Hidden Minimum-Norm Problems in Quadratic Programming

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Abstract: This note presents sufficient conditions for a convex quadratic program, or its dual, to be transformed into a minimum (Euclidean) norm problem, i.e. a problem of minimizing the norm of a linear transformation of an element of a polytope. These sufficient conditions are shown to be necessary under a suitable restriction on the class of transformations that are allowed. As part of the sufficient conditions, we characterize when the two linear inequality systems $Ax \geq b$ and $Az \leq b$ have simultaneous solutions. These results are used in conjunction with duality constructions to obtain two equivalent reformulations of a given quadratic program.

Key Words: Quadratic program, norm, minimum norm problem, gauge program.

Running Header: Hidden Norm Problems.

A minimum (Euclidean) norm problem over a polytope is a program of the form:

$$\text{NP:} \quad \underset{x}{\text{minimize}} \quad \|Cx + d\|$$

$$\text{subject to:} \quad Ex \geq f$$

where $\|\cdot\|$ denotes the Euclidean (l_2) norm.

The purpose of this note is to explore answers to and implications of the following question: when can a convex quadratic program or its dual be conveniently transformed into a minimum (Euclidean) norm program?

The standard convex quadratic program is given as

$$\text{QP:} \quad \underset{x}{\text{minimize}} \quad 1/2\ x^tQx + q^tx$$

$$\text{subject to:} \quad Ax \geq b \ ,$$

where Q is assumed to be symmetric and positive semi-definite. The standard dual of QP, see Dorn [1], is given by

$$\text{QD:} \quad \underset{y,\lambda}{\text{maximize}} \quad -1/2\ y^tQy + b^t\lambda$$

$$\text{subject to:} \quad -Qy + A^t\lambda = q$$

$$\lambda \geq 0$$

The concern herein lies in discovering properties of the problem data (Q,q,A,b) that allow the objective function in QP or QD to be replaced by a norm, thus transforming QP or QD into an instance of NP.

The rationale for exploring transformations of a quadratic program to the minimum norm problem is threefold. First, the minimum norm problem has an immediate geometric interpretation that may be useful in the analysis and the solution of a given quadratic program. Second, the minimum norm problem is a classical optimization problem, and has received extensive study, see for example, Luenberger [5]. Third, the author has recently investigated an alternate duality theory for the minimum norm problem [2], that is applicable to quadratic programs that can be so transformed.

If the matrix Q is symmetric and positive semi-definite, then Q can be written as $Q = M^tM$ for some matrix M; efficient procedures for constructing M are well-known, see e.g. Gill, Murray, and Wright [3].

<u>Proposition 1</u>. If the system of linear inequalities (1):

$$A^t\pi + Qs = q \qquad (1.1)$$

$$A^t\delta - Qs = -q \qquad (1.2)$$

$$b^t\pi + b^t\delta \geq 0 \qquad (1.3)$$

$$\pi \geq 0,\ \delta \geq 0, \qquad (1.4)$$

has a solution $(\bar{\pi}, \bar{\delta}, \bar{s})$, then the program QP is equivalent to the minimum norm problem

$$\text{NP1:} \quad \underset{x}{\text{minimize}} \quad \|Mx + M\bar{s}\|$$

$$\text{subject to:} \quad Ax \geq b,$$

where M is any matrix for which $M^tM = Q$.

<u>Proof</u>: For any x satisfying $Ax \geq b$, $1/2\ x^tQx + q^tx =$ $1/2\ x^tM^tMx + q^tx = 1/2\ x^tM^tMx + \bar{\pi}^tAx + \bar{s}^tM^tMx = 1/2\ \|Mx + M\bar{s}\|^2$ $- 1/2\ \bar{s}^tM^tM\bar{s} + \bar{\pi}^tAx \geq 1/2\ \|Mx + M\bar{s}\|^2 - 1/2\ \bar{s}^tM^tM\bar{s} + b^t\bar{\pi}$,

and similarly, using (1.2), $1/2\ x^tQx + q^tx \leq 1/2\ \|Mx + M\bar{s}\|^2 - 1/2\ \bar{s}^tM^tM\bar{s} - b^t\bar{\delta}$. But since $b^t\bar{\pi} \geq -b^t\bar{\delta}$, equality is obtained throughout, and $1/2\ x^tQx + q^tx =$

$1/2 \|Mx + M\bar{s}\|^2 - 1/2\ \bar{s}^t M^t M\bar{s} + b^t\bar{\pi}$. This expression is strictly increasing in $\|Mx + M\bar{s}\|$, and so the quadratic objective function can be replaced by $\|Mx + M\bar{s}\|$. [X]

Remark 1. If q lies in the column space of Q, i.e., $Qs = q$ has a solution, then (0, 0, s) solves (1). In particular, If Q is positive definite, $(\pi, \delta, s) = (0, 0, Q^{-1}q)$ solves (1).

Remark 2. For any solution $(\bar{\pi}, \bar{\delta}, \bar{s})$ to (1), $b^t\bar{\pi} = -b^t\bar{\delta}$, and any x that solves $Ax \geq b$ will satisfy $\bar{\pi}^tAx = \bar{\pi}^tb$ and $\bar{\delta}^tAx = \bar{\delta}^tb$. In particular, any index i, $1 \leq i \leq m$, for which $\bar{\pi}_i > 0$ or $\bar{\delta}_i > 0$ will be an always-active constraint of the system $Ax \geq b$.

Proposition 1 provides sufficient conditions for QP to be transformed into a minimum norm problem. Before turning to the question of necessary conditions, we first introduce some terminology. Given the matrix Q of the program QP, the norm $\|Cx + d\|$ is said to be derived from Q if $C^tC = Q$. A norm $\|Cx + d\|$ is said to be monotonically transformable in the objective function of QP if for any feasible points $\bar{x}$, $\tilde{x}$ of QP. $\|C\bar{x} + d\| > \|C\tilde{x} + d\|$ (respectively, $\geq$) if and only if $1/2\ \bar{x}^tQ\bar{x} + q^t\bar{x} > 1/2\ \tilde{x}^tQ\tilde{x} + q^t\tilde{x}$ (respectively, $\geq$).

Proposition 2. If QP is feasible, then QP can be monotonically transformed to a minimum norm problem with objective function $\|Cx + d\|$ derived from Q only if the system of linear inequalities (1) has a solution.

Proof: Suppose that system (1) has no solution. Then, by a theorem of the alternative, there exist vectors $\bar{x}$, $\tilde{x}$ and a nonnegative scalar $\bar{\theta}$ that satisfies:

$$A\bar{x} \geq b\bar{\theta}$$

$$A\tilde{x} \geq b\bar{\theta}$$

$$Q\bar{x} = Q\tilde{x}$$

$$q^t\bar{x} > q^t\tilde{x}$$

There are two cases to consider, depending on whether $\bar{\theta}$ is positive or zero.

<u>Case 1</u>: $\bar{\theta} > 0$. Without loss of generality, we can assume that $\bar{\theta} = 1$, and so $\bar{x}$ and $\tilde{x}$ satisfy:

$$A\bar{x} \geq b$$

$$A\tilde{x} \geq b$$

$$Q\bar{x} = Q\tilde{x}$$

$$q^t\bar{x} > q^t\tilde{x} \quad .$$

If there exists C and d such that $\|Cx + d\|$ is a strictly monotonic transformation of the objective function of QP and is derived from Q, then $\|C\bar{x} + d\| > \|C\tilde{x} + d\|$. But $Q\bar{x} = Q\tilde{x}$ and $Q = C^tC$ implies that $C\bar{x} = C\tilde{x}$, and so $\|C\bar{x} + d\| = \|C\tilde{x} + d\|$, a contradiction. Thus no such norm can be found.

<u>Case 2</u>: $\bar{\theta} = 0$. Because QP is feasible, there exists x' that satisfies $Ax' \geq b$. Therefore $(x' + \bar{x})$ and $(x' + \tilde{x})$ satisfy:

$$A(x' + \bar{x}) \geq b$$

$$A(x' + \tilde{x}) \geq b$$

$$Q(x' + \bar{x}) = Q(x + \tilde{x})$$

$$q^t(x' + \bar{x}) > q^t(x' + \tilde{x}) \quad ,$$

and the proof follows that of case 1, with $\bar{x}$ replaced by $(x' + \bar{x})$ and $\tilde{x}$ replaced by $(x' + \tilde{x})$. [X]

Turning to the dual quadratic program QD, our first result is:

Proposition 3. If the system of linear inequalities (2):

$$Av \geq b \qquad (2.1)$$

$$Az \leq b \qquad (2.2)$$

$$Qv - Qz = 0 \qquad (2.3)$$

$$q^t v - q^t z = 0 \ , \qquad (2.4)$$

has a solution $(\bar{v}, \bar{z})$, then the dual quadratic program QD is equivalent to the minimum norm problem

$$\text{NP2:} \quad \underset{y,\lambda}{\text{minimize}} \quad \|My - M\bar{z}\|$$

$$\text{subject to:} \quad -Qy + A^t\lambda = q$$

$$\lambda \geq 0 \ ,$$

where M is any matrix for which $Q = M^tM$.

Proof: For any (y, λ) that is feasible for QD, $-1/2\ y^tQy + b^t\lambda \leq -1/2\ y^tQy + \bar{v}^tA^t\lambda = -1/2\ y^tM^tMy + q^t\bar{v} + \bar{v}^tQy = -1/2\ \|My - M\bar{v}\|^2 + q^t\bar{v} + 1/2\ \bar{v}^tQ\bar{v}$. Similarly, using (2.2), we obtain $-1/2\ y^tQy + b^t\lambda \leq -1/2\ \|My - M\bar{z}\|^2 + q^t\bar{z} + 1/2\ \bar{z}^tQ\bar{z}$. However, $Q\bar{z} = Q\bar{v}$ implies that $M\bar{z} = M\bar{v}$, and combining this relation with $q^t\bar{z} = q^t\bar{v}$, we have $-1/2\ y^tQy + b^t\lambda = -1/2\ \|My - M\bar{z}\|^2 + q^t\bar{z} + 1/2\ \bar{z}^tQ\bar{z}$. This expression is strictly decreasing in $\|My - M\bar{z}\|$, and so we can replace the maximand by the minimand $\|My - M\bar{z}\|$. [X]

Remark 3. Proposition 3 is structurally the same as Proposition 1, applied to the dual. Proposition 3 is obtained by rewriting the dual QD in the format of the primal and then applying Proposition 1. In this sense, the two propositions are the same.

In order to prove a result about necessary conditions for QD to be transformed into a minimum norm problem, our notation must be amended. Given the matrix Q of the dual quadratic program QD, the norm $\|Cy + E\lambda + d\|$ is said to be derived from Q if $C^tC = Q$ and $E = 0$. A norm $\|Cy + E\lambda + d\|$ is said to be monotonically transformable in the objective function of QD if for any feasible points $(\bar{y}, \bar{\lambda})$, $(\tilde{y}, \tilde{\lambda})$ of QD, $\|C\bar{y} + E\bar{\lambda} + d\| > \|C\tilde{y} + E\tilde{\lambda} + d\|$ (respectively, $\geq$) if and only if $- 1/2\ \bar{y}^tQ\bar{y} + b^t\bar{\lambda} < 1/2\ \tilde{y}^tQ\tilde{y} + b^t\tilde{\lambda}$ (respectively, $\leq$).

Analogous to Proposition 2, we have:

Proposition 4. If QD is feasible, then QD can be monotonically transformed to a minimum norm problem with objective function $\|Cy + E\lambda + d\|$ derived from Q only if the system of linear inequalities (2) has a solution.

Proof: The proof exactly parallels that of Propostion 2. If QD is feasible, then if system (2) has no solution, there must exist vectors $\bar{\lambda}$, $\tilde{\lambda}$, $\bar{y}$ that satisfy

$$
\begin{aligned}
- Q\bar{y} + A^t\bar{\lambda} &= q \\
- Q\bar{y} + A^t\tilde{\lambda} &= q \\
\bar{\lambda} &\geq 0 \\
\tilde{\lambda} &\geq 0 \\
b^t\bar{\lambda} &> b^t\tilde{\lambda} .
\end{aligned}
$$

Thus if $\|Cy + E\lambda + d\|$ is strictly monotonic in the objective function of QD, and is derived from Q, then $\|C\bar{y} + E\bar{\lambda} + d\| < \|C\bar{y} + E\tilde{\lambda} + d\|$, which is clearly a contradiction, because $E = 0$. Thus no such norm can be found. [X]

Proposition 2 (and in the dual, Proposition 4) shows that the solvability of the system of linear inequalities (1) is necessary for QP to be transformed to the minimum norm program, provided that the transformation is restricted by the monotonicity condition and the condition that the norm be derived from Q. When these restrictions are relaxed, the solvability of the system (1) is no longer the necessary condition, as the following example shows. Let

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Then the inequalities (1) have no solution. However, this instance of QP is monotonically transformable to the minimum norm program with objective function $\|Cx + d\|$, where

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} -8 \\ -8 \end{bmatrix}.$$

To see this, note that for any x feasible in QP, $q^t x \geq 8$, and indeed, the optimal solution is $x^* = (3,5)^t$, with $q^t x^* = 8$. Also,

$$\|Cx + d\| = \sqrt{2(q^t x - 8)(q^t x - 8)} \quad ,$$

which is strictly increasing in $q^t x$ for $q^t x \geq 8$. Thus $\|Cx + d\|$ is monotonically transformable from QP, even though system (1) has no solution. However, C is not derived from Q in this example. The key to the above transformation was the judicious choice of d, based on a known lower bound on the optimal value of QP.

If the montonicity condition is relaxed, then prior knowledge of the set of optimal solutions to QP allows us to write any instance of QP as a minimum norm program. For example, if the optimal solution x^* of QP is unique (which it can be even if Q=0, i.e. QP is just a linear program), then QP is equivalent to the minimum norm program:

$$\begin{aligned} &\underset{x}{\text{minimize}} && \|x-x^*\| \\ &\text{subject to:} && Ax \geq b \quad . \end{aligned}$$

This transformation appears somewhat pointless in that the quadratic program has been solved before the transformation is even made. Nevertheless, the transformation can be accomplished in polynomial time, because linear and convex quadratic programming are solvable in polynomial time [4]. The question of necessary conditions for a quadratic program to be transformed into a norm program thus clearly depends on the class of transformations that are allowed.

Note that the pair of inequalities (2.1) and (2.2) of Proposition 3 are described by reflecting the halfspaces defined by the feasible region of QP. A curious issue raised in light of Proposition 3 is the simultaneous solvability of the system $Ax \geq b$ and $Az \leq b$. This issue is treated in the next proposition. Let $\text{aff}(\chi)$ and $\text{rec}(\chi)$ denote, respectively, the affine hull and the recession cone of a set χ, see Rockafellar [6]. A constraint $A_j x \geq b_j$ in the system $Ax \geq b$ is said to be <u>parallel redundant</u> if the constraint is redundant and $A_j x = d \geq b_j$ for every x that satisfies $Ax \geq b$, i.e., $A_j x$ is constant for every x satisfying $Ax \geq b$. The following proposition characterizes when the system $Ax \geq b$ and $Az \leq b$ both have a solution.

<u>Proposition 5</u>. Let $\chi = \{x \in R^n | Ax \geq b\}$ and $Z = \{z \in R^n | Az \leq b\}$, and furthermore assume that $\chi \neq \phi$ and that no constraint of the system $Ax \geq b$ is parallel redundant. Then $Z \neq \phi$ if and only if $\dim(\text{rec}(\chi)) = \dim(\chi)$.

Proof: Suppose $Z \neq \phi$. Then there exists $\bar{x}$ and $\bar{z}$ for which $A\bar{x} \geq b$ and $A\bar{z} \leq b$. Let $k = \dim(\chi)$, and note that $k \geq 0$. If $k=0$, then since $0 \in \text{rec}(\chi)$, $\dim(\text{rec}(\chi)) \geq 0 = k$. However, since $\dim(\text{rec}(\chi)) \leq \dim(\chi)$, equality must hold, i.e., $\dim(\text{rec}(\chi)) = \dim(\chi)$. If $k > 0$, there exist vectors $x^1,\ldots,x^k$, all elements of χ, for which $\dim \text{aff}\{\bar{x}, x^1,\ldots,x^k\} = k$. Therefore $\dim \text{aff}\{\bar{x}, x^1,\ldots,x^k, \bar{z}\} \geq k$, and so $\{x^1-\bar{z},\ldots,x^k-\bar{z}, \bar{x}-\bar{z}\}$ has k linearly independent elements. But $A(x^1-\bar{z}) \geq 0,\ldots,A(x^k-\bar{z}) \geq 0$, $A(\bar{x}-\bar{z}) \geq 0$, i.e. $\{x^1-\bar{z},\ldots,x^k-\bar{z}, \bar{x}-\bar{z}\}$ are elements of $\text{rec}(\chi)$. Thus $\dim \text{rec}(\chi) \geq k$, and hence equal to k, because $\dim(\text{rec}(\chi)) \leq \dim(\chi)$.

Conversely, suppose $Z = \phi$, and assume that $\dim(\text{rec}(\chi)) = \dim(\chi) = k \geq 0$. If the constraint matrix A has m rows, then the constraint index set $M = \{1,\ldots,m\}$ can be partitioned into disjoint sets α and β, where $\alpha \cup \beta = M$, such that $A_\alpha x = b_\alpha$ for every $x \in \chi$, and there exists an element $\bar{x}$ of χ that satisfies $A_\beta \bar{x} > b_\beta$ ($\bar{x}$ is any element of the relative interior of χ). Furthermore, rank $(A_\alpha) = n-k$. If $k=0$, χ is a singleton $\{\bar{x}\}$ and every index $j \in \beta$ is a parallel redundant constraint, whereby $\beta=\phi$. Thus $A\bar{x} = b$, and so $\bar{x} \in Z$, contradicting $Z=\phi$. Therefore, $\dim(\text{rec}(\chi)) < \dim(\chi)$. If $k > 0$, there exists linearly independent vectors $x^1,\ldots,x^k$ in $\text{rec}(\chi)$, i.e. that satisfy $Ax^i \geq 0$, $i=1,\ldots,k$, and $A_\alpha x^i = 0$, and $\text{aff}\{x^1,\ldots,x^k,0\} = \text{aff}(\text{rec}(\chi)) = \{x \in R^n | A_\alpha x = b_\alpha\}$. If there exists an index $j \in \beta$ for which $A_j x^i = 0$, $i=1,\ldots,k$, then A_j must be a nonnegative combination of the rows of A_α, i.e. there exists $\lambda_\alpha \geq 0$ for which $\lambda_\alpha^t A_\alpha = A_j$. Thus $\lambda_\alpha^t b_\alpha \geq b_j$, and $A_j x \geq b_j$ is a parallel redundant constraint, violating the hypothesis of the proposition. Thus $\dim(\text{rec}(\chi)) < \dim(\chi)$. [X]

It is curious that the solvability of $Ax \geq b$ and $Az \leq b$ in Proposition 5 can be characterized in terms of the dimension of χ and $\text{rec}(\chi)$, as opposed to a characterization in terms of dual multipliers via a theorem of the alternative.

As the following examples show, the primal, dual, neither, or both programs will satisfy the linear inequalities (1) and/or (2).

<u>Example 1</u> $$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 3 \\ -10 \end{bmatrix}.$$

In this example $(\pi, \delta, s) = (0, 0, 0)$ solves (1), whereby QP is a minimum norm program. However, $Az \leq b$ has no solution, and so the transformation of the dual QD by Proposition 3 cannot be accomplished.

<u>Example 2</u> $$Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

In this example, $\pi = 0$, $\delta = 0$, $s = (1, 0)^t$ solves (1), and $x = (1, 1)^t$, $z = (-1, -1)^t$ solves (2); therefore both QP and QD are transformable to norm programs.

<u>Example 3</u> $$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

In this example, it is straightforward to show that neither (1) nor (2) have a solution, and so the transformations of Propositions 1 and 3 cannot be applied.

An Application of Hidden Norms in Gauge Duality

One motivation for the study of hidden minimum norm programs in quadratic programming is the author's recent investigation of dual gauge programs [2], of which the minimum norm problem is a specific case. If QP can be converted (through Proposition 1) to the minimum norm problem NP1, then either the standard (Lagrange) dual or the gauge dual of NP1 can be constructed.

The gauge dual of NP1, see [2], is given by

$$\begin{aligned} \text{GNP1:} \quad & \underset{h,\lambda}{\text{minimize}} \quad \|h\|_2 \\ & \text{subject to:} \quad M^t h - A^t\lambda = 0 \\ & \qquad (\bar{s}^t A^t + b^t)\lambda = 1 \\ & \qquad \lambda \geq 0 \quad . \end{aligned}$$

Replacing the objective function by $1/2\ h^t h$ and then taking the standard (Lagrange) dual yields:

$$\begin{aligned} \overline{QP}: \quad & \underset{f,\theta}{\text{minimize}} \quad 1/2\ f^t Q f + \theta \bar{s}^t Q f + 1/2\ \bar{s}^t Q \bar{s} \theta^2 - \theta \\ & \text{subject to:} \quad Af - b\theta \geq 0. \end{aligned}$$

Because $\overline{QP}$ is obtained from QP by two consecutive duality constructions (first the gauge dual, then the Lagrange dual), we should expect QP and $\overline{QP}$ to be equivalent. The feasible region of $\overline{QP}$ is obtained by adding the extra variable θ that scales the right-hand-side b, and converts the constraints $Ax \geq b$ to the homogeneous system $Af - b\theta \geq 0$, replacing the polytope of QP by a polyhedral cone in one higher dimension. The sense of equivalence of QP and $\overline{QP}$ is shown in the next proposition, which shows how optimal solutions of QP and $\overline{QP}$ transform one to the other.

Proposition 6. Let $(\bar{\pi}, \bar{\delta}, \bar{s})$ be a solution to (1). Let $\bar{x}$ be an optimal solution to QP with optimal Karush-Kuhn-Tucker (K-K-T) multipliers $\bar{\eta}$, and define $\bar{t} = \bar{s}^tQ\bar{x} + \bar{s}^tQ\bar{s} + b^t\bar{\eta} + b^t\bar{\delta}$. Then

(i) if $\bar{t} \neq 0$, $(\bar{f}, \bar{\theta}) = (\bar{x}/\bar{t}, 1/\bar{t})$ solves $\overline{QP}$ with optimal K-K-T multipliers $\bar{\varepsilon} = (\bar{\eta} + \bar{\delta})/\bar{t}$.

(ii) if $\bar{t} = 0$, $Ax \geq b$, $Qx + Q\bar{s} = 0$ has a solution, and $\overline{QP}$ is unbounded.

Let $(\bar{f}, \bar{\theta})$ be an optimal solution to $\overline{QP}$ with optimal K-K-T multipliers $\bar{\varepsilon}$. Then:

(i) if $\bar{\theta} \neq 0$, $\bar{x} = \bar{f}/\bar{\theta}$ solves QP with optimal K-K-T multipliers $\bar{\eta} = \bar{\varepsilon}/\bar{\theta} + \bar{\delta}$.

(ii) if $\bar{\theta} = 0$, then QP is infeasible. [X]

The proof of this proposition follows from an examination of the K-K-T conditions and from direct substitution of the indicated transformations.

The program $\overline{QP}$ was obtained by taking the gauge dual of QP, followed by the standard (Lagrange) dual. If $(\bar{v}, \bar{z})$ solves (2), then the (Lagrange) dual QD is transformable to a minimum norm problem, and the order of the dualization can be reversed. Starting with the standard quadratic program dual QD, and converting QD to the minimum norm problem NP2, and then taking the gauge dual of NP2 yields:

$\overline{\overline{QP}}$: minimize$_w$ $\quad 1/2\ w^tQw$

subject to: $\quad Aw \geq 0 \qquad (\ell)$

$\quad (-q^t - \bar{z}^tQ)w = 1 \qquad (r)$

The feasible region of $\overline{\overline{QP}}$ is composed of the intersection of the recession cone of the feasible region of QP (described by the homogeneous constraints $Aw \geq 0$) with a hyperplane (described by

$(-q^t - \bar{z}^t Q)\, w = 1)$ that scales elements of the recession cone.

Analagous to Proposition 6, we have:

Proposition 7. Let $(\bar{v}, \bar{z})$ be a solution to (2). Let $\bar{x}$ be an optimal solution to QP with optimal K-K-T multipliers $\bar{\eta}$, and define $\bar{u} = -q^t\bar{x} + q^t\bar{z} - \bar{z}^tQ\bar{x} + \bar{z}^tQ\bar{z}$. Then

(i) if $\bar{u} \neq 0$, $\bar{w} = (\bar{x} - \bar{z})/\bar{u}$ solves $\overline{\overline{QP}}$ with optimal K-K-T multipliers $\bar{\ell} = \bar{\eta}/\bar{u}$, $\bar{r} = 1/\bar{u}$.

(ii) if $\bar{u} = 0$, $\overline{\overline{QP}}$ is infeasible.

Let $\bar{w}$ be an optimal solution to $\overline{\overline{QP}}$ with optimal K-K-T multipliers $\bar{\ell}$, $\bar{r}$. Then

(i) if $\bar{r} \neq 0$, $\bar{x} = \bar{w}/\bar{r} + \bar{v}$ solves QP with optimal K-K-T multipliers $\bar{\eta} = \bar{\ell}/\bar{r}$.

(ii) if $\bar{r} = 0$, QP is unbounded. [X]

It is hoped that the equivalences of QP to the programs $\overline{QP}$ and $\overline{\overline{QP}}$ given herein will be useful in applications of quadratic programming.

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