A Simple Model of Risk and Return

on Long-Lived Tangible Assets

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ABSTRACT

This essay considers stationary long-run equilibria. Dividend payments of a set of firms are assumed to follow linear, stationary stochastic processes. Assuming rational expectations, a specialization of Ross's Arbitrage Pricing Theory is used to obtain a simple securities market valuation formula. If real capital stocks are constant over time and a natural condition for equilibrium in tangible asset markets is satisfied, relations among average accounting rates of return are determined by a new measure of riskiness, based entirely on accounting data. Various implications of the model are discussed.
A Simple Model of Risk and Return on Long-Lived Tangible Assets

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In a variety of microeconomic applications, it is necessary to adjust (observed or expected) accounting rates of return or cash flows for riskiness. In many of these applications, securities market data cannot be usefully employed. Seventeen operating telephone companies are wholly-owned subsidiaries of AT&T, for instance. As these entities differ in a variety of ways, data on AT&T common stock cannot be used to determine whether their individual rates of return exceed or fall short of their costs of capital. Similarly, in antitrust proceedings involving divisions of diversified firms, measures of risk-adjusted profitability cannot be based with any confidence on the market's valuation of the securities of the firms involved. Industrial organization studies of the determinants of industry profitability often use Internal Revenue Service data, which do not permit the construction of cost of capital or risk premium estimates based on securities market information. Finally, capital budgeting decisions involving long-lived projects of substantial size may require the development of project-specific risk measures.

In all these cases, and in others, there is need for a measure of riskiness based on the sort of data furnished by accounting systems, not on securities market information. This essay develops a simple measure of this type. Until the model presented here has been appropriately generalized and systematically confronted with data, however, this measure should be taken as suggestive, not operational.

A number of studies have been made of the appropriate valuation of long-lived, risky tangible assets. In most of them, the appropriate risk adjustment is found to depend on the stochastic relation between the asset's ac-
counting rate of return and the securities market rate of return on a diversified portfolio. It is not clear how one justifies assuming that relations between returns in financial markets and yields of tangible assets are stable over time. It seems more plausible to postulate stable stochastic relations among returns on tangible assets employed in different lines of business. That approach is taken here.

It has also been taken in two recent studies. William Brock (1978) constructs an elegant dynamic general equilibrium model, but assumes throughout that yields on tangible assets are uncorrelated over time. R.C. Stapleton and M.G. Subrahmanyam (1978) consider market valuation of a set of risky cash flows with a general intertemporal correlation structure, but they assume normality and deal only with exponential utility functions. Neither paper obtains simple valuation formulae (or risk adjustments) that can be readily related to observable quantities. Such a formula is obtained here under rather different assumptions.

This essay also relates to another line of research. It seems clear that the riskiness of the securities market return on investment in a particular common stock, for instance, ought to depend on the stochastic properties of the earnings stream generated by the underlying tangible assets. An understanding of this linkage can be expected to shed important light on the workings of speculative markets and to have implications for business decision-making. A number of empirical studies of this relation have been made, but they have generally proceeded without a theoretical framework, and they have obtained mixed results. The theoretical analysis below has implications about the nature of the tangible asset -- securities market linkage.

Section I presents notation and key assumptions. In Section II, a specialization of Stephen Ross's (1976a, 1976b) Arbitrage Pricing Theory (APT)
is used to obtain security market valuations. The relation between this
theory and the more commonly employed Capital Asset Pricing Model (CAPM)
is discussed. Section III derives and discusses a new measure of riski-
ness that determines equilibrium relations among average accounting rates
of return in this model. The relations between this new measure and two
"beta coefficient" measures of riskiness are indicated.

I. Assumptions and Notation

In all that follows, we consider a one-good economy in which there
are N firms, where N is a large number, with constant real capital stocks. This
set of firms will be referred to from time to time as "the market," even
though there may be other enterprises in the economy. With only a single
good, the real capital stocks or book values of the firms can be defined un-
ambiguously. We thus assume away all problems of interpreting accounting
data.

Default risk is assumed away in all that follows. Initially,
the economy is assumed to be free of taxes, so that at any instant the mar-
ket value of any firm must be independent of its financial structure. The
analysis can thus be considerably simplified by assuming initially that all
firms are entirely equity-financed. Coupled with the no-growth assumption,
this implies that all earnings are paid as dividends. Shares of all firms
are assumed traded in a frictionless, competitive securities market. A dis-
crete time framework is employed, along with the following notation, where
i is understood to run from 1 to N:

\[ B_i^t = \text{book value (real capital stock) of firm } i, \]
\[ D_i^t = \text{dividends paid at the end of period } t \text{ by firm } i, \]
\[ V_i^t = \text{security market value of firm } i \text{ at the end of period } t, \]

after the payment of dividends,
qt = Vt^i / Bt^i = market/book value ratio for firm i at the end of period t,

r = one-period riskless rate of return,

δ = 1/(1+r) = one-period riskless discount factor,

rt = Dt^i / Bt^i = accounting rate of return of firm i in period t,

Rt = (Vt^i + Dt^i - Vt_{t-1}^i) / Vt_{t-1}^i = security market rate of return on firm i's stock during period t.

Let BM^i, DM^i, and VM_t be the totals of the corresponding firm-specific quantities. Then the market accounting rate of return, rt, and the security market rate of return on the market portfolio, R^M_t, are given by the obvious modifications of the last two definitions above.

This model is designed to depict a long-run, stationary, stochastic equilibrium in tangible asset markets. A natural and convenient assumption about returns in such a world is that the Dt^i (and thus D^M_t) follow stationary stochastic processes. That is, these cash flows have constant means and variances over time, and the covariance of Dt^i and D^j_{t+k} depends only on i, j, and k, not on t. It is convenient notationally to impose this assumption here by postulating constant B^i and stationary stochastic processes for the rt^i (and thus for r^M_t), though the assumption that real capital stocks are constant over time in equilibrium is not required until Section III.

Once it is postulated that r^M_t follows a stationary stochastic process, Herman Wold's (1954) decomposition theorem implies that little generality is lost if it is further assumed that r^M_t can be written as a linear process:

\[ r^M_t = r^M + \sum_{t=0}^{\infty} \alpha^M_t \epsilon^M_{t-t}, \]

where r^M and the α^M_t are constants, and the ε^M_{t-t} are independent, identically distributed random variables with mean zero. It is assumed that ε^M_t is observed
an instant before the D_{t} are paid to stockholders.

If one thinks of the \( \epsilon_{t} \) as shocks affecting the level of aggregate economic activity, the effects of which may persist over time, it is natural to assume that the individual \( r_{t}^{i} \) are affected by these same shocks. In addition, individual firm rates of return are affected by firm-specific random events, the effects of which may persist over time. We thus assume that the \( r_{t}^{i} \) are generated by stationary stochastic processes of the following form:

\[
rt^{i} = r^{i} + \sum_{\tau=0}^{\infty} \alpha_{\tau} \varepsilon_{t-\tau}^{i} + \sum_{\tau=0}^{\infty} \gamma_{\tau} \xi_{t-\tau}^{i}, \quad i = 1, \ldots, N,
\]

where \( r^{i} \), the \( \alpha_{\tau}^{i} \), and the \( \gamma_{\tau}^{i} \) are constants, the \( \varepsilon_{t-\tau}^{i} \) are as above, and the \( \xi_{t-\tau}^{i} \) are firm-specific independent, identically distributed random variables with mean zero, uncorrelated with the \( \varepsilon_{t-\tau}^{i} \).

In what follows, considerable notational economy is obtained by adopting the following conventions:

\[
\gamma_{T}^{M} = \xi_{T}^{M} = 0,
\]

for all \( T \). Thus, by letting the superscript \( i \) take on the value \( M \), equation (2) and others that follow can be made to serve for both the \( N \) firms and their aggregate.

It is assumed throughout that investor expectations are rational. Investors know the parameters of (2), and expectations about the future are derived from optimal forecasts based on those models. (Since book values are constant, forecasts of accounting rates of return translate immediately into forecasts of dividend payments.) It then follows that for \( k > 0 \),
Equation (4) provides for the systematic updating of expectations to reflect new information.

II. Securities Market Valuation

The version of Ross's (1976a, 1976b) APT used here assumes initially that the $R_i^t$ can be written as

$$R_i^t = E_{t-1}[R_i^t] + k_i u_t + v_i^t,$$

where the $k_i^t$ are non-stochastic, the $u_t$ are serially independent random variables with zero means, $N$ is large, and the $v_i^t$ are serially and contemporaneously independent and satisfy certain weak distributional assumptions. If arbitrage limits the expected return on portfolios with no systematic risk to $r$, Ross shows that the following relation holds:

$$E_{t-1}[R_i^t] = r + \lambda_i k_i^t,$$

where $\lambda_i$ is non-stochastic as of the start of period $t$. This constant functions like the market price of risk in the CAPM. It is easy to show (see Ross [1976a]) that if $k_i^M$ is positive and investors are risk-averse, then $\lambda_i$ must also be positive, since investors must be compensated for bearing the systematic or undiversifiable risk of the market portfolio.

In order to use this theory, one must provide for the determination of the $\lambda_i$. In stationary equilibrium, it would seem that these parameters must either be constant over time or follow a stationary stochastic process, with $\lambda_i$ determined by the $\xi_{t-\tau}$ and $\xi_{t-\tau}^i$ for $\tau > 1$. In order to specify such a stochastic process, one would have to model investor behavior explicitly. This essay
adopts the simpler course and assumes that $\lambda_t = \lambda$ for all $t$. This is a strong assumption, but it seems plausible in the context of stationary equilibrium, it enormously simplifies the analysis, and it may yield useful approximations if changes in $\lambda_t$ are small.\textsuperscript{12}

Using the definition of the $R_t^i$ given in Section I, the equations just above can be written in more useable form. Under the assumptions of the APT, which are adopted here, if

\begin{equation}
V_t^i + D_t^i = E_{t-1}[V_t^i + D_t^i] + K_t^i V_t + \zeta_t^i,
\end{equation}

where $K_t^i$ is non-stochastic, and $V_t$ and $\zeta_t^i$ are random variables as above, then,

\begin{equation}
V_{t-1}^i = \delta[E_{t-1}[V_t^i + D_t^i] - \lambda K_t^i],
\end{equation}

Equation (6) resembles the valuation formula produced by the CAPM; only the systematic risk, which affects all firms, is reflected in stock prices. (This resemblance is discussed further below.)

In the Appendix, it is shown that (5) and (6), together with the other assumptions made above, imply the valuation formula

\begin{equation}
q_t^i = V_t^i/B_t^i = [r_t^i/r] + \sum_{T=0}^{\infty} \delta^{T} \alpha_{t}^i + \sum_{k=0}^{\infty} \delta^{k-T} \alpha_{k}^i + \sum_{k=0}^{\infty} \delta^{k-T} \beta_{k}^i
\end{equation}

\begin{equation}
- \frac{\lambda}{r} \sum_{T=0}^{\infty} \delta^{T} \alpha_{t}^i,
\end{equation}

\begin{equation}
i = 1, \ldots, N, M.
\end{equation}

It is also shown there that this formula can be re-written as

\begin{equation}
V_t^i = \sum_{\tau=1}^{\infty} \delta^{\tau} E_t[D_{t+\tau}^i] - (\lambda B_t^i/r) \sum_{\tau=0}^{\infty} \delta^{\tau} \alpha_{t}^i,
\end{equation}

\begin{equation}
i = 1, \ldots, N, M.
\end{equation}
That is, market value at any instant is the present value, using the risk-
less rate of interest, of optimally forecast future dividends, minus an adjust-
ment for risk. In principle, equation (8) could be used to value investment
projects (of a particular sort) for purposes of capital budgeting.

As one would expect from the single-period CAPM, the risk adjust-
ment reflects only the influence of the $\varepsilon_t$, the random factors affecting all
firms. Risk associated with the firm-specific $\varepsilon^i_t$ is diversified away completely
in a fully efficient market. It will be useful to define the following poly-
nomial functions:

$$(9) \quad \alpha^i(x) = \sum_{t=0}^{\infty} x^t \alpha^i_t, \quad \text{and} \quad \gamma^i(x) = \sum_{t=0}^{\infty} x^t \gamma^i_t, \quad i = 1, \ldots, N, M.$$ 

Following Ross (1976a), we can assume without loss of generality that $\alpha^M(\delta)$ is
positive as long as investors are risk-averse. Since such investors will re-
quire a risk premium for holding the market portfolio, it follows that $\lambda$ in (7)
and (8) must also be positive. If $\alpha^i(\delta)$ is positive for any firm, the earnings
of that firm will tend to covary positively with those of the market as a whole,
and the firm's market value will be reduced as a consequence.

For $T > 1$ and $k > 0$, a natural measure of the sensitivity of $D^{i}_{t+T+k}$
to the random shocks affecting the market as a whole is given by

$$s^i(T+k, T; t) = \frac{\partial D^i_{t+T+k}}{\partial \varepsilon_{t+T}} = \frac{\alpha^i}{k} B^i.$$ 

Similarly, a natural measure of the sensitivity of firm's present value to all
future fluctuations in the $\varepsilon_t$ is given by the present value of all the $s^i$:

$$S^i(t) = \sum_{T=1}^{\infty} \sum_{k=0}^{\infty} \delta^{T+k} s^i(T+k, T; t) = B^i \sum_{T=1}^{\infty} \delta^{T} \left[ \sum_{k=0}^{\infty} \delta^k \alpha^i_k \right] = (B^i/r) \sum_{T=0}^{\infty} \delta^{T} \alpha^i_{\tau}. $$
Thus the risk adjustment in (8) is simply $\lambda$ times $S^i(t)$, and $\lambda$ can be interpreted as the market price of long-run, discounted sensitivity of earnings to $\varepsilon$.

Let us now consider the relation of equation (7) to the CAPM. First, it is necessary to assume that the $N$ stocks considered here are in fact the only risky investment opportunities available in the economy. Second, it is necessary to impose assumptions on the distributions of the $\varepsilon_t$ and the $\xi_t^i$ and/or the preferences and expectations of investors sufficient to justify working in mean-variance terms. Then the single-period CAPM states that

$$V^i_{t-1} = \delta(t)E_{t-1}[V_t + D^i_t] - P_t[Cov_{t-1}(V_t + D^i_t, V_t + D^M_t) + \sigma_{t-1}(V_t + D^M_t)],$$

where $Cov_{t-1}$ and $\sigma_{t-1}$ are the covariance and standard deviation, respectively, expected by investors at the end of period $t-1$, and $P_t$ is a positive constant under risk-aversion. It is straightforward but tedious to show that if (2) holds and expectations are rational, (7) satisfies (10) with $P_t$ equal to $\lambda$ divided by the standard deviation of $\varepsilon$. This suggests that with some minor changes in assumptions, we could have derived (7) from the CAPM.

There are problems with this approach, however, as an examination of the security market rates of return indicates. Using (7) and the definitions in Section I, we obtain

$$R^i_t = r + \frac{q^i_t + r^i_t - (1+r)q^i_{t-1}}{q^i_{t-1}}$$

$$= r + [\alpha^i(\delta)(\lambda + \xi_t^i) + \gamma^i(\delta) \xi_t^i]/q_{t-1}^i.$$

Note that the expected rate of return from investing in firm $i$'s stock exceeds (falls short of) the riskless rate if $\alpha^i(\delta)$ is positive (negative). Unless both $\alpha^i(\delta)$ and $\gamma^i(\delta)$ are zero, the distribution of $R^i_t$ depends on the value of
All else equal, a higher price of stock $i$ at the end of period $t-1$ translates into a lower expected volatility of that stock during period $t$, as the variance of $R^i_t$ is lower. This is broadly consistent with evidence on actual and expected common stock volatilities, respectively, presented by Fischer Black (1976) and by Richard Schmalensee and Robert Trippi (1978). But the dependence of the distribution of $R^i_t$ on $q_{t-1}^i$ causes problems within the CAPM framework. At the start of period $t-1$, $q_{t-1}^i$ is random. The realizations of the random variables $\varepsilon_{t-1}^i$ and $\xi_{t-1}^i$, for $i = 1, \ldots, N$, affect both the rates of return obtained during period $t-1$, and by (shifting the $q_{t-1}^i$) the set of securities market investment opportunities available during period $t$. Under the assumptions of the CAPM, rational investors will in general take this second effect into account in their decisions, and this hedging behavior may cause (10) not to hold. If no special assumptions are made about preferences, it can be shown that the CAPM applies in this model with constant $P$ only if for all $i$, $\alpha_t^i = \gamma_t^i = 0$ for $\tau \geq 1$. This will be referred to hereafter as the uncorrelated case. In this case, it follows from (7) that $q_t^i$ and $V_t^i$ are constant over time for all $i$. Since previously observed values of the random variables in the economy provide no information about future dividends, each period is exactly like all others and market values must be constant.

In the derivation of (7) and (8), the difficulties associated with the CAPM and equation (10) in a multi-period context were finessed by adopting the assumptions of the APT, thus obtaining a single-period valuation formula (equation (6) with the $t$ subscript restored to $\lambda$) that does not require realized returns in any one period to be independent of opportunities available on the securities market in subsequent periods. The additional assumption of constant $\lambda$ then permitted that formula to be employed recursively to value long-lived assets.
III. Long-Run Equilibrium in Tangible Asset Markets

Let us now explore the implications of the assumption that the economy modelled here is in long-run equilibrium with real capital stocks, the $B^i$, constant over time. In the non-stochastic context, James Tobin (1969) has stressed the importance of the long-run equilibrium condition that market values of tangible assets equal their replacement costs. In our one-commodity world, the ratio of market value to replacement cost for firm $i$ in period $t$ is $q^i_t$. As equation (7) shows, this quantity is not in general constant over time.

In the uncorrelated case, however, the $q^i_t$ are constant, and Tobin's condition can be applied directly. Equation (7) then yields

$$r^i = r - \lambda \alpha^i_0, \quad i = 1, \ldots, N, M. \tag{12}$$

Comparing any single firm to the market as a whole and eliminating $\lambda$, we obtain the equilibrium condition relating average accounting rates of return:

$$r^i = r + \left[ r^M - r \right] \left[ \alpha^i_0 / \alpha^M_0 \right], \quad i = 1, \ldots, N, M. \tag{13}$$

The ratio $[\alpha^i_0 / \alpha^M_0]$ gives a measure of the riskiness of firm $i$'s dividend stream in this case. The uncorrelated case seems very special, however: random events can have long-lasting consequences, and we do observe stock price fluctuations. Before discussing (13), it thus seems appropriate to investigate situations in which the $q^i_t$ do vary over time.

The most natural generalization of the Tobin condition to such situations would seem to require that the long-run averages of the $q^i_t$, given by

$$q^i = (1/r) [r^i - \lambda \alpha^i(\delta)], \quad i = 1, \ldots, N, M, \tag{7'}$$

be unity, so that the $q^i_t$ fluctuate around one and market values equal book values on average. If the $B^i$ must be fixed for all time at the start of period zero, with prior values of the random variables in the system unobservable, in-
vestment can be expected to proceed until market and book values are equal, with the ratios of these quantities given (7'). This is an extreme assumption, of course. If random events have long-lived effects on the profitability of tangible investments, one would presumably need to deal explicitly with irreversibilities and adjustment costs in order to construct a full-blown model of long-run stationary equilibrium with constant (or approximately constant) real capital stocks. Having not done this, I cannot claim to have derived \( q_i = 1 \) for all \( i \) as an equilibrium condition. But this condition has a number of interesting and relatively immediate implications, and it seems natural and plausible enough to make those implications worth presenting. This condition is thus assumed to hold in what follows.\(^\text{18}\)

If all the \( q_i \) are unity, one can eliminate \( \lambda \) in (7') and obtain a generalization of (13):

\[
(14) \quad r^i = r + [r^M - r] \theta^i, \quad i = 1, \ldots, N, M,
\]

where the accounting-based (or tangible asset market) risk measure, \( \theta \), is given by

\[
(15) \quad \theta^i = \alpha^i(\delta)/\alpha^M(\delta), \quad i = 1, \ldots, N, M.
\]

Note that this measure of riskiness is defined entirely in terms of cash flows provided by tangible assets; securities market data are not employed. It is clear from (15) that \( \theta^M = 1 \). Moreover, by the adding-up property that connects equations (1) and (2),

\[
(16) \quad \sum_{i=1}^{N} (B^i_t/B^M_t) \theta^i = 1,
\]

for all \( t \); the weighted average of the \( \theta^i \) using book value weights is unity.
It is relatively simple to modify equations (14) and (15) to take account of corporate income taxation and (exogenous) leverage, assuming total payments to security holders unchanged. With no default risk, each firm can borrow at the riskless rate of interest, r. Suppose that a fraction $\phi^i$ of firm i's assets are debt-financed, let the corporate tax rate be $\tau$, and assume that interest and dividends are paid at the same instants. Then firm i's dividends are given by

$D_t^i = (1-\tau)(r^i - \phi^i r)B_t^i = [(1-\tau)(r^i - \phi^i r)] + \sum_{\tau=0}^{\infty} [(1-\tau)\alpha^i]e_{t-\tau}B_t^i.$

Using (7) to value this dividend stream, setting that value equal to the book value of firm i's equity, $(1-\phi^i)B^i$, and eliminating $\lambda$ on the assumption that all firms face the same corporate tax rate, one obtains

$\frac{-i}{r^e} = r + [\frac{r^M - r}{r^e}]\theta^i_e,$

where the $\frac{-i}{r^e}$ are average after-tax returns on the book value of equity,

$\frac{-i}{r^e} = (1-\tau)(\frac{-i}{r^i} - \tau^i)/(1-\phi^i),$

with $\phi^M$ defined as the book-value-weighted average of the $\phi^i$, and the leverage-adjusted risk measure, $\theta^i_e$, defined by

$\theta^i_e = \frac{\alpha^i(\delta)/(1-\phi^i)}{\alpha^M(\delta)/(1-\phi^M)} = \theta^i[(1-\phi^M)/(1-\phi^i)],$

Note that $\theta^i_e = 1$ and that the weighted average of the $\theta^i_e$ using book values of equity, $(1-\phi^i)B^i$, is unity from (16) and (20).
The single-period CAPM has the well-known implication

\[ E_{t-1}[R^i_t] = r + \{E_{t-1}[R^M_t] - r\} \beta^i_t, \quad i = 1, \ldots, N, M, \]

where the "market beta" coefficients, \( \beta^i_t \), are defined by

\[ \beta^i_t = \frac{\text{Cov}_{t-1}(R^i_t, R^M_t)}{\text{Var}_{t-1}(R^M_t)}, \quad i = 1, \ldots, N, M. \]

(This can be derived from equation (10), above.) Because of the adding-up relation connecting the \( R^i_t \) and \( R^M_t \), it is easily seen that \( \beta_t^M = 1 \) for all \( t \), and

\[ \frac{1}{N} \sum_{i=1}^{N} \left( \frac{V^i_t}{V^M_t} \right)^{\beta^i_t} = 1, \]

for all \( t \); the weighted average of the market betas, using market value weights, is unity. Equations (21) and (23) can be directly compared to equations (14) and (16) above. Moreover, it is easily seen that if the \( R^i_t \) are given by (11), equations (21) and (22) are satisfied, with

\[ \beta^i_t = \left( q^i_t / q^M_{t-1} \right) \theta^i_t, \quad i = 1, \ldots, N, M. \]

Thus equations (21) and (23) are direct security market analogs of the tangible asset market equations (14) and (16) above.

Many authors have noted that market beta coefficients seem to change over time.\(^{19}\) Equation (24) predicts that under conditions of tangible asset market equilibrium, those changes will follow a relatively simple pattern: if a stock rises more rapidly than the market average, its beta coefficient should subsequently move away from zero. Further, equation (23) serves to connect the
measures of security market and real asset market risk in a relatively simple fashion in full equilibrium. With no leverage, one would expect estimates or time averages of market betas to be roughly proportional to the corresponding \( \theta_i \) in cross-section. \(^{20}\) If firms are levered, the correspondence is between market betas and the \( \theta_i \).

Finally, a number of studies have employed variants of the accounting-based "asset beta" measure of riskiness:

\[
(25) \quad a_{\beta_i}^t = \frac{\text{Cov}(r^i_t, r^M_t)}{\text{Var}(r^M_t)}, \quad i = 1, \ldots, N, M.
\]

Here \( \text{Cov} \) and \( \text{Var} \) refer to the sample covariance and variance, respectively, computed over some set of time periods. The mean of the asset beta coefficient in any finite sample is not easily evaluated, but the corresponding population coefficient can be directly obtained from (1) and (2):

\[
(26) \quad a_{\beta_i}^t = \frac{\sum_{t=0}^{\infty} \alpha^M_i \alpha_t^t}{\sum_{t=0}^{\infty} \alpha_t^t}, \quad i = 1, \ldots, N, M.
\]

Comparing equations (15) and (26), it is clear that \( \theta_i \) and \( a_{\beta_i}^t \) are equal in the uncorrelated case, but in general one would not expect them to be perfectly correlated in cross-section. This is at least consistent with the mixed results obtained by studies using asset betas to explain cross-section differences in market betas. \(^{21}\)
APPENDIX

Equation (7) in the text can be reached through an induction argument, assuming that the world ends at some time \( T \) and working backwards in time. It can then be verified by showing that it satisfies (5) and (6) in the text.

Define \( \bar{D} \) and \( c_i = B^{i} \gamma_T \) for all \( i \) and \( T \). Equation (2) in the text then implies

\[
\begin{align*}
D_t^i &= \bar{D} + \sum_{t=0}^{\infty} a_t^i e_t T, \\
&\quad + \sum_{t=0}^{\infty} c_t^i e_t T, \\
&\quad i = 1, \ldots, N, M.
\end{align*}
\]

and equation (4) becomes

\[
\begin{align*}
E_t^i [D_t+1] &= \bar{D} + \sum_{t=0}^{\infty} a_t^i e_t T, \\
&\quad + \sum_{t=0}^{\infty} c_t^i e_t T, \\
&\quad i = 1, \ldots, N, M.
\end{align*}
\]

Now consider any particular value of \( i \), and let us drop the superscript for the moment to avoid clutter. Suppose that the world is known to end at the end of period \( T \), just after \( D_T \) is paid. Then clearly \( V_T = 0 \), and (A1) implies

\[
V_T + D_T = D_T = [\bar{D} + \sum_{t=1}^{\infty} a_t^i e_t T, \\
&\quad + \sum_{t=1}^{\infty} c_t^i e_t T, \\
&\quad + \sum_{t=0}^{\infty} a_T e_T T, \\
&\quad + \sum_{t=0}^{\infty} c_T e_T T].
\]

As of the start of period \( T \) (end of period \( T-1 \)), everything on the right of this equation except \( e_T \) and \( e_T \) is non-stochastic. This equation is thus of the form of (5) in the text, since \( e_T \) is firm-specific, so that equation (6) can be used to yield

\[
\begin{align*}
V_{T-1} &= \{\delta D \} + \sum_{t=0}^{\infty} e_{T-1-t} (\delta a_{t+1}) \\
&\quad + \sum_{t=0}^{\infty} e_{T-1-t} (\delta c_{t+1}) - \lambda(\delta a_0).
\end{align*}
\]
Applying (A1) again yields

\[ V_{T-1} + D_{T-1} = [(1+\delta)D + \sum_{\tau=1}^{\infty} \xi_{T-1-\tau}(a_{\tau+1} + \delta a_{\tau+1}) + \sum_{\tau=1}^{\infty} \xi_{T-1-\tau}(c_{\tau+1} + \delta c_{\tau+1})] 
- \lambda \delta a_0 + (a_0 + \delta a_1) \xi_{T-1} + (c_0 + \delta c_1) \xi_{T-1} \].

The first term on the right is non-stochastic as of the start of period \(T-1\), so that (5) again applies, and (6) then yields

\[ (A4) \quad V_{T-2} = \{(\delta + \delta^2)D + \sum_{\tau=0}^{\infty} \xi_{T-2-\tau}(\delta a_{\tau+1} + \delta^2 a_{\tau+2}) + \sum_{\tau=0}^{\infty} \xi_{T-2-\tau}(\delta c_{\tau+1} + \delta^2 c_{\tau+2}) \} - \lambda \{ a_0(\delta + \delta^2) + a_1(\delta^2) \}. \]

Proceeding similarly, one obtains

\[ (A5) \quad V_{T-3} = \{(\delta + \delta^2 + \delta^3)D + \sum_{\tau=0}^{\infty} \xi_{T-3-\tau}(\delta a_{\tau+1} + \delta^2 a_{\tau+2} + \delta^3 a_{\tau+3}) + \sum_{\tau=0}^{\infty} \xi_{T-3-\tau}(\delta c_{\tau+1} + \delta^2 c_{\tau+2} + \delta^3 c_{\tau+3}) \} - \lambda \{ a_0(\delta + \delta^2 + \delta^3) + a_1(\delta^2 + \delta^3) + a_2(\delta^3) \}. \]

Considering equations (A3) - (A5) above, it is clear that for time \(t = T-k\), with \(k\) very large, we must have

\[ (A6) \quad V_t = \{(D/r) + \sum_{\tau=0}^{\infty} \xi_{t-\tau} \{ \sum_{k=T+1}^{\infty} \delta^{k-\tau} a_k \} + \sum_{\tau=0}^{\infty} \xi_{t-\tau} \{ \sum_{k=T+1}^{\infty} \delta^{k-\tau} c_k \} \} - (\lambda/r) \sum_{\tau=0}^{\infty} \delta^t a_{\tau}. \]

Dividing through by \(B^1\) and using the definitions above to convert to the notation in the text, equation (A6) becomes equation (7) there.

To complete the derivation of equation (7), we now show that (A6) satisfies (5) and (6) in the text. Dropping the superscripts in (A1), adding (A1) and (A6), and separating out the terms in \(\xi_t\) and \(\xi_t\), one obtains
(A7) \[ V_{t+D_t} = \delta^{-1}\left\{\frac{D}{r}\right\} + \sum_{\tau=1}^{\infty} \xi_{t-\tau} \left[ \sum_{k=\tau}^{\infty} \delta^{k-\tau+1} a_k \right] + \sum_{\tau=1}^{\infty} \xi_{t-\tau} \left[ \sum_{k=\tau}^{\infty} \delta^{k-\tau+1} c_k \right] \]

\[ - \delta(\lambda/r) \sum_{\tau=0}^{\infty} \delta^\tau a_\tau \]

\[ + \sum_{\tau=0}^{\infty} \delta^\tau c_k \xi_t. \]

Equation (5) clearly applies here, so that (6) can be used to yield

(A8) \[ V_{t-1} = \left\{\frac{D}{r}\right\} + \sum_{\tau=0}^{\infty} \xi_{t-1-\tau} \left[ \sum_{k=\tau+1}^{\infty} \delta^{k-\tau} a_k \right] + \sum_{\tau=0}^{\infty} \xi_{t-1-\tau} \left[ \sum_{k=\tau+1}^{\infty} \delta^{k-\tau} c_k \right] \]

\[ - (\lambda/r) \sum_{\tau=0}^{\infty} \delta^\tau a_\tau, \]

which is clearly (A6) applied to time \( t-1 \), as was to be shown.

To obtain equation (8) in the text, use (A2) to yield

(A9) \[ \sum_{m=1}^{\infty} \delta^m E_t \left[D_{t+m}\right] = \left\{\frac{D}{r}\right\} + \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \delta^m a_k \xi_{t+m-k} + \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \delta^m c_k \xi_{t+m-k}. \]

Setting \( \tau = k-m \) and collecting coefficients of the various random variables, one obtains the first of the two terms on the right of (A6).
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FOOTNOTES

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1. Some such studies have employed standard deviations of profit rates [William Shepherd (1972)] or of sales [Gloria Hurdle (1974)] as independent variables to control for differences in riskiness, but these variables have no real theoretical underpinnings.

2. See Stewart Myers (1976), Mark Rubinstein (1976), Sudipto Bhattacharya (1977), Stewart Myers and Stuart Turnbull (1977), and the references they cite.

3. Such implications are discussed by Edward Greenberg, William Marshall, and Jess Yawitz (1978) in the context of a one-period model; it is unfortunately not clear to what extent their results hold in a multi-period setting.

4. Many of these studies are discussed by Stewart Myers (1976); see also William Beaver and James Manegold (1975).

5. The basic CAPM, which assumes the existence of a riskless asset as is done in this essay, was developed by William Sharpe (1964), John Lintner (1965), and Jan Mossin (1966). Michael Jensen (1972) provides an excellent discussion of this model, along with related extensions and empirical tests.

6. In an earlier version of this paper, available on request from the author, real capital stocks are assumed to grow at the same constant rate. This generalization complicates the algebra considerably without yielding much additional insight.
7. This is, of course, the classic result of Franco Modigliani and Merton Miller (1958), which has been re-established by numerous subsequent authors.

8. Note that the number of non-zero $\alpha^M_t$ may be finite. In order for $r^M_t$ to have finite variance, the sum of $(\alpha^M_t)^2$ must be finite.

9. The remarks of the preceding footnote also apply to the $\alpha^i_t$ and to the $\gamma^i_t$.

10. Equation (4) yields minimum mean-squared error forecasts; see Box and Jenkins (1970, ch.5). In an interesting paper, Clive Granger (1975) explores the implications of this sort of optimal forecasting of dividend flows; he assumes that share prices equal discounted expected dividends, however, and thus does not deal with issues of risk.

11. These assumptions are developed in Ross (1976b); see also Ross (1978). The number of firms, N, is required to be large so that it is possible to diversify away the firm-specific risk represented by $\gamma^i_t$. With finite N, the APT is always an approximation, just as the theory of perfect competition is an approximation when the number of sellers in any market is finite. In Ross's papers, there are several independent random factors that, like $u_t$, affect all the $R^i_t$. Using this more general framework, the analysis of this section could be easily generalized to allow the $r^i_t$ to depend on $K > 1$ independent white noise processes that determine $r^M_t$: equations (7) and (8) below would then contain K risk-adjustment terms involving K different $\lambda$'s. The analysis of Section III would be greatly complicated by such a generalization, however, since it would be much harder to obtain expressions not involving the $\lambda$'s.

12. In the models of Stapleton and Subrahmanyam (1978), the market price of risk changes monotonically over time; this does not seem plausible in stationary equilibrium. On this general issue, see the work of Rubinstein (1976) and Robert Lucas (1978).
13. On the basic CAPM, see the references cited in footnote 5, above. Richard Roll (1977) and Ross (1978) discuss the assumptions on which it depends.

14. Equation (10) assumes that risk is measured in units of standard deviation instead of units of variance, but as Jensen (1972) notes, these assumptions are equivalent in a one-period context.

15. For some tests of a model resembling (11), see Menachem Brenner and Seymour Smidt (1977).

16. See Robert Merton (1973), Eugene Fama (1977), Ross (1978), and the references they cite. The assumption of logarithmic utility generally serves to rule out such hedging behavior; see Nils Hakansson (1971) and Merton (1973). On the issue of neglecting hedging behavior in a multi-period context, see Ross (1975).

17. This can be verified by working within Fama's (1977) framework with r and the market price of risk constant; one employs (4) and (11) and checks to see when the condition given below Fama's equation (33) is satisfied.

18. It is worth noting again that nothing in Section II depends on this assumption. Thus the implications of equation (8) for capital budgeting decisions are valid even if tangible asset markets are not in full equilibrium.

19. See, for instance, Robert Klemkosky and John Martin (1975), Richard Pettway (1978), and the references they cite.

20. By Jensen's inequality, the expectation of \[1/q_{t-1}^M\] will exceed one. Moreover, \(i_{t-1}\) and \(q_{t-1}^M\) are correlated, so that the expectation of \(i_{t-1}/q_{t-1}^M\) will depend on \(i\) in general. Thus the most that one can expect is rough proportionality between the \(\theta_i\) and market betas.

21. See the literature cited in footnote 4 above. Most of these studies
have in fact used earnings, not dividends, in computing asset betas. In this model the two are equal, and it would require additional work to consider explicitly cases where they are not. Granger (1975), for instance, shows that even if investors are concerned only with discounted expected dividends, earnings may nonetheless directly affect stock prices by providing information about future dividend payments.