condition (4.3.3), the feasible space for best response optimization problem (4.1) is a convex and closed set. Moreover, the objective function is continuously differentiable. Consider any point $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$. The convexity of the feasible space $\mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ ensures that the point $\mathbf{z}_i' + \theta \cdot (\mathbf{z}_i - \mathbf{z}_i')$ also belongs to the feasible space for all $\theta \in [0,1]$. We define function $\zeta(\theta) = -J_i(\mathbf{z}_i' + \theta \cdot (\mathbf{z}_i - \mathbf{z}_i'))$, for $\theta \in [0,1]$. Function $\zeta(\theta)$ achieves its minimum at $\theta = 0$. It follows that $\xi'(0) = -\nabla J_i(\mathbf{z}_i') \cdot (\mathbf{z}_i - \mathbf{z}_i') \geq 0$. It follows that if the feasible space is a closed and convex set, and the objective function is continuously differentiable, the solution to best response optimization problem (4.1) solves variational inequality problem (4.2). □

In Proposition 4.3.3 we prove the converse of Proposition 4.3.2 by showing that any solution to variational inequality problem (4.2) also solves best response optimization problem (4.1).

Proposition 4.3.3. *Under Condition 4.3.4, for any fixed $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$, any solution to the variational inequality problem (4.2) solves the best response optimization problem (4.1).*

Proof. The policy $(\mathbf{p}'_{-i}, \mathbf{d}'_{-i}, \mathbf{D}'_{-i})$ that solves variational inequality problem (4.2) satisfies

$$-\sum_t p_i^{t'}(d_i^t - d_i^{t'}) - \sum_t d_i^{t'}(p_i^t - p_i^{t'}) \geq 0 \quad \forall\ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i}). \quad (4.5)$$

From Condition 4.3.4 it follows that for any $\xi_i^t \in \mathcal{U}_i^t$

$$\left(-h_i^t(\hat{p}_i^t, p_{-i}^{t'}, \xi_i^t) + h_i^t(\check{p}_i^t, p_{-i}^{t'}, \xi_i^t)\right) \cdot (\hat{p}_i^t - \check{p}_i^t) \geq 0, \qquad \forall\ (\hat{p}_i^{t'}, \check{p}_i^t).$$

For a given p_i^t and $p_i^{t'}$, we pick ξ_i^t and $\xi_i^{t'}$ as follows:

$$\xi_i^t = \arg\min_{\xi_i^t \in \mathcal{U}_i^t} h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t) \quad (4.6)$$

$$\xi_i^{t'} = \arg\min_{\xi_i^t \in \mathcal{U}_i^t} h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^t) \quad (4.7)$$

From Condition 4.3.4 it follows that for any $\xi_i^{t'}$ and $\xi_i^t \in \mathcal{U}_i^t$

$$\left(-h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^t) + h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t)\right) \cdot (p_i^{t'} - p_i^t) \geq 0 \qquad \forall\ (p_i^{t'}, p_i^t) \quad (4.8)$$

$$\left(-h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^{t'}) + h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^{t'})\right) \cdot (p_i^t - p_i^{t'}) \geq 0 \qquad \forall\ (p_i^{t'}, p_i^t) \quad (4.9)$$

Consider two cases:

1. If $p_i^{t'} \geq p_i^t$,

$$h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t) \geq h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^t) \geq h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^{t'})$$

   The first inequality follows from (4.8) while the second inequality follows from (4.7). This implies that

$$\left(-h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^{t'}) + h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t)\right) \cdot (p_i^{t'} - p_i^t) \geq 0$$

2. If $p_i^t \geq p_i^{t'}$,

$$h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^{t'}) \geq h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^{t'}) \geq h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t).$$

The first inequality follows from (4.9) while the second inequality follows from (4.6). This also implies that

$$\left(-h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^{t'}) + h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t)\right) \cdot (p_i^{t'} - p_i^t) \geq 0.$$

In both cases,

$$\left(-h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^{t'}) + h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t)\right) \cdot (p_i^{t'} - p_i^t) \geq 0. \tag{4.10}$$

From Lemma 4.3.1, $d_i^{t'} = h_i^t(p_i^{t'}, p_{-i}^{t'}, \xi_i^{t'})$. Therefore,

$$\left(-d_i^{t'} + h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t)\right) \cdot (p_i^{t'} - p_i^t) \geq 0 \qquad \forall\ (p_i^{t'}, p_i^t).$$

Summing over all t

$$\sum_t \left(-d_i^{t'} + h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t)\right) \cdot (p_i^{t'} - p_i^t) \geq 0 \qquad \forall\ (p_i^{t'}, p_i^t) \tag{4.11}$$

Adding (4.5) and (4.11) we get that for all $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$,

$$\sum_t \left(p_i^{t'} d_i^{t'} - p_i^t h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t) + p_i^{t'} \left(h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t) - d_i^t\right)\right) \ \geq \ 0 \tag{4.12}$$

Note that for every $\mathbf{d}_i$ such that $\sum_t d_i^t \leq C_i$, there exist corresponding $\hat{\mathbf{D}}_i(\mathbf{d}_i)$ and $\hat{\mathbf{p}}_i(\mathbf{d}_i)$ given by

$$\begin{aligned}
\hat{D}_i^t(\mathbf{d}_i) &= C_i - \sum_{\tau=1}^{t} d_i^\tau, \text{ and} \\
\hat{p}_i^t(d_i^t) &= \max\left\{p_i^t \mid d_i^t \leq h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t)\right\}
\end{aligned}$$

respectively, for all $t \in \mathbf{T}$. It is easy to show that $d_i^t = h_i^t(\hat{p}_i^t(d_i^t), p_{-i}^{t'}, \xi_i^t)$ using arguments similar to those used in Lemma 4.3.1. By construction, $(\hat{\mathbf{p}}_i(\mathbf{d}_i), \mathbf{d}_i, \hat{\mathbf{D}}_i(\mathbf{d}_i)) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$. Substituting in (4.12), we get that

$$\sum_t \left(p_i^{t'} d_i^{t'} - \hat{p}_i^t(d_i^t)\ d_i^t\right) \ \geq \ 0$$

With a rearrangement of terms we get

$$\sum_t p_i^{t'} d_i^{t'} \geq \sum_t \hat{p}_i^t(d_i^t)\ d_i^t$$

Since $\hat{p}_i^t(d_i^t) \geq p_i^t$ for any feasible $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ by construction, it is easy to see that

$$\sum_t p_i^{t'} d_i^{t'} \geq \sum_t p_i^t d_i^t \quad \forall\ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$$

The variational inequality solution is thus an optimal policy for seller i and solves best response optimization problem (4.1). □

First in Proposition 4.3.4 we prove that there is a unique solution to the variational inequality problem (4.2) and hence best response optimization problem (4.1).

Proposition 4.3.4. *Under Condition 4.3.5, for any fixed $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$, any solution to the variational inequality problem (4.2) is a unique solution to the best response optimization problem (4.1).*

Proof. Under Condition (4.3.5), (4.11) in the proof of Proposition 4.3.3 becomes a strict inequality:

$$\sum_t \left(-d_i^{t'} + h_i^t(p_i^t, p_{-i}^{t'}, \xi_i^t) \right) \cdot (p_i^{t'} - p_i^t) > 0 \qquad \forall (p_i^{t'}, p_i^t),\ p_i^{t'} \neq p_i^t$$

and adding (4.5) and (4.11) we get that $\forall\ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\mathbf{p}'_{-i}, \mathbf{d}'_{-i}, \mathbf{D}'_{-i})$

$$\sum_t p_i^{t'} d_i^{t'} > \sum_t p_i^t d_i^t.$$

Thus, $(\mathbf{p}'_i, \mathbf{d}'_i, \mathbf{D}'_i)$ is the unique solution to the best response optimization problem (4.1). It is easy to show using results from Proposition 4.3.2 and Proposition 4.3.3 that the variational inequality also has a unique solution. □

4.3.3 Market equilibrium problem

We now formulate the market equilibrium problem using a quasi-variational inequality formulation. We combine variational inequality problems (4.2) for each seller $i \in \mathbf{I}$. We define the following feasible space for all sellers:

$$\mathcal{K} \ = \ \{\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_I) \mid \mathbf{z}_i \in \mathcal{K}_i(\mathbf{z}_{-i}),\ \forall i \in \mathbf{I}\}.$$

The joint quasi-variational inequality problem seeks to find a point $(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$ such that

$$\mathbf{F}(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*) \cdot \begin{pmatrix} \mathbf{p} - \mathbf{p}^* \\ \mathbf{d} - \mathbf{d}^* \\ \mathbf{D} - \mathbf{D}^* \end{pmatrix} \geq 0 \qquad \forall\ (\mathbf{p}, \mathbf{d}, \mathbf{D}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*), \qquad (4.13)$$

where $F_i(\mathbf{z}^*) = -\nabla J_i(\mathbf{z}_i^*)$, $\forall i \in \mathbf{I}$.

Proposition 4.3.5. *The policy arising from the joint quasi-variational inequality problem (4.13) and the policy arising from the solution of variational inequality problems (4.2) for each seller i simultaneously are the same.*

Proof. It is easy to show that a policy $(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$ that solves variational inequality problem (4.2), for each $i \in \mathbf{I}$ simultaneously, also solves the joint quasi-variational inequality problem (4.13). We will now show the converse: the solution to joint quasi-variational inequality problem (4.13) solves variational inequality problems (4.2) for each seller i simultaneously. That is, if $(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$ is a solution to joint quasi-variational inequality problem (4.13), then for each $i \in \mathbf{I}$, $(\mathbf{p}_i^*, \mathbf{d}_i^*, \mathbf{D}_i^*)$ solves variational inequality problem (4.2) with the competitor policy $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ given by $(\mathbf{p}_{-i}^*, \mathbf{d}_{-i}^*, \mathbf{D}_{-i}^*)$.

The joint quasi-variational inequality problem (4.13) can be rewritten as the following: Find $(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*) \in \mathcal{K}$ so that

$$-\sum_{i,t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{i,t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0 \quad \forall\ (\mathbf{p}, \mathbf{d}, \mathbf{D}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$$

Consider an individual seller i. Consider the policy $(\mathbf{p}, \mathbf{d}, \mathbf{D}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$ that has $p_{-i}^t = p_{-i}^{t*}$, $d_{-i}^t = d_{-i}^{t*}$ and $D_{-i}^t = D_{-i}^{t*}$ for all her competitors and general $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\mathbf{p}_{-i}^*, \mathbf{d}_{-i}^*, \mathbf{D}_{-i}^*)$ then,

$$-\sum_{t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0 \quad \forall\ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\mathbf{p}_{-i}^*, \mathbf{d}_{-i}^*, \mathbf{D}_{-i}^*)$$

This is the same as variational inequality problem (4.2). Repeating the argument for every $i \in \mathbf{I}$, it is easy to show that $(\mathbf{p}_i^*, \mathbf{d}_i^*, \mathbf{D}_i^*)$ solves variational inequality problem (4.2) for every seller when the corresponding competitor policy is given to be $(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i}) = (\mathbf{p}_{-i}^*, \mathbf{d}_{-i}^*, \mathbf{D}_{-i}^*)$. □

Lemma 4.3.2 proves that any solution to the joint quasi-variational inequality problem (4.13) satisfies a relation similar to that described in Lemma 4.3.1.

Lemma 4.3.2. *Let $(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$ be a solution to joint quasi-variational inequality problem (4.13). $(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$ satisfies the following relation:*

$$d_i^{t*} = h_i^t(p_i^{t*}, p_{-i}^{t*}, \xi_i^{t*}) \quad \forall\ i \in \mathbf{I},\ t \in \mathbf{T}$$

Proof. Proposition 4.3.5 shows that for every $i \in \mathbf{I}$, $(\mathbf{p}_i^*, \mathbf{d}_i^*, \mathbf{D}_i^*)$ solves the corresponding variational inequality problem (4.2). Lemma 4.3.1 shows that $(\mathbf{p}_i^*, \mathbf{d}_i^*, \mathbf{D}_i^*)$ follows the relation

$$d_i^{t*} = h_i^t(p_i^{t*}, p_{-i}^{t*}, \xi_i^{t*}) \quad \forall\ t \in \mathbf{T}$$

for some $\xi_i^{t*} \in \mathcal{U}_i^t$. The argument can be repeated for every $i \in \mathbf{I}$ and the result follows. □

Proposition 4.3.6 shows that the solution to the joint quasi-variational inequality problem (4.13) is an optimal policy for each seller i and hence defines a Nash equilibrium policy.

Proposition 4.3.6. *Under Condition 4.3.4 and Lemma 4.3.2, the policy that solves the joint quasi-variational inequality problem (4.13) is also a Nash equilibrium policy for each individual seller and optimizes her revenues given that her competitors do the same.*

Proof. The policy that solves variational inequality problem (4.13) satisfies the following condition,

$$-\sum_{i,t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{i,t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0 \quad \forall\ (\mathbf{p}, \mathbf{d}, \mathbf{D}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*).$$

Consider an individual seller i. Since the above holds for all feasible $(\mathbf{p}, \mathbf{d}, \mathbf{D})$. Consider the policy $(\mathbf{p}, \mathbf{d}, \mathbf{D}) \in \mathcal{K}(\mathbf{p}^*, \mathbf{d}^*, \mathbf{D}^*)$: $p_{-i}^t = p_{-i}^{t*}$, $d_{-i}^t = d_{-i}^{t*}$ and $D_{-i}^t = D_{-i}^{t*}$ for all her competitors and any $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\mathbf{p}_{-i}^*, \mathbf{d}_{-i}^*, \mathbf{D}_{-i}^*)$. For this seller,

$$-\sum_{t} p_i^{t*}(d_i^t - d_i^{t*}) - \sum_{t} d_i^{t*}(p_i^t - p_i^{t*}) \geq 0 \tag{4.14}$$

for all $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\mathbf{p}_{-i}^*, \mathbf{d}_{-i}^*, \mathbf{D}_{-i}^*)$. For a given p_i^t and p_i^{t*}, we pick the corresponding uncertainty parameters ξ_i^t and ξ_i^{t*} as follows:

$$\begin{aligned} \xi_i^t &= \arg\min_{\xi_i^t \in \mathcal{U}_i^t} h_i^t(p_i^t, p_{-i}^{t*}, \xi_i^t) \\ \xi_i^{t*} &= \arg\min_{\xi_i^t \in \mathcal{U}_i^t} h_i^t(p_i^{t*}, p_{-i}^{t*}, \xi_i^t). \end{aligned}$$

Under Condition 4.3.4 it can be shown, in a way similar to (4.10), that

$$\left(-h_i^t(p_i^{t*}, p_{-i}^{t*}, \xi_i^{t*}) + h_i^t(p_i^t, p_{-i}^{t*}, \xi_i^t)\right) \cdot (p_i^{t*} - p_i^t) \geq 0, \qquad \forall\ (p_i^{t*}, p_i^t).$$

From Lemma 4.3.2 it follows that $d_i^{t*} = h_i^t(p_i^{t*}, p_{-i}^{t*}, \xi_i^{t*})$.

$$\left(-d_i^{t*} + h_i^t(p_i^t, p_{-i}^{t*}, \xi_i^t)\right) \cdot (p_i^{t*} - p_i^t) \geq 0, \qquad \forall\ (p_i^{t*}, p_i^t).$$

Summing over all t

$$\sum_t \left(-d_i^{t*} + h_i^t(p_i^t, p_{-i}^{t*}, \xi_i^t)\right) \cdot (p_i^{t*} - p_i^t) \geq 0, \qquad \forall\ (p_i^{t*}, p_i^t). \tag{4.15}$$

Adding (4.14) and (4.15) we get that $\forall\ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\mathbf{p}_{-i}^*, \mathbf{d}_{-i}^*, \mathbf{D}_{-i}^*)$

$$\sum_t \left(p_i^{t*} d_i^{t*} - p_i^t h_i^t(p_i^t, p_{-i}^{t*}, \xi_i^t) + p_i^{t*}\left(h_i^t(p_i^t, p_{-i}^{t*}, \xi_i^t) - d_i^t\right)\right) \geq 0 \tag{4.16}$$

Note that for every $\mathbf{d}_i$ such that $\sum_t d_i^t \leq C_i$, there exist corresponding $\hat{\mathbf{D}}_i(\mathbf{d}_i)$ and $\hat{\mathbf{p}}_i(\mathbf{d}_i)$ given by

$$\begin{aligned} \hat{D}_i^t(\mathbf{d}_i) &= C_i - \sum_{\tau=1}^{t} d_i^\tau, \text{ and} \\ \hat{p}_i^t(d_i^t) &= \max\left\{p_i^t \mid d_i^t \leq h_i^t(p_i^t, p_{-i}^{t*}, \xi_i^t)\right\} \end{aligned}$$

respectively, for all $t \in \mathbf{T}$. It is easy to show that $d_i^t = h_i^t(\hat{p}_i^t(d_i^t), p_{-i}^{t*}, \xi_i^t)$ using arguments similar to those in Lemma 4.3.2. By construction, $(\hat{\mathbf{p}}_i(\mathbf{d}_i), \mathbf{d}_i, \hat{\mathbf{D}}_i(\mathbf{d}_i)) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$. Substituting in (4.16), we get that

$$\sum_t \left(p_i^{t*} d_i^{t*} - \hat{p}_i^t(d_i^t)\ d_i^t\right) \ \geq \ 0.$$

With a rearrangement of terms we obtain that

$$\sum_t p_i^{t*} d_i^{t*} \geq \sum_t \hat{p}_i^t(d_i^t)\ d_i^t$$

Since $\hat{p}_i^t(d_i^t) \geq p_i^t$ for any feasible $(\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$ by construction, it is easy to see that

$$\sum_t p_i^{t*} d_i^{t*} \geq \sum_t p_i^t d_i^t \quad \forall\ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}, \bar{\mathbf{d}}_{-i}, \bar{\mathbf{D}}_{-i})$$

The variational inequality solution is thus an optimal policy for seller i and solves the best response optimization problem (4.1). Repeating the argument for each seller $i \in \mathbf{I}$, the variational inequality solution is a Nash equilibrium policy for each individual seller and the result follows. □

Theorem 4.3.1 illustrates that a solution to the joint quasi-variational inequality problem (4.13) exists.

Theorem 4.3.1. *Under Condition 4.3.3, a solution to the joint quasi-variational inequality problem (4.13) exists. As a result, a market equilibrium exists.*

Proof. Notice that the feasible region in the joint quasi-variational inequality problem (4.13) is not empty for any feasible $(\mathbf{p}, \mathbf{d}, \mathbf{D})$. To prove that, consider the point $(\dot{\mathbf{p}}, \dot{\mathbf{d}}, \dot{\mathbf{D}})$ where for all $i \in \mathbf{I}$ and $t \in \mathbf{T}$, $(\dot{p}_i^t, \dot{d}_i^t, \dot{D}_i^t) = (p_{\min}, d_{\min}, C_i - tD_{\min})$. It is easy to show that $(\dot{\mathbf{p}}, \dot{\mathbf{d}}, \dot{\mathbf{D}})$ lies in the feasible space $\mathcal{K}(\mathbf{p}, \mathbf{d}, \mathbf{D})$. $\mathcal{K}(\mathbf{p}, \mathbf{d}, \mathbf{D})$ is thus non-empty. Condition 4.3.3 implies that the feasible space $\mathcal{K}(\mathbf{p}, \mathbf{d}, \mathbf{D})$ is a convex and compact set. We construct the set T required in Theorem 3.3.1 as

$$T = \left\{ (\mathbf{p}_i, \mathbf{d}_i, \mathbf{D}_i) \ \middle|\ \begin{array}{ll} d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t, \xi_i^t) & \forall \xi_i^t \in \mathcal{U}_i^t,\ \forall t \in \mathbf{T},\ \forall i \in \mathbf{I} \\ \sum_{\tau=1}^t d_i^\tau \leq C_i - D_i^t & \forall t \in \mathbf{T},\ \forall i \in \mathbf{I} \\ p_{i,\ \mathtt{min}}^t \leq p_i^t \leq p_{i,\ \mathtt{max}}^t & \forall t \in \mathbf{T},\ \forall i \in \mathbf{I} \\ d_i^t \geq d_{i,\ \mathtt{min}}^t & \forall t \in \mathbf{T},\ \forall i \in \mathbf{I} \end{array} \right\}$$

This satisfies condition **(a)** of Theorem 3.3.1. It is also easy to show that **(b)** is true. Furthermore, the quasi-variational inequality function $\mathbf{F}$ is continuous. Thus, a solution to the joint quasi-variational inequality problem exists. □

Chapter 5

Stochastic demand model

In this chapter we will assume that the demand is a stochastic function of the prices and the sellers adopt policies arising from optimizing expected payoffs. The analysis of the market equilibrium problem in this case is more difficult than the deterministic demand and the robust demand case. The primary reason is that the bilevel formulation of the best response problem for the stochastic demand case cannot be reduced to a single level optimization problem without any simplifying approximations. In this chapter we discuss results for some stochastic models with some additional approximations.

5.1 Analysis

For the stochastic demand model, we consider a relaxation of the inventory constraint and establish results in the linear demand case. In particular, we replace the inventory constraint with a penalty function. For this purpose, we allow sellers to sell the product in excess of their inventory levels and charge a per-unit penalty cost (K_i), for the excess sale (overbooking).

The demand is modelled as a linear function of the price levels involving the uncertainty parameter as follows: for all $t \in \mathbf{T}$,

$$h_i^t(p_i^t, p_{-i}^t, \xi^t) = \theta_i^t(p_i^t, p_{-i}^t) + \xi^t \mu_i^t(p_i^t, p_{-i}^t)$$

where the functions θ and μ are defined as

$$\begin{aligned} \mu_i^t(p_i^t, p_{-i}^t) &= \hat{d}_i^t - \beta_{ii}^t p_i^t + \sum_{j \in \mathbf{I},\ j \neq i} \beta_{ij}^t p_j^t \\ \theta_i^t(p_i^t, p_{-i}^t) &= \check{d}_i^t - \gamma_{ii}^t p_i^t + \sum_{j \in \mathbf{I},\ j \neq i} \gamma_{ij}^t p_j^t. \end{aligned}$$

We will assume that demand follows some basic rules. These allow us to characterize the demand function in every period t:

- The demand seen by each seller is non-increasing in the price level set by that seller. Thus,

$$\beta_{ii}^t, \gamma_{ii}^t \geq 0.$$

- The demand seen by each seller is non-decreasing in the price level set by any of that seller's competitor. Thus,

$$\beta^t_{ij}, \gamma^t_{ij} \geq 0.$$

- If the sellers set the same price level and increase (or decrease) this level simultaneously, the corresponding demand seen by each seller is non-increasing (or non-decreasing) with respect to this price level. That is,

$$\beta^t_{ii} \geq \sum_{j \neq i} \beta^t_{ij}$$

and

$$\gamma^t_{ii} \geq \sum_{j \neq i} \gamma^t_{ij}.$$

- The total demand should be non-increasing in any seller's price level. Thus,

$$\beta^t_{ii} \geq \sum_{j \neq i} \beta^t_{ji}$$

and

$$\gamma^t_{ii} \geq \sum_{j \neq i} \gamma^t_{ji}.$$

We essentially require that the demand function $h^t_i(p^t_i, \bar{p}^t_{-i}, \xi)$ increases with respect to $\bar{p}^t_{-i}$ (i.e. $\bar{p}^t_j$ for all $j \neq i$). This states that the demand for any seller's product increases with respect to the price of any other seller. This is a standard condition for substitute products. We also require that decreasing the price of any product results in a greater increase in the demand for that product for lower levels of the price of any other product; that is, the demand for any product is more sensitive to its price when any other product is more competitive by virtue of its lower price.

Under this structure, the best response problem can be formulated as the following convex optimization problem.

$$\begin{array}{lll} \max\limits_{(\mathbf{p}_i)} & \sum_{\xi \in \mathcal{U}} f_\xi \left[\sum_{t=1}^{T} p^t_i d^t_{\xi i} + \min(0, -K_i(\sum_t d^t_{\xi i} - C_i)) \right] & \\ \text{such that} & p^t_{i\min} \leq p^t_i \leq p^t_{i\max} & \forall\, t \in \mathbf{T} \end{array}$$

where $d^t_{\xi i} = h^t_i(p^t_i, \bar{p}^t_{-i}, \xi), \forall\, t \in \mathbf{T},\ \xi \in \mathcal{U}$. The objective function is a convex function of $\mathbf{p}_i$ for any given $\mathbf{p}_{-i}$. The strategy space is a compact and convex set. Using the following theorem attributed to Fan [32], Debreu [24] and Glicksberg [38], it can be shown that a pure strategy Nash equilibrium exists.

Theorem 5.1.1 (Fan, Debreu, Glicksberg). *Suppose that for each player the strategy space is compact and convex and the payoff function is continuous and quasi-concave with respect to each players own strategy. Then there exists at least one pure strategy NE in the game.*

A typical method for showing uniqueness of equilibrium in a game with smooth, twice differentiable payoff functions, is to use the following theorem attributed to Gale and Nikaido [36].

Theorem 5.1.2 (Gale and Nikaido). *Suppose the strategy space of the game is convex and all equilibria are interior. Then if the determinant $|H|$ is negative quasi-definite (i.e., if the matrix $H + H^T$ is negative definite) on the players strategy set, there is a unique NE.*

Note that for our game, the payoff functions are not differentiable since the objective of the best response problem involves the minimization of two functions. In this case, the same result can be shown by considering a slightly perturbed payoff function that is smooth and twice differentiable. We show that the Hessian H of the perturbed payoff function is indeed negative quasi-definite in the limit and arrive at conditions under which a unique equilibrium exists. The basic outline of the process is as follows.

Consider a function $h(x,y) = min(f(x,y), g(x,y))$ defined over $(x,y) \in \mathbf{S}$. Let $f(x,y)$ and $g(x,y)$ be smooth concave functions over $\mathbf{S}$. Clearly $h(x,y)$ is also a concave function that might not be double differentiable everywhere on $\mathbf{S}$. We consider a perturbation of $h(x,y)$ for every $\epsilon \geq 0$ as follows:

$$h_\epsilon = \frac{1}{2}\left(f + g - \sqrt{f^2 + g^2 - 2(1-\epsilon)fg}\right).$$

Note that $h_\epsilon(x,y) = h(x,y)$ at $\epsilon = 0$ and $h_\epsilon(x,y)$ is a smooth function for $\epsilon > 0$. We only need to consider the function at points of potential non-differentiability, i.e., at points where $f(x,y) = g(x,y)$. Using the Hessian matrix of the smooth function, it can be shown that a unique equilibrium exists under the conditions given above. We do not go deeper into the analysis since Topkis [65] gives results for a similar model. He does not require the demand function to be linear in the prices. Instead of using the concavity of the objective function, he shows that the payoffs are supermodular under certain conditions. See Section 4.4.1, Page 196 in Topkis [65] for more details.

Note that the above formulation does not require the use of the allocation variables. In that sense, we have moved away from the fixed-inventory model and used an approximation that is closer to the periodic production-review model.

There is another approximation we will introduce next in order to study a dynamic pricing model with competition and stochastic demand. It incorporates strict allocations for each period. We consider the special case of a multiplicative demand model and strict period-wise allocations. Before we introduce that approximate model for the multi-period problem, we consider very briefly the single period problem: For the single period problem, assuming that the inventory allocated for sale in the period is given by D_i, the revenue given price levels is

$$\pi_i(p_i, D_i, p_{-i}, D_{-i}, \xi) = p_i \min(h_i(p_i, p_{-i}, \xi), D_i)$$

where

$$h_i(p_i, p_{-i}, \xi) = \xi\mu(p_i, p_{-i}).$$

Taking the expectation over the uncertain parameters, we get that

$$\begin{aligned}\pi_i(p_i, D_i, p_{-i}, D_{-i}) &= E_\xi[\pi_i(p_i, D_i, p_{-i}, D_{-i}, \xi)] \\ &= \sum_\xi f(\xi)\pi_i(p_i, D_i, p_{-i}, D_{-i}, \xi).\end{aligned}$$

Using a result by Zabel [70], it can be shown that for $D_i > 0$, the optimal price

$$p_i^*(D_i) = \text{argmax}_{p_i} \pi_i(p_i, D_i, p_{-i}, D_{-i}),$$

is a continuously differentiable function of D_i and is the unique solution of the equation

$$E[\min\{\frac{D_i}{\mu(p_i)}, \xi\}] = \frac{-p_i\mu'(p_i)}{\mu(p_i)} \int_0^{\frac{D_i}{\mu(p_i)}} \xi d\phi(\xi).$$

Using the above result, we define

$$\pi_i(D_i, p_{-i}, D_{-i}) = \pi_i(p_i^*(D_i), D_i, p_{-i}, D_{-i}).$$

A result by Young [69] shows that if $log(f(\xi))$ is concave or ξ is Lognormal-distrbuted, then $\pi_i(D_i, p_{-i}, D_{-i})$ is concave and $p_i^*(D_i)$ is non-increasing in D_i. This condition is satisfied for the cases where ξ follows a uniform, beta, gamma, or weibull distribution. In this chapter we extend the result to the multi-period problem. First, we note that the payoff is given by

$$\pi_i(p_i, D_i, p_{-i}, D_{-i}, \xi) = \sum_t p_i^t \min(h_i^t(p_i^t, p_{-i}^t, \xi^t), D_i^t)$$

Therefore the best response problem for Seller i can be written as

$$\begin{aligned}\pi_i(p_i, D_i, p_{-i}, D_{-i}) &= E_\xi[\pi_i(p_i, D_i, p_{-i}, D_{-i}, \xi)] \\ &= \sum_\xi f(\xi)\pi_i(p_i, D_i, p_{-i}, D_{-i}, \xi)\end{aligned}$$

$$\begin{aligned}\sum_t D_i^t &\leq C_i \\ D_i^t &\geq 0 \\ p_i^t &\in [p_{min}, p_{max}]\end{aligned}$$

We note that this is an approximation since it does not allow unsold inventory from one period to be used in the future time periods. This would be a good approximation if prices are increasing over time; e.g. airline fare pricing since the allocations will be made such that probability of unsold inventory being sold in future periods is very low. This would be a bad approximation if prices are decreasing over time; e.g. retail markdown modelling.

Using the result from Young [69] mentioned above, we define the objective of the best response problem each seller solves as

$$\begin{aligned}
\pi_i(D_i, p_{-i}, D_{-i}, \xi) &= \sum_t p_i^{t*}(D_i^t) \min(d_i^{t*}, D_i^t) \\
\pi_i(D_i, p_{-i}, D_{-i}) &= E_\xi[\pi_i(D_i, p_{-i}, D_{-i}, \xi)] \\
&= \sum_\xi f(\xi)\pi_i(D_i, p_{-i}, D_{-i}, \xi).
\end{aligned}$$

We can finally show that $\pi_i(D_i, p_{-i}, D_{-i})$ is concave and continuously differentiable in D_i^t under previous assumptions. Transforming back to the price variables, we get that the $\pi_i(p_i, D_i^*(p_i), p_{-i}, D_{-i})$ is quasi-concave in p_i. The strategy space is compact and convex. Using Theorem 5.1.1 from Fan, Debreu, Glicksberg [32, 24, 38], it follows that a pure strategy Nash equilibrium exists.

Chapter 6

Comparison of user optimal and system optimal

As discussed in the literature survey in Chapter 1, there has been a lot of work recently in various fields comparing the system optimal costs and user optimal costs. That is, quantifying the loss in efficiency in a decentralized vs. a centralized system. Traditionally, the results derived in literature (See, for example, Perakis [53]) consider a centralized/de-centralized setting where cost is minimized. In the problem of this thesis, we consider a profit maximization problem problem. The analysis in literature does not directly extend. The primary reason is that the problems considered in the literature are cost-minimization problems and the conditions required for the objective functions do not hold for our problem. Also, unlike the setup considered in the literature, the competitors in an oligopoly are not infinitesimal. In this section we give some results for two competitive settings, a quantity competition model and a price competition model.

6.1 Quantity competition model

In this section, we consider a multi-period quantity competition model. As before, there is a set of sellers denoted by **I** competing through the sale of a single product over several time periods denoted by **T**. Each seller has an inventory, C_i, of the product and competes by setting quantities for sale at every period. Competing through quantity competition, it is different from price competition as we studied in previous chapters. For example, the resulting equilibrium policies are different from the ones obtained in a price competition model. We illustrate this with an example in Section 6.3. The total quantity of the product for sale in a period defines a market clearing price and the sellers sell the amount they allocate at that price. In this section, we consider a linear function for the market clearing price.

Let the market clearing price in period t be defined as

$$p^t(d_i^t, d_{-i}^t) = D^t - \beta_{ii}^t d_i^t - \sum_{j \neq i} \beta_{ij}^t d_j^t.$$

This kind of a structure can be used to model a market-clearing price function similar to that used by Murphy, Sherali, and Soyster [49] and Harker [39] in the single period setting with linear demand. That is,

$$p^t(Q^t) = D^t - \beta^t(Q^t),$$

where $Q^t = \sum_i d_i^t$ is the total quantity for sale in period t and $D^t, \beta_t \geq 0$ for all t. The best response problem is the decentralized (user optimal) problem each seller solves:

$$\begin{array}{lll} \max\limits_{\mathbf{d}_i} & \sum_t d_i^t p^t(Q_{-i}^t + d_i^t) & \\ \text{such that,} & \sum_t d_i^t \leq C_i & \\ & d_i^t \geq 0 & \forall t \in \mathbf{T}, \end{array}$$

where $Q_{-i}^t = \sum_{j \neq i} d_j^t$. For this linear market clearing price function, the optimization problem can be written as

$$\begin{array}{lll} \max\limits_{\mathbf{d}_i} & \sum_t d_i^t(D^t - \sum_{j \in \mathbf{I}} \beta_{ij}^t d_j^t) & \\ \text{such that,} & \sum_t d_i^t \leq C_i & \\ & d_i^t \geq 0 & \forall t \in \mathbf{T}. \end{array}$$

This can be cast as the following variational inequality problem for every seller i. Given $\bar{\mathbf{d}}_{-i}$ find $\mathbf{d}_i^* \in \mathcal{K}_i$ such that

$$\sum_t (-D^t + 2\beta_{ii}^t d_i^{t*} + \sum_{j \neq i} \beta_{ij}^t \bar{d}_j^t)(d_i^t - d_i^{t*}) \quad \geq \quad 0 \tag{6.1}$$

for all $\mathbf{d}_i \in \mathcal{K}_i$. The feasible set is defined as

$$\mathcal{K}_i = \left\{ \mathbf{d}_i \mid \sum_t d_i^t \leq C_i, \ d_i^t \geq 0 \ \forall t \in \mathbf{T} \right\}.$$

Considering the market equilibrium problem for all sellers gives rise to the following variational inequality: Find $\mathbf{d}^{\text{UO}} \in \mathcal{K}$ such that

$$\sum_i \sum_t (-D^t + 2\beta_{ii}^t d_i^{t^{\text{UO}}} + \sum_{j \neq i} \beta_{ij}^t d_j^{t^{\text{UO}}})(d_i^t - d_i^{t^{\text{UO}}}) \quad \geq \quad 0 \tag{6.2}$$

for all $\mathbf{d} \in \mathcal{K}$. The feasible set is defined as

$$\mathcal{K} = \left\{ \mathbf{d}_i \mid \sum_t d_i^t \leq C_i, \ d_i^t \geq 0 \ \forall t \in \mathbf{T}, \ \forall i \in \mathbf{I} \right\}.$$

Theorem 6.1.1. *For the multi-period quantity competition model given above,*

$$Z^{\text{UO}} - Z^{\text{SO}} + \sum_i \sum_t d_i^{t^{\text{SO}}} \sum_{j \neq i} \beta_{ij}^t (d_j^{t^{\text{UO}}} - d_j^{t^{\text{SO}}}) \quad \geq \quad 0,$$

where Z^{UO} and Z^{SO} are the user optimal and system optimal profits respectively.

Proof. Notice that the variational inequality (6.2) can be written as

$$\sum_i \sum_t (-D^t + \sum_{j \in \mathbf{I}} \beta_{ij}^t d_j^{t^{\mathrm{UO}}})(d_i^t - d_i^{t^{\mathrm{UO}}}) + \sum_i \sum_t \beta_{ii}^t d_i^{t^{\mathrm{UO}}}(d_i^t - d_i^{t^{\mathrm{UO}}}) \;\geq\; 0 \;\; \forall \mathbf{d} \in \mathcal{K}.$$

Considering the above for $\mathbf{d} = \mathbf{d}^{\mathrm{SO}}$ and noting that

$$Z^{\mathrm{UO}} = \sum_i \sum_t (D^t - \beta_{ii}^t \sum_{j \in \mathbf{I}} d_j^{t^{\mathrm{UO}}}) d_i^{t^{\mathrm{UO}}},$$

we get

$$Z^{\mathrm{UO}} + \sum_i \sum_t (-D^t + \sum_{j \in \mathbf{I}} \beta_{ij}^t d_j^{t^{\mathrm{UO}}})(d_i^{t^{\mathrm{SO}}}) + \sum_i \sum_t \beta_{ii}^t d_i^{t^{\mathrm{UO}}}(d_i^{t^{\mathrm{SO}}} - d_i^{t^{\mathrm{UO}}}) \;\geq\; 0. \tag{6.3}$$

or

$$Z^{\mathrm{UO}} + \sum_i \sum_t (-D^t + \sum_{j \in \mathbf{I}} \beta_{ij}^t d_j^{t^{\mathrm{UO}}})(d_i^{t^{\mathrm{SO}}}) + \sum_i \sum_t \beta_{ii}^t d_i^{t^{\mathrm{UO}}} d_i^{t^{\mathrm{SO}}} - \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\mathrm{UO}}})^2 \;\geq\; 0. \tag{6.4}$$

Note that for any $a_1, a_2 \geq 0$, since $(\sqrt{a_1} d_i^{t^{\mathrm{UO}}} - \sqrt{a_2} d_i^{t^{\mathrm{SO}}})^2 \geq 0$,

$$2\sqrt{a_1 a_2} d_i^{t^{\mathrm{UO}}} d_i^{t^{\mathrm{SO}}} \leq a_1 (d_i^{t^{\mathrm{UO}}})^2 + a_2 (d_i^{t^{\mathrm{SO}}})^2.$$

Using this result in inequality (6.4), we get that for $\{a_1 \geq 0,\ a_2 \geq 0,\ a_1 a_2 \geq \frac{1}{4}\}$

$$Z^{\mathrm{UO}} + \sum_i \sum_t (-D^t + \sum_{j \in \mathbf{I}} \beta_{ij}^t d_j^{t^{\mathrm{UO}}})(d_i^{t^{\mathrm{SO}}}) + a_2 \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\mathrm{SO}}})^2 + (a_1 - 1) \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\mathrm{UO}}})^2 \;\geq\; 0. \tag{6.5}$$

Also note that

$$Z^{\mathrm{SO}} = \sum_i \sum_t (D^t - \sum_{j \in \mathbf{I}} \beta_{ij}^t d_j^{t^{\mathrm{SO}}}) d_i^{t^{\mathrm{SO}}}.$$

Incorporating this in (6.5), we get

$$Z^{\mathrm{UO}} - Z^{\mathrm{SO}} + \sum_i \sum_t d_i^{t^{\mathrm{SO}}} \sum_{j \in \mathbf{I}} \beta_{ij}^t (d_j^{t^{\mathrm{UO}}} - d_j^{t^{\mathrm{SO}}}) + a_2 \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\mathrm{SO}}})^2 + (a_1 - 1) \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\mathrm{UO}}})^2 \;\geq\; 0.$$

This can be written as

$$Z^{\text{UO}} - Z^{\text{SO}} + \sum_i \sum_t d_i^{t^{\text{SO}}} \sum_{j \neq i} \beta_{ij}^t (d_j^{t^{\text{UO}}} - d_j^{t^{\text{SO}}}) + \sum_i \sum_t \beta_{ii}^t d_i^{t^{\text{SO}}} (d_i^{t^{\text{UO}}} - d_i^{t^{\text{SO}}})$$
$$+a_2 \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\text{SO}}})^2 + (a_1 - 1) \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\text{UO}}})^2 \geq 0.$$

Note again that for any $b_1, b_2 \geq 0$, since $(\sqrt{b_1} d_i^{t^{\text{UO}}} - \sqrt{b_2} d_i^{t^{\text{SO}}})^2 \geq 0$, we know that

$$2\sqrt{b_1 b_2} d_i^{t^{\text{UO}}} d_i^{t^{\text{SO}}} \leq b_1 (d_i^{t^{\text{UO}}})^2 + b_2 (d_i^{t^{\text{SO}}})^2$$

Using this result in inequality (6.4), we get that for $\{b_1 \geq 0,\ b_2 \geq 0,\ b_1 b_2 \geq \frac{1}{4}\}$,

$$Z^{\text{UO}} - Z^{\text{SO}} + \sum_i \sum_t d_i^{t^{\text{SO}}} \sum_{j \neq i} \beta_{ij}^t (d_j^{t^{\text{UO}}} - d_j^{t^{\text{SO}}})$$
$$+(a_2 + b_2 - 1) \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\text{SO}}})^2 + (a_1 + b_1 - 1) \sum_i \sum_t \beta_{ii}^t (d_i^{t^{\text{UO}}})^2 \geq 0.$$

Selecting $\{a_1, a_2, b_1, b_2\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$, we get that

$$Z^{\text{UO}} - Z^{\text{SO}} + \sum_i \sum_t d_i^{t^{\text{SO}}} \sum_{j \neq i} \beta_{ij}^t (d_j^{t^{\text{UO}}} - d_j^{t^{\text{SO}}}) \geq 0. \tag{6.6}$$

□

Note that in Theorem 6.1.1, we have not used any assumptions of symmetry of demand for all sellers or over time periods, nor have we assumed anything about the tightness of capacities. Inequality (6.6) holds very generally.

In what follows, we will derive bounds on the ratio of Z^{UO} and Z^{SO} under the assumption that the system optimal policy for a seller will be the same across all sellers (and the user optimal policy for a seller will be the same across all sellers). For example, this might hold under symmetry of demand (that is, same demand across sellers) and assuming that the capacity of each seller is the same.

Corollary 6.1.1. *If the system optimal policy for a seller is the same across all sellers, then*

$$\frac{4}{3 + \mathbf{I}} \leq \frac{Z^{\text{UO}}}{Z^{\text{SO}}} \leq 1$$

Proof. Under the assumption, $\beta_{ii}^t = \alpha^t$ and $\beta_{ij}^t = \beta^t$ for all i and j, $j \neq i$ with the condition that $\alpha^t \geq \beta^t$ for all t. Then (6.6) can be written as

$$Z^{\text{UO}} - Z^{\text{SO}} + \mathbf{I} \sum_t \beta^t d^{t^{\text{SO}}} (\mathbf{I} - 1)(d^{t^{\text{UO}}} - d^{t^{\text{SO}}}) \geq 0.$$

which implies that

$$Z^{\text{UO}} - Z^{\text{SO}} + \mathbf{I}(\mathbf{I} - 1) \sum_t \beta^t (d^{t^{\text{SO}}} d^{t^{\text{UO}}} - (d^{t^{\text{SO}}})^2) \geq 0.$$

Again, using the fact that $d^{t^{\mathrm{SO}}} d^{t^{\mathrm{UO}}} \leq (d^{t^{\mathrm{SO}}})^2 + \frac{1}{4}(d^{t^{\mathrm{UO}}})^2$, we get

$$Z^{\mathrm{UO}} - Z^{\mathrm{SO}} + \frac{\mathbf{I}(\mathbf{I}-1)}{4} \sum_t \beta^t (d^{t^{\mathrm{UO}}})^2 \ \geq \ 0. \tag{6.7}$$

Consider $\mathbf{d} = 0$ in the user optimal variational inequality (6.2). For the symmetric case, it implies that

$$\begin{aligned} Z^{\mathrm{UO}} &\geq \mathbf{I} \sum_t \alpha^t (d^{t^{\mathrm{UO}}})^2 \\ &\geq \mathbf{I} \sum_t \beta^t (d^{t^{\mathrm{UO}}})^2. \end{aligned}$$

Using this in (6.7), we get that

$$Z^{\mathrm{UO}} - Z^{\mathrm{SO}} + \frac{(\mathbf{I}-1)}{4} Z^{\mathrm{UO}} \ \geq \ 0.$$

This implies that

$$\frac{3+\mathbf{I}}{4} Z^{\mathrm{UO}} \geq Z^{\mathrm{SO}}.$$

Since $Z^{\mathrm{UO}} \leq Z^{\mathrm{SO}}$, it follows that

$$\frac{4}{3+\mathbf{I}} \leq \frac{Z^{\mathrm{UO}}}{Z^{\mathrm{SO}}} \leq 1$$

where $\mathbf{I}$ is the number of sellers. □

Remark: In the case where $\mathbf{I} = 1$, this result translates to the fact that the system optimal and the user optimal are the same. This is to be expected since there is only a single user. As the number of users becomes higher, the lower bound on $\frac{Z^{\mathrm{UO}}}{Z^{\mathrm{SO}}}$ falls lower and lower and goes to zero as the number of users tends to infinity.

Special case: Let us now compare this bound to the actual ratio of $\frac{Z^{\mathrm{UO}}}{Z^{\mathrm{SO}}}$ for the specific case where the market clearing price function is the same for all periods in the multi-period symmetric uncapacitated game. That is, $D^t = D$ and $\beta^t_{ij} = \beta$ for all i, j and $t \in \mathbf{T}$. In such a case, the system optimal solution is

$$d_i^{t^{\mathrm{SO}}} = \frac{D}{2\mathbf{I}\beta}$$

and the user optimal solution is

$$d_i^{t^{\mathrm{UO}}} = \frac{D}{(\mathbf{I}+1)\beta}$$

for all $i \in \mathbf{I}$ and $t \in \mathbf{T}$. The corresponding profits are

$$Z^{\mathrm{SO}} = \frac{TD^2}{4\beta}$$

and

$$Z^{\text{UO}} = \frac{TD^2}{4\beta} \cdot \frac{4\mathbf{I}}{(\mathbf{I}+1)^2}$$

The ratio

$$\frac{Z^{\text{UO}}}{Z^{\text{SO}}} = \frac{4\mathbf{I}}{(\mathbf{I}+1)^2}$$

is very closely approximated by the ratio obtained. This implies that the time-symmetric case is close to the worst case bound. The two are compared in Figure B-2 that illustrates how good the bound obtained is. The line on top is the actual ratio for the time- and seller- symmetric case while the bottom line is the bound obtained for the general seller- symmetric case.

6.2 Price competition model

In this section, we consider a seller-symmetric[1] game. As in the previous section, we will compare the system optimal and user optimal profit but for the price competition problem we studied in Chapters 3 of this thesis. We let $(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}})$ be the user optimal solution and $(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}})$ be the system optimal solution to the pricing problem. The user optimal solution solves the quasi-variational inequality given by

$$-\mathbf{d}^{\text{UO}}(\mathbf{p} - \mathbf{p}^{\text{UO}}) - \mathbf{p}^{\text{UO}}(\mathbf{d} - \mathbf{d}^{\text{UO}}) \;\geq\; 0 \tag{6.8}$$

for all $(\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}})$. The system optimal solution solves the quasi-variational inequality given by

$$-\mathbf{d}^{\text{SO}}(\mathbf{p} - \mathbf{p}^{\text{SO}}) - \mathbf{p}^{\text{SO}}(\mathbf{d} - \mathbf{d}^{\text{SO}}) \;\geq\; 0 \tag{6.9}$$

for all $(\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}})$. In expanded format, this can be written as the quasi variational inequality

$$-\sum_{i,t} d_i^{t\text{UO}}(p_i^t - p_i^{t\text{UO}}) - \sum_{i,t} p_i^{t\text{UO}}(p_i^t - p_i^{t\text{UO}}) \;\geq\; 0$$

for all $(\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}})$ and

$$-\sum_{i,t} d_i^{t\text{SO}}(p_i^t - p_i^{t\text{SO}}) - \sum_{i,t} p_i^{t\text{SO}}(p_i^t - p_i^{t\text{SO}}) \;\geq\; 0$$

for all $(\mathbf{p}, \mathbf{d}) \in \mathcal{K}(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}})$ respectively. We show that if $(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}}) \in \mathcal{K}(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}})$ and $(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}}) \in \mathcal{K}(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}})$, then the user optimal profit, Z^{UO}, is equal to the system optimal profit, Z^{SO}. Following that, we show that for the time-symmetric[2] case, this condition follows when the capacity constraint of each seller is tight in the system optimal and the user optimal solution.

[1]We assume that the game is symmetric with respect to all sellers, and between themselves, all sellers will have identical user optimal policies (and system optimal policies).

[2]The time-symmetric case is the case where the demand function parameters are the same in all periods.

Theorem 6.2.1. *If* $(\mathbf{p}^{SO}, \mathbf{d}^{SO}) \in \mathcal{K}(\mathbf{p}^{UO}, \mathbf{d}^{UO})$ *and* $(\mathbf{p}^{UO}, \mathbf{d}^{UO}) \in \mathcal{K}(\mathbf{p}^{SO}, \mathbf{d}^{SO})$, *then* $Z^{UO} = Z^{SO}$.

Proof. $(\mathbf{p}^{SO}, \mathbf{d}^{SO}) \in \mathcal{K}(\mathbf{p}^{UO}, \mathbf{d}^{UO})$ implies that

$$d_i^{t^{SO}} \le h_i^t(p_i^{t^{SO}}, p_{-i}^{t^{UO}})$$

for all $i \in \mathbf{I}, t \in \mathbf{T}$. Since $d_i^{t^{SO}} = h_i^t(p_i^{t^{SO}}, p_{-i}^{t^{SO}})$, this implies that

$$h_i^t(p_i^{t^{SO}}, p_{-i}^{t^{SO}}) \le h_i^t(p_i^{t^{SO}}, p_{-i}^{t^{UO}}).$$

Since the demand is increasing in the competitor's price, we get that for all $i \in \mathbf{I}, t \in \mathbf{T}$

$$p_i^{t^{SO}} \le p_i^{t^{UO}}. \tag{6.10}$$

$(\mathbf{p}^{UO}, \mathbf{d}^{UO}) \in \mathcal{K}(\mathbf{p}^{SO}, \mathbf{d}^{SO})$ implies that

$$d_i^{t^{UO}} \le h_i^t(p_i^{t^{UO}}, p_{-i}^{t^{SO}})$$

for all $i \in \mathbf{I}, t \in \mathbf{T}$. Since $d_i^{t^{UO}} = h_i^t(p_i^{t^{UO}}, p_{-i}^{t^{UO}})$, this implies that

$$h_i^t(p_i^{t^{UO}}, p_{-i}^{t^{UO}}) \le h_i^t(p_i^{t^{UO}}, p_{-i}^{t^{SO}}).$$

Since the demand is increasing in the competitor's price, we get that for all $i \in \mathbf{I}, t \in \mathbf{T}$

$$p_i^{t^{SO}} \ge p_i^{t^{UO}}. \tag{6.11}$$

Thus, from (6.10) and (6.11), we get $p_i^{t^{SO}} = p_i^{t^{UO}}$ and $d_i^{t^{SO}} = d_i^{t^{UO}}$ for all i, t and thus they coincide. □

In order to give some intuition as to when these conditions might hold, we consider the time- and seller- symmetric game. We show that in this case, if the capacity constraint is tight for the system optimal setting then $(\mathbf{p}^{SO}, \mathbf{d}^{SO}) \in \mathcal{K}(\mathbf{p}^{UO}, \mathbf{d}^{UO})$. Conversely, if the user optimal solution is tight then $(\mathbf{p}^{UO}, \mathbf{d}^{UO}) \in \mathcal{K}(\mathbf{p}^{SO}, \mathbf{d}^{SO})$. This is shown in the following two lemmas.

Lemma 6.2.1. *If the capacity constraint for every seller is tight for the system optimal solution, then* $(\mathbf{p}^{SO}, \mathbf{d}^{SO}) \in \mathcal{K}(\mathbf{p}^{UO}, \mathbf{d}^{UO})$.

Proof. Since the capacity constraint is tight for the system optimal solution, it follows that

$$\sum_t d_i^{t^{SO}} = C_i$$

for all $i \in \mathbf{I}$. We also know that since the user optimal solution is feasible,

$$\sum_t d_i^{t^{UO}} \le C_i$$

for all $i \in \mathbf{I}$. We thus have that

$$\begin{aligned}\sum_t h_i^t(p_i^{t^{\text{UO}}}, p_{-i}^{t^{\text{UO}}}) &\leq C_i \\ &= \sum_t d_i^{t^{\text{SO}}} \\ &= \sum_t h_i^t(p_i^{t^{\text{SO}}}, p_{-i}^{t^{\text{SO}}})\end{aligned}$$

For the game that is symmetric for all time periods, this implies that

$$h_i^t(\mathbf{p}^{\text{UO}}) \leq h_i^t(\mathbf{p}^{\text{SO}})$$

which implies that

$$\mathbf{p}^{\text{SO}} \leq \mathbf{p}^{\text{UO}}$$

which further implies that

$$\begin{aligned}h_i^t(\mathbf{p}_i^{\text{SO}}, \mathbf{p}_{-i}^{\text{UO}}) &\geq h_i^t(\mathbf{p}_i^{\text{SO}}, \mathbf{p}_{-i}^{\text{SO}}) \\ &= d^{\text{SO}}\end{aligned}$$

This implies that $(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}}) \in \mathcal{K}(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}})$ and completes the proof. □

Lemma 6.2.2. *If the capacity constraint for every seller is tight for the user optimal solution, then* $(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}}) \in \mathcal{K}(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}})$.

Proof. Since the capacity constraint is tight for the user optimal solution, we have that

$$\sum_t d_i^{t^{\text{UO}}} = C_i$$

for all $i \in \mathbf{I}$. We also know that since the system optimal solution is feasible,

$$\sum_t d_i^{t^{\text{SO}}} \leq C_i$$

for all $i \in \mathbf{I}$. We thus have that

$$\begin{aligned}\sum_t h_i^t(p_i^{t^{\text{SO}}}, p_{-i}^{t^{\text{SO}}}) &\leq C_i \\ &= \sum_t d_i^{t^{\text{UO}}} \\ &= \sum_t h_i^t(p_i^{t^{\text{UO}}}, p_{-i}^{t^{\text{UO}}}).\end{aligned}$$

For a symmetric setting for all time periods, it follows that

$$h(\mathbf{p}^{\text{SO}}) \leq h(\mathbf{p}^{\text{UO}})$$

which implies that

$$\mathbf{p}^{\text{UO}} \leq \mathbf{p}^{\text{SO}}.$$

This further implies that

$$\begin{aligned} h(\mathbf{p}_i^{\text{UO}}, \mathbf{p}_{-i}^{\text{SO}}) &\geq h(\mathbf{p}_i^{\text{UO}}, \mathbf{p}_{-i}^{\text{UO}}) \\ &= d^{\text{UO}} \end{aligned}$$

This implies that $(\mathbf{p}^{\text{UO}}, \mathbf{d}^{\text{UO}}) \in \mathcal{K}(\mathbf{p}^{\text{SO}}, \mathbf{d}^{\text{SO}})$ and completes the proof. □

6.3 Equilibrium in price and quantity competition models

We show using a 2-seller example that the formulation of the best response problem (whether it is price-competition or quantity-competition) does affect the nature of the resulting equilibrium solution. Intuitively, this is because a transformation of variables changes the definition of the Nash equilibrium for the problem: in the price competition case the equilibrium is defined in terms of keeping the price of competitors fixed while in the quantity competition case, it is the quantity of each competitor that is fixed. For more details see *Oligopoly Pricing*[3] by *Xavier Vives.* In the example that follows, the equations determining the demand-price relation in both models (price-competition and quantity-competition) are identical but the resulting (unique) equilibrium is not.

6.3.1 Price Competition

The demand is defined as:

$$d_i(p_i, p_{-i}) = 1 - \frac{1}{2}p_i + \frac{1}{4}p_{-i} \quad \forall i \in \{1, 2\}$$

The price competition will lead to the following equilibrium solution:

$$\begin{aligned} \pi_i &= p_i \cdot d_i(p_i, p_{-i}) \\ \frac{\partial \pi_i}{\partial p_i} &= 0 \\ \Rightarrow p_i^*(p_{-i}) &= 1 + \frac{p_{-i}}{4} \\ \Rightarrow p_i^* &= \frac{4}{3} \\ d_i^* &= \frac{2}{3} \\ \pi_i^* &= \frac{8}{9} \end{aligned}$$

[3]Section 5.2.4, Pages 132-136 and Section 7.1 Pages 185-187

6.3.2 Quantity Competition

The clearing price is defined as:

$$p_i(d_i, d_{-i}) = 4 - \frac{8}{3}d_i - \frac{4}{3}d_{-i} \quad \forall i \in \{1, 2\}$$

The quantity competition will lead to the following equilibrium solution:

$$\begin{aligned}
\pi_i &= p_i(d_i, d_{-i}) \cdot d_i \\
\frac{\partial \pi_i}{\partial d_i} &= 0 \\
\Rightarrow d_i^*(d_{-i}) &= \frac{3}{4} - \frac{d_{-i}}{4} \\
\Rightarrow d_i^* &= \frac{3}{5} \\
p_i^* &= \frac{8}{5} \\
\pi_i^* &= \frac{24}{25}
\end{aligned}$$

Chapter 7

Computation of equilibrium strategy

In this chapter we introduce an algorithm for computing the market equilibrium prices arising from the joint quasi-variational inequality formulation (3.8). A number of algorithms proposed for solving quasi variational inequalities exist in the literature (see for example, Pang and Fukushima [52], Morgan and Romaniello [48]). The algorithm we study is based on a simple intuitive process inspired by the concept of *fictitious play*, first introduced by Brown [13] and Robinson [57]. The tatônnement process described in Vives [66] is very similar in nature. It is shown to converge (tatônnement stability) for supermodular games in particular. Consequently, the result does not apply to the model in this thesis.

The basic idea for the algorithm is as follows. Imagine a market where none of the sellers are aware of the equilibrium policies. Instead, each seller observes the pricing policies of its competitors and would like to adopt a policy that optimizes her payoff. No seller has any information about the starting inventories of any competitor and can only observe the prices that are part of the market information. The entire multi period game is repeated a number of times. Each seller observes the policies of her competitors at every realization of the game and adapts her pricing policy myopically (i.e. optimizes her payoff given the prices from the previous realization of the game). As this process is repeated, one would expect the market to involuntarily approach the equilibrium state. We describe conditions under which this would happen.

There are two parts in the process. The first part involves the best response problem a seller needs to solve and the second involves how prices form in the market as a whole. In the following sections, we describe how the best response problem can be solved computationally, and then provide conditions that guarantee convergence of the algorithm for the multi-period pricing models (both deterministic demand and uncertain demand) we introduced in this paper. To provide some intuition, we discuss how these conditions can be interpreted for the linear demand case.

7.1 Iterative learning algorithm

Consider the market we described in Chapter 3, consisting of several sellers pricing a product in a multi-period setting. Assume that the process is repeated under the same conditions of initial inventory and demand. The sellers do not start with the equilibrium policies but rather follow a naive myopic optimization approach: They price using the best response policy given all competitors' prices from the previous instance of the process. The key question is that if this process is repeated sufficiently many times, under what conditions will the sellers' prices converge to the equilibrium prices, irrespective of the state that the sellers in the market started from.

The outline of the general algorithm is as follows. Start by considering an initial estimate for the solution denoted by $\mathbf{z}^0 \in \mathcal{K}$ and set $k = 1$. Compute $\mathbf{z}^k$ by solving the following set of separable variational inequality subproblems for each $i \in \mathbf{I}$:

$$F_i(\mathbf{z}_i^k) \cdot (\mathbf{z}_i - \mathbf{z}_i^k) \geq 0, \qquad \forall \mathbf{z}_i \in \mathcal{K}_i(\mathbf{z}_{-i}^{k-1}). \tag{7.1}$$

For our problem, this iteration step corresponds to each seller setting the best response policy to her competitors' strategies from the last iteration. This step in detailed in Subsection 7.2. We check for convergence (if the policies from two successive iterations are the same or ε-close to each other) and stop; otherwise we repeat with an incremented value for k. This algorithm is formally presented in Algorithm 1.

Algorithm 1

for $i = 1 \ldots N$ **do**
    ${}^0p_i^t \leftarrow p^t_{\text{initial}}$
end for
for $i = 1 \ldots N$ **do**
    ${}^1\mathbf{p}_i \leftarrow \mathcal{BR}_i({}^0\mathbf{p}_{-i})$
end for
$k \leftarrow 1$
while ${}^k\mathbf{p}_i \neq {}^{k-1}\mathbf{p}_i$ **do**
    for $i = 1 \ldots N$ **do**
        ${}^{k+1}\mathbf{p}_i \leftarrow \mathcal{BR}_i({}^k\mathbf{p}_{-i})$
    end for
    $k \leftarrow k + 1$
end while
$\mathbf{p}^* \leftarrow {}^k\mathbf{p}$
RETURN $\mathbf{p}^*$

7.2 Solving the best response problem

In this section we illustrate how we solve each step of Algorithm 1 for the deterministic demand case. The best response policy $\mathbf{p}^k = \mathcal{BR}(\mathbf{p}^{k-1})$ of each seller $i \in \mathbf{I}$, given all her competitors' policies $\mathbf{p}^{k-1}$, is obtained as the solution of the following optimization problem. This step is the main part of each iteration of Algorithm 1.

For each $i \in \mathbf{I}$, set $\bar{\mathbf{p}}_{-i} = p_{-i}^{k-1}$ and solve for $\mathbf{p}_i^k$ as the solution to:

$$\begin{array}{rll} \operatorname{argmax}_{\mathbf{d}_i, \mathbf{p}_i} & \sum_{t=1}^{T} d_i^t p_i^t & \\ \text{such that} & d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t) & \forall t \in \mathbf{T} \\ & \sum_{t=1}^{T} d_i^t \leq C_i & \\ & p_{\min}^t \leq p_i^t \leq p_{\max}^t & \forall t \in \mathbf{T} \\ & d_i^t \geq 0 & \forall t \in \mathbf{T}. \end{array}$$

For convenience, we denote the demand function $h_i^t(., \bar{p}_{-i}^t)$ by $\bar{h}_i^t(.)$. The function is invertible under Condition 3.3.5 and we denote $p_i^t = \breve{p}_i^t(d_i^t) = \bar{h}_i^{t^{-1}}(d_i^t)$ as a function of d_i^t. Under this notation, the problem can be formulated as:

$$\begin{array}{rll} \max_{\mathbf{d}_i} & \sum_{t=1}^{T} d_i^t \cdot \breve{p}_i^t(d_i^t) & \\ \text{such that} & \sum_{t=1}^{T} d_i^t \leq C_i & \\ & d_i^t \geq 0 & \forall t \in \mathbf{T}. \end{array}$$

This formulation is a convex optimization problem since it has a concave objective function that is maximized over a convex feasible space. There are well studied algorithms that can be used to solve such problems efficiently since there is an underlying network structure. One way to solve it is to consider two cases based on whether the inventory constraint is tight or not.

If the inventory constraint is not tight, the problem separates by time period. We denote the single period revenue by $\pi_i^t = \breve{p}_i^t(d_i^t) \cdot d_i^t$ and solve the following optimization problem:

$$\begin{array}{rll} \max_{\mathbf{d}_i} & \sum_{t=1}^{T} \pi_i^t & \\ \text{such that} & d_i^t \geq 0 & \forall t \in \mathbf{T}. \end{array}$$

This separates into simpler optimization problems corresponding to each period $t \in \mathbf{T}$, each of which is an easily solvable optimization problem in a single variable d_i^t:

$$\begin{array}{rl} \max_{\mathbf{d}_i^t} & \pi_i^t = d_i^t \cdot \breve{p}_i^t(d_i^t) \\ \text{such that} & d_i^t \geq 0 \end{array}$$

If the inventory constraint is tight, the problem is not separable by time period. However it can be formulated as a min-cost network flow problem as shown in Figure B-1. In the figure, the label on each arc refers to the flow variable and per unit flow cost function respectively. For this reformulation, efficient solution methods are well established in the transportation literature (See Florian and Hearn [35]). Under this structure the solution can be calculated using the following condition for optimality:

$$\frac{\partial \pi_i^t}{\partial d_i^t} \quad \begin{cases} = v_i \text{ if } d_i^t > 0 \\ \leq v_i \text{ if } d_i^t = 0 \end{cases}$$

$$\text{subject to} \quad \sum_{t=1}^{T} d_i^t = C_i$$
$$d_i^t \geq 0 \quad \forall t \in \mathbf{T}.$$

7.3 Convergence under the deterministic demand model

In this section we study the convergence of Algorithm 1. We first need to introduce the following conditions.

Condition 7.3.1. *For any given $\bar{\mathbf{p}}_i$, $\mathbf{h}_i(\bar{\mathbf{p}}_i, \mathbf{p}_{-i})$ is Lipschitz continuous with respect to $\mathbf{p}_{-i}$ with parameter $\mathcal{L}_h$.*

$$\|\mathbf{h}_i(\bar{\mathbf{p}}_i, \hat{\mathbf{p}}_{-i}) - \mathbf{h}_i(\bar{\mathbf{p}}_i, \check{\mathbf{p}}_{-i})\| \quad \leq \quad \mathcal{L}_h \; \|\hat{\mathbf{p}}_{-i} - \check{\mathbf{p}}_{-i}\|, \quad \forall \; (\check{\mathbf{p}}_{-i}, \; \hat{\mathbf{p}}_{-i})$$

Condition 7.3.2. *For any given $\bar{\mathbf{p}}_{-i}$, $-\mathbf{h}_i(\mathbf{p}_i, \bar{\mathbf{p}}_{-i})$ is strongly monotone with respect to $\mathbf{p}_i$ with parameter $\mathcal{A}_h$.*

$$(-\mathbf{h}_i(\hat{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i}) + \mathbf{h}_i(\check{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i})) \cdot (\hat{\mathbf{p}}_i - \check{\mathbf{p}}_i) \quad \geq \quad \mathcal{A}_h \; \|\hat{\mathbf{p}}_i - \check{\mathbf{p}}_i\|^2, \quad \forall \; (\check{\mathbf{p}}_i, \; \hat{\mathbf{p}}_i)$$

Condition 7.3.3. *There exists an $\epsilon_1 > 0$ such that $\frac{\mathcal{L}_h}{\mathcal{A}_h} \leq 1 - \epsilon_1$ where $\mathcal{A}_h$ and $\mathcal{L}_h$ are defined as above.*

Notice, for example, that for the two seller, linear demand case, the above conditions hold when for all $i \in \mathbf{I}$, the minimum sensitivity of seller i's demand to seller i's price over all periods, is greater than the maximum sensitivity of her demand to her competitor's price over all periods. In particular, if the demand for a two seller market is given by $h_i^t(p_i^t, p_{-i}^t) = D_{\text{base}i}^t - \beta_i^t p_i^t + \alpha_i^t p_{-i}^t$ then $\mathcal{A}_h = \min_t(\beta_i^t)$ and $\mathcal{L}_h = \max_t(\alpha_i^t)$.

We first define the inverse demand function $\breve{p}_i^t(d_i^t, p_{-i}^t)$ as follows.

$$p_i^t = \breve{p}_i^t(d_i^t, p_{-i}^t) \Leftrightarrow d_i^t = h_i^t(p_i^t, p_{-i}^t)$$

Since our demand function is strictly decreasing and concave, it is invertible and $\breve{p}_i^t$ is well defined. We impose the following conditions on

$$\breve{\mathbf{p}}(\mathbf{d}, \mathbf{p}) = \left[\breve{p}_i^t(d_i^t, p_{-i}^t)\right]_{i \times t}$$

.

Condition 7.3.4. *For any given $\bar{\mathbf{d}}$, $\breve{\mathbf{p}}(\bar{\mathbf{d}}, \mathbf{p})$ is Lipschitz continuous with respect to $\mathbf{p}$ with parameter $\mathcal{P}_{\breve{p}} \geq 0$.*

$$\|\breve{\mathbf{p}}(\bar{\mathbf{d}}, \hat{\mathbf{p}}) - \breve{\mathbf{p}}(\bar{\mathbf{d}}, \check{\mathbf{p}})\| \quad \leq \quad \mathcal{P}_{\breve{p}} \; \|\hat{\mathbf{p}} - \check{\mathbf{p}}\|, \quad \forall \; (\check{\mathbf{p}}, \; \hat{\mathbf{p}})$$

Condition 7.3.5. *For any given* $\bar{\mathbf{p}}$, $\breve{\mathbf{p}}(\mathbf{d}, \bar{\mathbf{p}})$ *is Lipschitz continuous with respect to* $\mathbf{d}$ *with parameter* $\mathcal{D}_{\breve{p}} \geq 0$.

$$\|\breve{\mathbf{p}}(\hat{\mathbf{d}}, \bar{\mathbf{p}}) - \breve{\mathbf{p}}(\check{\mathbf{d}}, \bar{\mathbf{p}})\| \leq \mathcal{D}_{\breve{p}} \|\hat{\mathbf{d}} - \check{\mathbf{d}}\|, \quad \forall\ (\check{\mathbf{d}},\ \hat{\mathbf{d}})$$

We also define the function $\delta(\mathbf{d}, \mathbf{p})$ as follows.

$$\delta(\mathbf{d}, \mathbf{p}) = \left[-d_i^t \frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^t, p_{-i}^t) \right]_{i \times t}$$

where $\breve{p}_i^t(d_i^t, p_{-i}^t)$ is the inverse of the demand function defined above. On this function, we impose the following conditions.

Condition 7.3.6. *For any given* $\bar{\mathbf{d}}$, $\delta(\bar{\mathbf{d}}, \mathbf{p})$ *is Lipschitz continuous with respect to* $\mathbf{p}$ *with parameter* $\mathcal{L}_\delta \geq 0$.

$$\|\delta(\bar{\mathbf{d}}, \hat{\mathbf{p}}) - \delta(\bar{\mathbf{d}}, \breve{\mathbf{p}})\| \leq \mathcal{L}_\delta \|\hat{\mathbf{p}} - \breve{\mathbf{p}}\|, \quad \forall\ (\breve{\mathbf{p}},\ \hat{\mathbf{p}})$$

Condition 7.3.7. *For any given* $\bar{\mathbf{p}}$, $\delta(\mathbf{d}, \bar{\mathbf{p}})$ *is strongly monotone with respect to* $\mathbf{d}$ *with parameter* $\mathcal{A}_\delta > 0$.

$$\left(\delta(\hat{\mathbf{d}}, \bar{\mathbf{p}}) + \delta(\check{\mathbf{d}}, \bar{\mathbf{p}})\right) \cdot (\hat{\mathbf{d}} - \check{\mathbf{d}}) \geq \mathcal{A}_\delta \|\hat{\mathbf{d}} - \check{\mathbf{d}}\|^2, \quad \forall\ (\check{\mathbf{d}},\ \hat{\mathbf{d}})$$

Condition 7.3.8. *For* $\mathcal{P}_{\breve{p}}$, $\mathcal{D}_{\breve{p}}$, $\mathcal{L}_\delta$ *and* $\mathcal{A}_\delta$ *defined as above, there exists an* $\epsilon_2 > 0$ *such that*

$$\mathcal{P}_{\breve{p}} + \mathcal{D}_{\breve{p}} \frac{\mathcal{L}_\delta}{\mathcal{A}_\delta} < 1 - \epsilon_2$$

For the two seller, linear demand case with $h_i^t(p_i^t, p_{-i}^t) = D_{\text{base}i}^t - \beta_i^t p_i^t + \alpha_i^t p_{-i}^t$ mentioned above, the values of the parameters are $\mathcal{P}_{\breve{p}} = \frac{\max_t(\alpha_i^t)}{\min_t(\beta_i^t)}$, $\mathcal{D}_{\breve{p}} = \frac{1}{\min_t(\beta_i^t)}$, $\mathcal{L}_\delta = 0$ and $\mathcal{A}_\delta = \frac{1}{\min_t(\beta_i^t)}$. This implies that Condition 7.3.8 is satisfied whenever Condition 7.3.3 is satisfied.

Theorem 7.3.1. *Under Conditions 7.3.3 and 7.3.8, Algorithm 1 converges to an equilibrium pricing policy.*

Proof. In Algorithm 1, we start with an initial guess for the pricing policies of all sellers, $\mathbf{p}^0$. Let us call this the $m = 0^{\text{th}}$ iteration. Given price levels at the m^{th} iteration ($\mathbf{p}^m$) we find the best response policy for each seller by solving the best response problem using the algorithm in Section 7.2. Essentially, we transform the problem into an equivalent problem for each seller i with variables ($\mathbf{d}_i$) instead of ($\mathbf{p}_i, \mathbf{d}_i$) as given below:

$$\begin{array}{lll} \max_{\mathbf{d}_i} & \sum_{t=1}^T d_i^t \cdot \breve{p}_i^t(d_i^t, \mathbf{p}^m) & \\ \text{such that} & \sum_{t=1}^T d_i^t \leq C_i & \\ & d_i^t \geq 0 & \forall t \in \mathbf{T}. \end{array}$$

The solution to this problem is $\mathbf{d}_i^{m+1}$ and we obtain the corresponding $\mathbf{p}_i^{m+1}$ using the relation

$$p_i^{t\,m+1} = \breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m)$$

thereby ensuring that $d_i^{t\,m+1} = h_i^t(p_i^{t\,m+1}, p_{-i}^{t\ m})$ for all $i \in \mathbf{I}$ and $t \in \mathbf{T}$. The equivalent variational inequality for step $(m+1)$ is thus to find a $\mathbf{d}_i^{m+1} \in \mathcal{K}_{d_i}$ such that

$$\sum_{t\in\mathbf{T}}(-\breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m) - d_i^{t\,m+1}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m+1}, \mathbf{p}^m))(d_i^t - d_i^{t\,m+1}) \geq 0 \tag{7.2}$$

for all $\mathbf{d}_i \in \mathcal{K}_{d_i}$ where $\mathcal{K}_{d_i} = \left\{\mathbf{d}_i \mid \sum_{t=1}^T d_i^t \leq C_i,\ d_i^t \geq 0\ \forall t \in \mathbf{T}\right\}$. Similarly, the variational inequality for step m is to find a $\mathbf{d}_i^m \in \mathcal{K}_{d_i}$ such that

$$\sum_{t\in\mathbf{T}}(-\breve{p}_i^t(d_i^{t\,m}, \mathbf{p}^{m-1}) - d_i^{t\,m}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m}, \mathbf{p}^{m-1}))(d_i^t - d_i^{t\,m}) \geq 0 \tag{7.3}$$

for all $\mathbf{d}_i \in \mathcal{K}_{d_i}$. Substituting $\mathbf{d}_i = \mathbf{d}_i^m$ in (7.2) and $\mathbf{d}_i = \mathbf{d}_i^{m+1}$ in (7.3) and adding the two we get,

$$\begin{aligned}
&\sum_{t\in\mathbf{T}}\left(\breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m) - \breve{p}_i^t(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) + \\
&\quad\sum_{t\in\mathbf{T}}\left(d_i^{t\,m+1}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m+1}, \mathbf{p}^m) - d_i^{t\,m}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) \ \geq\ 0
\end{aligned} \tag{7.4}$$

Adding the above for all $i \in \mathbf{I}$,

$$\begin{aligned}
&\sum_{i\in\mathbf{I}}\sum_{t\in\mathbf{T}}\left(\breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m) - \breve{p}_i^t(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) + \\
&\sum_{i\in\mathbf{I}}\sum_{t\in\mathbf{T}}\left(d_i^{t\,m+1}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m+1}, \mathbf{p}^m) - d_i^{t\,m}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) \ \geq\ 0
\end{aligned}$$

In vector notation, this can be written as

$$\begin{aligned}
&\left(\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\right)(\mathbf{d}^{m+1} - \mathbf{d}^m) + \\
&\quad\left(-\delta(\mathbf{d}^{m+1}, \mathbf{p}^m) + \delta(\mathbf{d}^m, \mathbf{p}^{m-1})\right)(\mathbf{d}^{m+1} - \mathbf{d}^m) \ \geq\ 0
\end{aligned} \tag{7.5}$$

We consider two exhaustive cases. From the inequality (7.5), we can conclude that at least one of the terms (the first or the second term) is greater than or equal to zero.

Case 1: The first term in (7.5) is non-negative, i.e.,

$$\left(\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\right)(\mathbf{d}^{m+1} - \mathbf{d}^m) \geq 0.$$

Note that

$$\begin{aligned}
d_i^{t\,m+1} &= h_i^t(p_i^{t\,m+1}, p_i^{t\,m}), \\
d_i^{t\,m} &= h_i^t(p_i^{t\,m}, p_i^{t\,m-1}), \\
\breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m) &= p_i^{t\,m+1}, \text{ and} \\
\breve{p}_i^t(d_i^{t\,m}, \mathbf{p}^{m-1}) &= p_i^{t\,m}.
\end{aligned}$$

Making these substitutions in the given inequality, we get that

$$\sum_{t\in\mathbf{T}}(p_i^{tm+1} - p_i^{tm})(h_i^t(p_i^{tm+1}, p_i^{tm}) - h_i^t(p_i^{tm}, p_i^{tm-1})) \;\geq\; 0.$$

The above can be written in vector notation as

$$\left(\mathbf{p}_i^{m+1} - \mathbf{p}_i^m\right) \cdot \left(\mathbf{h}_i(\mathbf{p}_i^{m+1}, \mathbf{p}_{-i}^m) - \mathbf{h}_i(\mathbf{p}_i^m, \mathbf{p}_{-i}^{m-1})\right) \;\geq\; 0. \tag{7.6}$$

Adding $\forall i \in \mathbf{I}$, gives rise to

$$\left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \cdot \left(\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1})\right) \;\geq\; 0. \tag{7.7}$$

Adding and subtracting $\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m)$ we get

$$\begin{aligned}\big(\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^m) + \mathbf{h}(\mathbf{p}^m, \mathbf{p}^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1})\big)& \\ \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \;\geq\;& 0.\end{aligned}$$

On rearranging terms we get

$$\begin{aligned}\left(\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1})\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \geq& \\ \left((-\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m)) - (-\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m))\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right).&\end{aligned} \tag{7.8}$$

Under Condition 7.3.2, the right hand side of (7.9) is non negative. Thus, both sides are non-negative and (7.9) becomes equivalent to

$$\begin{aligned}&\left|\left(\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1})\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right)\right| \\ &\qquad\geq \left|\left((-\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m)) - (-\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m))\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right)\right|.\end{aligned}$$

Under Conditions 7.3.1 and 7.3.2 it follows that:

$$\begin{aligned}&\mathcal{L}_h\|\mathbf{p}^m - \mathbf{p}^{m-1}\| \cdot \|\mathbf{p}^{m+1} - \mathbf{p}^m\| \\ &\qquad\geq \|\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1})\| \cdot \|\mathbf{p}^{m+1} - \mathbf{p}^m\| \\ &\qquad\geq \left(\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1})\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \\ &\qquad\geq \left((-\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m)) - (-\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m))\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \\ &\qquad\geq \mathcal{A}_h\|\mathbf{p}^{m+1} - \mathbf{p}^m\|^2.\end{aligned}$$

The first step follows from Lipschitz continuity. The second step follows from Cauchy's Inequality. The third step follows from (7.9). The fourth step follows from the strong monotonicity assumption. As a result,

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \;\leq\; \frac{\mathcal{L}_h}{\mathcal{A}_h} \cdot \|\mathbf{p}^m - \mathbf{p}^{m-1}\|.$$

Case 2: The second term in (7.5) is non-negative, ie.

$$\left(-\delta(\mathbf{d}^{m+1}, \mathbf{p}^m) + \delta(\mathbf{d}^m, \mathbf{p}^{m-1})\right)\left(\mathbf{d}^{m+1} - \mathbf{d}^m\right) \geq 0.$$

Adding and subtracting $\delta(\mathbf{d}^m, \mathbf{p}^m)$ in above, we get

$$\left(-\delta(\mathbf{d}^{m+1}, \mathbf{p}^m) + \delta(\mathbf{d}^m, \mathbf{p}^m) - \delta(\mathbf{d}^m, \mathbf{p}^m) + \delta(\mathbf{d}^m, \mathbf{p}^{m-1})\right)(\mathbf{d}^{m+1} - \mathbf{d}^m) \geq 0$$

which implies that

$$\begin{aligned}&\left(\delta(\mathbf{d}^{m+1}, \mathbf{p}^m) - \delta(\mathbf{d}^m, \mathbf{p}^m)\right)(\mathbf{d}^{m+1} - \mathbf{d}^m)\\ &\qquad \leq \left(\delta(\mathbf{d}^m, \mathbf{p}^{m-1}) - \delta(\mathbf{d}^m, \mathbf{p}^m)\right)(\mathbf{d}^{m+1} - \mathbf{d}^m).\end{aligned}$$

Under concavity of the demand function (and hence the inverse demand function) the left hand side of the above inequality is non-negative. Hence, both sides of the inequality are non negative and we can take absolute value of both sides without affecting the direction of the inequality. This gives us the following relation.

$$\begin{aligned}&\left|\left(\delta(\mathbf{d}^{m+1}, \mathbf{p}^m) - \delta(\mathbf{d}^m, \mathbf{p}^m)\right)(\mathbf{d}^{m+1} - \mathbf{d}^m)\right|\\ &\qquad \leq \left|\left(\delta(\mathbf{d}^m, \mathbf{p}^{m-1}) - \delta(\mathbf{d}^m, \mathbf{p}^m)\right)(\mathbf{d}^{m+1} - \mathbf{d}^m)\right|. \end{aligned} \tag{7.9}$$

Note that,

$$\begin{aligned}&\mathcal{L}_\delta \|\mathbf{p}^{m-1} - \mathbf{p}^m\| \cdot \|\mathbf{d}^{m+1} - \mathbf{d}^m\|\\ &\qquad \geq \|\delta(\mathbf{d}^m, \mathbf{p}^{m-1}) - \delta(\mathbf{d}^m, \mathbf{p}^m)\| \cdot \|\mathbf{d}^{m+1} - \mathbf{d}^m\|\\ &\qquad \geq \|(\delta(\mathbf{d}^m, \mathbf{p}^{m-1}) - \delta(\mathbf{d}^m, \mathbf{p}^m)) \cdot (\mathbf{d}^{m+1} - \mathbf{d}^m)\|\\ &\qquad \geq (\delta(\mathbf{d}^{m+1}, \mathbf{p}^m) - \delta(\mathbf{d}^m, \mathbf{p}^m)) \cdot (\mathbf{d}^{m+1} - \mathbf{d}^m)\\ &\qquad \geq \mathcal{A}_\delta \|\mathbf{d}^{m+1} - \mathbf{d}^m\|^2.\end{aligned}$$

The first step follows from Lipschitz continuity. The second step follows from Cauchy's Inequality. The third step follows from (7.9). The fourth step follows from the strong monotonicity assumption. As a result,

$$\mathcal{L}_\delta \|\mathbf{p}^{m-1} - \mathbf{p}^m\| \;\; \geq \;\; \mathcal{A}_\delta \|\mathbf{d}^{m+1} - \mathbf{d}^m\|. \tag{7.10}$$

Now consider that

$$\begin{aligned}&\|\mathbf{p}^{m+1} - \mathbf{p}^m\|\\ &\quad = \|\check{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\|\\ &\quad = \|\check{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m) + \check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m) - \check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\|\\ &\quad \leq \|\check{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m)\| + \|\check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m) - \check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\|\\ &\quad \leq \mathcal{D}_p \|\mathbf{d}^{m+1} - \mathbf{d}^m\| + \mathcal{P}_p \|\mathbf{p}^m - \mathbf{p}^{m-1}\|\\ &\quad \leq \mathcal{D}_p \frac{\mathcal{L}_\delta}{\mathcal{A}_\delta} \|\mathbf{p}^m - \mathbf{p}^{m-1}\| + \mathcal{P}_p \|\mathbf{p}^m - \mathbf{p}^{m-1}\|\\ &\quad = \left(\mathcal{D}_p \frac{\mathcal{L}_\delta}{\mathcal{A}_\delta} + \mathcal{P}_p\right) \|\mathbf{p}^m - \mathbf{p}^{m-1}\|.\end{aligned}$$

The first step follows from the definition of the inverse demand function. In the second step we add and subtract $\check{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m)$. The first inequality comes from the usual triangle vector inequality. The second inequality comes from the continuity conditions. The last inequality comes from the inequality (7.10) derived above. As a result,

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \ \leq \ \left(\mathcal{D}_p \frac{\mathcal{L}_\delta}{\mathcal{A}_\delta} + \mathcal{P}_p\right) \|\mathbf{p}^m - \mathbf{p}^{m-1}\|.$$

In both cases, we get

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \ \leq \ (1-\epsilon) \cdot \|\mathbf{p}^m - \mathbf{p}^{m-1}\|,$$

where $\epsilon = \min(\epsilon_1, \epsilon_2)$ and thus,

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \ \leq \ (1-\epsilon)^m \cdot \|\mathbf{p}^1 - \mathbf{p}^0\|. \tag{7.11}$$

It follows that sequence $\{\|\mathbf{p}^{m+1} - \mathbf{p}^m\|\}_m$ converges to zero at a geometric rate. As a result it follows that sequence $\{\mathbf{p}^m\}_m$ is a Cauchy sequence and thus is a convergent sequence to a stationary price $\mathbf{p}$. This is an equilibrium point since each seller's best response leads to the same point. □

Corollary 7.3.1. *Let $(\mathbf{p}^k)_k$ be the sequence generated by Algorithm 1. The number of iterations required to reach a solution within ε-distance from $\mathbf{p}^*$ is $O(\frac{ln(\frac{D}{\epsilon\varepsilon})}{ln(\frac{1}{1-\epsilon})})$ where D is the diameter of the feasible space $\mathcal{K}$ and ϵ is as defined above.*

Proof. Denote the diameter of the feasible policy space by $\mathcal{D}$, i.e. the maximum distance between any two points $\hat{\mathbf{p}}$ and $\check{\mathbf{p}}$ in $\mathcal{K}$.

$$\mathcal{D} = \sup_{\hat{\mathbf{p}}, \check{\mathbf{p}} \in \mathcal{K}} \|\hat{\mathbf{p}} - \check{\mathbf{p}}\|$$

Equation 7.11 can be written as

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \leq (1-\epsilon)^m \mathcal{D}$$

since $\mathcal{D} \geq \|\mathbf{p}^1 - \mathbf{p}^0\|$. Summing over $k = m, m+1, \cdots, \infty$ we get

$$\sum_{k=m}^{\infty} \|\mathbf{p}^{k+1} - \mathbf{p}^k\| \leq \sum_{k=m}^{\infty} (1-\epsilon)^k \mathcal{D}.$$

It follows from the triangle inequality that the left hand side of the equation is greater than the distance between $\mathbf{p}^m$ and $\mathbf{p}^*$. The right hand side sum is equal to $\frac{(1-\epsilon)^m \mathcal{D}}{\epsilon}$. Thus we get,

$$\|\mathbf{p}^m - \mathbf{p}^*\| \leq \frac{(1-\epsilon)^m \mathcal{D}}{\epsilon}.$$

Consequently, for

$$m \geq \frac{ln(\frac{D}{\epsilon\varepsilon})}{ln(\frac{1}{1-\epsilon})}$$

we get that $\|\mathbf{p}^m - \mathbf{p}^*\| \leq \varepsilon$. This proves the result and the number of iterations required for the algorithm to converge to a solution ϵ-close to the equilibrium is $O(\frac{ln(\frac{D}{\epsilon\varepsilon})}{ln(\frac{1}{1-\epsilon})})$. □

7.4 Convergence under the robust demand model

For the robust demand model, Algorithm 1 starts with an initial guess for the pricing policies of all sellers, $\mathbf{p}^0$. Let us call this the 0^{th} iteration. Given price levels at the m^{th} iteration ($\mathbf{p}^m$), we find the best response policy for each seller by solving the robust best response problem. We also define the inverse demand function $\breve{p}_i^t(d_i^t, p_{-i}^t, \xi)$ similar to before.

$$p_i^t = \breve{p}_i^t(d_i^t, p_{-i}^t, \xi) \Leftrightarrow d_i^t = h_i^t(p_i^t, p_{-i}^t, \xi)$$

Since our demand function is strictly decreasing and concave, it is invertible and $\breve{p}_i^t$ is well defined. We also denote in vector form,

$$\breve{\mathbf{p}}(\mathbf{d}, \mathbf{p}, \xi) = \left[\breve{p}_i^t(d_i^t, p_{-i}^t, \xi)\right]_{i \times t}$$

As before, we transform the problem into an equivalent problem for each seller i with variables $(\mathbf{d}_i)$ instead of $(\mathbf{p}_i, \mathbf{d}_i)$,

$$\begin{array}{lll} \max_{\mathbf{d}_i} & \sum_{t=1}^T d_i^t \cdot \left(\min_{\xi_i^t \in \mathcal{U}_i^t} \breve{p}_i^t(d_i^t, \mathbf{p}^m, \xi_i^t)\right) & \\ \text{such that} & \sum_{t=1}^T d_i^t \leq C_i & \\ & d_i^t \geq 0 & \forall t \in \mathbf{T}. \end{array}$$

Let us denote the robust inverse demand function by $\breve{p}_i^t(d_i^t, \mathbf{p}^m)$ as follows

$$\breve{p}_i^t(d_i^t, \mathbf{p}^m) = \min_{\xi_i^t \in \mathcal{U}_i^t} \breve{p}_i^t(d_i^t, \mathbf{p}^m, \xi_i^t).$$

The solution to this problem is $\mathbf{d}_i^{m+1}$ (with the corresponding ${\xi_i}^m$) and we obtain the corresponding $\mathbf{p}_i^{m+1}$ using the relation

$${p_i^t}^{m+1} = \breve{p}_i^t({d_i^t}^{m+1}, \mathbf{p}^m, {\xi_i^t}^m)$$

thereby ensuring that ${d_i^t}^{m+1} = h_i^t({p_i^t}^{m+1}, {p_{-i}^t}^m, {\xi_i^t}^m)$ for all $i \in \mathbf{I}$ and $t \in \mathbf{T}$. As in the previous section, we first introduce some required conditions.

Condition 7.4.1. *For any given $\bar{\mathbf{p}}_i$ and $\bar{\xi}$, $\mathbf{h}_i(\bar{\mathbf{p}}_i, \mathbf{p}_{-i}, \bar{\xi})$ is Lipschitz continuous with respect to $\mathbf{p}_{-i}$ with parameter $\mathcal{L}_h(\bar{\xi})$.*

$$\|\mathbf{h}_i(\bar{\mathbf{p}}_i, \hat{\mathbf{p}}_{-i}, \bar{\xi}) - \mathbf{h}_i(\bar{\mathbf{p}}_i, \breve{\mathbf{p}}_{-i}, \bar{\xi})\| \;\leq\; \mathcal{L}_h(\bar{\xi})\; \|\hat{\mathbf{p}}_{-i} - \breve{\mathbf{p}}_{-i}\|, \;\; \forall\; (\breve{\mathbf{p}}_{-i},\; \hat{\mathbf{p}}_{-i})$$

Condition 7.4.2. *For any given $\bar{\mathbf{p}}_{-i}$ and $\bar{\xi}$, $-\mathbf{h}_i(\mathbf{p}_i, \bar{\mathbf{p}}_{-i}, \bar{\xi})$ is strongly monotone with respect to $\mathbf{p}_i$ with parameter $\mathcal{A}_h(\bar{\xi})$.*

$$\left(-\mathbf{h}_i(\hat{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i}, \bar{\xi}) + \mathbf{h}_i(\breve{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i}, \bar{\xi})\right) \cdot (\hat{\mathbf{p}}_i - \breve{\mathbf{p}}_i) \;\geq\; \mathcal{A}_h(\bar{\xi})\; \|\hat{\mathbf{p}}_i - \breve{\mathbf{p}}_i\|^2, \;\; \forall\; (\breve{\mathbf{p}}_i,\; \hat{\mathbf{p}}_i)$$

Condition 7.4.3. *There exists an $\epsilon_1 > 0$ such that $\frac{\mathcal{L}_h(\xi)}{\mathcal{A}_h(\xi)} \leq 1 - \epsilon_1$, for all ξ, where $\mathcal{A}_h(\xi)$ and $\mathcal{L}_h(\xi)$ are defined as above.*

The conditions on the inverse demand function $\breve{p}_i^t(d_i^t, p_{-i}^t)$ are as follows.

Condition 7.4.4. *For any given* $\bar{\mathbf{d}}$, $\check{\mathbf{p}}(\bar{\mathbf{d}},\mathbf{p})$ *is Lipschitz continuous with respect to* $\mathbf{p}$ *with parameter* $\mathcal{P}_{\check{p}} \geq 0$.

$$\|\check{\mathbf{p}}(\bar{\mathbf{d}},\hat{\mathbf{p}}) - \check{\mathbf{p}}(\bar{\mathbf{d}},\check{\mathbf{p}})\| \;\; \leq \;\; \mathcal{P}_{\check{p}} \, \|\hat{\mathbf{p}} - \check{\mathbf{p}}\|, \;\; \forall \; (\check{\mathbf{p}}, \; \hat{\mathbf{p}})$$

Condition 7.4.5. *For any given* $\bar{\mathbf{p}}$, $\check{\mathbf{p}}(\mathbf{d},\bar{\mathbf{p}})$ *is Lipschitz continuous with respect to* $\mathbf{d}$ *with parameter* $\mathcal{D}_{\check{p}} \geq 0$.

$$\|\check{\mathbf{p}}(\hat{\mathbf{d}},\bar{\mathbf{p}}) - \check{\mathbf{p}}(\check{\mathbf{d}},\bar{\mathbf{p}})\| \;\; \leq \;\; \mathcal{D}_{\check{p}} \, \|\hat{\mathbf{d}} - \check{\mathbf{d}}\|, \;\; \forall \; (\check{\mathbf{d}}, \; \hat{\mathbf{d}})$$

We define the function $\delta(\mathbf{d},\mathbf{p})$ as follows.

$$\delta(\mathbf{d},\mathbf{p}) = \left[-d_i^t \frac{\partial \check{p}_i^t}{\partial d_i^t}(d_i^t, p_{-i}^t) \right]_{i \times t}$$

where $\check{p}_i^t(d_i^t, p_{-i}^t)$ is the inverse of the demand function defined above. On this function, we impose the following conditions.

Condition 7.4.6. *For any given* $\bar{\mathbf{d}}$, $\delta(\bar{\mathbf{d}},\mathbf{p})$ *is Lipschitz continuous with respect to* $\mathbf{p}$ *with parameter* $\mathcal{L}_{\delta} \geq 0$.

$$\|\delta(\bar{\mathbf{d}},\hat{\mathbf{p}}) - \delta(\bar{\mathbf{d}},\check{\mathbf{p}})\| \;\; \leq \;\; \mathcal{L}_{\delta} \, \|\hat{\mathbf{p}} - \check{\mathbf{p}}\|, \;\; \forall \; (\check{\mathbf{p}}, \; \hat{\mathbf{p}})$$

Condition 7.4.7. *For any given* $\bar{\mathbf{p}}$, $\delta(\mathbf{d},\bar{\mathbf{p}})$ *is strongly monotone with respect to* $\mathbf{d}$ *with parameter* $\mathcal{A}_{\delta} > 0$.

$$\Big(\delta(\hat{\mathbf{d}},\bar{\mathbf{p}}) - \delta(\check{\mathbf{d}},\bar{\mathbf{p}})\Big) \cdot (\hat{\mathbf{d}} - \check{\mathbf{d}}) \;\; \geq \;\; \mathcal{A}_{\delta} \, \|\hat{\mathbf{d}} - \check{\mathbf{d}}\|^2, \;\; \forall \; (\check{\mathbf{d}}, \; \hat{\mathbf{d}})$$

Condition 7.4.8. *Let parameters* $\mathcal{P}_{\check{p}}$, $\mathcal{D}_{\check{p}}$, $\mathcal{L}_{\delta}$ *and* $\mathcal{A}_{\delta}$ *are defined as above, there exists an* $\epsilon_2 > 0$ *such that*

$$\mathcal{P}_{\check{p}} + \mathcal{D}_{\check{p}} \frac{\mathcal{L}_{\delta}}{\mathcal{A}_{\delta}} < 1 - \epsilon_2.$$

Note, that for the two seller, linear demand case with rectangular uncertainty bounds the above conditions hold when for all $i \in \mathbf{I}$, the minimum sensitivity of seller i's demand to seller i's price over all periods, is greater than the maximum sensitivity of her demand to her competitor's price over all periods. In particular, if the demand for a two seller market is given by $h_i^t(p_i^t, p_{-i}^t, \xi) = D^t_{\text{base}i} - \beta_i^t p_i^t + \alpha_i^t p_{-i}^t$ and the uncertainty sets are of the form

$$\left\{ (D^t_{\text{base}i}, \beta_i^t, \alpha_i^t) \mid D^t_{\text{base}i} \in \left[\check{D}^t_{\text{base}i}, \hat{D}^t_{\text{base}i} \right], \; \beta_i^t \in \left[\check{\beta}_i^t, \hat{\beta}_i^t \right] \alpha_i^t \in \left[\hat{\alpha}_i^t, \check{\alpha}_i^t \right] \forall i, t \right\}$$

then $\mathcal{P}_{\check{p}} = \sup_{\mathcal{U}} \max_{i,t}(\frac{\alpha_i^t}{\beta_i^t})$, $\mathcal{D}_{\check{p}} = \sup_{\mathcal{U}} \max_{i,t}(\frac{1}{\beta_i^t})$, $\mathcal{L}_{\delta} = 0$ and $\mathcal{A}_{\delta} = \sup_{\mathcal{U}} \min_{i,t}(\frac{1}{\beta_i^t})$. Hence, for such a case, the conditions 7.4.3 and 7.4.8 hold.

Theorem 7.4.1. *Under Conditions 7.4.3 and 7.4.8, Algorithm 1 converges to an equilibrium pricing policy.*

Proof. The solution to Step $m+1$ of Algorithm 1 is $\mathbf{d}_i^{m+1}$ (with the corresponding ξ_i^m) and we obtain the corresponding $\mathbf{p}_i^{m+1}$ using the relation

$$p_i^{t\,m+1} = \breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m, \xi_i^m)$$

thereby ensuring that $d_i^{t\,m+1} = h_i^t(p_i^{t\,m+1}, p_{-i}^{t\;\,m}, \xi_i^m)$ for all $i \in \mathbf{I}$ and $t \in \mathbf{T}$. The equivalent variational inequality problem for step $(m+1)$ is thus to find a $\mathbf{d}_i{}^{m+1} \in \mathcal{K}_{d_i}$ such that

$$\sum_{t\in\mathbf{T}}(-\breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m) - d_i^{t\,m+1}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m+1}, \mathbf{p}^m))(d_i^t - d_i^{t\,m+1}) \geq 0 \tag{7.12}$$

for all $\mathbf{d}_i \in \mathcal{K}_{d_i}$ where $\mathcal{K}_{d_i} = \left\{\mathbf{d}_i \mid \sum_{t=1}^T d_i^t \leq C_i,\ d_i^t \geq 0\ \forall t \in \mathbf{T}\right\}$. Similarly, the variational inequality problem for step m is to find a $\mathbf{d}_i{}^m \in \mathcal{K}_{d_i}$ such that

$$\sum_{t\in\mathbf{T}}(-\breve{p}_i^t(d_i^{t\,m}, \mathbf{p}^{m-1}) - d_i^{t\,m}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m}, \mathbf{p}^{m-1}))(d_i^t - d_i^{t\,m}) \geq 0 \tag{7.13}$$

for all $\mathbf{d}_i \in \mathcal{K}_{d_i}$. Substituting $\mathbf{d}_i = \mathbf{d}_i{}^m$ in (7.12) and $\mathbf{d}_i = \mathbf{d}_i{}^{m+1}$ in (7.13) and adding the two gives rise to,

$$\begin{aligned}\sum_{t\in\mathbf{T}} &\left(\breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m) - \breve{p}_i^t(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) + \\ &\sum_{t\in\mathbf{T}}\left(d_i^{t\,m+1}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m+1}, \mathbf{p}^m) - d_i^{t\,m}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) \quad \geq \quad 0\end{aligned} \tag{7.14}$$

Adding the above for all $i \in \mathbf{I}$,

$$\begin{aligned}\sum_{i\in\mathbf{I}}\sum_{t\in\mathbf{T}} &\left(\breve{p}_i^t(d_i^{t\,m+1}, \mathbf{p}^m) - \breve{p}_i^t(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) + \\ &\sum_{i\in\mathbf{I}}\sum_{t\in\mathbf{T}}\left(d_i^{t\,m+1}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m+1}, \mathbf{p}^m) - d_i^{t\,m}\frac{\partial \breve{p}_i^t}{\partial d_i^t}(d_i^{t\,m}, \mathbf{p}^{m-1})\right)(d_i^{t\,m+1} - d_i^{t\,m}) \quad \geq \quad 0\end{aligned}$$

In vector notation, this can be written as

$$\begin{aligned}&\left(\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\right)(\mathbf{d}^{m+1} - \mathbf{d}^m) + \\ &\qquad \left(-\delta(\mathbf{d}^{m+1}, \mathbf{p}^m) + \delta(\mathbf{d}^m, \mathbf{p}^{m-1})\right)(\mathbf{d}^{m+1} - \mathbf{d}^m) \quad \geq \quad 0\end{aligned} \tag{7.15}$$

We consider two cases. From inequality (7.15), we can conclude that at least one of the two terms (the first or the second term) is greater than or equal to zero.

Case 1: The first term in (7.15) is non-negative and can be written as

$$\left(\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m, \xi^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1}, \xi^{m-1})\right)(\mathbf{d}^{m+1} - \mathbf{d}^m) \geq 0$$

Note that

$$\begin{aligned}
d_i^{tm+1} &= h_i^t(p_i^{tm+1}, p_i^{tm}, \xi_i^m), \\
d_i^{tm} &= h_i^t(p_i^{tm}, p_i^{tm-1}, \xi^{m-1}), \\
\breve{p}_i^t(d_i^{tm+1}, \mathbf{p}^m, \xi_i^m) &= p_i^{tm+1} \text{ and} \\
\breve{p}_i^t(d_i^{tm}, \mathbf{p}^{m-1}, \xi_i^{m-1}) &= p_i^{tm}
\end{aligned}$$

Making these substitutions in the given inequality, we get that

$$\sum_{t \in \mathbf{T}} (p_i^{tm+1} - p_i^{tm})(h_i^t(p_i^{tm+1}, p_i^{tm}, \xi_i^m) - h_i^t(p_i^{tm}, p_i^{tm-1}, \xi_i^{m-1})) \geq 0$$

The above can be written in vector notation as

$$\left(\mathbf{p}_i^{m+1} - \mathbf{p}_i^m\right) \cdot \left(\mathbf{h}_i(\mathbf{p}_i^{m+1}, \mathbf{p}_{-i}^m, \xi_i^m) - \mathbf{h}_i(\mathbf{p}_i^m, \mathbf{p}_{-i}^{m-1}, \xi_i^{m-1})\right) \geq 0 \quad (7.16)$$

Adding $\forall i \in \mathbf{I}$, this is equivalent to

$$\left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \cdot \left(\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m, \xi^m) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1}, \xi^{m-1})\right) \geq 0 \quad (7.17)$$

Here we need to perform an additional step compared to the proof of Theorem 7.3.1. We construct ξ as follows

$$\xi_i^t = \begin{cases} \xi_i^{t,m-1} & p_i^{t,m} \leq p_i^{t,m+1} \\ \xi_i^{t,m} & \text{Otherwise} \end{cases}$$

From the definition of ξ, it follows that componentwise

$$\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m, \xi) \geq \mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m, \xi^m),$$

and

$$\mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1}, \xi) \geq \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1}, \xi^{m-1}).$$

Hence, inequality (7.17) implies that

$$\left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \cdot \left(\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m, \xi) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1}, \xi)\right) \geq 0 \quad (7.18)$$

Adding and subtracting $\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m, \xi)$ we get

$$\begin{aligned}
&\left(\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m, \xi) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^m, \xi) + \mathbf{h}(\mathbf{p}^m, \mathbf{p}^m, \xi) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1}, \xi)\right) \\
&\qquad \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \geq 0.
\end{aligned}$$

On rearranging terms we get

$$\begin{aligned}
&\left(\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m, \xi) - \mathbf{h}(\mathbf{p}^m, \mathbf{p}^{m-1}, \xi)\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right) \geq \\
&\qquad \left((-\mathbf{h}(\mathbf{p}^{m+1}, \mathbf{p}^m, \xi)) - (-\mathbf{h}(\mathbf{p}^m, \mathbf{p}^m, \xi))\right) \cdot \left(\mathbf{p}^{m+1} - \mathbf{p}^m\right)
\end{aligned} \quad (7.19)$$

Under Condition 7.4.2, the right hand side of (7.20) is non negative. Thus, both sides are non-negative and (7.20) becomes equivalent to

$$\begin{aligned}&\left|\left(\mathbf{h}(\mathbf{p}^m,\mathbf{p}^m,\xi)-\mathbf{h}(\mathbf{p}^m,\mathbf{p}^{m-1},\xi)\right)\cdot\left(\mathbf{p}^{m+1}-\mathbf{p}^m\right)\right|\\ &\qquad\geq\left|\left((-\mathbf{h}(\mathbf{p}^{m+1},\mathbf{p}^m,\xi))-(-\mathbf{h}(\mathbf{p}^m,\mathbf{p}^m,\xi))\right)\cdot\left(\mathbf{p}^{m+1}-\mathbf{p}^m\right)\right|\end{aligned}$$

Under Conditions 7.4.1 and 7.4.2 it follows that:

$$\begin{aligned}&\mathcal{L}_h(\xi)\|\mathbf{p}^m-\mathbf{p}^{m-1}\|\cdot\|\mathbf{p}^{m+1}-\mathbf{p}^m\|\\ &\qquad\geq\|\mathbf{h}(\mathbf{p}^m,\mathbf{p}^m,\xi)-\mathbf{h}(\mathbf{p}^m,\mathbf{p}^{m-1},\xi)\|\cdot\|\mathbf{p}^{m+1}-\mathbf{p}^m\|\\ &\qquad\geq\left(\mathbf{h}(\mathbf{p}^m,\mathbf{p}^m,\xi)-\mathbf{h}(\mathbf{p}^m,\mathbf{p}^{m-1},\xi)\right)\cdot\left(\mathbf{p}^{m+1}-\mathbf{p}^m\right)\\ &\qquad\geq\left((-\mathbf{h}(\mathbf{p}^{m+1},\mathbf{p}^m,\xi))-(-\mathbf{h}(\mathbf{p}^m,\mathbf{p}^m,\xi))\right)\cdot\left(\mathbf{p}^{m+1}-\mathbf{p}^m\right)\\ &\qquad\geq\mathcal{A}_h(\xi)\|\mathbf{p}^{m+1}-\mathbf{p}^m\|^2\end{aligned}$$

The first step follows from Lipschitz continuity. The second step follows from Cauchy's Inequality. The third step follows from (7.20). The fourth step follows from the strong monotonicity assumption. As a result,

$$\|\mathbf{p}^{m+1}-\mathbf{p}^m\| \;\leq\; \frac{\mathcal{L}_h(\xi)}{\mathcal{A}_h(\xi)}\cdot\|\mathbf{p}^m-\mathbf{p}^{m-1}\|$$

Case 2: The second term in (7.15) is non-negative, ie.

$$\left(-\delta(\mathbf{d}^{m+1},\mathbf{p}^m)+\delta(\mathbf{d}^m,\mathbf{p}^{m-1})\right)\left(\mathbf{d}^{m+1}-\mathbf{d}^m\right)\geq 0.$$

Adding and subtracting $\delta(\mathbf{d}^m,\mathbf{p}^m)$ in above, we get

$$\begin{aligned}&\left(-\delta(\mathbf{d}^{m+1},\mathbf{p}^m)+\delta(\mathbf{d}^m,\mathbf{p}^m)-\delta(\mathbf{d}^m,\mathbf{p}^m)+\delta(\mathbf{d}^m,\mathbf{p}^{m-1})\right)\\ &\qquad\qquad\cdot(\mathbf{d}^{m+1}-\mathbf{d}^m)\geq 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\left(\delta(\mathbf{d}^{m+1},\mathbf{p}^m)-\delta(\mathbf{d}^m,\mathbf{p}^m)\right)\left(\mathbf{d}^{m+1}-\mathbf{d}^m\right)\\ &\qquad\leq\left(\delta(\mathbf{d}^m,\mathbf{p}^{m-1})-\delta(\mathbf{d}^m,\mathbf{p}^m)\right)\left(\mathbf{d}^{m+1}-\mathbf{d}^m\right).\end{aligned}$$

Under concavity of the demand function (and hence the inverse demand function) the left hand side of the above inequality is non-negative. Hence, both sides of the inequality are non negative. Taking the absolute value on both sides gives us the following relation.

$$\begin{aligned}&\left|\left(\delta(\mathbf{d}^{m+1},\mathbf{p}^m)-\delta(\mathbf{d}^m,\mathbf{p}^m)\right)\left(\mathbf{d}^{m+1}-\mathbf{d}^m\right)\right|\\ &\qquad\leq\left|\left(\delta(\mathbf{d}^m,\mathbf{p}^{m-1})-\delta(\mathbf{d}^m,\mathbf{p}^m)\right)\left(\mathbf{d}^{m+1}-\mathbf{d}^m\right)\right|\end{aligned}\tag{7.20}$$

Therefore,

$$\begin{aligned}&\mathcal{L}_\delta\|\mathbf{p}^{m-1}-\mathbf{p}^m\|\cdot\|\mathbf{d}^{m+1}-\mathbf{d}^m\|\\ &\qquad\geq\|\delta(\mathbf{d}^m,\mathbf{p}^{m-1})-\delta(\mathbf{d}^m,\mathbf{p}^m)\|\cdot\|\mathbf{d}^{m+1}-\mathbf{d}^m\|\\ &\qquad\geq\|(\delta(\mathbf{d}^m,\mathbf{p}^{m-1})-\delta(\mathbf{d}^m,\mathbf{p}^m))\cdot(\mathbf{d}^{m+1}-\mathbf{d}^m)\|\\ &\qquad\geq(\delta(\mathbf{d}^{m+1},\mathbf{p}^m)-\delta(\mathbf{d}^m,\mathbf{p}^m))\cdot(\mathbf{d}^{m+1}-\mathbf{d}^m)\\ &\qquad\geq\mathcal{A}_\delta\|\mathbf{d}^{m+1}-\mathbf{d}^m\|^2.\end{aligned}$$

The first step follows from Lipschitz continuity. The second step follows from Cauchy's Inequality. The third step follows from (7.20). The fourth step follows from the strong monotonicity assumption. As a result,

$$\mathcal{L}_\delta \|\mathbf{p}^{m-1} - \mathbf{p}^m\| \;\geq\; \mathcal{A}_\delta \|\mathbf{d}^{m+1} - \mathbf{d}^m\|. \tag{7.21}$$

Now consider that

$$\begin{aligned}
&\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \\
&\quad = \|\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\| \\
&\quad \leq \|\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\| \\
&\quad = \|\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m) + \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\| \\
&\quad \leq \|\breve{\mathbf{p}}(\mathbf{d}^{m+1}, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m)\| + \|\breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m) - \breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^{m-1})\| \\
&\quad \leq \mathcal{D}_p \|\mathbf{d}^{m+1} - \mathbf{d}^m\| + \mathcal{P}_p \|\mathbf{p}^m - \mathbf{p}^{m-1}\| \\
&\quad \leq \mathcal{D}_p \frac{\mathcal{L}_\delta}{\mathcal{A}_\delta} \|\mathbf{p}^m - \mathbf{p}^{m-1}\| + \mathcal{P}_p \|\mathbf{p}^m - \mathbf{p}^{m-1}\| \\
&\quad = \left(\mathcal{D}_p \frac{\mathcal{L}_\delta}{\mathcal{A}_\delta} + \mathcal{P}_p\right) \|\mathbf{p}^m - \mathbf{p}^{m-1}\|
\end{aligned}$$

The first step follows from the definition of the inverse demand function. In the second step we add and subtract $\breve{\mathbf{p}}(\mathbf{d}^m, \mathbf{p}^m)$. The first inequality comes from the usual triangle inequality. The second inequality comes from the continuity conditions. The last inequality comes from the inequality (7.21) derived above. As a result,

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \;\leq\; \left(\mathcal{D}_p \frac{\mathcal{L}_\delta}{\mathcal{A}_\delta} + \mathcal{P}_p\right) \|\mathbf{p}^m - \mathbf{p}^{m-1}\|$$

In both cases, we obtain that

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \;\leq\; (1 - \epsilon) \cdot \|\mathbf{p}^m - \mathbf{p}^{m-1}\|,$$

where $\epsilon = \min(\epsilon_1, \epsilon_2)$ and thus,

$$\|\mathbf{p}^{m+1} - \mathbf{p}^m\| \;\leq\; (1 - \epsilon)^m \cdot \|\mathbf{p}^1 - \mathbf{p}^0\|. \tag{7.22}$$

It follows that sequence $\{\|\mathbf{p}^{m+1} - \mathbf{p}^m\|\}_m$ converges to zero at a geometric rate. As a result it follows that sequence $\{\mathbf{p}^m\}_m$ is a Cauchy sequence and thus is a convergent sequence to a price $\mathbf{p}$ which is a stationary equilibrium point. □

Corollary 7.4.1. *Let $(\mathbf{p}^k)_k$ be the sequence generated by Algorithm 1. The number of iterations required to reach a solution within ϵ-distance from $\mathbf{p}^*$ is $O(\frac{ln(\frac{D}{\epsilon\epsilon})}{ln(\frac{1}{1-\epsilon})})$ where D is the diameter of the feasible space $\mathcal{K}$ and ϵ is as defined above.*

Proof. The proof is similar to that in Corollary 7.3.1. □

Chapter 8

Numerical results

In this chapter we illustrate some of the results obtained in this thesis through numerical examples. The purpose of this exercise is to show the general trends for the equilibrium pricing policies and revenues for each seller when different factors are varied, and demonstrate the convergence behavior of the computational algorithm. We also compare the performance of the robust policy with non-robust counterparts and experiment with the robustness budget for a seller. The goal is to demonstrate the results qualitatively using simple examples.

This chapter is divided into three parts. In the first part we study the general properties of the equilibrium policies that the model results in. In particular, we observe the changes in pricing policies when the following factors are varied:

1. Price sensitivity of demand in different time periods on the price levels in those periods
2. Starting inventory of the seller and the effect on her pricing policy
3. Starting inventory of a competitor and the effect on a seller's pricing policy
4. Number of sellers in the market.

In the second part, we study the convergence behavior of the algorithm. We track the number of iterations it takes to converge when different parameters of the problem are varied. The factors that potentially would affect the complexity of the problem are:

1. Relative sensitivities of demand to seller's and competitors' price
2. Starting estimate of policies of each seller
3. Number of sellers in the market
4. Number of periods in the market.

In the third part, we study the performance of robust pricing policies and analyze numerically the effect of varying the robustness budget for a seller. We also study the effect of a seller adopting a robust policy on the equilibrium policies of all sellers.

All computations in this thesis (including this example) were performed using MATLAB[1] Version 6.1.0.450 Release 12.1 on a IBM ThinkPad[2] with a Mobile Intel Pentium 4 Processor-M[3] 1.60GHz CPU and 256MB RAM on a Microsoft Windows 2000[4] platform.

8.1 General properties of the equilibrium policies

In this section, we use examples with deterministic demand to show the general properties of the policies at equilibrium. Note that the results presented hold for general demand functions though in this section we use the linear demand case for illustration. As mentioned above, we study the nature of the resulting equilibrium pricing policies when the following parameters are varied:

1. Price sensitivity of demand in different time periods
2. Starting inventory of the seller
3. Starting inventory of a competitor
4. Number of sellers in the market.

The results observed agree with intuition. In particular, we observe that:

1. The higher the inventory that any seller has available for sale over the entire horizon, the lower the prices that she sets. The revenue earned, however, is higher even though the prices set are lower.
2. Correspondingly, an increase in the inventory of a competitor results in lower revenues for the seller since the competitor reduces prices.
3. Prices are higher in periods with lower price sensitivities.
4. If the number of sellers is increased and the demand function adjusted for the new split, the total revenue earned remains the same. Each seller's share of the revenue is increasing in (but not proportional to) her starting inventory.

For illustration purposes, we consider a two-seller multi-period, symmetric linear demand example. For this example, $\mathbf{I} = \{1, 2\}$ and $\mathbf{T} = \{1, 2, \cdots, 10\}$. The demand is linear in prices and symmetric with respect to both sellers and varies with time: $\forall i \in \mathbf{I}$, the demand function $h_i^t = D_{\text{base}}^t - \beta^t p_i^t + \alpha^t p_{-i}^t$. For this example, we assume that the demand is symmetric for the sake of convenience. Note that the results hold in general for asymmetric demand. We model markets where customers with lower price sensitivities typically arrive in later periods. This is usually the case for airlines etc. As a result, the sensitivity of the demand to the seller's price (and also to her competitor's price) in the examples decreases towards the end of the time horizon.

In Table A.3 we study the trend in pricing policies with varying inventory balances. We consider three cases with different inventories for each of the two sellers. In the first

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case both sellers have excess inventory with $\{C1, C2\} = \{3000, 2000\}$ and the optimal equilibrium policy results in neither of them selling their entire inventory. This case is effectively equal to the uncapacitated case. Figure B-3 shows the evolution of the pricing policies as the algorithm iterates, the resulting equilibrium prices, the remaining inventory over the time horizon under the equilibrium prices, and the cumulative revenue from those sales. In the second case, only one of them is over-inventoried ($\{C1, C2\} = \{3000, 500\}$). Figure B-4 shows the results from this case. Note that the seller with less inventory sets prices higher than the seller with higher inventory. Even though the average price is lower for the latter, her total revenues are higher. The prices in general are also higher than in the previous case. Finally, in the third case, neither has sufficient inventory ($\{C1, C2\} = \{1000, 500\}$) so the demand supply imbalance results in a general price hike (Figure B-5).

In Table A.4 we study the effect of the number of sellers in the market. We start with three sellers, each having an inventory of 500 units (Figure B-7). We compare this to two other situations with only two sellers. In the first situation, we assume that sellers $C1$ and $C2$ combine to form a single seller (with inventory 1000) and seller $C3$ remains alone (Figure B-5). In the second situation, we assume that there are two new sellers with inventories of 750 each (Figure B-6). Of course, we need to adjust the demand function to reflect the presence or absence of a seller and make the markets comparable. We do that by adjusting the parameters so that the net demand in the market is comparable in all cases[5]. On doing that and solving for equilibrium, we find that the total revenue earned by the sellers collectively does not change. In a nutshell, a change in the number of sellers does not increase or decrease the competition in the market or lead to price slashing. We also note that each seller's share of this total revenue is increasing in her starting inventory, but not proportional to it. The seller with a lower starting inventory earns a higher payoff per unit inventory than the seller with a higher starting inventory.

In Table A.5 we study the effect of an asymmetric demand function on the policies. We compare the symmetric case from Figure B-6 in which each seller starts with an inventory of 750. We keep the demand function for Seller 1 the same as before and modify the demand function for Seller 2. The price sensitivities for both sellers are shown in Table A.5. Note that we have made the demand less favorable for Seller 2 by increasing the β's and decreasing the α's. The result is shown in Figure B-8. We note that Seller 2 has lower revenue than Seller 1. The total revenues are lower than the symmetric case since the total demand function has been tightened. In Table A.6 we consider an asymmetry only in the β's for Seller 1 and 2. In particular, we increase the β's for Seller 2 and decrease them for Seller 1. As expected, the revenue for Seller 2 is lower than that for Seller 1 (Figure B-9).

[5]If the demand for Seller i in a three seller market is $\hat{d}_i(p_i, p_{-i})$, $i = 1, 2, 3$, then the demand for a comparable two seller market $\check{d}_i(p_i, p_{-i})$, $i = 1, 2$ should be such that is satisfies the following condition:

$$\sum_{i=1,2,3} \hat{d}_i(p_i, p_{-i}) = \sum_{i=1,2} \check{d}_i(p_i, p_{-i})$$

when $p_1 = p_2 = p_3 = p$.

8.2 Convergence behavior of algorithm

In this part we examine the convergence behavior of Algorithm 1 numerically as the relative ratio of price sensitivities is varied and also as the initial estimate of prices used in the algorithm is varied. We use the same numerical example as in the previous section. In general, numerical experience led us to the following conclusions regarding the practical convergence of the algorithm:

1. The algorithm converges to the equilibrium policies rapidly in practice.
2. The numerical results verify the theoretical analysis regarding convergence of the algorithm to an equilibrium pricing policy when starting from different starting points.
3. The number of iterations taken to converge were dependent on the starting point. Convergence was tested by initializing the algorithms with different initial prices. In general, numerical experience led us to conclude that the number of iterations required to converge were smallest for cases where the starting prices were taken close to the equilibrium prices for all sellers. However, the rate of convergence did not depend on the starting point.
4. Changing the relative ratio of demand sensitivities to price affected the rate of convergence in accordance to Theorem 7.3.1. The prices converged to the equilibrium prices at a geometric rate roughly proportional to the theoretically predicted rate.

We have previously shown that the convergence of Algorithm 1 is geometric. Note that this only gives information about the number of iterations that would be required by the algorithm and does not imply anything about the time needed to complete a single iteration. The complexity of solving an iteration is the same as that of solving a best response problem for each seller. The best response problem, in general, is a non-linear optimization problem with a number of variables proportional to the number of periods. The complexity of solving a best response problem is thus dependent on the number of time periods. Consequently, the time taken to solve an iteration depends on the number of sellers and the number of time periods.

In Table A.7 we study the movement of pricing policies as Algorithm 1 iterates with varying initial estimates for starting prices. Figure B-10 shows how Algorithm 1 converges to the equilibrium pricing policy when starting from four different starting points. We consider prices which are constant over all time periods as our initial estimates. We find that convergence is faster when the starting point is close to the equilibrium price. In Table A.9 we look at the same issue by measuring the 2-norm distance between the price policy vector from successive iterations.

In Table A.8 we study the practical convergence behavior of Algorithm 1 with varying relative price sensitivities. Figure B-11 shows the 2-norm distance between the price vectors $\mathbf{p}$ in the current iteration and the previous iteration of Algorithm 1. The four cases correspond to the choice of different ratios of the sensitivity of seller's demand to her own price and her competitor's price. The fastest convergence (steepest line) occurs for the smallest ratio and vice versa.

In Table A.9 we study the practical convergence behavior of Algorithm 1 with varying initial estimates for starting prices. Figure B-12 shows the 2-norm distance between the price vectors $\mathbf{p}$ in the current iteration and the previous iteration of Algorithm 1. The four cases correspond to the choice of different initial estimates of the $\mathbf{p}$. We observe that the rate of convergence (downward slope of the line) is the same.

8.3 Performance of robust pricing policies

We will consider two numerical examples with four sellers and ten time periods, ie. $\mathbf{I} = \{1, 2, 3, 4\}$ and $\mathbf{T} = \{1, \cdots, 10\}$.

The starting inventories of the sellers in the first example (Robust demand example (1)) are given by $\mathbf{C} = \{55, 466, 636, 843\}$. The demand is linear and of the form:

$$h_i^t(p_i^t, p_{-i}^t, \xi_i^t) = D_{i_{\text{base}}}^t - \beta_i^t p_i^t + \sum_{j \in \mathbf{I},\ j \neq i} \alpha_j^t p_j^t \quad \forall\, i \in \mathbf{I}, t \in \mathbf{T}$$

where $\xi_i^t = (D_{i_{\text{base}}}^t, \beta_i^t, \alpha_{-i}^t)$ can take any value in an uncertainty set $\mathcal{U}_i^t$ given by

$$\mathcal{U}_i^t = \left\{ (D_{i_{\text{base}}}^t, \beta_i^t, \alpha_{-i}^t) \;\middle|\; \begin{array}{l} D_{i_{\text{base}}}^t = \bar{D}_{i_{\text{base}}}^t, \\ \beta_i^t \in (\beta_{i_{\min}}^t, \beta_{i_{\max}}^t), \\ \alpha_j^t \in (\alpha_{j_{\min}}^t, \alpha_{j_{\max}}^t) \quad \forall j \neq i \end{array} \right\}$$

The actual uncertainty set was generated randomly (see Table A.10) and shown graphically in Figure B-13. The values were generated so that the sensitivity to price in different periods shows a decreasing trend with time in order to model a market where customers with lower price sensitivity arrive in later periods. For the sake of simplicity we consider the symmetric demand with respect to the sellers in this numerical example (i.e. same price sensitivities across sellers' demand function) though it need not be so.

We use Algorithm 1 to compute the robust equilibrium policies. Figure B-14 shows the policies for each seller as the algorithm progresses. The robust equilibrium pricing policies found when the algorithm converges are given in Table A.11 and depicted graphically in Figure B-15. Figure B-16 shows the starting inventory level, the total inventory sold and the total payoff under robust equilibrium conditions for each seller.

We pick Seller 3 (chosen arbitrarily) and analyze the performance of different pricing and protection level policies under uncertainty. We compare two policies: the robust pricing and protection level policy of Seller 3 (given in Table A.12) which is obtained when the seller optimizes her policy using the robust best response problem; and the nominal policy that Seller 3 would have chosen if she had ignored the uncertainty in the demand parameters and had optimized after naively assuming the nominal values for the demand parameters (given in Table A.13). We call these policies the robust policy and the nominal policy respectively. We then generate 10,000 instances of the uncertain variables uniformly within the uncertain set and compute, for both policies, the payoff for Seller 3 for every generated instance. The histogram for the payoff in the trials for the robust policy and nominal policy is given in Figure B-17 and Figure B-18 respectively. Figure B-19 shows the comparison between the two. Since the assumption regarding uniformity of the uncertain parameters in the uncertain set is arbitrary, it makes sense to consider the range instead of the distribution of the payoffs. As shown in Figure B-20, the range of payoff from the robust policy ($[1208.55, 1217.85]$) is much narrower and the policy has much better worst-case performance than the the payoff from the nominal policy (Payoff range: $[514.72, 1393.43]$).

The numerical example shows that the robust pricing model behaves like the deterministic demand model in Perakis and Sood [54]. We find that the prices are typically higher in periods where the demand sensitivity to price is lower. Prices set by sellers with higher inventories tend to be typically lower than the prices set by sellers with lower inventories, though their overall profits still remain higher. Regarding the performance of the robust policies, we find that the payoffs are much less sensitive to the uncertain parameters compared to the payoffs when policies which ignore uncertainty in demand. In particular, the worst case performance of robust policies is much better. Finally, Algorithm 1 that is used to compute the robust policies converges to the equilibrium prices rapidly.

An interesting issue to consider is the effect of robustness on the equilibrium prices. Before we move to a more complex example, we study the following two seller example (robust demand example (2)). The demand function is symmetric and the starting inventory is the same (3000 units) for both sellers. The demand parameters are given in Table A.14 and shown graphically in Figure B-21. On running the algorithm (see Figure B-22) the equilibrium policy obtained (Table A.16 and Figure B-23) is symmetric with respect to the sellers since they are identical in all respects. The average iteration time for the algorithm was 0.2721 seconds per iteration (with a standard deviation of 0.0691 seconds.)

Now, we let Seller 2 adopt a policy that is robust towards the demand parameter set in Table A.15 while Seller 1 still considers a policy optimal for the nominal values of the demand parameters. The algorithm converges to a different equilibrium solution (see Figure B-24). This leads to lower prices and a reduction in the payoff to both sellers in general. The equilibrium prices obtained for both sellers are shown in Figure B-25 and given in Table A.17.

Let us now compare the distribution of the payoffs for the sellers in the two cases - one where both sellers assume the nominal values for the demand parameters; and two, where one seller adopts a robust policy and the other does not. Figure B-26 shows the distribution of the payoff to either seller (identical) when neither of them considers adopting a robust policy. The average payoff is $11,416$ with a standard deviation of 415 units. In the second case where Seller 2 considers a robust policy, the average payoff is Sellers 1 and 2 are $10,386$ units with a standard deviation of 339 units, and $9,713$ with a standard deviation of 40 respectively. Note that on adopting a robust policy, Seller 2's payoff has decreased more on average than Seller 1's, but with a much lower standard deviation. The comparison can be made in Figures B-27 and B-28.

We now consider another numerical example (Robust demand example (3)) with four sellers and ten time periods, ie. $\mathbf{I} = \{1, 2, 3, 4\}$ and $\mathbf{T} = \{1, \cdots, 10\}$. The starting inventories of the sellers are given by $\mathbf{C} = \{1000, 1300, 1500, 2800\}$. The demand is linear and of the same form as in the previous example. Table A.18 shows the range for uncertain parameters in the demand function for this robust demand example. This is also shown graphically in Figure B-32.

On running the algorithm for this problem, we note that results similar to the deterministic example are obtained. The equilibrium prices are given in Table A.19. Interestingly, we note that the skewed distribution of starting inventory between sellers leads to an equilibrium where Sellers 1, 2, and 3 manage to sell off their entire inventory while Seller 4 manages to sell only part of his inventory (see Figure B-31). This is similar to what was observed for the deterministic demand case. Figure B-30 shows the equilibrium policies found for the sellers. In this example, we will focus on the average payoff of a seller and its randomness when the robustness parameter is varied. We will do that by measuring the payoff when the uncertain demand parameters are assumed to take values uniformly in the uncertainty range given above but the seller adopts a policy that is robust to only a smaller range of parameters.

Figure B-29 shows the policies during successive iterations of the algorithm. The time taken for an iteration averaged 0.5668 seconds (standard deviation: 0.0726 seconds). We note, however, that most of this time was required by the graph-drawing routines rather than the optimization routines. The optimization problems in an iteration of the algorithm was not a bottleneck for problems of this size.

We now study the effect of robustness on the payoff. We select Seller 1 arbitrarily. The robust and nominal policies for Seller 1 are given in Table A.20 and Table A.21 respectively. The range of payoff for Seller 1 when the robust policy was adopted was from 2684.51 to 2702.01 while the same with the nominal policy was from 2303.80 to 4321.58. Instead of looking at only the range in these two extreme cases, let us now look at the distribution for varying levels of robustness. We vary the range of robustness for the seller and compute the corresponding robust policy. We then observe the distribution of payoffs under this policy assuming that the uncertain demand parameters take values randomly and uniformly within the original range. The results are shown in Figures B-34 (and B-33[6]). As we move from the graph on the top-left to the graph at the bottom-right (row-wise from left to right) we vary the robustness of the policy adopted from very optimistic (Seller assumes that the demand parameters are most favorable) to nominal (Center: Seller assumes that the demand parameters will take the nominal values) to very robust (Seller assumes that the demand parameters could be anywhere in the uncertain area). The graphs show the distribution of the payoff when the actual values are uniform over the uncertain set.

Figure B-34 shows the distribution of the payoff to Seller 1 when she changes her policy from optimistic (top left) to nominal (center) to robust (bottom right). We

[6]Figure B-33 is the same as B-34 except that the distributions are scaled to make them clearer.

note that the standard deviation reduces with decreasing optimism. Also, reduced standard deviation for progressively more robust policies comes with a slight sacrifice in terms of average payoff. Using this knowledge, a seller can adjust her budget of robustness. Figure B-35 shows this tradeoff between average payoff and the standard deviation for different values of robustness. The standard deviation of the payoff (y-axis) verses the expected payoff (x-axis) as the robustness of policies is varied for Seller 1. The right-most point corresponds to optimism (low mean and high std. deviation) and the left-most point corresponds to robustness (some sacrifice of mean with very low std. deviation). The nominal policy point is marked with a circle. The actual values of the mean payoff and the standard deviation are given in Table A.22.

Chapter 9

Conclusions

9.1 Contributions and future research directions

In this thesis we formulated a multi-period pricing model for an oligopoly where each seller has a pre-determined starting inventory and additional production is not an option. We showed that this problem does not have a structure that falls under the framework of game theoretic models such as quasi-concave games or supermodular games. We first focussed on the competitive aspect of the problem, and established existence and uniqueness of the best response policy and the existence of equilibrium pricing policies under deterministic demand. As traditional approaches do not apply to this problem, a key innovation of this thesis was the quasi-variational inequality reformulation. This reformulation allowed us to study existence and uniqueness of the best response policy and existence of equilibrium prices and did not require the payoff functions to be concave. We have shown cases (without the concavity condition on the demand function) for which the market equilibrium does not exist uniquely. We have also considered conditions guaranteeing uniqueness. However, the question of uniqueness under when this condition holds is an interesting open question for future research. To the best of our knowledge, no such analysis has been done before for multi-period price competition models for perishable products.

We addressed the issue of uncertainty in demand for the model via robust optimization. We established existence and uniqueness of the robust best response policy and existence of robust equilibrium policies under such uncertain demand. We have some restrictions on the nature of the uncertainty sets for the demand parameters which somewhat limits the kind of robustness that can be modelled. In particular, we assume that the uncertainty set for the joint set of demand variables from all time periods is a cartesian product of the uncertainty set for the set of demand variables from every single time period (see Chapter 4). This assumption is required in order to reduce the best response problem from the general bilevel program we presented to a single level optimization problem that is easier to analyze. In the absence of this restriction, the robust demand model faces the same kind of difficulties as the stochastic demand model. This would be an interesting future research direction.

In this thesis we presented equilibrium results for the deterministic and robust demand models along with results for two approximate models for the stochastic demand case. The line of analysis that was used for the deterministic and robust demand models could not be used as is for the stochastic case. We established equilibrium results for the special cases where sellers are faced with stochastic linear demand and each seller adopts policies that maximize their expected payoff.

The two approximations we have discussed for the stochastic demand model involve getting rid of either the fixed inventory constraint or the multi-period character of the model. Another challenging research direction is the analysis of the stochastic demand model without the approximations. This is closely linked to the future research proposed for the robust demand model without the current restrictions on the uncertainty set. This is done by assuming an over-sale penalty in the first approximation and by separating the different periods in the second approximation. In either case, we essentially moved away from the multi-period fixed-inventory model and in some sense, moved towards the periodic production review model. This makes the model a supermodular game and the analysis from literature on supermodularity can be applied. Although the approach followed for the deterministic demand model and the robust demand model does not work for the stochastic demand model, we see no reason why a completely different approach might not succeed.

We introduced and studied an algorithm for computing equilibrium pricing policies and analyzed its convergence. In our algorithm we cycle between sellers and solve their best response problems simultaneously at each iteration. There are several variants of the algorithm. One of these is where we could modify the iteration by solving the best response policies of sellers sequentially instead of simultaneously. This does not improve the complexity of the algorithm but practically could be faster than the aforementioned algorithm since we are incorporating information about sellers' policies faster into the market. A numerical or analytical comparison of such variants of the algorithm could be an interesting research direction for the future.

We established that under symmetric policies, (i.e. the same market conditions for all sellers and tight capacities) the equilibrium policies globally maximize the payoffs for all sellers. That is, we showed that the user-optimal and the system-optimal solutions for the model are the same. This result is specific to the structure of the problem we consider. For example, it is not clear if the same result would hold if there were other interactions between the sellers (for example, if the sellers could buy and sell products between themselves at market or pre-agreed rates). We also considered a variant of our model (quantity competition model) from the literature, extending it to a multi period setting, and studied the loss in the efficiency for this formulation. We feel that this is a promising new research field. Finally, we illustrated our results through numerical examples and compare the performance of robust policies with non-robust policies.

There are several extensions to this model that we propose for future research. These include:

- **Incorporating multiple products into the market**: In this thesis we have considered only a single product market. This could be extended to multiple products being sold by sellers, each having her own portfolio of products. The interactions between the demand for one product with another would give rise to interesting results. One could also study an extension where the sellers do not have an inventory of products as such, but has a fixed capacity of a resource (or multiple resources) that are shared by the products. Using such an extension one could model advanced characterization of some markets like the network nature of airlines for the purposes of fare pricing.

- **Stronger interaction between sellers**: In the current model, the only interaction between the sellers is through the demand function through which they influence each other's demand. One possible extension would be to allow stronger interaction, for example, by allowing sellers to buy inventory from each other at pre-determined or market-prevailing prices in each period.

- **Demand learning**: We have assumed that the seller has no knowledge of the demand faced by her competitors. In the real market, sellers try to guess the market share of competitors and the price sensitivity of competitor's demand over time by observing the pricing behavior of sellers over successive games. Such demand learning behavior could be a possible extension of this model.

Appendix A

Tables

To / From	New York, NY (LGA)	New York, NY (JFK)	Newark, NJ (EWR)
Boston, MA (BOS)	$168+	$231+	$241+
Manchester, NH (MHT)	$246+	$345+	$331+
Providence, RI (PVD)	$279+	$289+	$332+

Table A.1: Price variations for options from three Boston airports to three New York airports for flights on 26^{th} November, 2003 between 6 and 10 pm.

	American Airlines	Delta Air Lines	Continental Airlines	United Airlines	US Airways	Multiple airlines	Northwest Airlines
Nonstop	$168+	$231+	$241+	$241+	$241+	-	-
One stop	$338+	$285+	$331+	$256+	$287+	$296+	$1034+
Two stops	-	-	$429+	$629+	$566+	$345+	-

Table A.2: Price variations for different airlines and flight options from Boston to New York on 26^{th} November, 2003 between 6 and 10 pm.

Model parameters held constant	*Model parameters varied*
$D_{\text{base}} = \{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$	$\{C1, C2\} = \{3000, 2000\},$
$\beta = \{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$	$\{3000, 500\},$
$\alpha = \{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$	$\{1000, 500\}$

Table A.3: Deterministic demand example in Chapter 8: Trend in pricing policies with varying inventory balances.

Model parameters held constant	*Model parameters varied*
$D'_{\text{base}} = \frac{2}{3} \cdot D_{\text{base}},\ \beta' = \frac{2}{3} \cdot \beta,\ \alpha' = \frac{1}{3} \cdot \alpha$	$\{C1', C2', C3'\} = \{500, 500, 500\}$
$D_{\text{base}} = \{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$	$\{C1, C2\} = \{1000, 500\}, \{750, 750\}$
$\beta = \{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$	
$\alpha = \{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$	

Table A.4: Deterministic demand example in Chapter 8: Effect of the number of sellers. Demand function parameters are adjusted to reflect presence of third seller.

Model parameters held constant	*Model parameters varied*
	$\beta_1 = \{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$
$D_{\text{base}} = \{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$	$\beta_2 = \{1.3, 1.3, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5\}$
$\{C1, C2\} = \{750, 750\}$	$\alpha_1 = \{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$
	$\alpha_2 = \{0.9, 1.0, 0.9, 0.7, 0.7, 0.6, 0.4, 0.3, 0.3, 0.3\}$

Table A.5: Deterministic demand example in Chapter 8: Effect of asymmetric demand.

Model parameters held constant	*Model parameters varied*
$D_{\text{base}} = \{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$	$\beta_1 = \{1.1, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3\}$
$\alpha = \{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$	$\beta_2 = \{1.3, 1.3, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5\}$
$\{C1, C2\} = \{750, 750\}$	

Table A.6: Deterministic demand example in Chapter 8: Effect of asymmetric β in demand.

Model parameters held constant	*Model parameters varied*
$D_{\text{base}} = \{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$	Starting estimate of prices
$\beta = \{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$	$\forall i \in \mathbf{I}$ and $t \in \mathbf{T}$
$\alpha = \{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$	$p_i^t = 0, 150, 300, 450$
$\{C_1, C_2\} = \{1000, 500\}$	

Table A.7: Deterministic demand example in Chapter 8: Movement of pricing policies in iterations of Algorithm 1 with varying initial estimates for starting prices.

Model parameters held constant	*Model parameters varied*
$D_{\text{base}} = \{110, 105, 100, 95, 90, 85, 80, 75, 75, 85\}$ $\beta = \{1.2, 1.15, 1.1, 1.05, 1, .95, .9, .85, .8, 0.75\}$ $\{C1, C2\} = \{1000, 500\}$	$\alpha = k\beta$ where, $k = \frac{20}{32}, \frac{15}{32}, \frac{10}{52}, \frac{5}{32}$

Table A.8: Deterministic demand example in Chapter 8: Practical convergence behavior of Algorithm 1 with varying relative price sensitivities.

Model parameters held constant	*Model parameters varied*
$D_{\text{base}} = \{110, 105, 100, 95, 90, 85, 80, 75, 75, 85\}$ $\beta = \{1.2, 1.15, 1.1, 1.05, 1, .95, .9, .85, .8, 0.75\}$ $\alpha = \frac{15}{32}\beta$ $\{C1, C2\} = \{1000, 500\}$	Starting estimate of prices $\forall i \in \mathbf{I}$ and $t \in \mathbf{T}$ $p_i^t = 0, 50, 100, 150$

Table A.9: Deterministic demand example in Chapter 8: Practical convergence behavior of Algorithm 1 with varying initial estimates for starting prices.

t	$D^t_{i_{\text{base}}}$	$\beta^t_{i_{\min}}$	$\beta^t_{i_{\text{nominal}}}$	$\beta^t_{i_{\max}}$	$\alpha^t_{i_{\min}}$	$\alpha^t_{i_{\text{nominal}}}$	$\alpha^t_{i_{\max}}$
1	250	162.80	180.89	198.98	32.99	36.66	40.33
2	250	154.47	171.63	188.79	29.60	32.89	36.17
3	250	121.62	135.14	148.65	16.22	18.02	19.82
4	250	119.08	132.31	145.54	15.18	16.87	18.55
5	250	116.85	129.83	142.82	14.27	15.86	17.44
6	250	115.00	127.78	140.55	13.52	15.02	16.52
7	250	111.95	124.39	136.83	12.28	13.64	15.01
8	250	108.06	120.07	132.08	10.69	11.88	13.07
9	250	107.41	119.34	131.28	10.43	11.58	12.74
10	250	93.42	103.81	114.19	4.73	5.25	5.78

Table A.10: Robust demand example (1) in Chapter 8: The range for uncertain parameters in the demand function.

t	Seller 1	Seller 2	Seller 3	Seller 4
1	2.20	1.96	1.86	1.74
2	2.21	1.97	1.87	1.74
3	2.27	2.02	1.91	1.79
4	2.27	2.02	1.92	1.79
5	2.28	2.02	1.92	1.79
6	2.28	2.03	1.92	1.80
7	2.29	2.03	1.93	1.80
8	2.30	2.04	1.94	1.81
9	2.30	2.04	1.94	1.81
10	2.33	2.07	1.96	1.83

Table A.11: Robust demand example (1) in Chapter 8: Equilibrium prices.

t	Price	Protection Level
1	1.86	561.15
2	1.87	488.32
3	1.91	424.24
4	1.92	360.90
5	1.92	298.22
6	1.92	236.10
7	1.93	174.91
8	1.94	114.92
9	1.94	55.14
10	1.96	0.00

Table A.12: Robust demand example (1) in Chapter 8: Robust Policy for Seller 3.

t	Price	Protection Level
1	2.14	557.55
2	2.15	481.81
3	2.18	417.64
4	2.19	354.45
5	2.19	292.11
6	2.19	230.49
7	2.20	170.08
8	2.20	111.23
9	2.20	52.64
10	2.22	0.00

Table A.13: Robust demand example (1) in Chapter 8: Nominal Policy for Seller 3.

t	β_{nom}	α_{nom}
1	139.1299	89.1299
2	132.1407	82.1407
3	129.1937	79.1937
4	126.2097	76.2097
5	111.5432	61.5432
6	110.6843	60.6843
7	98.5982	48.5982
8	95.6468	45.6468
9	94.4703	44.4703
10	51.8504	1.8504

Table A.14: Robust demand example (2) in Chapter 8: The demand parameters for the non-robust competition.

t	β_{min}	β_{nom}	β_{max}	α_{min}	α_{nom}	α_{max}
1	111.3039	139.1299	166.9559	71.3039	89.1299	106.9559
2	105.7126	132.1407	158.5689	65.7126	82.1407	98.5689
3	103.355	129.1937	155.0324	63.355	79.1937	95.0324
4	100.9677	126.2097	151.4516	60.9677	76.2097	91.4516
5	89.2346	111.5432	133.8519	49.2346	61.5432	73.8519
6	88.5474	110.6843	132.8211	48.5474	60.6843	72.8211
7	78.8786	98.5982	118.3179	38.8786	48.5982	58.3179
8	76.5174	95.6468	114.7761	36.5174	45.6468	54.7761
9	75.5763	94.4703	113.3644	35.5763	44.4703	53.3644
10	41.4803	51.8504	62.2204	1.4803	1.8504	2.2204

Table A.15: Robust demand example (2) in Chapter 8: The range of demand parameters for the robust policies for Seller 2.

t	Seller 1	Seller 2
1	3.4716	3.4716
2	3.5616	3.5616
3	3.6017	3.6017
4	3.6436	3.6436
5	3.8722	3.8722
6	3.8869	3.8869
7	4.1115	4.1115
8	4.1720	4.1720
9	4.1968	4.1968
10	5.4821	5.4821

Table A.16: Robust demand example (2) in Chapter 8: The identical equilibrium prices for both sellers.

t	Seller 1	Seller 2
1	3.0515	2.7640
2	3.1473	2.8510
3	3.1901	2.8898
4	3.2349	2.9306
5	3.4807	3.1539
6	3.4965	3.1683
7	3.7403	3.3899
8	3.8063	3.4500
9	3.8335	3.4747
10	5.2764	4.7910

Table A.17: Robust demand example (2) in Chapter 8: The equilibrium prices for sellers when Seller 2 adopts a robust policy.

t	$D^t_{i_{\text{base}}}$	$\beta^t_{i_{\min}}$	$\beta^t_{i_{\text{nominal}}}$	$\beta^t_{i_{\max}}$	$\alpha^t_{i_{\min}}$	$\alpha^t_{i_{\text{nominal}}}$	$\alpha^t_{i_{\max}}$
1	250	107.66	119.62	131.58	27.19	30.21	33.24
2	250	88.16	97.95	107.75	19.25	21.39	23.53
3	250	69.18	76.86	84.55	11.52	12.80	14.08
4	250	65.95	73.28	80.61	10.20	11.34	12.47
5	250	65.44	72.71	79.98	9.99	11.10	12.21
6	250	55.82	62.02	68.22	6.07	6.75	7.42
7	250	52.28	58.09	63.90	4.63	5.15	5.66
8	250	50.67	56.30	61.93	3.98	4.42	4.86
9	250	43.77	48.64	53.50	1.17	1.30	1.43
10	250	42.91	47.68	52.45	0.82	0.91	1.00

Table A.18: Robust demand example (3) in Chapter 8: The range for uncertain parameters in the demand function.

t	Seller 1	Seller 2	Seller 3	Seller 4
1	2.13	1.80	1.58	1.52
2	2.32	1.99	1.77	1.70
3	2.57	2.23	2.01	1.94
4	2.62	2.28	2.06	1.99
5	2.63	2.29	2.07	2.00
6	2.81	2.46	2.23	2.17
7	2.89	2.54	2.30	2.24
8	2.92	2.57	2.34	2.27
9	3.09	2.73	2.50	2.43
10	3.12	2.76	2.52	2.45

Table A.19: Robust demand example (3) in Chapter 8: Equilibrium prices.

t	Price	Protection Level
1	2.13	896.88
2	2.32	791.88
3	2.57	688.18
4	2.62	585.13
5	2.63	482.2
6	2.81	382.27
7	2.89	283.89
8	2.92	186.31
9	3.09	92.85
10	3.12	0.00

Table A.20: Robust demand example (3) in Chapter 8: Robust Policy for Seller 1.

t	Price	Protection Level
1	4.40	822.86
2	4.37	680.32
3	4.33	571.46
4	4.32	468.32
5	4.32	366.09
6	4.29	280.94
7	4.28	202.06
8	4.27	126.04
9	4.23	62.26
10	4.22	0.00

Table A.21: Robust demand example (3) in Chapter 8: Nominal Policy for Seller 1.

t	Mean Payoff	Std. Dev.
1	2448.86	462.31
2	2653.21	462.98
3	2830.85	451.80
4	3007.85	448.98
5	3186.63	431.68
6	3346.20	413.21
7	3495.60	392.20
8	3635.83	373.32
9	3758.09	353.92
10	3875.65	321.42
11	3968.03	292.63
12	4045.08	247.43
13	4093.58	203.90
14	4121.41	157.04
15	4123.44	115.98
16	4106.84	80.15
17	4077.53	53.45
18	4039.17	36.58
19	3996.48	24.84
20	3951.66	17.86
21	3906.23	12.78
22	3860.59	8.69
23	3815.07	5.76
24	3769.83	3.26
25	3725.01	1.34

Table A.22: Robust demand example (3) in Chapter 8: Risk vs. Return for different values of robustness.

Appendix B

Figures

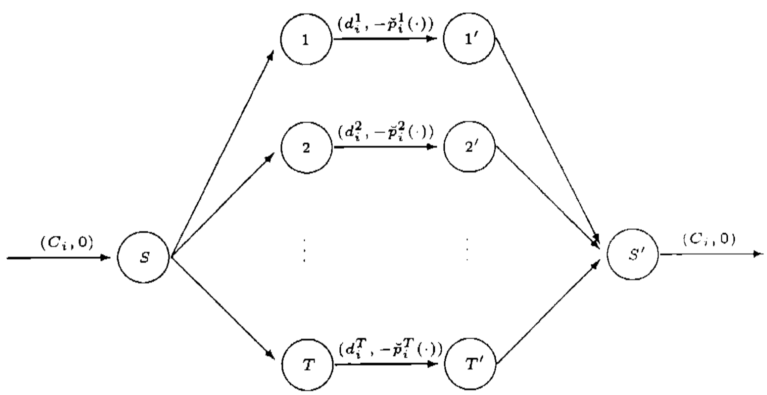

Figure B-1: Representation of the min-cost network flow problem for the best-response problem of Seller i.

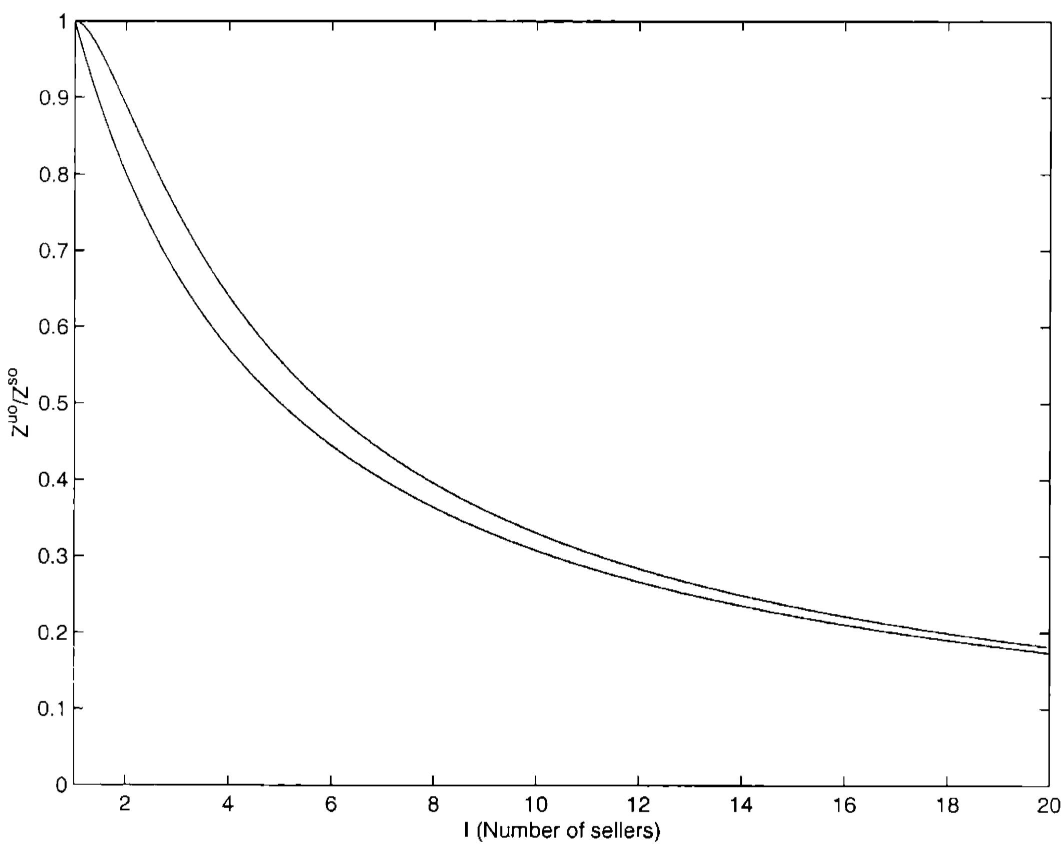

Figure B-2: Comparison of the general lower bound obtained for $\frac{Z^{\mathrm{UO}}}{Z^{\mathrm{SO}}}$ for a seller-symmetric case (bottom line) with the actual ratio (top line) for a time- and seller-symmetric quantity competition game.

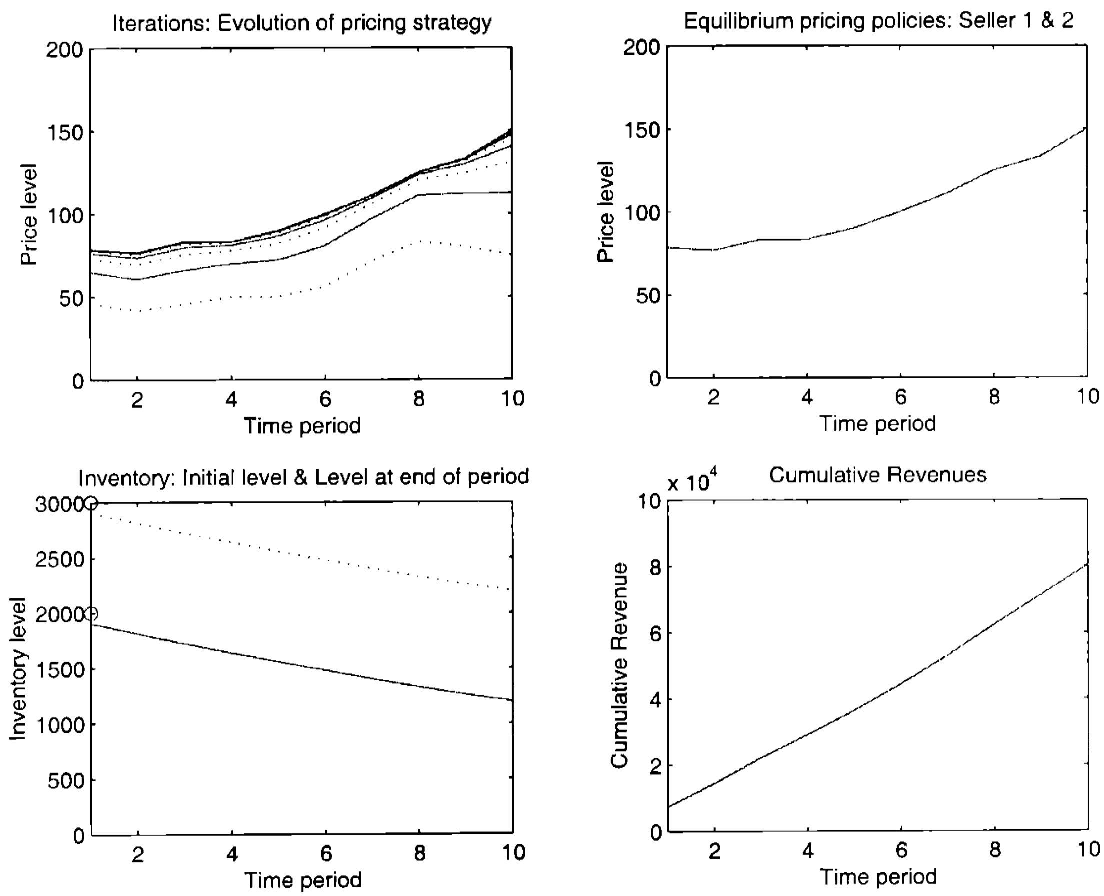

Figure B-3: Deterministic demand example from Chapter 8: Both sellers have excess inventory. $\{C1, C2\} = \{3000, 2000\}$

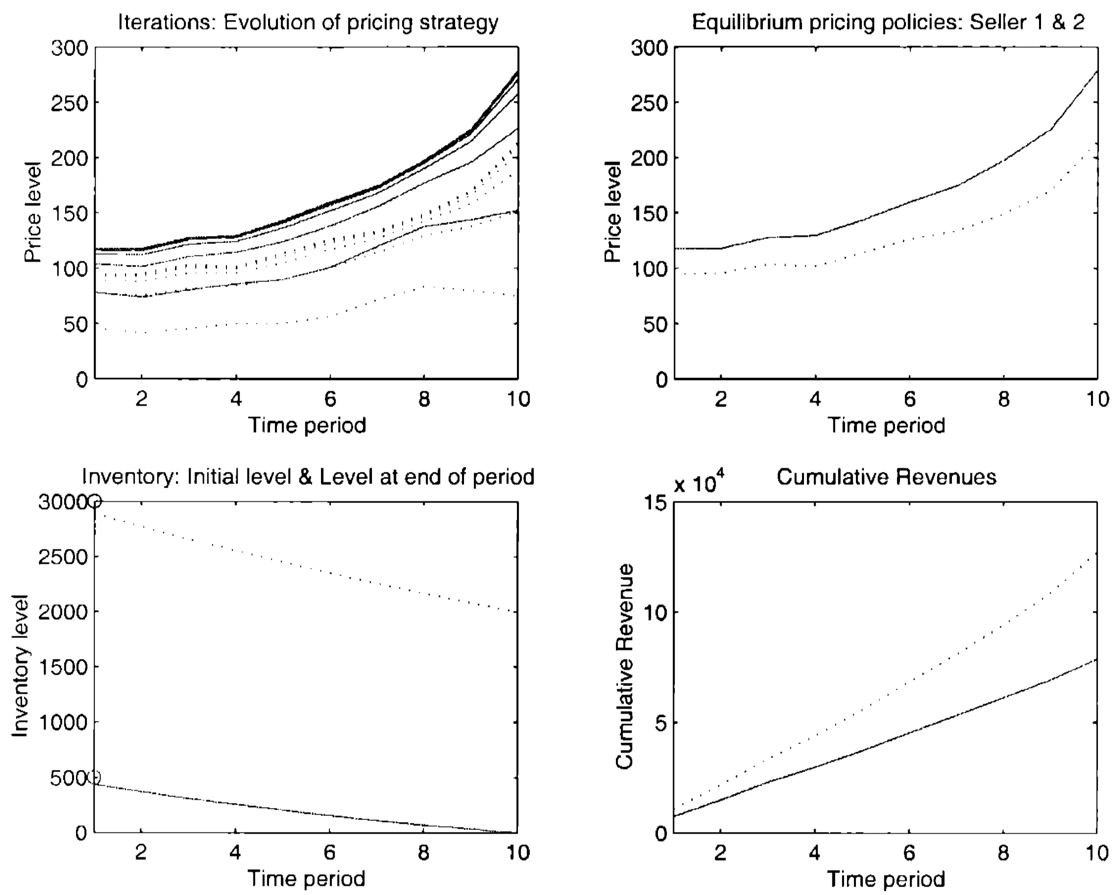

Figure B-4: Deterministic demand example from Chapter 8: One seller has excess inventory. $\{C1, C2\} = \{3000, 500\}$

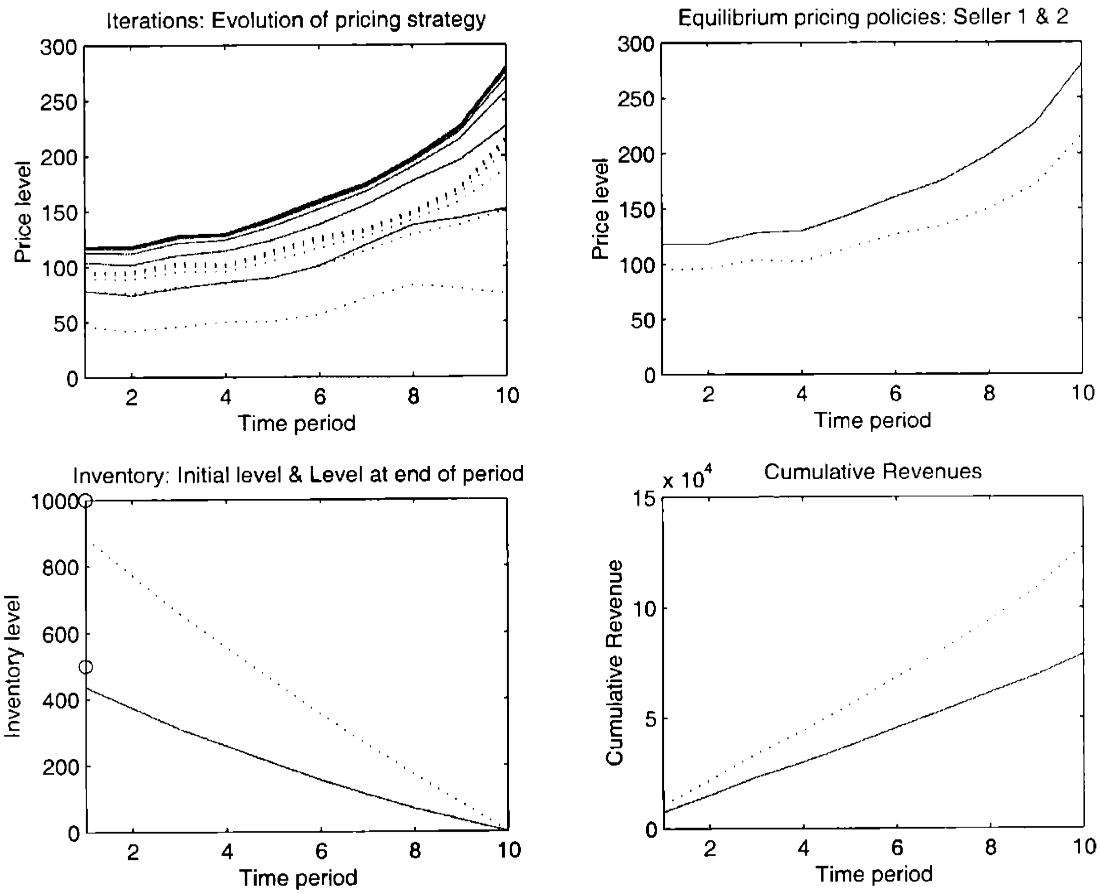

Figure B-5: Deterministic demand example from Chapter 8: Neither of the sellers have excess inventory. $\{C1, C2\} = \{1000, 500\}$

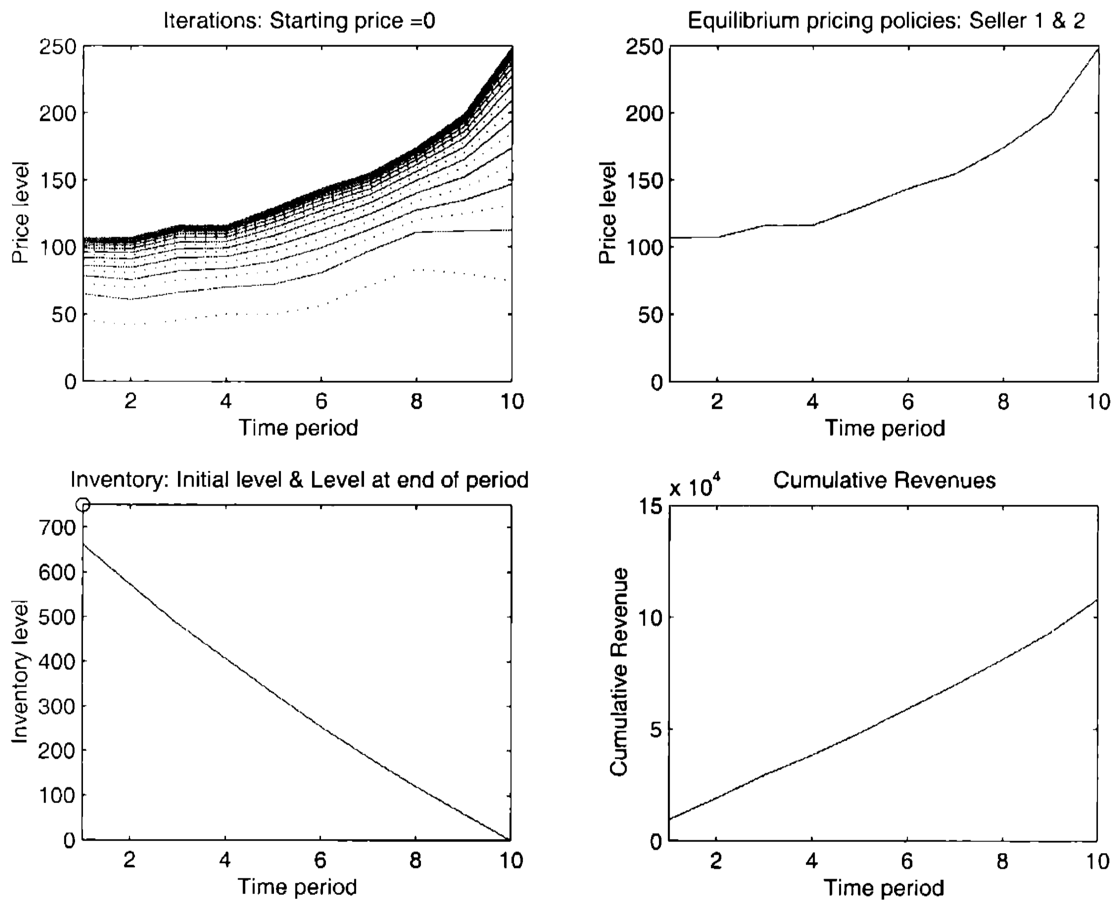

Figure B-6: Deterministic demand example from Chapter 8: A redistribution of inventory over sellers. $\{C1, C2\} = \{750, 750\}$

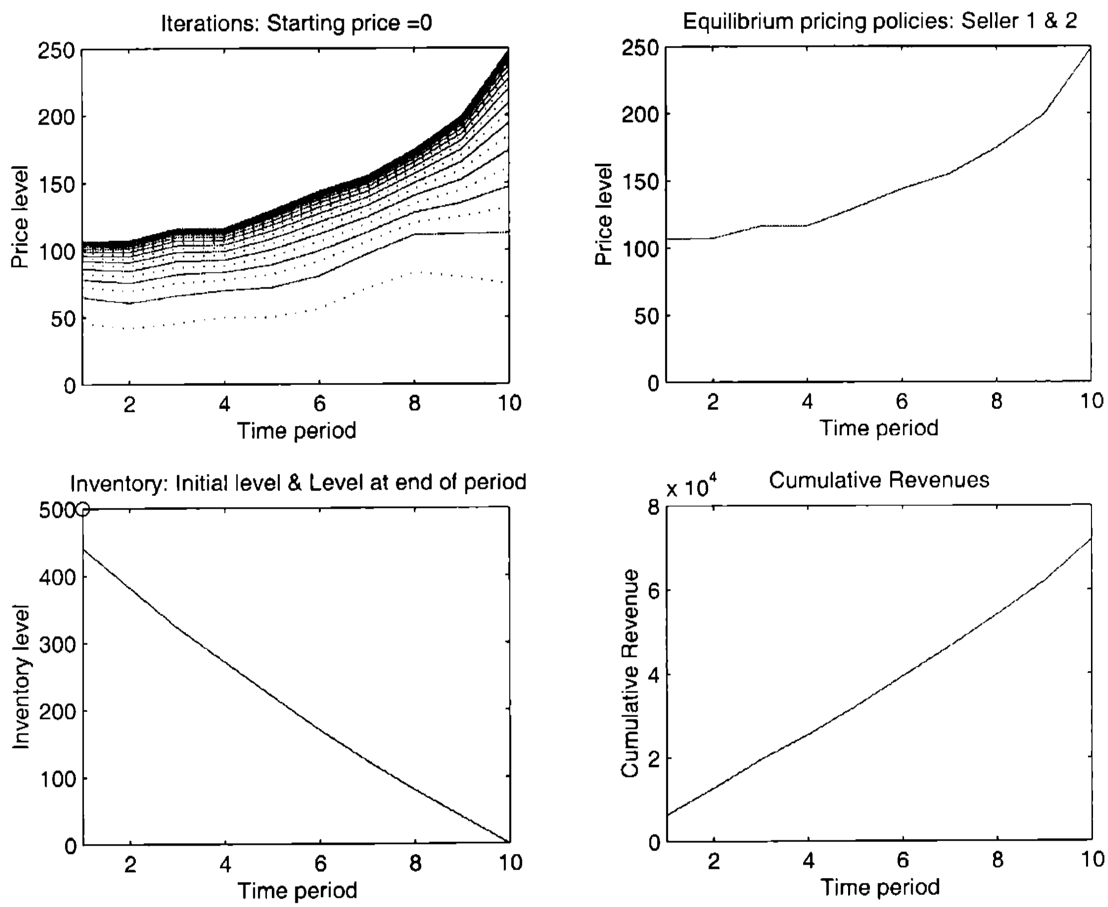

Figure B-7: Deterministic demand example from Chapter 8: Increasing the number of sellers to three. $\{C1, C2, C3\} = \{500, 500, 500\}$

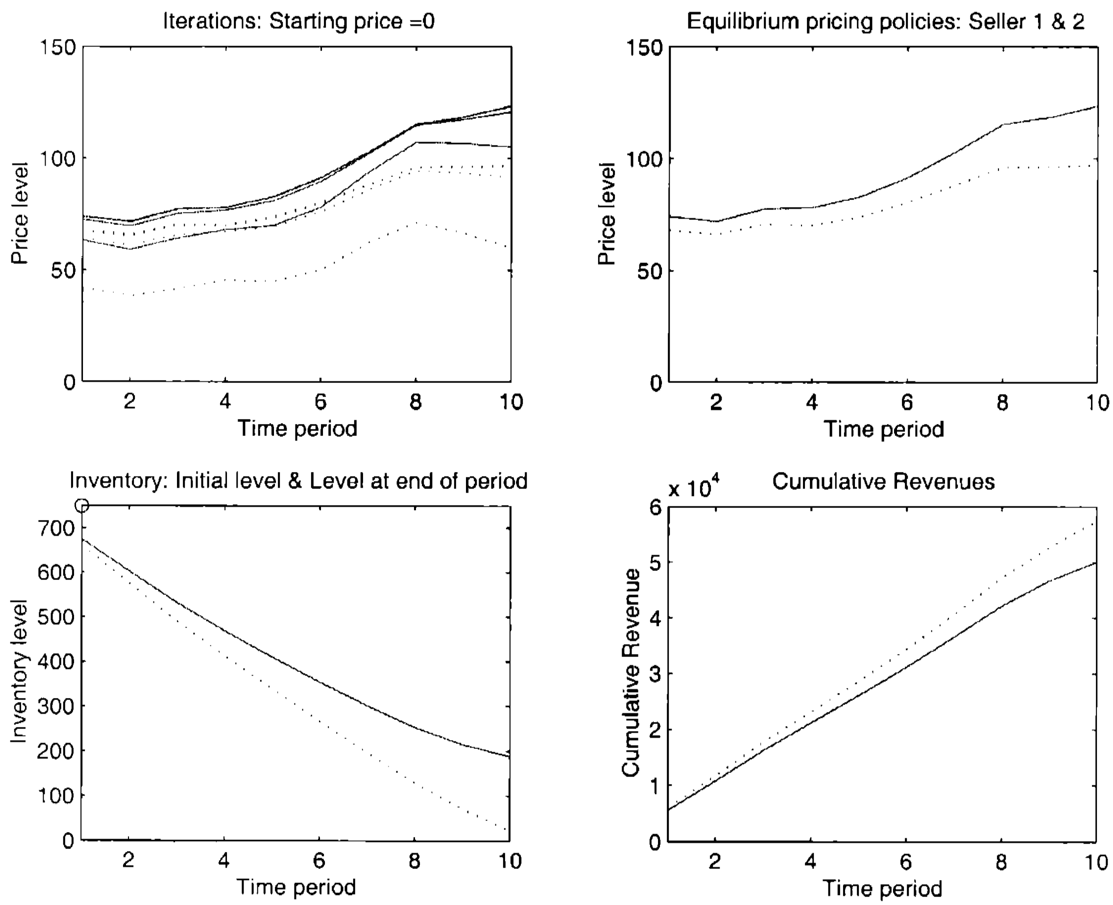

Figure B-8: Deterministic demand example from Chapter 8: Asymmetric demand function $\{C1, C2\} = \{750, 750\}$

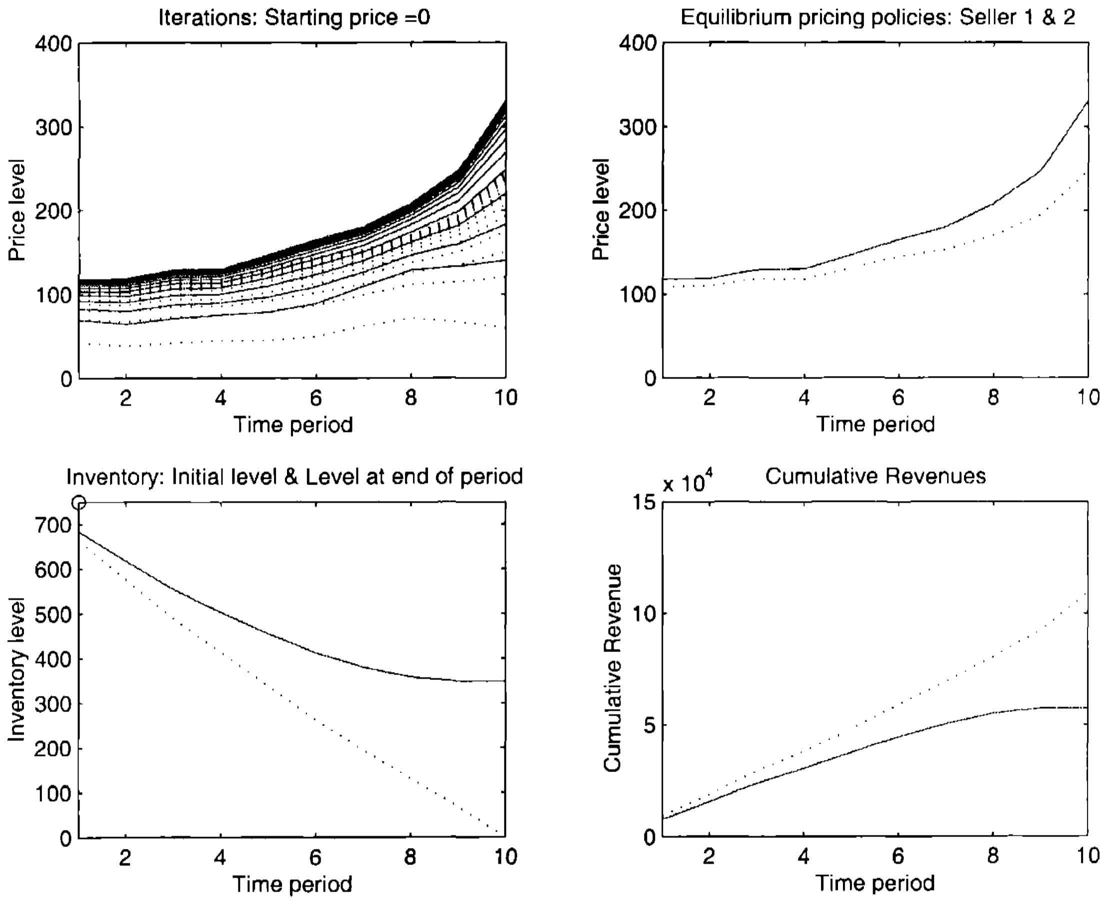

Figure B-9: Deterministic demand example from Chapter 8: Asymmetric β in the demand function $\{C1, C2\} = \{750, 750\}$

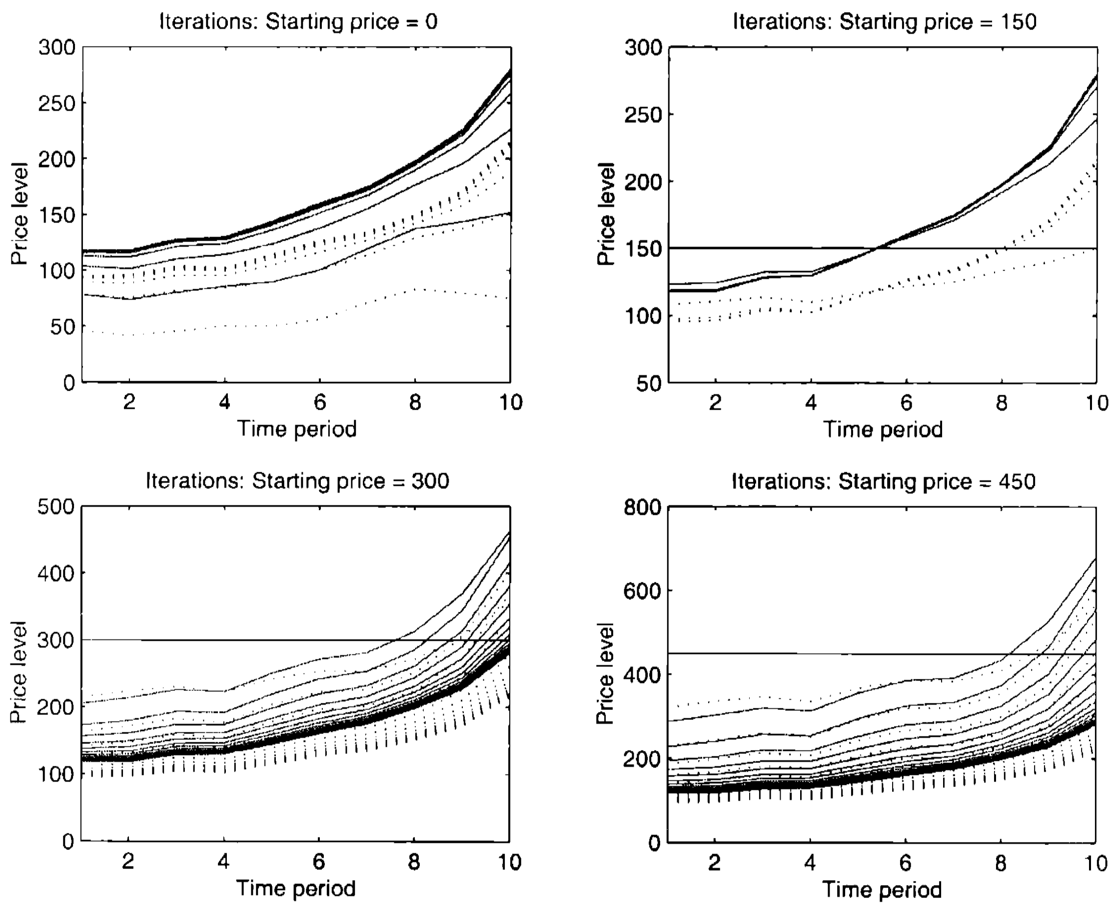

Figure B-10: Deterministic demand example from Chapter 8: Actual trend of pricing policies over successive iterations of Algorithm 1 when starting with different initial prices.

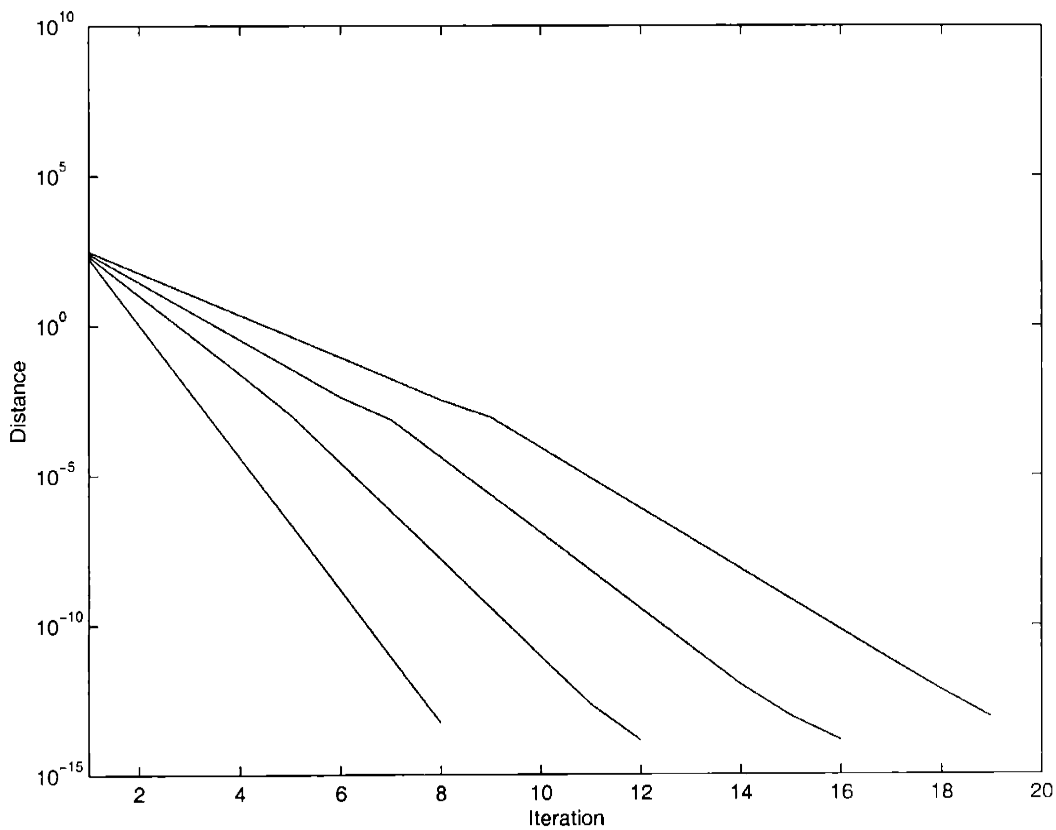

Figure B-11: Deterministic demand example from Chapter 8: Convergence starting with different α.

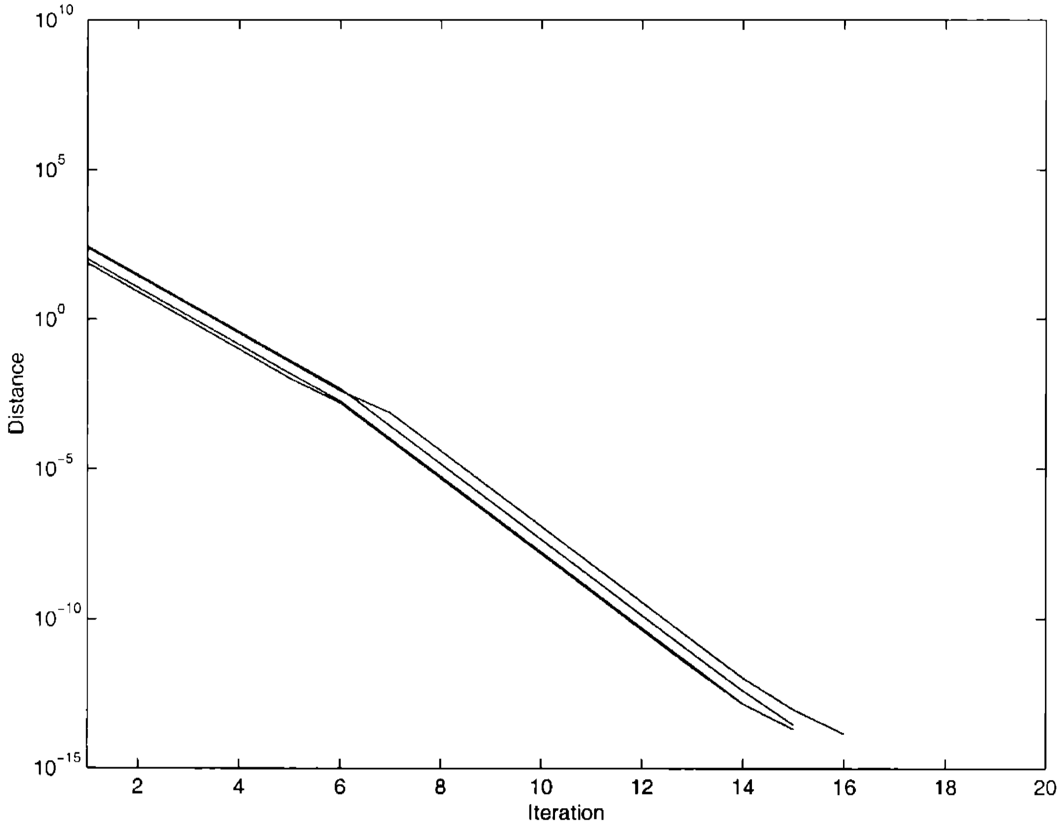

Figure B-12: Deterministic demand example from Chapter 8: Convergence behavior starting with different initial prices.

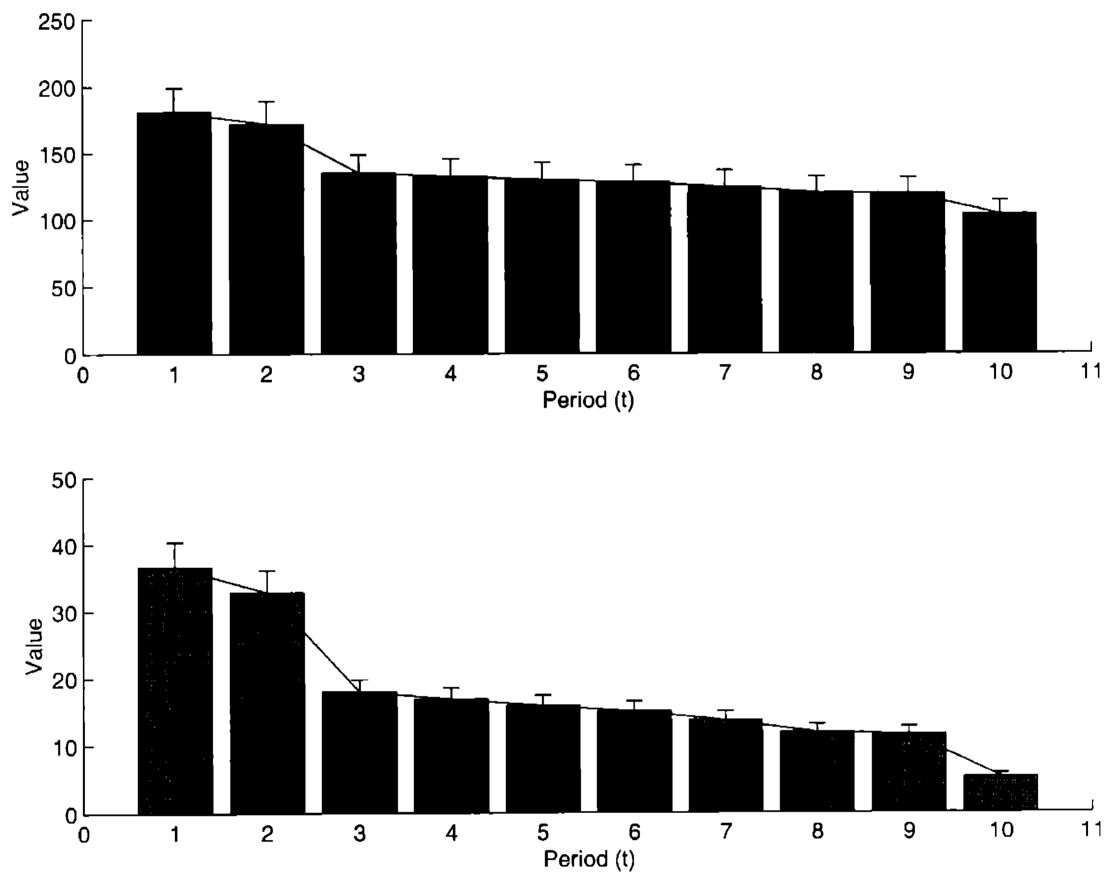

Figure B-13: Robust demand example (1) from Chapter 8: The value of the uncertain parameters β (top) and α (bottom) for any seller i with time period t. The error bars denote the uncertainty in the parameters.

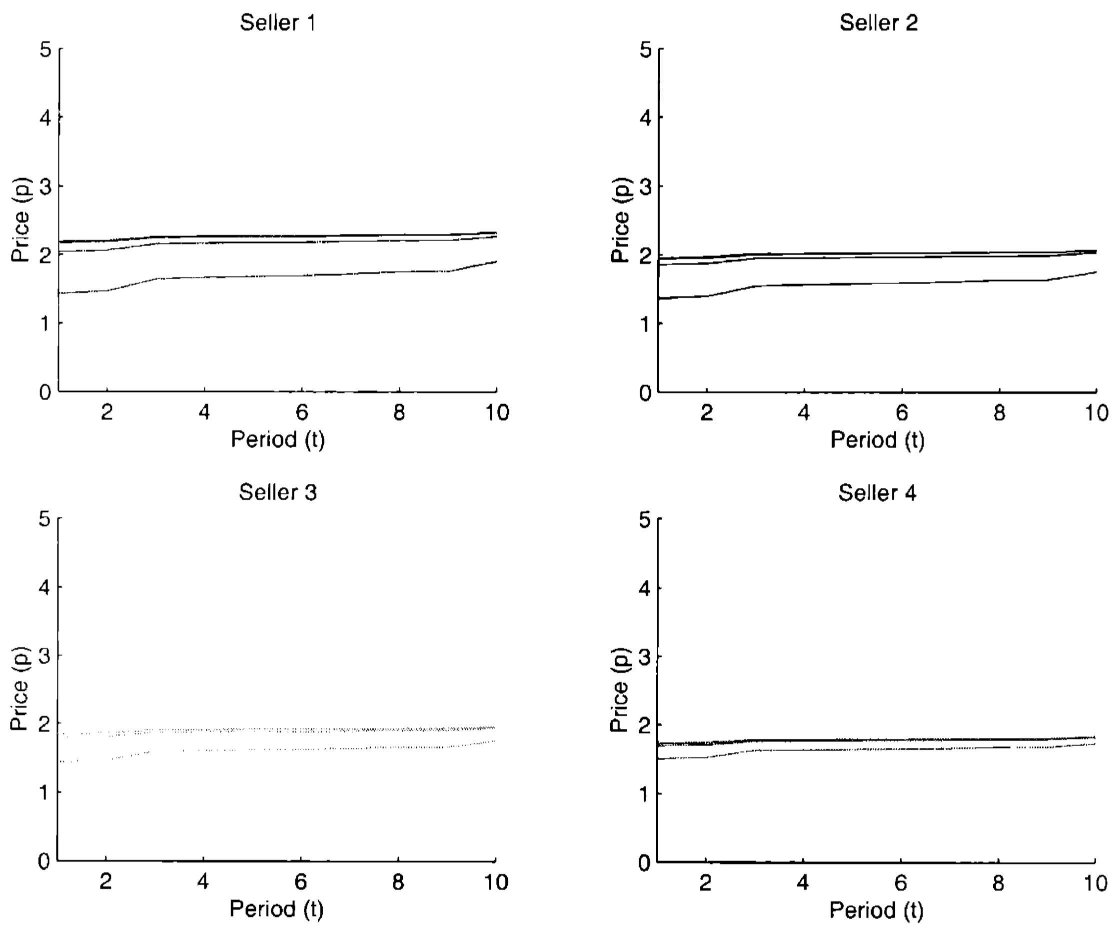

Figure B-14: Robust demand example (1) from Chapter 8: The pricing policies for all sellers over successive iterations of the algorithm.

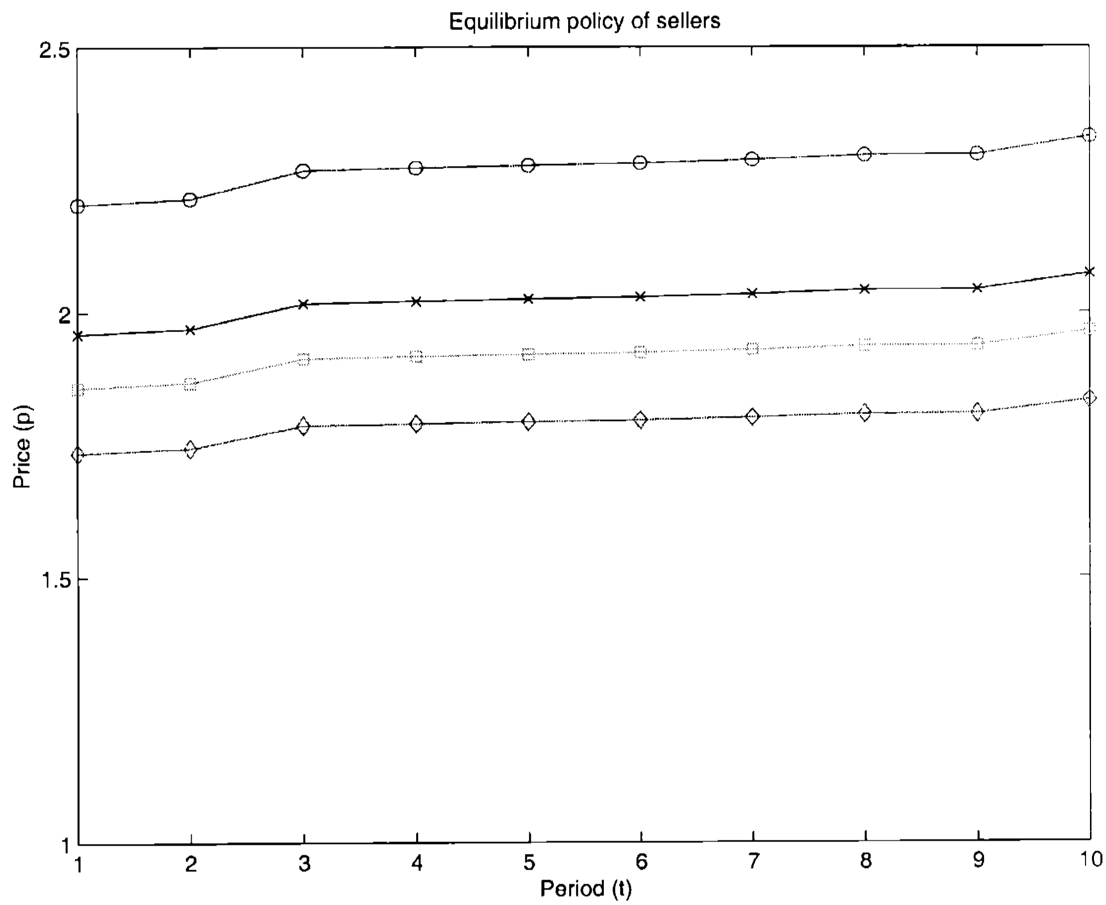

Figure B-15: Robust demand example (1) from Chapter 8: The equilibrium pricing policies for all sellers (From top to bottom are seller 1 though 4.)

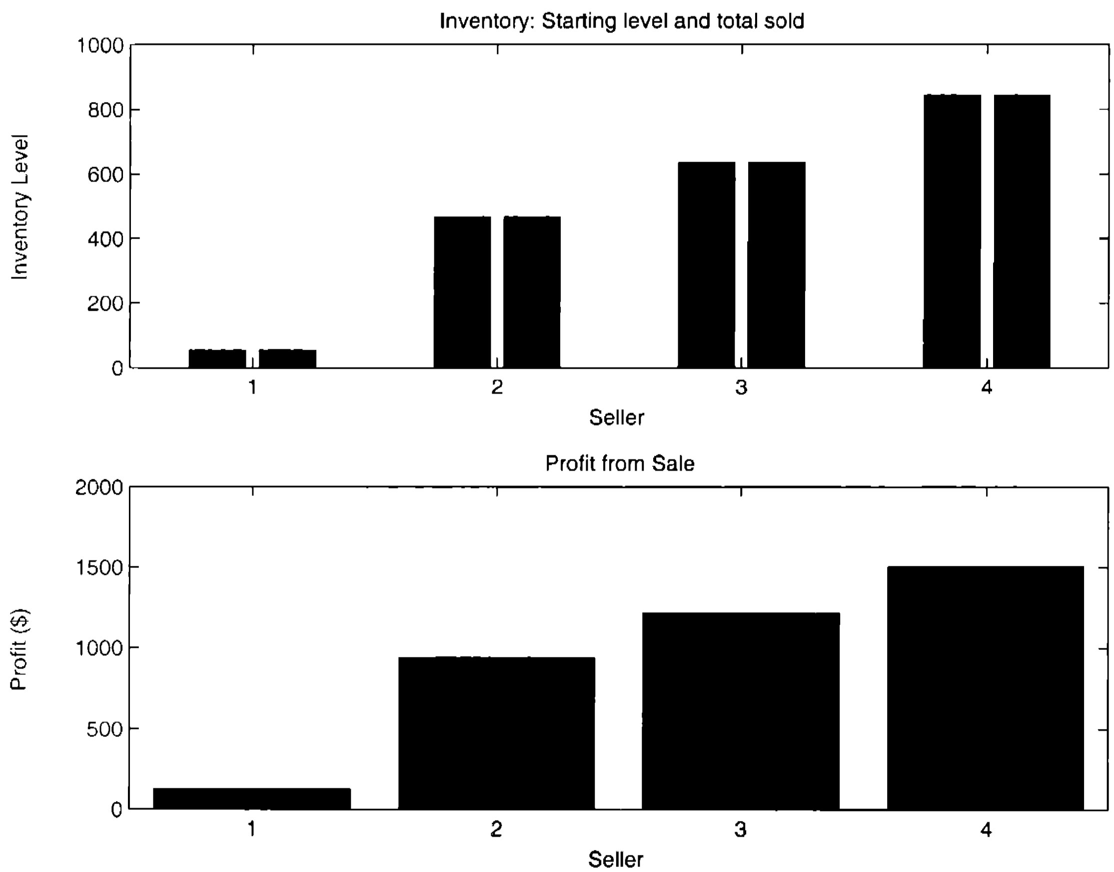

Figure B-16: Robust demand example (1) from Chapter 8: The starting inventory level for all sellers and the total amount sold (top graph) and the total payoff for each seller (bottom graph) under the equilibrium policies.

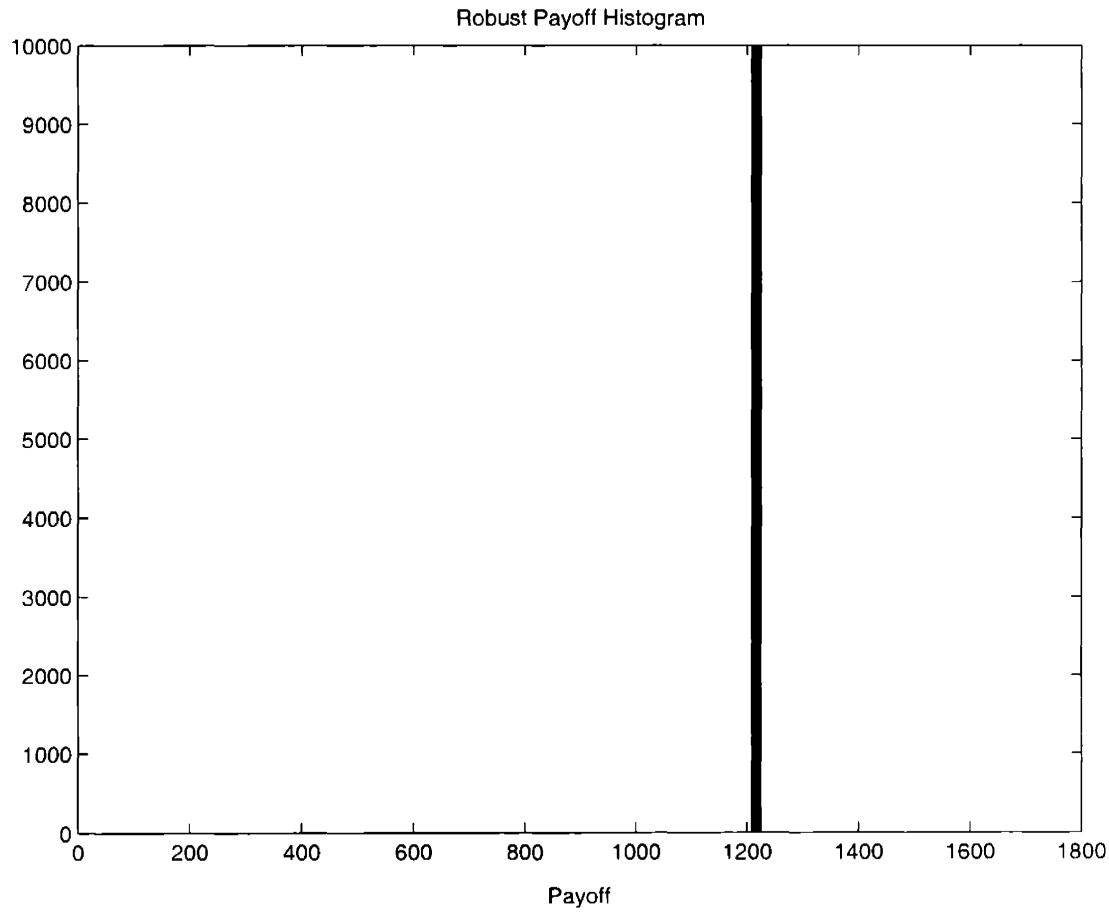

Figure B-17: Robust demand example (1) from Chapter 8: A histogram of payoffs for seller 3 from the robust policy when uncertain parameters are sampled uniformly from the uncertainty set.

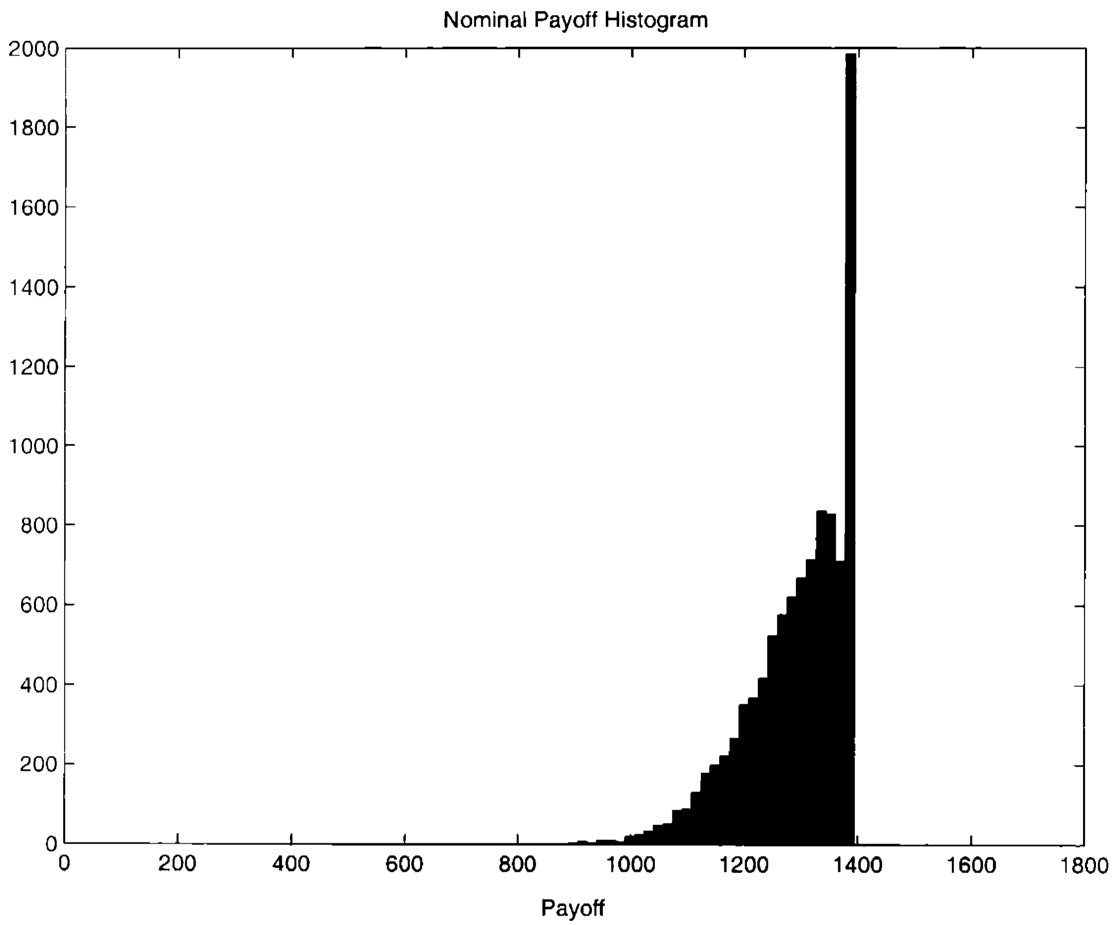

Figure B-18: Robust demand example (1) from Chapter 8: A histogram of payoffs for seller 3 from the nominal policy when uncertain parameters are sampled uniformly from the uncertainty set.

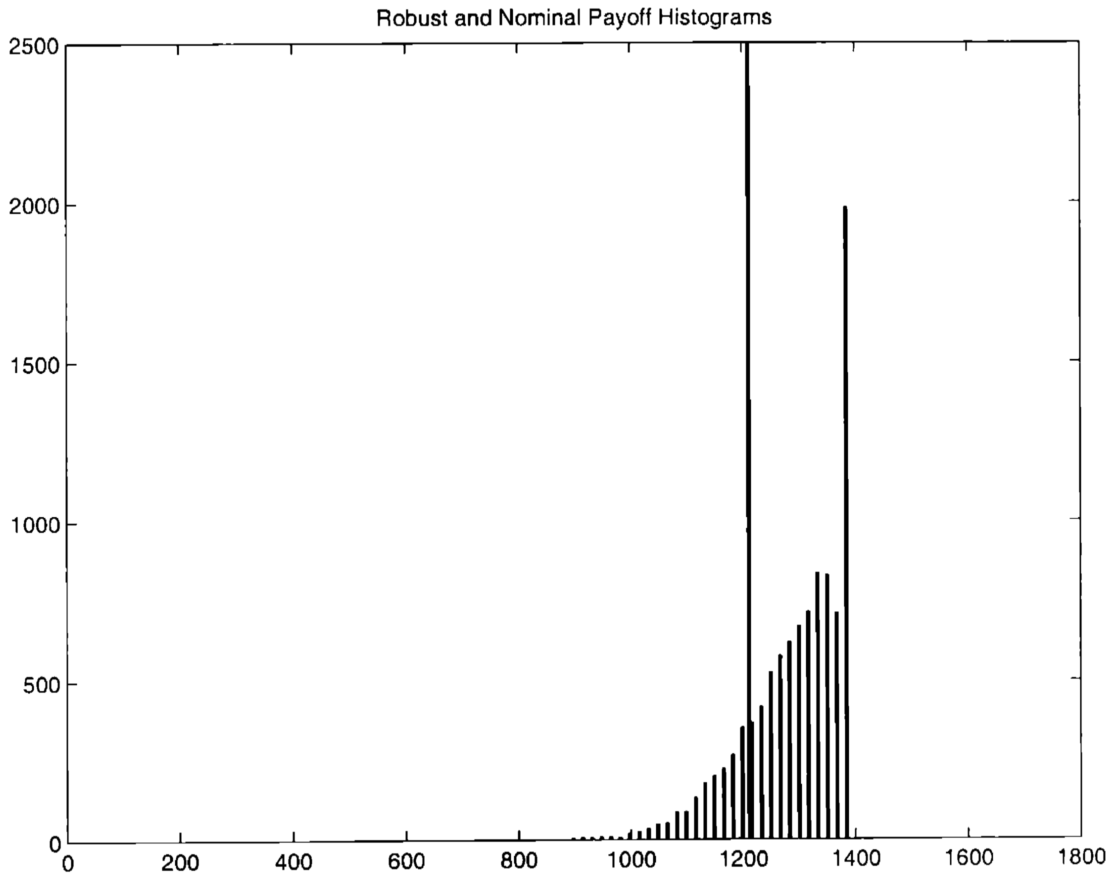

Figure B-19: Robust demand example (1) from Chapter 8: A comparison of the payoffs for seller 3 from the robust and the nominal policy when uncertain parameters are sampled uniformly from the uncertainty set.

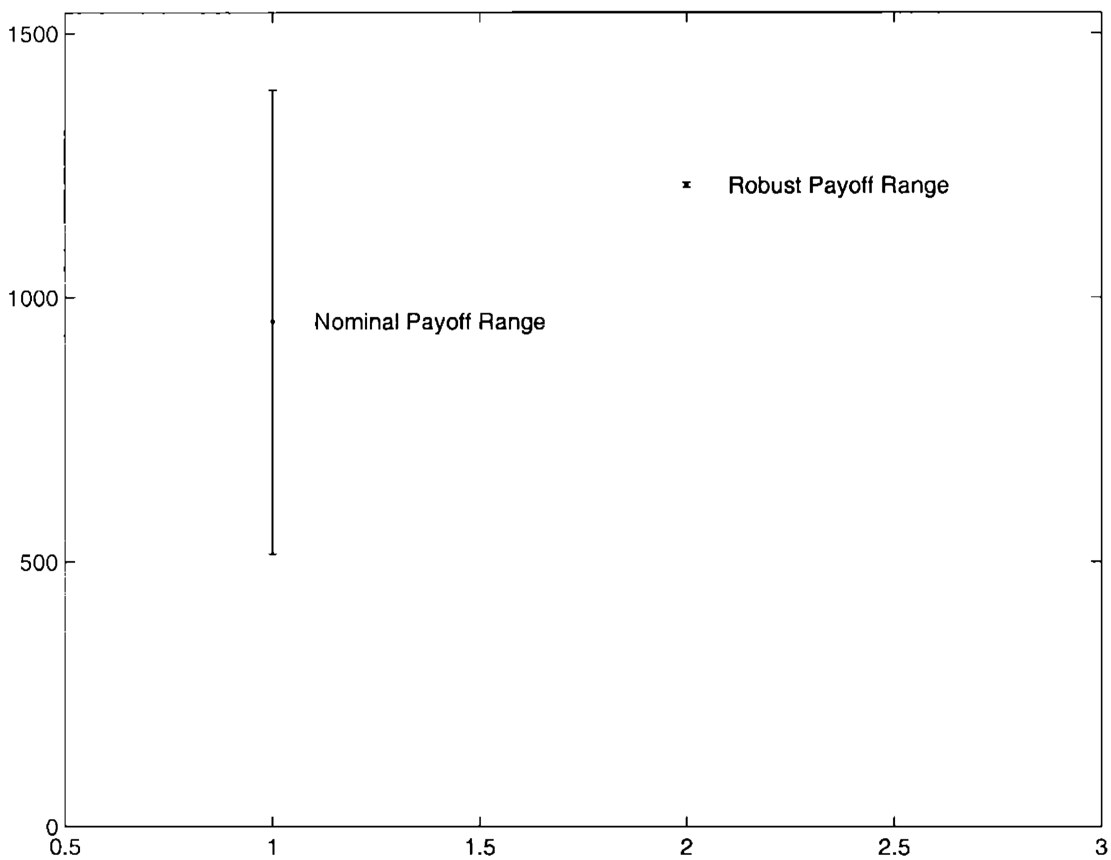

Figure B-20: Robust demand example (1) from Chapter 8: A comparison of the range of payoffs for seller 3 from the robust and the nominal policy under uncertainty.

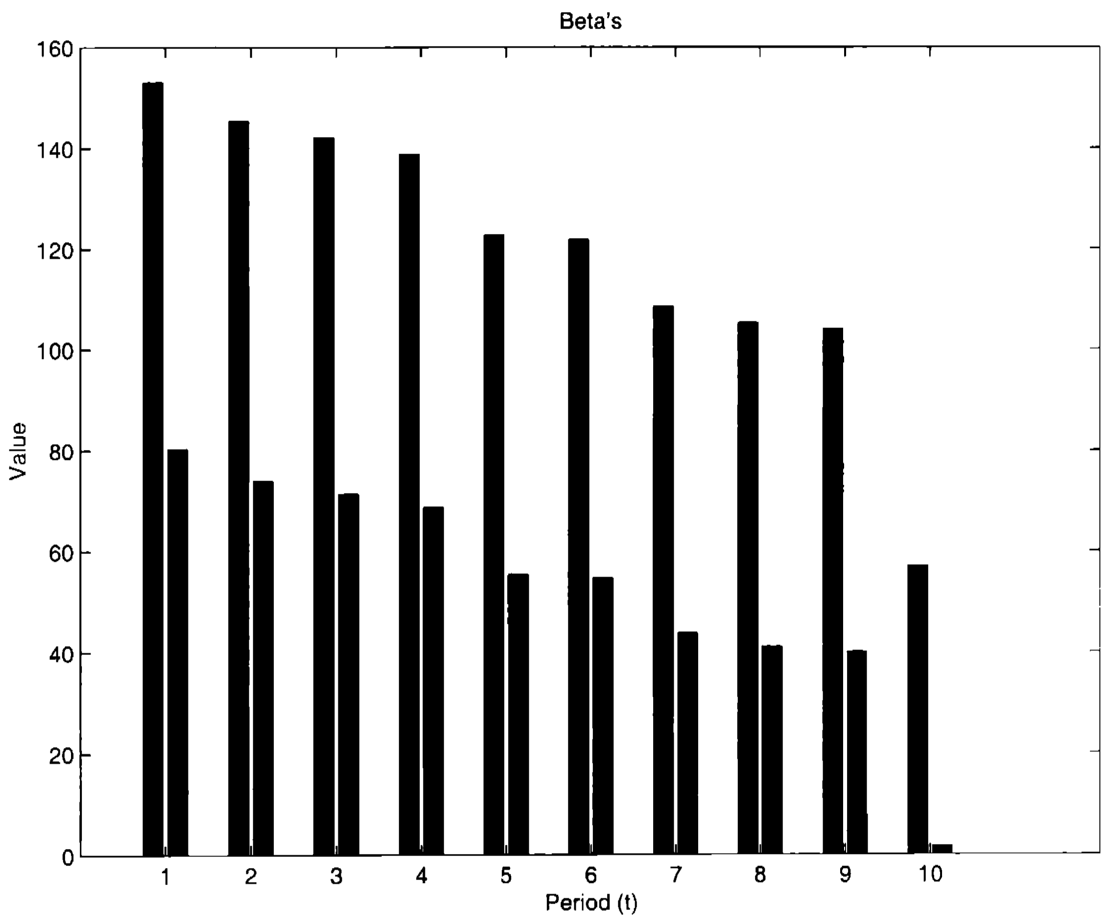

Figure B-21: Robust demand example (2) from Chapter 8: Values of the nominal demand function parameters (α and β).

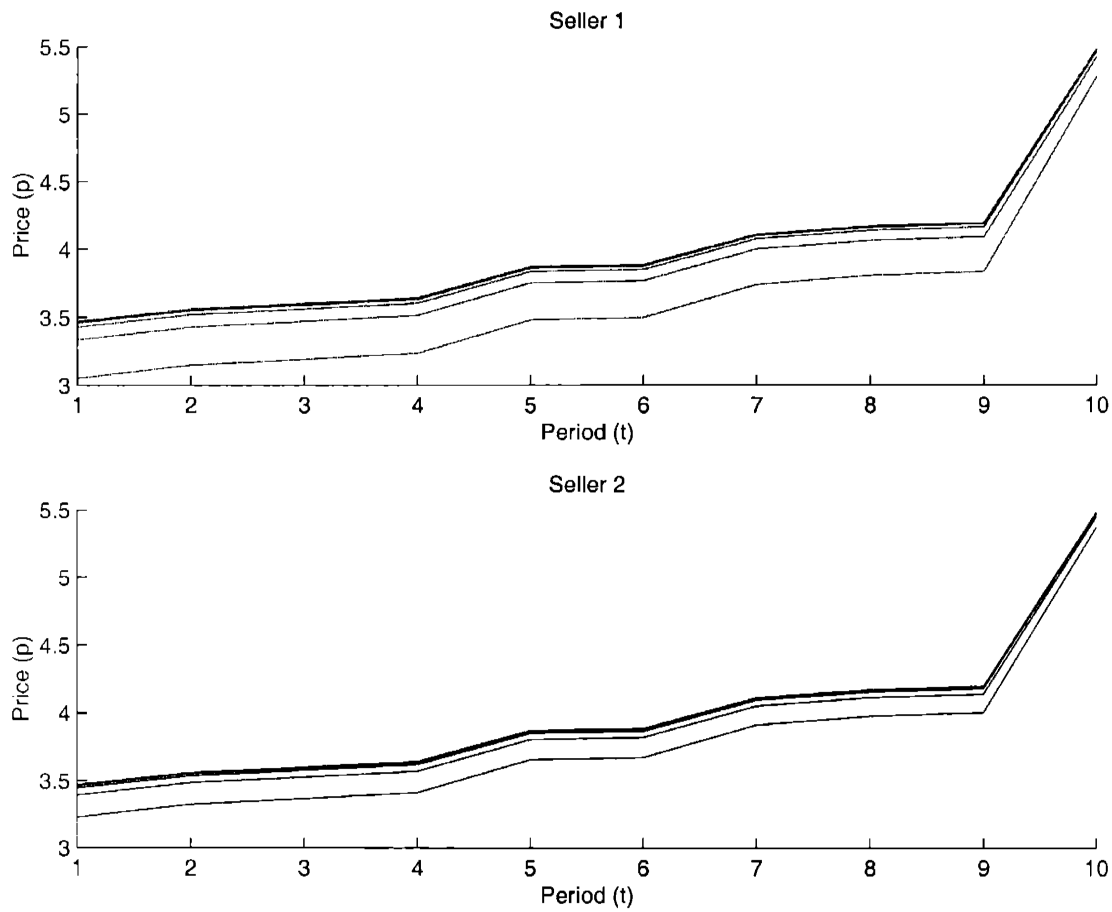

Figure B-22: Robust demand example (2) from Chapter 8: The pricing policies of both sellers over successive iterations of the algorithm as it converges to the equilibrium.

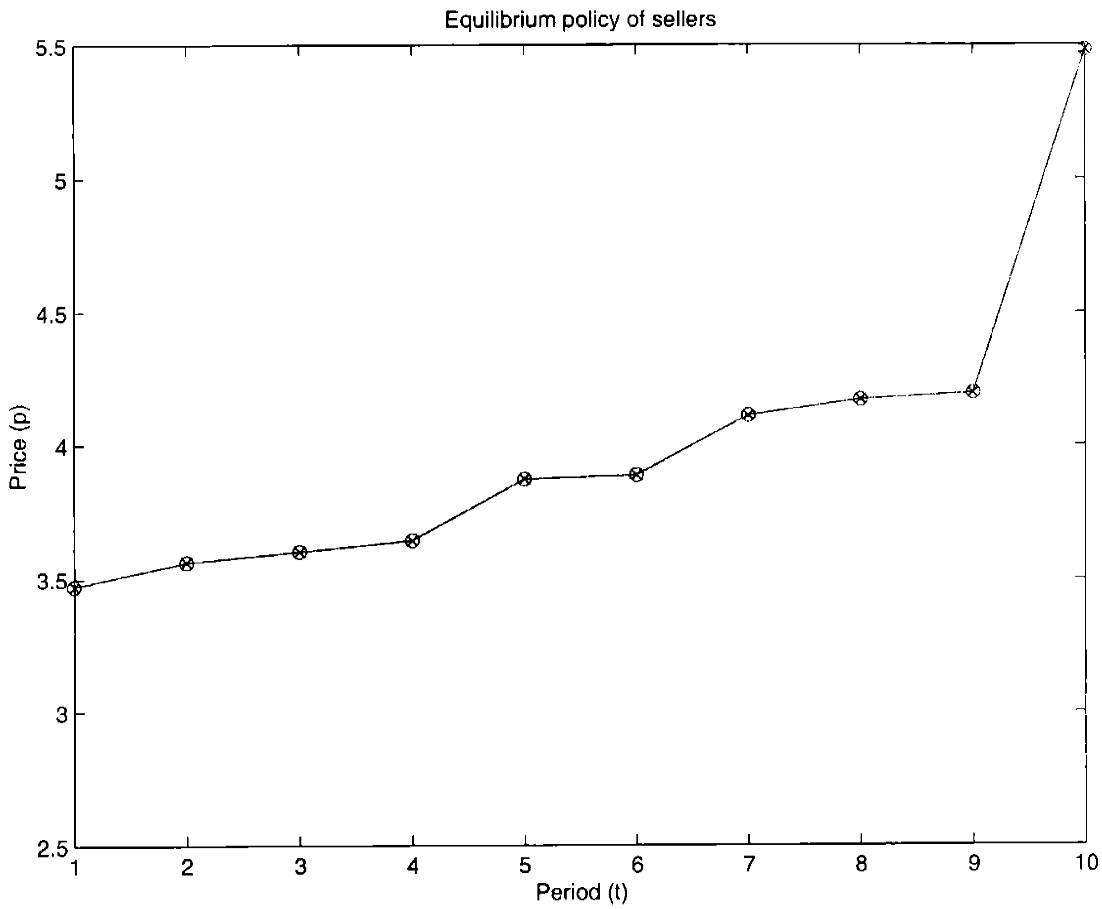

Figure B-23: Robust demand example (2) from Chapter 8: The equilibrium prices that the algorithm converges to. Note that the prices for Seller 1 and Seller 2 are identical.

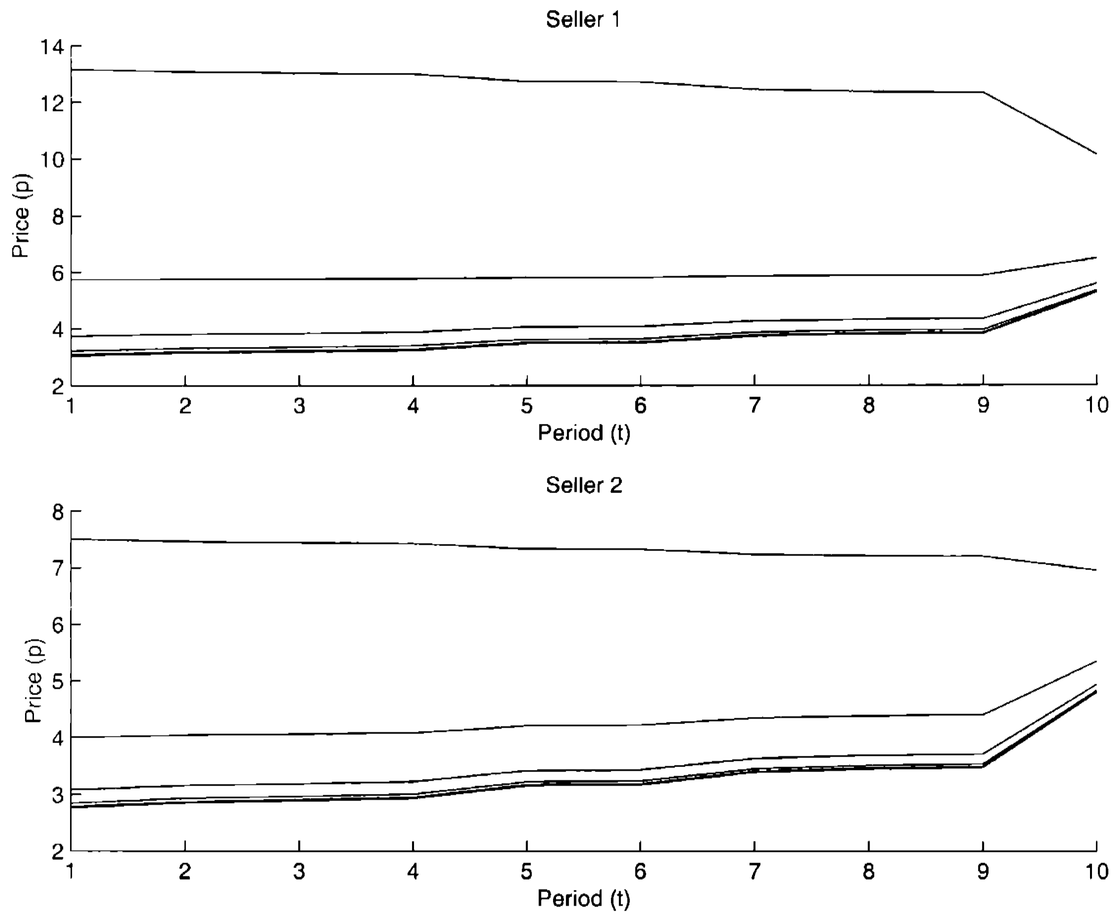

Figure B-24: Robust demand example (2) from Chapter 8: The pricing policies over successive iterations of the algorithm when Seller 2 adopts a robust policy.

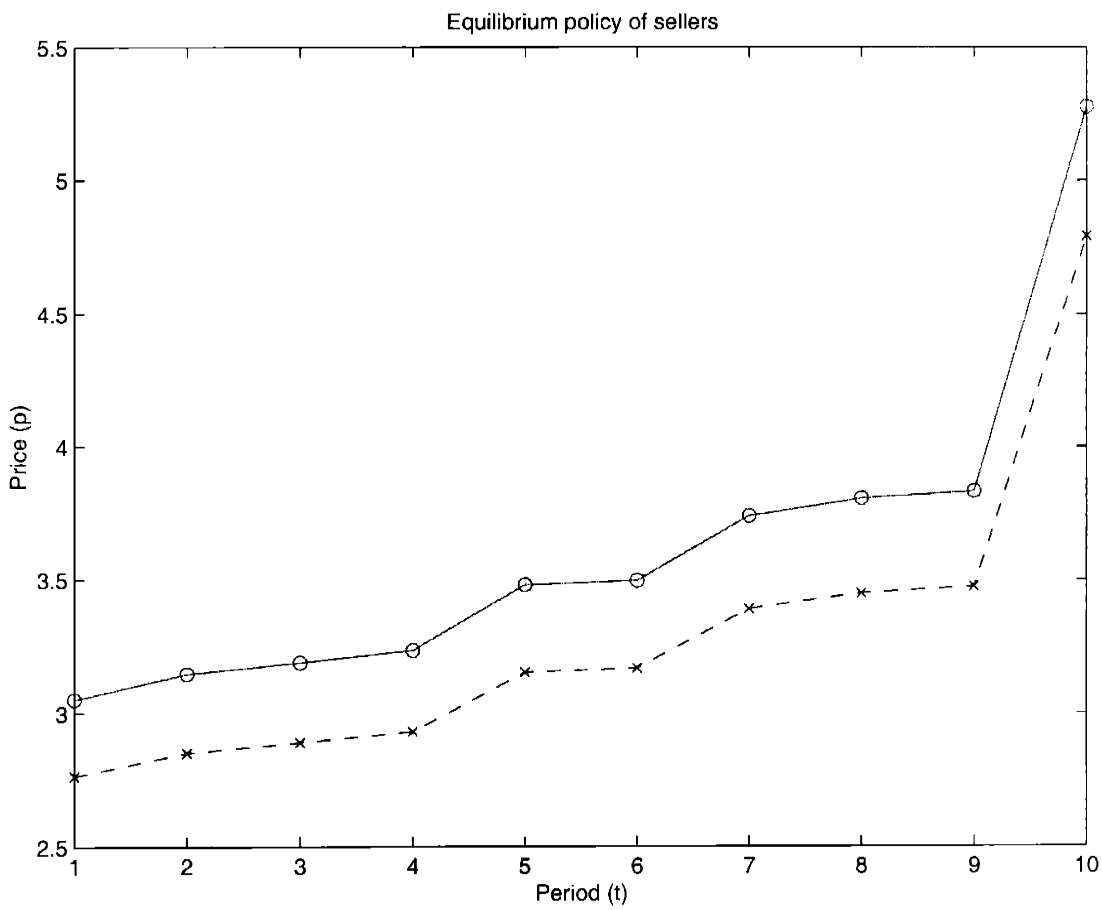

Figure B-25: Robust demand example (2) from Chapter 8: The equilibrium prices that the algorithm converges to. Note that the prices for Seller 2 (who adopts a robust policy) are lower than prices for Seller 1.

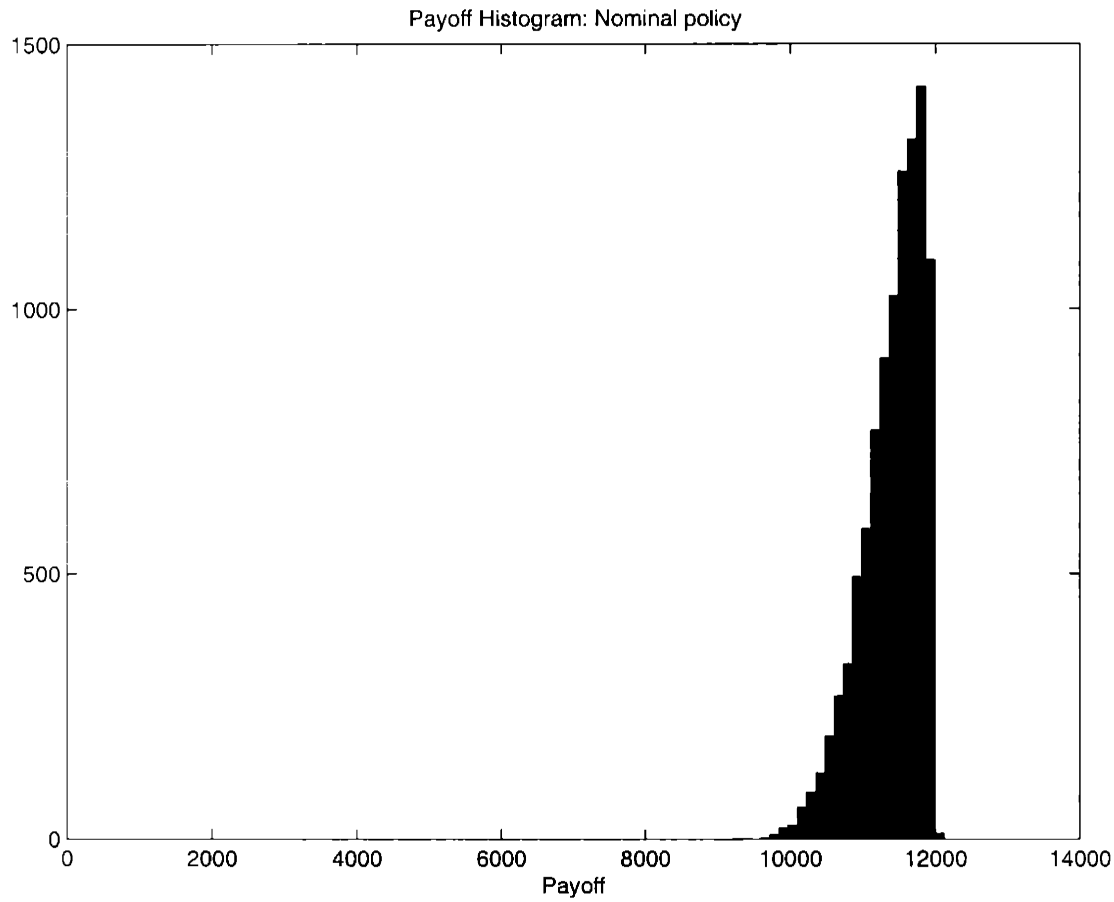

Figure B-26: Robust demand example (2) from Chapter 8: Distribution of payoff for either seller when both adopt nominal policies.

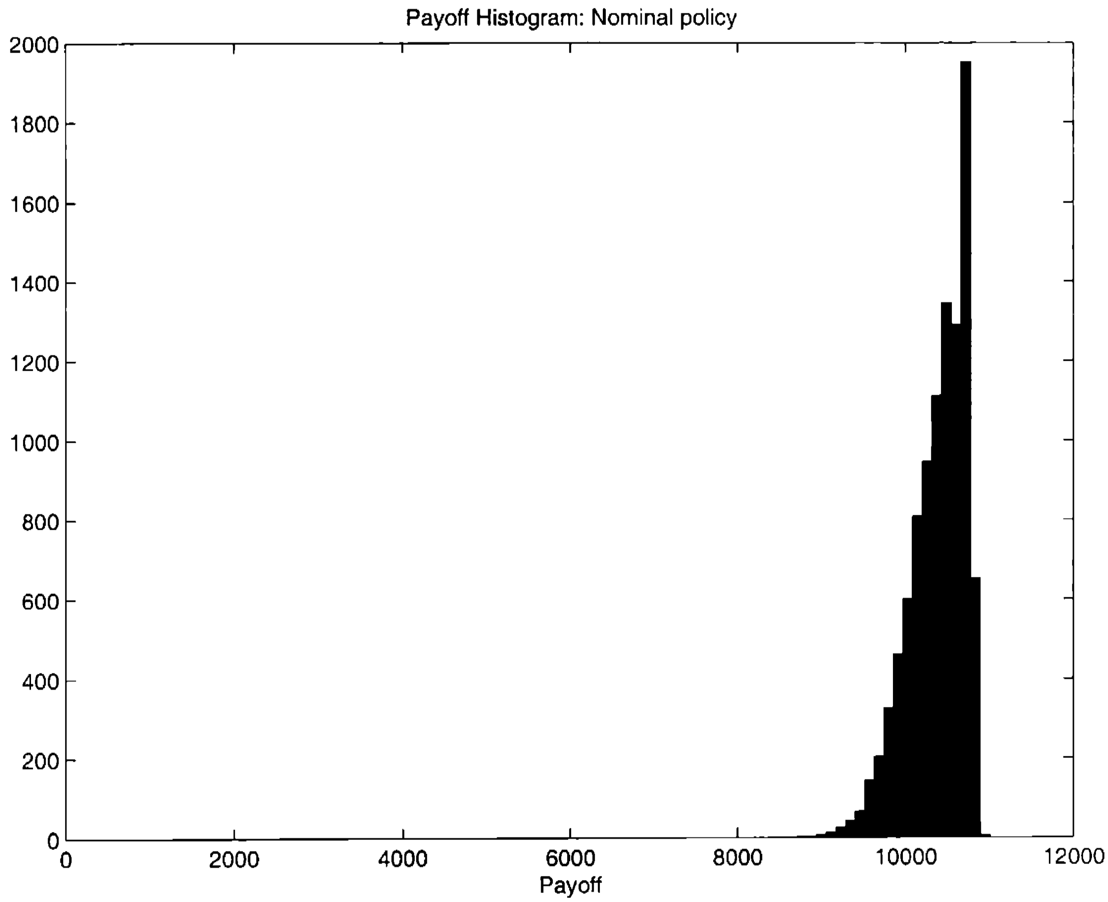

Figure B-27: Robust demand example (2) from Chapter 8: Distribution of payoff for Seller 1 when only Seller 2 adopts nominal policies.

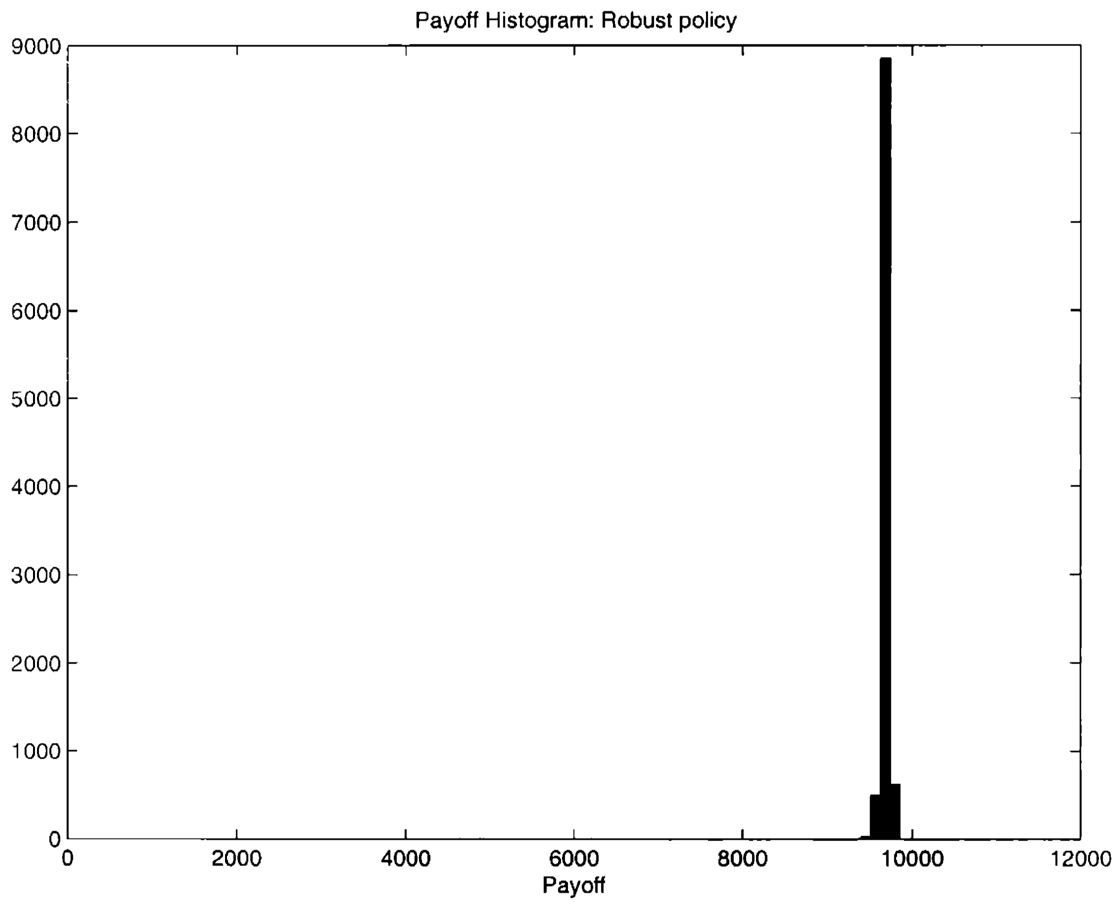

Figure B-28: Robust demand example (2) from Chapter 8: Distribution of payoff for Seller 2 when only Seller 2 adopts nominal policies.

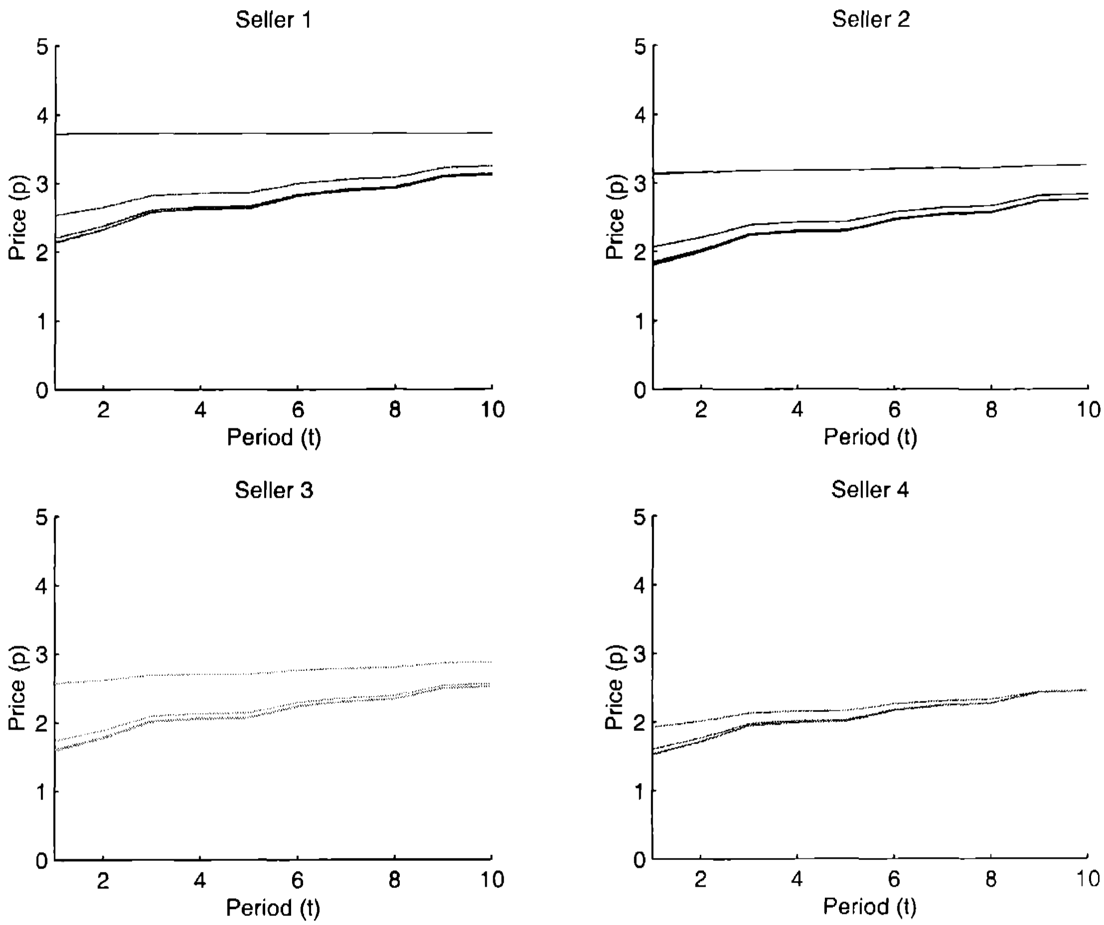

Figure B-29: Robust demand example (3) from Chapter 8: The pricing policies for all sellers over successive iterations of the algorithm.

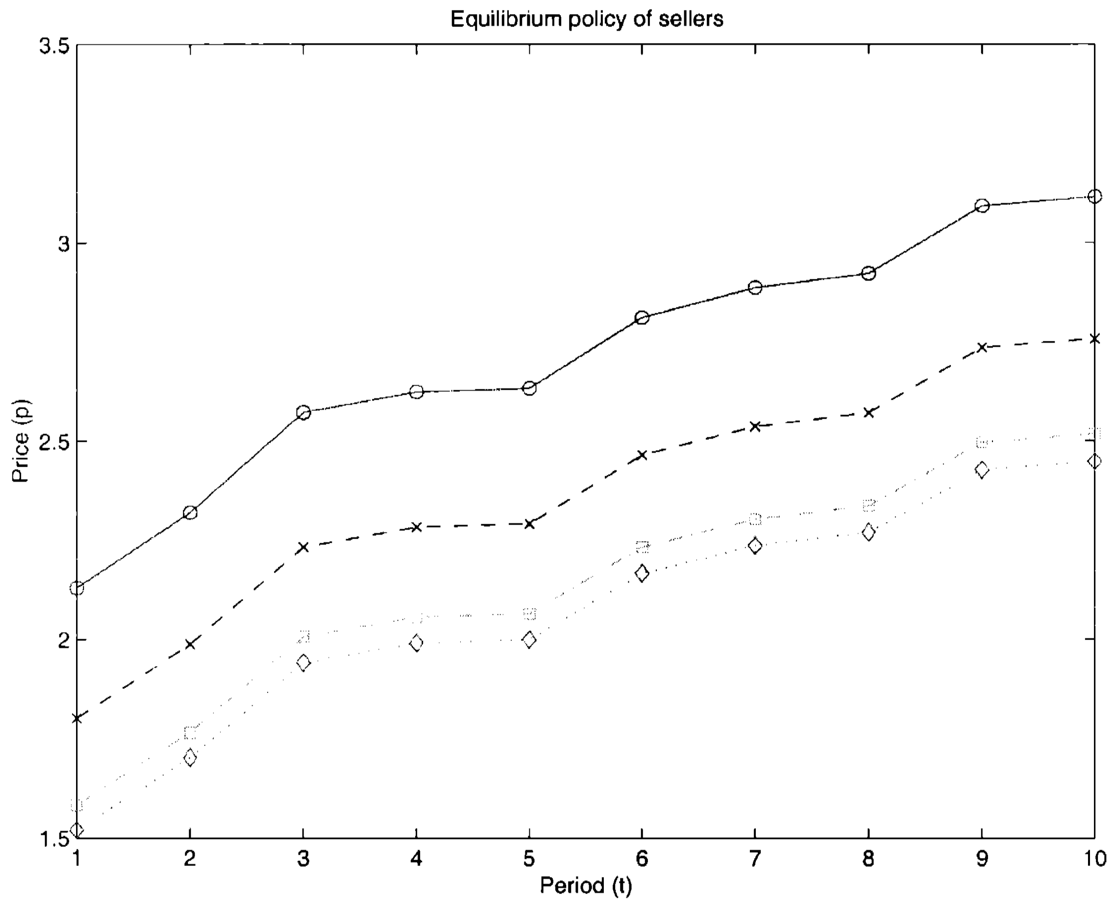

Figure B-30: Robust demand example (3) from Chapter 8: The equilibrium pricing policies for all sellers (From top to bottom are seller 1 though 4.

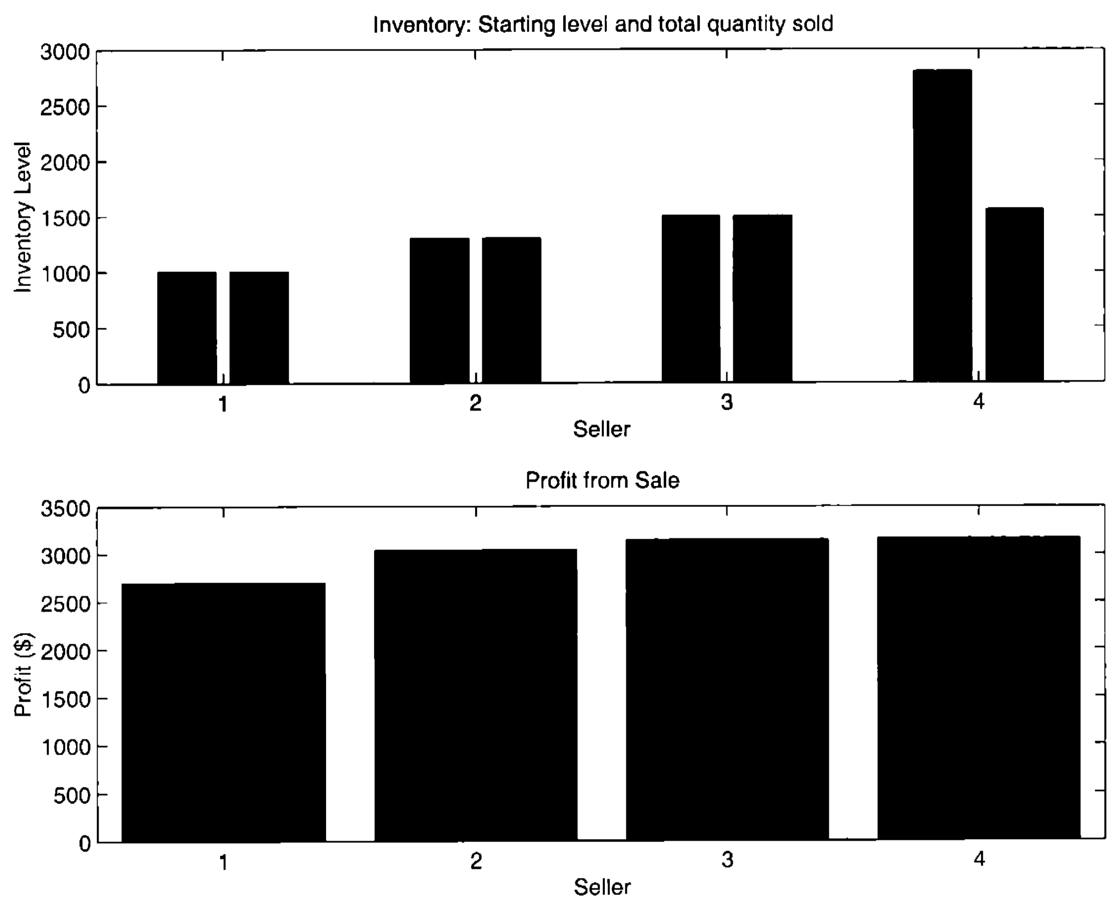

Figure B-31: Robust demand example (3) from Chapter 8: The starting inventory level for all sellers and the total amount sold (top graph) and the total payoff for each seller (bottom graph) under the equilibrium policies.

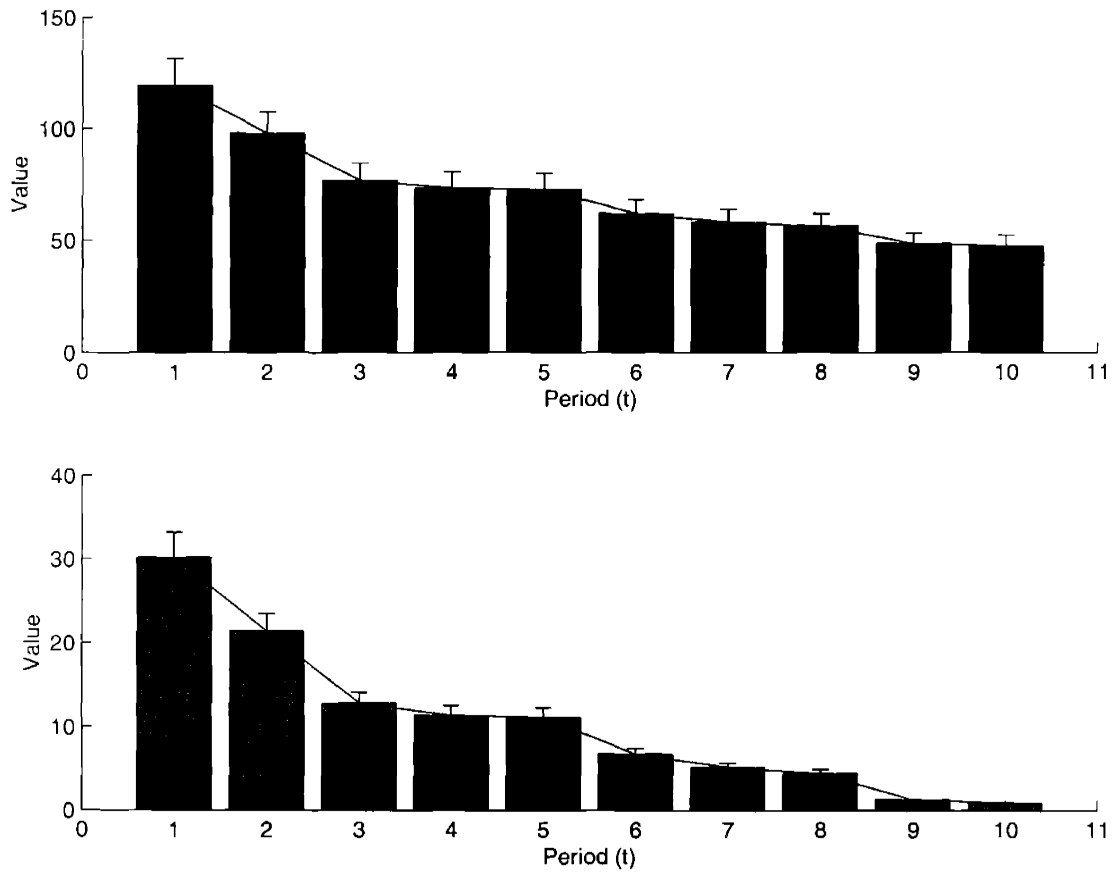

Figure B-32: Robust demand example (3) from Chapter 8: The value of the uncertain parameters β (top) and α (bottom) for any seller i with time period t. The error bars denote the uncertainty in the parameters.

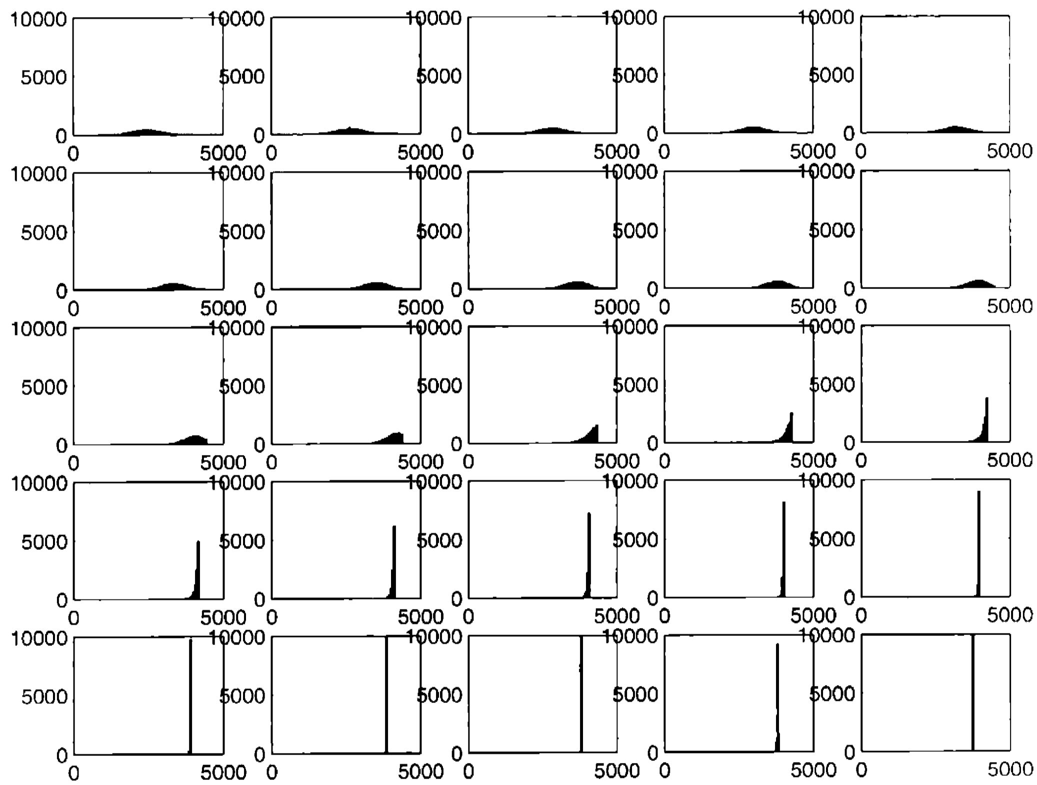

Figure B-33: Robust demand example (3) from Chapter 8: As we move from the graph on the top-left to the graph at the bottom-right (row-wise from left to right) we vary the robustness of the policy adopted from very optimistic (Seller assumes that the demand parameters are most favorable) to nominal (Center: Seller assumes that the demand parameters will take the nominal values) to very robust (Seller assumes that the demand parameters could take any values in the uncertain set). The graphs show the distribution of the payoff when the actual values are uniform over the uncertain set.

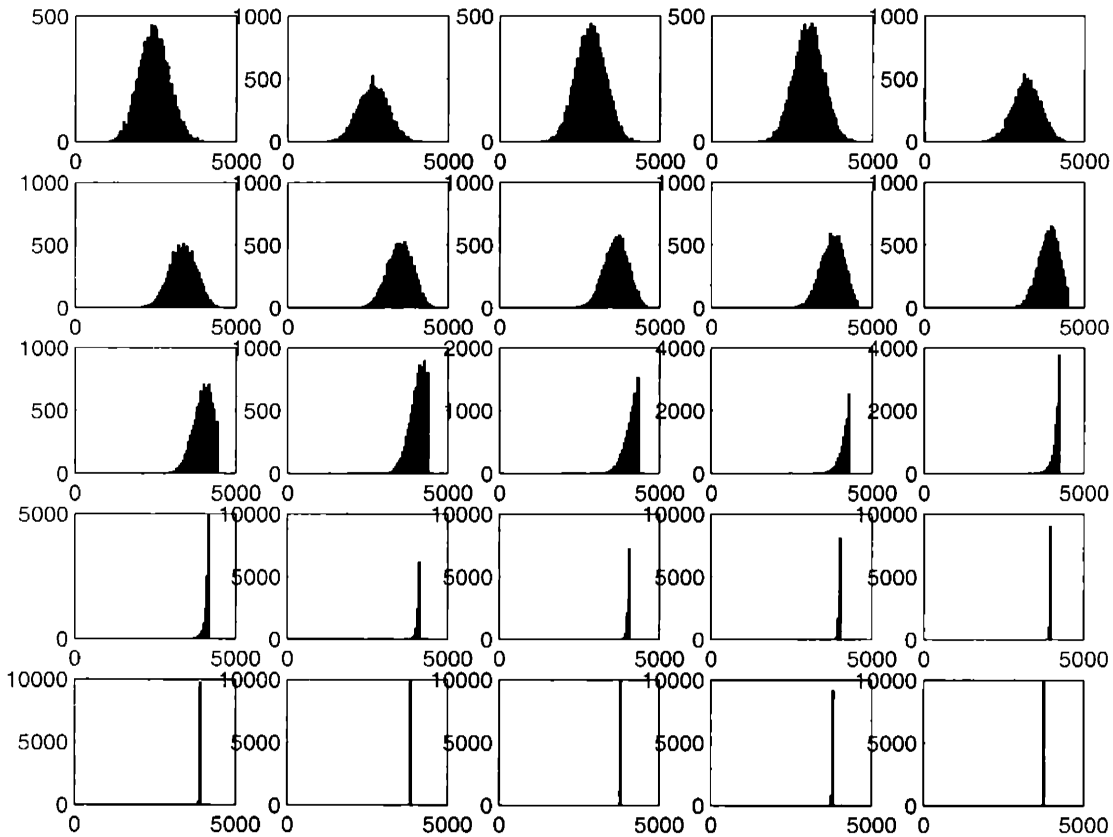

Figure B-34: Robust demand example (3) from Chapter 8: These graphs show the same distribution as Figure B-33. The y-axis in each graph has been scaled to show the shape of the distribution more clearly.

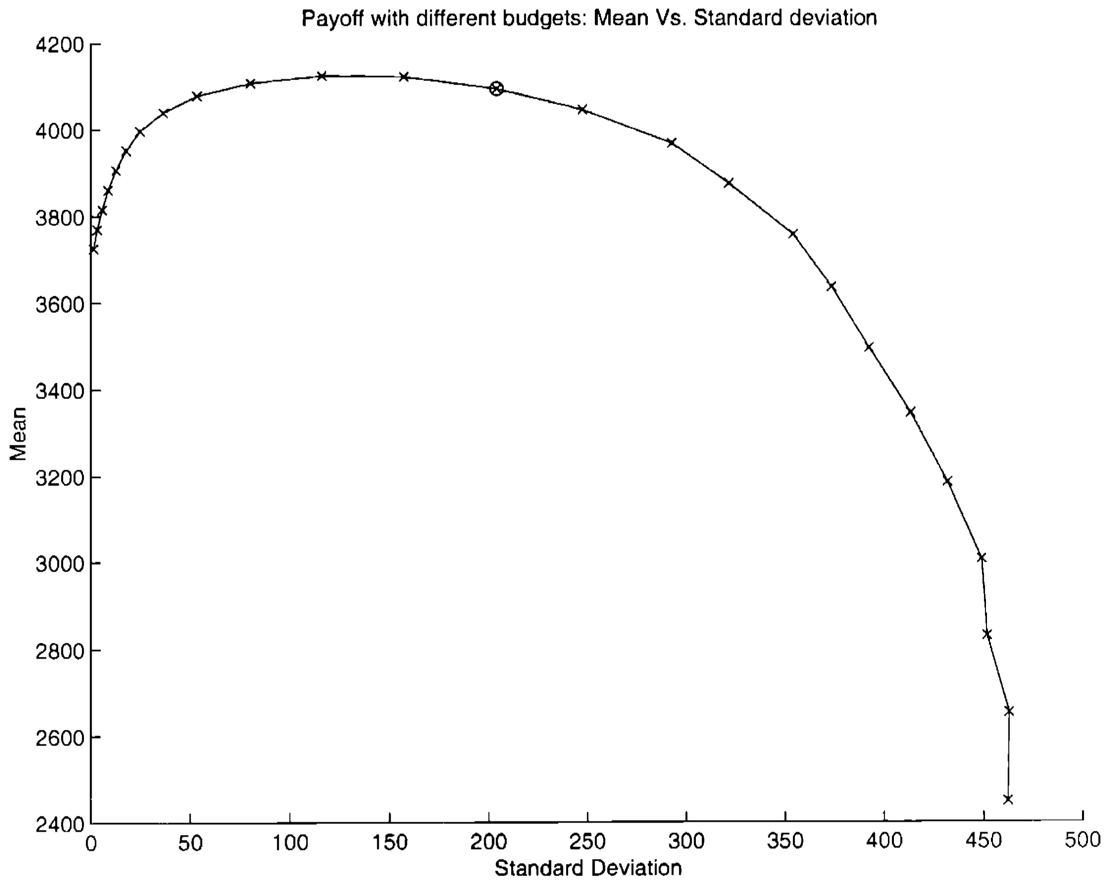

Figure B-35: Robust demand example (3) from Chapter 8: The standard deviation of the payoff (y-axis) verses the expected payoff (x-axis) as the robustness of policies is varied for Seller 1. The right-most point corresponds to optimism (low average payoff and high risk) and the left-most point corresponds to robustness (some sacrifice of average payoff with very low risk). The nominal policy point is marked with a circle.

Bibliography

[1] M. S. Bazaraa, H. D. Sherali, and C. Shetty. *Nonlinear Programming.* John Wiley & Sons, New York, 2 edition, 1993.

[2] A. Ben-Tal, L. El-Ghaoui, and A. Nemirovski. Robust semidefinite programming. In R. Saigal, L. Vandenberghe, and W. H., editors, *Semidefinite Programming and Applications.* Kluwer Academic Publishers, 2000.

[3] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23:769–805, 1998.

[4] A. Ben-Tal and A. Nemirovski. Robust solutions to uncertain linear programs. *Operations Research Letters*, 25:1–13, 1999.

[5] A. Ben-Tal and A. Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, 88:411–424, 2000.

[6] J. Bertrand. Theorie mathematique de la richesse sociale. *Journal des Savants*, 67:499–508, 1883.

[7] D. Bertsimas and M. Sim. The price of robustness. *Operations Research*, (to appear), 2002.

[8] D. Bertsimas and A. Thiele. Applications of robust optimization in supply chains and revenue management. *INFORMS Annual Meeting, Atlanta, Georgia*, 2003.

[9] G. Bitran and R. Caldentey. An overview of pricing models for revenue management. *Manufacturing and Service Operations Management*, (to appear), 2002.

[10] G. Bitran, R. Caldentey, and S. Mondschein. Coordinating clearance markdown sale of seasonal products in retail chains. *Operations Research*, 46:609–624, 1998.

[11] G. Bitran and S. Mondschein. Periodic pricing of seasonal products in retailing. *Management Science*, 43(1):64–79, 1997.

[12] D. Bracss. Uber ein paradoxon der verkehrsplanung. *Unternehmenforschung*, 12:258–268, 1968.

[13] G. Brown. Iterative solution of games by fictitious play. In *Activity Analysis of Production and Allocation.* John Wiley & Sons, New York, 1951.

[14] G. Cachon and S. Netessine. Game theory in supply chain analysis. In D. Simchi-Levi, S. D. Wu, and M. Shen, editors, *Supply Chain Analysis in E-business era*. Kluwer Academic Publishers, 2004.

[15] L. M. A. Chan, Z. J. M. Shen, D. Simchi-Levi, and J. Swann. Coordinating pricing, inventory, and production: A taxonomy and review. In *Handbook on Supply Chain Analysis in the eBusiness Era*. Kluwer Academic Publishers, 2001.

[16] L. M. A. Chan, D. Simchi-Levi, and J. Swann. Dynamic pricing strategies for manufacturing with stochastic demand and discretionary sales. *Working Paper, MIT*, 2002.

[17] C. K. Chau and K. M. Sim. The price of anarchy for non-atomic congestion games with symmetric cost maps and elastic demands. *Operations Research Letters*, 31, 2003.

[18] X. Chen and D. Simchi-Levi. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost:the finite horizon case. *Working Paper, #157, Center for E-business, MIT*, 2002.

[19] R. Cole, Y. Dodis, and T. Roughgarden. Pricing network edges for heterogeneous selfish users. *Conference version appeared in STOC*, 2003.

[20] J. Correa, A. Schulz, and N. Stier-Moses. Selfish routing in capacitated networks. *MIT Sloan Working Paper No. 4319-03. Also to appear in the IPCO Proceedings*, 2004.

[21] A. Cournot. Recherches sur les principes mathematics de la theorie de la richesse. *Hachette*, 1838.

[22] S. Dafermos. Traffic equilibria and variational inequalities. *Trasportation Science*, 14:42–54, 1980.

[23] S. Dafermos and F. Sparrow. The traffic assignment problem for a general network. *Journal of Research of the National Bureau of Standards - B*, 73B, 1969.

[24] G. Debreu. A social equilibrium existence theorem. *Proceedings of the National Academy of Science*, 38:886–893, 1952.

[25] P. Dubey. Inefficiency of nash equilibria. *Mathematics of Operations Research*, 11:1–8, 1986.

[26] F. Y. Edgeworth. *Mathematical Psychics: an Essay on the Application of Mathematics to the Moral Sciences*. Kegan Paul, London, 1881.

[27] L. El-Ghaoui, F. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM Journal of Optimization*, 9(1):33–52, 1999.

[28] J. Eliashberg and A. P. Jeuland. The impact of competitive entry in a developing market upon dynamic pricing strategies. *Marketing Science*, 5(1):20–36, 1986.

[29] J. Eliashberg and R. Steinberg. Marketing-production joint decision making. In J. Eliashberg and J. D. Lilien, editors, *Management Science in Marketing, Handbooks in Operations Research and Management Science.* Elsevier Science, North Holland, 1991.

[30] W. Elmaghraby and P. Keskinocak. Dynamic pricing in the presence of inventory considerations: Research overview, current practices and future directions. *Management Science*, 49(10):1287–1309, 2003.

[31] F. Facchinei and J. S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*, volume I and II. Springer Verlag, Berlin, 2003.

[32] K. Fan. Fixed point and minimax theorems in locally convex topological linear spaces. *Proceedings of the National Academy of Sciences*, 38(121-126), 1952.

[33] A. Federgruen and A. Heching. Combined pricing and inventory control under uncertainty. *Operations Research*, 47(3):454–475, 1999.

[34] Y. Feng and G. Gallego. Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management Science*, 41:1371–1391, 1995.

[35] M. Florian and D. Hearn. Chapter 6: Network routing. In M. Ball, T. Magnanti, C. Monma, and G. Neumhauser, editors, *Network equilibrium models and algorithms*, volume 8. Elsevier Science, 1995.

[36] D. Gale and H. Nikaido. The jacobian matrix and global univalence of mappings. *Mathematische Annalen*, 159:81–93, 1965.

[37] G. Gallego and G. van Ryzin. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, 40(8):999–1020, 1994.

[38] I. L. Glicksberg. A further generalization of the kakutani fixed point theorem, with application to nash equilibrium points. *Proceedings of the American Mathematical Society*, 3(1):170–174, 1952.

[39] P. T. Harker. A variational inequality approach for the determination of oligopolistic market equilibrium. *Mathematical Programming*, 30:105–111, 1984.

[40] D. Hearn and M. Ramana. Solving congestion toll pricing models. In P. Marcotte and N. S., editors, *Equilibrium and Advanced Transportation Modeling.*

[41] D. Hearn and M. Yildirim. A toll pricing framework for traffic assignment problems with elastic demand. In P. Marcotte and M. Gendreau, editors, *Transportation and Network Analysis - Current Trends.*

[42] R. Johari and J. N. Tsitsiklis. Network resource allocation and a congestion game. *MIT-LIDS Working Paper*, 2003.

[43] A. P. Kirman and M. J. Sobel. Dynamic oligopoly with inventories. *Econometrica: Journal of the Econometric Society*, 42(2):279–288, 1974.

[44] E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. In *STACS*, 1999.

[45] E. P. Lazear. Retail pricing and clearance sales. *The American Economic Review*, 76(1):14–32, 1986.

[46] P. Marcotte. Network design problem with congestion effects: A case of bilevel programming. *Mathematical Programming*, 34:142–162, 1986.

[47] J. I. McGill and G. J. Van Ryzin. Revenue management: Research overview and prospects. *Transportation Science: Focused Issue on Yield Management in Transportation*, 33(2), 1999.

[48] J. Morgan and M. Romaniello. Generalised quasi-variational inequalities and duality. *Journal of Inequalities in Pure and Applied Mathematics*, 4(2), 2003.

[49] F. H. Murphy, H. D. Sherali, and A. L. Soyster. A mathematical programming approach for determining oligopolistic market equilibrium. *Mathematical Programming*, 24:92–106, 1982.

[50] A. Nagurney. *Network Economics: A Variational Inequality Approach.* Kluwer Academic Publishers, Boston, 2000.

[51] J. Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951.

[52] J.-S. Pang and M. Fukushima. Quasi-variational inequalities, generalized nash equilibria, and multi-leader-follower games. *Working Paper, The Johns Hopkins University, Baltimore, Maryland*, 2003.

[53] G. Perakis. The price of anarchy when costs are non-separable and asymmetric. *Working Paper, MIT*, 2004.

[54] G. Perakis and A. Sood. Competitive multi-period pricing for perishable products. *Working Paper, Operations Research Center, MIT*, 2002.

[55] G. Perakis and A. Sood. Competitive multi-period pricing for perishable products; a robust optimization approach. *Working Paper, Operations Research Center, MIT*, 2003.

[56] N. C. Petruzzi and M. Dada. Pricing and the newsvendor problem: A review with extensions. *Operations Research*, 47(2):183–194, 1999.

[57] J. Robinson. An iterative method of solving a game. *Annals of Mathematics*, 54(2):296–301, 1951.

[58] J. B. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, 33(3):520–534, 1965.

[59] T. Roughgarden. *Selfish Routing.* Ph.d., Cornell University, 2002.

[60] T. Roughgarden and E. Tardos. Bounding the inefficiency of equilibria in nonatomic congestion games. *Journal of the ACM*, 49(2):236–259, 2002.

[61] M. J. Smith. The existence, uniqueness and stability of traffic equilibria. *Transportation Research*, 13B:295–304, 1979.

[62] S. A. Smith and D. D. Achabal. Clearance pricing and inventory policies for retail chains. *Management Science*, 44(3):285–300, 1998.

[63] A. L. Soyster. Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research*, 21:1154–1157, 1973.

[64] Y. Tanaka. Profitability of price and quantity strategies in an oligopoly. *Journal of Mathematical Economics*, 35(3):409–418, 2001.

[65] D. M. Topkis. *Supermodularity and complementarity.* Princeton University Press, Princeton, New Jersey, 1998.

[66] X. Vives. *Oligopoly Pricing - Old Ideas and New Tools.* MIT Press, Cambridge, 1999.

[67] J. G. Wardrop. Some theoretical aspects of road traffic research. In *Proceedings of the Institute of Civil Engineers, Part II*, pages 325–378, 1952.

[68] C. A. Yano and S. M. Gilbert. Coordinated pricing and production/procurement decisions: A review. In A. Chakravarty and J. Eliashberg, editors, *Managing Business Interfaces: Marketing, Engineering and Manufacturing Perspectives.* Kluwer Academic Publishers, 2004.

[69] L. Young. Price, inventory, and the structure of uncertain demand. *New Zealand Operations Research*, 6:157–177, 1978.

[70] E. Zabel. Monopoly and uncertainty. *The Review of Economic Studies*, 37(2):205–219, 1970.

[71] W. Zhao and Y. S. Zheng. Optimal dynamic pricing for pershable assets with nonhomogenous demand. *Management Science*, 46:375–388, 2000.

[72] P. Zipkin. *Foundations of Inventory Management.* McGraw-Hill, Burr Ridge, IL, 2000.