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A Theoretical Plasma Physics Model of the Plasmatron

by

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B.S. Engineering Physics
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ABSTRACT

An analysis was conducted to determine the behavior of a high pressure electric arc discharge inside the Plasmatron Fuel Reformer. The Plasmatron Fuel Reformer uses an electric arc to partially combust fuel in an internal combustion engine to increase efficiency and reduce emissions. Solutions of the conservation equations for the arc yield temperature, pressure and velocity profiles for arcs with 0.2-0.8 A currents. Acquired knowledge was used to predict arc radius and power delivered by the arc as a function of current.

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1. Introduction

This thesis is supporting research for a project at MIT named the Plasmatron. The Plasmatron is a fuel reformer that can be integrated into internal combustion engines (ICEs) to increase fuel efficiencies and reduce emissions.\(^1\) The basic tenant of the Plasmatron is the use of an electric arc to partially combust a portion of the fuel to be used in an ICE in a low oxygen environment, creating a H\(_2\) and CO\(_2\) mix. When remixed with the rest of the fuel, a hydrogen rich variant of fuel is produced. This hydrogen rich fuel allows the engine to run extremely lean, which increases engine efficiency and lowers operating temperatures.\(^2\) As NOx production is directly related to temperature, the lower operating temperatures reduce NOx production and ultimately NOx emission.\(^3\)

This thesis studies the characteristics of the arc used in the partial combustion and models the arc behavior using thermal conductivity and thermal equilibration between species as the primary methods of thermal transport. The primary goal of this thesis is to find values for key plasma parameters at the center of the arc and to describe these parameters as functions of current and radius. The secondary goal is to calculate the macroscopic parameters of voltage, resistance and power as functions of current and radius.

In this analysis, the arc is first described in terms of the conservation equations; Conservation of Energy, Conservation of Momentum, and Conservation of Mass. Next, each of the transport coefficients is carefully examined. An asymptotic expansion reduces the model to three equation and three unknowns. Next, a theory is proposed which determines the radius of the arc based on the thermal properties of the electrodes. With the analytical work completed, a brief overview is presented on how the model is input into the MATLAB BVP solver. Finally, the results are presented, first giving general results as functions of

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\(^1\) Bromberg, 2004.


\(^3\) Bromberg, 2004.
radius and current, and then specific results for the plasmatron. All units in this thesis are SI except temperature which is in units of eV.

This analysis shows that high pressure electric arcs can be adequately modeled using thermal conductivity and thermal equilibration between species as the primary methods of energy transport.

2. The Model

To start the analysis, we will look at a model of a steady state arc in a simple environment. The arc is a long cylindrical tube composed of stationary molecular gases at atmospheric pressure which are cooled at the outer radial limit. This three fluid model considers electrons, neutrals, and singly ionized ions in a steady state partially ionized arc. The model consists of the Conservation of Mass, Momentum, and Energy for each species as described below.

2.1. Conservation of Mass

For electrons:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \cdot n_e \cdot v_e \right) = -S_{\text{recombination}} + S_{\text{ionization}}$$

(2.1.1)

For ions:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \cdot n_i \cdot v_i \right) = -S_{\text{recombination}} + S_{\text{ionization}}$$

(2.1.2)

For neutrals:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \cdot n_n \cdot v_n \right) = 2 \cdot (S_{\text{recombination}} - S_{\text{ionization}})$$

(2.1.3)

2.2. Conservation of Momentum

Conservation of momentum has two non trivial components; the radial component and the axial component. The model includes effects due to the electric field, the pressure gradient force, and collisional momentum exchange between species. These equations neglect inertia because of the very slow velocities of the plasmatron particles in this analysis. Viscosity is neglected because the neutral and ion fluids are moving slowly and there are no surfaces for boundary layers to form. Also, the viscous heating is small. Gravity is neglected
because the gravitational force exerted on the species in the model is extremely small compared to the external electric field. The magnetic $\mathbf{V} \times \mathbf{B}$ force is neglected since there is no externally applied magnetic field, and the self induced magnetic field is very small for an arc.

Radial momentum for electrons:

$$0 = -e \cdot n_e \cdot E_r - \frac{\partial}{\partial r} p_r - m_e \cdot n_e \cdot \nu_{en} \cdot (v_{er} - v_{nr}) - m_e \cdot n_e \cdot \nu_{en} \cdot (v_{er} - v_r)$$

(2.1.4)

Radial momentum for ions:

$$0 = e \cdot n_i \cdot E_r - \frac{\partial}{\partial r} p_i - m_i \cdot n_i \cdot \nu_{mi} \cdot (v_{ir} - v_{nr}) + m_i \cdot n_i \cdot \nu_{mi} \cdot (v_{ir} - v_r)$$

(2.1.5)

Radial momentum for neutrals:

$$0 = -\frac{\partial}{\partial r} p_n + m_e \cdot n_e \cdot \nu_{en} \cdot (v_{er} - v_{nr}) + m_i \cdot n_i \cdot \nu_{mi} \cdot (v_{ir} - v_{nr})$$

(2.1.6)

Axial momentum for electrons:

$$0 = -e \cdot n_e \cdot E_z - m_e \cdot n_e \cdot \nu_{en} \cdot (v_{ez} - v_{nz}) - m_e \cdot n_e \cdot \nu_{en} \cdot (v_{ez} - v_z)$$

(2.1.7)

Axial momentum for ions:

$$0 = e \cdot n_i \cdot E_z - m_i \cdot n_i \cdot \nu_{mi} \cdot (v_{iz} - v_{nz}) + m_i \cdot n_i \cdot \nu_{mi} \cdot (v_{iz} - v_z)$$

(2.1.8)

Axial momentum for neutrals:

$$0 = m_e \cdot n_e \cdot \nu_{en} \cdot (v_{ez} - v_{nz}) + m_i \cdot n_i \cdot \nu_{mi} \cdot (v_{iz} - v_{nz})$$

(2.1.9)

2.3. Conservation of Energy

The Conservation of Energy equations include convection, compression, thermal conduction, ohmic heating, and energy equilibration between species. Radiation is neglected because the arc is optically thick.

For electrons:

$$\frac{3}{2} \cdot v_{er} \cdot \frac{\partial}{\partial r} p_r + \frac{p_r}{r} \cdot \frac{\partial}{\partial r} r \cdot v_{er} = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{es} \cdot \frac{\partial}{\partial r} (k_b \cdot T_e) \right] + \eta J^2$$

$$-\frac{3}{2} \cdot n_e \cdot \nu_{en}^E \cdot k_b \cdot (T_e - T_n) = \frac{3}{2} \cdot n_e \cdot \nu_{en}^E \cdot k_b \cdot (T_e - T)$$

(2.1.10)

For ions:

$$\frac{3}{2} \cdot v_{ir} \cdot \frac{\partial}{\partial r} p_i + \frac{p_i}{r} \cdot \frac{\partial}{\partial r} r \cdot v_{ir} = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{is} \cdot \frac{\partial}{\partial r} (k_b \cdot T_i) \right]$$

$$-\frac{3}{2} \cdot n_i \cdot \nu_{ni}^E \cdot k_b \cdot (T_i - T_n) = \frac{3}{2} \cdot n_e \cdot \nu_{en}^E \cdot k_b \cdot (T_e - T)$$

(2.1.11)

For neutrals:
\[
\frac{3}{2} \cdot v_{nr} \cdot \frac{\partial}{\partial r} p_n + \frac{p_n}{r} \cdot \frac{\partial}{\partial r} r \cdot v_{nr} = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{nn} \cdot \frac{\partial}{\partial r} (k_b \cdot T_n) \right] \\
+ \frac{3}{2} \cdot n_r \cdot \nu^e_{nn} \cdot k_b \cdot (T_r - T_n) + \frac{3}{2} \cdot n_r \cdot \nu^e_{nn} \cdot k_b \cdot (T_t - T_n)
\]

(2.1.12)

### 2.4. Maxwell's Equations

The two non-trivial Maxwell equations are Poisson’s equation and Faraday’s Law. In the low frequency limit, Poisson’s Equation is

\[
\rho = \varepsilon_0 \nabla \cdot \mathbf{E} \approx 0
\]

(2.2.1)

where \( \rho = \sum_{\alpha} n_{\alpha} q_{\alpha} \) is the summation of the charge density of all species.

Faraday’s Law is

\[
\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}
\]

(2.2.2)

### 3. Transport Coefficients

The results of this analysis are critically dependant upon the transport coefficients. These coefficients are summarized below.

#### 3.1. Ionization and Recombination

The most important quantities determining the physical characteristics of an arc are the ionization and recombination rates. The neutral molecules of the medium are ionized by excitation through electron impact.\(^4\) The ionization energy, \( E_i \), is the total amount of energy required to separate one electron from a molecule. Since each molecule already has a certain amount of thermal energy, the additional energy required to separate an electron varies from molecule to molecule.\(^5\) However, the typical thermal energies of neutral molecules in an arc are very small when compared to the ionization energy. Therefore, the energy contribution of the molecule's thermal energy is neglected in the calculation of

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\(^4\) Dover, p. 67.

\(^5\) Herzberg, p. 9.
the ionization energy. When an electron, with energy equal to or greater than
the ionization energy, strikes a neutral molecule, there is a certain likelihood that
an ionization event will occur. This probability is represented by the ionization
cross section. Integration of this ionization cross section across a Maxwellian
distribution of electrons gives an ionization rate which depends on the average
thermal energy of the electron population.\textsuperscript{6} Because the electron population has
a Maxwellian distribution and the average energy of electrons is much less than
the ionization energy, there is an exponential factor which describes the
proportion of electrons with energy sufficient to enable ionization.
Recombination occurs when an electron with energy less than the ionization
energy strikes an ionized molecule and is retained. The probability of this
downward transition is closely related to the probability of the ionization
transition. The main difference is that since almost all electrons have energies
less than the ionization energy, the exponential factor is not required.\textsuperscript{7} The
derivation of these rates is taken from Hutchinson’s Principals of Plasma
Diagnostics. In these derivations, Hutchinson accounts for both impact
transitions and transitions resulting from the interaction between the electric
field of a passing electron and the electric field of the target molecule. These two
processes have been evaluated across a Maxwellian distribution of electron
energies.
The detailed ionization cross section is\textsuperscript{8}:
\[
\langle \sigma \nu \rangle = A(B + C) \tag{3.1.1}
\]
\[
A = \pi \cdot a^2 \cdot \alpha \cdot c \cdot \frac{8}{\sqrt{\pi}} \cdot \frac{Ry}{\chi_i} \cdot \sqrt{\frac{Ry}{T_e}} \cdot e^{-u}
\]
\[
B = 1 - (1 + a) \cdot u \cdot e^{(s+u)} \cdot E_1(u + a \cdot u)
\tag{3.1.2}
\]
\[
C = f_{eg} \cdot \left\{ \ln \left( 4 \cdot \frac{s}{s + u} \right) + e^u \cdot E_1(u) - \frac{u}{s + u} \cdot e^{(s+u)} \cdot E_1(s + u) \right\}
\]
where \( u = \frac{\chi_i}{T_e} \) and \( a = 4 \).

\textsuperscript{6} Hutchinson, p. 239.

\textsuperscript{7} Hutchinson, p. 241.

\textsuperscript{8} Hutchinson, p. 240.
In these equations:

\( a_e = 10^{10} \) m is the radius of the target molecule
\( \alpha = 0.007297 \) is the fine structure constant
\( c = 2.998 \times 10^8 \) ms\(^{-1} \) is the speed of light
\( m_e = 9.109 \times 10^{-31} \) kg is the mass of the electron
\( R_y = 13.61 \) eV is the Rydberg energy
\( \chi = 15.58 \) eV is the ionization energy for \( \text{N}_2 \)
\( T_e \) is the electron temperature
\( f_{\text{eff}} \approx .28 \) is the effective oscillator strength per electron
\( s = .25 \) is a shape factor to fit empirical results

The following approximation for the exponential integral function is accurate to 0.5% for positive \( x \):\(^9\)

\[ e^x \cdot E_1(x) \approx \ln \left(1 + \frac{1}{x}\right) - \frac{0.56}{1 + 4.1 \cdot x + 0.9 \cdot x^2} \]  \( (3.1.3) \)

Also, in the regime where \( T_e \) is less than 2 eV, the following approximation is accurate to 0.4%.

\[ B + C \approx 0.0206 \cdot T_e \]  \( (3.1.4) \)

Evaluating fundamental constants and approximations, the ionization cross section reduces to

\[ \langle \sigma_v \rangle = C_I \cdot \sqrt{T_e} \cdot e^{\left[-\frac{15.58}{T_e}\right]} \frac{m^3}{s} \text{ where } C_I = 2.054 \cdot 10^{-14} \]  \( (3.1.5) \)

The recombination rate is found by eliminating the exponential factor.

\[ \langle \sigma_v \rangle = C_I \cdot \sqrt{T_e} \cdot \frac{m^3}{s} \text{ where } C_I = 2.054 \cdot 10^{-14} \]  \( (3.1.6) \)

The complete formulas are as follows:

\[ S_{\text{ionization}} = C_I \cdot n_e \cdot n_n \cdot T_e^{\frac{1}{2}} \cdot e^{-\left(\frac{15.58}{T_e}\right)} \frac{m^3}{s} \]  \( (3.1.7) \)

\[ S_{\text{recombination}} = C_I \cdot n_e^2 \cdot T_e^{\frac{1}{2}} \frac{m^3}{s} \]  \( (3.1.8) \)

---

\(^9\) Hutchinson, p. 240.
The following plots show ionization and recombination rates as a function of the electron temperature, $T_e$.

![Ionization Rate as a Function of Electron Temperature](image)

Figure 3-1
3.2. Resistivity

In the case of a simple long cylindrical arc, the total resistivity of the arc is a combination of the resistances due to electron collisions with both neutrals and ions. From analysis of the axial component of the Conservation of Momentum equations, which is found in Section 4.4, we find that the total resistivity of the arc is the sum of the resistivity of the ions and the neutrals.

\[ \eta = \eta_{\text{i}} + \eta_{\text{ne}} \] (3.1.9)

Each of these resistivities is proportional to the population densities and the collision times.\(^{10}\) In the electron-neutral interaction, the collisions are assumed to be elastic hard sphere collisions. In the electron-ion interaction, the collisions are coulomb collisions due to the charges on each of the species. The difference in the type of collision is reflected in the collision frequencies, where \(\nu_{\text{ne}}\) and \(\nu_{\text{ei}}\) are the electron-neutral and electron-ion momentum exchange collision frequencies.

\[ \eta_{\text{ne}} = \frac{m_e}{n_e \cdot e^2} \cdot \nu_{\text{ne}} \] (3.1.10)

\(^{10}\) Wesson, p. 70.
\[ \eta_{en} = \frac{m_e}{n_e \cdot e^2} \cdot \nu_e \]  

(3.1.11)

In the electron-neutral resistivity, \( \eta_{en} \), the average frequency of elastic collisions of electrons with neutrals\(^{11}\) is dependant on the neutral density, which is the target population, and the relative velocities. Since the electron velocities are so much greater than the velocities of the much more massive neutrals, the electron velocity can be taken as the relative velocity. Using \( \sigma_n \) as the hard sphere neutral cross section, and \( \nu_e \) as the electron thermal speed

\[ \nu_{en} = n_e \langle \sigma_n \cdot \nu_e \rangle \approx n_e \cdot \sigma_n \cdot \nu_e \]  

(3.1.12)

where the electron thermal speed is defined as

\[ \nu_e = \sqrt{\frac{3 \cdot k_e \cdot T_e}{m_e}} \]  

(3.1.13)

It must be noted that the cross section for elastic collisions of electrons on neutrals changes with the electron temperature, but in the specific regimes examined in this model, the neutral cross section may taken as an approximate constant, \( 4 \cdot 10^{-20} \text{m}^2 \). This gives the following formula for \( \eta_{en} \).

\[ \eta_{en} = C_{en} \cdot \frac{n_e}{n_{e_0}} \cdot \sqrt{T_e} \text{ ohm-m} \]  

(3.1.14)

where \( C_{en} = 1.026 \cdot 10^{-6} \)

Since the electron-neutral resistivity has a strong density dependence, the following plot uses density data from the solution of a code run. This gives a good approximation of the neutral resistivity as a function of electron temperature.

\(^{11}\) Chen, p. 157.
Figure 3-3

In the electron-ion resistivity, $\eta_{ii}$, the average frequency of coulomb collisions of electrons with ions$^{12}$ is based on a Debye shielded potential which limits the effects of the Coulomb force at ranges larger than the Debye length. Evaluation of the collision integral across impact parameters leaves a factor called the Coulomb logarithm.

$$\ln (\Lambda) = \ln \left( \frac{\lambda_d}{r_o} \right)$$  \hspace{1cm} (3.1.15)

where $\lambda_d$ is the Debye length, and $r_o$ is impact parameter for a 90 deg collision. This classical treatment of the logarithm holds for electron temperatures below 10 eV$^{13}$, where quantum mechanical effects are not apparent.

$$\lambda_d = \left( \frac{\varepsilon_0 \cdot k_b \cdot T_e}{n_e \cdot e^2} \right)^{\frac{1}{2}}$$  \hspace{1cm} (3.1.16)

$^{12}$ Wesson, p. 69.

$^{13}$ Wesson, p. 660.
\[ r_v = \frac{e_1 \cdot e_2}{12 \cdot \pi \cdot \varepsilon_o \cdot \hbar \cdot T_e} \quad (3.1.17) \]

This gives the following formula for the Coulomb logarithm,

\[ \ln(\Lambda) = \ln \left( C_\Lambda \cdot \frac{T_e^{\frac{3}{2}}}{n_e^{\frac{1}{2}}} \right) \quad \text{where} \quad C_\Lambda = 1.549 \cdot 10^{13} \quad (3.1.18) \]

For characteristic arcs in the plasmatron, the Coulomb logarithm is typically 7.
When the Coulomb logarithm is included, the electron-ion momentum exchange collision frequency is\(^\text{14}\),

\[ \nu_{ei} = \frac{n_e \cdot Z^2 \cdot e^4 \cdot \ln(\Lambda)}{3 \cdot (2 \cdot \pi \hbar)^{\frac{3}{2}} \cdot \varepsilon_o^{\frac{3}{2}} \cdot m_e^{\frac{1}{2}} \cdot (k_\Lambda \cdot T_e)^{\frac{3}{2}}} \quad (3.1.19) \]

For singly charged ions, Z=1, in a quasi-neutral plasma, \( n_e = n_i \), the electron-ion resistivity reduces to

\[ \eta_{ei} = \frac{C_{ei}}{T_e^{\frac{3}{2}}} \text{ ohm-m} \quad \text{where} \quad C_{ei} = 7.182 \cdot 10^{-4} \quad (3.1.20) \]

The following plot shows electron-ion resistivity as a function of the electron temperature, \( T_e \).

\(^{14}\) Wesson, p. 729.
Figure 3-4

As one can see in the two preceding graphs of electron-neutral and electron-ion resistivities, the neutral species resistivity dominates at electron temperatures up to 1.5 eV, because the plasma is only weakly ionized. Elastic collisions between electrons and neutrals dominate at lower temperatures, where the fractional ionization is low. The coulomb collisions between electrons and ions start to play a role in the resistivity as electron temperatures climb above 1.5 eV, and more of the plasma is ionized.

3.3. Ohmic Heating

In the energy balance equation, the primary energy source is ohmic heating. This energy is delivered to the electrons by the electric field, which is shown to be constant according to Faraday’s Law in Section 4. The ohmic heating density, is

\[ P_\Omega = \eta J^2 = \frac{E^2}{\eta} \quad (3.1.21) \]

Substituting for the resistivities detailed above,
\[ P_{th} = \frac{E^2}{C_{et} + C_{en} \cdot n_n \cdot \sqrt{T_e}} \frac{W}{m^3} \]

where

\[ C_{en} = 1.026 \cdot 10^{-6} \]
\[ C_{es} = 7.182 \cdot 10^{-4} \]

Using characteristic values for the electric field, species densities, and electron temperature,

\[ P_{th} = 5.5 \cdot 10^9 \frac{W}{m^3} \]

where

\[ n_e \approx 10^{30} \text{ m}^{-3} \]
\[ n_n \approx 10^{25} \text{ m}^{-3} \]
\[ T_e \approx 1.2 \text{ eV} \]

For a characteristic arc which has a length of 0.04 m and a radius of 0.001 m, total ohmic heating is approximately 690 W.

### 3.4. Thermal Conductivity

#### Electron Thermal Conductivity

Thermal conductivity transports heat generated in the center of the arc towards the cooler arc edges. The thermal conductivity process of electrons follows a simple random walk model\(^{15}\). The thermal conductivity coefficient expresses how much energy is transferred across a temperature gradient by collisions of like species\(^{16}\).

\[ \kappa_{ee} = n_e \cdot \lambda_{pp} \cdot \nu_{ee}^E = \frac{n_e \cdot v_e^2}{\nu_{ee}^E} \frac{W}{m \cdot J} \]

In the thermal diffusion term, \( \nu_{ee}^E \) is the heat exchange collision frequency between electrons and electrons due to coulomb collisions. The superscript E

\(^{15}\) Wesson, p. 88.

\(^{16}\) Holman, p. 7.
denotes the fact that this term is an energy exchange collision frequency as opposed to a momentum exchange collision frequency which does not have a superscript. This term is approximately equal to the collision frequency for momentum transfer for electron-ion collisions, \( \nu_{ei} \).

\[
\nu_{ee} \approx \nu_{ei} = \frac{n_e \cdot Z^2 \cdot e^4 \cdot \ln(A)}{3 \cdot (2 \cdot \pi)^{\frac{3}{2}} \cdot \varepsilon_0 \cdot m_e \cdot (k_b \cdot T_e)^{\frac{3}{2}}} \quad (3.1.25)
\]

Substituting for the heat exchange collision frequency, the thermal diffusion coefficient reduces to,

\[
\kappa_{ee} = C_{ee} \cdot T_e^{\frac{5}{2}} \frac{W}{m \cdot J} \quad \text{where} \quad C_{ee} = 2.607 \cdot 10^{22} \quad (3.1.26)
\]

This graph illustrates thermal diffusion coefficient as a function of electron temperature.

![Figure 3-5](image)

The electron thermal diffusion rate is a function of this coefficient and the electron temperature gradient.
Ion Heat Diffusion

The heat diffusion due to ions is calculated in a method similar to that of heat diffusion due to electrons. Since the ions in this model are singly ionized, there are only slight differences in the collision frequency calculation, such as a different mass dependence\footnote{Wesson, p. 730.}.

\[ \nu_{\text{iii}}^E = \frac{n_i \cdot Z^2 \cdot e^4 \cdot \ln(\Lambda)}{12 \cdot \pi^2 \cdot \varepsilon_{\text{a}}^2 \cdot m_i^{1/2} \cdot (k_B \cdot T_i)^{3/2}} \]  

(3.1.27)

This yields ion thermal diffusion of

\[ \kappa_{\text{ii}} = C_{\text{ii}} \cdot T_i^{5/2} \cdot \frac{W}{\text{m} \cdot \text{J}} \quad \text{where} \quad C_{\text{ii}} = 1.624 \cdot 10^{20} \]  

(3.1.28)

Comparing the ion thermal conductivity to the electron thermal conductivity, one finds that the ion thermal conductivity is much less due to the different mass dependencies.

\[ \kappa_{\text{ii}} \ll \kappa_{\text{ee}} \]  

(3.1.29)

The following graph shows the heat diffusion due to ions.
Figure 3-6

Neutral Heat Diffusion

Like the electron thermal diffusion, neutral thermal diffusion transports heat generated in the center of the arc towards the cooler arc edges. The thermal diffusion process of neutrals follows a simple random walk model. While the cross section of collisions of neutrals on neutrals, $\sigma_n$, varies slightly with temperature, it may approximated as a constant, $4 \cdot 10^{-20} \, \text{m}^2$.

$$ \kappa_{nn} = n_n \cdot \lambda_{mfp} \cdot \nu_{nn} = \frac{n_n \cdot \nu_n^2}{\nu_{nn}} = \frac{n_n}{\sigma_n} \quad (3.1.30) $$

which reduces to

$$ \kappa_{nn} = C_{nn} \cdot T_n^{1/2} \frac{W}{m \cdot J} \quad \text{where} \quad C_{nn} = 8.037 \cdot 10^{22} \quad (3.1.31) $$

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3.5. Thermal Equilibration

Heat Transfer from Electrons to Neutrals or Ions

Thermal equilibration transfers energy between a temperature difference between species. The first two terms to be examined are the heat transfer terms between electrons and neutrals and electrons and ions. The derivation of the two equations is similar, so it will be illustrated using the neutral term. The heat transfer is a function of the difference in temperature between the two species and the rate at which energy can be transferred via collisions.

\[ Q_{en} = \frac{3}{2} n_e \cdot \nu_{en}^E \cdot k_b \cdot (T_e - T_n) \]  \hspace{1cm} (3.1.32)

The heat exchange frequency, \( \nu_{en}^E \), in this equation is similar to the collision frequency, \( \nu_{en} \), in the resistivity equation, but it differs by a mass factor since we...
are now looking at both the collision frequency and the amount of energy transferred in each collision.\(^{18}\)

\[
\nu_{es}^E = \frac{2 \cdot m_e}{m_i} \cdot \nu_{es}
\]

Substituting for fundamental parameters in the heat exchange times for both neutrals and ions, we find that the equations can again be reduced to functions of temperature. However, in these terms, neutral and ion temperatures are also required.

For electrons and neutrals, we find a heat transfer rate of

\[
Q_{en} = C_{Q_{en}} \cdot n_e \cdot n_n \cdot T_e^{\frac{3}{2}} \cdot (T_e - T_n) \cdot \frac{W}{m^3} \text{ where } C_{Q_{en}} = 2.722 \cdot 10^{-37} \quad (3.1.34)
\]

And for electrons and ions, the heat transfer rate is

\[
Q_{ei} = C_{Q_{ei}} \cdot n_e^2 \cdot \frac{(T_e - T_i)}{T_e^{\frac{3}{2}}} \cdot \frac{W}{m^3} \text{ where } C_{Q_{ei}} = 1.905 \cdot 10^{-34} \quad (3.1.35)
\]

To get a feel for the sizes of the heat transfer rates, drop the neutral and ion temperatures because they are usually small when compared to the electron temperature. This plot uses characteristic densities to show the rate of energy equilibration as a function of electron temperature.

---

\(^{18}\) Wesson, p. 67.
While these energy equilibration rates are substantially different, they are both much smaller than the ion-neutral equilibration rate which is detailed below. This means that even though there may be some variations in the heat flow from electrons to ions and neutrals, the ions and neutrals will equilibrate fastest and their temperatures will be very close.

**Ion Heating by Electrons**

In the Ion Energy Equation, the electron heating term is the same as in the Electron Energy Equation, but the heat is entering the ion population as opposed to leaving the electron population.

\[ Q_{ie} = -Q_{ei} \]  \hspace{1cm} (3.1.36)
Neutral Heating by Electrons

As in the Ion Energy Equation, the electron heating term for the Neutral Energy Equation is the same as in the Electron Energy Equation, but the heat is entering the neutral population as opposed to leaving the electron population.

\[ Q_{ne} = -Q_{en} \tag{3.1.37} \]

Heat Transfer from Ions to Neutrals

For heat transfer between ions and neutrals, the collision time for energy transfer is approximately the same as the collision time for momentum transfer since the masses of the two species are approximately equal.

\[ \nu_{in}^E \approx \nu_{en} \tag{3.1.38} \]

where

\[ \nu_{in} = n_n \langle \sigma_n | u_i - u_n | \rangle \approx n_n \cdot \sigma_n \cdot v_i \tag{3.1.39} \]

This gives a collision frequency for energy transfer of

\[ \nu_{en}^E = n_n \cdot \sigma_n \cdot v_i \tag{3.1.40} \]

Again, while the cross section of collisions of ions on neutrals, \( \sigma_n \), varies with temperature, it may taken as an approximate constant, \( 4 \cdot 10^{-20} \text{m}^2 \). The heat transfer term is

\[ Q_{en} = \frac{3}{2} \cdot n_i \cdot \nu_{en}^E \cdot k_n \cdot (T_i - T_n) \tag{3.1.41} \]

which reduces to

\[ Q_{en} = C_{Q_{en}} \cdot n_i \cdot n_n \cdot \frac{T_i^2}{T_n^2} \cdot (T_i - T_n) \cdot \frac{W}{\text{m}^3} \tag{3.1.42} \]

where \( C_{Q_{en}} = 3.091 \cdot 10^{-36} \)

This graph uses characteristic densities to show the heat that would be transferred from ions to neutrals if the neutral temperature was dropped as negligible.
Figure 3-9

This graph shows that the energy equilibration rate between ions and neutrals is very large, even with very small differences in the temperatures of the two species. This means that the ions and neutrals will equilibrate very fast and their temperatures will be approximately the same.

3.6. Radiation

Radiation losses for arcs generally contribute very little to the energy balance, either because a negligible amount of energy is generated or because the optical density does not allow much energy to escape the arc. We show this by estimating the radiation power losses and comparing them to ohmic heating. Radiated power can be broken down into two main components; bremsstrahlung, and recombination radiation.
Bremsstrahlung

Bremsstrahlung is the radiation resulting from acceleration of electrons due to coulomb collisions. The derivation from Wesson\textsuperscript{19} calculates the radiated power density due to bremsstrahlung.

\[
P_{\text{br}em} = \frac{g \cdot e^6 \cdot Z^2 \cdot n_e \cdot n_i \cdot (k_B^2 \cdot T_e^{\frac{3}{2}})}{6 \cdot \left(\frac{3}{2}\right)^{\frac{1}{2}} \cdot \pi^{\frac{3}{2}} \cdot \varepsilon \cdot e^3 \cdot h \cdot m_e^2}
\]

(3.1.43)

where \( g \) is the Gaunt Factor, taken as 1.1 in our region of interest. This reduces to,

\[
P_{\text{br}em} = C_{\text{br}em} \cdot n_e \cdot n_i \cdot T_e^{\frac{3}{2}} \cdot \frac{W}{m^3}
\]

where \( C_{\text{br}em} = 1.715 \cdot 10^{-38} \) (3.1.44)

Substituting characteristic values,

\[
P_{\text{br}em} = 188 \ \frac{W}{m^3}
\]

where

\[
n_e \approx n_i \approx 10^{20} \text{ m}^{-3}
\]

\[
T_e \approx 1.2 \ \text{eV}
\]

(3.1.45)

This is negligible when compared with ohmic heating density of 5 GW/m\(^3\).

Line and Recombination Radiation

Line and recombination radiation is emitted when a molecule transitions from an excited or ionized state to the ground state. For transition events in the arc, energies are limited to the ionization energy of 15.58 eV or less. To show that the power loss due to this radiation is negligible, we will show that the arc is optically thick for even the most energetic of photons. Since the arc is optically thick and radiation inside the arc is reabsorbed, the arc will radiate as a blackbody at the neutral species temperature, \( T_a \).

To show that the arc is optically thick, one must calculate the intensity of radiation which escapes the plasma without being absorbed\textsuperscript{20}. This calculation is a function of the optical depth of the plasma, \( \tau (\lambda) \), where \( \lambda \) is the wavelength of

\textsuperscript{19} Wesson, p. 228.

\textsuperscript{20} Wesson, p. 519.
the radiation under consideration and \( I_B(\lambda) \) is the blackbody radiation intensity for that wavelength.

\[
I_{\text{lin}}(\lambda) = I_B(\lambda) \cdot \left(1 - e^{-\tau(\lambda)}\right) \frac{W}{m^2} \quad (3.1.46)
\]

When \( \tau(\lambda) \geq 1 \) a substantial portion of the radiation is reabsorbed and the plasma is considered optically thick\(^{21}\). The emergent radiation for an arc which is optically thick is the blackbody radiation.

The optical depth for line radiation\(^{22}\) is

\[
\tau(\lambda) = 1.33 \cdot 10^4 \cdot \lambda \cdot \left(\frac{m_u}{T_u}\right)^{\frac{1}{2}} \cdot n_u \cdot L \quad (3.1.47)
\]

where \( \lambda \) is the wavelength of the radiation and \( L \) is the physical path length of radiation in the arc.

The wavelength of photons with an energy of 15.58 eV is

\[
\lambda = \frac{h \cdot c}{k_B \cdot 15.58} = 7.96 \cdot 10^{-8} \text{ m} \quad (3.1.48)
\]

Substituting characteristic values and solving for optical depth,

\[\tau = 7.2 \cdot 10^5\]

where

\[n_u \approx 10^{25} \text{ m}^{-3}\]

\[T_u \approx 0.1 \text{ eV}\]

\[L \approx 0.001 \text{ m}\]

For 15.58 eV photons \( \tau \gg 1 \) and the arc is extremely thick. This holds for all photons with 15.58 eV or less. Since the transition energy in the arc is capped at 15.58 eV, the arc will radiate at blackbody levels. Using Stefan’s Law, we can find the total radiant intensity over all wavelengths\(^{23}\).

\(^{21}\) Wesson, p. 519.

\(^{22}\) NRL Formulary, p.57.

\(^{23}\) Krane, p. 61.
\[ I_B = \sigma \cdot T_{edge}^4 \]

where

\[ \sigma \approx 1.027 \cdot 10^9 \frac{W}{m^2 \cdot eV^4} \]
\[ T_{edge} \approx T_x \approx 0.025 \text{ eV} \]  

For a characteristic arc which has a length of 0.04 m and a radius of 0.001 m,

\[ P_B = I_B \cdot A = 0.1 \text{ W} \]

where

\[ A = 2.513 \cdot 10^{-4} \text{ m}^2 \]

Comparing this to the ohmic heating power of 690 W, blackbody losses are negligible.

4. The Reduced Model

As it now stands, there are 16 unknowns in the model, including densities, velocities, temperatures, and partial pressures. This section details the introduction of an asymptotic expansion in order to reduce the model to a set of three equations and three unknowns: \( T_r, T_u, \) and \( P_c \). The ordering keeps the maximal amount of physics in the model with the minimal amount of approximations required to reduce the model.

4.1. The Basic Ordering

To establish a basis for comparing terms, one must define a comparison factor which differentiates between large and small terms. We introduce a small ordering parameter \( \varepsilon \ll 1 \). In order to reduce this model, three ratios are useful: the mass ratio of electrons to ions; the ratio of the collision cross sections of hard sphere collisions to electron coulomb collisions; and the ratio of the density of electrons to the density of ions.

**Mass ratio**

The first small parameter is the mass ratio of electrons to nitrogen molecules, which is ordered as follows:
\[ \frac{m_e}{m_n} \sim \varepsilon \quad (4.1.1) \]

Substituting numerical values for the masses,
\[ \varepsilon \approx 10^{-5} \quad (4.1.2) \]

This can be used for practical comparisons.

**Collision Cross Section Ratio**

The next small parameter is the ratio of the hard sphere collision cross section to the coulomb cross section.
\[ \frac{\sigma_n}{\sigma_e} \sim \varepsilon \quad (4.1.3) \]

The collision cross section for hard sphere collisions is applicable for both electron-neutral collisions and ion-neutral collisions. Its typical value is the cross section of the neutral molecule.
\[ \sigma_n = 4 \cdot 10^{-20} \text{ m} \quad (4.1.4) \]

The coulomb collision cross section is applicable to collisions between charged particles and it can be estimated from collision frequencies. The collision frequency for electrons with ions is
\[ \nu_{ei} = \frac{\sqrt{2} \cdot n_i \cdot e^4 \cdot \ln(\Lambda)}{12 \cdot \pi^2 \cdot \varepsilon_e^2 \cdot m_e^2 \cdot (k_B \cdot T_e)^2} = n_i \cdot \sigma_e \cdot v_e \quad (4.1.5) \]

Solving for \( \sigma_e \),
\[ \sigma_e = \frac{\sqrt{2} \cdot e^4 \cdot \ln(\Lambda)}{12 \cdot \pi^2 \cdot \varepsilon_e^2 \cdot (k_B \cdot T_e)^2} \quad (4.1.6) \]

For characteristic electron temperatures of 1 eV, and a characteristic coulomb logarithm for electrons of 7,
\[ \sigma_e \approx 10^{-16} \quad (4.1.7) \]

Substituting numerical values for the coulomb cross section and the neutral cross section,
\[ \frac{\sigma_n}{\sigma_e} \approx 10^{-4} \quad (4.1.8) \]

One can see that the linear approximation with \( \varepsilon \) is appropriate.
**Characteristic Ionization**

The next ordering parameter to examine is the density ratio. Since the plasma is weakly ionized, the corresponding approximate ordering is

\[
\frac{n_i}{n_n} \approx \frac{n_i}{n_e} \sim \varepsilon
\]  
(4.1.9)

For a neutral density on the order of $10^{25}$ m$^{-3}$ this corresponds to an electron density on the order of $10^{20}$ m$^{-3}$, which is plausible, but difficult to verify experimentally.

**Temperature Ratio**

The final parameter of interest is the temperature ratio, and the approximate ordering is found to be

\[
\frac{T_i}{T_e} \approx \frac{T_n}{T_e} \sim \varepsilon^{\frac{1}{3}}
\]  
(4.1.10)

The ions and neutrals are much cooler than the electrons. The $\varepsilon^{\frac{1}{3}}$ scaling implies

\[
\frac{T_n}{T_e} \sim 10^{-1}
\]  
(4.1.11)

which is plausible, but difficult to measure.

There is one final point regarding the expansion of the neutral pressure. The correct representation is

\[
p_n(r) = p_\infty + \tilde{p}_n(r)
\]  
(4.1.12)

where $p_\infty$ is the constant background neutral pressure. In this representation, the change in neutral pressure is caused by increases in electron and ion pressures. However, in our ordering the ion pressure is small and may be neglected.

\[
\frac{p_i}{p_e} \sim \varepsilon^{\frac{1}{3}} \ll 1
\]  
(4.1.13)

This leads to an approximate ordering of $\tilde{p}_n(r)$ as

\[
\frac{\tilde{p}_n(r)}{p_e} \approx 1
\]  
(4.1.14)

These ordering assumptions
\[ \frac{m_n}{m_n} \approx \frac{\sigma_n}{\sigma_n} \approx \frac{n_e}{n_e} \approx \varepsilon \]
\[ \frac{T_n}{T_e} \approx \frac{p_n}{p_e} \approx \varepsilon^{\frac{1}{3}} \]
\[ \bar{p}_n(r) \approx 1 \]
\[ p_e \]

are sufficient to estimate all other quantities that appear in the model equations.

The ordering scheme, defined above, is introduced as a starting hypothesis.

Next, the model equations are analyzed in order to obtain a reduced model, and the ordering scheme hypothesis is shown to accurately represent the physics of interest.

4.2. Maxwell’s Equations

The two non-trivial Maxwell Equations are Poisson’s Equation and Ampere’s Law.

**Poisson’s Equation**

\[ \nabla \cdot \mathbf{E} = \frac{\varepsilon_0}{\varepsilon_n} (n_i - n_e) \]  \hspace{1cm} (4.2.1)

Examining the plasma Debye Length,

\[ \lambda_D = \left( \frac{\varepsilon_n \cdot k_B \cdot T_e}{n_e \cdot e^2} \right)^{\frac{1}{3}} \]  \hspace{1cm} (4.2.2)

and taking conservative characteristic values of \( n_e = 10^{20} \, \text{m}^{-3} \) and \( T_e = 1 \, \text{eV} \), one finds a typical Debye Length of \( \lambda_D = 10^{-6} \, \text{m} \). Given the model’s characteristic length, \( a = 10^{-3} \, \text{m} \), one finds that \( \lambda_D \ll a \). In this regime, it is well known that the plasma behaves like a quasi-neutral fluid.

\[ (n_i - n_e) \approx \frac{\varepsilon_0}{e} \nabla \cdot \mathbf{E} \approx 0 \]  \hspace{1cm} (4.2.3)

or

\[ n_e \approx n_i \]  \hspace{1cm} (4.2.4)

This also implies

\[ \nabla \cdot \mathbf{J} = 0 \]  \hspace{1cm} (4.2.5)
Faraday’s Law

Turning next to Faraday’s Law,
\[ \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \]  
(4.2.6)

Given steady state and a negligible B field, Faraday’s Law reduces to,
\[ \nabla \times \mathbf{E} = 0 \]  
(4.2.7)

In the cylindrical model, this allows one to set the E field as a constant in the axial direction.
\[ \mathbf{E}(r, \phi, z) = E_r(r) \hat{r} + E_\phi \hat{\phi} + E_z \hat{z} \]  
(4.2.8)

\( E_\phi \) is a constant related to the applied voltage. \( E_r \) is the radial electric field required to keep the radial particle diffusion ambipolar.

4.3. Conservation of Mass Equations

Analysis of the conservation of mass equations gives relations between the radial velocities of each species \( (v_{er}, v_{er}, v_{sr}) \) and one of the three basic equations in the reduced model.

From Poisson’s Equation, the plasma is quasi-neutral, so
\[ n_i \approx n_e \]  
(4.3.1)

Using this relation, and the fact that the source terms cancel exactly, subtract the ion mass equation from the electron mass equation.
\[ \frac{1}{r} \frac{\partial}{\partial r} r \cdot n_e \cdot v_{er} - \frac{1}{r} \frac{\partial}{\partial r} r \cdot n_e \cdot v_{er} = 0 \]  
(4.3.2)

The solution of this equation which is regular at the origin gives
\[ v_{er} = v_{er} \]  
(4.3.3)

Next, add the ion and electron equations and subtract twice the neutral mass equation, noting that the source terms again cancel exactly.
\[ \frac{1}{r} \frac{\partial}{\partial r} r \cdot n_e \cdot v_{en} + \frac{1}{r} \frac{\partial}{\partial r} r \cdot n_e \cdot v_{er} = 2 \frac{1}{r} \frac{\partial}{\partial r} r \cdot n_e \cdot v_{er} \]  
(4.3.4)

Solving for the neutral velocity and again using the solution which is regular at the origin, one finds:
\[ v_{nr} \approx \frac{-n_e}{n_n} v_{en} \sim -\varepsilon \cdot v_{er} \]  
(4.3.5)

This shows that the radial neutral flow velocity is very small when compared to the ambipolar flow velocity.
The final equation of interest is the electron Conservation of Mass equation.

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \cdot n_e \cdot v_{er}) = -S_{\text{recombination}} + S_{\text{ionization}}
\]  

(4.3.6)

In the radial momentum equations, we find a relation for \( v_{er} \) which allows diffusion to be expressed in terms of electron partial pressure. This gives one of the three equations in the reduced model.

### 4.4. Radial Momentum Equations

With these equations, one finds relations between species pressures, and a relation between pressure and ambipolar flow velocity, \( v_{pr} \).

First, add all three radial momentum equations, noting that the momentum exchange terms cancel exactly.

\[
\frac{\partial}{\partial r} (p_e + p_i + p_n) = 0
\]  

(4.4.1)

Integrating this equation and assuming that electron and ion pressures are negligible at the plasma edge, the integration constant is set to the ambient pressure outside the arc, \( p_\infty \):

\[
p_e (r) + p_i (r) + p_n (r) = p_\infty
\]  

(4.4.2)

or

\[
p_e (r) + p_i (r) = 0
\]  

(4.4.3)

This relation states that the total pressure across the radius of the arc is a constant, independent of \( r \).

Next, add the electron and ion radial momentum equations in order to cancel the electric field term.

\[
\frac{\partial}{\partial r} (p_e + p_i) + m_e \cdot n_e \cdot \nu_{en} \cdot (v_{er} - v_{nr}) + m_i \cdot n_i \cdot \nu_{in} \cdot (v_{ir} - v_{nr}) = 0
\]  

(4.4.4)

From the Conservation of Mass equations, we know that the electron and ion radial flow velocities are equal and that they are both much larger than the neutral flow velocity. Neglecting the neutral radial velocity and setting the radial ion flow velocity to the radial electron flow velocity, the equation reduces to:

\[
\frac{\partial}{\partial r} (p_e + p_i) = -n_e \cdot v_{er} \cdot (m_e \cdot \nu_{en} + m_i \cdot \nu_{in})
\]  

(4.4.5)

Next, compare \( m_e \cdot \nu_{en} \) and \( m_i \cdot \nu_{in} \).
\[ \frac{m_e \cdot \nu_{en}}{m_i \cdot \nu_{in}} \approx \frac{m_e}{m_i} \cdot \frac{\nu_{en}}{\nu_{in}} \approx \left( \frac{m_e}{m_i} \right)^{\frac{1}{2}} \cdot \left( \frac{T_e}{T_i} \right)^{\frac{1}{2}} \sim \varepsilon^{\frac{1}{2}} \ll 1 \]  

(4.4.6)

Since \( m_e \cdot \nu_{en} \) is small, and \( p_i \ll p_e \) the equation may be rewritten as:

\[ \frac{\partial}{\partial r} p_e = -n_e \cdot \eta_{en} \cdot m_i \cdot \nu_{in} \]  

(4.4.7)

Solving for \( n_e \cdot \nu_{ex} \):

\[ n_e \cdot \nu_{ex} = -\frac{1}{m_i \cdot \nu_{in}} \frac{\partial}{\partial r} p_e \]  

(4.4.8)

Later, this equation will be substituted for the diffusion coefficient in the electron conservation of mass equation to model particle diffusion.

The final piece of information contained in the radial momentum equation is an expression for the radial electric field.

\[ E_r = \frac{1}{\varepsilon \cdot n_e} \cdot \left( \frac{\partial}{\partial r} p_r + m_e \cdot n_e \cdot \nu_{en} \cdot \nu_{ex} \right) \]  

(4.4.9)

This expression comes directly from the electron radial momentum equation and uses the same assumptions detailed above.

### 4.5. Axial Momentum Equations

With these equations, one finds that the axial velocities of ions and neutrals are small in relation to electron axial velocities. Also, Ohm's Law for the plasma is derived.

First, assume that the total axial momentum in the arc is approximately zero.

\[ m_e \cdot n_e \cdot \nu_{ez} + m_i \cdot n_i \cdot \nu_{iz} + m_n \cdot n_n \cdot \nu_{nz} = 0 \]  

(4.5.1)

Neglecting the electron contribution, because both their mass and population density are small, and solving for the neutral axial velocity, one finds:

\[ \nu_{nz} \approx -\frac{m_i}{m_n} \cdot \frac{n_i}{n_n} \cdot \nu_{iz} \sim -\varepsilon \cdot \nu_{iz} \]  

(4.5.2)

One finds that the neutral axial velocity is very small and may be neglected.

Next, add the electron and ion axial momentum equations.

\[ 0 = -m_e \cdot n_e \cdot \nu_{en} \cdot (\nu_{ez} - \nu_{nz}) - m_i \cdot n_i \cdot \nu_{in} \cdot (\nu_{iz} - \nu_{nz}) \]  

(4.5.3)

Neglecting the neutral axial velocity, solve for the ion axial velocity.

\[ \nu_{iz} = -\left( \frac{m_e \cdot T_e}{m_i \cdot T_i} \right)^{\frac{1}{2}} \cdot \nu_{ez} \sim -\varepsilon^{\frac{1}{2}} \cdot \nu_{ez} \]  

(4.5.4)
This ordering shows that the current due to ions is small when compared with the current due to electrons, so one can well approximate the axial current using only electrons.

\[ J_z = -e \cdot n_e \cdot v_e \quad (4.5.5) \]

Returning to the electron axial momentum equation, solve for the electric field in terms of the axial current.

\[ E_z = \frac{m_e \cdot (\nu_{en} + \nu_{en})}{n_e \cdot e^2} \cdot J_z \quad (4.5.6) \]

Recalling Faraday’s Law, one finds that the axial electric field is constant across the radius of the arc, so this equation may be rewritten as:

\[ E_z = \eta \cdot J_z \quad (4.5.7) \]

where the total resistivity is the sum of the resistivities due to each species.

\[ \eta = \frac{m_e}{n_e \cdot e^2} \cdot (\nu_{en} + \nu_{en}) \quad (4.5.8) \]

Note that although \( E_z \) is a constant, its value is not known. Ultimately, it will be determined by the plasma current, \( I \), which is generally a known input to the problem.

4.6. Conservation of Energy Equations

The electron and neutral energy equations represent the final two basic equations in the reduced model. The ion energy equation yields a simple relation for species temperatures; \( T_e \), \( T_i \), and \( T_n \).

Electron Energy Equation

The analysis begins with the electron energy equation, which is repeated here for convenience.

\[ \frac{3}{2} \cdot v_e \cdot \frac{\partial}{\partial r} p_e + \frac{p_e}{r} \cdot \frac{\partial}{\partial r} r \cdot v_e = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{en} \cdot \frac{\partial}{\partial r} (k_e \cdot T_e) \right] + \eta J^2 \]

\[ -\frac{3}{2} \cdot n_e \cdot \nu^E_{en} \cdot k_e \cdot (T_e - T_n) - \frac{3}{2} \cdot n_e \cdot \nu^E_{en} \cdot k_e \cdot (T_e - T_i) \quad (4.5.9) \]

In the simplification of this equation of this term, note that \( \nu^E_{en} \sim \nu^E_{en} \) and both temperature equilibration terms must be maintained. Next, note that in the maximal ordering scheme, the heat conduction term is competitive with the
equilibrations terms. This requirement sets a scaling relationship for the plasma radius. These two terms scale as follows:

Heat Conduction:

\[
\frac{1}{r} \frac{\partial}{\partial r}\left[r \cdot \kappa_r \frac{\partial}{\partial r} k_b \cdot T_e\right] \approx \frac{k_b \cdot T_e \cdot v_e}{a^2 \cdot \sigma_e}
\]  

(4.5.10)

Electron-Ion Equilibration:

\[
\frac{3}{2} n_e \cdot \nu_{ei} \cdot k_b \cdot (T_e - T_i) \approx \frac{m_e}{m_i} \cdot n_e^2 \cdot v_e \cdot \sigma_e \cdot k_b \cdot T_e
\]

(4.5.11)

Their ratio gives the scaling for the radius, \(a\).

\[
a \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \cdot \frac{1}{n_e \cdot \sigma_e}
\]  

(4.5.12)

For typical values, \(a \sim 3 \text{ mm}\), which is consistent with experimental observations.

Now examine the ohmic heating term, which is the only source term. In order for the model to make physical sense, the ohmic heating must compete with the equilibration terms, and this sets the scale of the plasma.

Ohmic Power:

\[
\eta \cdot J^2 \approx \frac{m_e \cdot \nu_{ei}}{n_e \cdot e^2} \cdot \frac{I^2}{\pi^2 \cdot a^4}
\]  

(4.5.13)

The ratio of ohmic power to electron-ion equilibration sets the scaling for the plasma current.

\[
I \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \cdot \frac{e \cdot v_e}{n_e \cdot \sigma_e^2}
\]  

(4.5.14)

For typical values, \(I \sim 1 \text{ A}\), which is consistent with experimental values.

Finally, note that convection and compression are of the same order, and they scale as follows:

Convection and Compression:

\[
\frac{3}{2} \cdot n_e \cdot \frac{\partial}{\partial r} p_e \approx \frac{n_e \cdot \nu_{re} \cdot k_b \cdot T_e}{a}
\]  

(4.5.15)

The ratio of the convection or compression terms to the electron-ion equilibration terms is

\[
\frac{\text{Convection}}{\text{e-i Equilibration}} \approx \frac{m_e \cdot \nu_{re} \cdot e^2}{m_e \cdot n_e \cdot \sigma_e \cdot v_e \cdot a} \approx \frac{m_e \cdot n_e \cdot \sigma_e \cdot v_e}{m_i \cdot n_i \cdot \sigma_i \cdot v_i} \approx \epsilon^3 \ll 1
\]  

(4.5.16)
Convection and compression terms may be neglected as small terms.

With these simplifications, the electron energy equation reduces to:

\[
\frac{1}{r} \frac{\partial}{\partial r} r \cdot \kappa_e \cdot \frac{\partial}{\partial r} (k_b \cdot T_e) + \eta \cdot J^2 - \frac{3}{2} n_e \cdot \left( \nu_{em}^e + \nu_{en}^e \right) \cdot k_b \cdot T_e = 0
\]  

(4.5.17)

**Ion Energy Equation**

Now examine the ion energy equation. If the ion-neutral thermal equilibration term is taken as the reference, the other terms scale as follows:

\[
\begin{align*}
\text{ion-electron equilibration} & \sim \frac{\nu_{ei}^e}{\nu_{in}^e} \cdot T_e \sim 1 \\
\text{ion-neutral equilibration} & \sim \frac{\nu_{en}^e}{\nu_{in}^e} \cdot T_e \sim 1 \\
\text{ion thermal conductivity} & \sim \frac{\kappa_i}{a^2 \cdot n_i \cdot \nu_{in}^e} \sim \varepsilon \ll 1 \\
\text{convection, compression} & \sim \frac{\nu_{en}^e}{a \cdot \nu_{in}^e} \sim \varepsilon \ll 1
\end{align*}
\]

(4.5.18) (4.5.19) (4.5.20)

These scalings reduce the ion energy equation to

\[
T_i - T_n = \frac{\nu_{en}^e}{\nu_{in}^e} \cdot \frac{1}{3} \cdot T_e \sim \varepsilon^3 \cdot T_e
\]

(4.5.21)

**Neutral Energy Equations**

The final Conservation of Energy equation is the neutral energy equation. In this equation, the thermal equilibration terms have already been shown to be of the same order. The other terms in the equation scale as follows:

\[
\begin{align*}
\text{neutral convection} & \sim \frac{\bar{p}_n}{\nu_{in}^e} \sim \varepsilon^2 \\
\text{neutral compression} & \sim \bar{p}_n \sim \varepsilon^3 \\
\text{thermal conductivity} & \sim \frac{\kappa_{nn}}{a^2 \cdot n_i \cdot \nu_{in}^e} \sim 1 \\
\text{ion-neutral equilibration} & \sim \frac{n_n \cdot \nu_{en}^e}{a \cdot \nu_{in}^e} \sim \varepsilon^3 \ll 1
\end{align*}
\]

(4.5.22) (4.5.23) (4.5.24)

The neutral energy equation now stands as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \cdot \kappa_{nn} \cdot \frac{\partial}{\partial r} (k_b \cdot T_n) \right] + \frac{3}{2} n_e \cdot \nu_{en}^e \cdot k_b \cdot T_e \\
+ \frac{3}{2} n_i \cdot \nu_{in}^e \cdot k_b \cdot (T_i - T_n) = 0
\]

(4.5.25)
We can substitute the electron-ion equilibration term for the ion-neutral energy equilibration term to eliminate the \((T_i - T_n)\) term and obtain the final equation in the reduced model.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \cdot \kappa_{an} \cdot \frac{\partial}{\partial r} (k_i \cdot T_n) \right] + \frac{3}{2} \sigma_e \cdot \left( \nu_{en} + \nu_{ei} \right) \cdot k_i \cdot T_e = 0
\]  

(4.5.26)

This completes the reduction of the model.

### 4.7. Summary of Basic Equations

The reduced model now contains three equations and three unknowns, \(T_e, T_n,\) and \(p_e.\) The equations are summarized below with the applicable transport coefficients and boundary conditions.

**Conservation of Mass**

For electrons:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{m_i} \frac{\partial}{\partial r} p_e \right] = S_{recombination} - S_{ionization}
\]

(4.7.1)

**Conservation of Energy**

For electrons:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \cdot \kappa_{au} \cdot \frac{\partial}{\partial r} (k_i \cdot T_e) \right] + \frac{E_e^2}{\eta} - \frac{3}{2} \sigma_e \cdot \frac{m_e}{m_i} \left( \nu_{en} + \nu_{ei} \right) = 0
\]

(4.7.2)

For neutrals:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \cdot \kappa_{au} \cdot \frac{\partial}{\partial r} (k_i \cdot T_n) \right] + \frac{3}{2} \sigma_e \cdot \frac{m_e}{m_i} \left( \nu_{en} + \nu_{ei} \right) = 0
\]

(4.7.3)

**Transport Coefficients**

\[
\kappa_{au} = \frac{2 \cdot p_e}{m_i \cdot \nu_{ei}}
\]

(4.7.4)

\[
\kappa_{au} = \frac{2 \cdot p_{\infty}}{m_i \cdot \nu_{au}}
\]

(4.7.5)

\[
\eta = \frac{m_e \cdot \left( \nu_{en} + \nu_{ei} \right)}{n_e \cdot e^2}
\]

(4.7.6)
Collision Frequencies

\[ \nu_{ei} = n_e \cdot \sigma_e \cdot v_e = \frac{e^4 \cdot \ln(\Lambda)}{3 \cdot (2 \cdot \pi)^{\frac{3}{2}} \cdot \varepsilon_0^{\frac{2}{2}} \cdot m_e^{\frac{1}{2}} \cdot (k_b \cdot T_e)^{\frac{5}{2}}} \cdot \frac{p_e}{T_e} \]  

(4.7.7)

\[ \nu_{ni} = n_n \cdot \sigma_n \cdot v_n = \left( \frac{2}{m_e} \right)^{\frac{1}{2}} \cdot \frac{p_n \cdot \sigma_n \cdot T_e^{\frac{1}{2}}}{T_n} \]  

(4.7.8)

\[ \nu_{nn} = n_n \cdot \sigma_n \cdot v_n = \left( \frac{2}{m_e} \right)^{\frac{1}{2}} \cdot \frac{p_n \cdot \sigma_n}{T_n^{\frac{1}{2}}} \]  

(4.7.9)

\[ \nu_{nn} = n_n \cdot \sigma_n \cdot v_n = \nu_{nn} \]  

(4.7.10)

Ionization and Recombination

\[ S_{\text{ionization}} = n_n \cdot n_e \cdot \langle \sigma, \nu \rangle = n_n \cdot n_e \cdot A(B + C) \]

where

\[ A = \pi \cdot a^2 \cdot \alpha \cdot c \cdot \frac{8}{\sqrt{\pi}} \cdot \frac{R_y}{X_i} \cdot \sqrt{\frac{R_y}{T_e}} \cdot e^{\frac{1}{8}} \]  

(4.7.11)

\[ B = 1 - (1 + a) \cdot u \cdot e^{(s + u)} \cdot E_i (u + a \cdot u) \]

\[ C = f_{eff} \cdot \left( \ln \left( \frac{4 \cdot s}{s + u} \right) + e^u \cdot E_i (u) - \frac{u}{s + u} \cdot e^{(s + u)} \cdot E_i (u + u) \right) \]

This reduces to

\[ S_{nn} = \frac{4 \pi a^2 c}{5} \left( \frac{8 R_y}{\pi m_e c^2} \right) \left( \frac{R_y}{X_i} \right)^{\frac{1}{2}} \left( \frac{p_n}{p_e} \right) \left( \frac{p_e}{T_e} \right) \left( \frac{X_i}{T_n} \right) \left( \frac{1}{T_e} \right) e^{-\frac{1}{8}} \]  

(4.7.12)

\[ S_{nn} = \frac{n_n \cdot e^{\frac{8}{s}}}{{s + u} \cdot S_{nn}} = \frac{p_e \cdot T_e}{p_n \cdot T_n} \cdot e^{\frac{8}{s}} \cdot S_{nn} \]  

(4.7.13)

Boundary Conditions

The boundary conditions are determined by the physical environment of the arc. In the middle of the arc, the gradients of the partial pressure of electrons and species temperatures are zero.

\[ \frac{\partial}{\partial r} p_e (0) = 0 \]  

(4.8.1)

\[ \frac{\partial}{\partial r} (k_b \cdot T_e (0)) = 0 \]  

(4.8.2)
\[
\frac{\partial}{\partial r} \left( k \cdot T_n(0) \right) = 0
\]  

(4.8.3)

At the edge of the arc, species temperatures approach room temperature, and the electron pressure approaches zero since the electron density is very small.

\[
T_e(a) = T_{room}
\]  

(4.8.4)

\[
T_n(a) = T_{room}
\]  

(4.8.5)

\[
p_e(a) = 0
\]  

(4.8.6)

### 4.8. Normalized Equation

The sizes of various quantities in the model can be reasonably estimated through a normalization scheme. In this scheme, certain parameters are given and do not change over the entire operating regime of the plasmatron. These parameters are:

- **Background neutral pressure**
  \[ p_\infty = 1 \text{ atm} = 10^5 \text{ Pascals} \]  
  (4.8.7)

- **Background neutral temperature**
  \[ T_\infty = \text{room temperature} = 0.025 \text{ eV} \]  
  (4.8.8)

- **Hard sphere collision cross section**
  \[ \sigma_n = 4 \cdot 10^{-30} \text{ m}^2 \]  
  (4.8.9)

In addition to these given parameters, the plasma current and plasma radius are assumed to be specified. However, these values will vary with the experimental conditions. Typically \( I \sim 1 \text{ A} \) and \( a = 10^{-3} \text{ m} \).

We introduce the following normalized variables:

\[
T_e = e \cdot U_e
\]

\[
T_n = n \cdot U_n
\]

\[
p_e = e \cdot P_e
\]

\[
r = a \cdot x
\]  

(4.8.10)

The goal is to define the four normalization constants in terms of the five specified parameters. The is done by substituting the normalize terms into the electron and neutral energy equations and then setting all numerical coefficients to unity. After some algebra, the coefficients are reduced to three coupled equations.

Electron thermal conductivity and ohmic heating:
\[ T_{na}^{-2} \cdot T_{na} \cdot p_{na} = K_1 \cdot \frac{\dot{I}^2}{a^2} \]

where

\[ K_1 = \frac{9 \cdot m_e^2 \cdot p_{\infty} \cdot \sigma_n \cdot C}{\sqrt{2} \cdot 4 \cdot \pi^2 \cdot e^2} = 9.88 \]

\[ C = \frac{e^4 \cdot \ln(\Lambda)}{3 \cdot (2 \cdot \pi)^{\frac{3}{2}} \cdot \varepsilon_0^2 \cdot m_e^2} = 1.86 \cdot 10^{-39} \]  (4.8.11)

Electron-neutral thermal equilibration and ohmic heating:

\[ \frac{p_{na}^2}{T_{na}} = K_2 \cdot \frac{\dot{I}^2}{a^4} \]

where

\[ K_2 = \frac{3 \cdot m_e}{4 \cdot \pi^2 \cdot e^2} = 1.11 \cdot 10^4 \]  (4.8.12)

Neutral thermal conductivity and electron-neutral thermal equilibration:

\[ \frac{T_{na}^{-2}}{a^2} = K_3 \cdot a^2 \]

\[ p_{na} \cdot T_{na}^{-2} \]

where

\[ K_3 = 3 \cdot \left( \frac{m_n}{m_e} \right)^{\frac{1}{2}} \cdot p_{\infty} \cdot \sigma_n^2 = 1.17 \cdot 10^{-4} \]  (4.8.13)

In these coupled equations, a parameter \( \dot{I} \) has been introduced in place of \( E_u \).

\( \dot{I} \) is defined as:

\[ E_u = \frac{3 \cdot m_e}{\sqrt{2} \cdot \pi} \cdot \frac{p_{\infty} \cdot \sigma_n \cdot T_{na}^{-2}}{e^2 \cdot p_{na} \cdot T_{na} \cdot a^2} \cdot \dot{I} \]  (4.8.14)

The exact relation between \( I \) and \( \dot{I} \) is

\[ I = 3 \cdot \dot{I} \int_0^1 \frac{P_c \cdot U_u}{U_c^2 \cdot \left( 1 + \frac{\nu_c}{\nu_{eu}} \right)} \cdot x \cdot dx \]  (4.8.15)

In general, one specifies a value of \( \dot{I} \), solves the problem numerically, then finds \( I \) from the formula above.

The normalization constants are found by separating the three coupled equations. This gives:

\[ T_{na} = 0.811 \cdot \dot{I}^{\frac{6}{23}} \]  (4.8.16)
\[ T_{so} = 0.158 \cdot I^{14} \]  \hfill (4.8.17)

\[ p_{ro} = 94.9 \cdot 10^6 \cdot \frac{I^{32}}{a^2} \]  \hfill (4.8.18)

For typical values of \( I \sim 1 \text{ A} \) and \( a = 10^{-3} \text{ m} \), we find normalization constants of

\[ T_{ro} = 0.8 \text{ eV} \]
\[ T_{ro} = 0.16 \text{ eV} \]  \hfill (4.8.19)
\[ p_{ro} = 95 \text{ Pascals} \]

These values agree with typical experimental parameters.

Substituting the normalized variables into the three equations of the reduced model gives the following three normalized equations.

**Electron Energy:**

\[ \frac{1}{x} \frac{\partial}{\partial x} \left( x \cdot U_e^\frac{5}{2} \cdot \frac{\partial}{\partial x} U_e \right) + \frac{U_n \cdot P_e}{U_e^3} \cdot \frac{1}{\frac{U_e}{U_n} \cdot \left( 1 + \frac{\nu_{es}}{\nu_{en}} \right)} = 0 \]  \hfill (4.8.20)

**Neutral Energy:**

\[ \frac{1}{x} \frac{\partial}{\partial x} \left( x \cdot U_n^\frac{1}{2} \cdot \frac{\partial}{\partial x} U_n \right) + \frac{P_e \cdot U_e^\frac{1}{2}}{U_n} \cdot \left( 1 + \frac{\nu_{es}}{\nu_{en}} \right) = 0 \]  \hfill (4.8.21)

**Electron Mass:**

\[ \frac{1}{x} \frac{\partial}{\partial x} \left( x \cdot U_n^\frac{1}{2} \cdot \frac{\partial}{\partial x} P_e \right) = -S_{ion} \cdot \left( 1 - \frac{S_{rec}}{S_{ion}} \right) \]  \hfill (4.8.22)

In these equations,

\[ \frac{\nu_{es}}{\nu_{en}} = 0.344 \cdot \frac{I^{29}}{a^2} \cdot \frac{U_n \cdot P_e}{U_e^3} \]  \hfill (4.8.23)

\[ S_{ion} = 0.41 \cdot \frac{a^2}{I^{29}} \cdot \frac{P_e}{U_e^\frac{1}{2}} \exp \left[ 15.6 \cdot \left( 1 - \frac{1}{T_{ro} \cdot U_e} \right) \right] \]  \hfill (4.8.24)

\[ \frac{S_{rec}}{S_{ion}} = 1.10 \cdot 10^3 \cdot \frac{I^{29}}{a^2} \cdot \frac{P_e \cdot U_n}{U_e} \exp \left[ 15.6 \cdot \left( 1 - \frac{1}{T_{ro} \cdot U_e} \right) \right] \]  \hfill (4.8.25)

The boundary conditions for these normalized equations are:

\[ x = 0 \]

\[ U_e' (0) = U_n' (0) = P_e' (0) = 0 \]  \hfill (4.8.26)
\[ x = 1 \]

\[ U_s(t) = \frac{T_s}{T_{so}} = 0.0308 \frac{6}{79} \]

\[ U_v(t) = \frac{T_v}{T_{vo}} = 0.158 \frac{13}{29} \tag{4.8.27} \]

\[ P_r(1) = 0 \]

5. Determining the Radius

The radius of the arc is determined by the geometry and thermal properties of electrodes and the amount of heat delivered to the surface of the electrode by the arc.

As the arc is struck, the electron cascade is initiated by electrons emitted through field emission from the cathode. These electrons, emitted from a cold surface, require a large electric field to overcome the barrier potential of the electrode. For an arc generated by an AC power source, these electrons are alternately emitted from each electrode. The electrons are then heated as they travel along the arc. Electrons impacting on the electrodes transfer their heat to the surface of the electrode. As the electrodes are heated, (1) the effective potential barrier of the electrode is reduced, (2) the electron emission process changes from field emission to thermionic emission, and (3) the electric field required to sustain the arc is reduced. This is why the electric field required for breakdown is much higher than the electric field required for steady state operation.

In order for the electrode to emit electrons through thermionic emission, the surface of the electrode must conserve enough heat to remain at or above a critical temperature, \( T_c \). Nickel alloys and other electrode materials with work functions of 4.5 V become good thermionic emitters at temperatures around

---


In order for the electrode surface to remain at a temperature where it is a good thermionic emitter, the heat transferred to the electrode by the electrons striking the electrode surface must be equal to or greater than the heat conducted away from the electrode surface by thermal conductivity and convection. This requirement to conserve heat dictates the shape of the arc cross section. Any deviation from a circular profile reduces the ability of the arc to retain heat, both in the electrode and the body of the arc. On the electrode surface, any deviation from the circular profile will cool faster than other portions of the arc due to the increased surface area available for cooling. The arc quickly relaxes to a circular cross section.

In steady state, the energy delivered by the electrons is balanced by the thermal conduction in the electrode metal. If electrode heating exceeds thermal conduction in the surface of the electrode at the arc strike, the hot spot where thermionic emission is occurs will expand. If thermal conduction exceeds the arc's ability to heat the electrode, the radius of thermionic emission will contract. To find the temperature of the electrode surface, model the electrode as a rod which is heated by the arc at one end and maintains a temperature equal to the operating temperature of the plasmatron at the other end. The length, L, of the electrode in this problem approximates the physical geometry of the electrode in the plasmatron and is taken as 0.04 m.

---

Figure 5-1

---

The temperature of the rod at any point along the length, \( L \), may be found by solving the cylindrical heat equation.

\[
\frac{\partial^2}{\partial z^2} (k_y \cdot T_{rod}) = 0
\]  

(5.1.1)

There are two boundary conditions. The first boundary condition dictates the amount of heat that flows into the cylinder across the face that is heated by the arc. The second dictates the temperature at opposite end of the rod, \( T_{rod}(L) \).

For the first boundary condition, the amount of energy deposited by electrons on the surface of the electrode is obtained by multiplying the electron density, the axial electron velocity, and electron temperature and integrating across the radius of the arc.

\[
Q_{electrode} = 2 \cdot \pi \int_0^a r \cdot n_e \cdot v_{ez} \cdot k_y \cdot T_e \cdot dr
\]  

(5.1.2)

Approximating the radial electron temperature with a Gaussian profile, and setting heat deposited by electrons equal to the heat flux across the face of the rod,

\[
\kappa_{rod} \cdot \frac{\partial}{\partial z} (k_y \cdot T_{rod}) \bigg|_{z=0} \approx \frac{3}{2} \cdot \frac{I \cdot k_y \cdot T_e(0)}{\pi \cdot a^2 \cdot e}
\]  

(5.1.3)

In this equation, \( \kappa_{rod} \) is the thermal conductivity of the electrode material.

\[
\kappa_{rod} \approx 2.32 \cdot 10^5 \frac{W}{m \cdot eV} \text{ for nickel alloys}
\]  

(5.1.4)

For the second boundary condition, the face of the rod at \( z=L \) is held at the operating temperature of the plasmatron body.

\[
T_{rod}(L) \approx T_{plasmatron} \approx 0.05 \text{ eV}
\]  

(5.1.5)

The solution to the rod heat equation is

\[
T_{rod}(z) = T_{rod}(L) + \frac{3}{2} \cdot \frac{I \cdot k_y \cdot T_e(0)}{\pi \cdot a^2 \cdot e \cdot \kappa_{rod}} \cdot (L - z)
\]  

(5.1.6)

To determine the radius of the arc, solve for radius where \( T_{rod}(0) = 0.08 \text{ eV} \).

\[
a = 1.6 \cdot 10^{-3} \cdot [I \cdot T_e(0)]^{\frac{1}{2}} \text{ mm}
\]  

(5.1.7)

The following plot gives a rough estimate of radius using a typical electron temperature of 1.4 eV.
6. The Numerical Procedures

6.1. Using the MATLAB differential equation solver BVP4C

The mathematical suite MATLAB contains a differential equation problem solver for boundary value problems called BVP4C. This problem solver uses a 3-stage Lobatto IIIa collocation formula to solve a system of ordinary differential equations on the interval \([a,b]\). The solver inputs are the set of first order differential equations, parameters, a set of boundary conditions, and a guess for each equation. The resulting solutions are uniformly fourth-order accurate in the interval of integration\(^{27}\).

In the first part of this solution,

6.2. Creating a set of first order differential equations

The arc problem presents a set of three second order differential equations detailed in Section 4.

Conservation of Mass

For electrons:

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \cdot r \cdot \left( - \frac{1}{m_e \cdot \nu_{in}} \cdot \frac{\partial}{\partial r} p_r \right) = -S_{\text{recombination}} + S_{\text{ionization}}$$  \hspace{1cm} (6.2.1)

Conservation of Energy

For electrons:

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{ee} \cdot \frac{\partial}{\partial r} (k_b \cdot T_e) \right] = -\eta J^2 + \frac{3}{2} \cdot n_e \cdot k_b \cdot T_e \cdot \left( \nu_{en}^E + \nu_{ei}^E \right)$$  \hspace{1cm} (6.2.2)

For neutrals:

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{nn} \cdot \frac{\partial}{\partial r} (k_b \cdot T_n) \right] = -\frac{3}{2} \cdot n_e \cdot k_b \cdot T_e \cdot \left( \nu_{en}^E + \nu_{ei}^E \right)$$  \hspace{1cm} (6.2.3)

Boundary Conditions

$$\frac{\partial}{\partial r} p_r (0) = 0$$  \hspace{1cm} (6.2.4)

$$\frac{\partial}{\partial r} (k_b \cdot T_e (0)) = 0$$  \hspace{1cm} (6.2.5)

$$\frac{\partial}{\partial r} (k_b \cdot T_n (0)) = 0$$  \hspace{1cm} (6.2.6)

$$T_e (a) = T_{\text{room}}$$  \hspace{1cm} (6.2.7)

$$T_n (a) = T_{\text{room}}$$  \hspace{1cm} (6.2.8)

$$p_r (a) = 0$$  \hspace{1cm} (6.2.9)

To make each second order equation into a set of first order equations, nest two first order equations.

The Electron Equations

Starting with the electron energy equation,

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{ee} \cdot \frac{\partial}{\partial r} (k_b \cdot T_e) \right] = -\eta J^2 + \frac{3}{2} \cdot n_e \cdot k_b \cdot T_e \cdot \left( \nu_{en}^E + \nu_{ei}^E \right)$$  \hspace{1cm} (6.2.10)
Since the electron heat diffusion coefficient, \( \kappa_{ee} \), is primarily a function of \( T_e \), it may be brought inside the derivative.

\[
\kappa_{ee} \frac{\partial}{\partial r} (k_b \cdot T_e) = C_{ee} \cdot T_e^\frac{7}{2} \frac{\partial}{\partial r} (k_b \cdot T_e) \approx C_{ee} \cdot \frac{2}{7} k_b \cdot \frac{\partial}{\partial r} T_e^\frac{7}{2} \tag{6.2.11}
\]

The electron energy equation is now:

\[
\frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot C_{ee} \cdot \frac{2}{7} k_b \cdot \frac{\partial}{\partial r} T_e^\frac{7}{2} \right] = -\eta J^2 + \frac{3}{2} n_e \cdot k_b \cdot T_e \cdot (\nu_{ee} + \nu_{ei}) \tag{6.2.12}
\]

Set the first equation, \( y(1) \), as a function of \( T_e \) in the electron heat diffusion coefficient. To assist solver performance, electron temperature, \( T_e \), is normalized to the center of the arc, \( T_{eo} \).

\[
y(1) = \left( \frac{T_e}{T_{eo}} \right)^\frac{7}{2} \tag{6.2.13}
\]

The electron energy equation is now

\[
\frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot C_{ee} \cdot \frac{2}{7} k_b \cdot T_{eo}^\frac{7}{2} \frac{\partial}{\partial r} y(1) \right] = -\eta J^2 + \frac{3}{2} n_e \cdot k_b \cdot T_e \cdot (\nu_{ee} + \nu_{ei}) \tag{6.2.14}
\]

Set the second equation, \( y(2) \), as a function of the second derivative.

\[
y(2) = r \cdot C_{ee} \cdot \frac{2}{7} k_b \cdot T_{eo}^\frac{7}{2} \frac{\partial}{\partial r} y(1) \tag{6.2.15}
\]

Solving this for \( y(1) \) yields the first first order differential equation.

\[
\frac{\partial}{\partial r} y(1) = \frac{7}{2} \cdot \frac{y(2)}{r \cdot C_{ee} \cdot k_b \cdot T_{eo}^\frac{7}{2}} \tag{6.2.16}
\]

Substituting \( y(2) \) into the electron energy equation and solving for \( y(2) \) gives the second first order differential equation.

\[
\frac{\partial}{\partial r} y(2) = r \cdot \left( -\eta J^2 + \frac{3}{2} n_e \cdot k_b \cdot T_e \cdot (\nu_{ee} + \nu_{ei}) \right) \tag{6.2.17}
\]

The First Order Neutral Equations

Transformation of the neutral energy equation into a set of first order equations follows the same pattern. Starting with the neutral energy equation

\[
\frac{1}{r} \cdot \frac{\partial}{\partial r} \left[ r \cdot \kappa_{nn} \cdot \frac{\partial}{\partial r} (k_b \cdot T_n) \right] = -\frac{3}{2} n_e \cdot k_b \cdot T_e \cdot (\nu_{en} + \nu_{ei}) \tag{6.2.18}
\]

The neutral heat diffusion coefficient has a different dependence on \( T_n \).
\[ \kappa_{\text{in}} \cdot \frac{\partial}{\partial r} \left( k_b \cdot T_n \right) = C_{\text{in}} \cdot T_n^\frac{1}{3} \cdot \frac{\partial}{\partial r} \left( k_b \cdot T_n \right) \approx C_{\text{in}} \cdot \frac{2}{3} \cdot k_b \cdot \frac{\partial}{\partial r} T_n^\frac{3}{2} \]  \hspace{1cm} (6.2.19)

So the first equation is,

\[ y(3) = T_n^\frac{3}{2} \]  \hspace{1cm} (6.2.20)

Substituting back into the neutral heat diffusion term,

\[ \frac{1}{r} \frac{\partial}{\partial r} r \cdot C_{\text{in}} \cdot \frac{2}{3} \frac{\partial}{\partial r} y(3) = -\frac{3}{2} \cdot n_e \cdot k_b \cdot T_e \left( \nu_{\text{en}} + \nu_{\text{e}} \right) \]  \hspace{1cm} (6.2.21)

The second neutral equation is then,

\[ y(4) = r \cdot C_{\text{in}} \cdot \frac{2}{3} \cdot k_b \cdot \frac{\partial}{\partial r} y(3) \]  \hspace{1cm} (6.2.22)

Solving for \( y(3) \) gives the third differential equation.

\[ \frac{\partial}{\partial r} y(3) = \frac{3}{2} \frac{y(4)}{r \cdot C_{\text{in}} \cdot k_b} \]  \hspace{1cm} (6.2.23)

Substituting into the neutral energy equation gives the fourth differential equation.

\[ \frac{\partial}{\partial r} y(4) = r \cdot \left( -\frac{3}{2} \cdot n_e \cdot k_b \cdot T_e \left( \nu_{\text{en}} + \nu_{\text{e}} \right) \right) \]  \hspace{1cm} (6.2.24)

**First Order Electron Diffusion Equations**

\[ \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{r}{m_e} \cdot \frac{\partial}{\partial r} p_e \right] = -S_{\text{recombination}} + S_{\text{ionization}} \]  \hspace{1cm} (6.2.25)

Set the first equation equal to the electron pressure.

\[ y(5) = p_e \]  \hspace{1cm} (6.2.26)

Set the second equation equal to the function inside the derivative.

\[ y(6) = r \cdot \left( -\frac{1}{m_e \cdot \nu_{\text{en}}} \frac{\partial}{\partial r} y(5) \right) \]  \hspace{1cm} (6.2.27)

Solving for \( y(5) \),

\[ \frac{\partial}{\partial r} y(5) = -\frac{m_e \cdot \nu_{\text{en}}}{r} \cdot y(6) \]  \hspace{1cm} (6.2.28)

Substituting \( y(6) \) into the electron diffusion equation and solving for \( y(6) \),

\[ \frac{\partial}{\partial r} y(6) = r \cdot \left( -S_{\text{recombination}} + S_{\text{ionization}} \right) \]  \hspace{1cm} (6.2.29)

To improve solver performance, normalize the electron pressure equation by multiplying both sides by \( 10^{-13.5} \). This leaves the two first order electron diffusion equations as
\[
\frac{\partial}{\partial r} y(5) = -\frac{m_e \cdot \nu_m}{r} \cdot y(6) \cdot 10^{13.5} \\
\frac{\partial}{\partial r} y(6) = r \cdot (-S_{\text{recombination}} + S_{\text{ionization}}) \cdot 10^{-13.5}
\]

(6.2.30)  
(6.2.31)

6.3. Parameters and Boundary Conditions

In this solution, the only floating parameter is the core electron temperature of the arc, \( T_{e0} \). Given a set of six differential equations and one parameter, seven boundary conditions are required. For \( y(1) \), a function of electron temperature, the normalized core temperature was set at 1. The two equations involving species temperatures, \( y(1) \) and \( y(3) \), are required to meet a specified edge temperature at the outer radius of the arc. The electron pressure equation is required to go to a value close to zero. Finally, the three equations involving gradients, \( y(2) \), \( y(4) \), and \( y(6) \), are required to have a zero gradient at the core of the arc. The following table gives the boundary conditions used in the MATLAB BVP solver.

| \( y(1) \) | Core constraint, \( 1 \) | Edge constraint, \( (T_{\text{ext}}/T_{e0})^{1.5} \) |
| \( y(2) \) | 0 | \( T_{\text{ext}}^{1.5} \) |
| \( y(3) \) | 0 | \( T_{\text{ext}}^{1.5} \) |
| \( y(4) \) | 0 | 10^-8 |
| \( y(5) \) | 0 | \( 10^-8 \) |

7. Results

In this analysis, a reduced model has been developed to describe the behavior of a high pressure (1 atm) electric arc in air. This model has been solved using a MATLAB solver to produce numerical solutions of arcs at various radii and currents. The primary goal of these solutions is to find values for key parameters \( T_e(0), T_n(0), \) and \( P_e(0) \) at the center of the arc and to describe these parameters as functions of current and radius. The secondary goal is to calculate the
macroscopic parameters of voltage, resistance and power as functions of current and radius. These results are presented in the following order. First, representative profiles of temperature, pressure, and density are presented as functions of radius for arcs with various radii (1 mm, 1.25 mm, and 1.5 mm) and currents (1.25, 1, 0.75, 0.5, and 0.25 Amps). Next, the key parameters and macroscopic parameters for various electric fields are presented as functions of current and radius and compared with the analytical model. Finally, the arc parameters are presented as functions of current and predicted radius for the plasmatron electrode materials. All values presented are for typical plasmatron arc lengths of 4 cm.

**Radial Profiles for Arcs**

The following profiles are presented in the following figures.

- **Figure 7-1**  Current = 1.25 A  Radius = 1.00 mm
- **Figure 7-2**  Current = 1.00 A  Radius = 1.00 mm
- **Figure 7-3**  Current = 1.00 A  Radius = 1.25 mm
- **Figure 7-4**  Current = 0.75 A  Radius = 1.00 mm
- **Figure 7-5**  Current = 0.75 A  Radius = 1.25 mm
- **Figure 7-6**  Current = 0.75 A  Radius = 1.50 mm
- **Figure 7-7**  Current = 0.50 A  Radius = 1.00 mm
- **Figure 7-8**  Current = 0.50 A  Radius = 1.25 mm
- **Figure 7-9**  Current = 0.50 A  Radius = 1.50 mm
- **Figure 7-10** Current = 0.25 A  Radius = 1.00 mm
- **Figure 7-11** Current = 0.25 A  Radius = 1.25 mm
- **Figure 7-12** Current = 0.25 A  Radius = 1.50 mm
Figure 7-1

Radius = 1 mm
Current = 1.24 Amps
Power = 1336 Watts
Resistance = 872 Ohms
Voltage = 1080 Volts (for a typical 4 cm arc)
Figure 7-2

Radius = 1 mm
Current = 0.97 Amps
Power = 1160 Watts
Resistance = 1242 Ohms
Voltage = 1200 Volts (for a typical 4 cm arc)
Figure 7.3

Radius = 1.25 mm
Current = 1.00 Amps
Power = 1043 Watts
Resistance = 1043 Ohms
Voltage = 1040 Volts (for a typical 4 cm arc)
Figure 7.4

Radius = 1 mm
Current = 0.78 Amps
Power = 1029 Watts
Resistance = 1694 Ohms
Voltage = 1320 Volts (for a typical 4 cm arc)
Figure 7.5
Radius = 1.25 mm
Current = 0.73 Amps
Power = 872 Watts
Resistance = 1650 Ohms
Voltage = 1200 Volts (for a typical 4 cm arc)
Figure 7-6

Radius = 1.5 mm
Current = 0.73 Amps
Power = 792 Watts
Resistance = 1472 Ohms
Voltage = 1080 Volts (for a typical 4 cm arc)
Figure 7-7

Radius = 1 mm
Current = 0.90 Amps
Power = 1111 Watts
Resistance = 1383 Ohms
Voltage = 1240 Volts (for a typical 4 cm arc)
Figure 7-8

Radius = 1.25 mm
Current = 0.50 Amps
Power = 715 Watts
Resistance = 2899 Ohms
Voltage = 1440 Volts (for a typical 4 cm arc)
Figure 7.9

Radius = 1.5 mm
Current = 0.48 Amps
Power = 640 Watts
Resistance = 2722 Ohms
Voltage = 1320 Volts (for a typical 4 cm arc)
Figure 7-10

Radius = 1 mm
Current = 0.25 Amps
Power = 580 Watts
Resistance = 9273 Ohms
Voltage = 2320 Volts (for a typical 4 cm arc)
Figure 7-11

Radius = 1.25 mm
Current = 0.25 Amps
Power = 511 Watts
Resistance = 8466 Ohms
Voltage = 2080 Volts (for a typical 4 cm arc)
Figure 7-12

Radius = 1.5 mm
Current = 0.25 Amps
Power = 467 Watts
Resistance = 7576 Ohms
Voltage = 1880 Volts (for a typical 4 cm arc)
Key Parameters
The following graphs show values for arc parameters $T_0(0)$, $n_e(0)$, $T_n(0)$, $n_n(0)$ and $P_e(0)$.

Figure 7-13
Macroscopic Arc Parameters

With known profiles for $T_e$, $T_a$, and $P_e$, one can derive other arc parameters. Calculation of the resistance involves one aspect of the arc that is discussed in the conclusions, the length of the arc, $L$. This calculation uses an observed characteristic length of 0.04 m.

$$R = \frac{E_o \cdot L}{I}$$  \hspace{1cm} (7.1.1)

The power in the arc is dissipated through ohmic heating and is calculated using Joule’s Heating Law.

$$P = I^2 \cdot R$$  \hspace{1cm} (7.1.2)
8. Conclusions

The Plasmatron Fuel Reformer is a viable product for increasing efficiencies of internal combustion engines and reducing NOx emissions. Past research has established that enriching gasoline with a hydrogen rich gas is an effective method of increasing the efficiency of an internal combustion engine.\(^{28}\) The hydrogen additive allows a stable burn with large amounts of excess air (lean burns). Additionally, the additive increases the octane of the fuel, allowing operation with a high compression ratio. The combination of these two factors decreases the engine size required for constant power. Use of this process has been limited by the difficulties of storing or producing hydrogen on vehicles. The Plasmatron Fuel Reformer is a compact device able to produce on-demand hydrogen rich additive for gasoline. In current designs, the plasmatron partially combusts 25% of total fuel, producing a H\(_2\) and CO mix which is used to enrich the remaining portion of the fuel.\(^{29}\) This process releases 15% of the chemical energy in the processed fuel, but it increases the gross fuel efficiency by approximately 40%.\(^{30}\) When the chemical energy lost in the partial combustion and the energy required by the plasmatron are taken into account, the net increase in fuel efficiency is 30%.

Additionally, one side effect of the leaner operation of the engine is a reduced operating temperature. Since production of NOx gases is proportional to engine temperatures, a corresponding reduction in emission is also achieved.

This model was developed in support of the Plasmatron in order to predict characteristics of the arc used in the fuel reformation process. The model predicts heating power provided by the arc as a function of current, as well as critical temperatures for electrons and neutrals. The model also provides a plausible prediction of arc radius. These characteristic parameters are key factors in optimization of the reformation process.

\(^{28}\) Tully, 2003.

\(^{29}\) Bromberg, 1999.

\(^{30}\) Bromberg, 1999.
In conclusion, this technology is simple, effective, and it can be rapidly implemented while development of alternative energy sources continues.
9. References


