STATISTICAL ANALYSIS OF A HIGH ACCURACY POINTING AND TRACKING SYSTEM

by

GEORGE W. PFEIFFER

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Signature of Author

Department of Aeronautics and Astronautics

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ABSTRACT

The tracking accuracy of a postulated anti-aircraft weapon is investigated using statistical techniques. The weapon possesses a zero time of flight projectile as would a laser anti-aircraft. Models of the gimbal dynamics, disturbance torques, aircraft motion and measurement uncertainties are presented. A simple automatic control law is developed and evaluated on a deterministic basis. The complete system contains an extended Kalman filter to process measurements and produce state estimates. The state estimates are used to calculate the applied control. The tracking accuracy of this system is evaluated on a statistical basis. Results are presented for the nominal system with an optimal Kalman filter. The effects of changes in various parameters are shown. Also presented are the results when a miss-matched Kalman filter is used.

Thesis Supervisor: Wallace E. Vander Velde
Title: Professor of Aeronautics and Astronautics
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CHAPTER I

INTRODUCTION

The development of high energy lasers within the past few years has stirred interest in a number of weapon systems which would use the beam itself as the kill mechanism. Among these is a ground based laser anti-aircraft weapon.

One of the advantages of such a device is its zero time of flight "projectile". The control system for the weapon, whether manual, automatic or a combination of the two, requires no anticipation or lead. This may greatly reduce the effectiveness of standard target evasive maneuvers. However, to be effective the laser weapon will require extraordinary pointing and tracking accuracies. It is not uncommon to see accuracy requirements of fractions of a foot at ranges of 5 miles. Further, for the device to effect a kill, this accuracy is required to be held for some period of time, usually in the lower seconds. Factors such as aircraft motion (both pilot and disturbance induced), disturbance torques on the tracking gimbal and measurement uncertainty in target position make the accuracy requirements difficult to meet.

Manual control will probably not yield sufficient frequency response and accuracy to fulfill the requirements. Conventional automatic control systems require critical filtering of measurement signals to reduce jitter due to high loop gains necessary to maintain accuracy. Further conventional control systems may lack measurements of quantities desired to implement a good control law.

In this thesis an attempt is made to minimize the deficiencies of a standard control system by applying Kalman filtering to make the "best" use
of noisy measurements. The state estimates produced by the filter are used, via a simple control law, to produce a feedback control.

In order to analyze the performance of such a control scheme, models of the pointing system dynamics, disturbance torques, and aircraft motion were developed. These models were incorporated into two computer programs. The first program allows evaluation of the system in the absence of disturbances and state estimate uncertainties and errors. It was used to establish the adequacy of, and gain values used in, the simple control law when it was applied to the system models. Results of the program are presented in Chapter IV.

The second program is considerably more comprehensive. It contains the differential equations for the propagation of the second order statistics of the state variables and the statistics of the state estimates produced by the Kalman filter. Equations to update these statistics after the Kalman filter has processed a set of discrete measurements are also programmed. These equations are developed in Chapter V. The program provides a tool for the statistical evaluation of system performance in the presence of measurement noise and disturbance torques. Further, the effects of non-optimal Kalman filter output can be evaluated. Such a condition results from the mismatch of parameters of the filter system model and the corresponding parameters of the actual system model, or from implementation of non-optimal (perhaps pre-computed) filter gains. Program results are presented in Chapter VI.

Some care must be exercised when interpreting these results. They represent the control system's performance when it is applied to simple system models. These models neglect effects which will certainly be present in a real system. Some of the effects are noted as models are
described. The more important of these as yet unmodeled phenomena may degrade system performance considerably from the results presented.

The evaluation of the laser weapon as a whole must await a more complete analysis. It must include not only a thorough pointing and tracking accuracy analysis but also an analysis of beam effects such as bending and blooming. Device power figures will have to be coupled with the beam effects and accuracy to produce an evaluation of kill probability. This thesis is intended to be only a preliminary synthesis and analysis of an automatic tracking system. While it is deficient in many respects it is hoped that analytical tools have been developed that will prove useful in further work.
CHAPTER II

SYSTEM MODEL DESCRIPTION

2.1 Mount Dynamics Model

The directive mirror of the laser is assumed to be double gimballed. The rotational axes are elevation and azimuth. The dynamics of each axis are assumed to be independent of the other. Because of this decoupling assumption each axis of the mount will have the same model but may have different parameter values.

The mirror is rotated by a direct drive torque motor. Since it is direct drive the usual non-linear effects of gear trains are absent. The coupling between mirror and torquer is rigid and the mirror itself is modeled as a rigid body with constant rotational inertia. The pivot and motor are frictionless. The complete model of the torquer and gimbal is presented in Figure 2.1.

The input rate command, \( U \) is the control. It is the voltage applied to the direct drive electric torquer and is limited to prevent destruction of the motor windings. The induced back e.m.f. of the motor is subtracted and the difference divided by the motor electrical resistance to produce the winding current. The current when multiplied by the torque constant produces the torque applied to the gimbal by the motor. This is summed with the disturbance torque to yield the total torque on the gimbal. Multiplying by the inverse of the inertia and integrating once gives the gimbal angular rate. This is also the motor rate and when it is multiplied by the back e.m.f. constant produces the induced back e.m.f. Integration of the angular rate produces gimbal angle. Sensors provide measurements of both rate and angle which are available for processing. Viscous friction in either the
FIGURE 2.1

Simple Mount and Drive Model
(either axis)
motor or the pivot has not been explicitly included because its effects are identical to back e.m.f.

The above is a description of the model which will be used for simulation purposes. The same model without the non-linear saturation is employed to implement the Kalman filter controller and is used in Chapter VI in the statistical analysis of the controller's performance.

Appendix A presents realistic values for the mount and drive parameters. The values are intended to represent hardware which might be chosen or designed for a prototype system. These parameter values were used as inputs to the two computer programs whose results are presented in Chapter IV and VI.

This particular model was chosen because of its simplicity and low number of state variables. However, in a critical pointing and tracking problem several effects, neglected above, may become important. Two examples are the structural dynamics of the gimbal and the inductive nature of the motor electrical load. Figure 2.2 is a model incorporating both effects. Notice that the number of state variables has increased by three. It is presented to indicate the complexity of models which may be used. In further studies it may be found that the Kalman filter can produce adequate state estimates by using only the simple model of Figure 2.1. The filter also may be restricted to a simple model due to constraints on computation time. In either case, the performance of the control system should be evaluated by applying it to a rather more complex model such as is in Figure 2.2.

Other effects which might be found to be of some importance are, motor cogging torques, breakaway friction, granularity of sensor outputs and non-constant gimbal inertia caused by motion of the gimbal in the other
FIGURE 2.2
Mount Model Including Motor Inductance and Gimbal Dynamics

beam angle

$\psi_b$

$\dot{\psi}_b$

$\dot{\psi}_m$

$\tau_b$

$K_s$

$1/s$

$D_s$

$1/J_s$

$K_r$

$1/L_s$

$R$

$C_s$

$u$

to sensor
2.2 Mount Disturbance Torque Model

The disturbance torques experienced by an actual mount will be highly dependent upon the exact configuration of the mount and its shroud. In the absence of a hard mount design some crude approximations will be made in order to develop a disturbance torque model.

Assume that the chief contribution to the total disturbance torque arises from winds impinging on the pointing mirror. The wind induced torque may be written as

\[ \tau_w = K \frac{\rho |V| V}{2} C_D S \delta \]  \hspace{1cm} (2.1)

where \( \rho \) is the air density, \( V \) is the wind velocity, \( C_D \) is the coefficient of drag, \( S \) is the frontal area of the mirror, \( \delta \) is the wind force moment arm and \( K \) is a reduction factor to account for the effect of the shroud.

The non-linear function \( |V| V \) is inconvenient for analytic purposes. The function \( |V| V \) will be linearized by

\[ |V| V \equiv c V \]  \hspace{1cm} (2.2)

The value of \( c \) will be chosen to be the standard deviation of the wind velocity.

Realistic values for the constants of equations (2.1) and (2.2) are presented in Appendix B.

An appropriate model for low level wind velocities is easy to develop. Power spectral densities for winds have been taken. An example from reference (5), P. 264, is plotted in Figure 2.3. The root mean square wind velocity \( \sigma_w \) is obtained by taking the square root of the integral of the power spectral density. From the shape of the sample spectrum it appears that the wind could be approximated well by white noise passed
Figure 2.3
Wind Power Spectral Density at O'Neill Neb.
Aug. 31, 1953 - at a height of 6 meters

Legend:
- □ - 8 15 A.M.
- ○ - 2 26 P.M.
- △ - 4 23 P.M.
through a first order lag. This filter can be designed with a gain and time constant such that the root mean square output of the filter matches that obtained from the sample spectra.

Let noise with a spectrum \( G_{xx}(\omega) \) drive a system with a transfer function \( Y(s) \).

The spectrum of the output will be

\[
G_{yy}(\omega) = |Y(j\omega)|^2 G_{xx}(\omega)
\]

and the mean squared output is

\[
\overline{\dot{y}^2} = \int G_{yy}(\omega) \, d\omega
\]

In this case

\[
G_{xx}(\omega) = 1 \quad \text{and} \quad Y(s) = \frac{H}{1 + Ts}
\]

Therefore

\[
\overline{\dot{y}^2} = \int \left| \frac{1}{1 + T j\omega} \right|^2 d\omega = H^2 \int \frac{d\omega}{1 + T^2 \omega^2} = \frac{\pi H^2}{2T}
\]

The value of \( H \) is fixed by the requirement that the sample data and the simulation have the same root mean squared velocity.

\[
H = \sigma_\omega \sqrt{\frac{2T}{\pi}}
\]

The wind velocity input \( (V) \) to the disturbance torque model is finally obtained by passing unit white noise through a filter of the form of \( Y(s) \) in equation (2.5) where the value of \( H \) is given by equation (2.7) and the value of \( T \) is chosen to give a good match with the sample spectra of Figure 2.3. In Appendix B, values for the constants of the spectral model are presented.

Figure 2.4 is a diagram of the complete disturbance torque model. The equation for the disturbance torque is
FIGURE 2.4
Mount Disturbance Torque Model
\[ \gamma_0 = W\sigma\sqrt{\frac{2T}{\pi}} \frac{1}{1+T_s^2} \frac{1}{2} K \frac{L}{C_p} S D C \]

(2.8)

where \( W \) is the unit white noise.

Again, as with the mount model, the disturbance torque model may be an over simplification. For example, uneven bearing friction may contribute torques on the mount which, because of their unpredictable nature may be considered to be disturbance torques. Any plumbing attached to the mirror, perhaps for cooling, will contribute further disturbance torques. Although these and other disturbances have not been modeled here, they may prove to be significant sources of error and should be considered further.

2.3 Aircraft Motion Model

The aircraft in the absence of wind gusts and pilot evasive actions, will have a constant velocity vector. The vertical component of the vector will be small compared to its magnitude. The response of the aircraft to wind gusts and pilot actions will be characterized as small perturbations about this nominal trajectory.

The dynamics of the response of the aircraft to wind gusts is quite complex. Further, most sophisticated aircraft are equipped with autopilots whose effects and response dynamics cannot be ignored. In the interests of simplicity and economy of state variables a rather crude model of the dynamics of the aircraft and autopilot will be developed.

First, define a rectangular coordinate system \((x_n', y_n', z_n')\) which travels along the nominal trajectory. The \(x_n\) axis projects in the direction of travel, \(y_n\) out the right wind and, \(z_n\) projects out the bottom of the aircraft. The aircraft will experience disturbance accelerations \((a_x, a_y, a_z)\) in these directions. Equations for these accelerations will be developed.
The largest accelerations on the aircraft will be caused by wind
gusts in the \( z_n \) direction. The gusts produce changes in the angle of attack, \( \alpha \). This can be expressed by

\[
\alpha = \alpha_n - \frac{W_z}{V}
\]  

(2.9)

where \( \alpha_n \) is the nominal (no wind gust) angle of attack, \( W_z \) is the velocity of the gust in the \( z_n \) direction and \( V \) is the magnitude of the aircraft's nominal velocity vector. This expression is correct if \( W_z \ll V \) which is certainly the case for high performance aircraft.

Changes in the angle of attack will produce two accelerations, one in the axial \( (x_n) \) direction and the other in the vertical \( (-z_n) \) direction. The acceleration in the \( x_n \) direction \( (a_{x_n}) \) is developed first.

The wind induced change in \( \alpha \) will be small since \( W_z \ll V \). Also any autopilot controlling velocity will have a very long response time compared to the characteristic time for wind gusts, and its effects can be ignored. As long as \( \alpha \) does not become negative \( a_{x_n} \) is given by

\[
a_{x_n} = \frac{\rho V^2 S}{2 m} \ C_{DA} \Delta \alpha
\]  

(2.10)

where \( \rho \) is the air density, \( S \) is the wing area, \( m \) is the mass of the plane, \( C_{DA} \) is the change in the coefficient of drag caused by changes in angle of attack and \( \Delta \alpha \) is the wind induced change in angle of attack. \( \Delta \alpha \) is simply \(-\frac{W_z}{V}\). This is the final form for axial accelerations; a model for the wind gusts will be developed in the following section.

In the \( z_n \) direction the action of an autopilot must be considered. The response of the autopilot will be modeled as a second order system with natural frequency \( \omega \) and damping ratio \( \zeta \). With the autopilot disabled the aircraft response in the \( z_n \) direction is quite similar to the \( x_n \) direction.
where \( C_{L\alpha} \) is the change in lift coefficient produced by changes in angle of attack. \( V, S, m, \rho \) and \( \Delta \alpha \) are defined as before. The autopilot will compensate for the low frequency components of \( a'_{zn} \) but will do nothing to the high frequency components. Let the true acceleration of the plane be characterized by a simple filtering of \( a'_{zn} \) of the form

\[
a_{zn} = \frac{a'_{zn}}{s^4 + 2\omega s + \omega^2}
\]

To satisfy both the low and high frequency characteristics of the autopilot \( m \) must equal two. This formulation has several advantages. Only two state variables are used to describe the autopilot. Further, by rearranging the form of \( a_{zn} \) the autopilot off condition can be handled within the same framework.

Figure 2.5 is a block diagram of the rearranged system. Notice that when \( K \) is set to one the full autopilot condition is realized and when \( K \) is small the autopilot off case is represented. Also, by setting \( K \) to a small value and changing the spectrum and magnitude of \( W_z \) a crude approximation can be made to pilot evasive maneuvers.

The third component of acceleration, \( a_y \) is assumed to be zero.

Finally, the acceleration required as state variables by the Kalman filter are not in the \( x_n, y_n, z_n \) directions. The directions required are, \( x \) along the nominal ground track, \( y \) out the left wind and \( z \) vertical and pointing upward. However, since the \( z \) component of the nominal trajectory is small it is approximately true that,

\[
a_x = a_n, \quad a_y = a_{n}, \quad \text{and} \quad -a_z = a_{zn}
\]
FIGURE 2.5
Aircraft Vertical Motion Model
Typical values for the constants used in equations 2.10, 2.11 and 2.12 are contained in Appendix C along with the definition of the nominal target trajectory.

2.4 Wind Gust Model

The aircraft response functions require a model for vertical wind gusts. Reference (2) p. 318 represents a one dimensional spatial spectral density function given by

$$\phi(\Omega) = \frac{\sigma L}{\pi} \frac{1+3\Omega^2 L^2}{(1+\Omega^2 L^2)^2}$$

(2.14)

where $L$ is a characteristic length, $\sigma$ is a measure of strength and $\Omega$ is the positive spatial frequency. Such a spectrum can be obtained by passing white noise with power spectral density equal to one through a filter given by

$$Y(\Omega) = \sigma \sqrt{\frac{L}{\pi}} \frac{1+\sqrt{3}\Omega L}{(1+L \Omega)^2}$$

(2.15)

The power spectral density produced by this filter is plotted in Figure 2.6.

The filter response can be approximated quite well by a first order lag of the form

$$Y(\Omega) = \frac{H}{1+L \Omega}$$

(2.16)

The standard deviation produced by this filter is given by

$$\sigma' = H \sqrt{\frac{\pi}{2L}}$$

(2.17)

as is shown in the section on Mount Disturbance Torques. For the filter outputs to have the same standard deviation,

$$\sigma = \sigma' = H \sqrt{\frac{\pi}{2L}}$$

or

$$H = \sigma \sqrt{\frac{2L}{\pi}}$$

(2.18)
FIGURE 2.6
Non-dimensionalized Wind Gust Power Spectral Density

\[ \frac{\Phi(\Omega)}{\Phi_0} = \frac{\sigma^2 L}{\pi} \left( \frac{1 + 3 \Omega^2 L^4}{(1 + \Omega^2 L^4)^3} \right) \]

Key
- analytic function
- approximation

\[ \Phi(\Omega) = \frac{2\sigma^2 L}{\pi} \left( \frac{1}{1 + \Omega^2 L^2} \right) \]

Logarithmic scale for frequency and power density.
The spectral density produced by passing unit white noise through this filter is also plotted in Figure 2.6 for comparison.

Finally the filter must be changed from spatial frequencies to temporal. This is done by dividing the characteristic length $L$ by the aircraft velocity yielding the following filter

$$F(s) = \sigma \frac{2L}{\nu V} \frac{1}{1 + sL/V}$$  \hspace{1cm} (2.20)

where $V$ is the aircraft velocity. Therefore, if unit white noise is passed through the temporal filter given in equation (2.20) the approximate wind gust spectrum is generated.

Appendix C presents values for the constants $\sigma$ and $L$. The crude pilot maneuver model is formed by decreasing the value of $L$ and increasing $\sigma$. These values are also contained in Appendix C.

2.5 Coordinate System Definition

It is obvious from the preceding model development that there are two coordinate systems, each convenient for describing different parts of the problem. The natural coordinate system for describing beam motion is spherical as was indicated in the mount model. The aircraft equations of motion were expressed in rectilinear coordinates. Figure 2.7 defines the coordinate convention established for analysis purposes. The spherical coordinates of the beam motion are; $\psi$ the azimuth angle, $\theta$ the elevation angle and $R$ the range. $X$, $Y$, and $Z$ are the rectangular coordinates for the aircraft.

Conversion from one coordinate system to the other is a highly nonlinear process. The equations for the propagation of the statistics of state variables and estimates in Chapter V must therefore be linearized at the point when a coordinate transformation is required. Also the Kalman filter
FIGURE 2.7
Co-ordinate Definition
employed in the system will be an extended Kalman filter where state derivatives are calculated by expansion about the estimated state values.

The point at which the coordinate transformation is made was chosen to be conversion of target rectilinear accelerations to spherical accelerations. Since all measurements are presumed to be of velocities and positions in the spherical coordinate frame (see next section), this choice of transformation point enables all measurements to be processed linearly. The same is true of the feedback controls. The derivation of the transformation equations is a straightforward though tedious process. The final transformation equations are presented in Appendix D.

2.6 Measurement Definition

The measurements available to the Kalman filter have yet to be defined. In this section the postulated measurements are presented.

The measurements of primary importance will probably be tracking error, measured as an angular quantity. These measurements could be generated by an optical system bore sighted with the laser beam. A sensor processor might then detect an infra-red hot spot in the field of view. This would be the aircraft jet exhaust. After an appropriate offset was added, either by the processor or an operator, for example, a measure of angular miss distance would be formed. This measurement in both elevation and azimuth will be processed by the Kalman filter.

A host of important considerations are contained in the formulation of this measurement. How well can the center of the hot spot be determined? What happens if the hot spot is outside the field of view or cannot be detected? How accurately can the offset be determined? What about the effect of possible sensor output granularity? How well can the sensor be bore sighted with the beam? These are a sample of some of the questions that might be
asked about the measurement. For the purpose of this paper these will not be considered, and the measurement will be assigned an uncertainty and processed as are all the other measurements.

Measurements of gimbal angles and rates are easy to produce. Rate and angle measurements for each axis will be processed.

Finally, if an auxiliary radar is included in the system, perhaps for target acquisition, then target range and perhaps range rate will be available. Provision for both has been made in the computer program.

This completes the set of postulated measurements. Chapter VI contains the results of computer runs which attempt to identify the critical measurements by varying the uncertainties assigned to them.

2.7 State Variable Selection

Rather than choose the beam and target angular positions and rates to be state variables, a simple linear combination was chosen. Let $\Delta \psi$ and $\Delta \dot{\psi}$ be defined by

$$\Delta \psi = \psi - \psi_b \quad \text{and} \quad \Delta \dot{\psi} = \dot{\psi} - \dot{\psi}_b \quad (2.21)$$

A $T$ subscript indicates a target quantity and a $B$ subscript indicates a beam quantity. Corresponding definitions can be formed for the elevation errors $\Delta \theta$ and $\Delta \dot{\theta}$.

These error quantities are of primary interest because they represent the performance of the tracking system. They were chosen as state variables as well as the corresponding angular beam quantities.

Because no focusing control was postulated for the device the corresponding range errors are not of direct importance. The range state variables are target range and target range rate.

Other state variables are necessary to form random disturbances
from white noise. One state variable is required for the disturbance
torques of each axis. Three state variables are required to form the target
wind gust induced accelerations.
CHAPTER III

MEASUREMENT FILTER

AND

CONTROL LAW DESCRIPTION

The model of the system has been defined. The dynamic response of the system to both stochastic disturbance inputs and to deterministic controls is specified. Imperfect (noisy) measurements of states and combinations of states of the system are available. The problem is now to supply a method for calculating the control such that the pointing error remains within acceptable limits and the control itself does not become saturated. This is similar to the optimal stochastic control problem where a control, \( U \), is sought to minimize a performance index \( J \) defined by

\[
J = E \left[ x(t_f) \Sigma(t_f) x(t_f) + \int_{t_0}^{t_f} [x(t)^T \Sigma(t) x(t) + u(t)^T \Sigma(t) u(t)] dt \right]
\] (3.1)

where \( x(t) \) is the state vector, \( \Sigma(t) \) and \( \Sigma(t) \) are weighting matrices, and \( t_f \) is the terminal time.

It has been shown that the solution to this problem is a cascade of two functions. First, a Kalman filter to process the measurements and produce an estimate of the state, \( \hat{x}(t) \). Second, a linear controller to produce the control vector from the state estimate by the equation

\[
u^*(t) = C(t) \hat{x}(t)
\] (3.2)

where \( C(t) \) is a set of optimal control gains.

It is logical to assume that a control system employing this type of cascade might work well even if \( C(t) \) is non-optimal. This is the approach which was used.
The equations for the Kalman filter are derived in many texts (see Reference (3), Chapter IV) and will not be derived here. However, the equations for a continuous system with discrete measurements are summarized below.

Let the system be defined by:

$$\dot{x} = Fx + Gw + Lu$$

(3.3)

where $w$ is a stationary, zero mean, white noise disturbance. The intensity of $w$, $Q$, is given by

$$Q \delta(\Delta t) = \overline{w(t) w(t+\Delta t)^T}$$

(3.4)

where $\delta$ is the standard unit impulse function and $t$ is time. Since $w$ is stationary $Q$ is not a function of time.

The $k$'th measurement is given by:

$$z_k = Hx + v_k$$

(3.5)

where $v$ is a stationary, zero mean random variable. The intensity of $v$, $R$, is defined by

$$R = \overline{v v^T}$$

(3.6)

The covariance of estimation errors is defined by

$$P = E[(\hat{x} - x)(\hat{x} - x)^T]$$

(3.7)

where $\hat{x}$ is the state estimate.

The propagation equation for $P$ between measurements is

$$\dot{P} = FP + PF^T + GQG^T$$

(3.8)

and the corresponding equation for the state estimate is

$$\dot{\hat{x}} = F\hat{x} + Lu$$

(3.9)

Let a - subscript indicate a quantity just prior to a measurement and a + subscript indicate the quantity just after a measurement. Then the update
equations across a measurement are
\[ \hat{x}_t = \hat{x}_t + K(z - H\hat{x}_t) \]  \hspace{1cm} (3.10)

and
\[ P_t = (I - KH)P \]  \hspace{1cm} (3.11)

The filter gain matrix, \( K \), is calculated from
\[ K = PH(HPH^T + R)^{-1} \]  \hspace{1cm} (3.12)

These equations define the operation of the Kalman filter. The calculation of the optimal gain matrix \( C(t) \) in equation (3.2) is much more difficult. In general, it involves the integration of a matrix differential equation backward in time starting at \( t_u \). Further, it is not only a function of time but also a function of the nominal trajectory of the target. This comes about because of the non-linear nature of the problem. Instead of attempting to calculate the optimal \( C(t) \) a constant gain matrix \( C \) was derived and the control system's performance analyzed using it.

The block diagram of the azimuthal angle of the gimbal system is drawn in Figure 3.1. For the purpose of the derivation of the control law, assume that the disturbance torque, \( \gamma \), is zero. Also assume that the Kalman filter produces perfect estimates of the gimbal angular rate, \( \hat{\psi}_b \), the miss angle \( \Delta \psi = \psi_t - \psi_b \), and the miss rate \( \Delta \dot{\psi} = \dot{\psi}_t - \dot{\psi}_b \). If the element of \( C \) which multiplies the gimbal rate estimate is set equal to \(-K_b\), the natural damping of the gimbal torquer is cancelled and the system block diagram can be redrawn as in Figure 3.2. The gains \( A \) and \( B \) multiplying \( \Delta \hat{\psi} \) and \( \Delta \psi \) respectively remain to be chosen. If \( B \) is chosen to be
\[ B = \frac{\omega^2 R J}{K_T} \]  \hspace{1cm} (3.13)
System Block Diagram - Azimuthal Axis

FIGURE 3.1
FIGURE 3.2
Control Law Block Diagram
(no disturbances or state uncertainties)
and \( A \) is chosen to be
\[
A = \frac{2\xi \omega RJ}{K_T} \tag{3.14}
\]
then the system transfer function can be written as
\[
\frac{\Delta \psi}{\psi_T} = \frac{1}{s^2 + 2\xi \omega s + \omega^2} \tag{3.15}
\]
The constants \( \omega \) and \( \xi \) must be chosen to give acceptable performance without causing excessive control saturation.

To summarize, there are only three non-zero control constants. The first \( C_1 \) multiplies the gimbal rate estimate \( \widehat{\psi} \) and is given by
\[
C_1 = K_b \tag{3.16}
\]
The second constant \( C_2 \) multiplies the miss rate estimate, \( \Delta \psi \) and is
\[
C_2 = \frac{2\xi \omega RJ}{K_T} \tag{3.17}
\]
The third \( C_3 \) multiplies the miss estimate \( \Delta \psi \) and is given by
\[
C = \frac{\omega RJ}{K_T} \tag{3.18}
\]
The constants \( \omega \) and \( \xi \) are free to be chosen to optimize control performance. The state estimates are produced by a Kalman filter which is processing measurements. In the absence of disturbances and estimation errors the control system transfer function is given by equation (3.15).

A similar control law can be developed for the elevation axis.
CHAPTER IV

SYSTEM PERFORMANCE IN THE ABSENCE OF

DISTURBANCES AND STATE ESTIMATE UNCERTAINTIES

The performance of the control law formulated in the preceding chapter can be analyzed on a deterministic basis. To do this the random disturbance torques and random target accelerations must be set to zero. Further, it must also be assumed that the Kalman filter produces perfect estimates of the state vector. In order to conserve computer time the performance will be investigated in one axis only, azimuth.

With the restrictions of the preceding paragraph the system can be drawn as in Figure 4.1. A subscript of $B$ indicates beam quantities and $T$, target quantities. A superscript of * indicates parameter values used in the control gains. Ideally the asterisked quantities will be equal to the corresponding quantities without superscripts. In practice this will not be the case; some errors will occur. The three $\epsilon$ quantities are to indicate possible scale factor errors in the state estimates produced by the filter.

A computer program was written to simulate the system diagrammed in Figure 4.1. The listing of the program is presented in Appendix E.

The program was run with the nominal parameters defined in Appendix A. For the first runs no scale factor errors ($\epsilon's = 0.$) or parameter mismatch (asterisked quantities equal those without) were used. The resulting pointing error history is plotted in Figure 4.2. The program was run with two loop gain values, $\omega = 10 \text{ rad/sec}$ and $\omega = 20 \text{ rad/sec}$. As would be expected the case with the higher gain has better performance. In both cases the only control saturation occurred in the first two seconds of the trajectory. This is caused by a large initial pointing error of about $6^\circ$. 
FIGURE 4.1
Block Diagram of Simplified System
The value of $\omega$ could be increased still further until control saturation occurred somewhere along the trajectory. However, it was felt that this might lead to overly optimistic performance when used in the statistical program described in Chapter VI. This could come about because the program of Chapter VI requires that the system respond to relatively high frequency disturbances without consideration of control saturation. Such high frequency disturbances could drive the control into saturation if $\omega$ is too large. For this reason an $\omega$ of 20 rad/sec was chosen as a baseline loop gain.

Another means of reducing the tracking error is to change the control law. The primary source of error for these no disturbance conditions comes from angular acceleration of the target. Examination of equation (3.15) shows that for constant $\ddot{\psi}_T$ and steady state conditions,

$$\Delta \psi = \dot{\psi}_T / \omega$$  \hspace{1cm} (4.1)

A quick calculation of $\dot{\psi}_T$ using the equations of Appendix D verifies that practically all of the tracking error exhibited in Figure 4.2 can be accounted for by equation (4.1). An obvious remedy is to have the control law calculate an estimate of $\dot{\psi}_T$ from the Kalman filter state estimates, scale the estimate properly and add it into the command, thus cancelling the steady state error of equation (4.1). Such a scheme could be quite effective but would have to be analyzed to ensure system stability and freedom from control saturation.

The target trajectory which was used to generate Figure 4.2 has the target $y$ distance set at 1000 feet. Three other trajectories were run with $\omega$ equal to 20 rad/sec. They have the target $y$ distances of 500, 2500 and 5000 feet. The results for all four trajectories are plotted in Figure 4.3. Trajectories with higher $y$ target positions have larger angular accelera-
tions early in the encounter but smaller accelerations as the target \( x \) position approaches zero. This is reflected in the tracking accuracy.

The performance of the control system was also investigated when the target is executing evasive maneuvers or jinks. A crude maneuver model was used. The target accelerates at a constant 0.5 g's for one second alternately in the plus and minus \( y \) direction. The resulting pointing error history is plotted in Figure 4.4. The effect of the maneuvers is clearly present. No control saturation took place except in the initial transient. Again, practically all the maneuver induced error is accounted for by equation (4.1). The value of the angular acceleration contributed by the target \( y \) acceleration can be computed from the equations of Appendix D.

Finally, as can be seen in Figure 4.1, the effect of scale factor errors in the measurement of \( \Delta \psi \) and \( \Delta \dot{\psi} \) is the same as a change in the control parameters \( \omega \) and \( \zeta \). The same is true for errors in the assignments of values to \( R^* \), \( J^* \) and \( K_y^* \). For example, a 10% error in the value of \( R^* \) is equivalent to changing the values of \( \omega \) and \( \zeta \) by about 3.2%. Therefore, the performance of the control law is not critically dependent upon the accuracy of the determination of these values. The same statement cannot be made about the value used for \( K_b \) and for scale factor errors in the measurement of the beam rate \( E_3 \). The program was run with a value of -0.1 for \( E_3 \), the beam rate scale factor error. The resulting added tracking error is plotted in Figure 4.5. The extra error is not large and therefore the performance of the control law is not severely compromised by small scale factor errors in the measurement of gimbal rate or by small errors in the value of \( K_b \). This comes about because the control gain applied to the estimated \( \Delta \psi \) is much greater than that applied to \( \dot{\psi}_b \).

If the value of \( \omega \) is lowered or if the gimbal inertia is reduced or the torquer
FIGURE 4.5
Incremental Error from a -10% Scale Factor
Error in Beam Rate Measurement

Incremental Miss (ft)

Time (sec)
changed, the control law performance may become much more dependent upon accurate cancellation of motor damping.

This concludes the analysis of the control law performance on a deterministic basis. If it is found that the errors are unacceptably large, the performance could be improved by increasing the gain $\omega$ or by a modification to the control law to include a feed forward target acceleration signal.
CHAPTER V

DERIVATION OF SENSITIVITY EQUATIONS

The covariance of state estimation errors calculated by a Kalman filter will not be the covariance of the true estimation errors if the filter is sub-optimal. The non-optimality will result from any difference between the behavior of the model of the system used by the filter and the behavior of the actual system. Under these conditions the filter will produce more optimistic (smaller) values for estimation uncertainties than is actually the case.

Further, the state estimates calculated by the filter may not be unbiased. It is therefore important to be able to assess the performance of the system when controls are being generated from the state estimates of a suboptimal filter.

First the equations are derived for statistical propagation of the state and actual estimation uncertainties between measurements.

The equation for the state is given by

\[ \dot{\mathbf{x}} = F\mathbf{x} + G\mathbf{w} + L\mathbf{u} \quad (5.1) \]

where \( \mathbf{w} \) is a zero mean white noise forcing function and \( \mathbf{u} \) is a deterministic control.

The derivative of the estimate of the state \( \hat{\mathbf{x}} \) is

\[ \dot{\hat{\mathbf{x}}} = F\hat{\mathbf{x}} + G\hat{\mathbf{w}} + L^*\mathbf{u} \quad (5.2) \]

where a superscript * indicates the quantity actually implemented in the filter.

The control vector \( \mathbf{u} \) is generated by the filter from the equation

\[ \mathbf{u} = C\hat{\mathbf{x}} \quad (5.3) \]

Substitution of this into equations (5.1) and (5.2) yields

\[ \dot{\hat{\mathbf{x}}} = F\mathbf{x} + G\mathbf{w} + LC\hat{\mathbf{x}} \quad (5.4) \]

and
\[ \dot{\hat{x}} = \dot{x} - \dot{\hat{x}} \] (5.9)

which when combined with equations (5.7) and (5.8) yields

\[ \dot{\hat{x}} = (\dot{F}^* + \dot{L}^* C)(\dot{x} + \dot{\hat{x}}) - F \dot{x} - G \dot{w} - L \dot{C}(\dot{x} + \dot{\hat{x}}) \] (5.10)

Letting

\[ \Delta F = F^* - F \quad \text{and} \quad \Delta L = L^* - L \] (5.11)

equation (5.10) can be rewritten as

\[ \dot{\hat{x}} = (\Delta F + \Delta L C)\dot{x} + (F^* + \Delta L C)\dot{\hat{x}} - G \dot{w} \] (5.12)

Define a new vector \( \hat{y} \) as

\[ \hat{y} = \begin{bmatrix} \hat{x} \\ \hat{o} \end{bmatrix} \] (5.13)

The differential equation for this new vector can be written as

\[ \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{o}} \end{bmatrix} = \begin{bmatrix} F^* + \Delta L C & \Delta F + \Delta L C \\ L C & F^* + L C \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{o} \end{bmatrix} + \begin{bmatrix} -G \dot{w} \\ G \dot{w} \end{bmatrix} \] (5.14)

or

\[ \dot{y} = \hat{y} + \omega \] (5.15)

This differential equation has the form of a linear system driven by white noise. The differential equation for the covariance of such a system is derived in section 11.4 of reference (1). Because of the unique
character of white noise the differential equation for the covariance of $y$ is given by

$$\frac{d}{dt}E[yy^T] = YE[yy^T] + E[yy^T]Y^T + \Omega$$

(5.16)

Where $\Omega$ is the intensity of $\omega$.

This covariance may be partitioned as follows

$$E[yy^T] = \begin{bmatrix} P & V^T \\ V & U \end{bmatrix}$$

(5.17)

where

$$P = E[x\; x^T]$$

(5.18)

and

$$V = E[x\; x^T]$$

(5.19)

and

$$U = E[x\; x^T]$$

(5.20)

Breaking the differential equation (5.16) into its partitions yields the following three differential equations.

$$\dot{P} = (F^* + \Delta L C)P + P(F^* + \Delta L C)^T + (\Delta F^* + \Delta L C) + V^T + V\Delta F + \Delta L C)^T + GG^T$$

(5.21)

$$\dot{V} = LCP + (F + LC)V + V(F^* + \Delta L C)^T + U(\Delta F + \Delta L C)^T - GG^T$$

(5.22)

$$\dot{U} = LCV^T + (F + LC)U + V(LC)^T + U(F + LC)^T + GG^T$$

(5.23)

where $\Omega$ is defined as the intensity of $\omega$ as defined in equation (3.4).

It should be noted that both $P$ and $U$ are symmetric whereas $V$ is not.

Initial conditions for $U$, $V$ and $P$ must now be specified. Assume that the initial state can be expressed as

$$x(0) = \bar{x} + \alpha$$

(5.24)

and the initial state estimate as

$$\hat{x}(0) = \bar{x} + \beta$$

(5.25)
where \( \Phi \) and \( \Psi \) are uncorrelated, zero mean random variables and \( \overline{x} \) is the mean initial condition of the system. Individual elements of \( U \) are now given by

\[
U_{ij}(0) = E[x_i x_j] = E[(\overline{x}_i + n_i)(\overline{x}_j + n_j)] = E[\overline{x}_i \overline{x}_j + n_j \overline{x}_i + n_i \overline{x}_j + n_i n_j]
\]

Since \( \Phi \) has zero mean, the middle two terms are zero and \( U_{ij}(0) \) can be expressed as

\[
U_{ij}(0) = \begin{cases} 
\overline{x}_i \overline{x}_j, & i \neq j \\
\overline{x}_i^2 + \overline{n}_i^2, & i = j
\end{cases}
\]

The initial estimation error is \( \overline{x}(0) \) which can be written as

\[
\overline{x}(0) = \Psi - \Phi
\]

(5.28)

By reasoning similar to that above

\[
P_{ij}(0) = \begin{cases} 
0, & i \neq j \\
\frac{\overline{n}_i^2}{\overline{m}_i^2 + \overline{n}_i^2}, & i = j
\end{cases}
\]

and

\[
V_{ij}(0) = \begin{cases} 
0, & i \neq j \\
-\overline{n}_i^2, & i = j
\end{cases}
\]

(5.29)

(5.30)

E,quations must now be derived to update the sensitivity matrices across a measurement. In the following equations a subscript of \(-\) indicates the quantity just prior to a measurement and a + subscript indicates the quantity just after a measurement.

The update equation for \( U \) is simple. It is clear that a measurement cannot affect the state of the actual system, but only affect the estimate of the state. Therefore the covariance of the state is not changed and

\[
U_+ = U_-
\]

(5.31)

The equation which relates a measurement \( \mathbf{Z} \) to the state of the system is given by
\[ Z_t = Hx + v_t \]  
\[ (5.32) \]

where \( v_t \) is a stationary, zero mean random variable. Since \( v \) is stationary the update equations will not depend on \( k \) and the subscript will be dropped.

The Kalman filter state estimate update equation is given by (reference (3), p. 110)
\[ \hat{x}_t = \hat{x}_{t-1} + K(z - H^x \hat{x}_{t-1}) \]  
\[ (5.33) \]

where \( K \) is the filter gain matrix actually used by the filter. \( K \) may have been computed by the Kalman filter update equations or derived in any suboptimal manner. \( H^x \) is the state to measurement transformation matrix used by the filter. Substituting equation (5.32) into (5.33) and rearranging yields
\[ \hat{x}_t = (I - KH^x) \hat{x}_{t-1} + K(Hx + v) \]  
\[ (5.34) \]

Recalling equation (5.6) for \( \hat{x} \) and substituting it into equation (5.34) gives
\[ \hat{x}_t = (I - KH^x) \hat{x}_{t-1} + K(Hx + v) \]  
\[ (5.35) \]

Again using equation (5.6), solving for \( \hat{x}_{t-1} \), substituting into (5.35) gives
\[ \hat{x}_t = (I - KH^x)(\hat{x}_{t-1} + x) + K(Hx + v) - x \]  
\[ (5.36) \]

When terms are collected the following equation for \( \hat{x}_t \) results:
\[ \hat{x}_t = (I - KH^x) \hat{x}_{t-1} - K\Delta Hx + Ky \]  
\[ (5.37) \]

where \( \Delta H \) is defined by
\[ \Delta H = H^x - H \]  
\[ (5.38) \]

If both sides of equation (5.37) are post multiplied by their transpose and the expectation of the result is taken, the following expression results.
\[
E(\tilde x, \tilde x') = E\left((I-KH')\tilde x, \tilde x'(I-KH')^\dagger\right) +
E\left((K\Delta H\tilde x\tilde x'\Delta H'K')^\dagger\right) -
E\left(K\Delta H\tilde x\tilde x'(I-KH')^\dagger\right) -
E\left((I-KH^\dagger)\tilde x'\Delta H'K'\right) -
E\left(K\tilde x\tilde x'(I-KH')^\dagger\right) +
E\left(K\Delta H\tilde x\tilde x'(I-KH')^\dagger\right) +
E\left(K\Delta H\tilde x\tilde x'(I-KH')^\dagger\right)
\]

(5.39)

Since \( \psi \) is zero mean random variable and is uncorrelated with \( \tilde x \) or \( \tilde x' \), the expectation of the terms involving their product is zero. Recalling equations (5.18), (5.19) and (5.20) equation (5.39) may now be written as

\[
P_\psi = (I-KH^\dagger)P(I-KH^\dagger)^\dagger + K\Delta HU\Delta H'K' + KRK' -
(I-KH^\dagger)V\Delta H'K' - K\Delta HV(I-KH^\dagger)^\dagger
\]

(5.40)

where \( R \) is the intensity matrix of \( \psi \) as was defined in equation (3.6)

Using equation (5.37) and the expectation operator,

\[
E(\tilde x, \tilde x') = E(\tilde x, \tilde x'(I-KH^\dagger)^\dagger) - E(\tilde x\tilde x'\Delta H'K') + E(\tilde x\tilde x'K')
\]

(5.41)

The final term is zero and therefore

\[
V_\psi = V(I-KH^\dagger)^\dagger - U\Delta H'K'
\]

(5.42)

This completes the required update equations.

They are summarized as follows.

Between measurements

\[
P = (F^\dagger + \Delta LC)P + P(F^\dagger + \Delta LC)^\dagger + (\Delta F + \Delta LC)V + V^\dagger(\Delta F + \Delta LC)^\dagger +
GQG^\dagger
\]

(5.21)
\[ \dot{V} = LCP + (F + LC)V + V(F^2 + \Delta LC)^T + U(DF + \Delta LC)^T - GQG \] (5.22)

\[ \dot{U} = LCV^T + (F + LC)U + V(LC)^T + U(F + LC)^T + GQG \] (5.23)

and across a measurement

\[ P_t = (I - KH^s)P_t(I - KH^s)^T + KDHUH^TK^T + KRK^T - (I - KH^s)V^T \Delta H^TK^T - KDHV(I - KH^s)^T \] (5.40)

\[ V_t = V_t(I - KH^s)^T - U\Delta H^TK \] (5.42)

\[ U_t = U_t \] (5.31)
CHAPTER VI

SYSTEM PERFORMANCE IN THE PRESENCE OF DISTURBANCES,

MEASUREMENT UNCERTAINTY AND PARAMETER MISMATCH

The equations developed in Chapter V coupled with the system model from Chapter II and the Kalman filter equations from Chapter III are the analytic tools used to evaluate the statistical performance of the system. A computer program was written incorporating these features. The listing of the program is contained in Appendix F. In order to conserve computer time, the program was written to evaluate system performance in the azimuthal angular direction only, thus eliminating the four elevation state variables, $\Delta \theta, \Delta \phi, \theta, \phi$. The remaining state variables were defined in Section 2.7. They are the four azimuthal quantities, $\Delta \psi, \Delta \dot{\psi}, \psi, \dot{\psi}$, range and range rate (R, $\dot{R}$), one variable for disturbance torque generation and three for target motion, making a total of ten.

The basic flow of the program is diagramed in Figure 6.1 and described as follows:

The parameter values used to simulate the system and the corresponding values used by the filter are read. Also read are the initial conditions for the four correlation matrices $U, P, V, P^*$. The first three matrices were defined in Chapter V. The matrix $P^*$ is the covariance matrix produced by the Kalman filter. It is only needed to produce the filter gain matrix. Next a measurement is processed. The Kalman filter equations of Chapter III are evaluated to produce a filter gain matrix. The filter gain matrix is used to update the correlation matrices $P$ and $V$ via the equations of Chapter V and $P^*$ via equations of Chapter III. This completes the processing of one set of measurements. Now the two derivative matrices $F$
FIGURE 6.1
Basic Program Flow

Read Initial Conditions and Parameter Values

Execute Kalman Filter Equations to Generate the Filter Gain Matrix

Update the Correlation Matrices P, V, and P*

Calculate the Derivative Matrices F and F*

Integrate U, P, V, and P* Matrices by Second Order Runge-Kutta Technique

Is it Time for Next Measurement?

Yes

At Completion Time?

Yes
and F* are determined by linearization about the nominal target trajectory. The derivatives of the four matrices are calculated and new P, U, V, and P* matrices formed by a second order Runge-Kutta integration. If it is now time to process another measurement the filter gains are calculated and P, V, and P* updated as before, otherwise another integration step is taken. This process continues until the completion time is reached. At any point just before a new measurement is to be processed, the four matrices may be punched. They can then be used as initial conditions for another run.

The program was run for the entire nominal trajectory with the parameter values given in the first four appendices. The non-maneuvering set of target values and the low disturbance torque values were used. Also, there was no parameter mismatch. The filter uses the same values as are in the system model. The nominal measurement uncertainties are given in Appendix G. The results of the run are presented in Figures 6.2 and 6.3.

There are several features of the plots that require some comment. The first feature, and the most troublesome, is the dip at the end of the encounter at about 23 seconds. This seems to be caused by an integration problem in the program rather than a real phenomenon in the system. The dip occurs at the point where the square root of the beam angle diagonal element of the U matrix goes through 90 degrees. In the deterministic runs the beam angle is 90° at about 25 seconds, when the target angle is 90°. Further, examination of the square root of the target range diagonal element of U reveals that it is larger than the deterministic range and that the difference gets bigger as time progresses. Both of these observations point to integration errors. Tests were made where parts of the trajectory were integrated with a time step one fourth the size used for generating Figures 6.2 and 6.3. Only very small differences between the runs with different
FIGURE 6.2 System Performance, Nominal Conditions

\[ \Delta = \text{Miss Distance} \]

\[ \left[ \frac{\Delta}{\Delta} \right]^{\frac{1}{2}} \]

\[ \left[ \frac{\Delta - \Delta}{\Delta} \right]^{\frac{3}{2}} \]

deterministic results

---

Miss Distance (ft) vs. Time (sec)
FIGURE 6.3
System Performance, Nominal Conditions

\[ \Delta = \text{Miss Rate} \left( \frac{\Delta F}{\sqrt{\Delta^2}} \right) \left( C \right)^\frac{1}{2} \left( D \right) \]
step sizes occurred. The nominal integration step size appears to be adequate.

A possible explanation is that the premature dip is caused by the accumulation of errors experienced as a result of linearizing the equations. Unfortunately time does not permit further examination of the problem. The remaining computer runs will be limited to the portion of the encounter prior to 20 seconds where accuracy appears to be better.

The spike at the beginning of the encounter is caused by an information transient. The filter initially has relatively poor "knowledge" of the state of the system. As measurements are processed the uncertainty rapidly gets smaller. The transient is reflected in both the system error (curves A and C) and the estimation error (curves B and D) in Figures 6.2 and 6.3.

Since there were no parameter mismatches the Kalman filter covariance matrix, P*, should equal the actual estimation error matrix, P. This was found to be the case. Any run which has nominal parameter values used by the filter will produce this P* matrix history even if the system model does not have the same parameter values.

Examination of Figure 6.2 reveals that the actual system performance is considerably worse than was predicted in Chapter IV. This was expected. The difference is caused by the inclusion of measurement noise and disturbances. Measurement noise causes noisy state estimates to be formed which in turn yields a noisy control signal. Disturbances contribute further to the state estimate uncertainties. They also degrade the performance of the control law as was shown in Chapter IV.

Miss distances of large fractions of a foot may not reduce the effectiveness of the weapon as long as the miss is constant. The velocity of the laser spot on the target must be small. Figure 6.3 is a plot of the statistics
of miss rate. If the miss distance is small (curve A) but the rate is high (curve C) the presence of a lot of jitter is indicated. On the other hand if miss distance is large but the rate is low, a constant miss is present.

The remaining figures are the results of computer runs made with other than nominal conditions. In each a five second slice of the encounter was run rather than the full trajectory. A feel for the effect of the change under consideration can be derived and computer run time is conserved. The four matrices generated with nominal conditions were used as the initial values for cases with starting times other than zero.

Figure 6.4 and 6.5 show the effect of reducing the measurement rate by a factor of two. The initial transient is slightly higher and longer lived but not drastically so. In an operational system, if the computer requires more than 0.02 seconds to process a measurement, the measurement rate could be dropped without a serious reduction in system performance. Although they have not been plotted, the estimation error curves show the same tendency as the error curves themselves.

The effect of increasing the uncertainty of the miss angle measurement is shown in Figures 6.6 and 6.7. They show the same general character as the previous two figures. The performance of the system suffered a somewhat larger degradation from the increase in measurement uncertainty than from the decrease in measurement rate.

The case of high disturbance torques is investigated next. The parameter values of Appendix B for high disturbance torques were used to generate Figures 6.8 and 6.9. The initial time is 10 seconds. The extra torques have had quite a large effect, particularly in miss rate. This indicates the addition of some jitter to the beam as might be expected.

Target maneuvers are introduced next. The parameters of Appendix
FIGURE 6.5
Effect of a Lower Measurement Rate
FIGURE 6.6  
Effect of a Larger Miss Angle

Measurement Uncertainty

\[ \Delta \psi \text{ meas. unc.} \]

\[ \Delta \psi \text{ meas. unc.} \]

\[ \Delta \psi \text{ (nominal)} \]

Miss Distance (ft)
FIGURE 6.7
Effect of a Larger Miss Angle Measurement Uncertainty

- Δψ meas., unc. 1 x 10^-3 radians
- Δψ meas., inc. 3 x 10^-3 radians

Time (sec) vs. Miss Rate (ft/sec)
FIGURE 6.9
Effect of Higher Disturbance Torques

Miss Rate (ft/sec)

Time (sec)

High disturbance torques
Nominal

\[ \sqrt{\Delta^2} \]

\[ \sqrt{(\Delta - \hat{\Delta})^2} \]
C were used. The results are plotted in Figure 6.10 and 6.11. The miss distance has not been increased greatly but the miss rate was increased by a somewhat larger fraction. It should be remembered that target motion in the y direction has not been modeled. This is the direction of target motion which causes the greatest errors in the azimuthal tracking for the early part of the trajectory. The modeled target motion will have a larger effect in the elevation tracking which is sensitive to z motions.

Range rate measurements will probably be expensive to implement. To ascertain the value of these measurements a run was made with essentially no range rate measurement. This is done by setting the rate measurement uncertainty to a large number (1000 ft/sec). The run started at 5 seconds and ended at 10. The results were larger than the nominal in only the third significant figure. Therefore, range rate measurements can probably be eliminated without seriously compromising system performance.

The previous runs were all made with an optimal filter in the sense that the filter has an exact model of the system. The remaining computer runs are made with a miss-matched filter.

Figure 6.12 represents the system performance when the filter has a +10% error in the value of the scale factor used in the evaluation of the miss angle measurement. The rate curves have not been plotted since they are essentially identical to the nominal. The net result of the error is that a small angle error has been introduced without increasing the rate error.

A similar run was made with a 10% error in the scale factor of the beam rate measurement. The run started at 15 seconds. The results are not plotted since they are practically the same as the nominal. Apparently the beam angle measurement is good enough to compensate for the error in measuring beam rate. Also any errors in the estimate of beam rate will not
FIGURE 6.10
Effect of Target Maneuvers

- Target maneuvers
- No maneuvers (nominal)

Miss Distance (ft)

Time (sec)
FIGURE 6.11
Effect of Target Maneuvers

--- No maneuver (nominal)

\[ \sqrt{\Delta^2} \]

\[ (\hat{\Delta} - \Delta)^2 \]

Time (sec)

Miss Rate (ft/sec)
be strongly transmitted through the control law because of the relatively low gain applied to beam rate. Later in the encounter, as beam rates become large the effects of this error may become more significant.

The remaining figures present the results after a 10% error in the value of the motor torque constant was included. Extreme transient oscillations were introduced for both a plus and minus error as can be seen in Figures 6.13 to 6.16. Similar results are obtained early in the encounter. This case is plotted in Figures 6.17 and 6.18. The oscillations begin to decay after the five seconds analyzed. The frequency of the oscillation is by no means constant but it is far below the 20 radians/second used in the control law. Apparently the now sub-optimal filter is interacting with the system through the control law to produce the oscillations. In other words, the filter has introduced new dynamics into the system. These results indicate that some care must be taken when parameter values for the filter are chosen. If the values do not represent the true system, instability may result. Also the system model used by the filter will have to be chosen with care.

This concludes the discussion of the results of individual computer runs. One other feature of the program remains to be presented. Chapter IV showed that on a deterministic basis the major source of the tracking error was the failure to account for target accelerations. The error was given by equation 4.1 which is

$$\Delta \psi = \frac{\ddot{\psi}_r}{\omega_r^2}$$  \hspace{1cm} (6.1)

It was suggested that if the target acceleration can be estimated, this error can be eliminated. A target acceleration estimate can be calculated from the state vector by the following;

$$\hat{\psi}_r = C^T \hat{\Phi}$$  \hspace{1cm} (6.2)
Figure 6.14
Effect of an Error in Torque Constant

- +10% error in $K_T$
- no error (nominal)

Time (sec)

Miss Rate (ft/sec)

$\frac{(\Delta - \overline{\Delta})^2}{\overline{\Delta}^2}$
FIGURE 6.15
Effect of an Error in Torque Constant

-10% error in $K_T$
no error (nominal)

Miss Distance (ft)

Time (sec)
FIGURE 6.16
Effect of an Error in Torque Constant

-10% error in $K_T$
--- no error (nominal)
Figure 6.17: Effect of an Error in Torque Constant

-10% error in $K_T$

- no error (nominal)

Miss Distance (ft)

Time (sec)
FIGURE 6.18
Effect of an Error in Torque Constant

-10% error in $K_T$
--- no error (nominal)

Miss Rate (ft/sec)

Time (sec)

$[\Delta^2]^{1/2}$

$[(\Delta-\Delta^*)^{1/2}]$
The uncertainty in this estimate can now be calculated by

\[
(\hat{\Omega}_T - \hat{\Omega}_T)^2 = \mathbf{C}^T \mathbf{P} \mathbf{C}
\]  

(6.3)

where \( \mathbf{P} \) is the covariance matrix of the state estimation error. The uncertainty in miss distance resulting from the uncertainty in the estimate of the target acceleration is

\[
\left[ (\delta - \delta)^2 \right]^{1/2} = \frac{R}{\omega} \left[ \mathbf{C}^T \mathbf{P} \mathbf{C} \right]^{1/2}
\]  

(6.4)

Equations (6.3) and (6.4) were programmed and the results printed. The result for the nominal case is always below \( 1 \times 10^{-3} \) feet which is quite small when compared with the deterministic tracking error. This indicates that the majority of the target acceleration induced tracking error could be eliminated by the inclusion of a control generated from the acceleration estimate. This possibility should be investigated further. Particular attention should be paid to cases involving a miss-matched filter, especially in the light of the possible stability problems indicated in Figures 6.13 through 6.18. The inclusion of the new control signal could make this situation worse.
CHAPTER VII

CONCLUSIONS

The system performance indicated by the preceding chapter is encouraging. The tracking accuracy is at least in the neighborhood of a weapon requirement. However, the analysis that generated the results is by no means complete. The system model was very simple. Many effects, some of which may prove to be quite important, were ignored or greatly simplified. A more thorough analysis is required to completely assess their impact.

However, some general conclusions can be drawn from the analysis;

1) Range rate measurements increase system performance by very little and can probably be eliminated.

2) Parameter mismatch may generate stability problems.

3) The performance of the control law may be improved by the generation of a control signal proportional to estimated target acceleration. The augmented system must be analyzed to ensure stability, particularly when parameter mismatch is present.

4) The measurement most critical to system performance is miss angle.

Further studies of the problem should include more complete modeling as was indicated in Chapter II. Eventually both the azimuth and elevation tracking should be analyzed together. The practical considerations of implementing a Kalman filter should be evaluated. Real time processing may be difficult to achieve at the measurement rates postulated. Finally the entire system should be simulated completely and the performance evaluated by Monte Carlo techniques. A hybrid computer might be effectively used for this task.
APPENDIX A

MOUNT AND DRIVE MODEL PARAMETER VALUES

Table A.1 presents the parameter values used as the nominal conditions for the computer runs. The torque motor parameter values come from a large Inland torque motor, model number T-36001. A smaller motor might be successfully used especially if the inertia estimate is high.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_T$</td>
<td>Torque Constant</td>
<td>Ft. lb/amp.</td>
<td>53.0</td>
</tr>
<tr>
<td>$R$</td>
<td>Electrical Resistance</td>
<td>Ohms</td>
<td>2.3</td>
</tr>
<tr>
<td>$K_b$</td>
<td>Back EMF Constant</td>
<td>Volts/rad/sec</td>
<td>72.0</td>
</tr>
<tr>
<td>$C_s$</td>
<td>Maximum Applied Command</td>
<td>Volts</td>
<td>130.0</td>
</tr>
<tr>
<td>$J$</td>
<td>Gimbal Inertia</td>
<td>Slug Ft$^2$</td>
<td>1700.</td>
</tr>
</tbody>
</table>
APPENDIX B

MOUNT DISTURBANCE TORQUE MODEL PARAMETER VALUES

Table B.1 summarizes the parameter values used in the disturbance torque model. Two sets of values are presented to reflect the range of values which would yield low and high torques.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
<th>Low Value</th>
<th>High Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>Gimbal Frontal Area</td>
<td>$\text{Ft}^2$</td>
<td>16.</td>
<td>25.</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Air Density</td>
<td>$\text{Slug/Ft}^3$</td>
<td>$2.38 \times 10^{-3}$</td>
<td>$2.38 \times 10^{-3}$</td>
</tr>
<tr>
<td>$K$</td>
<td>Shroud Reduction Factor</td>
<td>-</td>
<td>0.1</td>
<td>1.0</td>
</tr>
<tr>
<td>$C_\alpha$</td>
<td>Coefficient of Drag</td>
<td>-</td>
<td>1.0</td>
<td>1.3</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Moment Arm</td>
<td>$\text{Ft}$</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>$C$</td>
<td>Linearization Constant</td>
<td>$\text{Ft/sec}$</td>
<td>2.3</td>
<td>20.0</td>
</tr>
<tr>
<td>$\sigma_\omega$</td>
<td>Wind Gust Strength</td>
<td>$\text{Ft/sec}$</td>
<td>2.3</td>
<td>20.0</td>
</tr>
<tr>
<td>$T$</td>
<td>Wind Filter Time Constant</td>
<td>$\text{Rad/sec}$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Table C. 1 is a summary of the parameter values used in the computer runs.

The aircraft parameters represent an F-4 fighter.

The wind parameters are contained in reference (2). The wind standard deviation represents a moderately gusty day. The values of \( \sigma \) and \( L \) used to represent pilot maneuvers were generated by assuming that the pilot could pull vertical accelerations with a standard deviation of 32 feet/second\(^2\) with a frequency break point of 10 radians/second.

The reference trajectory has a constant target \( y \) position fixed at 1000 feet. The initial \( x \) position is 25000 feet and the \( x \) velocity is constant at -1000 feet/second. The target \( z \) position is calculated to keep the target elevation angle (\( \theta_z \)) a constant 10 degrees.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
<th>No Maneuver Value</th>
<th>Maneuver Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>Air Density</td>
<td>Slug/Ft(^3)</td>
<td>2.38x10(^{-3})</td>
<td>2.38x10(^{-3})</td>
</tr>
<tr>
<td>( S )</td>
<td>Wing Area</td>
<td>Ft(^2)</td>
<td>530.</td>
<td>530.</td>
</tr>
<tr>
<td>m</td>
<td>Aircraft Mass</td>
<td>Slug</td>
<td>1365.</td>
<td>1365.</td>
</tr>
<tr>
<td>( C_{pa} )</td>
<td>Drag Coefficient slope</td>
<td>1/Radian</td>
<td>-.44</td>
<td>-.44</td>
</tr>
<tr>
<td>( C_{la} )</td>
<td>Lift Coefficient slope</td>
<td>1/Radian</td>
<td>3.65</td>
<td>3.65</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Autopilot Damping Ratio</td>
<td>-</td>
<td>.7</td>
<td>-</td>
</tr>
<tr>
<td>( \omega )</td>
<td>Autopilot Natural Frequency</td>
<td>Rad/sec</td>
<td>10.</td>
<td>-</td>
</tr>
<tr>
<td>( V )</td>
<td>Aircraft Velocity</td>
<td>Ft/sec</td>
<td>1000.</td>
<td>1000.</td>
</tr>
<tr>
<td>( K )</td>
<td>Autopilot Gain</td>
<td>-</td>
<td>1.</td>
<td>0.</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Wind Standard Deviation</td>
<td>Ft/sec</td>
<td>5.</td>
<td>20.</td>
</tr>
<tr>
<td>L</td>
<td>Wind Characteristic Length</td>
<td>Ft</td>
<td>1000.</td>
<td>100.</td>
</tr>
</tbody>
</table>
APPENDIX D

COORDINATE TRANSFORMATION EQUATIONS

The transformation equations required to convert the target linear acceleration \((\ddot{x}, \ddot{y}, \ddot{z})\) to spherical accelerations \((\ddot{\theta}, \ddot{\psi}, \ddot{R})\) are given in equations (D.1), (D.2), and (D.3). Notice that even if the target accelerations are zero, the angular accelerations are not. This is of importance in Chapter IV where angular accelerations were found to be the significant source of error in the control law performance.

The partial derivatives of equations (D.1), (D.2), and (D.3) with respect to \(\dot{x}, \ddot{y}, \ddot{z}, \dot{\theta}, \ddot{\theta}, \dot{\psi}, \ddot{\psi}\), \(\dot{R}\) and \(\ddot{R}\) are the derivative matrix elements necessary to implement the extended Kalman filter.

\[
\ddot{\theta} = \frac{1}{R}(\ddot{z}\cos\theta - \dot{x}\sin\theta\cos\psi - \dot{y}\sin\theta\sin\psi - 2\dot{R}\dot{\theta}) - \dot{\psi}\frac{\sin\theta}{\cos\theta}
\]

(D.1)

\[
\ddot{\psi} = \frac{1}{R\cos\theta} \left( \dot{y}\cos\psi - \dot{x}\sin\psi - 2\dot{\psi}(\dot{R}\cos\theta - R\dot{\theta}\sin\theta) \right)
\]

(D.2)

\[
\ddot{R} = \dot{x}\cos\theta\sin\psi + \dot{y}\cos\theta\sin\psi + \ddot{z}\sin\theta + R(\dot{\theta} + \dot{\psi}\cos\theta)
\]

(D.3)
APPENDIX E

Listing of Deterministic Program

The following pages are the listing of the computer program used for a deterministic evaluation of the control law. The results of the program were presented in Chapter IV. The program is written in FORTRAN. The subroutines WHERE, SETUP, and READIN are in the Avco Systems Division Computer Center subroutine library. They allow convenient free format input of the variables used in the calls to SETUP.
IMPLICIT REAL*A (A-H,O-Z)
REAL*4 TABLE
DIMENSION TABLE(7,20)
DIMENSION TORK(2),RES(2),BACK(2),GINIT(2),TACK(2),X(3),V0(3),
AT(3),ERI(2),XT(3),VT(3),B(6),HER(3)

CALL WHERE(TABLE)
J=9
K=4
CALL SETUP(8HTORK J,TORK 2)
CALL SETUP(8HRES J,RES 2)
CALL SETUP(8HBACK J,BACK 2)
CALL SETUP(8HGINIT J,GINIT 2)
CALL SETUP(8HTACK J,TACK 2)
CALL SETUP(8HCOMW J,COMW 1)
CALL SETUP(8HCOMZ J,COMZ 1)
CALL SETUP(8HXO J,X0 3)
CALL SETUP(8HV0 J,V0 3)
CALL SETUP(8HAT J,AT 3)
CALL SETUP(8HHER J,HER 3)
CALL SETUP(8HUSAT J,USAT 1)
CALL SETUP(8HERI J,ERI 2)
CALL SETUP(8HINOM K,INOM 1)
CALL SETUP(8HICNTU K,ICNTU 1)
CALL SETUP(8HNUMM K,NUMM 1)
CALL SETUP(8HIMP K,IMP 1)
CALL SETUP(8HTAT J,DT 1)
CALL SETUP(8HTHETA J,THETA 1)

PI=DATAN2(1.DO,1.DO)*4.DO
DTR=PI/180.DO

NEW CASE

2 CONTINUE
CALL READIN(K)
IF (ICNTU NE 0) GO TO 20

COMPLETE RESET — SKIPPED IF CASE IS CONTINUED

TANT=DTAN(DTR*THETA)
DO 3 J=1,3
XT(J)=X0(J)
VT(J)=V0(J)
3 CONTINUE
IF (INOM EQ 2) GO TO 10
TORK(2)=0.0
RES(2)=0.0
BACK(2)=0.0
GINIT(2)=0.0
TACK(2)=0.0
10 CONTINUE
TIME=0.0
CALL BCALC(XT,VT,AT,TANT,B(5),B(6),R)
IF (THETA .GE. 0.0) CALL ZCALC(XT,VT,AT,TANT,B(5),B(6),R)
B(1)=ERF(1)
B(2)=ERF(2)
B(3)=B(5)-B(1)
B(4)=B(6)-B(2)

20 CONTINUE
DTO2=.50*D*T*DT
ISATF=0
IP=0

UMAX=-1.0
UAF=TORK(1)/(RES(1)*GINIT(1))
DBAF=BACK(1)+TACK(1)+TORK(1)/(RES(1)*GINIT(1))

99 FORMAT(*1*)

INTEGRATION LOOP

DO 50 J=1,NUMM
TIME=TIME+DT
US=DMM*R(1)+DRM*B(2)+BRM*B(4)
UA=DABS(U)
IF (UA .GT. UMAX) UMAX=UA
US=U
IF (U .LT. USAT) GO TO 12
US=USAT
GO TO 14
12 IF (U .GT. -USAT) GO TO 15
US=-USAT
14 ISATF=ISATF+1
15 CONTINUE
RA=UAF*US-DBAF*B(4)
B(3) = B(3) + DT*B(4) + DTO2*BA
B(4) = R(4) + DT*BA
XT(1) = XT(1) + DT*VT(1) + DTO2*AT(1)
XT(2) = XT(2) + DT*VT(2) + DTO2*AT(2)
VT(1) = VT(1) + DT*AT(1)
VT(2) = VT(2) + DT*AT(2)
IF (THETA > 0.00) GO TO 49
XT(3) = XT(3) + DT*VT(3) + DTO2*AT(3)
VT(3) = VT(3) + DT*AT(3)
CALL BCALC(XT, VT, AT, TANT, B(5), B(6), R)
GO TO 48
49 CONTINUE
CALL ZCALC(XT, VT, AT, TANT, B(5), B(6), R)
48 CONTINUE
B(1) = B(5) - B(3)
B(2) = R(6) - B(4)
IP = IP + 1
IF (IP < LT IMP) GO TO 50
IP = 0
DMISS = B(1)*R
RMISS = B(2)*R
WRITE(6, 100) TIME, DMISS, RMISS, B(1), B(2), R, UMAX, US, ISATF
100 FORMAT(*,,F6.2, 8G13.5, 15)
UMAX = -1.0
ISATF = 0
50 CONTINUE
GO TO 2
END
SUBROUTINE ZCALC(X, XD, XDD, TANT, TP, TPD, R)
IMPLICIT REAL*8 (A-H, O-Z)
DIMENSION X(3), XD(3), XDD(3)
T=X(1)**2+X(2)**2
U=DSQRT(T)
X(3)=TANT*U
CONTINUE
TP=DATAN2(X(2), X(1))
TPD=(X(1)*XD(2)-X(2)*XD(1))/T
R=DSQRT(X(3)**2+T)
RETURN
ENTRY RCALC(X, XD, XDD, TANT, TP, TPD, R)
T=X(1)**2+X(2)**2
GO TO 1
END
APPENDIX F

Listing of Statistical Program

The listing of the statistical program described in Chapter VI is presented in the following pages. The program is written in FORTRAN. The functions of the subroutines WHERE, SETUP, and READIN are described in Appendix E. The symmetric matrix inversion subroutine, DSINV, was taken from the IBM FORTRAN scientific subroutine package.
KALMAN FILTER ERROR ANALYSIS PROGRAM -AZIMUTH ONLY-

IMPLICIT REAL*8 (A-H,O-Z)
REAL*4 TABLE

DIMENSION TABLE(7,45),TORK(2),RES(2),TACK(2),BACK(2),GINIT(2),
1 WGTCT(2),WGFGT(2),CDA(2),WA(2),DEN(2),SIGW(2),UNC(5),
2 TORG(2),TLIN(2),HER(5),XO(3),V0(3),ACM(2),CI(10),XI(10),
3 CC(10),CLA(2),AUTOG(2),AUTOW(2),AUTOZ(2),AAC(10),QZ(10),QX(10)
4 DIMENSION PARY(6),URY(6),PFARY(6),QARY(6)

COMMON /FILT/ P(10,10),V(10,10),U(10,10),PF(10,10),F(10,10),
1 FF(10,10),DF(10,10),H(5,10),HF(5,10),DH(5,10),FG(10,5),X(10),
2 EL(10),ELF(10),DEL(10),RDOT,T,TDOT,XT(3),VT(3),
3 FAD(2),AXG(2),RN(5,5),Q(10,10),WLEN(2),ELC(10,10),ELC(10,10),
4 DT1,DT12,DT12,AT(3),HFT(10,5),TANT,AT1G(2),AZ2G(2),AXZ(2)

COMMON /IFILT/ NS,NM,NC,NIS

ONE TIME ONLY INITIALIZATIONS

CALL WHERE(TABLE)

J=4
K=8
CALL SETUP(SHTORK) K,TORK *,2
CALL SETUP(8HRES) K,RES *,2
CALL SETUP(8HTACK) K,TACK *,2
CALL SETUP(8HBACK) K,BACK *,2
CALL SETUP(8HGINIT) K,GINIT *,2
CALL SETUP(8HWGTCT) K,WGTCT *,2
CALL SETUP(8HWGFGT) K,WGFGT *,2
CALL SETUP(8HCDA) K,CDA *,2
CALL SETUP(8HWA) K,WA *,2
CALL SETUP(8HACM) K,ACM *,2
CALL SETUP(8HDEN) K,DEN *,2
CALL SETUP(8HWLEN) K,WLEN *,2
CALL SETUP(8HSIGW) K,SIGW *,2
CALL SETUP(8HUNCM) K,UNC *,5
CALL SETUP(8HTORG) K,TORG *,2
CALL SETUP(8HTLIN) K,TLIN *,2
CALL SETUP(8HHER) K,HFR *,5
CALL SETUP(8HXO) K,XO *,3
CALL SETUP(8HVO) K,V0 *,3
CALL SETUP(8HAT) K,AT *,3
CALL SETUP(BHXT) *K,XT  3
CALL SETUP(BHVT) *K,VT  3
CALL SETUP(BHTheta) *K,THETA
CALL SETUP(BHCLA) *K,CLA  2
CALL SETUP(BHAURO) *K,AURO  2
CALL SETUP(BHAUAW) *K,AUROW  2
CALL SETUP(BHAUOW) *K,AUOW  2
CALL SETUP(BHAUOW) *K,AUOWZ  2
CALL SETUP(BAHAOW) *K,HAOW  2
CALL SETUP(BAHCOM) *K,COM  2
CALL SETUP(BHCOM) *K,COMZ  2
CALL SETUP(BHPEO) *K,PEO
CALL SETUP(BHEVE) *K,VE  
CALL SETUP(BHDTM) *K,DTM  
CALL SETUP(BHTIME) *K,TIME  
CALL SETUP(BHNIIS) *J,NIS  
CALL SETUP(BHINOM) *J,INOM  
CALL SETUP(BHTEXT) *J,TEXT  
CALL SETUP(BHNUMM) *J,NUMM  
CALL SETUP(BHIMP) *J,IMP  
CALL SETUP(BHICNTU) *J,ICNTU  
CALL SETUP(BHICNTU) *J,ICNTU  
CALL SETUP(BHIPUNCH) *J,IPUNCH  
CALL SETUP(BHIMATIN) *J,IMATIN  

C
NS=10
NM=5
NC=1
ZR=0.0
PI=3.14159 *4.0
DTR=PI/180.0
IMP=1
ICNTU=0
IPUNCH=0
IMATIN=0

C
DO 10 J=1,NS
  EL(J)=ZR
  DEL(J)=ZR
  ELF(J)=ZR
  CC(J)=ZR
DO 14 K=1,NS
  FF(K,J)=ZR
  DF(K,J)=ZR
  F(K,J)=ZR
  O(K,J)=ZR
14 CONTINUE
DO 15 K=1,NM
  HFT(J,K)=ZR
  H(K,J)=ZR
  HF(K,J)=ZR
15 CONTINUE
`DH(K,J)=ZR
15 CONTINUE
10 CONTINUE

C
DO 11 J=1,NM
HER(J)=ZR
DO 11 K=1,NM
RN(K,J)=ZR
11 CONTINUE
DO 12 J=1,3
AT(J)=ZR
X0(J)=ZR
VO(J)=ZR
12 CONTINUE

C
NEW CASE STARTS HERE
C
2 CONTINUE
CALL READIN(K)
IF (ICNTU .NE. 0) GO TO 13
DO 20 J=1,3
XT(J)=X0(J)
VT(J)=VO(J)
20 CONTINUE
13 CONTINUE
TANT=DTAN(THETA*DTR)
DTI=DTM/NIS
DTI2=.5D0*DTI
DTII2=DTI2*DTI

C
NOMINAL CONDITIONS IF REQUESTED
C
IF (INOM .EQ. 0) GO TO 8
TORK(2)=ZR
RES(2)=ZR
TACK(2)=ZR
BACK(2)=ZR
GINIT(2)=ZR
WGTCT(2)=ZR
WGFGT(2)=ZR
CDA(2)=ZR
CLA(2)=ZR
AUTOG(2)=ZR
AUTOW(2)=ZR
AUTOZ(2)=ZR
WA(2)=ZR
DERIVITIVE MATRIXES

F(1,2)=1.0
FF(1,2)=1.0
F(3,4)=1.0
FF(3,4)=1.0
F(7,8)=1.0
FF(7,8)=1.0
F(5,5)=1.0
FF(5,5)=1.0
FF(6,6)=VE/(WLEN(1)*(1.0+D0+WLEN(2)))
AXG(1)=CDA(1)*SIGW(1)*WA(1)*5.0*DEN(1)
AXG(2)=AXG(1)*(1.0+D0+CDA(2))*(1.0+SIGW(2))*(1.0+WA(2))*
1.0*(1.0+D0+DEN(2))*VE*VE*DSORT(2.0*D0+WLEN(1)*(1.0+D0+WLEN(2)))/
2.0*(PI*VE)/(ACM(1)*(1.0+D0+ACM(2))*WLEN(1)*(1.0+D0+WLEN(2)))
AXG(1)=AXG(1)*DSORT(2.0*D0+WLEN(1)/PI)/(ACM(1)*WLEN(1))
AXZ(1)=AXG(1)*CLA(1)/CDA(1)
AXZ(2)=AXG(2)*CLA(1)*(1.0+CDA(1))/(CDA(1)*(1.0+CDA(2)))
FAD(1)=(TACK(1)+BACK(1))*TORK(1)/RES(1)/GINIT(1)
FAD(2)=(TACK(1)*(1.0+TACK(2))+BACK(1)*(1.0+BACK(2)))*
1.0*TORK(1)*(1.0+TORK(2))/(RES(1)*(1.0+RES(2)))*GINIT(1)*
2.0*(1.0+GINIT(2))
DF(4,4)=FAD(2)-FAD(1)
F(4,4)=FAD(1)
FF(4,4)=FAD(2)
F(4,4)=FG(4,4)-F(4,4)
F(4,5)=WFGT(1)*TORG(1)*TLIN(1)/(WGTCT(1)*GINIT(1))
FF(4,5)=WFGT(1)*(1.0+FG(4,5))
TLIN(1)*(1.0+FG(4,5))*TORG(1)*(1.0+TORG(2))*
F(2,5)=F(4,5)
FF(2,5)=F(4,5)
FF(2,5)=F(2,5)-F(2,5)
F(9,6)=1.0
FF(9,6)=1.0
F(9,9)=-2.0*D0*AUTOZ(1)*AUTOW(1)
FF(9,9)=F(9,9)*(1.0*D0+AUTOW(2))*(1.0+D0+AUTOZ(2))
DF(9,9)=FF(9,9)-F(9,9)
F(10,9)=1.00
FF(10,9)=1.00
F(9,10)=0.00
F(9,10)=F(9,10)*F(9,10)*F(9,10)
DF(9,10)=DF(9,10)-F(9,10)
AZIG(1)=-AZX(1)*AZOG(1)*AZOZ(1)*AUTOW(1)
AZIG(2)=-AZX(2)*
1 AUTOG(1)*(1.00+AUTOG(2))**2+D0*AUTOZ(1)*AUTOW(1)**2
2 AUTOW(1)**(1.00+AUTOW(2))
AZ2G(1)=-AZX(1)*AUTOG(1)*AUTOW(1)**2
AZ2G(2)=-AZX(2)*
1 AUTOG(1)*(1.00+AUTOG(2))**2*(AUTOW(1)**(1.00+AUTOW(2))**2

CONTROL MATRIXES

EL(4)=TORK(1)/(RES(1)*GINIT(1))
EL(4)=EL(4)
ELF(4)=TORK(1)*(1.00+TORK(2))/(RES(1)**(1.00+RES(2))**2
GINIT(1)**(1.00+GINIT(2))
ELF(2)=ELF(4)
DEL(4)=ELF(4)-EL(4)
DEL(2)=DEL(4)

COMMAND CONSTANT

CC(4)=BACK(1)*(1.00+BACK(2))**TACK(1)**(1.00+TACK(2))
CC(1)=COMW**COMW*GINIT(1)**(1.00+GINIT(2))**RES(1)**(1.00+RES(2))**2
TORK(1)**(1.00+TORK(2))
CC(2)=2.00*COMZ**COMW*GINIT(1)**(1.00+GINIT(2))**RES(1)**
(1.00+RES(2))/(TORK(1)**(1.00+TORK(2))
CALM PRD(EL,CC,ELC,NS,1,NS)
CALM PRD(DEL,CC,DELNC,NS,1,NS)

SYSTEM DRIVING NOISE

0(5,5)=1.00
0(6,6)=1.00

MEASUREMENT NOISE

DO 21 J=1,NM
RN(J,J)=UNCM(J)**2
21 CONTINUE
MEASUREMENT MATRIXES

\[
\begin{align*}
H(1,1) &= 1.00 - H(1) \\
HF(1,1) &= 1.00 \\
HFT(1,1) &= 1.00 \\
DH(1,1) &= H(1) \\
H(2,3) &= 1.00 - H(2) \\
HF(2,3) &= 1.00 \\
HFT(2,3) &= 1.00 \\
DH(2,3) &= H(2) \\
H(3,4) &= 1.00 - H(3) \\
HF(3,4) &= 1.00 \\
HFT(3,4) &= 1.00 \\
DH(3,4) &= H(3) \\
H(4,7) &= 1.00 - H(4) \\
HF(4,7) &= 1.00 \\
HFT(4,7) &= 1.00 \\
DH(4,7) &= H(4) \\
H(5,8) &= 1.00 - H(5) \\
HF(5,8) &= 1.00 \\
HFT(5,8) &= 1.00 \\
DH(5,8) &= H(5)
\end{align*}
\]

INITIAL CONDITIONS

IF (ICNTU .NE. 0) GO TO 30
CALL ZCALC(XT,VT,AT,TANT)
CI(1) = UNCME(1)
CI(2) = UNCME(1)*10.00
CI(3) = UNCME(2)
IF (CI(3) .LT. 0.00) CI(3) = CI(1)
CI(4) = UNCME(3)
CI(5) = 1.00/FF(5,5)
CI(6) = 1.00/FF(6,6)
CI(7) = UNCME(4)
IF (CI(7) .LT. 0.00) CI(7) = 1000.00
CI(8) = UNCME(5)
IF (CI(8) .LT. 0.00) CI(8) = 500.00
CI(9) = CI(6)/FF(9,9)
CI(10) = CI(6)/FF(9,10)
A = XT(1)**2 + XT(2)**2
X(1) = PEO
X(2) = 0.00
X(3) = DATAN2(XT(2), XT(1)) - PEO
X(4) = (XT(1)*VT(2) - XT(2)*VT(1)) / A
X(5) = 0.00

A
XI(6)=0,D0
XI(7)=DSQRT(A+XT(3)**2)
XI(8)=(XT(1)*VT(1)+XT(2)*VT(2)+XT(3)*VT(3))/XI(7)
XI(9)=0,D0
XI(10)=0,D0
DO 31 J=1,NS
   DO 31 K=1,NS
      A=0,D0
      B=XT(J)*XI(K)
      IF (J NE K) GO TO 35
      A=CI(J)**2
      B=B+A
   35 CONTINUE
   U(K,J)=B
   V(K,J)=-A
   P(K,J)=A
   PF(K,J)=A
   31 CONTINUE
WRITE(6,100)
100 FORMAT(*1*)
TIME=0,D0
30 CONTINUE
   IF (IMATIN EQ 0) GO TO 48
   READ(5,201) U,P,V,PF
   WRITE(6,100)
   48 CONTINUE
IMPC=1

TO PROCESS NUMM MEASUREMENTS

DO 50 IJK=1,NUMM
   TIME=TIME+DTM
   CALL PROPO
   CALL KFILT(IJK, IEXT)
   IF (IMP GT IMPC) GO TO 49
   K=1
   DO 93 J=1,6
      IF (J EQ 5) K=7
      A=U(K,K)
      UARY(J)=DSIGN(DSQRNT(DABS(A)),A)
      A=P(K,K)
      PARY(J)=DSIGN(DSQRNT(DABS(A)),A)
      A=PF(K,K)
      PFARY(J)=DSIGN(DSQRNT(DABS(A)),A)
      K=K+1
   93 CONTINUE
RR=DSQRT(XT(1)**2+XT(2)**2+XT(3)**2)
QARY(1)=UARY(1)*RR
QARY(2)=UARY(2)*RR
QARY(3)=PARY(1)*RR
QARY(4)=PARY(2)*RR
QARY(5)=PFARY(1)*RR
QARY(6)=PFARY(2)*RR
AAC(1)=F(2,1)
AAC(2)=F(2,2)
AAC(3)=AAC(1)
AAC(4)=AAC(2)
AAC(5)=0.D0
AAC(6)=F(2,6)
AAC(7)=F(2,7)
AAC(8)=F(2,8)
AAC(9)=0.D0
AAC(10)=0.D0
CALL MPAR(AAC,P,QZ,1,NS,NS)
CALL MPAR(QZ,AAC,AAE,1,NS,1)
CALL MPAR(AAC,QZ,1,NS,NS)
CALL MPAR(QZ,AAC,AAU,1,NS,1)
AAE=DSIGN(DSQR(DABS(AAE)),AAE)
AAU=DSIGN(DSQR(DABS(AAU)),AAU)
AAED=RR*AAE/(COMW*COMW)
AAUD=RR*AAU/(COMW*COMW)
WRITE(6,102) TIME,RR,QARY,UARY,PARY,PFARY,AAED,AAUD,AAE,AAU
102 FORMAT(10,F7,2,F19.0,3X,6G16.6,4(/,21X,6G16.6))
IMPC=0
49 IMPC=IMPC+1
50 CONTINUE
IF (IPUNCH.EQ.0) GO TO 2
WRITE(7,200) XT,VT,TIME,THETA
200 FORMAT(1X,3G20.10,/,4X,3G20.10,/,1 ICNTU 1 ,1 IMATIN 1 ,1 IMATL 1 ,1 TIME ,4X,3G20.10,/,THETA ,3G20.10)
WRITE(7,201) U,P,V,PF
201 FORMAT(4X,9A8,4X)
GO TO 2
END
SUBROUTINE NLIN
IMPLICIT REAL*8 (A-H,O-Z)
COMMON /FILT/ P(10,10), V(10,10), U(10,10), PF(10,10), F(10,10),
1 FF(10,10), DF(10,10), H(5,10), HF(5,10), DH(5,10), FG(10,5), X(10),
2 FL(10), ELF(10), DEL(10), R, RDDT, T, TDDT, XT(3), VT(3),
3 FAD(2), AXG(2), RN(5,5), O(10,10), WLEN(2), DELC(10,10), FLC(10,10),
4 DTI, DTL, DTI2, AT(3), HFT(10,5), TANT, AZIG(2), AZ2G(2), AXZ(2)

CONVERSION CONSTANTS

D=XT(1)**2+XT(2)**2
PD=(XT(1)*VT(2)-VT(1)*XT(2))/D
RT=DSRT(D+XT(3)**2)
D=DSRT(D)
RI=1.0D0/RT
ST=XT(3)*RI
CT=D*RI
D=1.0D0/D
SP=XT(2)*D
CP=XT(1)*D
RD=(XT(1)*VT(1)+XT(2)*VT(2)+XT(3)*VT(3))*RI
RCTI=RI/CT
TD=(VT(3)-RD*ST)*RCTI
D2=VT(1)**2+VT(2)**2+VT(3)**2
D=DSRT(D2)
T=-RCTI*SP
TDT=D2*DSRT(1.0D0/D)
TT=AXG(1)*TDT

PHI DOUBLE DOT

F(2,6)=T*TT
FF(2,6)=T*AXG(2)
DF(2,6)=FF(2,6)-F(2,6)
T=2.0D0*(TD*ST/CT-RD*RI)
F(2,2)=T
FF(2,2)=T
F(2,4)=FAD(1)+T
FF(2,4)=FAD(2)+T
F(6,6)=-D/WLEN(1)
DF(6,6)=FF(6,6)-F(6,6)
T=2.0D0*PD*RI
F(2,8)=T
FF(2,8)=T
T=RCTI*(-AT(1)*CP AT(3)*SP)
F(2,1)=T
FF(2,1)=T
F(2,3)=T
FF(2,3)=T
T=RI*(RCTI*(AT(1)*SP-AT(2)*CP)+2*DO*RI*RD*PD)
F(2,7)=T
FF(2,7)=T

RANGE DOUBLE DOT

D=CT*CP
FF(8,6)=D*AXG(2)+AXZ(2)*ST
F(8,6)=D*TT+AXZ(1)*ST*TD
DF(8,6)=FF(R,6)-F(8,6)
TT=CT*CT*PD
T=TD*TD*PD*TT
F(8,7)=T
FF(8,7)=T
T=2*RT*TT
F(8,2)=T
FF(8,2)=T
F(8,4)=T
FF(8,4)=T
T=D*AT(2)-CT*SP*AT(1)
F(8,1)=T
FF(8,1)=T
F(8,3)=T
FF(8,3)=T
TDT=TDT*ST
F(8,9)=AZ1G(1)*TDT
F(8,10)=AZ2G(1)*TDT
FF(8,9)=AZ1G(2)*ST
FF(8,10)=AZ2G(2)*ST
DF(8,9)=FF(8,9)-F(8,9)
DF(8,10)=FF(8,10)-F(8,10)
RETURN
END
DOUBLE PRECISION SYMMETRIC INVERSION

SUBROUTINE DSINV(A,N, EPS, IER)
DIMENSION A(1)
REAL*8 A, AINV, DSUM, DPIV, EPS, TOL

FACTORIZE GIVEN MATRIX
A = TRANSPOSE(T) * T

IER=0

INITIALIZE DIAGONAL LOOP
KPIV=0
DO 110 K=1,N
KPIV=KPIV+K
IND=KPIV
LEND=K-1

CALCULATE TOLERANCE
TOL=DAABS(EPS*A(KPIV))

START FACTORIZATION-LOOP OVER K-TH ROW
DO 110 I=K,N
DSUM=0.DO
IF (LEND) 20,40,20

START INNER LOOP
20 DO 30 L=1,LEND
LANF=KPIV-L
LIND=IND-L
30 DSUM=DSUM+A(LANF)*A(LIND)
END OF INNER LOOP

TRANSFORM ELEMENT A(IND)
40 DSUM=A(IND)-DSUM
IF (I-K) 100,50,100

TEST FOR NEGATIVE PIVOT ELEMENT AND FOR LOSS OF SIG
50 IF (DSUM-TOL) 60,60,90
60 IF (DSUM) 120,120,70
70 IF (IER) 80,80,90
80 IER=K-1

COMPUTE PIVOT ELEMENT
90 DPIV=DGSORT(DSUM)
A(KPIV)=DPIV
DPIV=1.0D0/DPIV
GO TO 110

C CALCULATE TERMS IN ROW
100 A(IND)=DSUM*DPIV
110 IND=IND+1

C END OF DIAGONAL LOOP
GO TO 200
120 IER=-1
200 CONTINUE
IF (IER) 9,1,1

C INVERT UPPER TRAANGULAR MATRIX T
C PREPARE INVERSION LOOP
1    IPIV=N*(N+1)/2
    IND=IPIV

C INITIALIZE INVERSION LOOP
DO 6 I=1,N
   DIN=1.0DO/A(IPIV)
   A(IPIV)=DIN
   MIN=N
   KEND=I-1
   LAF=N-KEND
   IF (KEND) 5,5,2
2    J=IND

C INITIALIZE ROW LOOP
DO 4 K=1,KEND
   WORK=0.0DO
   MIN=MIN-1
   LHR=IPIV
   LV=J

C START INNER LOOP
DO 3 L=LAF,MIN
   LV=LV+1
   LHR=LHR*L
3   WORK=WORK+A(LV)*A(LHR)

C END OF INNER LOOP
C A(J)=-WORK*DIN
4    J=J-MIN

C END OF ROW LOOP
C 5 IPIV=IPIV-MIN
6 IND=IND-1
C END OF INVERSION LOOP
CALCULATE INVERSE(A) BY MEANS OF INVERSE(T)

INVERSE(A) = INVERSE(T) * TRANSPOSE(INVERSE(T))

INITIALIZE MULTIPLICATION LOOP

DO 8 I=1,N
  IPIV=IPIV+1
  J=IPIV

INITIALIZE ROW LOOP

DO 9 K=1,N
  WORK=0.0DO
  LHOR=J

START INNER LOOP

DO 7 L=K,N
  LVER=LHOR+K-1
  WORK=WORK+A(LHOR)*A(LVER)
  7 LHOR=LHOR+L

END OF INNER LOOP

  A(J)=WORK

  8 J=J+K

END OF ROW AND MULTIPLICATION LOOP

RETURN

END
SUBROUTINE PROPO

IMPLICIT REAL*8 (A-H, O-Z)

COMMON /FILT/ P(10, 10), V(10, 10), U(10, 10), PF(10, 10), F(10, 10),
1 FF(10, 10), DF(10, 10), H(5, 10), HF(5, 10), DH(5, 10), FG(10, 5), X(10),
2 EL(10), ELF(10), DEL(10), R, RDOT, T, TDOT, XT(3), VT(3),
3 FAO(2), AXG(2), RN(5, 5), Q(10, 10), WLEN(2), DELC(10, 10), ELC(10, 10),
4 DT(1), DT(2), DT(3), HFT(10, 5), TANT, AZ1G(2), AZ2G(2), AXZ(2)

COMMON /IFILT/ NS, NM, NC, NIS

DIMENSION W3(100), VS(5, 5)
EQUIVALENCE (W3(1), ST(1, 1)), (VS(1, 1), VI(1, 1))

DIMENSION PFD(10, 10), PD(10, 10), UD(10, 10), VD(10, 10), ST(10, 10),
1 SU(10, 10), SV(10, 10), PI(10, 10), UI(10, 10), VI(10, 10),
2 PFI(10, 10), S1(10, 10), S2(10, 10), S3(10, 10), S4(10, 10), PFJ(10, 10)

DO 30 IJK=1, NIS

CALL ZCALC(XT, VT, AT, TANT)
CALL NLIN

DERIVATIVES OF COVARIANCE MATRIXES
PFDDT = FF*PF + PF*FFT + Q
LET A = FF + DELC AND B = DF + DELC AND C = F + ELC
THFN
PDADT = A*P + P*AT + B*V + VT*BT + Q
VDDT = ELC*P + C*V + V*AT + U*BT = O
UDADT = ELC*VT + C*U + V*ELCT + U*CT + O

INTERMEDIATE V MATRIX
DO 11 J=1, NS
DO 11 K=1, NS
A=DELC(K, J)
ST(J, K)=DF(K, J)+A
SU(J, K)=FF(K, J)+A
SV(K, J)=F(K, J)+ELC(K, J)
11 CONTINUE
CALL MPRD(ELC,P,S1,NS,NS,NS)
CALL MPRD(SV,V,S2,NS,NS,NS)
CALL MPRD(VS,SU,S3,NS,NS,NS)
CALL MPRD(U,ST,S4,NS,NS,NS)
DO 12 J=1,NS
DO 12 K=1,NS
A=S1(K,J)+S2(K,J)+S3(K,J)+S4(K,J)=Q(K,J)
VD(K,J)=A
B=V(K,J)
C=B+A*DTI
VI(K,J)=C
PFJ(J,K)=C
PD(J,K)=B
12 CONTINUE

INTERMEDIATE P, PF, AND U MATRIXES

CALL MPRD(P,SU,S1,NS,NS,NS)
CALL MPFD(PD,ST,S2,NS,NS,NS)
CALL MPRD(ELC,PD,S3,NS,NS,NS)
CALL MPRD(SV,U,S4,NS,NS,NS)
CALL MPFD(FF,PF,ST,NS,NS,NS)
DO 13 J=1,NS
DO 13 K=1,J
B=Q(K,J)
C
A=S1(K,J)+S1(J,K)+S2(K,J)+S2(J,K)+B
PD(K,J)=A
A=P(K,J)+A*DTI
PI(K,J)=A
PI(J,K)=A
C
A=S3(K,J)+S3(J,K)
C=S4(K,J)+S4(J,K)
A=A+C+B
UD(K,J)=A
A=U(K,J)+A*DTI
UI(K,J)=A
UI(J,K)=A
C
A=ST(K,J)+ST(J,K)+B
PFJ(K,J)=A
A=PF(K,J)+A*DTI
PFJ(K,J)=A
PFJ(J,K)=A
C
13 CONTINUE
FAITHFULLY INTEGRATE TARGET NOMINAL POSITION

DO 17 J=1,3
XT(J)=XT(J)+DTI2*AT(J)+DTI*VT(J)
VT(J)=VT(J)+AT(J)*DTI
17 CONTINUE

DERIVATIVE MATRICES AT INTERMEDIATE POINT

CALL ZCALC(XT,VT,AT,TANT)
CALL NLIN

FINAL V MATRIX

DO 14 J=1,NS
DO 14 K=1,NS
A=DELV(K,J)
ST(J,K)=DF(K,J)+A
SU(J,K)=FF(K,J)+A
SV(K,J)=F(K,J)+ELC(K,J)
14 CONTINUE

CALL MPRD(ELC,PI,S1,NS,NS,NS)
CALL MPRD(SV,VI,S2,NS,NS,NS)
CALL MPRD(VT,SU,S3,NS,NS,NS)
CALL MPRD(UI,ST,S4,NS,NS,NS)
DO 15 J=1,NS
DO 15 K=1,NS
A=S1(K,J)+S2(K,J)+S3(K,J)+S4(K,J)-Q(K,J)
B=V(K,J)
V(K,J)=B+DTI2*(VD(K,J)+A)
15 CONTINUE

FINAL P, PF, AND U MATRIXES

CALL MPRD(PI,SU,S1,NS,NS,NS)
CALL MPRD(PFJ,ST,S2,NS,NS,NS)
CALL MPRD(ELC,PFI,S3,NS,NS,NS)
CALL MPRD(SV,UI,S4,NS,NS,NS)
CALL MPRD(FF,PFI,ST,NS,NS,NS)
DO 16 J=1,NS
DO 16 K=1,NS
B=Q(K,J)
16 CONTINUE
A=S1(K,J)+S1(J,K)+S2(K,J)+S2(J,K)+B  
C=P(K,J)+DTI2*(PD(K,J)+A)  
P(K,J)=C  
P(J,K)=C

A=S3(K,J)+S3(J,K)  
C=S4(K,J)+S4(J,K)  
A=A+C+B  
C=U(K,J)+DTI2*(UD(K,J)+A)  
U(K,J)=C  
U(J,K)=C

A=ST(K,J)+ST(J,K)+R  
C=PF(K,J)+DTI2*(PFD(K,J)+A)  
PF(K,J)=C  
PF(J,K)=C

16 CONTINUE  
30 CONTINUE  
RETURN

PROCESS NEW MEASUREMENT AND UPDATE STATE AND COVARIANCE

ENTRY KFILT(NMES, IEXT)

COMPUTE NEW FILTER GAINS - FG = P*HT*(H*P*HT+R)**(-1)

CALL MPRD(PF, HFT, PFI, NS, NS, NM)  
CALL MPRD(HF, PFI, VS, NM, NS, NM)  
L=1  
DO 51 J=1, NM  
DO 51 K=1, J  
W3(L)=.5*(VS(K,J)+VS(J,K))*RN(K,J)  
L=L+1  
51 CONTINUE  
CALL DINV(W3, NM, I, D-S, IER)  
IF (IER .EQ. -1) GO TO 31  
IF (IER .NE. 0) GO TO 32  
33 L=1  
DO 52 J=1, NM  
DO 52 K=1, J  
A=W3(L)
VS(J,K)=A
VS(K,J)=A
L=L+1

52 CONTINUE
CALL MPRD(PFI,VS,FG,NS,NM,NM)

UPDATE PLUS - PPLUS=(I-FG*H)*P*(I-FG*H)'+FG*R*FGT OR PPLUS=(I-FG*H)*P

CALL MPRD(FG,HF,PFI,NS,NM,NM)
DO 42 J=1,NS
DO 42 K=1,NS
A=-PFI(K,J)
IF (J .EQ. K) A=1.0 DO +A
UI(K,J)=A
UD(J,K)=A
42 CONTINUE
CALL MPRD(UI,PF,PFI,NS,NS,NS)
L=1
DO 62 J=1,NS
DO 62 K=1,NM
W3(L)=FG(J,K)
L=L+1
62 CONTINUE
IF (NMES .LE. IEXT) GO TO 65
DO 46 J=1,NS
DO 46 K=1,J
A=500*(PFI(J,K)+PFI(K,J))
PF(K,J)=A
PF(J,K)=A
46 CONTINUE
70 CONTINUE

UPDATE OTHER COVARIANCE MATRIXES

LET UI=(I-FG*HF) AND UD=UIT AND PI=FG*DH AND VI=PII THEN

P = UI*P*UD - UI*VT*DHT*FGT - FG*DH*V*UD + FG*DH*U*DHT*FGT + FG*RN*FGT
V = V*UD - U*DHT*FGT
U = U
CALL MPRD(RN,W3,SV,NM,NM,NS)
CALL MPRD(U1,P,V1,NS,NS,NS)
CALL MPRD(V1,UD,VD,NS,NS,NS)
CALL MPRD(FG,DH,PI,NS,NM,NS)
CALL MPRD(V,UD,ST,NS,NS,NS)
CALL MPRD(P1,ST,PFI,NS,NS,NS)
DO 61 J=1,NS
DO 61 K=1,NS
VI(K,J)=PI(J,K)
61 CONTINUE
CALL MPRD(U,V1,PD,NS,NS,NS)
CALL MPRD(P1,PD,V1,NS,NS,NS)
CALL MPRD(FG,SV,UI,NS,NM,NS)
DO 64 J=1,NS
DO 66 K=1,J
A=VD(K,J)-PFI(K,J)-PFI(J,K)+VI(K,J)+VI(J,K)
P(K,J)=A
P(J,K)=A
66 CONTINUE
DO 67 K=1,NS
V(K,J)=ST(K,J)-PD(K,J)
67 CONTINUE
64 CONTINUE
RETURN

EXTENDED UPDATE EQUATION FOR STARTUP

65 CALL MPRD(PFI,UD,S1,NS,NS,NS)
CALL MPRD(FG,RN,PFI,NS,NM,NM)
CALL MPRD(PFI,W3,VI,NS,NM,NS)
DO 68 J=1,NS
DO 68 K=1,J
A=5D0*(S1(K,J)+S1(J,K)+VI(K,J)+VI(J,K))
PFI(K,J)=A
PFI(J,K)=A
68 CONTINUE
GO TO 70

POSSIBLY BAD MATRIX

32 WRITE(6,102) IER
102 FORMAT(' LOSS OF SIG, IN KALMAN INVERSION, ROW NO, = ',I2)
GO TO 33

31 WRITE(6,100)
100 FORMAT(* NON-POSITIVE DEFINITE MATRIX AT KALMAN INVERSION*)
END
SUBROUTINE ZCALC(X,XD,XDD,TT)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(3),XD(3),XDD(3)
T=X(1)**2+X(2)**2
U=DSQRT(T)
V=1.0/U
W=X(1)*XD(1)+X(2)*XD(2)
X(3)=TT*U
XD(3)=TT*V*W
XDD(3)=TT*V*(XD(1)**2+XD(2)**2+X(1)*XDD(1)+X(2)*XDD(2)-W*W*T)
RETURN
END
SUBROUTINE MPD(X,Y,Z,II, JJ, KK)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(1),Y(1),Z(1)

MPD PERFORMS A MATRIX MULTIPLICATION Z=X*Y
II = NUMBER OF ROWS IN X
JJ = NUMBER OF COLUMNS IN X AND NUMBER OF ROWS IN Y
KK = NUMBER OF COLUMNS IN Y
Z HAS DIMENSION Z(II,KK) II ROWS AND KK COLUMNS

K=1
JS=1

LOOP TO ESTABLISH EACH COLUMN OF Z

DO 30 M=1,KK

LOOP TO ESTABLISH EACH ROW OF Z

DO 20 L=1,II
    J=JS
    I=L
    A=0

LOOP TO DO PRODUCTS

    DO 10 MN=1, JJ
        A=A+X(I)*Y(J)
        I=I+II
        J=J+1
    10 CONTINUE
    Z(K)=A
    K=K+1

20 CONTINUE
    JS=JS+JJ
30 CONTINUE
RETURN
END
APPENDIX G

Numerical Values of Measurement Uncertainties

The nominal values of measurement uncertainties are presented in the following table.

<table>
<thead>
<tr>
<th>Measurement</th>
<th>Units</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miss Angle</td>
<td>Radians</td>
<td>$1 \times 10^{-5}$</td>
</tr>
<tr>
<td>Beam Angle</td>
<td>Radians</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>Beam Rate</td>
<td>Radians/sec</td>
<td>$6 \times 10^{-3}$</td>
</tr>
<tr>
<td>Target Range</td>
<td>Ft</td>
<td>100.</td>
</tr>
<tr>
<td>Target Range Rate</td>
<td>Ft/sec</td>
<td>10.</td>
</tr>
<tr>
<td>Measurement Rate</td>
<td>Measurements/sec</td>
<td>50.</td>
</tr>
</tbody>
</table>
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