PALEY-WIENER THEOREMS AND SURJECTIVITY OF INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES AND LIE GROUPS

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1. Introduction. The principal result of this paper is that if \( D \) is an invariant differential operator on a symmetric space \( X \) of the noncompact type then, for each function \( f \in C^\infty(X) \), the differential equation \( Du = f \) has a solution \( u \in C^\infty(X) \). This is proved by means of a Paley-Wiener type theorem for the Radon transform on \( X \). As a consequence we also obtain a Paley-Wiener theorem for the Fourier transform on \( X \), that is an intrinsic characterization of the Fourier transforms of the functions in \( C^\infty_c(X) \). In [2], Eguchi and Okamoto characterized the Fourier transforms of the Schwartz space on \( X \). Invoking in addition the division theorem of Hörmander [16] and Lojasiewicz [18] we obtain by the method of [11] the surjectivity of \( D \) on the space of tempered distributions on \( X \).

Finally, as a consequence of a structure theorem of Harish-Chandra [8] for the bi-invariant differential operators on a noncompact semisimple Lie group \( G \), we obtain a local solvability theorem for each such operator.

2. The range of invariant differential operators. Let \( X \) be a symmetric space of the noncompact type, that is a coset space \( G/K \) where \( G \) is a connected, noncompact semisimple Lie group with finite center and \( K \) a maximal compact subgroup. Let \( D(X) \) denote the set of differential operators on \( X \), invariant under \( G \) and let \( C^\infty(X) \) denote the set of all \( C^\infty \) functions on \( X \) and \( C^\infty_c(X) \) the set of \( f \in C^\infty_c(X) \) of compact support.

**Theorem 2.1.** Let \( D \neq 0 \) in \( D(X) \). Then
\[
DC^\infty_c(X) = C^\infty_c(X).
\]

As in Malgrange's proof of an analogous theorem for constant coefficient operators on \( \mathbb{R}^n \) ([3], [20]) our proof proceeds by proving that if \( V \) is a closed ball in \( X \) then
\[
\text{supp}(Df) \subset V \quad \text{implies} \quad \text{supp}(f) \subset V,
\]
\( \text{supp} \) denoting support. This is proved by means of Theorem 2.2 below.
for the Radon transform \([10]\) \(f \to \hat{f}\) on \(X\). If \(\xi\) is a horocycle in \(X\) then \(\hat{f}(\xi)\) is the integral of \(f\) over \(\xi\). The following Paley-Wiener type theorem for the Radon transform is the analog for \(X\) of Theorem 2.1 in \([12]\). The proof is however quite different and is in part based on Harish-Chandra’s expansion for general Eisenstein integrals on \(G\) \([7]\). I am indebted to Harish-Chandra for communicating to me this expansion which has not been published, but will appear in \([22]\). It is a generalization of the asymptotic expansion for the spherical functions in \([6]\).

**Theorem 2.2.** Let \(L \in C_c^\infty(X)\) and let \(V\) be a closed ball in \(X\). Assume \(f(\xi) = 0\) whenever the horocycle \(\xi\) in \(X\) is disjoint from \(V\). Then \(f(x) = 0\) for \(x \notin V\).

**Remark.** Instead of assuming \(f \in C_c^\infty(X)\) it suffices to assume that the function \(g \to f(gK)\) belongs to the Schwartz space on \(G\) in the sense of \([9, \text{p. 19}]\).

The analog of Theorem 2.1 for left invariant differential operators \(D\) on a Lie group \(L\) is in general false. In fact, it was proved to me by Hörmander in 1964 (independently proved in Cerezo-Rouvière \([1]\)) that if for a given \(L\) one assumes local solvability for every \(D\) then either \(L\) is abelian or has an abelian normal subgroup of codimension 1. However for each bi-invariant (i.e., left and right invariant) operator on the semi-simple group \(G\) we have the following local solvability result.

**Theorem 2.3.** There exists an open neighborhood \(V\) of \(e\) in \(G\) with the following property: For each bi-invariant differential operator \(D \neq 0\) on \(G\),

\[
DC^\infty(V) \supset C_c^\infty(V).
\]

The proof is easily deduced from a structure theorem for \(D\) (Harish-Chandra \([8\, \text{p. 477}]\)) combined with Proposition 1.4 in Rais \([21]\) which deals with nilpotent groups.

3. **The Fourier transform on \(X\).** Let \(G = KAN\) be an Iwasawa decomposition of \(G\), \(A\) and \(N\) being abelian and nilpotent, respectively. Let \(\mathfrak{a}\) denote the Lie algebra of \(A\), \(\mathfrak{a}^*\) its dual and \(\mathfrak{a}_c^*\) the complexification of \(\mathfrak{a}^*\). If \(\lambda \in \mathfrak{a}_c^*\) let \(\text{Im} \lambda\) denote its imaginary part. Let \(|\lambda|\) denote the norm on \(\mathfrak{a}^*\) given by the Killing form of the Lie algebra of \(G\). If \(H \in \mathfrak{a}\) the map \(X \to [H, X]\) is an endomorphism of the Lie algebra \(n\) of \(N\) whose trace we denote \(2p(H)\). Let \(M\) be the centralizer of \(A\) in \(K\), put \(B = K/M\) and let \(db\) be the \(K\)-invariant measure on \(B\) with total measure 1. For \(x \in X\), \(b = kM \in B\), let \(A(x, b) = a\) be determined by \(n \in N\), \(x = kn \exp A(x, b)K\).

Fixing a \(G\)-invariant measure \(dx\) on \(X\) the Fourier transform \(\hat{f}\) of a function \(f\) on \(X\) is defined by

\[
\mathfrak{a}_c^* \ni \lambda \to \int_B e^{i\lambda x} db.
\]

It satisfies

\[
(1) \quad \int_B e^{i\lambda x} db
\]

for \(f \in C_c^\infty(X)\), an mapping \(f \to \hat{f}\).
Theorem 1. The mapping \( f \to \tilde{f} \) is a bijection of \( C_c^\infty(X) \) onto the space of holomorphic functions of uniform exponential type satisfying (1).

For the case when \( f \) is assumed \( K \)-invariant this reduces to a known result ([4, p. 434], for \( \text{SL}(2, \mathbb{R}) \), [14], [5], [15, p. 37]). Finally, let \( \mathcal{S}'(X) \) denote the dual space of the Schwartz space \( \mathcal{S}(X) \). Its elements are distributions on \( X \), the tempered distributions. In the manner indicated in the introduction we obtain an extension of Theorem 4.2 in [11].

Theorem 3.2. Let \( D \neq 0 \) in \( D(X) \). Then

\[
D\mathcal{S}'(X) = \mathcal{S}'(X).
\]

References


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