1. Introduction. The result to be proved in this article is that if \( u \) is a bounded harmonic function on a symmetric space \( X \) and \( x_0 \) any point in \( X \) then \( u \) has a limit along almost every geodesic in \( X \) starting at \( x_0 \) (Theorem 2.3). In the case when \( X \) is the unit disk with the non-Euclidean metric this result reduces to the classical Fatou theorem (for radial limits). When specialized to this case our proof is quite different from the usual one; in fact it corresponds to transforming the Poisson integral of the unit disk to that of the upper half-plane and using only a homogeneity property of the Poisson kernel. The kernel itself never enters into the proof.

2. Harmonic functions on symmetric spaces. Let \( G \) be a semisimple connected Lie group with finite center, \( K \) a maximal compact subgroup of \( G \) and \( g \) and \( \mathfrak{g} \) their respective Lie algebras. Let \( B \) denote the Killing form of \( \mathfrak{g} \) and the corresponding orthogonal complement of \( \mathfrak{f} \) in \( \mathfrak{g} \). Let \( \text{Ad} \) denote the adjoint representation of \( G \). As usual we view as the tangent space to the symmetric space \( X = G/K \) at the origin \( o = \{ K \} \) and accordingly give \( X \) the \( G \)-invariant Riemannian structure induced by the restriction of \( B \) to \( \mathfrak{p} \times \mathfrak{p} \). Let \( \Delta \) denote the corresponding Laplace-Beltrami operator.

Fix a maximal abelian subspace \( \mathfrak{a} \subset \mathfrak{p} \) and let \( M \) denote the centralizer of \( \mathfrak{a} \) in \( K \). If \( \lambda \) is a linear function on \( \mathfrak{a} \) and \( \lambda \neq 0 \) let \( \mathfrak{g}_\lambda = \{ \mathfrak{X} \in \mathfrak{g} \mid [H, \mathfrak{X}] = \lambda(H)\mathfrak{X} \text{ for all } H \in \mathfrak{a} \} \); \( \lambda \) is called a restricted root if \( \mathfrak{g}_\lambda \neq 0 \). Let \( \mathfrak{a}' \) denote the open subset of \( \mathfrak{a} \) where all restricted roots are \( \neq 0 \). Fix a Weyl chamber \( \mathfrak{a}^+ \) in \( \mathfrak{a} \), i.e. a connected component of \( \mathfrak{a}' \). A restricted root \( \alpha \) is called positive (denoted \( \alpha > 0 \)) if its values on \( \mathfrak{a}^+ \) are positive. Let the linear function \( \rho \) on \( \mathfrak{a} \) be determined by \( 2\rho = \sum_{\alpha > 0} (\dim \mathfrak{g}_\alpha)\alpha \) and denote the subalgebras \( \sum_{\alpha > 0} \mathfrak{g}_\alpha \) and \( \sum_{\alpha > 0} \mathfrak{g}_-\alpha \) of \( \mathfrak{g} \) by \( \mathfrak{n} \) and \( \mathfrak{h} \) respectively. Let \( \mathfrak{N} \) and \( \mathfrak{N} \) denote the corresponding analytic subgroups of \( G \).

By a Weyl chamber in \( \mathfrak{p} \) we understand a Weyl chamber in some maximal abelian subspace of \( \mathfrak{p} \). The boundary of \( X \) is defined as the set \( B \) of all Weyl chambers in the tangent space \( \mathfrak{p} \) to \( X \) at \( o \); since this boundary is via the map \( kM \to \text{Ad}(k)\mathfrak{a}^+ \) identified with \( K/M \), which by the Iwasawa decomposition \( G = KAN \) equals \( G/MAN \), this defi-

---

1 This work was supported by the National Science Foundation, GP 7477 and GP 6155.
nition of boundary is equivalent to Furstenberg's [2] (see also [6] and [4]). In particular the group $G$ acts transitively on $B$ as well as on $X$. The two actions will be denoted $(g, b) \rightarrow g(b)$ and $(g, x) \rightarrow g \cdot x$ ($g \in G$, $b \in B$, $x \in X$). Let $db$ denote the unique $K$-invariant measure on $B$ normalized by $\int_B db = 1$. Then according to Furstenberg [2], the mapping $f \rightarrow u$ where

\[ u(g \cdot o) = \int_B f(g(b)) db \quad (g \in G), \]

is a bijection of the set $L^\infty(B)$ of bounded measurable functions on $B$ onto the set of bounded solutions of Laplace's equation $\Delta u = 0$ on $X$. The function $u$ in (1) is called the Poisson integral of $f$.

If $g \in G$ let $k(g) \in K$, $H(g) \in a$ be determined by $g = k(g) \exp H(g) n$ ($n \in N$). Observe that if $g^h$ denotes $hgh^{-1}$ for $h \in G$ then $k(g^h) = k(g)^h$, $H(g^h) = H(g)$ for $\tilde{n} \in \tilde{N}$, $m \in M$. According to Harish-Chandra [3, Lemma 44], the mapping $\tilde{n} \rightarrow k(\tilde{n}) M$ is a bijection of $\tilde{N}$ onto a subset of $K/M$ whose complement is of lower dimension and if $f$ is a continuous function on $B$, then

\[ \int_B f(b) db = \int_{\tilde{N}} f(k(\tilde{n}) M) \exp (-2\pi(H(\tilde{n}))) d\tilde{n} \]

for a suitably normalized Haar measure $d\tilde{n}$ on $\tilde{N}$. If $a \in A$ we have $ak(\tilde{n}) M A N = k(\tilde{a}^o) M A N$ whence

\[ a(k(\tilde{n}) M) = k(\tilde{a}^o) M \]

so the action of $a$ on the boundary corresponds to the conjugation $\tilde{n} \rightarrow \tilde{a}^o \tilde{n}$ on $\tilde{N}$.

Let $E_1, \cdots, E_r$ be a basis of $\tilde{n}$ such that each $E_i$ lies in some $\tilde{g}_{-\alpha_i}$, say $\tilde{g}_{-\alpha_i}$. Since the map $\exp: \tilde{n} \rightarrow \tilde{N}$ is a bijection we can, for each $H \in a^+$, consider the function $\tilde{n} \rightarrow |\tilde{n}|_H$ defined as follows: If $\tilde{n} = \exp(\sum_i a_i E_i)$ ($a_i \in R$) we put

\[ |\tilde{n}|_H = \max \{ |a_i|^{1/\alpha_i(M)} \mid 1 \leq i \leq r \} \]

Since

\[ \tilde{n} \exp t H = \exp \left( \sum_{i=1}^r a_i \exp(-\alpha_i(H)t) E_i \right) \]

we have

\[ |\tilde{n} \exp t H|_H = e^{-t} |\tilde{n}|_H \quad \text{for } \tilde{n} \in \tilde{N}, \quad t \in R, \quad H \in a^+. \]

For $r > 0$ let $B_{H,r}$ denote the set $\{ \tilde{n} \in \tilde{N} \mid |\tilde{n}|_H < r \}$ and let $V_{H,r}$ denote the volume of $B_{H,r}$ (with respect to the Haar measure on $\tilde{N}$).
Lemmas. Let $f \in L^n(B)$ and $u$ the Poisson integral (1) of $f$. Put $F(\bar{n}) = f(k(\bar{n})M)$ for $\bar{n} \in \bar{N}$. Fix $\bar{n}_0 \in \bar{N}$ and $H \in a^+$ and assume

$$\int_{B_{H,r}} |F(\bar{n}_0 \bar{n}) - F(\bar{n}_0)| \, d\bar{n} \to 0$$

for $r \to 0$. Then

$$\lim_{r \to +0} u(k(\bar{n}_0) \exp tH(\cdot)) = f(k(\bar{n}_0)M).$$

Proof. By the Iwasawa decomposition we can write $\bar{n}_0 = k(\bar{n}_0) \cdot (a_1 n_1)^{-1} (a_1 \in A, n_1 \in N)$ so

$$u(k(\bar{n}_0) \exp tH \cdot o) = u(\bar{n}_0 a_1 n_1 \exp tH \cdot o) = u(\bar{n}_0 \exp tH a_1 n_1 \exp (-tH) \cdot o).$$

But $G = ANK$ so $n_1 \exp (-tH) = a(t) n(t) k(t)$, each factor tending to $e$ as $t \to +\infty$. If $H_i \in a$ is determined by $\exp tH_i = \exp tH a_1 a(t)$

we have

$$u(k(\bar{n}_0) \exp tH \cdot o) = u(\bar{n}_0 a_1 \exp tH_i \exp tH \cdot o).$$

The function $f'(b) = f(\bar{n}_0 \bar{n}(t) \exp tH_i(b))$ has Poisson integral $u'(x) = u(\bar{n}_0 \bar{n}(t) \exp tH_i \cdot x)$; using (1) on $u'$ and $f'$ with $g = \exp tH_i$ we get from (2) and (3)

$$u(k(\bar{n}_0) \exp tH \cdot o) - f(k(\bar{n}_0)M) = \int_{\bar{N}} (F(\bar{n}_0 \bar{n}(t) \exp tH_i \exp tH) - F(\bar{n}_0)) \exp(-2\rho(H(\bar{n}))) \, d\bar{n}$$

so

$$|u(k(\bar{n}_0) \exp tH \cdot o) - f(k(\bar{n}_0)M)|$$

$$\leq \int_{\bar{N}} |F(\bar{n}_0 \bar{n} \exp tH_i) - F(\bar{n}_0)| \exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n}))) \, d\bar{n}.$$ 

Now if $c > 0$ let $N_\epsilon$ denote the "square"

$$N_\epsilon = \left\{ \exp \left( \sum_{1}^{r} a_i E_i \right) \mid a_i \leq c, 1 \leq i \leq r \right\}.$$ 

The integral on the right in (7) equals the sum

$$\sum_{\bar{n}} |F(\bar{n}_0 \bar{n} \exp tH_i) - F(\bar{n}_0)| \exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n})))$$

where $d$ and $d_4$ are

$$2||F||\int_{\bar{N} - N_\epsilon}$$

On the other hand, the second term in (8)
\[ \int_{\mathcal{N}_0} \left| F(\tilde{n}_0 \exp tH) - F(\tilde{n}_0) \right| \exp(-2\rho(H(\tilde{n}(t)^{-1}\tilde{n}))) \, d\tilde{n} \]

\[ + \int_{\mathcal{N} - \mathcal{N}_0} \left| F(\tilde{n}_0 \exp tH) - F(\tilde{n}_0) \right| \exp(-2\rho(H(\tilde{n}(t)^{-1}\tilde{n}))) \, d\tilde{n}. \]

Since \( \rho(H(\tilde{n})) \geq 0 \) for all \( \tilde{n} \in \mathcal{N} \) ([3, p. 287]) and since the mapping \( \tilde{n} \to \tilde{n} \exp H \) has Jacobian \( \exp(-2\rho(H)) \) (cf. (4)) we see that

\[ \int_{\mathcal{N}_0} \left| F(\tilde{n}_0 \exp tH) - F(\tilde{n}_0) \right| \exp(-2\rho(H(\tilde{n}(t)^{-1}\tilde{n}))) \, d\tilde{n} \leq \exp(2\rho(tH)) \int_{\mathcal{N}_0} \left| F(\tilde{n}_0) - F(\tilde{n}_0) \right| \, d\tilde{n}. \]

Now \( \tilde{n} \in \mathcal{N}_0 \exp tH \) if and only if

\[ \tilde{n} = \exp(\sum a_i E_i) \exp(tH) \quad \text{where} \quad |a_i| \leq c \]

and \( tH_i - tH \) is bounded (for fixed \( \tilde{n}_0 \) and \( H \)). It follows that

\[ \mathcal{N}_0 \exp tH \subseteq B_{H, c, t} \quad \text{for all} \quad t \geq 0, \]

where \( d = d(H, \tilde{n}_0, c) \) being a constant. But since the map \( \exp: \mathfrak{n} \to \mathcal{N} \) is measure-preserving it is clear that

\[ V_{H, c, t} = \exp(-2\rho(H)t) d_1 \quad \text{for} \quad t \geq 0 \]

where \( d_1 = d_1(H, \tilde{n}_0, c) \) is another constant. Also

\[ \exp(2\rho(tH)) \leq \exp(2\rho(tH)) d_2 \]

where \( d_2(H, \tilde{n}_0) \) is a constant. Thus the right hand side of (9) can be majorized for all \( t \geq 0 \):

\[ \exp(2\rho(tH)) \int_{\mathcal{N}_0 \exp tH} |F(\tilde{n}_0 \tilde{n}) - F(\tilde{n}_0)| \, d\tilde{n} \leq d_2 \frac{1}{V_{H, c, t}} \int_{B_{H, c, t}} |F(\tilde{n}_0 \tilde{n}) - F(\tilde{n}_0)| \, d\tilde{n} \]

where \( d \) and \( d_2 \) are constants depending on \( H, \tilde{n}_0 \) and \( c \).

On the other hand, if \( || \cdot ||_\infty \) denotes the uniform norm on \( \mathcal{N} \) the second term in (8) is majorized by

\[ 2 ||F||_\infty \int_{\mathcal{N} - \mathcal{N}_0} \exp(-2\rho(H(\tilde{n}(t)^{-1}\tilde{n}))) \, d\tilde{n} \]

\[ = 2 ||F||_\infty \left( 1 - \int_{\mathcal{N}_0 \exp tH} \exp(-2\rho(H(\tilde{n}))) \, d\tilde{n} \right). \]
Now given $\epsilon > 0$ we first choose $c$ so large that

$$2\|F\|_\infty \left(1 - \int_{N_0} \exp(-2\rho(H(\tilde{n}))) d\tilde{n}\right) < \epsilon/2;$$

since $\tilde{n}(t) \to e$ for $t \to +\infty$ we can choose $t_1$ such that $\tilde{n}(t) N_0 \supset N_0$ for $t \geq t_1$. Then the expression in (11) is $< \epsilon/2$ for $t \geq t_1$; by our assumption (6) we can choose $t_2$ such that the right hand side of (10) is $< \epsilon/2$ for $t > t_2$. In view of (7) and (8) this proves the lemma.

The next lemma shows that, for a fixed $H$, the assumption of Lemma 2.1 actually holds for almost all $\tilde{n}_0 \in \tilde{N}$.

**Lemma 2.2.** Let $F \in L^\infty(\tilde{N})$ and fix $H \in \alpha^+$. Then

$$\lim_{r \to 0} \frac{1}{V_{H,r}} \int_{B_{H,r}} |F(\tilde{n}_0) - F(\tilde{n}_0)| d\tilde{n} = 0$$

for almost all $\tilde{n}_0 \in \tilde{N}$.

The proof of this result is essentially in the literature: In [1] Edwards and Hewitt give all the necessary arguments for the case of a discrete sequence tending to 0 and everything they do remains trivially valid in the case $r \to 0$. The result in the exact form required here was also proved by E. M. Stein independently of [1] (cf. his expository article [6]).

**Theorem 2.3.** Let $u$ be a bounded solution of Laplace's equation $\Delta u = 0$ on the symmetric space $X$. Then for almost all geodesics $\gamma(t)$ starting at $o$

$$\lim_{t \to \infty} u(\gamma(t)) \quad \text{exists.}$$

**Proof.** Let $S^+ = \{ H \in \alpha^+ \mid B(H, H) = 1 \}$. Then the mapping $(kM, H) \mapsto \text{Ad}(k)H$ is a bijection of $(K/M) \times S^+$ onto a subset of the unit sphere $S$ in $p$ whose complement has lower dimension. Since $\dim(K/M - k(\tilde{N})/M) < \dim K/M$ the mapping $(\tilde{n}, H) \mapsto \text{Ad}(k(\tilde{n}))H$ is a bijection of $\tilde{N} \times S^+$ onto a subset of $S$ whose complement in $S$ has lower dimension. If $\tilde{N}_H$ denotes the set of $\tilde{n}_0$ for which (12) holds (with $F(\tilde{n}) = f(k(\tilde{n})M)$) and if $S_0 = \cup_{H \in S^+} \text{Ad}(k(\tilde{N}_H))H$ it follows from the Fubini theorem that $S - S_0$ is a null set. This concludes the proof.

**Remarks.** (i) If $f$ is continuous the limit relation

$$\lim_{t \to +\infty} u(k \exp tH \cdot o) = f(kM) \quad (H \in \alpha^+, kM \in K/M)$$

follows immediately; convergence theorem

Weyl chamber in $p$.

(ii) In the case $w$ has proved (13), ev
follows immediately from (1), (2) and (3), by use of the dominated convergence theorem. (See also [4, Theorem 18.3.2.]) In particular, $u$ has the same limit along all geodesics from $o$ which lie in the same Weyl chamber in $\mathfrak{p}$.

(ii) In the case when $X$ has rank one ($\dim \mathfrak{a} = 1$) A. W. Knapp [5] has proved (13), even under the weaker assumption that $f \in L^1(B)$.

REFERENCES

5. A. W. Knapp, unpublished manuscript.