

CONTINUOUS-TIME PORTFOLIO THEORY
AND THE PRICING OF CONTINGENT CLAIMS

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Continuous-Time Portfolio Theory and the Pricing of Contingent Claims

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I. Introduction

The theory of portfolio selection in continuous-time has as its foundation two assumptions: (1) the capital markets are assumed to be open at all times, and therefore economic agents have the opportunity to trade continuously and (2) the stochastic processes generating the state variables can be described by diffusion processes with continuous sample paths.^{1/}

If these two assumptions are accepted, then the continuous-time mode of analysis has proved fruitful in solving some of the basic problems in portfolio selection and capital market equilibrium theory. Since I have summarized many of these solutions in an earlier paper [5], I will only briefly highlight the main results.

In solving the intertemporal consumption-portfolio selection problem for the individual, the continuous-time approach leads to many of the analytical simplicities of the classic Markowitz-Tobin mean-variance model, but without their objectionable assumptions. In general, the intertemporal nature of the problem induces a "derived" utility function for the investor which is a function of other variables in addition to "end-of-period" wealth. Hence, there are other sources of uncertainty in addition to end-of-period asset price uncertainty. These other uncertainties systematically affect the optimal portfolio demands by investors for assets, and hence lead to different demand functions than would be

derived from either the mean-variance model or from maximizing the expected utility of terminal wealth. However, like the mean-variance model, there are separation or "mutual fund" theorems although, in general, more than two "funds" will be required to span the space of investors' optimal portfolios.

In an analogous fashion to the derivation of the Capital Asset Pricing Model^{2/} from the mean-variance model, the demand functions of individual investors can be aggregated; market-clearing conditions imposed; and a structure of the equilibrium expected returns on assets derived. However, because of the multi-dimensional nature of the uncertainties that affect investors' portfolio allocations, there will be more than one dimension to the measure of a security's risk. Hence, instead of a Security Market Line equation defining this equilibrium structure, there will be a Security Market Plane equation.

Like the mean-variance Capital Asset Pricing Model, the continuous-time model can be used to derive equilibrium asset prices. However, to derive the asset prices by this method requires estimates of the price of risk, the covariance of the asset's cash flows with the market, and the expected cash flows. These numbers are often difficult to estimate. Moreover, as will be shown, it is not always necessary to have these numbers to price an asset.

In a seminal paper, Black and Scholes [1] used the continuous-time analysis to derive a formula for pricing common stock call options.^{3/} Although their derivation uses the same assumptions and analytical tools used in the continuous-time portfolio analysis, the resulting formula expressed in terms of the price of the underlying stock requires as inputs

neither expected returns or cash flows, the price of risk, nor the covariance of the returns with the market. In effect, all these variables are implicit in the stock's price. Moreover, while their pricing formula is consistent with the continuous-time portfolio model's equilibrium price structure, it does not assume equilibrium pricing for its validity.

The essential reason that the Black-Scholes pricing formula requires so little information as inputs is that the call option is a security whose value on a specified future date is uniquely determined by the price of another security (the stock). As such, a call option is an example of a contingent claim. While call options are very specialized financial instruments, Black and Scholes and others^{4/} recognized that the same analysis could be applied to the pricing of corporate liabilities generally where such liabilities were viewed as claims whose values were contingent on the value of the firm. Moreover, whenever an asset's return structure is such that it can be described as a contingent claim, the same technique is applicable.

In section II, I derive a general formula for a price of an asset whose value under specified conditions is a known function of the value of another asset. In section III, the Modigliani-Miller Theorem [9] that the value of the firm is invariant to its capital structure is proved even when there is a positive probability of bankruptcy. In section IV, the formula is applied in a nonstandard fashion to derive the value of capital assets and human capital in an economy with a single source of uncertainty. This example is designed to illustrate the potential of this approach to asset pricing in a simple economic structure and is not meant to be descriptive of a "real world" economy.

II. A General Derivation of a Contingent Claim Price

To develop the contingent-claim pricing model, I make the following assumptions:

(A.1) "Frictionless" Markets:

There are no transactions costs or taxes. Trading takes place continuously in time. Borrowing and shortselling are allowed without restriction. The borrowing rate equals the lending rate.

(A.2) Riskless Asset:

There is a riskless asset whose rate of return per unit time is known and constant over time. Denote this return rate by r .

(A.3) Asset #1:

There is a risky asset whose value at any point in time is denoted by $V(t)$. The dynamics of the stochastic process generating $V(\cdot)$ over time is assumed to be describable by a diffusion process with a formal stochastic differential equation representation of:

$$dV = [\alpha V - D_1(V,t)]dt + \sigma V dZ$$

where α = instantaneous expected rate of return on the asset per unit time;

σ^2 = instantaneous variance per unit time of the rate of return

$D_1(V,t)$ = instantaneous payout to the owners of the asset per unit time

$dZ \equiv$ standard Wiener process

α can be generated by a stochastic process of a quite general type. σ^2 is restricted to be at most a function of V and t .

(A.4) Asset #2:

There is a second risky asset whose value at any date t is denoted by $W(t)$ with the following properties:

For $0 \leq t < T$, its owners will receive an instantaneous payout per unit time, $D_2(V,t)$.

For any $t(0 \leq t < T)$, if $V(t) = \bar{V}(t)$, then the value of the second asset is given by: $W(t) = f[\bar{V}(t),t]$, where f is a known function

For any $t(0 \leq t < T)$, if $V(t) = \underline{V}(t)$, then the value of the second asset is given by: $[V(t) < \bar{V}(t)]$

$$W(t) = g[\underline{V}(t),t], \text{ where } g \text{ is a known function}$$

For $t = T$, the value of the second asset is given by: $W(T) = h[V(T)]$.

Asset #2 will be called a contingent claim, contingent on the value of Asset #1.

(A.5) Investor Preferences and Expectations

It is assumed that investors prefer more to less. It is assumed that investors agree upon σ^2 , but it is not assumed that they necessarily agree on α .

(A.6) Other

There can be as many or as few other assets or securities as one likes. Market prices need not be equilibrium prices. The constant interest rate and most of the "frictionless" market assumptions are not essential to the development of the model but are chosen for expositional convenience. The critical assumptions are continuous-trading opportunities and the dynamics description for Asset #1.

The derivation of the pricing formula takes place in two steps: First, we assume that the value of asset #2 can be written as a twice-continuously differentiable function of the price of asset #1 and time, and derive what the function must be. Second, we show that if such a pricing function can be found, then asset #2 must be priced according to that function.

Suppose $W(t) = F[V(t), t]$ for $0 \leq t \leq T$ and for $\underline{V}(t) \leq V(t) \leq \bar{V}(t)$.

If F is sufficiently smooth, the dynamics for $W(t)$ can be written in stochastic differential equation form as

$$(1) \quad dW = [\alpha_F W - D_2(V, t)] dt + \sigma_F W dZ_F$$

where all the symbols have the same definition as in (A.3) except they apply to asset #2.

Consider a portfolio strategy that invests X_1 dollars in asset #1, X_2 dollars in asset #2, and X_3 dollars in the riskless asset. However, unlike a standard portfolio, require that

$$(2) \quad X_1 + X_2 + X_3 = 0$$

I.e., the net investment in the portfolio is zero. This is possible by using the proceeds of short sales and borrowing to finance purchases. If the portfolio is continuously revised, then the dynamics for the dollar return on the portfolio, dY , can be written in stochastic differential equation as

$$(3) \quad \begin{aligned} dY &= X_1 \left[\frac{dV + D_1 dt}{V} \right] + X_2 \left[\frac{dW + D_2 dt}{W} \right] + X_3 r dt \\ &= X_1 \left[\frac{dV + D_1 dt}{V} - r dt \right] + X_2 \left[\frac{dW + D_2 dt}{W} - r dt \right] \end{aligned}$$

where, in the second line, X_3 is eliminated by substitution from the constraint (2).

By using Itô's Lemma,^{5/} an alternative representation to (1) for the dynamics of W can be written as

$$(4) \quad dW = \left[\frac{1}{2} \sigma^2 V^2 F_{11} + [\alpha V - D_1] F_1 + F_2 \right] dt + F_1 \sigma V dZ$$

where subscripts on F denote partial derivatives with respect to its two explicit arguments, V and t.

Matching terms in the two representations (1) and (4), we have that

$$(5a) \quad \alpha_F W \equiv D_2(V, t) + \frac{1}{2} \sigma^2 V^2 F_{11} + [\alpha V - D_1] F_1 + F_2$$

$$(5b) \quad \sigma_F W \equiv F_1 \sigma V$$

$$(5c) \quad dZ_F \equiv dZ.$$

Substituting for the dynamics of V from (A.3) and for the dynamics of W given in (1), and using (5c), we can rewrite (3) as

$$(6) \quad dY = \{ [X_1(\alpha - r)] + [X_2(\alpha_F - r)] \} dt + [X_1 \sigma + X_2 \sigma_F] dZ.$$

Let us further restrict our choice of portfolios such that

$$(7) \quad X_1 \sigma + X_2 \sigma_F = 0$$

at each point in time. If the set of investment choices (X_1, X_2) that satisfy (7) are denoted by (X_1^*, X_2^*) and if the portfolio returns using such investment choices are denoted by dY^* , then from (6) and (7), we have that

$$(8) \quad \begin{aligned} dY^* &= [X_1^*(\alpha - r) + X_2^*(\alpha_F - r)] dt \\ &= X_2^* \left[\frac{-\sigma_F(\alpha - r)}{\sigma} + (\alpha_F - r) \right] dt. \end{aligned}$$

By inspection of (8), the dollar return on such portfolios, dY^* , is not uncertain, and hence is riskless. But, these portfolios require no net investment by construction. Therefore, to avoid arbitrage, $dY^* = 0$. However, X_2^* can be chosen arbitrarily. Hence, to avoid arbitrage opportunities, it must be that the assets are priced such that

$$(9) \quad \frac{\alpha - r}{\sigma} = \frac{\alpha_F - r}{\sigma_F}$$

Substituting from (5a) and (5b) and collecting terms, we can rewrite (9) as

$$(10) \quad 0 = \frac{1}{2}\sigma^2 V^2 F_{11} + [rV - D_1]F_1 - rF + F_2 + D_2$$

Equation (10) is a linear partial differential equation of the parabolic type that must be satisfied by the price of asset #2 (i.e., the contingent claim) to avoid arbitrage opportunities. Inspection of (10) shows that in addition to V and t , F will depend on σ^2 and r . However, F does not depend on the expected return on asset #1, α , and it does not depend on the characteristics of other assets available in the economy. Moreover, investors' preferences do not enter the equation either. Finally, while the absence of arbitrage opportunities is a necessary condition for an equilibrium, it is not sufficient. Therefore, it is not required that V and r be equilibrium prices for (10) to be valid.

To solve (10), boundary conditions must be specified. From (A.4), we have that

$$(11a) \quad F[\bar{V}(t), t] = f[\bar{V}(t), t]$$

$$(11b) \quad F[\underline{V}(t), t] = g[\underline{V}(t), t]$$

$$(11c) \quad F[V, T] = h[V].$$

While the function f , g , and h are required to solve for F , they are generally deducible from the terms of the specific contingent claim being priced. For example, the original case examined by Black and Scholes is a common stock call option with an exercise price of E dollars and an expiration date of T . If V is the value of the underlying stock, then the boundary conditions can be written as

$$(12a) \quad F/V \leq 1 \quad \text{as} \quad V \rightarrow \infty$$

$$(12b) \quad F[0,t] = 0$$

$$(12c) \quad F[V,T] = \text{Max} [0, V-E]$$

where (12a) is a regularity condition which replaces the usual boundary condition when $\bar{V}(t) = \infty$. Both (12a) and (12b) follow from limited liability and from the easy-to-prove condition that the underlying stock is always more valuable than the option. (12c) follows from the terms of the call option which establish the exact price relationship between the stock and option on the expiration date.

Hence, (10) together with (11a) - (11c) provide the general equation for pricing contingent claims. Moreover, if the contingent claim is priced according to (10) and (11), then it follows that there is no opportunity for intertemporal arbitrage. I.e., the relative prices (W, V, r) are intertemporally consistent.

Suppose there exists a twice-continuously differentiable solution to (10) and (11). Since the derivation of (10) depends on the assumption that the pricing function satisfies this condition, it may be possible that some other solution exists which does not satisfy this differentiability condition. Indeed, in discussing the Black-Scholes solution to the call

option case, Smith^{6/} points out that there are an infinite number of solutions to equation (10) and (12) which have discontinuous derivatives at only one interior point although the Black-Scholes solution is the only smooth solution. He goes on to state that there is no obvious economic justification for the smoothness assumption.

The following is a direct proof that if a twice-continuously differential solution to (10) and (11) exists, then it must be the pricing function.

Let F be the formal twice-continuously differentiable solution to equation (10) with boundary conditions (11). Consider the continuous-time portfolio strategy where the investor allocates the fraction $w(t)$ of his portfolio to asset #1 and $[1 - w(t)]$ to the riskless asset. Moreover, let the investor make net "withdrawals" per unit time (for example, for consumption) of $C(t)$. If $C(t)$ and $w(t)$ are right-continuous functions and $P(t)$ denotes the value of the investor's portfolio, then I have shown elsewhere^{7/} that the portfolio dynamics will satisfy the stochastic differential equation

$$(13) \quad dP = \{[w(\alpha - r) + r]P - C\}dt + w\sigma P dZ.$$

Suppose we pick the particular portfolio strategy with

$$(14) \quad w(t) = F_1[V, t]V(t)/P(t)$$

where F_1 is the partial derivative of F with respect to V , and the "consumption" strategy,

$$(15) \quad C(t) = D_2(V, t).$$

By construction, F_1 is continuously-differentiable, and hence, is a right-continuous function. Substituting from (14) and (15) into (13), we have that

$$(16) \quad dP = F_1 dV + \{F_1(D_1 - rV) + rP - D_2\}dt$$

where dV is given in (A.3).

Since F is twice-continuously differentiable, we can use Ito's Lemma to express the stochastic process for F as

$$(17) \quad dF = \left[\frac{1}{2} \sigma^2 V^2 F_{11} + (\alpha V - D_1) F_1 + F_2 \right] dt + F_1 \sigma V dZ.$$

But F satisfies equation (10). Hence, we can rewrite (17) as

$$(18) \quad dF = F_1 dV + \{F_1(D_1 - rV) + rF - D_2\}dt.$$

Let $Q(t) \equiv P(t) - F[V(t), t]$. Then, from (16) and (18), we have that

$$(19) \quad \begin{aligned} dQ &= dP - dF \\ &= r(P - F)dt \\ &= rQdt. \end{aligned}$$

But, (19) is a non-stochastic differential equation with solution

$$(20) \quad Q(t) = Q(0)e^{rt}$$

for any time t and where $Q(0) \equiv P(0) - F[V(0), 0]$. Suppose the initial amount invested in the portfolio, $P(0)$, is chosen equal to $F[V(0), 0]$. Then from (20) we have that

$$(21) \quad P(t) = F[V(t), t].$$

By construction, the value of asset #2, $W(t)$, will equal F at the boundaries $\underline{V}(t)$ and $\bar{V}(t)$ and at the termination date T . Hence, from (21), the constructed portfolio's value, $P(t)$, will equal $W(t)$ at the boundaries. Moreover, the interim "payments" or withdrawals available to the portfolio strategy, $D_2[V(t),t]$, are identical to the interim payments made to asset #2.

Therefore, if $W(t) > P(t)$, then the investor could short-sell asset #2; proceed with the prescribed portfolio strategy including all interim payments; and be guaranteed a positive return on zero investment. I.e., there would be an arbitrage opportunity. If $W(t) < P(t)$, then the investor could essentially "short-sell" the prescribed portfolio strategy; use the proceeds to buy asset #2; and again be guaranteed a positive return on zero investment. If institutional restrictions prohibit arbitrage, then a similar argument could be developed on the principle that no security should be priced so as to "dominate" another security.^{8/} Hence, $W(t)$ must equal $F[V(t),t]$.

While this method of proof may appear very close to the original derivation, it was not assumed that asset #2 had a smooth pricing function. Rather it was proved that if a smooth solution to (10) and (11) exists, then this solution must be the pricing function.

II. On the Modigliani-Miller Theorem with Bankruptcy

In an earlier paper [10, p. 460], I proved that in the absence of bankruptcy costs and corporate taxes, the Modigliani-Miller theorem [9] obtains even in the presence of bankruptcy. In a comment on this earlier paper, Long [11] has asserted that my method of proof was "logically incoherent." Rather than debate over the original proof's validity, the method of derivation used in the previous section provides an immediate alternative proof.

Let there be a firm with two corporate liabilities: (1) a single homogeneous debt issue and (2) equity. The debt issue is promised a continuous coupon payment per unit time, C , which continues until either the maturity date of the bond, T , or until the total assets of the firm reach zero. The firm is prohibited by the debt indenture from issuing additional debt or paying dividends. At the maturity date, there is a promised principal payment of B to the debtholders. In the event the payment is not made, the firm is defaulted to the debtholders, and the equityholders receive nothing. If $Q(t)$ denotes the value of the firm's equity and $D(t)$ the value of the firm's debt, then the value of the (levered) firm, $V_L(t)$, is identically equal to $Q(t) + D(t)$. Moreover, in the event that the total assets of the firm reach zero, $V_L(t) = Q(t) = D(t) = 0$ by limited liability. Also, by limited liability, $D(t)/V_L(t) \leq 1$.

Consider a second firm with identical initial assets and an identical investment policy to the levered firm. However, the second firm is all-equity financed with total value equal to $V(t)$. To ensure the identical investment policy including scale, it follows from the well known accounting identity that the net payout policy of the second firm must be the same as

for the first firm. Hence, let the second firm have a dividend policy that pays dividends of C per unit time until either date T or until the value of its total assets reach zero (i.e. $V = 0$). Let the dynamics of the firm's value be as posited in (A.3) where $D_1(V,t) = C$ for $V > 0$ and $D_1 = 0$ for $V = 0$.

Let $F[V,t]$ be the formal twice-continuously differentiable solution to equation (10) subject to the boundary conditions: $F[0,t] = 0$; $F[V,t]/V \leq 1$; and $F[V(T),T] = \text{Min}[V(T),B]$. Consider the dynamic portfolio strategy of investing in the all-equity firm and the riskless asset according to the "rules" (14) and (15) of section II where $C(t)$ is taken equal to C . If the total initial amount invested in the portfolio, $P(0)$, is equal to $F[V(0),0]$, then from (20), $P(t) = F[V(t),t]$.

Because both the levered firm and the all-equity firm have identical investment policies including scale, it follows that $V(t) = 0$ if and only if $V_L(t) = 0$. And it also follows that on the maturity date T , $V_L(T) = V(T)$.

By the indenture conditions on the levered firm's debt, $D(T) = \text{Min}[V_L(T),B]$. But since $V(T) = V_L(T)$ and $P(T) = F[V(T),T]$, it follows that $P(T) = D(T)$. Moreover, since $V_L(t) = 0$ if and only if $V(t) = 0$, it follows that $P(t) = F[0,t] = D(t) = 0$ in that event.

Thus, by following the prescribed portfolio strategy, one would receive interim payments exactly equal to those on the debt of the levered firm. Moreover, on a specified future date, T , the value of the portfolio will equal the value of the debt. Hence, to avoid arbitrage or dominance, $P(t) = D(t)$.

The proof for equity follows on similar lines. Let $f[V,t]$ be the formal solution to equation (10) subject to the boundary conditions: $f[0,t] = 0$, $f[V,t]/V \leq 1$; and $f[V(T),T] = \text{Max}[0,V(T) - B]$. Consider the dynamic portfolio strategy of investing in the all-equity firm and the riskless

asset according to the "rules" (14) and (15) of section II where $C(t)$ is taken equal to zero. If the total initial amount invested in this portfolio, $p(0)$, is equal to $f[V(0),0]$, then from (20), $p(t) = f[V(t),t]$.

As with debt, if $V(t) = 0$, then $p(t) = Q(t) = 0$, and at the maturity date, $p(T) = \text{Max}[0, V(T) - B] = Q(T)$.

Thus, by following this prescribed portfolio strategy, one would receive the same interim payments as those on the equity of the levered firm. On the maturity date, the value of the portfolio will equal the value of the levered firm's equity. Therefore, to avoid arbitrage or dominance, $p(t) = Q(t)$.

If one were to combine both portfolio strategies, then the resulting interim payments would be C per unit time with a value at the maturity date of $V(T)$. I.e., both strategies together are the same as holding the equity of the unlevered firm. Hence, $f[V(t),t] + F[V(t),t] = V(t)$. But it was shown that $f[V(t),t] + F[V(t),t] = Q(t) + D(t) \equiv V_L(t)$. Therefore, $V_L(t) = V(t)$, and the proof is completed.

IV. On the Pricing of Capital Assets and Human Capital: A Simple Example

Consider a single commodity world with one source of economy-wide risk.^{9/} Let V denote the market value of all assets in the economy (including human capital). Because there is only one source of uncertainty, all unanticipated changes in asset values must be perfectly linked. Hence, the unanticipated changes in V can serve as an instrumental variable for this uncertainty. Let α denote the expected rate of return for the economy as a whole (including both income flows and asset value changes) and let σ be its instantaneous standard deviation. I.e.,

$$(22) \quad dV = (\alpha V - C)dt + \sigma V dZ$$

describes the dynamics of the economy with aggregate consumption equal to net withdrawals from the system.

Let there be n business firms (which could also be treated as factors of production) with the i^{th} such firm having net payouts per unit time of $Q_i(V)$; ^{10/} a termination date T_i and a "salvage" value at that date of $S_i(V)$ where $Q_i(0) = S_i(0) = 0$. Let $F^i(V, t)$ denote the value of firm i . If there is "free disposal," then $F^i \geq 0$. Substituting into (10) and (11), we have that F^i must satisfy

$$(23) \quad \frac{1}{2} \sigma^2 V^2 F_{11}^i + [rV - C(V)]F_1^i - rF^i + F_2^i + Q_i(V) = 0$$

subject to the boundary conditions:

$$(23a) \quad F^i/V \leq 1 \quad \text{as} \quad V \rightarrow \infty$$

$$(23b) \quad F^i[0, t] = 0$$

$$(23c) \quad F^i[V, T_i] = S_i(V)$$

In the special case when σ^2 is constant and aggregate consumption is a constant fraction of national wealth (i.e., $C(V) = cV$), then the solution to (23) can be written as

$$(24) \quad F^i(V, t) = \int_0^{\tau_i} e^{-rs} ds \int_0^{\infty} Q_i(Vy) p(y, s; r, \sigma^2, c) dy \\ + e^{-r\tau_i} \int_0^{\infty} S_i(Vy) p(y, \tau_i; r, \sigma^2, c) dy$$

where $\tau_i \equiv T_i - t$ and

$$p(x, s; r, \sigma^2, c) \equiv \frac{1}{x\sqrt{2\pi\sigma^2 s}} \exp \left[\frac{1}{2\sigma^2 s} (\log(x) - (r - c - \frac{1}{2}\sigma^2)s)^2 \right]$$

While (24) may seem formidable, depending upon the functions Q_i and S_i , it may have a closed-form solution. Moreover, even when such closed-form solutions cannot be found, it lends itself to numerical integration rather nicely.

To illustrate more specifically, consider the following example for the evaluation of an individual's human capital.

Let τ be the length of time until death which is assumed to be an exponentially-distributed random variable with parameter λ which corresponds to the event of death being Poisson-distributed.

Let the individual's wages per unit time, $Q(V)$, be given by

$$(25) \quad Q(V) = aV - bV^\gamma, \quad \text{for } V \geq \bar{V} \\ = 0 \quad \text{for } V < \bar{V}$$

where $0 < a < 1$; $b > 0$; $0 < \gamma < 1$; and for $V \leq \bar{V} \equiv (b/a)^{\frac{1}{1-\gamma}}$, he is unemployed.^{12/}
 The "salvage" value at death is zero. If σ^2 is constant, $C(V) = cV$, and the event of his death is "diversifiable" in an economy-wide sense,^{13/} then from (24), the value of his human capital can be written as

$$(26) \quad F(V) = E \left\{ \int_0^{\tau} e^{-rs} ds \int_0^{\infty} Q(Vy)p(y,s;r,\sigma^2,c)dy \right\}$$

where "E" is the expectation operator over the random variable τ .

Using the assumed properties for τ , we can rewrite (26) as

$$(26') \quad F(V) = \int_0^{\infty} e^{-(r+\lambda)s} ds \int_0^{\infty} Q(Vy)p(y,s;r,\sigma^2,c)dy$$

By substitution, it is easy to show that $F(V)$ is given by

$$(27) \quad \begin{aligned} F(V) &= f_u(V) && \text{for } V \geq \bar{V} \equiv \left(\frac{b}{a}\right)^{\frac{1}{1-\gamma}} \\ &= f_l(V) && \text{for } 0 \leq V \leq \bar{V} \end{aligned}$$

where f_u is the solution to

$$(28) \quad \frac{1}{2}\sigma^2 V^2 f_u'' + (\delta V)f_u' - \rho f + aV - bV^\gamma = 0$$

subject to the boundary conditions:

$$(28a) \quad \frac{f_u}{V} \leq 1 \quad \text{as } V \rightarrow \infty$$

$$(28b) \quad f_u(\bar{V}) = f_l(\bar{V})$$

$$(28c) \quad f_u'(\bar{V}) = f_l'(\bar{V})$$

where primes denote derivatives and $\delta \equiv r - c$, a constant, and $\rho \equiv r + \lambda$, a constant, and where f_l is the solution to

$$(29) \quad \frac{1}{2} \sigma^2 V^2 f_{\ell}'' + \delta V f_{\ell}' - \rho f_{\ell} = 0$$

subject to the boundary conditions:

$$(29a) \quad f_{\ell}(0) = 0$$

$$(29b) \quad f_{\ell}(\bar{V}) = f_u(\bar{V})$$

$$(29c) \quad f_{\ell}'(\bar{V}) + f_u'(\bar{V})$$

In essence, f_u is the value of his human capital while employed and f_{ℓ} is its value when he is unemployed for different levels of the economy. The solution of the coupled set of differential equations (28) and (29) is straightforward, and the solutions are:

$$(30a) \quad \text{For } V \geq \bar{V} \equiv \left(\frac{b}{a}\right)^{\frac{1}{1-\gamma}}$$

$$F(V) = f_u(V) = \frac{1}{\left(\frac{b}{a}\right)^{\frac{1}{1-\gamma}}} \left[\frac{a}{(c+\lambda)} [1-k_1] + \frac{(k_1-\gamma)\eta a}{b} \right] \left(\frac{V}{\bar{V}}\right)^{k_2} + \frac{aV}{(c+\lambda)} - \eta V$$

and for $0 \leq V \leq \bar{V}$,

$$(30b) \quad F(V) = f_{\ell}(V) = \frac{1}{\left(\frac{b}{a}\right)^{\frac{1}{1-\gamma}}} \left[\frac{a}{c+\lambda} [1-k_2] + \frac{(k_2-\gamma)\eta a}{b} \right] \left(\frac{V}{\bar{V}}\right)^{k_1}$$

where

$$\eta \equiv \frac{b}{[(1-\gamma) \left[r + \frac{\gamma \sigma^2}{2} \right] + \lambda + c\gamma]} > 0$$

$$k_1 \equiv \frac{-(\delta - \frac{1}{2}\sigma^2) + \sqrt{(\delta + \frac{\sigma^2}{2})^2 + 2\sigma^2(c+\lambda)}}{\sigma^2} > 1$$

$$k_2 \equiv \frac{-(\delta - \frac{1}{2} \sigma^2) - \sqrt{(\delta + \frac{\sigma^2}{2}) + 2\sigma^2(c + \lambda)}}{\sigma^2} < 0$$

The examples in this section are simple illustrations of this technique for asset evaluation, and they were designed more to show the potential breadth of the technique than for their substantive content. The technique can be expanded in a straightforward fashion to include multiple sources of uncertainty. Research is in progress to develop efficient numerical methods for solving the fundamental partial differential equation when a closed form solution cannot be found.^{13/}

FOOTNOTES

*Professor of Finance. The paper was presented in seminars at Yale and Brown Universities in April 1976 and at the EIASM Workshop in Management Science, Bergamo, Italy in October 1976. I thank the participants for their helpful comments. Aid from the National Science Foundation is gratefully acknowledged.

1. For references to the mathematics of diffusion processes and their applications in economics, see the bibliographies in Merton [3] and [4].
2. For an excellent survey article on the Capital Asset Pricing Model, see Jensen [2].
3. A call option gives its owner the right to buy a specified number of shares of a given stock at a specified price (the "exercise price") on or before a specified date (the "expiration date").
4. The literature based on the Black-Scholes analysis has expanded so rapidly that rather than attempt to list individual published articles and works-in-progress, I refer the reader to an up to date survey article by Smith [8].
5. See Merton [3] for a discussion of Ito's Lemma and stochastic differential equations.
6. See Smith [8, p. 23, footnote 21].
7. See Merton [3, p. 379].
8. See Merton [4, p. 143] and Smith [8, p. 7] for a discussion of "dominance" in this context.

9. While the formal analysis assumes a single uncertainty source, essentially the same formulas will apply if there are other sources of uncertainty, but these sources can be "diversified" away for the economy at large.
10. While, for the purpose of this analysis, the $Q_1(V)$ are exogenous, they could be deduced from value maximizing behavior by firms. The Q_1 can be negative in which case they are interpreted as net additions to the firm's capital from outside sources.
11. For the particular wage schedule chosen in (25), his unemployment could be "voluntary" in the sense that "net" wages below this level are negative. As an example, negative net wages from working can occur if unemployment compensation or welfare payments exceed the wage rate.
12. An individual death is generally viewed as a source of individual uncertainty that has no impact on economy-wide risk. Hence, as shown in Merton [6] for such diversifiable risks, the correct formula can be determined by first solving the model conditional on a given death date and then taking the expected value of this conditional formula over the distribution of possible death dates.
13. Cf. Parkinson [7].

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