Adjustment Cost, Demand Uncertainty, and the Behavior of the Firm*

by

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Abstract

Demand uncertainty is characterized by letting the market demand function shift randomly but continuously through time according to a stochastic process. Thus the firm knows its current demand, but does not know what demand will be in the future. The firm can adjust some factor inputs freely in response to stochastic changes in demand, but other factors are "quasi-fixed" in that adjustment costs are incurred when they are changed. We show that the risk-neutral firm will (in expected value terms) produce more (less) if marginal adjustment costs are rising at an increasing (decreasing) rate, and that risk aversion causes the firm to produce more under demand uncertainty. These results hold whether the firm is competitive or monopolistic, whether or not the firm holds inventories, and whether uncertainty affects demand additively or multiplicatively.
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1. Introduction

This paper takes a new look at the behavior of the firm facing demand uncertainty. As in earlier studies, we are concerned with the effects of uncertainty on the firm's output and pricing decisions. However, our characterization of uncertainty is quite different from the usual one in which demand is simply not known at the time an output (or pricing) decision is made.

We model demand uncertainty in a dynamic context by assuming that the market demand function shifts randomly but continuously through time according to a stochastic process. Thus, although today's demand is known exactly, future demand may be larger or smaller, and has a variance that increases with the time horizon. We combine this characterization of uncertainty with a dynamic model of the firm in which some factor inputs can be adjusted freely in response to stochastic demand changes, but other factors are "quasi-fixed" in that adjustment costs are incurred when they are changed.

In such a world the firm's capacity, price, and level of sales will be random processes, but there are a number of questions to be asked about the firm's behavior in expected value terms. In particular, should the presence of uncertainty cause competitive or monopolistic firms to produce more or less than they would otherwise? How are the effects of uncertainty influenced by the presence of risk aversion? And how would the use of inventories alter the impact of demand uncertainty on capacity and sales?
These question have of course been addressed by others in the past. Among the earliest and best known studies of demand uncertainty are those of Mills (1959, 1962) and Karlin and Carr (1962). Mills examined a single-period monopolistic firm that sets both output and price in the face of additive demand uncertainty (i.e. the demand function is of the form $q = q(p) + u$ where $u$ is a random variable), and showed that the uncertainty leads to a lower price if marginal cost is constant (so that the firm reduces the expected loss from discarding unsold production). Karlin and Carr confirmed, both for the static case and the multi-period case with inventory carry-over, that additive uncertainty tends to reduce the price and increase the output of a risk-neutral firm, while multiplicative uncertainty does just the opposite.

Several papers have appeared since, and have been concerned with such issues as the way in which the error term enters the demand function (additively, multiplicatively, or nonlinearly), the implications of choosing price *ex ante* instead of output, the implications of risk aversion, and the use of inventories. For example, Sandmo (1971) and Baron (1970) showed that a risk-averse competitive firm will produce less when the price is a random variable or subject to an additive error term, but a risk-neutral firm will not alter its production.¹ Leland (1972) extended these results

¹ - These papers ignored the firm's choice of factor inputs under uncertainty and the implications of that choice for output. Batra and Ullah (1974) showed that Sandmo's results hold if all factor inputs are chosen at the same time (before price is observed). However, Hartman (1976) showed that if one ("quasi-fixed") input is chosen before price is observed while the second is chosen afterwards, nonlinearities in the production function can lead to a higher or lower output for a risk-neutral firm. Also, Young (1979) recently argued that demand should be proportionately more uncertain the greater the elasticity of expected demand (and the more competitive the market), and showed that (with risk-neutral firms) this can lead to prices that are higher in competitive than monopolistic markets if inventory and shortage costs are high enough.
to a monopolistic firm whose demand can depend in a general way on a random error term (e.g. \( q = q(p,u) \)), and showed that uncertainty reduces the production of a risk-averse firm that sets quantity \textit{ex ante} and price \textit{ex post}, and can raise (lower) the price of a firm that sets price \textit{ex ante} if marginal cost is increasing (decreasing). Finally, Zabel (1972) worked with a multi-period monopoly model with inventory carry-over, and showed that (as in the Karlin and Carr paper) if a firm faces additive demand uncertainty in each period output tends to be lower, while with multiplicative uncertainty it tends to be higher.\(^2\)

In all of these models, whether single- or multi-period, current demand is unknown (i.e. is subject to an error term) each time an output or pricing decision is made. While actual firms do in fact face some uncertainty over their current demand functions, they face much more uncertainty over the future evolution of demand. As time passes new competitive substitutes may appear on the market, the prices of existing competitive (and complementary) goods are likely to fluctuate, income and population will grow at unpredictable rates, and of course tastes are likely to change, again unpredictably. Thus the demand function that the firm faces today is relatively certain compared to the one it will face one or two years from now, and the firm's uncertainty over future demand is greater the farther into the future it looks.

Of course if the firm can adjust its capacity freely, uncertainty over future demand should not influence its current behavior. But for most firms output flexibility is limited in the short run, and changes in capacity involve significant adjustment costs. While the use of inventories might reduce the need to make capacity adjustments, it is unlikely to eliminate it, so that most firms must regularly make investment decisions

\(^2\) Amihud and Mendelson (1979) recently developed a model similar to Zabel's, but with additive error terms affecting output as well as demand in each period.
that take into account short-run fluctuations in demand as well as the (even greater) uncertainties affecting long-run demand.

The model presented in this paper is therefore an inherently dynamic one, and captures these basic characteristics of demand uncertainty and adjustment costs. The results obtained from the model are different from those of earlier studies, and tend to be quite robust. For example, we show that risk-neutral firms will (in expected value terms) produce more (less) if marginal adjustment costs are rising at an increasing (decreasing) rate. We also show that the risk-averse firm will produce more under demand uncertainty. These results hold whether the firm is competitive or monopolistic, whether or not the firm holds inventories, and whether uncertainty affects demand additively or multiplicatively.

The basic model is set forth in the next section. In Section 3 we briefly review the behavior of the model when there is no uncertainty; this will make it easier to understand the stochastic solution. The stochastic version of the model is solved and discussed in Sections 4 and 5 for the case where the firm holds no inventories, and in Section 6 we discuss the use of inventories and show that it does not change the basic results. The last section contains a summary of the results and some concluding remarks.

2. The Basic Model

We model demand uncertainty by letting the demand function be driven by a stochastic process with independent increments (an Ito process). In particular, we write demand as

\[ p = p[q, \theta(t)] \] (1)
with \( \partial p/\partial q < 0, \partial p/\partial \theta > 0 \), and \( \theta(t) \) a stochastic process of the form

\[
d\theta = \sigma(\theta) d\omega = \sigma(\theta) \varepsilon(t) \sqrt{dt}
\]  

(2)

where \( \varepsilon(t) \) is a serially uncorrelated normal random variable with zero mean and unit variance (i.e. \( d\omega \) describes a Wiener process). 3

Equations (1) and (2) imply that uncertainty about demand grows with the time horizon, and that fluctuations in demand occur continuously over time. No jumps in \( \theta(t) \) are possible (although over any finite time period any change in \( \theta \) of finite size is possible). Also, note that eqn. (1) puts no restrictions on the way in which \( \theta \) enters the demand function. For a monopolistic firm it can enter additively, multiplicatively, or nonlinearly. For a competitive firm eqn. (1) becomes \( p = a + a \theta(t) \) (for which \( p = 0 \) is a special case), and \( \theta \) can be additive, multiplicative, or both.

The firm can react to stochastic fluctuations in demand by changing its factor inputs and thus its capacity. However, for at least some factors this will involve adjustment costs. In this paper we assume that labor is a "flexible" input (i.e. can be adjusted freely), but capital is "quasi-fixed," so that the purchase and installation of "usable" capital at a rate \( I \) involves a cost \( vI + C(I) \), where \( v \) is the purchase price of a unit of capital equipment, and \( C(I) \) is the full adjustment cost, with \( C'(I) > 0 \) and \( C''(I) > 0 \). Here \( C(I) \) includes the cost of installing the capital, training workers to use it, etc. Since this takes time, \( C''(I) > 0 \), i.e. it is more costly to increase the stock of usable capital quickly than

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3 - Equation (2) is the limiting form as \( h \to 0 \) of the discrete-time difference equation \( \theta(t+h) - \theta(t) = \sigma(\theta) \varepsilon(t) \sqrt{h} \), and \( E[d\theta] = 0 \) and \( \text{Var}[d\theta] = \sigma^2(\theta) dt \). If \( \sigma(\theta) = \sigma \), then \( \theta \) follows a simple random walk. If \( \sigma(\theta) = \sigma \theta \), \( \theta \) will be log-normally distributed (and always positive). Note that eqn. (2) contains no deterministic drift component; although most firms would anticipate some deterministic component of demand growth, we are interested here only in the implications of stochastic fluctuations in demand. For an introduction to stochastic processes of the form of (2), see Karlin and Taylor (1975).
Ignoring for the moment the possibility of holding inventories, we can write the firm's instantaneous profit as:

\[ \Pi(t) = p(q, \theta)q - wL - vI - C(I) \] (3)

where \( w \) is the wage rate, and \( q = F(K, L) \) a strictly concave production function, i.e. \( F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0, \) and \( F_{KL} > 0. \) The firm's capital stock is given by

\[ K = I - \delta K \] (4)

where \( \delta \) is the depreciation rate.

For the risk-neutral firm the problem is to choose \( L(t) \) and \( I(t) \) to minimize the sum of discounted profits:

\[ \max_{L,I} \int_0^\infty (\Pi(t) - rt) e^{-rt} dt, \] (5)

whereas the risk-averse firm maximizes the integral of discounted flow of utility \( U(\Pi) \), with \( U'(\Pi) > 0 \) and \( U''(\Pi) < 0.5,6 \) In either case the maximi-

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4 - Lucas (1967a) showed that this description of adjustment costs is consistent with the flexible accelerator model of investment demand. Alternatively (and equivalently) we could have represented adjustment costs by writing the production function as \( q = F(K, L, I) \), with \( F_L < 0 \) and \( F_{II} < 0 \), i.e. firms must devote internal resources (mainly labor) to the installation of and adaptation to newly purchased capital. This representation was used by Lucas (1967b) to analyze the effects of adjustment costs on competitive supply. The deterministic model that we discuss in the next section is similar to his, but more general in that we do not impose constant returns to scale.

5 - These objective functions ignore financial markets. An alternative objective is to maximize the equilibrium market value of the firm taking into account the market value of risk. Meyer (1976) examined the effects of demand uncertainty on a multi-product monopolist in a static context, and showed that market value could be expressed as

\[ V = \frac{1}{r} \left\{ E(\Pi) - R \beta \sigma_\Pi \right\} \]

where \( r \) is the riskless interest rate, \( \sigma_\Pi \) is the standard deviation of profit, \( R \) is the market value of unit of risk as measured by \( \sigma_{\Pi}^2 \), and \( \beta \) is the correlation of the firm's profits \( \Pi \) with the overall market return.

6 - (next page)
zation is subject to the ordinary differential equation (4), the stochastic
differential equation (2), and the conditions $L(t), I(t) > 0$.\textsuperscript{7}

In Section 6 of this paper we add inventories to the model. If
$N$ is the stock of inventory and $k$ the storage cost, profit becomes

$$\Pi(t) = p(q,θ)q - wL - vI - C(I) - kN \quad (3')$$

with the inventory stock given by

$$N = X - q \quad (6)$$

where $q$ is sales and $X = F(K,L)$ is production. The firm now maximizes the
sum of discounted profit (or utility) by choosing $L(t), I(t)$ and $q(t)$, so
that $p$ adjusts to clear the market.\textsuperscript{8} The maximization is subject to eqn. (6)
as well as (2) and (4), and the added conditions that $N \geq 0$ and $q \geq 0$.

\textsuperscript{6} Note that this is an equilibrium model, i.e. price adjusts so that there
is never any excess supply or demand. A number of interesting disequilibrium
models have appeared recently, notably in papers by Carlton (1978) and Gould
(1978). There the emphasis is on price inflexibility, and the demand for a
good depends both on its price and on the probability that it can be pur-
chased. In such a case "market clearing" implies that there will always be
some customers unable to purchase the good.

\textsuperscript{7} Note that the competitive (monopolistic) firm takes price (the demand
function) as \textit{exogenously given} in performing the maximization. Any shifts in
price or demand are purely stochastic and cannot be predicted.

\textsuperscript{8} Note that this is equivalent to choosing $L(t), I(t)$ and $p(t)$, with $q(t)$
adjusting to clear the market. Thus there is no longer any distinction be-
tween a quantity-setting and a price-setting firm, as in Leland (1972).
3. The Deterministic Solution

Before solving the stochastic optimization problems outlined above, let us briefly review the characteristics of the solution for the deterministic case, i.e. for which \( \theta = 0 \). We ignore inventories at first, and consider a risk-neutral firm that maximizes (5) subject to (4) and the conditions that \( L, I > 0 \). By straightforward application of the Maximum Principle it is easy to show that the optimal levels of employment and investment are given by

\[
MR \cdot F_L = w
\]

(7)

where \( MR = p - q(\partial p/\partial q) \) is marginal revenue, and

\[
\dot{I} = \frac{1}{C''(I)} \left\{ (r + \delta)[v + C'(I)] - MR \cdot F_K \right\}
\]

(8)

Note that if adjustment costs were zero, i.e. \( C''(I) = 0 \), the capital stock would be adjusted instantaneously so that its marginal revenue product would always equal its service price.

Eqn. (7) implies that \( L^* = L^*(K, w, p) \), so that by substitution \( L \) can be eliminated from (8). The resulting equation, together with (4), describes the dynamics of \( I \) and \( K \). This is easiest to see from the phase diagram of Figure 1. The \( I = 0 \) isocline is downwards sloping, and is shallower the larger is \( C''(I) \). If \( C''(I) = 0 \) the isocline becomes a vertical line at \( K^* \), so that \( K \) adjusts instantaneously to this equilibrium level. If \( C''(I) > 0 \), the isocline will be nearly horizontal, with \( I(t) \) always close to its equilibrium value \( \overline{I}^* = \delta K^* \), and adjustment of \( K \) to \( K^* \) taking place more slowly. But in general, if \( K(0) < K^* \), \( I^*(t) > \overline{I}^* \) and \( \dot{I}^* < 0 \) (as in trajectory A), and the opposite for \( K(0) > K^* \).
Suppose the firm is competitive so that MR = p, and in addition F(K,L) exhibits constant returns to scale so that \( F_K = F_L(K/L) * \) is independent of K. Then the \( I = 0 \) isocline is a horizontal line at \( I = I^* \). As can be seen from the phase diagram of Figure 2, \( I^* = 0 \) always, and the optimal investment policy is simply to set \( I(t) = \delta K \). Note, however, that even though \( F(K,L) \) exhibits constant returns, the long-run total cost function of the firm exhibits decreasing returns. The reason is that a higher equilibrium level of output (and larger equilibrium capital stock) requires a higher maintenance level of investment (\( I = \delta K \)), which implies higher costs of installation, etc.9

Note from eqn. (8) that whether or not \( F(K,L) \) exhibits constant returns, the \( I = 0 \) isocline shifts upward if price or marginal revenue increases, and shifts downwards if \( v, w, r \) or \( \delta \) increase. Now let us see how price, output, and investment evolve in response to a sudden unanticipated increase in demand. This is illustrated for the competitive firm with constant returns in \( F(K,L) \) in Figures 3a and 3b. The short-run supply curve is highly inelastic (only the labor input can be increased in the short run) so that price, and therefore investment, immediately rise to levels higher than their long-run equilibrium values. Then as the capital stock begins increasing further, price falls, the \( I = 0 \) isocline falls, and investment falls, asymptotically approaching its equilibrium value \( I = \delta K \).10

9 - To have full constant returns (i.e. a horizontal long-run marginal cost curve) we would write the adjustment cost function as \( C(I) = \phi(I/K)I \), with \( \phi' > 0 \). However, this would complicate the algebra in the stochastic solution without changing any of the basic results.

10 - The rate at which investment falls from \( I_n \) to \( I^* \) is found by differentiating both sides of the equation defining the \( I = 0 \) isocline with respect to time:

(continued)
Figure 1 - Phase Diagram for Deterministic Model

Figure 2 - Competitive Firm and Constant Returns
Now suppose the firm can hold inventories. In this case the maximization is subject to eqn. (6), with \( N \geq 0 \), in addition to eqn. (4). The solution is again obtained from straightforward application of the Maximum Principle. As before, eqn. (8) describes the dynamics of investment. In addition, the existence of inventories limits the rate at which marginal revenue (or for a competitive market, price) can rise:

\[
\dot{MR} \leq rMR + k \tag{9}
\]

and \( N = 0 \) unless (9) holds with equality. If \( N > 0 \), eqn. (9) simply says that the total user cost of holding a unit of inventory is zero. That total user cost is just the storage cost \( k \), plus the amortized capital cost (in terms of foregone marginal revenue) \( rMR \), less the rate of capital gain \( MR \).

In steady-state equilibrium \( MR = 0 \) and therefore no inventory is held. (We will see in Section 5 that this is not the case when demand is subject to stochastic fluctuations.) If in a competitive market demand increases suddenly so that price jumps (as in Figure 3b), inventories cannot be accumulated, and certainly would not be held while price is falling. However if demand falls suddenly (so that price drops but later rises as capital is decumulated), inventories may be held so that price later rises at a rate no faster than \( rp + k \). However what is important with respect to the stochastic model is that while movements in price (sometimes moderated by inventories) cause movements in the \( I = 0 \) isocline, eqn.

\[
\dot{I} = \left( \frac{E^2}{\frac{\delta}{\delta + \delta}} \right) \frac{\partial^2 I}{\partial (g)} (I - \delta K) < 0
\]

Thus \( I \) falls more rapidly the steeper is the demand curve, and less rapidly (from a lower level) the larger are adjustment costs.
(8) still describes the dynamics of investment given any price.

4. The Stochastic Solution

We now solve the stochastic version of the model for the case of no inventories. Recall that the maximization of (5), with profits given by (3), is now subject to the ordinary differential equation (4) and the stochastic differential equation (2). We obtain a solution using stochastic dynamic programming.\(^{11}\)

Define the optimal value function:

\[
J = J(K, \theta, t) = \max_{L, I} E_t \int_t^\infty J_d(\tau) d\tau
\]  \hspace{1cm} (10)

where \(J_d(\tau) = \Pi(\tau) e^{-\lambda t}\). Note that \(J\) is a function of the state variable \(K\) and the stochastic process \(\theta\), so the fundamental equation of optimality is: \(^{12}\)

\[
0 = \max_{L, I} \left[ \Pi_d(t) + \frac{1}{dt} E_t \frac{dJ}{dt} \right]
\]

\[
= \max_{L, I} \left[ \Pi_d(t) + J_t + (I - \delta K) J_K + \frac{1}{2} \sigma^2(\theta) J_{\theta \theta} \right]
\]  \hspace{1cm} (11)

Maximization with respect to \(L\) gives \(\partial J_d / \partial L = 0\), so that

\[
MR = F_L = w
\]  \hspace{1cm} (12)

\(^{11}\) - This paper makes use of stochastic dynamic programming and Ito's differentiation rule for functions of stochastic processes. For a discussion of these techniques, with applications to problems in economics, see Merton (1971) and Chow (1979). Kushner (1967) provides a detailed treatment. The solution approach used in this paper is similar to that in Pindyck (1980).

\(^{12}\) - We use the notation \(J_K = \partial J / \partial K\), etc. \((1/dt)E_t d()\) is Ito's differential operator. For a discussion, see Kushner (1967), Merton (1971), or Chow (1979). A clear derivation of the fundamental equation of optimality in stochastic dynamic programming is provided by Dreyfus (1965).
as in the certainty case. Maximization with respect to \( I \) gives

\[-\partial \Pi_d / \partial I = J_K \] (13)

i.e. the undiscounted shadow price of an additional unit of capital should just equal the marginal cost of purchase and adjustment for the unit.

Equation (12) and (13) could be substituted back into equation (11) to yield a partial differential equation for \( J(K, \theta, t) \). Theoretically one could solve that equation for \( J \) and then determine the optimal investment trajectory \( I^*(t) \) explicitly from (13). In practice, however, the solution of such a partial differential equation is usually not feasible. Instead our approach is to eliminate \( J \) from the problem.

First, differentiate eqn. (11) with respect to \( K \):

\[ \frac{\partial \Pi_d}{\partial K} + J_{Kt} + (I - \delta K) J_{K} - \delta J_{K} + \frac{1}{2} \sigma^2(\theta) J_{\theta \theta} = 0 \] (14)

and using Ito's Lemma note that this can be re-written as:

\[ \frac{\partial \Pi_d}{\partial K} - \delta J_{K} + (1/\partial t) E_t d(J_K) = 0 \] (15)

We cannot differentiate both sides of eqn. (13) with respect to time since both \( \Pi_d \) and \( J \) are functions of the stochastic process \( \theta \), so their time derivatives do not exist. Instead we use Ito's Lemma and apply the differential operator \((1/\partial t) E_t d(\cdot)\):

\[-(1/\partial t) E_t d(\partial \Pi_d / \partial I) = (1/\partial \theta) E_t d(J_{\theta}) \] (16)

Now combine eqns. (13), (15) and (16) to eliminate \( J_K \):

\[ (1/\partial t) E_t d(\partial \Pi_d / \partial I) = \partial \Pi_d / \partial K + \delta \partial \Pi_d / \partial I \] (17)
Equation (17) is a stochastic version of the Euler equation from the calculus of variations. It is easiest to interpret in its integral form:

\[
\frac{\partial \Pi_d}{\partial I} = E_t \int_t^\infty \left[ \frac{\partial \Pi_d(\tau)}{\partial K} + \delta \frac{\partial \Pi_d(\tau)}{\partial I} \right] d\tau
\] (17')

which says that the marginal purchase and installation cost of a unit of capital should equal the expected sum of all discounted future increases in revenues from that unit were it not to depreciate less the expected sum of future discounted costs associated with maintaining that unit as it depreciates.

We can now use eqn. (17) to determine the expected dynamics of investment. Substitute the expression for discounted profit into (17) and divide through by \( e^{-rt} \):

\[
r[v + C'(I)] - (1/dt)E_t dC'(I) = MR \cdot F_K - \delta v - \delta C'(I)
\] (18)

Now expand \( dC'(I) \) using Ito's Lemma:

\[
dC'(I) = C''(I)dI + \frac{1}{2} C'''(I)(dI)^2
\] (19)

Remember that \( I = I^*(K,e) \) along the optimal trajectory, so that (expanding \( dI \) using Ito's Lemma) \( E_t [(dI)^2] = \sigma^2(\theta)I_0^2 dt \), and

\[
E_t dC'(I) = C''(I)E_t dI + \frac{1}{2} \sigma^2(\theta)I_0^2 C'''(I)dt
\] (20)

Substituting (20) into (18) and rearranging, we obtain an equation describing the expected dynamics of investment that is analogous to eqn. (8) from the deterministic case:

\[
\frac{1}{dt} E_t dI = \frac{1}{C''(I)} \{ (r + \delta)[v + C'(I)] - MR \cdot F_K - \frac{1}{2} \sigma^2(\theta)I_0^2 C'''(I) \} 
\] (21)
5. The Effects of Uncertainty

Although price, production, and investment will fluctuate stochastically in this model, we can see from eqn. (21) that the expected rate of change of investment will be the same as in the certainty case if \( C''(I) = 0 \). Furthermore, since the dynamics of \( K \) are still given by eqn. (4), the expected equilibrium capital stock, labor demand, and output will also be the same as in the certainty case if \( C''(I) = 0 \).

However, if \( C''(I) > 0 \), i.e. if marginal adjustment cost is rising at an increasing (decreasing) rate, then the expected equilibrium capital stock and output level will be higher (lower). To see this, suppose \( C''(I) > 0 \). We do not have an analytical expression for \( I \), but clearly \( I^2 > 0 \). Thus an increase in \( \sigma \) (i.e. more uncertainty over future demand) has the same effect on price as an increase in marginal revenue - it shifts the \( (1/dt)E_t dI = 0 \) isocline upwards and to the right, as shown in Figure 4. Note that with \( \sigma > 0 \), \( K \) and \( I \) fluctuate stochastically around (and may drift away from) their equilibrium values. But the expected equilibrium capital stock, and thus output level, are clearly larger.

This deviation from the certainty case occurs for a simple reason. Suppose the firm is competitive, \( C''(I) > 0 \), and random increases and decreases in \( \theta \) occur that balance each other out, so that price does not drift from its average value. These fluctuations in \( \theta \) will cause fluctuations in \( I \), but with \( C''(I) > 0 \), increases in \( I \) raise marginal adjustment costs more than decreases in \( I \) lower them. The firm therefore has an incentive to maintain

\[13 \text{ In fact we know that } I^2 > 0 \text{ since an increase in } \theta \text{ raises the marginal revenue product of capital, the desired capital stock, and both the short-run and equilibrium levels of investment.}\]
a larger stock of capital and thereby reduce the amount by which $K$ (and thus $I$) must be increased when $\theta$ increases, so that this component of adjustment costs is reduced. \textsuperscript{14}

This is illustrated graphically in Figure 5, and mathematically by eqn. (22) below:

$$\frac{MR*F}{K} + \frac{1}{2} \sigma^2(\theta)I^2 e^{C''(I)} = (r + \delta)[v + C'(I)]$$

This equation defines the $\frac{1}{t} E \frac{dI}{dt} = 0$ isocline, and says that in equilibrium (i.e. where the only changes in $I$ and $K$ are from stochastic fluctuations in demand), the user cost of a marginal unit of capital (the right hand side of the equation) should be equated to a marginal benefit that now has two components. That benefit equals the marginal revenue product of capital, plus the expected reduction in marginal adjustment costs that result from the extra unit of capital.

Long-run marginal and average costs are shown for $\sigma = 0$ ($MC^0$) and $\sigma > 0$ ($MC^1$) in Figure 5. As explained above, with $C''(I) > 0$ stochastic fluctuations create a positive expected "quasi-fixed cost" of adjustment, which is reduced by maintaining a higher capital stock, but not eliminated. Thus stochastic fluctuations reduce long-run marginal cost, but increase average cost, so that the firm produces more, but earns a smaller profit. \textsuperscript{15}

\textsuperscript{14} Of course if the firm knew that increases and decreases in $\theta$ were going to balance out and leave price unchanged on average, it could simply keep $I$ and $K$ fixed and avoid the whole problem. But since fluctuations in $\theta$ are in fact stochastic so that price may drift, the firm (if it is behaving optimally) must adjust to every change in $\theta$.

\textsuperscript{15} Note that just the opposite occurs if $C''(I) < 0$, in which case stochastic fluctuations lead to an expected net reduction in adjustment cost, an increase in long-run marginal cost, and a reduction in output.
Figure 4 - Effect of Uncertainty, $C''(I) > 0$

Figure 5 - Expected Output and Cost, $C''(I) = 0$
So far we have assumed that the firm is risk-neutral. Let us now examine the impact of uncertainty if the firm is risk-averse, i.e. it maximizes the integral of the flow of discounted utility, \( U(\Pi)e^{-rt} = U_d(t) \), with \( U' > 0 \) and \( U'' < 0 \). We assume that \( U \) is quadratic, i.e. \( U''' = 0 \), and that \( p \) is linear in \( \theta \), i.e. uncertainty is additive and/or multiplicative.

The solution in this case is similar to that of Section 4, and is presented in Appendix A. (\( \Pi_d \) is replaced by \( U_d \) in the equation analogous to (17).) There we show that the equation defining the \( (1/dt)E_t dI = 0 \) isocline is given by:

\[
U'(\Pi)MR^*F_K + U''(\Pi)[v + C'(I)]MR^*F_K(I - \delta K)
\]

\[
+ \frac{1}{2} \sigma^2(\theta)I^2_0[U'(\Pi)C'''(I) - 3U''(\Pi)C''(I)[v + C'(I)]] = (r + \delta)U'(\Pi)[v + C'(I)]
\]

(23)

Now note that since \( U'' < 0 \), if \( C'''(I) = 0 \), the effect of increasing uncertainty is unambiguously to shift the \( (1/dt)E_t dI = 0 \) isocline upwards, and thus increase the expected equilibrium capital stock and output level. (If \( C'''(I) > 0 \) it reinforces this effect from risk aversion, and if \( C'''(I) < 0 \) it counteracts it.)

Thus if marginal adjustment costs are rising at a constant or increasing rate, the risk-averse firm will always maintain a larger capital stock and higher output level under uncertainty. The reason for this is similar to that for the risk-neutral firm with \( C'''(I) > 0 \). If \( C'''(I) = 0 \), the fluctuations in \( I \) resulting from zero-mean fluctuations in \( \theta \) will not (on average) raise marginal adjustment cost, but with \( U'' < 0 \) they will cause a loss in marginal utility, since gains in marginal utility from decreases in \( I \) (which reduce adjustment costs) will be more than offset by the losses from increases in \( I \).
The firm again has an incentive to maintain a larger stock of capital and thereby reduce this loss of marginal utility. As can be seen from eqn. (23), in equilibrium (where \( I - \delta K = 0 \)), the marginal benefit from the last unit of capital again has two components, the marginal utility coming from the marginal revenue product of the unit, and the expected gain in marginal utility resulting from reduced upward adjustments in the capital stock.

6. The Use of Inventories

Inventories, of course, provide another means by which a firm can respond to stochastic demand fluctuations, and depending on storage costs and the size of the fluctuations, drawing down or adding to inventories may be preferable to changing output capacity significantly. Since the availability of inventories is likely to reduce the extent to which the firm must continually adjust its output capacity in response to demand fluctuations, the extent to which the optimal equilibrium output capacity is altered by uncertainty should likewise be reduced. Here we show that while those alterations in equilibrium output are indeed reduced by the availability of inventories (especially if inventory holding costs are small relative to capacity adjustment costs), the alterations should still occur, and they will still have the same signs as they do without inventories.

To include the use of inventories in our model we consider a risk-neutral firm that produces at a rate \( X = F(L,K) \), and sells at a rate \( q \), so that \( X - q \) represents the rate of flow into the stock of inventories \( N \). Assuming that the firm faces a constant inventory storage cost of \( k \)

16 - The results are basically the same for a risk-averse firm, but the algebra is more complicated and no more enlightening.
per unit, discounted profits are:

$$\Pi_d(t) = [p(q, \theta)q - wL - vI - C(I) - kN]e^{-rt}$$  \hspace{1cm} (24)

The firm's maximization problem is now:

$$\max_{L, I, q} \mathbb{E}_0 \int_0^\infty \Pi_d(t) dt$$  \hspace{1cm} (25)

subject to

$$\dot{K} = I - \delta K$$  \hspace{1cm} (26)

$$\dot{N} = X - q$$  \hspace{1cm} (27)

and

$$d\theta = \sigma(\theta)dz = \sigma(\theta)\varepsilon(t)\sqrt{dt}$$  \hspace{1cm} (28)

with $L, I, q, N > 0$.

Note that the firm now chooses its level of sales $q$ in addition to its factor inputs (which in turn determine its output). Since price and quantity are related by the demand function and the market is assumed to clear, this is equivalent (for a monopolistic firm) to choosing price. Thus, the distinction made by Leland (1972) between a price-setting and a quantity-setting firm does not apply here.

The solution to this problem follows the same approach used in Section 4, and is presented in Appendix B. There we show that the expected dynamics of investment is again given by eqn. (21), and also that the expected rate of change of marginal revenue (price for the competitive firm) is given by

$$(1/dt)\mathbb{E}_t dMR \leq rMR + k$$  \hspace{1cm} (29)

with $N = 0$ if the inequality holds, and $N > 0$ if (29) holds with equality. Eqn. (29) is a standard arbitrage condition which says that if the expected capital gain from a unit of inventory exceeds the holding cost of the unit,
additional units should be added to inventory rather than sold (which in turn will raise the price and reduce the expected rate of growth of marginal revenue until (29) holds with equality). Conversely, no inventory should be held if its holding cost exceeds the expected rate of capital gain.

In the deterministic case no inventories will be held unless an increase in price is anticipated (e.g. inventories may be accumulated and later decumulated during a period of rapid demand growth), so that in equilibrium the inventory stock is always zero. With stochastic demand fluctuations, however, an inventory stock may be held even in equilibrium. We show in Appendix B that a non-zero mean inventory stock is held once the variance of $\theta$ exceeds a critical value. At that critical value the adjustment cost savings from the use of inventories as a buffer against demand fluctuations (equivalent to a rate of capital gain) is just equal to the holding cost of the inventory.

If inventories are held in equilibrium the optimal response of investment to stochastic changes in demand will be reduced, i.e. $I_\theta$ will be smaller (since inventories provide a partial substitute for capital stock adjustments). However, $I_\theta$ will still exceed zero, so that eqn. (21) will still have the same implications for the effect of uncertainty on the equilibrium capital stock and output level; if $C''(I) > (\leq) 0$, demand uncertainty will lead to an increase (decrease) in expected output (although the increase or decrease will be smaller than it would without inventories). Furthermore, it can be shown that the basic results from the last section regarding risk aversion still apply if the firm holds inventories, i.e. with $C'''(I) \geq 0$, risk aversion will lead to a higher level of output if there is demand uncertainty.
7. Conclusions

This paper has treated the behavior of the firm under demand uncertainty as a problem that is inherently dynamic, both in terms of the nature of the uncertainty itself, and in terms of the constraints faced by the firm. The uncertainty that we have been concerned with here is an uncertainty over future demand, not current demand, with the degree of uncertainty growing with the time horizon. As we have seen, this kind of uncertainty can be characterized by allowing the demand function to be driven over time by a stochastic process. Furthermore, the firm's optimal behavior is affected by this uncertainty because of the adjustment costs associated with changes in factor input levels, and therefore the effects of uncertainty must be studied in the context of a dynamic model of the firm - whether or not one assumes that the firm is permitted to carry inventories.

This approach is quite different from that of most earlier papers, and our results are also quite different. We have seen that the effects of demand uncertainty on the firm's behavior depend strongly on the characteristics of adjustment costs. In particular, we have shown that a risk-neutral firm will (in expected value terms) maintain a larger (smaller) capital stock and produce more (less) on average if marginal adjustment costs are rising at an increasing (decreasing) rate with the level of capital accumulation. In addition, demand uncertainty causes a risk-averse firm to produce more on average, even if marginal adjustment costs rise at a constant rate. Furthermore these results are quite robust, and hold whether the firm is competitive or monopolistic, whether or not the firm holds inventories, and whether uncertainty affects demand additively or multiplicatively.
Of course one might argue about the importance of the type of demand uncertainty considered in this paper. In practice most firms must indeed worry about uncertainty over current demand as well as future demand, and they may face stochastic fluctuations in demand that are not always continuous in time. However, at the very least future demand uncertainty should be as important to the firm as current uncertainty, and it is therefore important to understand how it is likely to affect the firm's behavior.

As for stochastic fluctuations that are discontinuous in time, a natural extension of this paper would be to allow the demand function to be driven by a "jump" process instead of (or in addition to) a continuous-time process.

An important limitation of this paper is its assumption that markets clear instantaneously, so that there is never any excess demand or supply of the good. Several recent papers have examined the characteristics of market equilibrium when consumers face some probability of being unable to purchase the good, but these papers do not consider uncertainty (current or future) over the demand function itself.\(^{17}\) It would be useful to extend the model developed here to allow for market clearing that is slow or incomplete.

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Appendix

A. Behavior of the Risk-Averse Firm

Here we show that if the firm maximizes a quadratic utility function \( U'' = 0 \) and if \( p \) is linear in \( \theta \), eqn. (23) defines the \((l/dt)E_t dI = 0\) isocline for the risk-averse firm. First, defining the optimal value function

\[
J(K,\theta,t) = \max_{L,I} E_t \int_t^\infty U[I(\tau)]e^{-\tau t} d\tau,
\]

(A.1)

and then going through the same steps as in Section 4, it is easy to show that eqn. (17) now becomes

\[
(l/dt)E_t (\partial U_\xi/\partial I) = \partial U_\xi/\partial K + \partial U_\xi/\partial I
\]

(A.2)

with \( U_\xi = U(\Pi)e^{-rt} \). Taking the partial derivatives and substituting into (A.2), we have

\[
rU'(\Pi)[v + C'(I)] - v(l/dt)E_t dU'(\Pi) - (l/dt)E_t d[U'(\Pi)C'(I)]
\]

\[
= U'(\Pi)MRF_K - U'(\Pi)\delta[v + C'(I)]
\]

(A.3)

Next the differentials \( dU'(\Pi) \) and \( d[U'(\Pi)C'(I)] \) must be expanded using Ito's Lemma:

\[
dU' = U''MRF_K(I - \delta K)dt - U''[v + C']dI - \frac{1}{2}U''''(dI)^2
\]

(A.4)

and

\[
d(U'C') = C''MRF_K(I - \delta K)dt - [C''''(v + C') - U'C''']dI
\]

\[
- \left[ \frac{1}{2}C'C''u + U''(v + C') - \frac{1}{2}U'''' \right](dI)^2
\]

(A.5)

Noting that \( E_t(dI)^2 = \sigma^2(\theta)I_0^2 dt \), substituting (A.4) and (A.5) into (A.3) and rearranging, we obtain the equation (analogous to eqn.(21)) describing the expected dynamics of investment:
\[
[U'C'' - (v + C')^2U'](1/dt)E_t dI = (x + \delta)U'(v'' C') - U'MR*F_K
- U'(v + C')MR*F_K I - \delta K - \frac{1}{2} \sigma^2(\theta)I_\theta^2[U'C'' - 3C''(v + C')]
\]

Eqn. (A.6)

Eqn. (23) follows from setting \((1/dt)E_t dI = 0\).

**B. The Use of Inventories**

It is easy to show that eqns. (21) and (29) hold when the firm's maximization problem is given by eqns. (24) to (28). Defining the optimal value function \(J = J(K,N,\theta,t)\) as before, the fundamental equation of optimality is:

\[
0 = \max_{L,I,q} \left[ \Pi_d(t) + J_t + (I - \delta K)J_K + (X - q)J_N + \frac{1}{2} \sigma^2(\theta)J_\theta^2 \right]
\]

(B.1)

Maximizing with respect to \(L\) gives \(w = MR \cdot F_L\) as before. To see that eqn. (21) again holds, go through exactly the same steps as in Section 4. Eqn. (17) will again hold, and substituting for \(\partial \Pi_d / \partial K\) and \(\partial \Pi_d / \partial I\) and expanding \(dC'(I)\) using Ito's Lemma again leads to eqn. (21) (although \(I_\theta\) will be smaller in magnitude if an inventory stock is maintained in equilibrium).

To obtain eqn. (29), maximize (B.1) with respect to \(q\):

\[
\partial \Pi_d / \partial q = J_N
\]

(B.2)

i.e. the marginal profit from selling a unit of output should just equal the sum of all expected future discounted increases in profit resulting from adding the unit to inventory. Now differentiate eqn. (B.1) with respect to \(N\) and note that the resulting equation can be written as:
\[
\frac{\partial \Pi}{\partial N} + \left( \frac{1}{dt} \right) E_t d(J_N) = 0 \tag{B.3}
\]

Apply the differential operator \( \left( \frac{1}{dt} \right) E_t d(\cdot) \) to both sides of (B.2) and combine the resulting equation with (B.3) to eliminate \( J_N \):

\[
\left( \frac{1}{dt} \right) E_t d\left( \frac{\partial \Pi}{\partial q} \right) + \frac{\partial \Pi}{\partial N} = 0 \tag{B.4}
\]

Now simply substitute the partial derivatives of \( \Pi_t \) into (B.4) to obtain eqn. (29), and note that the inequality results from the constraint that \( N \geq 0 \).

In the deterministic case the firm will hold no inventories in equilibrium (since demand, and hence price, are fixed). However a firm facing stochastic demand fluctuations may hold inventories in equilibrium if the variance of the fluctuations is large enough. To see this, consider a competitive firm and assume for simplicity that \( d^i C(I)/dI^i = 0 \) for \( i \geq 4 \). Now expand \( p = p(q, \theta) \) using Itô's Lemma to obtain

\[
\frac{1}{dt} E_t dp = \left( \frac{\partial p}{\partial q} \right) \frac{1}{dt} E_t dq + \frac{1}{2} \sigma^2(\theta) \frac{\partial^2 p}{\partial \theta^2} \tag{B.5}
\]

But \( q = q^*(K, N, \theta) \), so that

\[
\left( \frac{1}{dt} \right) E_t dq = F_K (I - \delta K) + \frac{1}{2} \sigma^2(\theta) q_{\theta \theta} \tag{B.6}
\]

Now substitute (B.5) and (B.6) into (29), and note that \( I = \delta K \) in equilibrium:

\[
\frac{1}{2} \sigma^2(\theta) \left[ q_{\theta \theta} (\partial p/\partial q) + \partial^2 p/\partial \theta^2 \right] \leq rp + k \tag{B.7}
\]

\( N > 0 \) only if (B.7) holds with equality, and this in turn provides the minimum value of \( \sigma^2(\theta) \) above which inventories will be held in equilibrium. 18

18 - Unfortunately we could not obtain an analytical expression for \( q_{\theta \theta} \) (we only know that \( q_{\theta \theta} < 0 \) since \( C''(I) > 0 \)), so that (B.7) really only says that a particular minimum value of \( \sigma^2 \) exists, and that it depends positively on \( r, p, \) and \( k \) and inversely on the terms in the brackets.
References


