FUNDAMENTAL SOLUTIONS OF INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES

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1. Introduction and notation. Let $S$ be a Riemannian globally symmetric space, $G$ the largest connected group of isometries of $S$ in the compact open topology. We assume that $S$ is of the noncompact type, that is, $G$ is semisimple and has no compact normal subgroup $\not\{e\}$. Let $o$ be any point in $S$, $K$ the isotropy subgroup of $G$ at $o$, $f$ and $g$ their respective Lie algebras, and $p$ the orthogonal complement of $f$ in $g$ with respect to the Killing form $B$ of $g$. Let $a$ be any maximal abelian subspace of $p$ and let $A = \exp(a)$. For each $X$ in the dual space of $a$ (which we identify with $a$, via $B$) let $g_X = \{X \in g \mid [H, X] = \lambda(H)X \text{ for all } H \in a\}$. Let $d_\lambda = \dim(g_X)$. Choose some order on $a$ and let

$$
\pi = \prod_{\lambda > 0} \lambda^{d_\lambda},
$$

and let $\pi$ denote the product of the distinct prime factors in $\pi'$. Then we have the Iwasawa decompositions $g = f + a + n$, $G = KAN$ where $N$ is the nilpotent group $\exp(n)$. Given $g \in G$, let $H(g)$ denote the unique element in $a$ for which $g \in K \exp H(g)N$. Let $W$ denote the Weyl group $M'/M$ where $M$ and $M'$, respectively, denote the centralizer and normalizer of $a$ in $K$.

For each $\lambda \in a$ consider the spherical function

$$
\phi_\lambda(x) = \int_K e^{(i\lambda - \rho)(H(xk))}dk \quad (x \in G)
$$

$dK$ being the normalized Haar measure on $K$. Let $c(\lambda)$ denote Harish-Chandra's function on $a$ which occurs in the leading term of the asymptotic expansion of $\phi_\lambda$ [2, p. 283], i.e.,

$$
\phi_\lambda(\exp H) \sim \sum_{s \in W} c(s\lambda)e^{(i\lambda - \rho)(H)}
$$

where $\lambda$ and $H$ are suitably restricted in $a$.

Each $x \in G$ can be written uniquely in the form $x = k \exp X(k \in K, X \in p)$. We put $|X| = (B(X, X))^{1/2}$ and $\omega(X) = \det \sinh ad X/ad X_p^{1/2}$ where the subscript $p$ indicates restriction to $p$ of the linear transformation of $\sinh

Let $D(G)$ and $D(S)$ denote the differential operators over $a, S$, respectively, decreasing together. Note the set of $\tau$ isomorphic to a closed subspace [2, Theorem 1, p. 586], $I_0(G) \subset I_0(A)$. Let $\phi_\alpha(f)$ denote

$$
\phi_\alpha(f) = \int_a f(\alpha a)\phi_\alpha(a) \alpha^{-1} d\alpha
$$

where $\alpha \in S(a)$ an $\alpha^{-1}$ and $I_0(G)$ denote the set of regular elements on $A$. We define $I_0(A)$ by convolution over $I_0(G)$. The $I_0(A)$ are isomorphic.

LEMMA 1. $I_0(G) \subset I_0(A)$

Under the rest isomorphic conditions on $A$. We can say over $I_0(A)$ by convolution 2. Transmutation of functions on $A$. [2, p. 265] there is a set of regular elements $f$. The $\omega$ with the radial $(D_r = d/dr)$.
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Riemannian globally of isometries of is the noncompact normal subgroup of \( G \). Let \( a \) be any \( \lambda \) in the

\[ \prod_{\lambda \neq 0} \lambda^d \lambda \]

prime factors in \( \pi' \).

\[ X^a + n, G = KAN \]
\( \simeq G \), let \( H(g) \) denote \( J \). Let \( W \) denote the

\( \mathbb{C}(\Lambda) \) denote Harish-Chandra term of the

\[ x = k \exp X(k \in K, \Lambda) = \{ \text{det} (\sinh ad X) \text{ of the linear} \]

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\text{transformation of } g \text{ given by}

\[ \sinh \text{ad } X/\text{ad } X = \sum_{n \geq 0} (\text{ad } X)^{2n}/(2n + 1)! \]

Let \( D(G) \) and \( D(S) \) denote the set of left invariant (resp. \( G \)-invariant) differential operators on \( G \) (resp. \( S \)). Let \( \mathcal{S}(a) \) denote the symmetric algebra over \( a \), \( \mathcal{S}(a) \) the space of \( C^\infty \) functions on \( a \) which are rapidly decreasing together with all their derivatives. Let \( I(a) \) and \( \mathcal{S}(a) \) denote the set of \( W \)-invariants in \( \mathcal{S}(a) \) and \( \mathcal{S}(a) \), respectively; \( \mathcal{S}(a) \) is taken with its usual locally convex topology [7, p. 90] and then \( \mathcal{S}(a) \) is a closed subspace. According to a theorem of Harish-Chandra (cf. [2, Theorem 1, p. 260], also [3, p. 432]) there exists an isomorphism \( \Gamma \) of the algebra \( D(S) \) onto \( I(a) \). Let \( I(G) \) denote the set of \( C^\infty \) functions \( f \) on \( G \) which are bi-invariant under \( K \) and for each integer \( q \geq 0 \) and each \( D \in D(G) \) satisfy

\[ \tau_{D,a}(f) = \sup_{H \in a} (1 + |H|)^{q} \omega(H) \frac{Df}{2n} (\exp H) | < \infty. \]

Let \( I_0(G) \) denote the space of functions of the form

\[ \phi_a(x) = \int_{a} \pi(\lambda) a(\lambda) \phi_{a}(x) d\lambda \quad (x \in G) \]

where \( a \in \mathcal{S}(a) \) and \( d\lambda \) is the Euclidean measure on \( a \). Then, by [2, p. 586], \( I_0(G) \subset I(G) \) (it can be shown using [1] that \( \pi = \pi_0 \)). The seminorms \( \tau_{D,a} \) turn \( I_0(G) \) and \( I(G) \) into locally convex spaces.

Lemma 1. \( I_0(G) \) is an algebra under convolution on \( G \).

Under the restriction from \( G \) to \( A \), \( I_0(G) \) and \( I(G) \) are mapped isomorphically onto spaces \( I_0(A) \) and \( I(A) \) of \( W \)-invariant \( C^\infty \) functions on \( A \). We carry the algebraic and topological structure of \( I_0(G) \) over on \( I_0(A) \) by means of this mapping. The space \( \mathcal{S}(a) \) is an algebra under convolution on \( a \).

2. Transmutation operators. A function \( f \) on the space \( S = G/K \) is called a radial function if \( f(k \cdot p) = f(p) \) for all \( k \in K, p \in S \). The set of continuous (resp. \( C^\infty \)) radial functions on \( G/K \) is in one-to-one correspondence \( f \rightarrow \tilde{f} \) with the set of all continuous (resp. \( C^\infty \)) \( W \)-invariant functions on \( A \). Here \( f(aK) = \tilde{f}(a) \) for \( a \in A \). Let \( D \in D(S) \); then by [2, p. 265] there exists a unique differential operator \( \delta'(D) \) on \( A' \) (the set of regular elements in \( A \)) such that \( (Df)^{-} = \delta'(D) \tilde{f} \) for all \( C^\infty \) radial functions \( f \). The operator \( \delta'(D) \) is called the radial part of \( D \) in analogy with the radial part \( D_r^2 + (n - 1)/r D_r \) of the Laplacian on \( R^n \). (\( D_r = d/dr \)). It is known [5] that there exists an isomorphism \( X \)
("transmutation operator") of the vector space of even $C^\infty$ functions on $\mathbb{R}$, onto itself, under which the singular operator $D_r^2 + (n - 1)/r \, D_r$ corresponds to $D_r^2$. The operators $\delta'(D)$ ($D \in D(S)$) are singular when considered as differential operators on $A$ but Theorem 1 shows that they have a simultaneous transmutation operator $X$ under which they correspond to differential operators on the Euclidean space $a$ with constant coefficients.

Given a $W$-invariant function $\phi$ on $A$, let $\phi$ denote the corresponding radial function on $S$. For $\phi \in I(A)$, put

$$(X\phi)(H) = e^{\phi(H)} \int_N \phi((\exp H) \cdot o) \, dn \quad (H \in a)$$

where $dn$ is a suitably normalized invariant measure on $N$. As proved by Harish-Chandra [11, p. 595], $X$ is a continuous mapping of $I(A)$ into $\mathfrak{s}(a)$.

**Theorem 1.** The mapping $X$ is a topological isomorphism of the algebra $I_0(A)$ onto the algebra $\mathfrak{s}(a)$. Moreover, if $D \in D(S)$ then

$$X\delta'(D)\phi = \Gamma(D)X\phi, \quad \phi \in I_0(A).$$

Here $\Gamma(D)$ is considered as a differential operator on $a$.

The proof is based on the Plancherel formula for functions in $I_0(G)$, proved by Harish-Chandra [2]. It also uses the recent result of Gindikin and Karpelevič [1] according to which the function $c(\lambda)$ above can be expressed in terms of Gamma functions.

**Remarks.** At the end of [2], Harish-Chandra states the following two conjectures which would imply that $I_0(A)$ contains all the $W$-invariant $C^\infty$ functions on $A$ with compact support.

(I) There exists a polynomial $p \in S(a)$ such that $|c(\lambda)\pi(\lambda)p(\lambda)| \geq 1$ for all $\lambda \in a$. (Here we have used the fact that $\pi = \pi_0$.)

(II) The mapping $X$ is one-to-one on $I(A)$.

Now (I) can be verified on the basis of the mentioned result of Gindikin and Karpelevič. Theorem 1 shows that (II) is equivalent to $I_0(G) = I(G)$. On the other hand, (II) is easily implied by the Plancherel formula for the functions in $I(G)$. This formula is not proved in [2] but I understand that Harish-Chandra has proved it in recent, as yet unpublished, work. In the next section we shall therefore assume that $I_0(G) = I(G)$.

**3. Fundamental solutions.** Let $C_c^\infty(S)$ denote the space of $C^\infty$ functions on $S$ with compact support. Let $\delta$ denote the distribution on $S$ given by $\delta(f) = f(0)$ for $f \in C_c^\infty(S)$.

**Theorem 2.** $E_\delta$ on the symmetric space $G/K$ is a fundamental solution of the differential equation

$$Du = f$$

by putting $f = E_\delta$.

**Added in proof.** Theorem 2 was proved by Harish-Chandra [11, p. 595].

Here $Exp$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ and $\mathfrak{p}$ and

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given by $\delta(f) = f(\omega)$ where $f \in C^\infty_c(S)$. Let $D \in D(S)$. A distribution $T$ on $S$ is called a fundamental solution of $D$ if $DT = \delta$. If $f \in C^\infty_c(S)$, then a fundamental solution $T$ of $D$ gives a solution of the equation $Du = f$ by putting $u = f * T$ where $*$ is the operation on distributions on $G/K$ induced by the convolution product of distributions on $G$.

**Theorem 2.** Each invariant differential operator $D \in D(S)$ $(D \neq 0)$ on the symmetric space $S$ has a fundamental solution.

This is a consequence of Theorem 1 and the fact that a nonzero differential operator on $R^n$ with constant coefficients always has a tempered fundamental solution [4; 6].

Added in proof. In the case when $G$ is complex the following formula (which is a simple consequence of Lemma 55 in [2]) gives a simpler proof of Theorem 2.

$$ (DF) \circ \text{Exp} = \frac{1}{\omega} \lambda(D)(\omega(F \circ \text{Exp})) \quad (D \in D(S)). $$

Here Exp is the usual Exponential mapping of $p$ onto $S$, $\lambda$ is a certain isomorphism of $D(S)$ onto the algebra of $\text{Ad}(K)$-invariant polynomials on $p$ and $F$ is any radial function on $S$.

**References**


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