OPTIMAL STABILIZATION POLICIES UNDER DECENTRALIZED CONTROL AND CONFLICTING OBJECTIVES

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This paper derives methods for the calculation of optimal stabilization policies under the assumption that monetary and fiscal control are exercised by separate authorities who may have different objectives. Each authority minimizes its own quadratic cost functional subject to the constraint of a linear econometric model. Nash solution strategies are calculated for this discrete-time differential game, both in the context of open-loop and closed-loop behavior (in the closed-loop framework each authority can continually revise his policy in response to the evolving strategy of the other authority). The results are applied to a small econometric model, and show how the degree of fiscal or monetary control depends on the particular conflict situation, and how conflicting policies are "sub-optimal" in comparison with coordinated policies.
Optimal Stabilization Policies Under Decentralized Control and Conflicting Objectives

1. Introduction

There have recently appeared several applications of optimal control theory to economic stabilization policy which involve the calculation of time paths ("trajectories") for one or more policy variables in order to minimize some macroeconomic cost functional. These studies have attempted to show how policy variables such as the level of government expenditures, one or more tax rates, and the money stock can be manipulated in order to best attain objectives relating to GNP, employment, prices, etc. Previous work by this author [18, 19, 20] demonstrated how optimal policies could be calculated in a deterministic framework using a linear (or linearized) econometric model and a quadratic cost functional which penalized for deviations of target variables and policy variables from a set of nominal (i.e. "ideal") paths. Others have generalized the problem by allowing the objective function to be piece-wise quadratic [10, 11], by allowing the model to be non-linear [4, 9, 14, 16], or by working in a stochastic framework in which the model contained additive error terms and/or random coefficients [4, 5, 6, 7, 14, 15].

All of these studies have been based on the assumption that there is a single authority (or "controller") making and implementing policy based on a single set of objectives. Macroeconomic policy in the United States, however, is the product of a decentralized control process in which different authorities control different sets of policy instruments. The most obvious example of decentralization in this country is the separation of monetary and fiscal policy.¹ The formulation and execution of monetary policy is in

¹ - Fiscal policy is itself decentralized to some extent. Tax schedules, transfer schedules, and trade tariffs and quotas may, for example, be controlled, or at least influenced, to a greater or lesser extent by different Congressional committees and by the President.
the domain of the Federal Reserve System, and is largely independent of the formulation of fiscal policy and the control of fiscal instruments. In the past two or three decades monetary and fiscal policy have not always been coordinated and based on the same objectives. There have been periods in which the objectives differed, and there have been periods in which the objectives were the same but the world views on which policies are based differed. It is not clear what influence this has had in the past on our overall ability to control the economy, but some people have argued that the economy could be controlled more easily and effectively if monetary and fiscal policy were better coordinated or even put in the hands of a single authority.

From the point of view of optimal control, there is reason to believe that the separation of monetary and fiscal policy may be important in limiting our ability to stabilize the economy. Monetary policy operates with long lags and fiscal policy with short lags, and as a result the proper phasing of the two is critical. We have seen from optimal stabilization policies calculated under the assumption of centralized control that the relative timing of monetary and fiscal expansion is just as important as the relative amounts of expansion.\(^2\) Thus monetary and fiscal policies that are not well coordinated may turn out to be very sub-optimal. And monetary and fiscal policies designed with different objectives in mind may, because of the difference in lag time, result in economic performance that is far from either objective.

Our objective is to calculate optimal stabilization policies under the assumption that monetary and fiscal control are exercised by different authorities. Before we do this, however, we must start with a model for the decentralized control process, i.e. a description of how and why the

\(^2\) - See, for example, Pindyck [19], Chapter 6.
two authorities will arrive at policies different from those that would result in the centralized case. We could consider the following deterministic possibilities:

1. Each authority arrives at its policy using the same econometric model (i.e. each has the same view of the way the world works), but has a different set of objectives.

2. The two authorities have the same set of objectives, but each exercises control based on policies arrived at using a different econometric model.

3. The two authorities have the same set of objectives and use the same econometric model, but each has a different set of information available to it. Each, for example, might receive data on different variables at different times.

4. A combination of (1), (2), and (3), i.e. each authority has a different set of objectives, a different econometric model, and a different set of information.

In this paper we will concentrate only on the decentralized control process represented by (1) above. This process, besides being representative of at least part of the reason for the occasional lack of coordination of monetary and fiscal policy, can be viewed as a non-zero sum differential game (in discrete time), and the associated mathematics should therefore be reasonably tractable. The process represented by (2) may also be representative of the real world, but unfortunately this author knows of no game theory (or other theory) that he can turn to for guidance in finding a solution. The process represented by (3) is interesting, but also difficult to analyze. This process falls under the rubric of team theory, and although static team theory has been fairly well developed, a dynamic theory of teams is almost nonexistent. Finally, the process represented by (4), like (2) and (3), will be left for others to grapple with.

3 - For an overview of differential game theory, see Ho [2].
4 - See, for example, Marschak and Radner [17], and Ho and Chu [13].
Process (1) can be divided into two alternative modes of behavior:

(a) Each authority designs its optimal policy (based on its own objectives) at the beginning of the planning period, and then sticks to that policy throughout the entire planning period. This is called an open-loop strategy.

(b) Each authority designs a control rule at the beginning of the planning period, and then uses that control rule, together with observations of the state of the economy, to continuously revise his policy. This is called a closed-loop strategy, but should not be confused with the notion of a closed-loop optimal control as in the centralized case. We are viewing the economy as decentralized but deterministic, so that the closed-loop strategy allows adaptation not to the impact of random shocks, but to the evolving strategy of the other authority.

In this paper we will obtain both open-loop and closed-loop optimal strategies for a deterministic linear economy in which the controls are decentralized. Solution algorithms will be derived under the assumption that the monetary and fiscal authorities each attempt to minimize their own individual quadratic cost functionals. By applying the solutions to a small econometric model, we will try to gain some insight into what kinds of policies and economic behavior result when the two authorities attach different relative costs to different macroeconomic objectives. We will also examine how different these policies (and their economic implications) are from those that would result under an optimal centralized control. We hope to learn, for example, whether small differences in objectives result in large differences in policy.
2. Formulation of the Problem

We begin with a linear econometric model represented in state-variable form:

\[ x_{t+1} - x_t = A x_t + B_1 u_{1t} + B_2 u_{2t} + C z_t \]  

(1)

with initial condition \( x_0 = \xi \). Here \( x_t \) is a vector of \( n \) state variables, \( u_{1t} \) and \( u_{2t} \) are vectors of \( r_1 \) and \( r_2 \) control (policy) variables, each of which can be manipulated by a different authority (presumably monetary and fiscal), and \( z_t \) is a vector of \( s \) uncontrollable exogenous variables whose present and future values are known or can be predicted. \( A, B_1, B_2, \) and \( C \) are \( n \times n, n \times r_1, n \times r_2, \) and \( n \times s \) matrices respectively.

Each authority must choose an optimal trajectory (which we call an "strategy") for its own set of control variables over the time period \( t = 0, 1, \ldots, N-1 \). These controls, together with the trajectory for the exogenous variables \( z_t \) and the initial state \( x_0 \) determine the trajectory for the state variables over the time \( t = 1, 2, \ldots, N \). The first authority chooses its strategy \( \{u_{1t}\} \) to minimize its cost functional:

\[
\begin{align*}
J_1 &= \frac{1}{2} (x_N - \hat{x}_{1N})'Q_1 (x_N - \hat{x}_{1N}) + \frac{1}{2} \sum_{t=0}^{N-1} \left( (x_t - \hat{x}_t)'Q_1 (x_t - \hat{x}_t) ight. \\
&\quad \left. + (u_{1t} - \hat{u}_{1t})'R_{11} (u_{1t} - \hat{u}_{1t}) + (u_{2t} - \hat{u}_{2t})'R_{12} (u_{2t} - \hat{u}_{2t}) \right) 
\end{align*}
\]

(2)

The second authority chooses its strategy \( \{u_{2t}\} \) to minimize its cost functional:

\[
\begin{align*}
J_2 &= \frac{1}{2} (x_N - \hat{x}_{2N})'Q_2 (x_N - \hat{x}_{2N}) + \frac{1}{2} \sum_{t=0}^{N-1} \left( (x_t - \hat{x}_t)'Q_2 (x_t - \hat{x}_t) ight. \\
&\quad \left. + (u_{1t} - \hat{u}_{1t})'R_{21} (u_{1t} - \hat{u}_{1t}) + (u_{2t} - \hat{u}_{2t})'R_{22} (u_{2t} - \hat{u}_{2t}) \right) 
\end{align*}
\]

(3)

5 - For an introduction to the state-variable form of an econometric model, see Pindyck [19], Chapter 5 and the Appendix.
Here $\hat{x}_{1t}$ and $\hat{x}_{2t}$ represent nominal (desired) values for the state variables from the points of view of authorities 1 and 2 respectively, and similarly $\hat{u}_{1t}$ and $\hat{u}_{2t}$ represent nominal values of the control variables for each authority. The matrices $Q_1$ and $Q_2$ represent, for authorities 1 and 2 respectively, the relative weights assigned to deviations from the nominal paths for each state variable, and $R_{11}$ and $R_{22}$ designate the relative weights that each authority assigns to deviations from the nominal path for its own control variables. $R_{12}$ and $R_{21}$ designate the relative weights that each authority assigns to deviations from the nominal path for the other authority's control variables; a non-zero element in one of these matrices might indicate, for example, that the monetary authority considers it somewhat important that the fiscal authority keep government spending close to the target path for government spending specified by the fiscal authority. Thus these matrices indicate how important it is for each authority that the other authority stay close to its policy variable targets. As one would expect, these matrices are relevant only to the closed-loop strategies, where each authority must continually revise its own policy in response to policy changes by the other authority. If $R_{12}$ is large (relative to $Q_1$ and $R_{11}$) then authority 1 will design its strategy so as to force authority 2 to keep its policy variables close to their nominal paths. This has no relevance, of course, to an open-loop control strategy (where neither authority can influence the policy of the other), and as we will see $R_{12}$ and $R_{21}$ do not appear in the open-loop solution.

Some restrictions must be placed on the matrices $Q_1$, $Q_2$, $R_{11}$, and $R_{22}$. We will assume that $Q_1$ and $Q_2$ are positive semi-definite, and that $R_{11}$ and $R_{22}$ are positive definite. We put no restrictions on $R_{12}$ and $R_{21}$. For most economic problems all of these matrices will be diagonal, although it is not essential that this be the case.
The problem as stated above is essentially a discrete-time differential game. On the following pages we will find, for both the open-loop and closed-loop cases, Nash solutions to this problem. The Nash solutions \((u^*_1, u^*_2)\) are defined as satisfying the conditions

\[
J_1(u^*_1, u^*_2) \leq J_1(u_1, u_2) \tag{4}
\]

and \(J_2(u^*_1, u^*_2) \leq J_2(u^*_1, u_2) \tag{5}\)

for all possible \(u_1\) and \(u_2\).

3. Solution of the Open-Loop Problem

We seek a set of optimal controls \(u^*_1\) and \(u^*_2\) defined over \([0,N-1]\). These controls are determined by each authority at the beginning of the planning period, and are adhered to over the whole period.

3.1. Necessary Conditions

We write the two Hamiltonians:

\[
H_1(x_t, P_{1,t+1}, u_{1t}, u_{2t}) = \frac{1}{2} (x_t - \dot{x}_{1t})^T Q_1 (x_t - \dot{x}_{1t}) + \frac{1}{2} (u_{1t} - \hat{u}_{1t})^T R_{11} (u_{1t} - \hat{u}_{1t})
\]

\[+
\frac{1}{2} (u_{2t} - \hat{u}_{2t})^T R_{12} (u_{2t} - \hat{u}_{2t}) + p'_{1,t+1}(Ax_t + B_1 u_{1t} + B_2 u_{2t} + Cz_t) \tag{6}
\]

\[
H_2(x_t, P_{2,t+1}, u_{1t}, u_{2t}) = \frac{1}{2} (x_t - \dot{x}_{2t})^T Q_2 (x_t - \dot{x}_{2t}) + \frac{1}{2} (u_{1t} - \hat{u}_{1t})^T R_{21} (u_{1t} - \hat{u}_{1t})
\]

\[+
\frac{1}{2} (u_{2t} - \hat{u}_{2t})^T R_{22} (u_{2t} - \hat{u}_{2t}) + p'_{2,t+1}(Ax_t + B_1 u_{1t} + B_2 u_{2t} + Cz_t) \tag{7}
\]

6 For a discussion of the Nash equilibrium in a differential game, see Starr and Ho [21].
We can apply the discrete-time minimum principle directly. Letting $x_t^*$, $u_{1t}^*$, $u_{2t}^*$, $p_{1t}^*$, and $p_{2t}^*$ represent optimal values for the state, control, and co-state vectors, we have as necessary conditions for authority 1:

$$x_{t+1}^* - x_t^* = \frac{\partial H_1}{\partial p_{1,t+1}^*} = Ax_t^* + B_1 u_{1t}^* + B_2 u_{2t}^* + Cz_t$$  \hspace{1cm} (8)

$$p_{1,t+1}^* - p_{1,t}^* = - \frac{\partial H_1}{\partial x_t^*} = - Q_1 (x_t^* - \hat{x}_1 t) - A' p_{1,t+1}^*$$  \hspace{1cm} (9)

The difference equations (8) and (9) are subject to the split boundary conditions:

$$x_0 = \xi$$  \hspace{1cm} (10)

and $p_{1,N}^* = Q_1(x_N^* - \hat{x}_1 N)$  \hspace{1cm} (11)

Equation (11) is a result of the transversality condition. Finally, the optimal control $\{u_{1t}^*\}$ must minimize the Hamiltonian:

$$\frac{\partial H_1}{\partial u_{1t}^*} = 0$$  \hspace{1cm} (12)

or, $u_{1t}^* = -R_{11}^{-1} B_{11} p_{1,t+1}^* + \hat{u}_{1t}$  \hspace{1cm} (13)

Similarly, we have as necessary conditions for authority 2:

$$x_{t+1}^* - x_t^* = \frac{\partial H_2}{\partial p_{2,t+1}^*} = Ax_t^* + B_1 u_{1t}^* + B_2 u_{2t}^* + Cz_t$$  \hspace{1cm} (14)

$$p_{2,t+1}^* - p_{2,t}^* = - \frac{\partial H_2}{\partial x_t^*} = - Q_2 (x_t^* - \hat{x}_2 t) - A' p_{2,t+1}^*$$  \hspace{1cm} (15)

7 - See Pindyck [19], Chapter 2.
3.2. Obtaining a Solution

We assume that the optimal co-states $p^*_{1,t}$ and $p^*_{2,t}$ are of the form

$$p^*_{1,t} = K_{1,t}^t x^* + g_{1,t}$$  \hspace{1cm} (19)

and

$$p^*_{2,t} = K_{2,t}^t x^* + g_{2,t}$$  \hspace{1cm} (20)

We will soon see that these linear relationships indeed result in a solution. Now substitute (19) into (13) and (20) into (18):

$$u^*_{1t} = -R^{-1}_{11} B (K_{1,t}^t x^* + g_{1,t+1}) + \hat{u}_{1t}$$  \hspace{1cm} (21)

$$u^*_{2t} = -R^{-1}_{22} B (K_{2,t+1}^t x^* + g_{2,t+1}) + \hat{u}_{2t}$$  \hspace{1cm} (22)

We now substitute (19), (20), (21), and (22) into (8), (9), and (15):

$$x^*_{t+1} - x^* = A x^* - B R^{-1} B^t K_{1,t+1}^t x^* + B R^{-1} B^t g_{1,t+1} + B_1 \hat{u}_{1t}$$

$$- B_2 R_2^{-1} B_2^t K_{2,t+1}^t x^* + B_2 R_2^{-1} B_2^t g_{2,t+1} + B_2 \hat{u}_{2t} + C z_t$$  \hspace{1cm} (23)

$$p^*_{1,t+1} - p^*_{1,t} = -Q_1(x^*_{t+1} - x^*_{t}) - A'(K_{1,t+1}^t x^*_{t+1} + g_{1,t+1})$$  \hspace{1cm} (24)

$$p^*_{2,t+1} - p^*_{2,t} = -Q_2(x^*_{t+1} - x^*_{t}) - A'(K_{2,t+1}^t x^*_{t+1} + g_{2,t+1})$$  \hspace{1cm} (25)
Now rearrange (23):

\[
(I + B_1 R^{-1} B_1' K_{111111,t+1} + B_2 R^{-1} B_2' K_{222222,t+1}) x^*_{t+1} = (I+A) x^*_t
\]

\[
= B_1 R^{-1} B_1' g_{1,t+1} - B_2 R^{-1} B_2' g_{2,t+1} + B_1 \hat{u}_{1t} + B_2 \hat{u}_{2t} + C z_t
\]

(26)

We now substitute (19) into the left-hand side of (24) and (20) into the left-hand side of (25), and rearrange:

\[
(I+A)' K_{11,t+1} x^*_t + Q_1 x^*_t - K_{1t} x^*_t = -(I+A)' g_{1,t+1} + g_{1,t} + Q_1 \hat{x}_{1t}
\]

(27)

\[
(I+A)' K_{22,t+1} x^*_t + Q_2 x^*_t - K_{2t} x^*_t = -(I+A)' g_{2,t+1} + g_{2,t} + Q_2 \hat{x}_{2t}
\]

(28)

Now define the n x n matrix \( E_t \):

\[
E_t = I + B_1 R^{-1} B_1' K_{111111,t+1} + B_2 R^{-1} B_2' K_{222222,t+1}
\]

(29)

For now we will assume that \( E_t \) is non-singular so that \( E_t^{-1} \) always exists.

Later we will see that this is indeed the case since \( K_{1t} \) and \( K_{2,t} \) must always be positive semi-definite. Now we can re-write (26) as:

\[
x^*_{t+1} = E^{-1}_{t+1} (I+A) x^*_t - E^{-1}_{t+1} B_1 R^{-1} B_1' g_{1,t+1} - E^{-1}_{t+1} B_2 R^{-1} B_2' g_{2,t+1}
\]

\[
+ E^{-1}_{t+1} B_1 \hat{u}_{1t} + E^{-1}_{t+1} B_2 \hat{u}_{2t} + E^{-1}_{t+1} C z_t
\]

(30)

Now substitute (30) into (27):

\[
(I+A)' K_{11,t+1} [E^{-1}_{t+1} (I+A) x^*_t - E^{-1}_{t+1} B_1 R^{-1} B_1' g_{1,t+1}
\]

\[
- E^{-1}_{t+1} B_2 R^{-1} B_2' g_{2,t+1} + E^{-1}_{t+1} B_1 \hat{u}_{1t} + E^{-1}_{t+1} B_2 \hat{u}_{2t} + E^{-1}_{t+1} C z_t]
\]

\[
+ Q_1 x^*_t - K_{1t} x^*_t = -(I+A)' g_{1,t+1} + g_{1,t} + Q_1 \hat{x}_{1t}
\]

(31)

which, after rearranging, yields
\[ [Q_1 + (I+A)'K_{1,t+1}E^{-1}_{t+1}(I+A)] x_t^* \]

\[ - (I+A)'K_{1,t+1}E^{-1}_{t+1}(B_1R_{11}^{-1}g_1,t+1 + B_2R_{22}^{-1}g_2,t+1) + (I+A)'g_1,t+1 \]

\[ + (I+A)'K_{1,t+1}E^{-1}_{t+1}(B_1\hat{u}_1,t + B_2\hat{u}_2,t + Cz_t) - Q_1\hat{x}_{1t} = K_{1,t}x_t^* + g_1,t \]  

Equations (32) and (33) are results of the necessary conditions and therefore must hold for any initial state \( \xi \). As we will see, however, the matrices \( K_{1t} \) and \( K_{2t} \) are determined only by the matrices \( A, B_1, B_2, C, Q_1, Q_2, R_{11}, \) and \( R_{22} \), and do not depend in any way on \( \xi \). Thus equations (32) and (33) must hold for all \( x_t^* \), so that for each we can equate the coefficients of the left-hand and right-hand sides. Doing so yields the following four equations:

\[ K_{1,t} = Q_1 + (I+A)'K_{1,t+1}E^{-1}_{t+1}(I+A) \]  

\[ K_{2,t} = Q_2 + (I+A)'K_{2,t+1}E^{-1}_{t+1}(I+A) \]
\[
g_1, t = -(I+A)'K_1, t+1 E_{t+1}^{-1} (B_1 R_1^{-1} B_1' g_1, t+1 + B_2 R_2^{-1} B_2' g_2, t+1) \\
+ (I+A)'g_1, t+1 + (I+A)'K_1, t+1 E_{t+1}^{-1} (B_1 \hat{u}_{1t} + B_2 \hat{u}_{2t} + C z_{t}) - Q_1 \hat{x}_{1t} \quad (36)
\]

\[
g_2, t = -(I+A)'K_2, t+1 E_{t+1}^{-1} (B_1 R_1^{-1} B_1' g_1, t+1 + B_2 R_2^{-1} B_2' g_2, t+1) \\
+ (I+A)'g_2, t+1 + (I+A)'K_2, t+1 E_{t+1}^{-1} (B_1 \hat{u}_{1t} + B_2 \hat{u}_{2t} + C z_{t}) - Q_2 \hat{x}_{2t} \quad (37)
\]

From equations (11) and (19) we have

\[
p^{*}_{1, N} = Q_1 (x^{*}_{N} - \hat{x}_{1N}) = K_1, N x^{*}_{N} + g_1, N 
\]

This must hold for any \( x^{*}_{N} \), so that

\[
K_1, N = Q_1 
\]

and \( g_1, N = p^{*}_{1, N} - K_1, N x^{*}_{N} = - Q_1 \hat{x}_{1N} \) \quad (40)

Similarly, from (17) and (20):

\[
K_2, N = Q_2 
\]

and \( g_2, N = - Q_2 \hat{x}_{2N} \) \quad (42)

Equations (34) and (35) are the **joint Riccati equations** and (36) and (37) are the **joint tracking equations** for this open-loop decentralized control problem. Together with equation (29) for \( E_t \) and the boundary conditions (39), (40), (41), and (42), they can be solved for \( K_{1t}, K_{2t}, g_{1t}, \) and
\( g_{2t}, \quad t = 1, \ldots, N \). The optimal controls are determined by equations (21) and (22). Substituting (30) for \( x^*_{t+1} \) into (21) and (22), we obtain:

\[
\begin{align*}
    u^*_{1t} &= -R_{11}^{-1}B'_1K_{11}E_{11}^{-1}([I+A] x^*_{t} - B_1R_{11}^{-1}B'_1g_{1,t+1} - B_2R_{22}^{-1}B'_2g_{2,t+1} + B_1\hat{u}_{1t} + B_2\hat{u}_{2t} + Cz_t) - R_{11}^{-1}B'_1g_{1,t+1} + \hat{u}_{1t} \\
    \text{and,} \\
    u^*_{2t} &= -R_{22}^{-1}B'_2K_{22}E_{22}^{-1}([I+A] x^*_{t} - B_1R_{11}^{-1}B'_1g_{1,t+1} - B_2R_{22}^{-1}B'_2g_{2,t+1} + B_1\hat{u}_{1t} + B_2\hat{u}_{2t} + Cz_t) - R_{22}^{-1}B'_2g_{2,t+1} + \hat{u}_{2t}
\end{align*}
\]

Equations (43) and (44) determine the optimal controls \( u^*_{1t} \) and \( u^*_{2t} \) in terms of the present optimal state \( x^*_{t} \) and the solutions to the joint Riccati equations and joint tracking equations. The solution to all of these equations require calculations of the inverted matrix \( E_t^{-1} \). We can show that the matrix \( E_t \) is indeed invertible (see equation (29)) by demonstrating that all of the Riccati matrices \( K_{1t} \) and \( K_{2t} \) are positive semi-definite.

Since \( Q_1 \) and \( Q_2 \) were specified to be positive semi-definite \( K_{1N} \) and \( K_{2N} \) must be positive semi-definite (from equations (39) and (41)), and therefore \( E_N \) is positive definite. Now examine equations (34) and (35). Since \( K_{1N}', \quad K_{2N}', \quad Q_1', \quad \text{and} \quad Q_2' \) are all positive semi-definite, \( K_{1,N-1} \) and \( K_{2,N-1} \) must be positive semi-definite. And similarly \( K_{1,N-2}', \ldots, K_{1,1} \text{ and } K_{2,N-2}', \ldots, K_{2,1} \)
are all positive semi-definite. We can thus be certain that the matrix $E_t$ is always non-singular and has an inverse.

### 3.3 Computing the Inverse Matrix $E_t^{-1}$

The expression for $E_t$ in equation (29) is an $n \times n$ matrix, and repeatedly obtaining its inverse (for each time period $t$) will be computationally costly if the dimensionality of the econometric model (i.e. the number of state variables) is at all large. Fortunately equation (29) can be transformed into an expression that involves inverting a matrix of dimension $r$, where $r = r_1 + r_2$ is the total number of control variables and is likely to be considerably smaller than $n$. We use the matrix identity

$$\left( I_n + ST' \right)^{-1} = I_n - S(I_r + T'S)^{-1}T'$$

(45)

where $I_n$ and $I_r$ are identity matrices of order $n$ and $r$ respectively, and $S$ and $T$ are both $n \times r$ matrices. Now we partition $S$ and $T$ into two parts:

$$S = (S_1, S_2) \quad \text{and} \quad T = (T_1, T_2)$$

where $S_1$ and $T_1$ are $n \times r_1$ and $S_2$ and $T_2$ are $n \times r_2$, so that

$$\left( I_n + ST' \right)^{-1} = \left( I_n + S_1T_1' + S_2T_2' \right)^{-1}$$

(46)

Now letting $S_1 = B_1$, $S_2 = B_2$, $T_1' = R_{11}^{-1}B_1'K_{11}I_{1t}$ and $T_2' = R_{22}^{-1}B_2'K_{22}I_{2t}$, we have

$$E_t^{-1} = I_n - (B_1, B_2)_r \left[ \begin{pmatrix} R_{11}^{-1}B_1'K_{11}I_{1t} & R_{11}^{-1}B_1'K_{11}I_{2t} \\ R_{22}^{-1}B_2'K_{22}I_{1t} & R_{22}^{-1}B_2'K_{22}I_{2t} \end{pmatrix} \right]^{-1} \left[ \begin{pmatrix} R_{11}^{-1}B_1'K_{11}I_{1t} \\ R_{22}^{-1}B_2'K_{22}I_{2t} \end{pmatrix} \right]$$

(47)
The matrix to be inverted is now of dimension $r$.

3.4. Summary of the Open-Loop Solution

In order to obtain the open-loop decentralized optimal control solution we must begin with a specification of the econometric model in state-variable form (i.e. the matrices $A$, $B_1$, $B_2$, and $C$), the trajectories of the exogenous variables $z_t$, the nominal (target) state and control trajectories for each controller ($\hat{x}_{1t}$, $\hat{x}_{2t}$, $\hat{u}_{1t}$, and $\hat{u}_{2t}$), and the matrices of the cost functionals $Q_1$, $R_{11}$, and $R_{22}$ (note that $R_{12}$ and $R_{21}$ are not needed for the solution, and in fact are irrelevant to the open-loop problem). A solution can then be obtained by the following steps:

1. Solve the joint Riccati equations (34) and (35) with boundary conditions (39) and (41), together with equation (47) for $E_t^{-1}$, backwards in time to obtain values for $K_{1t}$, $K_{2t}$, and $E_t^{-1}$, $t = 1, \ldots, N$. Store the $2N$ $n \times n$ product matrices $K_{1t}E_t^{-1}$ and $K_{2t}E_t^{-1}$.

2. Using the matrices calculated above, solve the joint tracking equations (36) and (37) with boundary conditions (40) and (42) backwards in time to obtain values for $g_{1t}$ and $g_{2t}$, $t = 1, \ldots, N$. Store the resulting $2N$ $n$-vectors.

3. Compute the optimal control $u_{1t,0}^*$ from equation (43) and $u_{2t,0}^*$ from equation (44) using $x_{0}^* = \xi$. Then compute $x_{1}^*$ from equation (1), the system equation. Now $x_{1}^*$ can be used in equations (43) and (44) to compute $u_{11}^*$ and $u_{21}^*$, these can be used in equation (1) to compute $x_{2}^*$, etc. This is continued until all of the $u_{1t}^*$ and $u_{2t}^*$, $t = 0, 1, \ldots, N-1$, and all of $x_t^*$, $t = 1, \ldots, N$, have been computed.

4. The optimal costs $J_1^*$ and $J_2^*$ can be computed from equations (2) and (3).
3.5. Interpretation of the Co-State Variables

Two sets of co-state variables exist for our problem, one set for each controller. Their interpretation is not very different from that of the centralized case, except that now each set of co-states determines a marginal "cost-to-go" for a particular controller. Let \( J_1^*(x_t, t) \) and \( J_2^*(x_t, t) \) be the costs to controllers 1 and 2 respectively that would accrue if each followed an optimal strategy from time \( t \) to the terminal time \( N \). Clearly, these costs are a function of \( x_t \), the particular state that happens to exist at time \( t \).

Since each controller has obtained his optimal strategy by applying the minimum principle to his particular cost functional, we know that the co-states at time \( t \) must be given by:

\[
p_{1t} = \frac{3}{\partial x_t} J_1^*(x_t, t) \tag{48}
\]

and

\[
p_{2t} = \frac{3}{\partial x_t} J_2^*(x_t, t) \tag{49}
\]

Thus each co-state variable at time \( t \) is the marginal cost to one particular controller resulting from a small change in the value of the corresponding state variable. If at time \( t \) the state of the system were changed by a small amount \( \Delta x_t \) and the system is controlled optimally (by each controller) until the terminal time \( t = N \), then the additional cost to each controller is given by

\[
\Delta J_1^*(x_t, t) = p_{1t}^* \Delta x_t \tag{50}
\]

and

\[
\Delta J_2^*(x_t, t) = p_{2t}^* \Delta x_t \tag{51}
\]
Note that it is possible for these additional costs to be negative. A random shock might change the state of the system in a way that would be to the advantage of one controller but at the expense of the other.

3.6. Generalization to More Than Two Controllers

It is easy to generalize our results to the case of \( k \) controllers.

Write the model in state-variable form as

\[
\begin{align*}
    x_{t+1} - x_t &= A x_t + B_1 u_{1t} + B_2 u_{2t} + \ldots + B_k u_{kt} + C z_t \\
    \end{align*}
\]

and the cost functional for controller \( j \) as

\[
\begin{align*}
    J_j &= \frac{1}{2} (x_N - \hat{x}_j N)'Q_j (x_N - \hat{x}_j N) + \frac{1}{2} \sum_{t=0}^{N-1} (x_t - \hat{x}_{jt})'Q_j (x_t - \hat{x}_{jt}) \\
    &\quad + (u_{1t} - \hat{u}_{1t})'R_{j1} (u_{1t} - \hat{u}_{1t}) + \ldots + (u_{jt} - \hat{u}_{jt})'R_{jj} (u_{jt} - \hat{u}_{jt}) + \ldots \\
    &\quad + (u_{kt} - \hat{u}_{kt})'R_{jk} (u_{kt} - \hat{u}_{kt}) \\
    \end{align*}
\]

It is straightforward to determine that the optimal open-loop strategies \( u_{1t}^*, u_{2t}^*, \ldots, u_{kt}^* \), \( t = 0, 1, \ldots, N-1 \), are found by solving the \( k \) joint Riccati equations

\[
K_{jt} = Q_j + (I + A)'K_{jt+1}E_{t+1}^{-1}(I + A), \quad (j = 1, \ldots, k)
\]

with boundary conditions

\[
K_{jN} = Q_j, \quad (j = 1, \ldots, k)
\]

solving the \( k \) joint tracking equations.
\[ g_{jt} = -(I+A)'K_{j,t+1}E^{-1}_{t+1} \left( \sum_{j=1}^{k} B_j R^{-1}_{jj} B'_j g_{j,t+1} \right) + (I+A)'g_{j,t+1} \]

\[ + (I+A)'K_{j,t+1}E^{-1}_{t+1} \left( \sum_{j=1}^{k} B_j \hat{u}_{jt} + Cz_t \right) - Q_j \hat{x}_{jt}, \quad (j = 1, \ldots, k) \] (56)

with boundary conditions

\[ g_{jN} = -Q_j \hat{x}_{jN}, \quad (j = 1, \ldots, k) \] (57)

and using the solutions in the k optimal control equations

\[ u^*_j = -R^{-1}_{jj} B'_j K - E^{-1}_{t+1} [(I+A) x^*_t - \sum_{j=1}^{k} B_j R^{-1}_{jj} B'_j g_{j,t+1} \]

\[ + \sum_{j=1}^{k} B_j \hat{u}_{jt} + Cz_t] - R^{-1}_{jj} B'_j g_{j,t+1} + \hat{u}_{jt}, \quad (j = 1, \ldots, k) \] (58)

Note that the matrix \( E_t \) is now given by

\[ E_t = I + \sum_{j=1}^{k} B_j R^{-1}_{jj} B'_j K_j, t \] (59)

and its inverse can be calculated from

\[ E^{-1}_t = I_n - S(I_r + T'S)^{-1}T' \] (60)

with \( S = (B_1, B_2, \ldots, B_k) \) (61)

and

\[ T' = \begin{bmatrix} R^{-1}_{11} B'_1 K_{1t} \\ \vdots \\ R^{-1}_{kk} B'_k K_{kt} \end{bmatrix} \] (62)

Again, \( r \) is the total number of control variables.
4. Solution of the Closed-Loop Problem

If we assume that the two authorities are operating in a closed-loop mode, then we must allow for them to adapt their policies (optimally) to changes in the state of the economy. Note that they are not adapting to the impact of random shocks on the economy, but rather each is adapting to the evolving strategy of the other authority. Each authority will use a control rule, designed at the beginning of the planning period and applied to observations of the state of the economy, to continuously revise his policy.

The solution of the closed-loop problem involves a modification of the solution of the open-loop problem. The necessary conditions are the same, except that now the difference equations describing the evolution of the co-states must allow for changes in $u_{1t}$ and $u_{2t}$ in response to changes in $x_t$. We replace equation (9), for example, with

$$p_{1,t+1}^* - p_{1,t}^* = -\frac{\partial H_1}{\partial x_t}^* = -Q_1(x_t - \hat{x}_t) - \left(\frac{\partial u_{1t}}{\partial x_t}\right)'R_{11}(u_{1t} - \hat{u}_{1t})$$

$$- \left(\frac{\partial u_{2t}}{\partial x_t}\right)'R_{12}(u_{2t} - \hat{u}_{2t}) - A'p_{1,t+1}^* + \left(\frac{\partial u_{1t}}{\partial x_t}\right)'B_{11}^*p_{1,t+1}^* - \left(\frac{\partial u_{2t}}{\partial x_t}\right)'B_{12}^*p_{1,t+1}^* (63)$$

Since the Hamiltonians are the same as in the open-loop case, equations (13) and (18) still hold, and if we again assume (and later show to be true) that the co-state variables are related linearly to the state variables, then equations (21) and (22) also hold. Differentiating (21) and (22) with respect to $x_t$, we obtain

---

8 - It should be pointed out, however, that under certainty equivalence the closed-loop stochastic strategy is the same as the closed-loop deterministic strategy. Thus the closed-loop control rules that we will derive are also optimal in the case of additive serially uncorrelated errors.
\[
\frac{\partial u_{1t}}{\partial x_t} = -K_{11}^{-1}B_{11}'K_{11,1,t+1}
\]  
(64)

and
\[
\frac{\partial u_{2t}}{\partial x_t} = -K_{22}^{-1}B_{22}'K_{22,2,t+1}
\]  
(65)

Now substituting these derivatives into (63), we have
\[
P_{t+1} - P_t = -Q_1(x^*_t - \hat{x}_t) + K_{11,t+1}B_{11}^{-1}(u_{1t} - \hat{u}_{1t}) + K_{12,t+1}B_{22}^{-1}R_{22}^{-1}R_{21}(u_{2t} - \hat{u}_{2t})
\]

\[- A'p_{1,t+1} + K_{11,t+1}B_{11}^{-1}B_{11}'p_{1,t+1} + K_{21,t+1}B_{22}^{-1}B_{21}'p_{1,t+1}
\]  
(66)

And similarly, for \( p_{2,t} \)
\[
P_{2,t+1} - P_{2,t} = -Q_2(x^*_2 - \hat{x}_2) + K_{21,t+1}B_{22}^{-1}(u_{2t} - \hat{u}_{2t}) + K_{22,t+1}B_{22}^{-1}R_{22}^{-1}R_{21}(u_{2t} - \hat{u}_{2t})
\]

\[- A'p_{2,t+1} + K_{22,t+1}B_{22}^{-1}B_{22}'p_{2,t+1} + K_{22,t+1}B_{22}^{-1}B_{22}'p_{2,t+1}
\]  
(67)

Equation (23) still holds, but equations (24) and (25) are now replaced by:
\[
P_{1,t+1} - P_{1,t} = -Q_1(x^*_1 - \hat{x}_1) - A'(K_{11,t+1}x^*_1 + g_1,t+1)
\]

\[- K_{21,t+1}B_{22}^{-1}R_{22}^{-1}R_{21}(K_{22,t+1}x^*_2 + g_2,t+1)
\]

\[+ K_{22,t+1}B_{22}^{-1}B_{22}'(K_{11,t+1}x^*_1 + g_1,t+1)
\]  
(68)

and
\[
P_{2,t+1} - P_{2,t} = -Q_2(x^*_2 - \hat{x}_2) - A'(K_{22,t+1}x^*_2 + g_2,t+1)
\]

\[- K_{11,t+1}B_{11}^{-1}R_{11}^{-1}R_{12}^{-1}B_{11}'(K_{11,t+1}x^*_1 + g_1,t+1)
\]

\[+ K_{11,t+1}B_{11}^{-1}B_{11}'(K_{22,t+1}x^*_2 + g_2,t+1)
\]  
(69)
These equations are simply the result of substituting equations (19), (20), (21), and (22) into equations (66) and (67).

Now substitute equation (19) into the left-hand side of (68), (20) into the left-hand side of (69), and rearrange:

\[(I+A)'K_{1,t+1}x_{t+1} + K_{1,t+1}'B_{R}^{-1}R_{12}^{-1}B_{K_{1,t+1}}x_{t+1} + Q_{1}x_{t+1} = -(I+A)'g_{1,t+1} - K_{2,t+1}'B_{R}^{-1}R_{12}^{-1}B_{g_{2,t+1}}\]

\[+ K_{2,t+1}'B_{R}^{-1}B_{g_{1,t+1}} + g_{1,t} + Q_{1}x_{t+1}\]  

(70)

\[(I+A)'K_{2,t+1}x_{t+1} + K_{2,t+1}'B_{R}^{-1}R_{12}^{-1}B_{K_{2,t+1}}x_{t+1} + Q_{2}x_{t+1} = -(I+A)'g_{2,t+1} - K_{1,t+1}'B_{R}^{-1}R_{12}^{-1}B_{g_{1,t+1}}\]

\[+ K_{1,t+1}'B_{R}^{-1}B_{g_{2,t+1}} + g_{2,t} + Q_{2}x_{2t}\]  

(71)

We define \(E_{t}\) as before in equation (29), so equation (30) still holds. Substituting (30) into (70) and rearranging,

\[[(I+A)'K_{1,t+1}' + K_{2,t+1}'B_{R}^{-1}R_{12}^{-1}B_{1}K_{2,t+1} + Q_{1}x_{t+1}\]

\[+ [(I+A)'K_{1,t+1}' + K_{2,t+1}'B_{R}^{-1}R_{12}^{-1}B_{K_{1,t+1}}x_{t+1} + Q_{1}x_{t+1}\]

\[+ [(I+A)'K_{1,t+1}' + K_{2,t+1}'B_{R}^{-1}R_{12}^{-1}B_{g_{1,t+1}} + g_{1,t} + Q_{1}x_{t+1}\]

\[+ [(I+A)'K_{1,t+1}' + K_{2,t+1}'B_{R}^{-1}R_{12}^{-1}B_{g_{2,t+1}} + g_{2,t} + Q_{2}x_{2t}\]

\[E_{t+1}'(I+A) + O_{1}x_{t+1} + O_{2}x_{2t}\]  

(72)
Similarly, substituting (30) into (71) and rearranging,

\[
[(I+A)K_{2,t+1}^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 K_{1,t} + t + 1 B_1 R_2^{-1} B_1 K_{2,t+1}] E_{t+1}^{-1} (I+A) x_t^* + Q_2 x_t^* - [(I+A)K_{2,t+1}^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 K_{2,t+1}^{+K_1}]
\]

\[
- E_{t+1}^{-1} (B_1 R_2^{-1} B_1 g_1, t+1 + B_2 R_2^{-1} B_1 g_2, t+1)
\]

\[
+ (I+A)'g_2, t+1 + K_1^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 g_2, t+1 - K_1^{+K_1'} + B_1 R_2^{-1} B_1 g_2, t+1
\]

\[
+ ([I+A)'K_{2,t+1}^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 K_{2,t+1}^{+K_1}]
\]

\[
- E_{t+1}^{-1} (B_1 u_{1t} + B_2 u_{2t} + Cz_t) - Q_2 x_{2t} = K_2, t^{x_2} + g_2, t
\]

Since equations (72) and (73) must hold for all \(x_t^*\), we can equate the coefficients of the left-hand and right-hand sides. Doing so yields the following equations.

\[
K_1, t = Q_1 + (I+A)'K_{1,t+1}^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 K_{1,t} + t + 1 E_{t+1}^{-1} (I+A)\]

\[
+ (K_1^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 K_{2,t+1}^{+K_1'}) E_{t+1}^{-1} (I+A) \quad (74)
\]

\[
K_2, t = Q_2 + (I+A)'K_{2,t+1}^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 K_{1,t} + t + 1 E_{t+1}^{-1} (I+A)\]

\[
+ (K_1^{+K_1'} + B_1 R_2^{-1} R_1^{-1} B_1 K_{2,t+1}^{+K_1'}) E_{t+1}^{-1} (I+A) \quad (75)
\]
The transversality conditions expressed by equations (11') and (17) still hold, and therefore the boundary conditions in equations (39), (40), (41), and (42) still hold. Equations (74) and (75) are the joint closed-loop Riccati equations, and with boundary conditions (39) and (41) they can be solved together with equation (47) for $K_{1t}$, $K_{2t}$, and $E_{t}^{-1}$, $t = 1, \ldots, N$. Equations (76) and (77) are the joint closed-loop tracking equations, and together with boundary conditions (40) and (42) and the matrices $E_{t}^{-1}$, $K_{1t}$ and
\( K_2t \), they can be solved for \( g_1t \) and \( g_2t \), \( t = 1, \ldots, N \). Once the matrices \( E^{-1}_t, K_1t, \) and \( K_2t \) and vectors \( g_1t \) and \( g_2t \) have been calculated, the optimal closed-loop control rules are given by equations (43) and (44). For example, the optimal controls \( u^{*}_{1,0} \) and \( u^{*}_{2,0} \) are computed from equations (43) and (44) using \( x^{0} = \xi \). Then \( x^{1} \) is computed using equation (1) and is used in equations (43) and (44) to compute \( u^{*}_{1,1} \) and \( u^{*}_{2,1} \), etc.\(^9\)

The co-state variables have the same interpretation as in the open-loop problem; equations (48) and (49) still apply, so that each set of co-states determines the marginal "cost-to-go" for a particular controller resulting from a small change in the value of the state variables. Again, it is possible for the co-states to be negative, as a random shock might affect the system in a way that is advantageous to one controller.

5. Decentralized Stabilization Policies Using a Small Model

In order to examine the characteristics of decentralized stabilization policies, we will apply our open-loop and closed-loop solutions to a small, linear econometric model of the United States. Several experiments will be performed, all of which are designed to demonstrate the economic effects of conflict between monetary and fiscal objectives.

The model, which was constructed and used by this author in earlier studies of optimal stabilization policies, is described in detail elsewhere \([19,20]\). It contains nine behavioral equations together with a tax relation and an income identity. Fiscal policy is provided for through exogenous government expenditures \( G \) and a surtax \( T_0 \), and monetary policy through the money supply \( M \) (currency plus demand deposits). GNP and its components are in real terms, and total investment is disaggregated, so that separate equations explain

\(^9\) If additive noise is present, then the \( x^{*} \) are determined by measurement, and used directly in equations (43) and (44) to determine the optimal controls.
consumption C, fixed nonresidential investment INR, residential investment IR, and change in inventories IIN. The remaining behavioral equations explain short and long-term interest rates (R, RL), the price level P, the unemployment rate UR, and the money wage rate W. The equations themselves are listed in the Appendix. By adding new variables to replace variables with lags greater than one period, the model can easily be expressed in the state-variable form of equation (1).

All of the policy experiments pertain to a time horizon of twenty quarters, beginning with the first quarter of 1957 and ending with the first quarter of 1962. It is assumed that the nominal trajectories \( \{x_{1t}\} \) and \( \{x_{2t}\} \) are the same for both the fiscal and monetary authorities, and that differences in objectives are expressed by the weighting matrices \( Q_1 \) and \( Q_2 \). Thus both authorities are assumed to agree on what an "ideal" unemployment rate and "ideal" rate of inflation are, but one authority might place greater importance on unemployment while the other places greater importance on inflation. Nominal trajectories are also specified for the control variables; it is assumed that the fiscal authority \( (u_{1t}) \) desires a zero surtax and steady growth in government expenditures and the monetary authority \( (u_{2t}) \) desires steady growth of the money supply. All of the nominal trajectories are shown in Table 1.  

5.1 Open-Loop Policies

The open-loop policy experiments are summarized in Table 2 in terms of the diagonal coefficient values in the weighting matrices \( Q_1 \) and \( Q_2 \) (the off-diagonal coefficients in the Q's and R's are all zero). In all of the experiments the fiscal authority can manipulate only government spending (the coefficient in
Table 1  NOMINAL TRAJECTORIES FOR ENDOGENOUS VARIABLES AND CONTROL VARIABLES*

<table>
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<th>t</th>
<th>( \hat{C}_t )</th>
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<th>IRₜ</th>
<th>IₜN</th>
<th>( \hat{R}_t )</th>
<th>( \hat{R}_L )</th>
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<th>( \hat{U}_t )</th>
<th>( \hat{W}_t )</th>
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* The money supply appears in the model in differenced form, and therefore the change in the money supply is used as the monetary control variable.
R_{11} corresponding to T_0 is made large enough so that that variable is forced to follow its nominal path, i.e. the surtax is always zero).

Weights are assigned to deviations from the nominal paths of both government spending (R_{11}(G) = 30) and the money supply (R_{22}(\Delta M) = 150), so that each authority has a trade-off between macroeconomic targets and the use of its control variable to reach those targets. The fiscal authority, for example, might wish to decrease the unemployment rate, but this must be balanced against the (political) cost of very large increases in government spending. Finally, R_{12} and R_{21} are both zero; these matrices do not enter into the solution of the open-loop problem.

In the first set of experiments (Runs 1, 2, and 3) there are two target variables, the price level and the unemployment rate. In Run 1 both authorities have the same objectives, a two percent rate of inflation and a two percent unemployment rate, and both attach the same weights to these objectives (Q_1 = Q_2). The resulting optimal policy solution will provide a frame of reference for Runs 2 and 3, since it corresponds to the conventional case of centralized control. In Run 2 each authority has only one target variable, the price level for the fiscal authority and the unemployment rate for the monetary authority. The conflicting objectives are reversed in Run 3, with the fiscal authority attempting to achieve a low unemployment rate and the monetary authority attempting to achieve a low rate of inflation. The optimal policy results for Runs 1, 2, and 3 are shown graphically in Figures 1 to 9.

---

Note that the coefficient values in Q_1, Q_2, R_{11}, and R_{22} must be interpreted in terms of the relative magnitudes of the endogenous and control variables. Percent deviations from the nominal path of government spending are penalized 20 times as heavily as those for the money supply.

Both the price level and unemployment rate are weighted approximately equally in terms of percent deviations from their nominal paths, and twice as heavily as government spending. Of course an inflation rate of two percent and an unemployment rate of two percent are mutually inconsistent targets, even in this case where both authorities agree on objectives.
Table 2
OPEN-LOOP POLICY EXPERIMENTS

(In all cases, $R_{12} = R_{21} = 0$, $R_{11}(T_0) = 1 \times 10^5$, $R_{11}(G) = 30$, and $R_{22}(\Delta M) = 150$.)

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Observe that in Run 1 the optimal policy calls for both government spending and the quarterly change in the money supply to be above their nominal paths throughout the planning period. The result is growth in disposable income of about 5% per year, and an unemployment rate of between 3 1/2 and 3 percent. The policy is less effective in maintaining a slow rate of growth in prices than in maintaining a low unemployment rate; the rate of inflation is about 5% during most of the planning period. This is characteristic of the model's wage-price dynamics; it is considerably easier to reduce the unemployment rate than it is to reduce the inflation rate, so that when both are weighted equally in the cost functionals the optimal policy favors the unemployment rate. Another characteristic of the model is that the government spending multiplier is much larger than that for the money supply. Although deviations from the nominal path for government spending are penalized twenty times as heavily as those for the money supply, government spending provides much of the increase in disposable income (the interest rates rise above their nominal paths, so that residential investment grows only during the first eight quarters).

In Run 2 the fiscal authority initially sets government spending $13 billion below its nominal value, and keeps government spending below the nominal path.

When interpreting the results it is important to keep in mind that the cost functionals accumulate penalties only over a finite time period - namely, twenty quarters. This will influence the optimal paths for some variables during the last few quarters of the planning period. Changes in the money supply, for example, affect the short-term interest rate immediately, but at least two quarters must elapse before there is any impact on residential investment and hence GNP. Therefore, if the cost functional $J_2$ does not penalize directly for interest rate deviations from the nominal, the optimal quarterly change in the money supply will always be equal to the nominal value ($1.4$ billion) during the last two quarters of the planning period.

The unbalanced weights for government spending and the money supply in $R_{11}$ and $R_{22}$ are chosen to allow for more monetary action in the solutions.
until the end of the planning period. This results in an immediate drop in disposable income (and an increase in the unemployment rate to 8%), and a rate of inflation that is below 3% for the first two years. As expected the monetary authority increases the money supply extremely rapidly, particularly during the beginning of the planning period (rapid increases in the money supply are less "cost-effective" later because of the lag in monetary policy). The impact on aggregate demand, however, occurs after several periods (as can be seen from residential and nonresidential investment and disposable income), so that the unemployment rate falls below 5% only after six quarters.

In Run 3 the fiscal authority simply maintains a level of government spending that is well above the nominal path throughout the planning period. Monetary policy is restrictive, with the money supply contracting during the first year of the planning period. The higher interest rates do not, however, compensate for the increased government spending, so that disposable income is high throughout the period. The unemployment rate remains below 3%, and the rate of inflation stays at about 6%. The fiscal authority is far more effective in reaching its objective than is the monetary authority, but this is partly due to the fact that a low unemployment rate is easier to achieve (in the model) than a low rate of inflation. The money supply is closer to its nominal path in Run 3 than it is in Run 2 because deviations from the nominal path are not as "cost-effective" when the price level is the target variable (observe also that government spending is closer to its nominal path in Run 2 than it is in Run 3).

Note that the short-term interest becomes negative at the beginning of the planning period. This is a result of the linearity of the model.
A comparison of Runs 2 and 3 shows that overall the fiscal authority has more control over the economy than does the monetary authority. Observe in Figures 6 and 7 that both the unemployment rate and the price level are closer to their nominal paths when they are the target of the fiscal authority than when they are the target of the monetary authority. This difference is largely due to the lag in monetary policy. As we will see, this lag gives the fiscal authority an even greater advantage in the closed-loop case. In Runs 4, 5, and 6 residential investment is added as a third target variable (the performance of the housing market is often an objective of macroeconomic policy, both fiscal and monetary). This target variable is worth considering because it is highly dependent on interest rates, and therefore more influenced by monetary policy. The optimal policy results are shown in Figures 10 to 16.

Run 4 corresponds to the case of centralized control in that there is no conflict between fiscal and monetary objectives. The target variables and their weightings in $Q_1$ and $Q_2$ are the same as in Run 1, except that now residential investment is also assigned a weight in both matrices such that percent deviations from its nominal path are penalized twice as heavily as those for the price level and the unemployment rate. Observe that the optimal trajectory for government spending is close to that of Run 1, and that the increase in residential investment is achieved largely through a rapid expansion of the money supply. The rate of growth of the money supply is high to begin with, and increases, reaching a peak towards the end of the fourth year (the drop in the fifty year is due to the finite planning horizon and the lag between monetary expansion and growth in residential investment). The result
is that residential investment grows much more rapidly than was the case in Run 1. In addition, the unemployment rate is very low, falling from 3 1/2 percent to little more than 2 percent, while the rate of inflation is approximately 5 1/2 percent throughout the planning period (again it is easier to achieve a low unemployment rate than a low rate of inflation).

In Runs 5 and 6 both authorities still have the unemployment rate and the price level as targets, but there is a conflict over residential investment. In Run 5 residential investment is an additional target for the fiscal authority (but not the monetary authority), and in Run 6 it is an additional target for the monetary authority. Observe that the results for Run 6 are almost identical to those for Run 4. The optimal policy for Run 4 calls for a division of objectives, with government spending used to achieve a low unemployment rate and the money supply used to reduce the interest rate so that residential investment can grow. In Run 6 the fiscal authority has, in effect, the same objectives, and the monetary authority can manipulate the money supply in roughly the same way.

In Run 5 the behaviour of the fiscal authority is close to what it was in Runs 4 and 6. The only way that the fiscal authority can reduce interest rates is by reducing government spending and thus total income so that the demand for money falls. The drop in income, however, would have a negative impact on residential investment, canceling part of the effect of the lower interest rate. By itself, then, the fiscal authority has only limited control over residential investment, and its best policy is to pursue its unemployment and inflation objectives. (The fiscal
authority also knows that some moderation of interest rates will occur in any case as a result of the optimal policy of the monetary authority. Although the monetary authority does not have residential investment as a target, the only way that it can pursue its unemployment objective is through monetary expansion. Thus the money supply in Run 5 grows rapidly, though not as rapidly as in Runs 4 and 6.

5.2 Closed-loop Policies

In the closed-loop mode each authority will continually revise its policy during the planning period to adapt to the evolving strategy of the other authority (and each authority will design and revise its own strategy knowing that the other authority will respond to it as time goes on). One would expect that the closed-loop mode would favor the fiscal authority, since the time lag between changes in government spending and changes in GNP is smaller than that for the money supply. Since the fiscal response to changes in monetary policy is faster than the monetary response to changes in fiscal policy, the fiscal authority should gain more of an advantage from a closed-loop strategy.

Note that the matrices $R_{12}$ and $R_{21}$ appear in the closed-loop solution. If policies can be revised then it is possible for one authority to influence the policy of the other. In particular, each authority can (if it desires) act to force the other authority to move its policy variables closer to (or farther from) the nominal paths.

Two closed-loop policy experiments are performed using the solution algorithm derived in Section 4. The first (Run 2A) is a repetition of Run 2, but in the closed-loop mode, and the results permit a direct comparison
between open and closed-loop behavior. The second experiment (Run 2B) is the same as the first except that a value of 300 is assigned to the coefficient in $R_{21}$ corresponding to government spending. This means that the monetary authority (in addition to desiring a low unemployment rate) would like government spending to be close to its nominal path. (While the economic relevance of this experiment is questionable, it helps to illustrate the nature of the closed-loop solution.) The closed-loop policy results are shown in Figures 17 to 25, together with the open-loop results from Run 2.

Let us first examine Run 2A ($R_{12} = R_{21} = 0$). As can be seen in the figures, the closed-loop mode indeed favors the fiscal authority. The optimal trajectory for the price level (the fiscal target) rises less rapidly in the closed-loop case than in the open-loop case, while the unemployment rate (the monetary target) is higher throughout the planning period. What is interesting is that this is a result not of a shift in fiscal policy, but a shift in monetary policy. The closed-loop and open-loop trajectories for government spending differ only slightly, with the closed-loop trajectory lower than the open-loop trajectory during the first four quarters, and higher during the remainder of the period. The closed-loop money supply, on the other hand, grows much less rapidly than the open-loop money supply during the first three years. As a result, interest rates are higher, and investment and income are lower.

To see why this smaller expansion of the money supply is optimal for the monetary authority, consider how each authority would react to policies of the other. The fiscal authority would react to a large increase in the

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Percent deviations for government spending are weighted five times as heavily as those for the unemployment rate in the monetary authority's cost functional.
money supply by a decrease in government spending. The fiscal contraction, however, could occur several periods after the monetary expansion and still cancel out any expansion in the GNP. Both authorities would accumulate an additional cost by having their policy variables deviate farther from the nominal paths, but the cost would be lower for the fiscal authority because of the delay before the beginning of the fiscal contraction. Extreme policies are thus more costly to the monetary authority than the fiscal authority (note that the fiscal authority begins with a level of government spending that is below that of the open-loop case). As a result, the optimal closed-loop strategy for the monetary authority is to moderate its initial increases in the money supply.

Now let us turn to Run 2B, for which $R_{21}(G) = 30$. The optimal strategy for the monetary authority is now quite different, with the quarterly change in the money supply initially small but rapidly increasing. The optimal fiscal policy, given this monetary policy, calls for government spending to begin at a higher level but rise less rapidly than it did before. The result is that government spending is closer to its nominal path throughout the planning period, and the price level is lower, so that the fiscal authority is clearly better off. The unemployment rate is higher during most of the planning period, but this added cost for the monetary authority is more than offset by the gains from the higher level of government spending and the smaller increases in the money supply. (Although the increase in government spending is small, that variable is weighted heavily in the cost functional of the monetary authority.)
6. Concluding Remarks

Despite the simplicity of the econometric model used in the experiments, the results of the last section do demonstrate some of the implications of conflict in the objectives of stabilization policy. We found, for example, that when the conflict is between unemployment and inflation, the fiscal objectives will be more nearly met. This is particularly the case in the closed-loop mode, and is a result of the longer lag inherent in monetary policy. Certain targets, however, can be reached only by a particular authority; we saw that rapid increases in residential investment will not be achieved unless it is an objective shared by the monetary authority or is an indirect result of some other monetary objective (as in Run 2). Finally, the results seem to indicate that the "sub-optimality" resulting from a conflict situation is severe only in the first four to six quarters of the planning period. After six quarters a "compromise" behavior occurs where neither authority is as close to its targets as it would be in a cooperative situation, but there are no wide deviations from targets as a result of time-phasing problems arising from the conflict. In Run 2, for example, the unemployment rate goes above 8 percent during the first year of the planning period (as a result of the monetary lag), but then settles to within a range of four to five percent during the remaining four years.

Of course our results are subject to the limitations of the model, including the limitation of linearity. Unfortunately, computational solutions to differential games are extremely difficult to obtain for non-linear models (nonlinearities are far more difficult to handle than is the case in
a conventional centralized optimal control problem). The solution algorithms presented in this paper can, however, be applied to other linear or linearized models that are larger (and perhaps more acceptable). Here we have only tried to show that decentralized optimal control solutions can provide another tool for the analysis of macro-economic policy.
APPENDIX

THE ECONOMETRIC MODEL

The equations of the model are listed below. All of the behavioral equations were estimated over the period 1955-I to 1967-IV, using two-stage least squares, in combination with a Hildreth-Lu autoregressive correction. Standard errors are shown in parenthesis beneath each estimated coefficient. Also shown are the $R^2$, the standard error of the regression (SER), the F statistic, the Durbin-Watson (DW), and the value of $\rho$ used in the autoregressive transformation.

**GNP:**

$$\text{GNP} = C + INR + IR + IIN + G$$  
(A.1)

**Disposable Income, YD:**

$$YD = .85\text{GNP} - T_0$$  
(A.2)

**Consumption, C:**

$$C = .4152YD - .2819YD_{-1} + 8.1743W_{-1} -2.3676\Delta P + .7596C_{-1} + 5.2998.$$  
(A.3)

\[ R^2 = .9991, \quad \text{SER} = 1.594, \quad F = 10,280, \quad DW = 1.95, \quad \rho = -.400 \]

**Non-residential Investment, INR:**

$$INR = .1569AYD + .0443AYD_{-3} - 1.3563\Delta R_l_{-1} + .3397\Delta INR_{-1}$$  
(A.4)

\[ R^2 = .718, \quad \text{SER} = .744, \quad F = 29.3, \quad DW = 1.98, \quad \rho = -.335 \]

**Residential Investment, IR:**

$$IR = .0127YD - .550(R_2 + R_3) + .603IR_{-1} + 6.65.$$  
(A.5)

\[ R^2 = .992, \quad \text{SER} = .582, \quad F = 184.9, \quad DW = 1.64, \quad \rho = .700. \]
Inventory Investment, IIN:
\[ IIN = 0.0113YD + 0.4647\Delta_2YD - 0.6002\Delta_2C + 0.4219IIN_{-1} - 2.4615. \]  
\[ \text{(A.6)} \]
\[ R^2 = 0.740, \quad \text{SER} = 2.228, \quad F = 33.5, \quad DW = 2.28, \quad \rho = 0.400. \]

Short-term Interest Rate, R:
\[ R = 0.0071YD + 0.0233\Delta YD - 0.1648\Delta M + 0.4791\Delta P + 0.374R_{-1} - 1.4734. \]  
\[ \text{(A.7)} \]
\[ R^2 = 0.883, \quad \text{SER} = 0.336, \quad F = 68.2, \quad DW = 1.90, \quad \rho = 0.400. \]

Long-term Interest Rate, RL:
\[ RL = 0.0598R + 0.0055\Delta_2YD + 0.8715RL_{-1} + 0.3126. \]  
\[ \text{(A.8)} \]
\[ R^2 = 0.941, \quad \text{SER} = 0.1358, \quad F = 292, \quad DW = 1.97, \quad \rho = 0.250. \]

Price Level, P:
\[ P = 6.281W_{-1} + 0.195(YD_{-1} - YDP_{-1}) - 0.0328IIN_{-2} - 0.0156YD \]
\[ + 0.8040P_{-1} + 14.552. \]  
\[ \text{(A.9)} \]
\[ R^2 = 0.984, \quad \text{SER} = 0.195, \quad F = 710.3, \quad DW = 2.47, \quad \rho = 0.500. \]

Unemployment Rate, UR:
\[ UR = -0.00043\Delta YD - 0.00032\Delta YD_{-1} + 0.0024W_{-1} - 0.00014(YD_{-1} - YDP_{-1}) \]
\[ + 0.8047UR_{-1} + 0.0065. \]  
\[ \text{(A.10)} \]
\[ R^2 = 0.953, \quad \text{SER} = 0.0023, \quad F = 185.5, \quad DW = 1.11, \quad \rho = 0.50. \]

Money Wage Rate, W:
\[ W = 0.0105P_{-3} + 0.0011YD_{-1} + 0.0012\Delta YD - 0.8277UR_{-4} + 0.6269W_{-1} - 0.6850. \]  
\[ \text{(A.11)} \]
\[ R^2 = 0.9992, \quad \text{SER} = 0.0117, \quad F = 11,940, \quad DW = 1.87, \quad \rho = 0.0200. \]

1 \( \Delta_2YD = YD - YD_{-2} = \Delta YD + \Delta YD_{-1} \).

2 The exogenous variable YDP represents potential disposable income. It is based on a potential GNP trend line.
References


FIGURE 1 DISPOSABLE INCOME FOR RUNS 1, 2, AND 3

FIGURE 2 NONRESIDENTIAL INVESTMENT FOR RUNS 1, 2, AND 3
FIGURE 3 RESIDENTIAL INVESTMENT... RUNS 1, 2, AND 3

NOMINAL PATH
RUN 1
RUN 2 OPTIMAL
RUN 3 PATH

FIGURE 4 SHORT-TERM INTEREST RATE... RUNS 1, 2, AND 3

NOMINAL PATH
RUN 1
RUN 2 OPTIMAL
RUN 3 PATH
FIGURE 5  LONG-TERM INTEREST RATE ....RUNS 1, 2, AND 3

FIGURE 6  UNEMPLOYMENT RATE ....RUNS 1, 2, AND 3
FIGURE 9 QUARTERLY CHANGE IN THE MONEY SUPPLY
RUNS 1, 2, AND 3

FIGURE 10 DISPOSABLE INCOME... RUNS 4, 5, AND 6
FIGURE 11 RESIDENTIAL INVESTMENT...RUNS 4, 5, AND 6

NOMINAL PATH
RUN 4
RUN 5 OPTIMAL
RUN 6 PATH

FIGURE 12 SHORT-TERM INTEREST RATE...RUNS 4, 5, AND 6

NOMINAL PATH
RUN 4
RUN 5 OPTIMAL
RUN 6 PATH

...
FIGURE 13 UNEMPLOYMENT RATE: RUNS 4, 5, AND 6

FIGURE 14 PRICE LEVEL: RUNS 4, 5, AND 6
FIGURE 15 GOVERNMENT SPENDING—RUNS 4, 5, AND 6

- NOMINAL PATH
- RUN 4
- RUN 5 OPTIMAL
- RUN 6 PATH

FIGURE 16: QUARTERLY CHANGE IN THE MONEY SUPPLY—RUNS 4, 5, AND 6

- NOMINAL PATH
- RUN 4
- RUN 5 OPTIMAL
- RUN 6 PATH
FIGURE 17 DISPOSABLE INCOME...RUNS 2, 2A AND 2B

FIGURE 18 NONRESIDENTIAL INVESTMENT
RUNS 2, 2A, AND 2B
FIGURE 19 RESIDENTIAL INVESTMENT
RUNS 2, 2A, AND 2B.

FIGURE 20 SHORT-TERM INTEREST RATE
RUNS 2, 2A, AND 2B
FIGURE 21 LONG-TERM INTEREST RATE
RUNS 2, 2A, AND 2B

FIGURE 22 UNEMPLOYMENT RATE... RUNS 2, 2A, AND 2B
FIGURE 25 QUARTERLY CHANGE IN MONEY SUPPLY
RUNS 2, 2A, AND 2B

- NOMINAL PATH
- - - RUN 2
- - - - RUN 2A
- - - - - RUN 2B

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