TRANSIENT HEAT CONDUCTION IN
SHOCK SUPPORTS IN CRYOGENIC APPARATUS

by

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In Cryogenic Apparatus

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ABSTRACT

This work deals with the phenomena of transient heat conduction in cryogenic apparatus. Thermal losses in transient heat conduction for surfaces of different temperatures brought into contact suddenly are investigated.

The analysis is based on a one-dimensional model of transient heat flux in two semi-infinite slabs of finite length, in the direction of heat conduction, at different temperatures. The hot slab is exposed to the atmosphere and the cold one to a cryogenic fluid.

The main concern is to find the amount of heat flux into the cryogenic fluid, after contact with the hot slab. The change in the internal energy of the cold slab is also examined. Those quantities are calculated for a number of time intervals until steady state is established.

The cases of perfect thermal contact and thermal contact with resistance at the interface are investigated. This is done for a number of thermal conductances at the interface, which are a function of the interface pressure. The effect of the ratio of the lengths of the two slabs in contact is also considered.

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Nomenclature

\begin{align*}
a & \text{ constant length of cold slab} \\
A & \text{ Area of contact} \\
B & \text{ length ratio of the length of the hot slab to the cold one} \\
h & \text{ thermal contact resistance at the interface} \\
K & \text{ thermal conductivity} \\
K_i & \text{ eigenvalue of the theoretical analysis} \\
L & \text{ total length of both slabs} \\
R_i & \text{ discontinuous-weighting factor} \\
R & \text{ resistance} \\
T_{1} & \text{ temperature distribution of cold slab} \\
T_{2} & \text{ temperature distribution of hot slab} \\
T_0 & \text{ Initial temperature of hot slab} \\
T_0 & \text{ Initial temperature of cold slab} \\
C_p & \text{ Specific heat} \\
U & \text{ dimensionless internal energy of cold slab} \\
B_i & \text{ Biot number} \\
\alpha & \text{ thermal diffusivity} \\
\tau & \text{ Fourier number} \\
\theta_1 & \text{ dimensionless temperature distribution of cold slab} \\
\theta_2 & \text{ dimensionless temperature distribution of hot slab} \\
\eta & \text{ non-dimensionless length parameter} \\
\rho & \text{ density}
\end{align*}
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A. INTRODUCTION

The main concern of this work is to study the thermal behavior of two solid bodies at different temperatures after coming in contact. The problem is studied for various time intervals before the steady state is established. The cold body is in contact with a heat sink which in our case, is a cryogenic fluid. The hot body is in contact with a heat source. The problem of the high heat flux into the cold slab and into the fluid, at any instant of time, while the bodies are in contact is examined. Also, the increase in interval energy of the cold slab and the heat flux into the liquid after the contact is over, are of interest.

The problem of high heat leak into a body at very low temperature, when coming in contact with another body at a higher temperature is often encountered in storing, transporting and transferring cryogenic fluids in dewars and transfer lines. It also arises in superconducting generators.

In dewars, and trailers the inner vessel can come in contact with the outer, one either due to a tilted position or due to a violent movement. To lower the area of contact and to be used as structural supports in such case, bumper supports, (Fig. 1), located in the vacuum annulus between the warm and the cold walls, are used extensively. Bumper stops are also used to prevent the filling and emptying lines inside the dewar to touch the inner vessel. In transfer lines, carrying cryogenic fluids at very low temperatures, bumpers are used to prevent the outer line from touching the cold inner one [7]. In superconducting generators, the problem arises when the damper shield collapses due to electrical loads and it touches the casing of the winding which is cooled by helium, thus resulting in a high heat leak. The model used to study those phenomena is that of one-dimensional transient
heat conduction in two infinite slabs of finite length with different initial temperatures in sudden contact. It is an application of the transient boundary problem of heat conduction in a solid, consisting of many parallel layers. There are many theoretical works on this subject [1], [2], [3] introducing various methods to solve the problem. The method used here is introduced by Title [3], and it is a technique for orthogonal expansion of functions over a one-dimensional, multilayer region.

Both the cases of perfect thermal contact and thermal contact with resistance at the interface are treated here. Values for the contact resistance can be found in reference [4]. Any prediction based on a model, which does not include the interface effect would be unreliable, especially in cases of high heat flux as in ours. Our interest is primarily for cases when the interface is in vacuum under conditions of negligible radiation, and the values of thermal conductance used are for this special case. The thermal conductance is a function of pressure and thus the force with which the two bodies come in contact is taken into account.

This work is an effort in the direction of obtaining an idea of the amount of heat leak into the liquid for various time intervals and different values of contact resistance. The presentation of the results in dimensionless form, with time, length ratio, Fourier number and Biot number as the independent variables, makes the results applicable for every material, thermal contact resistance and slab length within certain limits.
B. MODELS AND ANALYTIC SOLUTIONS

When two infinite slabs of finite length at different temperatures, come in contact for a specified amount of time, the models adopted to calculate the heat influx into the cold slab, are the following:

a) Perfect Thermal Contact model.

b) Thermal Contact Resistance model.

The first is a model of one-dimensional transient heat conduction between two infinite slabs of finite length in perfect thermal contact between them. Fig. (2). Before the contact, they are both at different uniform temperatures and this serves as the initial condition. The outer boundaries are kept at constant temperatures, equal to the initial temperatures throughout the process. The hot slab is assumed to be in contact with a heat reservoir supplying the heat necessary to sustain the temperature at the hot boundary constant. The cold outer boundary is in direct contact with the cryogenic fluid. This acts as a heat sink absorbing the heat that reaches the cold boundary, trying to increase the temperature there. Thus the outer end of the cold slab is assumed to remain at constant temperature continuously. At the interface equality of fluxes and temperatures is assumed for the case of perfect contact.

The second model has exactly the same boundary conditions at the outer boundaries. But, it takes into account the contact resistance between the two slabs at the interface. The contact resistant, \( h \) is defined as: (Fig. 3).

\[
h = \frac{q/A}{T_2 - T_1} \tag{2.1}
\]

The boundary conditions at the interface change and we have equality of fluxes and

\[-k_1 \frac{\partial T_1}{\partial x} = h (T_1 - T_2) \tag{2.2}\]
where \( K_i \) is the thermal conductivity of the slab and \( T_1, T_2 \) are the temperature distributions along the two slabs.

Both models assume that: a) the two slabs are of the same material and b) the properties of the slabs are independent of the temperature changes and remain unchanged throughout the process.

**Perfect Thermal Contact Model**

The one-dimensional transient heat conduction equations are:

\[
\frac{\partial^2 \theta_1}{\partial \eta^2} = \frac{\partial \theta_1}{\partial \eta} \quad 0 \leq \eta \leq 1, \tau > 0
\]  
\[
(2.3)
\]

\[
\frac{\partial^2 \theta_2}{\partial \eta^2} = \frac{\partial \theta_2}{\partial \eta} \quad 1 \leq \eta \leq \frac{L}{a}, \tau > 0
\]  
\[
(2.4)
\]

The boundary conditions are:

a) \( \eta = 0 \quad \theta_1 = 0 \)

b) \( \eta = \frac{L}{a} \quad \theta_2 = 1 \)

c) \( \eta = 1 \quad \theta_1 = \theta_2 \)

d) \( \eta = 1 \quad \frac{\partial \theta_1}{\partial \eta} = \frac{\partial \theta_2}{\partial \eta} \)

The initial conditions are: a) \( \theta_1(\tau) = 0 \) b) \( \theta_2(\tau) = 1 \) \( \tau = 0 \)  
\[
(2.5)
\]

Since the boundary conditions are non-homogeneous the solution is a sum of the transient and steady state solutions, given as

\[
\theta_1(\eta, \tau) = \theta_{ss1}(\eta, \infty) + \sum_{n=1}^{\infty} A_n \sin Y_n \eta \left[ \frac{L}{a} \cos Y_n \tau \right] \quad 0 \leq \eta \leq 1
\]

\[
(2.7)
\]

\[
\theta_2(\eta, \tau) = \theta_{ss2}(\eta, \infty) + \sum_{n=1}^{\infty} A_n \left[ \sin Y_n \eta - \tan Y_n \frac{L}{a} \cos Y_n \right] \left[ \frac{L}{a} \cos Y_n \tau \right] \quad 1 \leq \eta \leq \frac{L}{a}
\]

\[
(2.8)
\]

where \( \theta_{ss1}, \theta_{ss2}, A_n \) and \( C_n \) are specified in the Appendix (B).

**Thermal Contact Resistance Model**

The equations and the initial conditions remain the same. The boundary conditions change with the introduction of the thermal contact resistance at the interface and they become; (Ref. 8).
a) \[ \eta = 0 \quad \theta_1 = 0 \]

b) \[ \eta = \frac{L}{a} \quad \theta_2 = 1 \]

c) \[ \eta = 1 \quad \frac{\partial \theta_1}{\partial \eta} = \frac{\partial \theta_2}{\partial \eta} \] (2.9)

d) \[ -\frac{\partial \theta_1}{\partial \eta} = (Bi) (\theta_1 - \theta_2) \]

Thus the eigenvalues \( y_n \), and the steady state part of the temperature distribution change. The temperature distribution is given by equations;

\[
\theta_1 (\eta, \tau) = \theta_{ss1} (\eta, \infty) + \sum_{n=1}^{\infty} A_n \sin y_n \eta \ e^{-y_n^2 \tau} \] (2.10)

\[
\theta_2 (\eta, \tau) = \theta_{ss2} (\eta, \infty) + \sum_{n=1}^{\infty} C_2 \cos y_n \eta e^{-y_n^2 \tau} \] (2.11)

where \( \theta_{ss1}, \theta_{ss2}, A_n, C_2 \) are specified in Appendix (C).
C. RESULTS AND CONCLUSIONS

The results are presented in a dimensionless form (Charts 1-25) with time, length ratio, the Biot number and the Fourier number as the independent variables.

The model assumed is a realistic one. The assumption that the outer boundaries of the two slabs remain at constant temperature throughout, is a valid one because; a) the outer boundary of the hot slab is always in contact with a heat source, usually the atmosphere and b) the outer boundary of the cold slab is cooled off continuously by a cryogenic fluid which absorbs the heat that reaches this end and thus, keeps the temperature there constant. The fluid can be modelled as a heat sink at low temperature in contact with the cold slab keeping its temperature constant.

When a perfect thermal contact exists between the two slabs the results hold true for every combination of length of the slabs, time and thermal properties of the materials used.

When a thermal contact resistance exists between the two slabs at the interface, the results hold true for every combination of length of the slabs, time and thermal properties of the materials, as long as the thermal properties of the slabs are identical. If the slabs are of different materials the results hold only as a good approximation. In the special case where; (Fig. 2)

\[
\begin{align*}
    a) \quad & \frac{K_1}{a} = \frac{K_2}{L-a} \\
    b) \quad & \rho_1 \ C_{p_1} \ a = \rho_2 \ C_{p_2} \ (L-a)
\end{align*}
\]

hold true, the results are good in general. This is the only limitation of the results presented in charts (1-25).

The results of the model are realistic. They are derived by keeping the length of the cold slab constant and changing the length of the hot one to
find the effect of the ratio of the length on the heat flux into the cold slab. This was done for a number of thermal contact resistances at the interface at every instant of time until steady state is established.

As the value of the thermal contact resistance decreases, the heat flux into the cold body increases until we get to the case of perfect thermal contact where the heat flux in the cold body and, as a result, its internal energy increase and the heat flux into the helium become a maximum. Using the different values of h, we take care of the different pressures existing between the two bodies in contact since the contact conductance is a function of pressure. (Ref. 4).

Now, the time effect and the size effect for the same value of h, should be studied and compared to the results obtained from the integral technique and the exact solution for semi-infinite bodies. (Appendix D).

For a small, initial time interval, after the contact takes place, the heat flux into the helium is zero or very near zero, which means that for this time interval, the temperature distribution, at the outer boundary of the cold slab remains unaffected by the applied boundary condition at the other end of the slab. At the time that the temperature there is affected by the contact, there is a jump in the heat flux going into the helium, which is easily seen in the curves. This conclusion agrees with the results found from the use of the integral technique, which predicts that the temperature at the outer boundary will be unaffected by the contact, for a certain amount of time, until the thermal-layer reaches the surface. The time needed for this to happen seems to be independent of the change in length of the hot slab, which agrees again with the results of the integral technique method. (Appendix D).
For the next time interval, after the thermal-layer has reached the outer boundary of the cold slab but not that of the hot one, when the hot slab is bigger than the cold, the heat flux into the cold slab should be the same, independent of the length of the hot slab, which acts as if it were a semi-infinite body. Our results agree with this theory very well and only when the ratio of the lengths gets to be very high (i.e. 10:1) we get a deviation from the theory, and that only for high values of contact resistance.

As the contact at the interface is prolonged, the difference in the ratio of the lengths of the two slabs (the hot over the cold one) starts to play an important role. As the heat flux into the helium and the internal energy of the cold slab increase, we see that they both increase more rapidly for those slab ratios that are bigger up to a certain time and then, gradually, the trend reverses its direction and remains unchanged until a steady state is established, when the heat flux into the cold slab goes entirely into the helium, while the internal energy of the cold slab remains unchanged.

As we approach the steady state, the resistance of the hot slab becomes more dominant and since at the steady state, the resistance, $R$, is found to be;

$$ R = \frac{L}{KA} \quad (3.2) $$

therefore, the bigger the hot slab, the smaller the heat influx into the cold slab (Appendix D).

The results for the heat flux going into the helium and the heat stored in the cold slab, show that they both keep increasing until the steady state is established, when the hot slab is smaller or equal to the cold slab. This means that the heat flux into the cold slab is always greater than the heat flux going into the helium. On the contrary, when the hot slab is bigger
than the cold one, the curves show a peak which means that a) the heat flux into the cold slab, after the peak is reached is smaller than the heat flux into the helium, until they get to be equal at the steady state. As a result, the internal energy stored in the cold slab reaches a maximum before the steady state is established. b) the slope of the temperature distribution at the outer boundary of the cold slab, and thus the heat flux into the helium is not maximum at the steady state (Charts 1-25).

In the charts (1-25), the results for the slope of the temperature distribution at the outer boundary of the cold slab, \( \frac{\partial \Theta}{\partial \eta} \), and for the area under the curve of the temperature distribution of the cold slab, \( U \), are presented in dimensionless form. \( U \) is defined as:

\[
U = \int_{0}^{1} \Theta d\eta 
\]  

(3.3)

The heat flux into the helium and the internal energy of the cold slab can be found respectively. The temperature is non-dimensionalized by a constant temperature factor, which in our case is the atmospheric temperature of 298°K. The length along the x-coordinate is non-dimensionalized by, \( a \), the constant length of the cold slab. The dimensionless parameters used are:

a) The length ratio \( \frac{L}{a} \)

b) The Biot number \( Bi = \frac{ha}{K} \)  

(3.4)

c) The Fourier number \( Fo = \frac{\alpha t}{a^2} \)

where the quantities are defined as; \( k \), the thermal conductivity of the material, \( h \), the thermal contact resistance at the interface, \( \alpha \), the thermal diffusivity of the material, \( t \), time of contact, \( a \), the length of the cold slab \( L \), the length of both slabs.
The nature of the results makes it difficult to state any general rules about the shape of the bumpers, because the Fourier number and the Biot number change for every case. But we can draw some general conclusions: a) If the Fourier number is greater than a certain value, depending on the Biot number, the hot slab should be as big as possible compared to the cold slab. b) If the ratio of the hot slab to the cold slab varies between, 5 to 4 the hot slab should be made as big as possible independently of the Fourier and Biot numbers. If the ratio becomes bigger than that, this rule holds true for Fourier numbers bigger than a certain value depending on the Biot number. c) For a Fourier number below .150 the difference in heat flux into the cold slab, because of the variation in the length ratio, for the same Biot number is negligible for our range of values except for high values of the Biot number and high length ratios. d) For a Fourier number below .050 the difference in the amount of heat flux into the cold slab, because of the variation of the Biot number, for the same length ratio, is negligible.
Recommendations for Further Research

Two major assumptions of this work can be relaxed in a future work: a) The fact that the thermal properties of the slabs are dependent on temperature, should be taken into consideration. Ref. (5), (6), (9) and (10) b) The assumption that both slabs are made of the same material, can be relaxed by including the discontinuous-weighting factor, \( R_i \), (A.1), into the analysis [3].

The work can be carried further by an analysis of the behavior of the two slabs when they touch and separate repeatedly for a certain amount of time.
\[ \frac{L}{a} = 1.5 \]

Diagram showing the relationship between \( \frac{\partial \theta}{\partial \eta} \) and \( \frac{\alpha t}{a} \) with different values of \( \frac{h_a}{k} \) ranging from 0 to 0.24 and \( h_a/k \) values of 1.5, 2.0, 3.0, and 3.97.
\[ \frac{L}{a} = 2 \]

\[ \frac{h}{k} = 0 \]

\[ \frac{\partial \theta}{\partial \eta} \]

\[ \frac{\alpha t}{a} \]

19
\( \frac{L}{a} = 11 \)

\( \frac{h a}{k} = 0 \)

\( \frac{\alpha t}{a} \)
\( \frac{L}{a} = 1.5 \)

\[
\frac{h_a}{k} = 0
\]

\[
0.24
\]

\[
0.466
\]

\[
0.88
\]

\[
1.5
\]

\[
2.0
\]

\[
3.0
\]

\[
3.97
\]
\[ \frac{L}{a} = 3 \]
\[ \frac{L}{a} = 4 \]

\[ \frac{h_a}{k} = 0 \]
\[ 0.24 \]
\[ 0.466 \]
\[ 0.88 \]
\[ 1.5 \]
\[ 2.0 \]
\[ 3.0 \]
\[ 3.97 \]

\[ \frac{\alpha t}{a} \]
\( \frac{1}{a} = 11 \)

\( \frac{h a}{k} = \)

\( 0 \)

\( 0.24 \)

\( 0.466 \)

\( 0.88 \)

\( 1.5 \)

\( 2.0 \)

\( 3.0 \)

\( 3.97 \)

\( \frac{\alpha t}{a} \)
\[ \frac{L}{a} = 1.5 \]

\[ \frac{ha}{k} = 0, 0.24, 0.466, 0.88, 1.5, 2.0, 3.0, 3.97 \]

\[ u \]

\[ \alpha t \]
\[ \frac{L}{a} = 3.4 \]

\[ \frac{h_a}{k} = 0 \]

\[ u \]

\[ \frac{a t}{a} \]

36
\( \frac{L}{a} = 1.5 \)

\( \frac{h}{a} = 0 \)

\( u \)

\( \frac{\alpha t}{a} \)

Values:
- \( 0.24 \)
- \( 0.466 \)
- \( 0.88 \)
- \( 1.5 \)
- \( 2.0 \)
- \( 3.0 \)
- \( 3.97 \)
\[ \frac{L}{a} = 2 \]

\[ \frac{h_a}{k} = 0 \]

\[ \frac{u}{a} \]

\[ (.5, .07) \]

\[ 0.24 \]

\[ 0.466 \]

\[ 0.88 \]

\[ 1.5 \]

\[ 2.0 \]

\[ 3.0 \]

\[ 3.97 \]
\[ \frac{L}{a} = 4 \]
\[ \frac{L}{a} = 11 \]

Graph with axes labeled as follows:
- Y-axis: \( u \)
- X-axis: \( \frac{\alpha t}{a} \)

Key points and lines:
- \( \frac{h a}{k} = 0 \)
- \( \frac{h a}{k} = 0.2 \)
- \( \frac{h a}{k} = 0.4 \)
- \( \frac{h a}{k} = 0.8 \)
- \( \frac{h a}{k} = 1.5 \)
- \( \frac{h a}{k} = 2.0 \)
- \( \frac{h a}{k} = 3.0 \)
- \( \frac{h a}{k} = 3.5 \)
References


Appendix A

Theoretical Analysis

The method used for the solution of the problem is based on the orthogonal expansion technique of a function over a multilayer region. It is an extension of ordinary orthogonality, by means of introducing a new orthogonality factor, \( R_i \), called discontinuous-weighting function, which is unchanged whether there is a perfect thermal contact or a linear contact resistance at the outer face. The method was first introduced by Tittle [3]. He suggested a discontinuous-weighting factor \( R_i \),

\[
R_i = \left( \frac{K_i}{\alpha_i} \right)^{\frac{1}{2}} \quad \text{(A.1)}
\]

Consider a one-dimensional composite region \( x_1 < x < xm + 1 \) involving in parallel layers in which the thermal properties for each layer are uniform but discontinuous at the interface.

The differential equation of the transient heat conduction for rectangular coordinates is given by

\[
\frac{\partial^2 T_i(x,t)}{\partial x^2} = \frac{1}{\alpha_i} \frac{\partial T_i(x,t)}{\partial t} \quad \text{(A.2)}
\]

The temperature function \( T \) is separated into space and time functions;

\[
T(x,t) = \sum_{n=1}^{\infty} a_n \sin(n \pi \frac{x}{x_m}) \cdot Y(t) \quad x_1 \leq x \leq x_m + 1 \quad \text{(A.3)}
\]

The boundary conditions needed are:

a) at \( x = x_1 \), the outer boundary, \( t > 0 \)

b) at \( x = x_m + 1 \), the outer boundary, \( t > 0 \) \quad \text{(A.4)}

c) \( K_i \left( \frac{\partial T_i(x_{i+1},t)}{\partial x} \right) = K_{i+1} \frac{\partial T_{i+1}(x_{i+1},t)}{\partial x} \)

at the interface, \( t > 0 \), \( i = 1,2, \ldots (x_m-1) \)

d) \( T_i(x_{i+1},t) = T_{i+1}(x_{i+1},t) \)

at the interface, for perfect thermal contact.

e) \( -K_i \frac{\partial T_i(x_{i+1},t)}{\partial x} = h [T_i(x_{i+1},t) - T_{i+1}(x_{i+1},t)] \)

at the interface for thermal contact resistance.
For layers of the same material the four boundary conditions are enough to specify the eigenvalues and eigenvectors, $k_n$ and $X_n$ respectively of the assumed solution (A.2).

For layers of different materials we need one more equation. Assuming that there is no energy storage in the infinitesimal thickness of the interface, the time behavior of the temperature at the interface should be the same on either side of the interface. This is satisfied if we have that

$$a_{i-1} K^2 (i-1)n = a_i K^2 in \quad i = 2, 3, \ldots, m \quad (A.7)$$

But for a region involving more than one layer the eigenvectors $X_n (x)$ are not orthogonal with respect to the weighting function for the same region Sturm-Liouville problem so, by using $R_i$, we construct an orthonormal set

$$H_n (x) = R_i - X_n (x) \quad (A.8)$$

The initial condition function, $F (x)$ can be expressed as

$$F(x) = \sum_{i=1}^{m} F_i (x) \quad (A.9)$$

$$F_i(x) = F(x) \text{ in layer } i \text{ and } F_i (x) = 0 \text{ in other layers} \quad (A.10)$$

$$F_i(x) = \sum_{n=1}^{\infty} C_{in} H_n (x) = \sum_{n=1}^{\infty} C_{in} R_i X_n (x)$$

and $C_{in}$ can be determined by generalized Fourier analysis as

$$C_{in} = \frac{\int_{\text{layers}}^{m} F_i(x) \cdot H_n(x) \cdot r(x) \, dx}{\int_{\text{layers}}^{m} H_n^2(x) \cdot r(x) \, dx} \quad (A.11)$$

$$C_{in} = \frac{R_i \int_{\text{layer}}^{l} F (x) - X_n (x) r (x) \, dx}{\sum_{i=1}^{m} R_i \int_{\text{layer}}^{l} X_n^2 \cdot r(x) \, dx} \quad (A.12)$$

where $r(x)$ is the one region weighting function.

Therefore $A_n$ becomes

$$A_n = \sum_{i=1}^{m} R_i C_{in} \quad (A.13)$$

45
\[
A_{n} = \frac{\sum_{i=1}^{m} R_{i}^{2} \int_{i}^{layer} F(x) \chi_{in}(x) r(x) \, dx}{\sum_{i=1}^{m} R_{i} \int_{i}^{layer} \chi_{in}^{2}(x) r(x) \, dx}
\]  

(A.14)

so the coefficient \(A_{n}\) has been determined and can be substituted in equation (A.3).
Appendix B

Perfect Thermal Contact Model

The system is modelled as two infinite slabs of finite length being initially at different temperatures uniformly. We assume a one-dimensional boundary problem of transient heat conduction in rectangular coordinates. The outer boundaries of the slabs remain at the initial temperatures throughout the process. The contact between the two slabs is perfect.

The two slabs are made of the same material, with thermal properties independent of temperature (Fig. 2).

The dimensionless quantities used are

\[ Bi = \frac{ha}{K} \quad , \quad Fo = t = \frac{\alpha t}{a^2} \quad , \quad \eta = \frac{x}{a} \]

\[ \theta_1 = \frac{T_1 - T_0}{T_i} \quad , \quad \theta_2 = \frac{T_2 - T_0}{T_i} \quad , \quad \eta_n = Kjna \]

where \( T_i \) is the initial temperature of the hot slab and \( T_0 \) is the initial of the cold slab.

The equations of transient heat conduction are;

\[ \frac{\partial^2 \theta_1}{\partial \eta^2} = \frac{\partial \theta_1}{\partial \tau} \quad 0 < \eta \leq 1, \quad \tau > 0 \quad (B.1) \]

\[ \frac{\partial^2 \theta_2}{\partial \eta^2} = \frac{\partial \theta_2}{\partial \tau} \quad 1 < \eta \leq \frac{L}{a} , \quad \tau > 0 \quad (B.2) \]

The boundary conditions are:

a) \( \eta = 0 \) \( \theta_1 = 0 \)
b) \( \eta = \frac{L}{a} \) \( \theta_2 = 1 \)
c) \( \eta = 1 \) \( \theta_1 = \theta_2 \)
d) \( \eta = 1 \) \( \frac{\partial \theta_1}{\partial \eta} = \frac{\partial \theta_2}{\partial \eta} \quad (B.3) \)

The initial conditions are:

\( \alpha \)\( \theta_1 \dot{\alpha} = 0 \quad , \quad \beta \)\( \theta_2 \dot{\alpha} = 1 \quad \tau = 0 \quad (B.4) \)

Since the boundary conditions are non-homogeneous at the outer boundary surfaces, the solutions for each region are expressed as the sum of the steady
and transient solutions.

\[ \theta_j (\eta, \tau) = \theta_{ss} + \sum_{n=1}^{\infty} A_n X_j \eta e^{-Y^2 n \tau} \]  \hspace{1cm} (B.5)

where \( j = 1 \) \( 0 < \eta \leq 1 \)

\( j = 2 \) \( 1 < \eta < \frac{L}{a} \)

\[ \theta_{ss1} = \frac{\eta}{L/a} \quad 0 < \eta \leq 1 \]  \hspace{1cm} (B.6)

\[ \theta_{ss2} = \frac{\eta}{L/a} \quad 1 < \eta < \frac{L}{a} \]  \hspace{1cm} (B.7)

In order to find the eigenfunctions \( X_j n \), we solve the homogeneous problem, where;

\[ \theta_1 = 0 \quad \eta = 0 \]  \hspace{1cm} (B.8)

\[ \theta_2 = 0 \quad \eta = \frac{L}{a} \]

and assuming the solution,

\[ X_{1n} = \sin Y_n \eta + D_{1n} \cos Y_n \eta \]  \hspace{1cm} (B.9)

\[ X_{2n} = C_{2n} \sin Y_n \eta + D_{2n} \cos Y_n \eta \]  \hspace{1cm} (B.10)

we find that,

\[ X_{1n} = \sin Y_n \eta \]  \hspace{1cm} (B.11)

\[ X_{2n} = C_{2n} \left[ \sin Y_n \eta - \tan Y_n \frac{L}{a} \cos Y_n \eta \right] . \]  \hspace{1cm} (B.12)

From B.C. (c) we get

\[ (4) \quad C_{2n} = \frac{\cos Y_n a}{\cos Y_n a + \tan Y_n \frac{L}{a} \sin Y_n a} \]  \hspace{1cm} (B.13)

From B.C. (d) we get the eigenvalues \( K_{jn} \), which for the dimensionless case become \( Y_n \).

\[ \sum_{n=1}^{\infty} A_n \sin Y_n \eta e^{-Y^2 n \tau} = \]  \hspace{1cm} (B.14)

\[ = \sum_{n=1}^{\infty} A_n C_{2n} \left[ \sin Y_n \eta - \tan Y_n \frac{L}{a} \cos Y_n \eta \ e^{-Y^2 n \tau} \right] \]

which can be simplified to be

\[ \tan Y_n \left( \frac{L}{a} - 1 \right) + \tan Y_n = 0 \]

Positive roots of this equation give the eigenvalues.
The initial condition functions can be specified as follows:

\[ F_1 (\eta) = -\theta s_{s1} = -\frac{\eta}{L/a} \]

\[ F_2 (\eta) = 1 - \theta s_{s2} = 1 - \frac{\eta}{L/a} \]

Therefore, \( A_n \) are specified by equation (A.13);

\[
A_n = \frac{\int_0^{L/a} \frac{1 - \eta}{L/a} \sin Y_n \eta \, d\eta + \int_0^{L/a} (1 - \eta) \, C_2 n \, [\sin Y_n \eta - \tan Y_n \frac{L}{a} \cos Y_n \eta] \, d\eta}{\int_0^{L/a} \sin^2 Y_n \eta \, d\eta + \int_0^{L/a} C_2 n \, [\sin Y_n \eta - \tan Y_n \frac{L}{a} \cos Y_n \eta]^2 \, d\eta} \quad (B.16)
\]

Therefore the temperature distribution in the two layer slab would be.

\[ \theta_1 (\eta, \tau) = \theta s_{s1} (\eta, \infty) + \sum_{n=1}^{\infty} A_n \sin Y_n \eta \, e^{-Y_n^2 \tau} \quad 0 \leq \eta \leq 1 \quad (B.17) \]

\[ \theta_2 (\eta, \tau) = \theta s_{s2} (\eta, \infty) + \sum_{n=1}^{\infty} A_n \cos Y_n \frac{L}{a} \cos Y_n \eta \, e^{-Y_n^2 \tau} \quad 1 \leq \eta \leq \frac{L}{a} \quad . \]
Appendix C

Thermal Contact Resistance

The only thing that changes in this model, is the boundary condition (C), which with the introduction of thermal contact resistance becomes

\[
(C) \quad -\frac{\partial \theta_1}{\partial \eta} = (\text{Bi}) \ (\theta_1 - \theta_2) \quad \text{(C.1)}
\]

where Bi is the Biot number.

The Steady State part of the solution becomes

\[
\theta_{ss1} = \frac{\eta}{\frac{1}{\text{Bi}} + \frac{L}{a}} \quad 0 \leq \eta \leq 1 \quad \text{(C.2)}
\]

\[
\theta_{ss2} = \frac{1}{\frac{1}{\text{Bi}} + \frac{L}{a}} \left( \frac{\frac{L}{a} - \eta}{\frac{L}{a}} \right) \quad 0 \leq \eta \leq \frac{L}{a} \quad \text{(C.3)}
\]

and the eigenvectors remain the same

\[
X_1 \eta (x) = \sin \eta \eta 
\]

\[
X_2 \eta (x) = C_2 \eta \ [\sin \eta \eta - \tan \eta \frac{L}{a} \cos \eta \eta] 
\]

where \( C_2 \eta \) is given by equation (4).

The eigenvalues, \( \lambda \eta \), change since the boundary condition (c) changes, and they are determined by the following equation;

\[
-\lambda \eta = \text{Bi} \left[ \tan \eta \left( \frac{L}{a} - 1 \right) + \tan \eta \right] \quad \text{(C.7)}
\]

Positive roots of this equation give the eigenvalues.

The initial condition functions can be specified as follows:

\[
F_1(\eta) = \theta_{ss1} = -\frac{\eta}{\frac{1}{\text{Bi}} + \frac{L}{a}} \quad 0 \leq \eta \leq 1
\]

\[
F_2(\eta) = \theta_{ss2} = \frac{\frac{L}{a} - \eta}{\frac{1}{\text{Bi}} + \frac{L}{a}} \quad 1 \leq \eta \leq \frac{L}{a}
\]

Therefore \( An \) are specified by equation (A.13).
\[
A_n = \frac{\int_0^1 - \frac{\eta}{\frac{L}{Bi} + \frac{L}{a}} \sin Y_n \eta \, d\eta + \int_1^{L/a} \frac{L/a - \eta}{\frac{L}{Bi} + \frac{L}{a}} C_2 \eta \left[ \frac{L}{\eta} - \tan Y_n \frac{L}{a} \cos Y_n \eta \right]}{\int_0^1 (\sin Y_n \eta)^2 \, d\eta + \int_1^{L/a} \frac{L/a - \eta}{\frac{L}{Bi} + \frac{L}{a}} C_2 \eta \left[ \frac{L}{\eta} - \tan Y_n \frac{L}{a} \cos Y_n \eta \right]^2 \, d\eta} \quad (C.10)
\]

The temperature distribution in the two-layer slab with contact resistance at the interface is:

\[
\theta_1(\eta, \tau) = \theta_{s1}(\eta, \infty) + \sum_{n=1}^{\infty} A_n \sin Y_n \eta e^{-Y_n^2 \tau} \quad 0 \leq \eta \leq 1 \quad (C.11)
\]

\[
\theta_2(\eta, \tau) = \theta_{s2}(\eta, \infty) + \sum_{n=1}^{\infty} A_n C_2 \eta \left[ \frac{L}{\eta} - \tan Y_n \frac{L}{a} \cos Y_n \eta \right] e^{-Y_n^2 \tau} \quad 1 \leq \eta \leq \frac{L}{a} \quad (C.11)
\]
Appendix D

Comparison of Results with Integral Method

The integral method was introduced by Goodman ([5],[6]). A thermal layer \( \delta(t) \) is defined as the distance from the origin beyond which the initial temperature distribution remains unaffected by the applied boundary condition, and therefore there is no heat flow in the region beyond \( \delta(t) \), Figure (4). The differential equation of heat conduction is integrated over \( \delta(t) \) and called heat-balance integral equation. A suitable profile is assumed for the temperature distribution over the thermal layer and the boundary conditions and initial conditions applied. For a finite region problem of heat conduction, the treatment of the problem is similar to that of a semi-infinite region until \( \delta(t) \) reaches the end of the slab. Then, \( \delta(t) \) has no longer a physical significance.

At \( x = \delta \) the following boundary conditions always hold true

\[
\frac{\partial T}{\partial x} = 0 \tag{D.1}
\]

\[
T = T_{\text{initial}} \quad x = \delta \tag{D.2}
\]

\[
\frac{\partial T^2}{\partial x^2} = 0 \tag{D.3}
\]

Our problem can be stated as follows:

\[
\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \quad 0 < x < a \quad t > 0 \tag{D.4}
\]

\[
T(x,t) = T_0 \quad x = 0 \quad t > 0 \tag{D.5}
\]

\[
T(x,t) = T_i \quad 0 < x < L/a \quad t = 0 \tag{D.6}
\]

We solve the problem until the time that \( \delta(t) \) reaches \( a \).

Integrating equation (D.4) and using the Boundary conditions (D.1), (D.2) we get

\[
- \alpha \frac{\partial T}{\partial x} \bigg|_{x=\delta} = \frac{\partial}{\partial t} \left[ \int_{x=0}^{\delta} T \, dn - T_i \delta \right] \tag{D.7}
\]
Assuming an approximate temperature distribution
\[ T = b + cx + dx^2 + ex^3 \quad 0 \leq x \leq \delta(t) \tag{D.8} \]
and using the boundary conditions (D.5), (D.1), (D.2) and (D.3) we obtain a temperature distribution
\[ \frac{T - T_i}{T_0 - T_i} = \left(1 - \frac{x}{\delta}\right)^3 \quad 0 \leq x \leq \delta \tag{D.9} \]
Using for initial condition
\[ \delta = 0 \quad \text{at} \quad t = 0 \quad \text{we get that} \]
\[ \delta = \sqrt{24\alpha t} \tag{D.11} \]
and the temperature distribution becomes
\[ \frac{T - T_i}{T_0 - T_i} = \left(1 - \frac{x}{24\alpha t}\right)^3 \quad 0 \leq x \leq \delta \tag{D.12} \]
and from equation (D.11) substituting for \( \delta = a \) we find the time needed for \( \delta(t) \) to reach the other end of the slab.
\[ t = \frac{a^2}{24\alpha} \tag{D.13} \]

For our comparisons we would use two slabs, a) the cold one, of constant length of \( a \), equal to lcm; b) the hot one of changing length, \( L \). Figure (4). The material is stainless steel 303 and the diffusivity \( \alpha \) is 0.04826 cm²/sec at \( T = 100^\circ\text{K} \).

First, we see that according to the integral method, a finite time is needed for the heat flux to reach the outer end of the cold slab and this is estimated by (D.13) to be equal to .863 sec. This is also seen in our results, using the exact method, in the "jump" observed in our curves for heat flux going into the helium. It means that, the temperature distribution at the outer boundary of the cold slab needed a certain time to be affected by the heat flux from the hot slab. This time is found to be for the perfect contact case, .900 sec. which is very near to the value found by the integral method.
This value is only dependent on the length of the cold slab, as long as the length of the hot slab is greater than or equal to it. As a result, the value is going to be the same for all different lengths of the hot slab. This also coincides with the results that we get with our exact method. This is going to hold even when there is contact resistance at the interface between the two slabs, as long as the contact resistance is the same for all cases.

Comparing two systems where the lengths of the hot slabs differ during the time that it takes for the boundary layer to reach the outer end of the smaller hot slab, the systems should behave the same. We can approximately find this time with the integral technique.

It was found that for the perfect contact resistance case, the temperature of the interface comes immediately at 149°K and stays approximately constant for the first 5.5 sec. Therefore by using the integral technique as before, we find that the thermal layer needs;

a) .215 sec for L = 1.5cm
b) .863 sec for L = 2cm
c) 3.45 sec for L = 3cm
d) 7.3 - 7.7 sec for L = 4cm

to reach the outer boundary of the hot slab. Comparing our results of any two systems within these time limits we see that they are the same for both of them. The only discrepancy found is for the ratio 10:1 and this only at high values of thermal contact resistance. The discrepancy varies up to 15%.

When we consider the change in internal energy of the cold slab we see that; a) It increases steadily until the steady state is established if the hot slab is equal to or less than the cold slab. b) It reaches a peak and then it decreases until the steady state is established. This can be seen to be the same with the integral technique, since the internal energy of the temperature distribution. (Figure (5,6).
In the steady state, we see that the heat flux into the cold slab decreases with increased length of the hot one which now acts as a resistance

\[ R = \frac{L}{KA} \]

and this is obvious in the results with the integral technique method. Observing Fig. (5,6), we see that the slope of the temperature distribution at the interface is greater when the length of the hot slab is smaller.
**Figure 1** Shock Bumpers

**Figure 2** Model of One-Dimensional Transient Heat Conduction
Figure 3  Thermal Contact Resistance

Figure 4  Model for the Integral Technique
Figure 5  Transient and Steady-State
Temperature Distributions, \( \frac{L}{a} = 2 \)

Figure 6  Transient and Steady-State
Temperature Distributions, \( \frac{L}{a} = 4 \)