DUALITY AND RADON TRANSFORM FOR
SYMMETRIC SPACES

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1. The dual space of a symmetric space. Let $S$ be a symmetric space (that is a Riemannian globally symmetric space), and let $I_0(S)$ denote the largest connected group of isometries of $S$ in the compact open topology. It will always be assumed that $S$ is of the noncompact type, that is $I_0(S)$ is semisimple and has no compact normal subgroup $\neq \{e\}$. Let $l$ denote the rank of $S$; then $S$ contains flat totally geodesic submanifolds of dimension $l$. These will be called planes in $S$.

Let $\mathfrak{o}$ be any point in $S$, $K$ the isotropy subgroup of $G=I_0(S)$ at $\mathfrak{o}$ and $\frak{t}_0$ and $\frak{g}_0$ their respective Lie algebras. Let $E$ be any plane in $S$ through $\mathfrak{o}$, $\mathfrak{a}_0$ the corresponding maximal abelian subspace of $\frak{g}_0$ and $A$ the subgroup $\exp(\mathfrak{a}_0)$ of $G$. Let $C$ be any Weyl chamber in $\mathfrak{a}_0$. Then the dual space of $\mathfrak{g}_0$ can be ordered by calling a linear function $X$ on $\mathfrak{a}_0$ positive if $X(H)>0$ for all $H \subset C$. This ordering gives rise to an Iwasawa decomposition of $G$, $G=KAN$, where $N$ is a connected nilpotent subgroup of $G$. It can for example be described by

$$N = \left\{ z \in G \middle| \lim_{t \to -\infty} \exp(-tH)z \exp(tH) = e \right\},$$

$H$ being an arbitrary fixed element in $C$. The group $N$ depends on the triple $(\mathfrak{o}, E, C)$. However, well-known conjugacy theorems show that if $N'$ is the group defined by a different triple $(\mathfrak{o}', E', C')$ then $N'=gNg^{-1}$ for some $g \in G$.

DEFINITION. A horocycle in $S$ is an orbit of a subgroup of the form $gNg^{-1}$, $g$ being any element in $G$.

Let $t \mapsto \gamma(t)$ ($t$ real) be any geodesic in $S$ and put $T_t = s_t^*s_0$ where $s_\tau$ denotes the geodesic symmetry of $S$ with respect to the point $\gamma(\tau)$. The elements of the one-parameter subgroup $T_t$ ($t$ real) are called transvections along $\gamma$. Two horocycles $\xi_1$, $\xi_2$ are called parallel if there exists a geodesic $\gamma$ intersecting $\xi_1$ and $\xi_2$ under a right angle such that $T_t\cdot\xi_1 = \xi_2$ for a suitable transvection $T_t$ along $\gamma$. For each fixed $g \in G$, the orbits of the group $gNg^{-1}$ form a parallel family of horocycles.

Let $M$ and $M'$, respectively, denote the centralizer and normalizer of $A$ in $K$. The group $W=M'/M$, which is finite, is called the Weyl group.

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Proposition 1.1. The group $G$ acts transitively on the set of horocycles in $S$. The subgroup of $G$ which maps the horocycle $N \circ o$ into itself equals $MN$.

Let $\hat{S}$ denote the set of horocycles in $S$. Then we have the natural identifications

$$S = G/K, \quad \hat{S} = G/MN$$

the latter of which turns $\hat{S}$ into a manifold, which we call the dual space of $S$.

Proposition 1.2.

(i) The mapping

$$\phi: (kM, a) \to kaK$$

is a differentiable mapping of $(K/M) \times A$ onto $S$ and a regular $w$-to-one mapping of $(K/M) \times A'$ onto $S'$.

(ii) The mapping

$$\hat{\phi}: (kM, a) \to kaMN$$

is a diffeomorphism of $(K/M) \times A$ onto $\hat{S}$.

In statement (i) which is well known, $w$ denotes the order of $W$, $A'$ is the set of regular elements in $A$ and $S'$ is the set of points in $S$ which lie on only one plane through $o$.

Proposition 1.3. The following relations are natural identifications of the double coset spaces on the left:

(i) $K \backslash G / K = A / W$;

(ii) $MN \backslash G / MN = A \times W$.

Statement (i) is again well known; (ii) is a sharpening of the lemma of Bruhat (see [6]) which identifies $MAN \backslash G / MAN$ with $W$.

The proofs of these results use the following lemma.

Lemma 1.4.

(i) Let $s_0$ denote the geodesic symmetry of $S$ with respect to $o$ and let $\theta$ denote the involution $g \to s_0 g s_0$ of $G$. Then

$$(N\theta(N)) \cap K = \{e\}.$$ 

(ii) Let $C$ and $C'$ be two Weyl chambers in $a_0$ and $G = KAN$, $G = KAN'$ the corresponding Iwasawa decompositions. Then

$$(NN') \cap (MA) = \{e\}.$$ 

2. Invariant differential operators on the space of horocycles. For any manifold $V$, $C^\infty(V)$ and $C^\infty_c(V)$ shall denote the spaces of $C^\infty$
functions on \( V \) (respectively, \( C^\infty \) functions on \( V \) with compact support). Let \( D(S) \) and \( D(\hat{S}) \), respectively, denote the algebras of all \( G \)-invariant differential operators on \( S \) and \( \hat{S} \). Let \( S(a_0) \) denote the symmetric algebra over \( a_0 \) and \( J(a_0) \) the set of \( W \)-invariants in \( S(a_0) \). There exists an isomorphism \( \Gamma \) of \( D(S) \) onto \( J(a_0) \) (cf. \cite[Theorem 1, p. 260]{7}, also \cite[p. 432]{9}). To describe \( D(\hat{S}) \), consider \( \hat{S} \) as a fibre bundle with base \( K/M \), the projection \( p: \hat{S} \to K/M \) being the mapping which to each horocycle associates the parallel horocycle through 0. Since each fibre \( F \) can be identified with \( A \), each \( \psi \in S(a_0) \) determines a differential operator \( U_F \) on \( F \). Denoting by \( f|_F \) the restriction of a function \( f \) on \( \hat{S} \) to \( F \) we define an endomorphism \( D_U \) on \( C^\infty(S) \) by
\[
(D_U f)|_F = U_F(f|_F) \quad f \in C^\infty(\hat{S}),
\]
\( F \) being any fibre. It is easy to prove that the mapping \( U \to D_U \) is a homomorphism of \( S(a_0) \) into \( D(\hat{S}) \).

**Theorem 2.1.** The mapping \( U \to D_U \) is an isomorphism of \( S(a_0) \) onto \( D(\hat{S}) \). In particular, \( D(\hat{S}) \) is commutative.

Although \( G/MN \) is not in general reductive, \( D(\hat{S}) \) can be determined from the polynomial invariants for the action of \( MN \) on the tangent space to \( G/MN \) at \( MN \) (cf. \cite[Theorem 10]{8}). It is then found that the algebra of these invariants is in a natural way isomorphic to \( S(a_0) \), whereupon Theorem 2.1 follows. Let \( \hat{\Gamma} \) denote the inverse of the mapping \( U \to D_U \).

3. The Radon transform. Let \( \xi \) be any horocycle in \( S \), \( ds_\xi \) the volume element on \( \xi \). For \( f \in C^\infty(S) \) put
\[
\hat{f}(\xi) = \int_\xi f(s)ds_\xi, \quad \xi \in \hat{S}.
\]
The function \( \hat{f} \) will be called the Radon transform of \( f \).

**Theorem 3.1.** The mapping \( f \to \hat{f} \) is a one-to-one linear mapping of \( C^\infty(S) \) into \( C^\infty(\hat{S}) \).

Now extend \( a_0 \) to a Cartan subalgebra \( h_0 \) of \( g_0 \); of the corresponding roots let \( P_+ \) denote the set of those whose restriction to \( a_0 \) is positive (in the ordering defined by \( C \)). Put \( \rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha \) and let \( \rho \to \hat{\rho} \) denote the unique automorphism of \( S(a_0) \) given by \( \hat{\rho} = H - \rho(H) \) (\( H \in \mathfrak{a}_0 \)) (cf. \cite[p. 260]{7}).

**Theorem 3.2.** Let \( \mathcal{D}(\hat{S}) \) be given by
\[
\mathcal{D}(\hat{S}) = \{ E \in D(\hat{S}) \mid \hat{\rho}(E) \in J(a_0) \},
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and let \( D \to \mathcal{D} \) denote

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**Theorem 3.3.** Then

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and let $D→Ω$ denote the isomorphism of $D(S)$ onto $'D(S)$ such that

$'(Ω(Ω))=Ω(D), \quad D ∈ D(S).$

Then

$$(Df) ^{\bigcirc} = Ω f \quad \text{for } f ∈ C^∞(S).$$

In view of the duality between points and horocycles there is a
atural dual to the transform $f→Ω$. This dual transform associates to
each function $ψ∈C^∞(S)$ a function $ψ'∈C^∞(S)$ given by

$$ψ'(p) = \int_{x∈p} ψ(x) \, dm(x), \quad p ∈ S,$$

where the integral on the right is the average of $ψ$ over the (compact)
set of horocycles passing through $p$. We put

$$I_ψ = (ψ) ^{\bigcirc}, \quad f ∈ C^∞(S)$$

and wish to relate $f$ and $I_ψ$.

**THEOREM 3.3.** Suppose the group $G=I₀(S)$ is a complex Lie group.
Then

$$(1) \quad \square I_ψ = cf, \quad f ∈ C^∞(S),$$

where $c$ is a constant $≠0$ and $\square$ is a certain operator in $D(S)$, both inde-
dependent of $f$.

We shall now indicate the definition of $\square$. Let $J$ denote the
complex structure of the Lie algebra $g₀$. Then the Cartan subalgebra $h₀$
above can be taken as $a₀+Ja₀$ and can then be considered as a com-
plex Cartan subalgebra of $g₀$ (considered as a complex Lie algebra).
Let $Δ'$ denote the corresponding set of nonzero roots and for each
$α∈Δ'$ select $H'_α$ in $h₀$ such that $B'(H'_α, H)=α(H) (H ∈ h₀)$ where $B'$
denotes the Killing form of the complex algebra $g₀$. Then $H'_α ∈ a₀$ and
the element $\prod_{α∈Δ'} H'_α$ in $S(a₀)$ is invariant under the Weyl group $W$.
Then $\box$ is the unique element in $D(S)$ such that

$$\Gamma(\square) = \prod_{α∈Δ'} H'_α.$$

The proof of Theorem 3.3 is based on Theorem 3 in Harish-Chandra
[5] (see also Gelfand-Nalmark [4, p. 156]), together with the Dar-
boux equation for $S$ ([9, p. 442]). In the case when $S$ is the space of
positive definite Hermitian $n×n$ matrices a formula closely related
to (1) was given in Gelfand [1]. Radon's classical problem of representing a function in $\mathbb{R}^n$ by means of its integrals over hyperplanes was solved by Radon [13] and John [10]. Generalizations to Riemannian manifolds of constant curvature were given by Helgason [8], Semyanistyi [15] and Gelfand-Graev-Vilenkin [3].

4. Applications to invariant differential equations. We shall now indicate how Theorem 3.3 can be used to reduce any $G$-invariant differential equation on $S$ to a differential equation with constant coefficients on a Euclidean space. The procedure is reminiscent of the method of plane waves for solving homogeneous hyperbolic equations with constant coefficients (see John [11]).

DEFINITION. A function on $S$ is called a plane wave if there exists a parallel family $\mathcal{E}$ of horocycles in $S$ such that (i) $S = \bigcup_{\xi \in \mathcal{E}} \xi$; (ii) For each $\xi \in \mathcal{E}$, $f$ is constant on $\xi$.

Theorem 3.3 can be interpreted as a decomposition of an arbitrary function $f \in C_c^\infty(S)$ into plane waves.

Now select $g \in G$ such that $\mathcal{E}$ is the family of orbits of the group $gN_g^{-1}$. The manifold $gAg^{-1} \cdot o$ intersects each horocycle $\xi \in \mathcal{E}$ orthogonally. A plane wave $f$ (corresponding to $\mathcal{E}$) can be regarded as a function $f^*$ on the Euclidean space $A$. If $D \in D(S)$, then $Df$ is also a plane wave (corresponding to $\mathcal{E}$) and $(Df)^* = D_Af^*$, where $D_A$ is a differential operator on $A$. Using the fact that $aNa^{-1} \subseteq N$ for each $a \in A$ it is easily proved (cf. [7, Lemma 3, p. 247] or [12, Theorem 1]) that $D_A$ is invariant under all translations on $A$. Thus an invariant differential equation in the space of plane waves (for a fixed $\mathcal{E}$) amounts to a differential equation with constant coefficients on the Euclidean space $A$. Using Theorem 3.3, and the fact that $\Box$ commutes elementwise with $D(S)$, an invariant differential equation for arbitrary functions on $S$ can be reduced to a differential equation with constant coefficients (and is thus, in principle, solvable).

EXAMPLE: THE WAVE EQUATION ON $S$. For an illustration of the procedure above we give now an explicit global solution of the wave equation on $S$ ($\mathcal{U}_0(S)$ assumed complex).

Let $\Delta$ denote the Laplacian on $S$ and let $f \in C_c^\infty(S)$. Consider the differential equation

$$\Delta u = \frac{\partial^2 u}{\partial t^2}$$

with initial data

1. $u(p, 0) = 0$;
2. $ \left\{ \frac{\partial}{\partial t} u(p, t) \right\}_{t=0} = f(p) \quad (p \in S).$
Let $\Delta_A$ denote the Laplacian on $A$ (in the metric induced by $E$), $||p||$ the length of the vector $p$ in §3. Given $a \in A$, let $\log a$ denote the unique element $H \in a_0$ for which $\exp H = a$. For simplicity, let $e^s$ denote the function $a \rightarrow e^{s(\log a)}$ on $A$. Let $\xi$ denote the horocycle $N \cdot a$.

Given $x \in G$, $k \in K$, consider the function

$$F_{k,x}(a) = \int_{\xi} f(xka \cdot s) ds,$$

and the differential equation on $A \times R$,

$$(\Delta_A - ||p||^2) V_{k,x}^t = \frac{\partial^2}{\partial t^2} V_{k,x},$$

with initial data

$$V_{k,x}^0 = 0; \quad \left\{ \frac{\partial}{\partial t} V_{k,x}^t \right\}_{t=0} = e^{s} F_{k,x}.$$

Equation (3) is just the equation for damped waves in the Euclidean space $A$ and is explicitly solvable (see e.g. [14, p. 88]). The solution of (1) is now given by

$$u(p, t) = c \Box_p (V(p, t)),$$

where

$$(4) \quad V(xK, t) = \int_K V_{k,x}(e) dk.$$

Here $dk$ is the normalized Haar measure on $K$ and $c$ is the same constant as in Theorem 3.3. It is not hard to see that the integral in (4) is invariant under each substitution $x \rightarrow xu$ ($u \in K$) so the function $V(p, t)$ is indeed well defined.

**References**


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### Upper Bounds

**1. Introduction.** The entries is either 0, or the permanent of $A$ is $c$ where the sum of metric group $S_n$. For permanents $p(A)$ is the ob been conjectured with exactly $k$ on $n!(k/n)^n$ [1, p. 59] in the class of all each row and column to the permanent entries are 1. In t for the permanent upper bound which the affirmative.

**2. Results.**

**Lemma.** If $r_1, \ldots, r_n$ with equality if an

**Proof.** Let $E_k$ the numbers $1/r_i$

$$0 \leq \prod_{i=1}^n (1 - \frac{1}{r_i})$$

with equality if a $\ldots$

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1 This work was ...