Contributions To Higher Recursion Theory

by

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Abstract

Forcing over E-closed inadmissible structures is studied. Some effective versions of Shelah's work in ZFC proper forcing [11] is developed.

Thesis Supervisor: Gerald Enoch Sacks
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0.1 Acknowledgements

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With dedication, to my parents, Harry and Luba Marcus.

Chapter 1

Fundamentals of E-Recursion

1.1 Preliminaries

1.1.1 The Normann Schemes

The E-recursive functions are generated by schemas as for the rudimentary functions:

- $\{e\}(x_1.., x_n) = x_i$ if $e = \langle 1, n, i \rangle$.

- $\{e\}(x_1, ..x_n) = x_i - x_j$ if $e = \langle 2. n, i, j \rangle$.

- $\{e\}(x_1, ..x_n) = \{x_i, x_j\}$ if $e = \langle 3, n, i, j \rangle$.

- $\{e\}(x_1, ..x_n) \simeq \bigcup\{\ \{c\}(y, x_2, .., x_n \mid y \in x_1\ \}$ if $e = \langle 4, n, c \rangle$.

- $\{e\}(x_1, ..x_n) \simeq \{c\}(\{d_1\}(x_1, ..x_n), \ldots, \{d_n\}(x_1, ..x_n))$

  if $e = \langle 5, n, m, c, d_1, .., d_m \rangle$.

together with a diagonalization scheme

- $\{e\}(c, x_1, ..x_n, y_1, .., y_m) \simeq \{c\}(x_1.., x_n)$ if $e = \langle 6, n, m \rangle$.

as developed by [7]. The Normann schemes for E-recursion extend Kleene [5] schemes to objects of arbitrary type.

A partial function from V to V is called *partial E-recursive* if there exists an e such that $f(x) \simeq \{e\}(x)$ for all $x \in V$.

1.1.2 E-Closed Structures

A transitive set M is *E-closed* if for every partial E-recursive function f

$$\vec{x} \in M \text{ and } f(\vec{x}) \downarrow \Longrightarrow f(\vec{x}) \in M.$$

The *E-closure* of x is the least E-closed structure M such that $x \in M$. For example $E(\emptyset) = HF$ and $E(\omega) = L(\omega_1^{CK})$.

1.1.3 The Universal Computation Tree

The $n+1$-tuple of the form $\langle e, \vec{x} \rangle$ is a computation instruction that computes $\{e\}(\vec{x})$. We say that b is an *immediate subcomputation instruction of a,* and write $b <_U a$, if b is used as a subcomputation in computing a. If $c <_U b <_U a$, then c is a *subcomputation instruction* of a.

Associated with $\langle e, \vec{x} \rangle$ is a tree-like object $T(e, x)$. This consists of $\langle e, \vec{x} \rangle$ and the part of $<_U$ below $\langle e, x \rangle$, and is called the *computation tree* of $\{e\}(\vec{x})$.

Claim 1: $\{e\}(x) \downarrow$ if and only if $T(e, x)$ is wellfounded. In this case $T(e, x)$ can be uniformly computed from e and x.

Proof: See [8].

♦

The height of $T(e, x)$, denoted as $|T(e, x)|$, is defined to be ∞ if $T(e, x)$ is not well founded. If $T(e,x)$ is well founded, then

$$|T(e,x)| = \bigcup \{\ T(c,y)+1 \mid \langle c,y\rangle <_U \langle e,x\rangle\ \}.$$

$|T(e,x)| = |\{e\}(\vec{x})|$ is called the *length of computation* of $\langle e,\vec{x}\rangle$.

Claim 2: If $\{e\}(x)\downarrow$, $|T(e,x)|$ can be computed uniformly from e and x.

Claim 3: The predicates $|\{e\}(x)| < \gamma$ and $|\{e\}(x)| = \gamma$ are E-recursive.

Proofs can be found in [8].

1.1.4 The Constructible Hierarchy

We consider $L(\delta, X)$ as an E-recursive function of two variables. See [8] for more details. A key lemma used throughout this work is as follows:

Lemma 1: If $\{e\}(x)\downarrow$ and $|\{e\}(x)| \geq \omega$, then $T(e,x)$ and $\{e\}(x)$ are first order definable over $L(|\{e\}(x)|+1, TC(\{x\}))$.

1.1.5 E-Reducibility

Definition 1: $y \leq_E \vec{x}$ if and only if $y = \{e\}(\vec{x})$ for some index e.

For example, it can be shown that if $x, y \subseteq \omega$ then y is hyperarithmetic in x if and only if $y \leq_E x$.

We define

- $\kappa_0^x = \cup\{\ \gamma \mid \gamma \leq_E x\ \}$.
- $\kappa^x = \{\ \gamma \mid \exists \vec{a} \in TC(\{x\})\ x \leq_E \vec{a}\ \}$.

It is easy to see that $E(x) = L(\kappa^x, TC(\{x\}))$.

We say A is *r.e. in x* if and only if $A = \{\ y \mid \{e\}(x,y)\downarrow\ \}$.

1.2 Effective Transfinite Recursion

Theorem 1: (Fixed Point Theorem) [4] If f be a total recursive function on the natural numbers, then there exists an $\{e\}$ such that $\{f(e)\} = \{e\}$.

Theorem 2: Effective Transfinite Recursion (ETR)

Let $\prec$ be a well ordering (not necessarily recursive) of some set X. Let $I : \omega \longrightarrow \omega$ be a total recursive function. Suppose for all $e < \omega$ and x in X that

$$\forall y \prec x \ \{e\}(y) \downarrow \Longrightarrow \{I(e)\}(x) \downarrow .$$

If c is such that $\{c\} \simeq \{I(c)\}$ (as guaranteed by Kleene) then $\forall x \in X \ \{c\}(x) \downarrow$.

Proof: Suppose x is the smallest element in X such that $\{c\}(x)$ is not defined. Then $\{c\}(y)$ is defined for all $y < x$, so by assumption $\{I(c)\}(x) \simeq \{c\}(x)$ is defined. Contradiction.

♦

1.3 Selection Theorems

Selection Theorems in recursion theory allow us to pick out elements of an r.e. set uniformly recursively in an index for the r.e. set and another parameter. The techniques used to prove Gandy selection are used throughout E-recursion. We will therefore provide the main ideas of the proof in this section.

Theorem 3: [2] There exists a partial E-recursive function $\pi(e,x)$ such that for all $e \in \omega$ and all x

1. $\exists n \ n \in \omega[\{e\}(n,x) \downarrow] \iff \pi(e,x) \downarrow$.

2. $\pi(e,x) \downarrow \iff \{e\}(\pi(e,x),x) \downarrow$.

Gandy selection selects an element from an E-r.e. set uniformly in an index for that r.e. set and another parameter.

There are two claims needed in the proof.

Claim 4: Suppose $\{d\}(x)\downarrow$ or $\{e\}(y)\downarrow$, then min $|\{d\}(x),\{e\}(y)| \leq_E x,y$ uniformly.

Claim 5: The predicates $|\{e\}(x)| < \gamma$ and $|\{e\}(x)| = \gamma$ are E-recursive.

For proofs see [8]

Proof: (Gandy) Define by Effective Transfinite Recursion,

$$\{I(r)\}(e,k) = \{r\}(e,k+1)+1 \text{ if } \{r\}(e,k+1)\downarrow$$

and

$$|\{r\}(e,k+1)| \leq |\{e\}(k)|$$

or

$$\{I(r)\}(e,k) = 0 \text{ if } \{e\}(k)\downarrow$$

and

$$|\{e\}(k)| < |\{r\}(e,k+1)|$$

By Kleene's fixed point theorem for all e and k,

$$\{c\}(e,k) = \{I(c)\}(e,k)$$

Now,

$$\forall k[\{e\}(k)\downarrow \Longrightarrow \{c\}(e,k)\downarrow]$$

and

$$\forall k[\{c\}(k+1)\downarrow \Longrightarrow \{c\}(e,k)\downarrow]$$

Hence,

$$\forall k[\{e\}(k)\downarrow \Longrightarrow \{c\}(e,0)\downarrow]$$

So,

$$\{c\}(e,0)\downarrow \Longrightarrow \exists k[|\{e\}(k)| < |\{c\}(e,k+1)|]$$

Hence it is enough to show

$$\exists k[\{e\}(k)\downarrow] \Longrightarrow \{e\}(\{c\}(e,0))\downarrow$$

which we may do by a backwards recursion.

♦

The Normann selection theorem [7] was developed as a result of his work on Post's Problem for $E(2^{\omega})$. It is used here in the generalization of forcing arguments in $E(\kappa)$ where κ is a regular cardinal. First, some definitions.

Definition 2: P satisfies the ρ chain condition in $L(\kappa)$ if every antichain in $L(\kappa)$ has cardinality less than ρ.

Definition 3: Let γ be a cardinal in the sense of $L(\kappa)$. $L(\kappa)$ is said to obey less than γ selection if there exists a partial E-recursive (in γ and some other parameter) function f such that for all $e \in \omega, \delta \in \gamma$ and $p \in L(\kappa)$

$$\exists x < \delta[\{e\}(p,x)\downarrow] \Longrightarrow [f(e,\delta,p)\downarrow \text{ and } \{e\}(p,f(e,\delta,p))]\downarrow$$

Hence, Normann selection implies that $E(\gamma)$ obeys less than γ selection whenever γ is a regular cardinal in the sense of L.

Theorem 4: [7] Let x be an unbounded subset of ρ, where ρ is a regular cardinal in the sense of $E(x)$. If $\delta < \rho$ and $C \subseteq \delta$ is nonempty and E-r.e. in x, then some element of C is E-recursive in δ, x uniformly.

1.4 Reflection

Definition 4: Let $X = TC\{x\}$. δ is *x-reflecting* iff for all $\Sigma_1(x)$ sentences $\pi(x)$.

$$L(\delta, X) \models \pi(x) \Longrightarrow L(\kappa_0^x, X) \models \pi(x),$$

and $\delta \geq \kappa_0^x$.

Remark: If

$$L(\kappa_0^x, X) \models \pi(x),$$

where π is a $\Sigma_1(x)$ sentence. then

$$\forall\, \delta \geq \kappa_0^x \; L(\delta, X) \models \pi(x).$$

In fact, δ is x-reflecting iff the $\Sigma_1(x)$ sentences false in $L(\kappa_0^x, X)$ are all still false in $L(\delta, X)$.

Claim 6: There exists a largest x-reflecting ordinal, κ_r^x, and $\kappa_r^x \leq \kappa_x$.

Proof: Let $\pi(x, M)$ be the formula "$x \in M$ and M transitive and M is E-closed." Then, by Proposition X.3.1 of [8] the sentence "$\exists M\ \pi(x, M)$" can be written as a sentence which is $\Sigma_1(x)$. This sentence must be false in $L(\delta, X)$ for all $\delta \leq \kappa^x$, as $L(\kappa^x, X)$ is the least E-closed transitive set with x as a member. But

$$L(\kappa^x, X) \in L(\kappa^x + 1, X)$$

and so

$$L(\kappa^x + 1, X) \models \exists\ M \pi(x, M).$$

Hence $\kappa^x + 1$ is not x-reflecting, so κ_r^x exists and $\kappa_r^x \leq \kappa^x$.

♦

Claim 7: "$\delta = \kappa_r^x$", "$\delta < \kappa_r^x$", and "$\delta > \kappa_r^x$" are E-recursive assertions about δ in parameter x.

Proof: Let $\pi(x)$ be Σ_1 such that

$$L(\kappa_0^x, X) \models \neg\pi(x),$$

$$L(\kappa_r^x + 1, X) \models \pi(x).$$

Σ_1 truth is upwards absolute so

$$\begin{aligned}
\delta > \kappa_r^x &\iff L(\kappa, X) \models \pi(x), \\
\delta \leq \kappa_r^x &\iff L(\delta, X) \models \neg\pi(x), \\
\delta = \kappa_r^x &\iff (\delta \leq \kappa_r^x) \wedge (\delta + 1 > \kappa_r^x).
\end{aligned}$$

♦

[13] has shown this cannot be done uniformly in x.

Claim 8: The relation $\kappa_r^{x,y} \leq \kappa_r^x$ is co-re.

Proof: Using the fact (from [8]) that

$$\kappa_r^{x,y} \leq \kappa_r^x \iff \kappa_0^{x,y} \leq \kappa_r^x,$$

it remains to be proved that the relation

$$\kappa_r^x < \kappa_0^{x,y}$$

is E-r.e. Now

$$\kappa_r^x < \kappa_0^{x,y} \iff \exists\, \delta \leq_E x, y \;\; \kappa_r^x < \delta,$$

where the relation $\kappa_\tau^x < \delta$ is E-recursive. By Gandy selection we may find an index $\{e\}$ such that

1. $\kappa_\tau^x < \kappa_0^{x,y} \iff \{e\}(x,y)\downarrow$
2. If $\{e\}(x,y)\downarrow$ then $\{e\}(x,y) = \delta$ with $\kappa_\tau^x < \delta$.

$\blacklozenge$

1.5 Admissibility Implies E-Closure

We work with the following abbreviated version of the E-schemas.

It should be clear how to extend the result to all E-schemas.

$$\begin{aligned}
\{2^m3^n\}(\vec{x}) &\equiv \{\ \{n\}(y) \mid y \in \{m\}(x)\ \}, \\
\{5\}(x) &\equiv x, \\
\{7\}(x,\vec{y}) &\equiv \{x\}(\vec{y}).
\end{aligned}$$

First we show that E-computation is Σ_1 in V, in the following sense.

Theorem 5: $\{f\}(\vec{z})\downarrow= y$ in σ steps iff there is G satisfying the following list of Δ_0 conditions.

- $G(f,\vec{z}) = (y,\sigma)$.
- G is a function, with domain a subset of $\omega \times V$ and range a subset of $V \times On$.
- $\text{dom}(G)$ consists of pairs $(e,\vec{x})$, where e is an integer of the form 2^m3^n or 5 or 7.

- Let $(e, \vec{x}) \in \mathrm{dom}(G)$. Then
  - If $e = 5$, then $\mathrm{lh}(\vec{x}) = 1$, say $\vec{x} = x$, and $G(e, x) = (x, 0)$.
  - If $e = 7$ and $\vec{x} = (x, \vec{y})$ then $(x, \vec{y}) \in \mathrm{dom}(G)$ and $G(7, (x, \vec{y})) = (v, \sigma + 1)$ where $G(x, \vec{y}) = (v, \sigma)$.
  - If $e = 2^m 3^n$ then $(m, \vec{x}) \in \mathrm{dom}(G)$, for every $y \in G(m, \vec{x})$ we have $(n, y) \in \mathrm{dom}(G)$, and $G(e, \vec{x}) = (v, \sigma)$ where

$$v = \{\, G(n, y)_0 \mid y \in G(m, \vec{x})_0 \,\}$$

and $\sigma = \sup \{\, G(n, y)_1 + 1 \mid y \in G(m, \vec{x})_0 \,\} \cup (G(m, \vec{x})_0 + 1)$.

Proof: Easy, by induction on the length of computation σ. Note that any two functions satisfying the above clauses agree on their common domain.

♦

Theorem 6: Let M be admissible, let $x \in M$, let $\{e\}(x)$ converge to y in σ steps. Then there is $G \in M$ satisfying the Δ_0 clauses given above, so by uniqueness $(y, \sigma) = G(e, x) \in M$ and we have shown M is E-closed.

Proof: Again, by induction on the length σ. NB this is **external** induction on a tree which is well-founded in V, namely $T_{(e,x)}$. The only interesting case is when $e = 2^m 3^n$, here we use Δ_0-replacement to glue functions together.

♦

1.6 Kechris Basis Theorem

Theorem 7: [8] Let $y \leq_E x$ and let A be E-r.e in x. Suppose that $y - A \neq \emptyset$ then $\exists b \in y - A$ such that $\kappa_r^{x,b} \leq \kappa_r^x$.

Proof: We use two facts proved in [8].

1. $\kappa_r^{x,b} \leq \kappa_r^x \iff \kappa_0^{x,b} \leq \kappa_r^x$.

2. "$\kappa_r^x < \kappa_0^{x,b}$" is E-r.e. in x, b.

By fact 1, it is sufficient to find a $b \in y - A$ such that $\kappa_0^{x,b} \leq \kappa_r^x$. For a contradiction, suppose there is no such b with this property.

Hence

$$y \subseteq A \cup \{\, b \mid \kappa_r^x < \kappa_0^{x,b} \,\}.$$

By fact 2, for all $b \in y$ either $b \in A$ or

$$\exists \delta \; \delta \leq_E x, b \text{ and } \delta \text{ is not x-reflecting.}$$

Gandy selection implies that

$$\{\, b \mid \exists \delta \; \delta \leq_E x, b \text{ and } \delta \text{ is not x-reflecting} \,\}$$

is r.e. in x.

Thus, y is contained in the union of these two E-r.e. in x sets. Therefore, for each $b \in y$, there is a $\delta_b \leq_E x, b$ such that EITHER

1. δ_b is the length of a computation that puts $b \in A$ OR

2. δ_b is not x-reflecting.

By Gandy Selection, δ_b can be construed as a partial E-recursive function of x, b defined for all $b \in y$. Let

$$\beta = \bigcup \{\, \delta_b \mid b \in y \,\},$$

then the bounding with union scheme results in

$$\beta \leq_E x,$$

so that

$$\beta < \kappa_r^x.$$

If condition 2 holds then

$$\delta_b > \kappa_r^x > \beta.$$

Hence condition 2 never holds, and $y \subseteq A$. Contradiction.

♦

1.7 Moschovakis Witnesses

Definition 5: Suppose that $\{e\}(x) \uparrow$. A Moschovakis witness is an infinite descending chain through $>_U$, the universal computation tree below $\langle e, x\rangle$. To be more precise, a Moschovakis witness to the divergence of $\{e\}(x)$ is a function $\lambda n\ t(n)$ such that $t(0) = \langle e, x\rangle$ and $t(n+1)$ is an immediate subcomputation of $t(n)$ for each n.

We first start with a claim needed to prove the main result in this section.

Claim 9: Assume some well ordering of TC(x) is E-recursive in x. Suppose that $\langle c, y\rangle$ is an immediate subcomputation instruction of $\langle e, x\rangle$. Then some well ordering of TC(y) is E-recursive in x and y. Furthermore, if $\{c\}(y) \downarrow$, then some well ordering of $\{c\}(y)$ is E-recursive in x and y.

The following is a key result central to the development of E-recursion theory, and was shown by Harrington.

Theorem 8: [3] Let $\leq$ be a well ordering of the transitive closure of $\{x\}$ which is E-recursive in x. If $\{e\}(x) \uparrow$, then there exists a Moschovakis witness first order definable over $L(\kappa_r^x, TC\{x\})$.

Proof: Let $\langle e_0, x_0 \rangle = e, x$.

Fix i. The Moschovakis witness is built by recursion on $i \in \omega$.

Suppose we have found $\langle e_i, x_i \rangle$ such that

1. $\{e_i\}(x_i) \uparrow$.

2. $x_i \in L(\kappa_r^x, tc\{x\})$.

3. $\kappa_r^x \geq \kappa_r^{x_0 \ldots, x_i}$.

We use scheme T and let $e_i = 2^{m_i} 3^{n_i}$.

Clause clearly implies that

$$\kappa_r^x \geq \kappa_r^{x_0 \ldots, x_n} \geq \kappa_0^{x_0, \ldots, x_n} \geq \kappa_0^{x_n}.$$

From a previous result we know that if

$$\{e\}(x) \downarrow \text{ and } |\{e\}(x)| \geq \omega$$

then

$$\{e\}(x) \in L(|\{e\}(x)| + 1, TC\{x\}).$$

Hence, anything E-recursive in x_i is in $L(\kappa_r^x, TC(\{x\}))$. So one can look down from height κ_r^x to see whether or not $\{m_i\}(x_i)$ converges or diverges. There are now 2 cases:

1. $\{m_i\}(x_i) \downarrow$.

2. $\{m_i\}(x_i) \uparrow$.

If case 2 occurs, let

$$\langle m_i, x \rangle = \langle e_{i+1}, x_{i+1} \rangle.$$

Otherwise, by our previous claim, there is a wellordering $\leq_i$ of $\{m_i\}(x_i)$ E-recursive in x_i. $\leq_i$ can be computed uniformly by Gandy selection. Define

$$e_{i+1} = n,$$

$$x_{i+1} = \leq_i \text{ least } u \text{ where } u \in \{m_i\}(x_i) \text{ and} |\{n_i\}(u)| \geq \kappa_r^x.$$

Since $\{m_i\}(x_i) \in L(\kappa_r^x, TC(x))$ and $x_{i+1} \in \{m_i\}(x_i)$, $\{x_{i+1}\}$ is first order definable over $L(\kappa_r^x, TC(\{x\})$.

By the Kechris Basis theorem, we know that there is a $y \in \{m_i\}(x_i)$ such that $\{n\}(y) \uparrow$ and $\kappa_r^{x_0,..x_i} \geq \kappa_r^{x_0,..x_i,y}$. Let z be the $<_i$-least element such that this property holds.

It is sufficient to show that $z = x_{i+1}$. Clearly $x_{i+1} \leq_E z$ since if $\{n\}(z) \uparrow$ then $|\{n_i\}(z)| \geq \kappa_r^x$. Suppose for a contradiction that $x_{i+1} <_i z$. For each $a <_i z$ either

1. $\{n_i\}(a) \downarrow$

2. $\kappa_r^{x_0,..x_i} < \kappa_r^{x_0,..x_i,a}$

If the second condition holds, then

$$\kappa_r^{x_0,..x_i} < \kappa_0^{x_0,..x_i,a},$$

so that

$$\kappa_r^{x_0,..x_i} < \{e\}(x_0, ..x_i, a)$$

for some e. By Gandy selection, there exists a partial E-recursive function f such that $\forall a <_i z$

$$|\{n_i\}(a)| = f(x_0, ..x_i, a)$$

or

$$\kappa_r^{x_0,..x_i} < f(x_0, ..x_i, a).$$

Let

$$\gamma = \bigcup_{a <_i z} f(x_0, \ldots, a).$$

By the bounding with union scheme,

$$\gamma \leq_E x_0, \ldots x_i, z.$$

As in the Kechris Basis Theorem, this shows that the first case always holds, in particular $\{n_i\}(x_{i+1}) \downarrow$. But now

$$|\{n_i\}(x_{i+1})| \geq \kappa_r^x \geq \kappa_r^{x_0..x_i} \geq \kappa_r^{x_0..x_{i+1}} > \gamma \geq |\{n_i\}(x_{i+1})|,$$

which is absurd.

So $x_{i+1} = z$, and we are done since z has the required properties.

♦

1.8 Admissibility - Divergence Split

The class of E-closed structures can be split into two disjoint classes; those that admit Moschovakis witnesses and those that are Σ_1-admissible.

Proofs of the following can be found in [8]

Theorem 9: Let x be a set of ordinals. The following are equivalent:

1. $E(x)$ is not Σ_1 admissible.

2. $\kappa_r^{x,y} \in E(x)$ for all $y \in E(x)$.

3. $E(x)$ admits Moschovakis witnesses.

Theorem 10: Let x be a set of ordinals. The following are equivalent:

1. $E(x)$ is Σ_1 admissible.

2. For all $A \subseteq E(x)$ A is Σ_1 definable over $E(x)$ if and only if A is r.e. on $E(x)$.

1.9 Facts About E-Closed Structures

To end this chapter, we described some results about $E(\omega_1)$, to give the reader a feel for the some of the properties of an E-closed structure before we pursue generalizations.

Claim 10:

$$E(\omega_1) = \{ \ \{e\}(\alpha_1, \ldots \alpha_n, \omega_1) \mid \vec{\alpha} < \omega_1, \{e\}(\vec{\alpha}, \omega_1) \downarrow \}$$

Proof: There is an E-recursive bijection G between finite tuples of ω_1 and ω_1 [Goedel's pairing function for ordinals gives a recursive bijection between $\omega_1 \times \omega_1$ and ω_1; now code up finite tuples. So actually,

$$E(\omega_1) = \{ \ \{e\}(\delta, \omega_1) \mid e < \omega \text{ and } \delta < \omega_1 \ \}$$

i.e. $x \in E(\omega_1)$ iff $\exists \delta < \omega_1$ such that x is E-recursive in δ and ω_1.

A note on parameters: ω_1 is often used as a parameter especially in [8], although this is not always made explicit. In this treatment, we will try to make it so.

♦

Claim 11: If $\gamma < \delta < \omega_1$, then γ is E-recursive in δ and ω_1.

Proof: $\exists \ f \in L$ such that f is a bijection between ω and δ. By condensation, $\exists \ f \in L(\omega_1)$ such that f is a bijection between ω and δ. Now $L(\omega_1)$ is E-recursive in ω_1 as is the standard well ordering of ω_1. The statement "f is a bijection between ω and δ" is an E-recursive assertion about f in parameter δ.

So the $< -$ least f such that f is a bijection between ω and δ is E-recursive in δ and ω_1.

♦

[1] If f, is that function, then $\gamma = f(n)$ for some n, so γ is E-recursive in f and n, so is E-recursive in δ and ω_1.

We assume that $V = L$ in the following claim.

Claim 12: $E(\omega_1)$ is closed under the formation of ω sequences of its elements.

Proof: Let x be such a sequence. Then $x(n)$ can be written as $\{e_n\}(\omega_1, \vec{\alpha}_n)$. By condensation if we define a sequence y by $y(n) = (e_n, \langle\alpha\rangle_n)$ then $y \in L(\omega_1)$. So y and ω_1 are both in $E(\omega_1)$, which can use them to reconstruct x.

♦

Corollary 1: $E(\omega_1)$ has Moschovakis Witnesses, hence is not Σ_1 Admissible.

Proof: A Moschovakis witness is just an ω-sequence from $E(\omega_1)$.

♦

The same proof works for $E(\kappa)$ as long as κ has uncountable cofinality.

As Slaman [13] remarks, absoluteness gives that V agrees that $E(\omega_1^L)$ is not admissible.

Claim 13: Selection holds over γ relative to an ordinal x iff $\kappa_r^{x,\gamma}$ is the height of $E(\gamma)$.

[1] If R wellorders X and A is an E-recursive predicate in parameter y s.t. $\exists\, x \in X\ A(x)$, then the R-least x in X s.t. $A(x)$ is E-recursive in R,X,y

Proof:

The above applies to any ordinal γ which is a cardinal in it's E-closure. One direction is a reworking of what [8], p 243 ("For a dynamic view of κ_r") says about reflection versus selection on $E(\omega_1)$. We may take it that the parameter x is an ordinal less than γ.

Assume $\kappa_r^{x,\gamma} = \kappa^\gamma$. As in [8], p 247, if $e(x,\delta)\downarrow$ for some $\delta < \gamma$ then a computation which shows that for some $\delta_0 < \gamma$ we have $e(x,\delta_0)\downarrow$ is in $L(\beta)$ for some $\beta \leq_E x, \gamma$. Now we can easily find the least δ_0 for which there is such a computation in $L(\beta)$, and by Gandy Selection we can take it that β is computed by uniform means from γ, x, so we can select uniformly in γ, x.

As for the other direction, the idea is this: if $\kappa_r^{x,\gamma} < \kappa^\gamma$ then $E(\gamma)$ has a straightforward procedure for determining whether an ordinal δ is equal to $\kappa_r^{x,\gamma}$. Namely this; there is a Σ_1 formula F with parameters x, γ which first becomes true at level $\kappa_r^{x,\gamma}+1$, so given x and γ and a δ which we wish to check we just ask if $L(\delta) \models \neg\phi$ and $L(\delta+1) \models \phi$, which can only happen for $\delta = \kappa_r^{x,\gamma}$. (See for example, [8], exercise 4.12.) So now it makes sense to talk about computing an index for $\kappa_r^{x,\gamma}$. The predicate "$\{g\}(\gamma,\nu)$ converges to $\kappa_r^{x,\gamma}$" is an r.e. one, and we cannot select a ν which satisfies it E-recursively in γ, x. Because if we could we'd have $\kappa_r^{x,\gamma} \leq_E \gamma, \nu \leq_E \gamma, x$ which cannot happen.

♦

Claim 14: If $E(\omega_1^L)$ satisfied selection over ω_1^L, then the r.e. predicates would be closed under existential quantification over all of $E(\omega_1^L)$.

Proof:

Just like corollary 4.4. on p. 247 of [8]

♦

Claim 15: Any function which is Σ_1 over $E(\omega_1^L)$ would also be E-recursive in whatever parameters were needed to do the selection.

Proof: Let F be Σ_1, given by $(x, y) \in F$ iff $\exists z \; \phi(x, y, z)$ with ϕ being Δ_0. We could use selection to find a partial function ϕ which would take x and select (whenever possible) some (y, z) which would work, but then the first component of this selector would give us F (by computing the unique y associated with x).

♦

But then the E-closure of $E(\omega_1^L)$ would imply its Σ_1-admissibility.

We can now prove that we have enough replacement; in fact it can be done directly from selection, in a manner very similar to 4.6 on p. 247 of [8].

Chapter 2

Fundamentals of Forcing over E-closed structures

2.1 Summary of Forcing over E-closed Structures

Let $L(\kappa)$ be E-closed. Let $P \in L(\kappa)$ be a poset. We also have a notion of what a condition is saying about x, a generic subset of η, the greatest cardinal in $L(\kappa)$, where $\eta = \mathrm{gc}(\kappa)$. That is, we have an E-recursive way of seeing if

$$p \Vdash \hat{r} \in \dot{x}$$

or

$$p \Vdash \hat{r} \notin \dot{x}$$

for $p \in P, r \in \mathrm{gc}(\kappa)$.

This is what [8] calls "ground level forcing facts" .

For example, if P is Cohen forcing then

$$p \Vdash n \in x \text{ if } n \in dom(p) \text{ and } p(n) = 1.$$

$$p \Vdash n \notin x \text{ if } n \notin domain(p) \text{ and } p(n) = 0.$$

The point is just so that when we have a filter G on P when can translate it to $x_G \subseteq \eta$. That is

$$x_G = \{ \ r \mid \exists p \in G \ p \ \Vdash \hat{r} \in \dot{x} \ \} = \eta - \{ \ r \mid \exists p \in G p \Vdash \hat{r} \notin \dot{x} \ \}$$

We want to set it up so that G is recoverable from x. For example, in Cohen forcing,

$$G = \{ \ p \mid p = 1_{x_G} \restriction dom(p) \ \}$$

The model we are interested in is

$$L(\kappa, G) = L(\kappa, x)$$

for generic $x \subseteq \eta$.

We know that

$$L(\kappa, G) = \{ \ \{e\}(a, G) \mid |\{e\}(a, G)| < \kappa \ \}$$

The problem with using $\{e\}(a, G)$ as a name is that we do not know if it diverges or converges in greater or equal to κ many steps.

We have a notion of terms t in $L(\kappa)$ that describe computations from elements of $L(\kappa)$ and G. We define by effective transfinite recursion on $\sigma < \kappa$ the forcing relation $\Vdash$ induced by P. The following sets and relations are defined simultaneously.

1. A relation $p \Vdash |t| = \sigma$.

2. A set $T(p, t, \sigma)$ of terms adequate for naming elements of t, if $p \Vdash |t| = \sigma$.

3. If $p \Vdash |t| = \sigma$ and $s \in T(p, t, \sigma), q \leq p$ then we will define a relation $''q \Vdash s \in t''$.

These relations are all uniformly E-recursive in P, σ.

Definition 6: G is generic if and only if G meets all dense sets in $L(\kappa+1)$.

Theorem 11: (Truth Lemma)

Let G be generic.

1. $|t^G| = \tau \iff \exists p \in G\, p \Vdash |t| = \tau$.
2. If $|t^G| = \tau$ and $p \Vdash |t| = \tau$ and and$p \in G$ then every element of t^G is named by some $s \in T(p, t, \tau)$.
3. If $|t^G| = \tau, p \in G, p \Vdash |t| = \tau, s \in T(p, t, \tau)$ then $s^G \in t^G \iff \exists q\, q \leq p\, (q \Vdash s \in t \wedge q \in G)$

Proof: see next section.

♦

Definition 7: The Tree of Possibilities. In the same spirit as the universal computation tree discussed in Chapter 1, we introduce the tree of possibilities. $>_V$ is the forcing counterpart to $>_U$. A node on $>_V$ is a triple $\langle p, e, t\rangle$ where p is a forcing condition , $e \in \omega$ and t is a term of $L(\kappa)$. We say

$$\langle p, e, t\rangle >_V \langle q, n, s\rangle$$

if and only if

$$p \geq q \text{ and } q \Vdash^* [\langle e, t\rangle >_u \langle n, s\rangle]$$

Definition 8: P satisfies effective bounding if for $P \in L(\kappa)$, if

$$p \Vdash^* \exists \sigma\, |t| = \sigma$$

then

$$p \Vdash^* |t| \leq \gamma$$

for some $\gamma \leq_E p, t, P$

Lemma 2: If $>_v$ is wellfounded below $\langle p, e, t\rangle$ and

$$p \Vdash^* \exists\sigma\ |\{e\}(t)| = \sigma$$

then P satisfies effective bounding.

Claim 16: Let G be generic, a set of ordinals bounded in κ. Then $L(\kappa, G)$ is E-closed under the following conditions.

1. If $L(\kappa)$ is admissible, then $L(\kappa, G)$ is E-closed [12]
2. If P is countably closed and $L(\kappa)$ is inadmissible, then $L(\kappa, G)$ is E-closed. [9]
3. If P satisfies the ρ chain condition and $L(\kappa)$ is inadmissible and satisfies $<\ \rho$-selection, then $L(\kappa, G)$ is E-closed. [9]

Claim 17: If $L(\kappa) = E(\omega_1)$, then $L(\kappa, G)$ is not E-closed.

Proof: Let $P = Coll(\omega, \omega_1)$. By a density argument , there is a real t coding a well order of ω_1 such that $t \equiv_E G$. Suppose $L(\kappa, G) = E(\omega_1, G)$ is E-closed. Then $t \in L(\kappa, G)$. As $G \leq_E t$ and $\omega_1 \leq t$, $L(\kappa, G) \subseteq E(t)$ and so $L(\kappa, G) = E(t)$. Contradiction. as then κ would be admissible. So $L(\kappa, G)$ is not E-closed.

♦

We want to generalize Sacks' forcing properties to encompass the ρ-cc case. We develop a property of local proper forcing which is an effective version of Shelah's proper forcing.

2.2 Proof of Truth Lemma for E-Recursion

In this section, we prove the truth lemma for E-recursive functions. For the sake of definiteness, we work with the ground model $L(\kappa) = E(\omega_1)$, $P = \{ f \mid f \to 2, |dom f| < \omega \}$. We describe a set of terms:

1. $\bar{a}$ for $a \in L(\kappa)$ (to denote a).

2. $\dot{G}$ (to denote G.)

3. $\{e\}(\bar{a}, \dot{G})$ (to denote $\{e\}(a, G)$ if that converges in less than κ steps)

More complex terms will occur below, but we assume that the composition scheme is used to reduce them to the form $\{e\}(\bar{a}, \dot{G})$. Via some suitable codings, we can think of terms as a Δ_0 class of $L(\kappa)$. We adopt an abbreviated form of the E-schemas.

$$\{7\}(x, y) = y$$

$$\{2^m 3^m\}(x) = \{ \{n\}(y) \mid y \in \{m\}(x) \}$$

We define by induction of $\sigma < \kappa$ the following sets and relations. For the moment we work in V; later we look at definability over $L(\kappa)$.

1. $p \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{\sigma}$

2. $T(p, e, a, \sigma)$, a set of terms, defined if condition (1) holds.

3. $q \Vdash s \in \{e\}(\bar{a}, \dot{G})$ to be defined if $q \leq p$ and $s \in T(p, e, a, \sigma)$.

For the moment we are adopting a minimalist approach and "$|\{e\}(\bar{a}, \dot{G})| = \bar{\sigma}$" and "$s \in \{e\}(\bar{a}, \dot{G})$" are the only expressions in the forcing language and can legitimately appear to the right of "$\Vdash$".

At level 0 we define,

1. $p \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{0} \iff e = 7$ for all p and a.

2. $T(p,7,a,0) = \{ \bar{n} \mid n \in \omega \}$

3. If $q \leq p$ and $\bar{n} \in T(p,7,a,0)$ then $q \Vdash \bar{n} \in \{7\}(\bar{a}, \dot{G})$ if and only if $n \in dom(q)$ and $q(n) = 1$.

We have already handled $\{7\}$. So for $\sigma > 0$, if

(a) $p \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{\sigma} \iff e = 2^m 3^n$ for some n,m and where the following hold (where every set and relationship mentioned have already been defined by induction on σ.)

(b) There is some $\gamma < \sigma$ such that $p \Vdash |\{m\}(\bar{a}, \dot{G})| = \gamma$.

(c) $\forall s \in T(p, m, a, \gamma)\ \forall\ q \leq p$

if $q \Vdash s \in \{m\}(\bar{a}, \dot{G})$

then $\exists r\ r \leq q\ \exists\ \tau < \sigma\ r \Vdash |\{n\}(s)| = \tau$ Note that $\{n\}(s)$ will have to be rewritten using the composition scheme, but this is a detail.

(d) $\forall \tau < \sigma\ \forall q\ q \leq p$

$\exists r\ r \leq q$ such that either $r \Vdash |\{m\}(\bar{a}, \dot{G})| = \bar{\tau}$ or

$\exists s \in T(p, m, a, \gamma)\ r \Vdash s \in \{m\}(\bar{a}, \dot{G})$ and $\forall\ w \leq r \forall \eta < \tau\ \ \neg\ w \Vdash |\{n\}(s)| = \eta$

1. If $p \Vdash |\{2^m 3^n\}(\bar{a}, \dot{G})| = \bar{\sigma}$,

   define $T(p, 2^m 3^n, a, \sigma) = \{ \{n\}(s) \mid s \in T(m, a, p, \gamma) \}$

2. Finally let $q \leq p$, let $\{n\}(s) \in T(p, 2^m 3^n, a, \sigma)$, so $\{n\}(s) \in T(p, m, a, \gamma)$ and $q \Vdash \{n\}(s) \in \{2^m 3^n\}(\bar{a}, \dot{G}) \iff q \Vdash s \in \{m\}(\bar{a}, \dot{G})$ and $\exists\ \tau < \sigma\ q \Vdash |\{n\}(s)| = \tau$.

A standard proof by effective transfinite recursion shows that all of these sets and relations are uniformly E-recursive. As $L(\kappa)$ is E-closed, everything is Σ_1 definable over $L(\kappa)$. This is a key remark, as it dictates the definition of generic filter below. Before showing the truth lemma, we show

Theorem 12: If

$$p \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{\sigma} \text{ and } q \leq p$$

then

$$q \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{\sigma}$$

Proof: The proof is by induction carrying along hypotheses saying that (2) and (3) behave well.

Induction hypothesis at σ:

If $p \Vdash |\{e\}(\bar{a}, \dot{G})| = \sigma$ then

$$q \leq p \Longrightarrow q \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{\sigma}$$

and

$$T(q, e, a, \sigma) = T(p, e, a, \sigma)$$

That is,

$$(r \leq q \leq p \text{ and } s \in T(p, e, a, \sigma) \text{ and } q \Vdash s \in \{n\}(\bar{a}, \dot{G}) \Longrightarrow r \Vdash s \in \{e\}(\bar{a}, \dot{G}))$$

If $\sigma = 0$, then none of this is problematic.

Let $\sigma > 0$ and $e = 2^m 3^n$. Suppose that

$$* \qquad p \Vdash |\{e\}(\bar{a}\dot{G})| = \sigma$$

Let $r \leq q \leq p$, $t \in T(p, e, a, \sigma)$ and $q \Vdash t \in \{e\}(\bar{a}, \dot{G})$. We check first that

$$q \Vdash |\{e\}(\bar{a}, \dot{G})| = \sigma$$

We check (1a),(1b),(1c). By (*)

$$\exists \gamma < \sigma \; p \Vdash |\{m\}(\bar{a}, \dot{G})| = \bar{\gamma}.$$

By induction,

$$q \Vdash |\{m\}(\bar{a}, \dot{G})| = \bar{\gamma}.$$

(1a) is clear.

(1b): Let $s \in T(q, m, a, \gamma)$. By induction, $s \in T(p, m, a, \gamma)$. Let

$$r \leq q,\ r \Vdash \{s\} \in \{m\}(\bar{a}, \dot{G}).$$

By $(*)$, there is

$$w \leq r \text{ and } \tau < \sigma\ w \Vdash |\{n\}(s)| = \tau$$

(1c). Let $\tau < \sigma$, $r \leq q$, then $r \leq p$, by $(*)$, there is a

$$w \leq r,\ w \Vdash |\{m\}(\bar{a}, \dot{G})| = \bar{\tau}$$

or

$$\exists\, s \in T(p, m, a, \gamma) = T(q, m, a, \gamma),\ \ w \Vdash s \in \{m\}(\bar{a}, \dot{G})$$

This verifies that

$$q \Vdash |\{2^m 3^n\}(\bar{a}, \dot{G})| = \bar{\sigma}.$$

By induction, $T(p, m, a, \gamma) = T(q, m, a, \gamma)$, so

$$T(p, e, a, \gamma) = T(q, e, a, \gamma).$$

Finally, $t \in T(p, 2^m 3^n, a, \sigma)$ and let $t \equiv \{n\}(s)$, $s \in T(p, m, a, \gamma)$. We are given that

$$q \Vdash \{n\}(s) \in \{2^m 3^n\}(\bar{a}, \dot{G})$$

which means by definition that

$$q \Vdash s \in \{m\}(\bar{a}, \dot{G})$$

and

$$\exists \tau < \sigma \; q \Vdash |\{n\}(s)| = \bar{\tau}$$

By induction, as $r \leq q$ and $s \in T(q, m, a, \gamma)$, $r \Vdash s \in \{m\}(\bar{a}, \dot{G})$. Similarly, because $\tau < \sigma$ we have,

$$r \Vdash |\{n\}(s)| = \bar{\tau}$$

So,

$$r \Vdash \{n\}(s) \in \{e\}(\bar{a}, \dot{G})$$

♦

Definition 9: $G \subseteq \omega$ is generic if and only if the filter

$$F_G = \{ \; p \in P \mid p = 1_G \restriction dom(p) \; \}$$

meets every dense subset of P first order definable with parameters over $L(\kappa)$.

Note that if G generic and $p \in F_G$ and E is definable and dense below p, then E meets F_G below p.

We adopt the following definitions for our abbreviated schemas for the lengths of computations.

$$|\{7\}(x, y)| = 0$$

If $\{2^m 3^n\}(x, y) \downarrow$ then

$$|\{2^m 3^n\}(x, y)|$$

is the supremum of $|\{m\}(x, y)|$ and $\{ \; |\{n\}(s)| \mid s \in \{m\}(x, y) \; \}$. We now prove the truth lemma for E-recursion.

Theorem 13: (Truth Lemma) If G is generic then $F = F_G$

1. $\exists p \in F\ p \Vdash |\{e\}(\bar{a}, \dot{G})| = \sigma \iff |\{e\}(\bar{a}, \dot{G})| = \sigma$.

2. If $p \in F$ and $p \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{\sigma}$, then for every element of $\{e\}(a, G)$, there is a term $\{f\}(\bar{b}, \dot{G}) \in T(p, e, a, \sigma)$ such that $\{f\}(b, G) \downarrow= x$.

3. If $p \in F$ and $p \Vdash |\{e\}(\bar{a}, \dot{G})| = \bar{\sigma}$ and $s = \{f\}(\bar{b}, \dot{G}) \in T(p, e, a, \sigma)$ then

$$\{f\}(b, G) \in \{e\}(a, G) \iff \exists q\ q \leq p\ q \in F\ q \Vdash s \in \{e\}(a, G).$$

Proof: If $\sigma = 0$ then all is clear.

If $\sigma > 0$, suppose first that $p \in F$ and $p \Vdash |\{2^m 3^n\}(\bar{a}, \dot{G})| = \bar{\sigma}$. By definition,

$$\exists \gamma < \sigma\ p \Vdash |\{m\}(\bar{a}, \dot{G})| = \gamma$$

. By induction,

$$\{m\}(a, G) \downarrow \text{ and } |\{m\}(a, G)| = \gamma.$$

Let $x \in \{m\}(a, G)$. By induction,

$$x = \{f\}(b, G) \text{ where } \{f\}(\bar{b}, \dot{G}) = s \in T(p, m, a, \gamma)$$

and for some $q \leq p\ q \in F$,

$$q \Vdash s \in \{e\}(\bar{a}, \dot{G}).$$

At this point, we start to use the power of the genericity hypothesis.

Claim 18:

$$\{\ r \mid \exists \tau < \sigma\ r \Vdash |\{n\}(s)| = \tau\ \}$$

is dense below q.

Proof: Let $r_0 \leq q$. $r_0 \Vdash s \in \{m\}(\bar{a}, \dot{G})$. The rest of the proof is clear.

♦

So we have $r \in F$, $r \leq q$ and $\tau < \sigma$ where

$$r \Vdash |\{n\}(s)| = \bar{\tau}.$$

By induction hypothesis, $\{n\}(\{f\}(b, G)) \downarrow$ and has length τ. So far we have that $\{e\}(a, G) \downarrow$ and $|\{e\}(a, G)| \leq \sigma$. To finish the proof we need to the that the lengths of the subcomputations are unbounded in σ. The definition of (1c) tells us precisely that the set of r with certain properties is dense below p. We get $r \in F$, $r \leq p$ such that either

$$r \Vdash |\{m\}(\bar{a}, \dot{G})| = \tau.$$

This case happens if $\tau = \gamma$. Or more interestingly,

$$\exists s \in T(p, m, a, \gamma)\; r \Vdash s \in \{m\}(\bar{a}, \dot{G})$$

and

$$\forall\, u \leq r \forall\, \eta < \tau\; \neg w \Vdash |\{n\}(s)| = \eta,\;\; r \leq p, \gamma \leq \sigma.$$

So by induction, if we let $s \equiv \{f\}(\bar{b}, \dot{G})$ say, $\{f\}(b, G) \in \{m\}(a, G)$. The above argument showed that the computation, $\{n\}(\{f\}(b, G))$ converges in less than σ steps. Suppose it converges in η steps. By induction and the fact that any two elements of F are compatible, we can find a $w \leq r$ such that

$$w \Vdash |\{n\}(s)| = \eta.$$

By construction, $\tau \leq \eta \leq \sigma$. We've shown that subcomputations of $\{e\}(a, G)$ have length unbounded in σ, that is, $|\{e\}(a, G)| = \sigma$. In case (2)

Let $p \in F$ and $p \Vdash |\{2^m 3^n\}(\bar{a}, \dot{G})| = \bar{\sigma}$. Let $x \in \{a\}(G)$ then, $x = \{n\}(y)$ where $y \in \{m\}(a, G)$ We know that

$$p \Vdash |\{m\}(\bar{a}, \dot{G})| = \bar{\gamma}$$

for some $\gamma < \sigma$. By induction,

$$y = \{f\}(b, G) \text{ and } \{f\}(\bar{b}, \dot{G}) \in T(p, m, a, \gamma),$$

and $\{n\}(\{f\}(\bar{a}, \dot{G})) \in T(p, 2^m 3^n, a, \sigma)$ and works as a name for x.

In case (3) Let $p \in F$, $p \Vdash |\{2^m 3^n\}(\bar{a}, \dot{G})| = \bar{\sigma}$ and $t \equiv \{n\}(s) \in T(p, 2^m 3^n, a, \sigma)$. As usual we assume that

$$p \Vdash |\{m\}(\bar{a}, \dot{G})| = \bar{\gamma}, s \in T(p, m, a, \gamma), s \equiv \{f\}(\bar{b}, \dot{G}).$$

Suppose

$$\{n\}(\{f\}(b, G)) \in \{2^m 3^n\}(a, G)$$

By definition, this happens only if $\{f\}(b, G) \in \{m\}(a, G)$, which by induction, happens if

$$\exists q \leq p \; q \in F \text{ and } q \Vdash \{f\}(\bar{b}, \dot{G}) \in \{m\}(\bar{a}, \dot{G})$$

This shows that

$$\exists \bar{q} \leq q \; \bar{q} \in F \bar{q} \Vdash |\{n\}(\{f\}(\bar{b}, \dot{G}))| < \sigma$$

so by definition,

$$\bar{q} \Vdash s \in \{2^m 3^n\}(\bar{a}, \dot{G}).$$

The converse is similar.

$\blacklozenge$

Chapter 3

Effective Proper Forcing

3.1 Theory of Reflection

Understanding the κ_r theory of E-closed structures is of major importance in forcing arguments. The reason is that if we see a convergent computation witnessing some fact by level κ_r, then we see a convergent computation witnessing that fact by level κ_0. Below level κ_0 it is often possible to use ideas like Gandy selection to get information about definability.

The ordinals which preserve the κ_r spectrum are known as the η sequence. Let

$$\eta_0 = \{ \ \delta \mid \kappa_r = \kappa_r^\delta \ \}$$

$$\eta_1 = \{ \ \delta \mid \kappa_r^\delta \leq \kappa_r^{\eta_0} \ \} = \{ \ \delta \mid \kappa_r^\delta = \kappa_r \vee \kappa_r^\delta = \kappa_r^{\eta_0} \ \}$$

and in general for successors

$$\eta_{\alpha+1} = \{ \ \delta \mid \kappa_r^\delta \leq \kappa_r^{\eta_\alpha} \ \} = \{ \ \delta \mid \kappa_r^\delta = \exists \, \zeta \leq \alpha \ \kappa_r^\alpha = \kappa_r^{\eta_{g^z}} \ \}.$$

In the case of λ limit,

$$\eta_\lambda = \{ \ \delta \mid \kappa_r^\delta < \kappa_r^{\eta_\alpha} \text{ for some } \ \alpha < \lambda \ \} = \bigcup_{\delta<\lambda} \eta_\delta$$

Note that $\kappa_r < \kappa_r^{\eta_0} < \ldots \kappa_r^{\eta_n}$.

We now have a corresponding "κ_r spectrum."

$$\kappa_{r,0} = \kappa_r$$

$$\kappa_{r,1} = \kappa_r^{\eta_0}$$

And for the general successor case $\alpha + 1$,

$$\kappa_{r,\alpha+1} = \kappa_r^{\eta_\alpha}$$

And in the case of λ limit,

$$\kappa_{r,\lambda} = \bigcup_{\alpha<\lambda} \kappa_{r,\alpha}$$

Claim 19: $\kappa_r^{\eta_\nu} \leq_E \eta_{\nu+1}$.

Proof:

$$\eta_0 = \{ \ \delta \mid \kappa_r = \kappa_r^\delta \ \}$$

So, $\kappa_r < \kappa_0^{\eta_0}$. In fact by recursiveness of " $\eta = \kappa_r$", we can show that $\kappa_r \leq_E \eta_0$. In the general case,

$$\eta_{\nu+1} = \{ \ \delta \mid \kappa_r^\delta \leq \kappa_r^{\eta_\nu} \ \} = \{ \ \delta \mid \kappa_0^\delta \leq \kappa_r^{\eta_\nu} \ \}.$$

And so,

$$\kappa_r^{\eta_\nu} < \kappa_0^{\eta_{\nu+1}}$$

which directly implies

$$\kappa_r^{\eta_\nu} \leq_E \eta_{\nu+1}$$

We have not shown this uniform, because "$\eta = \kappa_r^\delta$" is not uniformly recursive in δ.

♦

Slaman [13] has introduced an alternative κ_r spectrum.

$$\lambda_0 = \kappa_r^0$$

$$\lambda_1 = \kappa_r^{\lambda_0}$$

for α successor,

$$\lambda_{\alpha+1} = \kappa_r^{\lambda_\alpha}.$$

For ν limit,

$$\lambda_\nu = \bigcup_{\zeta<\nu} \lambda_\zeta$$

We first show that our theory and the Slaman theory are equivalent. By definition,

$$\kappa_{r,0} = \kappa_r = \lambda_0$$

Claim 20: $\kappa_{r,n} = \lambda_n$ for every n

Proof: By definition,

$$\kappa_{r,n} = \kappa_r^{\eta_{n-1}} \text{ and } \lambda_n = \kappa_r^{\lambda_{n-1}}.$$

By induction, We assume $\kappa_{r,n} = \lambda_n$. To show

$$\kappa_{r,n+1} = \lambda_{n+1},$$

or equivalently,

$$\kappa_r^{\eta_n} = \kappa_r^{\lambda_n}$$

To show

$$\kappa_{r,n+1} \leq \lambda_{n+1},$$

we know from Claim 1 that

$$\kappa_{r,n} \leq_E \eta_n$$

$$\kappa_0^{\kappa_{r,n},\eta_n} = \kappa_0^{\eta_n} \leq \kappa_r^{\eta_n}$$

Hence,

$$\kappa_r^{\kappa_{r,n}\eta_n} \leq \kappa_r^{\eta_n}$$

and so $\kappa_{r,n} = \lambda_n$

$$\kappa_r^{\lambda_n} \leq \kappa_r^{\eta_n}$$

For the converse,we use the following fact from [8]

$$\forall\, \lambda < \kappa^{\rho}\ \exists \mu < \rho \text{ such that } \lambda, \rho \equiv_E \mu, \rho$$

In particular,

$$\kappa_r, \rho \equiv_E \mu, \rho$$

where $\mu < \rho$. So easily. $\kappa_r^{\kappa_{r,n}} > \kappa_r$ and we know how κ_r^{μ} behaves as a function of μ for $\mu < \rho$, that is

$$\kappa_r^{\mu} = \kappa_r, \quad \mu < \eta_0$$

and

$$\kappa_r^{\mu} = \kappa_r^{\eta_0}, \quad \mu_0 \leq \mu < \mu_1.$$

So as $\kappa_r^{\mu} > \kappa_r$,

$$\kappa_r^{\mu} \geq \kappa_r^{\eta_0}.$$

That is,

$$\kappa_r^{\kappa_r} \geq \kappa_r^{\eta_0}.$$

♦

Claim 21: Suppose we have shown that for all n $\kappa_r, n = \lambda_n$, then $\kappa_r^{\eta_\omega} = \kappa_r^{\lambda_\omega}$.

Proof: To show $\kappa_r^{\lambda_\omega} \geq \kappa_r^{\eta_\omega}$, we know that $\kappa_r^{\eta_\omega} > \kappa_r^{\eta_n}$ for all n. So,

$$\kappa_r^{\eta_\omega} \geq \kappa_{r,\omega} = \lambda_\omega.$$

By the same line of argument as in the last half of the previous proof, we know that

$$\kappa_r^{\lambda_\omega} > \lambda_\omega.$$

Also,

$$\kappa_r^{\lambda_\omega} = \kappa_r^{\zeta}$$

for some $\zeta < \rho$. $\kappa_r^{\zeta} > \kappa_r^{\eta_n}$ for all n. Therefore, $\zeta > \eta_n$ for all n. $\zeta \geq \eta_\omega$. $\kappa_r^{\zeta} \geq \kappa_r^{\eta_\omega}$. That is

$$\kappa_r^{\lambda_\omega} \geq \kappa_r^{\eta_\omega}.$$

♦

3.2 Co-Re Substructures

Lemma 3:

$$\kappa_r^{x,y} \leq \kappa_r^{x} \iff \kappa_0^{x,y} \leq \kappa_r^{x}$$

Proof: $\Longrightarrow$ is clear. To show the converse, it is sufficient to show that $\kappa_r^{x,y}$ is x-reflecting. By assumption,

$$L(\kappa_r^{x,y}) \models \phi(x)$$

By reflection,

$$L(\kappa_0^{x,y}) \models \phi(x)$$

By upwards absoluteness of Σ_1 formulas,

$$L(\kappa_r^{x}) \models \phi(x)$$

So,

$$L(\kappa_0^{x}) \models \phi(x)$$

♦

Lemma 4: Let $L(\kappa, G)$ be E-closed, then $\kappa_r^G - Spectrum \subseteq \kappa_r - Spectrum$

Proof: Recall that $\kappa_r^{\omega_1,G} < \kappa$ and $\kappa_r - Spectrum$ is cofinal in κ. We also shown that $\kappa_r^{x,y} \leq \kappa_r^x$ if and only $\kappa_0^{x,y} \leq \kappa_r^x$. Also recall the fact that for x and y ordinals, $\kappa_r^x \leq \kappa_r^{x,y}$. Suppose for a contradiction that

$$\kappa_{r,\delta}^{\omega_1} < \kappa_r^{G,\omega_1} < \kappa_r^{\delta+1}$$

Letting $\delta = 0$ for simplicity we have,

$$\kappa_{r,0}^{\omega_1} < \kappa_r^{G,\omega_1} < \kappa_{r,1}^{\omega_1}$$

We've shown that for η a limit ordinal and $\delta < \eta$ the statement "$\delta = \kappa_r^{\omega_1}$" is

Δ_0 definable in $L(\eta)$. Now,

$$L(\kappa_r^{G,\omega_1}) \models \exists\delta\, \delta = \kappa_r^{\omega_1}$$

So in fact

$$L(\kappa_0^{G,\omega_1}) \models \exists\delta\, \delta = \kappa_r^{\omega_1}$$

So, $\kappa_r^{\omega_1} < \kappa_0^{G,\omega_1}$. $\exists\eta$ E-recursive in G, ω_1 such that $\kappa_r^{\omega_1} < \eta$, and we can compute $\kappa_r^{\omega_1}$ from η by searching through η for a δ such that $\delta = \kappa_r^{\omega_1}$. So $\kappa_r^{\omega_1}$ is E-recursive in η

Thus, $\kappa_r^{\omega_1}$ is E-recursive in G and ω_1. Now, $\kappa_r^{\kappa_{r,0}^{\omega_1}} \leq \kappa_r^{G,\omega_1}$ But $\kappa_r^{\kappa_{r,0}^{\omega_1}} = \kappa_{r,1}^{\omega_1}$ Contradiction.

♦

Definition 10: I is a co-re substructure if and only if

1. I is closed under E.

2. Given $a, b \in I$

$$[\exists x \in b\, \{e\}(x,a) \uparrow] \Longrightarrow [\exists x \in b \cap I\ \{e\}(x,a) \uparrow].$$

A co-re substructure is not necessarily transitive.

3.3 Genericity in E-Closed Structures

Let $E(\rho) = L(\kappa)$ be inadmissible. Let ρ be regular in the sense of $E(\rho)$. By [Normann], we have $< \rho$ selection in $E(\rho)$. The aim is to extend the result of [Sacks] as follows:

Theorem 14: Let $P \in L(\kappa)$ be a locally proper partial order and let G be P-generic. Then $L(\kappa, G)$ is E-closed.

We first begin with a few preliminaries. In this section we use $E(\rho) = L(\kappa)$ interchangeably. ρ is suppressed as a parameter.

Theorem 15: Let I be a successor co-re substructure. Let P be a partial order.

The following three definitions are equivalent.

1. q is generic over I. That is, $\forall a \in I \; q \Vdash |a| = \delta$ or $q \Vdash a$ has a Moschovakis Witness first order definable over $I[G]$.

2. $q \Vdash I[G]$ is a co-re substructure of $L(\kappa, G) \wedge I \cap ORD = I[G] \cap ORD$.

3. $q \Vdash \sup I \in \kappa_r^G$ spectrum and $I \bigcap ORD = I[G] \bigcap ORD$.

Proof: Note: We are letting $I = I_0$ for clarity of exposition.

$2 \Longrightarrow 1$. Let $q \in G$. We are given $a \in I$ which is a name for some computation $a = \{e\}(b, \dot{G})$ for $b \in I$. In $I[G]$, consider the computation $\{e\}(b, G)$ where $b \in I[G]$ and $G \in I[G]$. If $\{e\}(b, G) \downarrow$ then $|\{e\}(b, G)| \in I[G] \bigcap ORD$. Note that $I[G]$ is E-closed easily. If $\{e\}(b, G) \uparrow$ then by hypothesis there exists a Moschovakis Witness first order definable over $I[G]$.

$1. \Longrightarrow 3$. Let $q \in G$. Then we need to show that $\kappa_r = \kappa_r^G$. Suppose not for a contradiction; then $\kappa_r < \kappa_r^G$. By a reflection argument as used previously, κ_r is E-recursive in ω_1 and G. So, $\kappa_r = \{e\}(\omega_1, G)$, where there exists a name $a \in I$ for $\{e\}(\omega_1, G)$. Now $a^G \downarrow = \kappa_r$ but $\kappa_r \notin I \bigcap ORD$, contradiction.

$3. \Longrightarrow 2$. In the equation below, we are replacing arbitrary elements of $I[G]$ with $x \in \omega_1$

$$L(\kappa, G) \models \exists x < \omega_1 \; \{e\}(G, x) \uparrow$$

By an application of the Kechris Basis Theorem in $L(\kappa, G)$,

$$\exists x \ \kappa_r^x \leq \kappa_r^{\omega_1, G, x} \leq \kappa_r^{\omega_1, G} = \kappa_r$$

As x preserves the $\kappa_r - \mathit{Spectrum}$, $x \in I_0$

♦

For a limit co-re substructure I_λ where λ is a limit ordinal, we have the following definition of generic:

Definition 11: q is I_λ generic $\iff$ $\forall$ dense sets D $\Sigma_1 \vee \Pi_1$ definable over I_λ

$$D \cap I_\lambda \text{ is predense below } p.$$

Lemma 5: The property of being a co-re substructure is E-recursive.

Proof: $\forall e \in \omega \ \forall x \in I$ If $\{e\}(x) \downarrow$ then $T(e, x)$ is E-recursive in e and x so $T(e, x) \in I$. If $\{e\}(x) \uparrow$ then a Moschovakis witness for $\{e\}(x) \uparrow$ is first order definable over I.

Claim 22: There is an index e such that

$$\forall \ I\{e\}(I) \downarrow = \{ \ t \mid t \subseteq I, t \text{ is first order definable over I} \ \}$$

The proof of this fact is analogous to the proof that the construction of L is an E-recursive procedure. The procedure for seeing that I is a co-re substructure of $L(\kappa)$ is as follows: Given some I, compute the set described above. Now compute

$$\bigcup_{x \in I, e \in \omega} F(e, x) = G(I)$$

where the E-recursive function $F : \omega \times I \longrightarrow \{0,1\}$ is defined by

$$F(e,x) = \begin{cases} 0, & \text{if there is a Moschovakis Witness for } \{e\}(x)\uparrow \\ & \text{or a tree } T(e,x) \in I; \\ 1, & \text{otherwise.} \end{cases}$$

If we find a Moschovakis Witness then check that the co-re property holds. If we find a tree then check that $\{e\}(x) = z$ for $z \in I$ Note then that $G(I) = 0$ if I is a co-re substructure and 1 otherwise.

♦

Claim 23: $\gamma < \delta < \rho \Longrightarrow \kappa_r^{\gamma,\delta} = \kappa_r^{\delta}$.

Proof: Suppose that $\kappa_r^{\gamma,\delta} \neq \kappa_r^{\delta}$, then let $C = \{ \mu < \delta \mid \kappa_r^{\mu,\delta} > \kappa_r^{\delta} \}$. By Normann selection, there is $\mu \in C$ such that μ is E-recursive in δ. Then, $\kappa_0^{\mu,\delta} = \kappa_0^{\delta} \leq \kappa_r^{\delta}$, so $\kappa_r^{\mu,\delta} = \kappa_r^{\delta}$. Contradiction.

♦

Hence, Normann selection implies that the κ_r functions are "well behaved" on ρ.

We now define an increasing sequence of elements of ρ. In fact, we are measuring the increase of κ_r^{γ} with γ.

Let $\eta_0 = \{ \delta \mid \kappa_r^{\delta} = \kappa_r \}$. η_0 is an initial segment of ρ by the above claim. Notice that $\eta_0 < \rho$, because for some δ $\kappa_0^{\delta} > \kappa_r$. Now as $E(\rho)$ is inadmissible, $\kappa_r^{\eta_0} < \kappa^{\rho}$.

We can define

$$\eta_1 = \{ \delta \mid \kappa_r^{\delta} \leq \kappa_r^{\eta_0} \} = \{ \delta \mid \kappa_r^{\delta} = \kappa_r \text{ or } \kappa_r^{\delta} = \kappa_r^{\eta_0} \} = \{ \delta \mid \kappa_r^{\delta,\eta_0} = \kappa_r^{\eta_0} \}.$$

Similarly we can define

$$\eta_{n+1} = \{ \delta \mid \kappa_r^\delta \leq \kappa_r^{\eta_n} \},$$

and

$$\eta_\lambda = \bigcup_{\nu<\lambda} \eta_\nu$$

for λ limit.

We define η_ν for as long as $\eta_\nu < \kappa$. Eventually we reach limit λ with $\eta_\lambda = \kappa$ and we stop. The length of construction depends on the Σ_1 cofinality of ρ.

Definition 12: Let X be a set.

$$E - \text{hull}(X) = \{ \{e\}(\vec{a}) \mid \vec{a} \in X, \{e\}(\vec{a}) \downarrow \}.$$

We are now in a position to define an important class of structures.

Definition 13: $H_\delta = E - \text{Hull}(\eta_\delta \bigcup \{\rho\})$.

Notice that for λ limit.

$$H_\lambda = \bigcup_{\delta<\lambda} H_\delta.$$

The H_δ are the canonical examples of co-re substructures.

Claim 24: $H_{\delta+1}$ is a co-re substructure.

Proof: By construction, $H_{\delta+1}$ is closed under E. Assume $\exists\, x \in b\ \{e\}(x, a) \uparrow$. Let $a, b \in H_{\delta+1}$. We know $a \leq_E \lambda, \rho$ and $b \leq_E \mu, \rho$ where $\mu, \lambda \in \eta_{\delta+1}$. By the definition of $\eta_{\delta+1}$ we know that $\kappa_r^\lambda, \kappa_r^\mu \leq \kappa_r^{\eta_\delta}$. Also, there exists f which is an injection from b into ρ. Then use the Kechris Basis Theorem to find a $\eta < \rho$ such that $\{e\}(f^{-1}(\eta), a) \uparrow$ and $\kappa_r^{\eta,\lambda,\mu} \leq \kappa_r^{\lambda,\mu}$.

♦

It is now easy to see H_λ is a co-re substructure for λ limit.

Also observe that $\lambda, \mu \in \eta_0 \Longrightarrow \kappa_r^{\lambda,\mu} = \kappa_r$. To see this, we may assume without loss of generality that $\lambda < \mu$. By the above claim, $\kappa_r^{\lambda,\mu} = \kappa_r^{\mu} \leq \kappa_r$.

We now develop the properties of H_0, a typical successor co-re substructure.

Claim 25: $H_0 \bigcap \rho = \eta_0$.

Proof: It is clear that $\eta_0 \subseteq H_0 \bigcap \rho$. For the other direction, let $\alpha < \rho$, where $\alpha = \{e\}(\eta, \rho)$ for some $\eta < \eta_0$. Then

$$\kappa_0^\alpha \leq \kappa_0^\eta \leq \kappa_r^\eta = \kappa_r.$$

Therefore $\kappa_r^\alpha \leq \kappa_r$, so $\alpha < \eta_0$.

♦

Claim 26: $\bigcup(H_0 \bigcap ORD) = \kappa_r$.

Proof: First let us show that $H_0 \cap On \subseteq \kappa_r$. Let $\alpha = \{e\}(\eta, \rho)$ for some $\eta \in \eta_0$. Then

$$\alpha < \kappa_0^\eta \leq \kappa_r^\eta = \kappa_r.$$

For the other direction, a close inspection of Harrington's theorem on Moschovakis Witnesses shows that if $\{e\}(\rho) \uparrow$ then there exists a Moschovakis Witness $\lambda n \ \langle e(n), \lambda(n)\rangle$ where $\langle e(n), \lambda(n)\rangle \in H_0$ and $\lambda n \ \langle e(n), \lambda(n)\rangle$ is first order definable over H_0. By a result in [8], this implies that $H_0 \bigcap On$ is is unbounded in κ_r.

♦

Claim 27: $H_0 = \{\ x \in L(\kappa_r) \mid \kappa_r^x = \kappa_r\ \}$.

Proof: First we show $H_0 \subsetneq \{ x \in L(\kappa_r) \mid \kappa_r^x = \kappa_r \}$. Let $x \in H_0$, where $x = \{e\}(\rho, \delta)$ and $\kappa_r^{\delta,\rho} = \kappa_r^\rho$. Now, $x \in L(|\{e\}(\rho, \delta)| + 1)$ and

$$|\{e\}(\rho, \delta)| + 1 < \kappa_0^{\rho,\delta} \leq \kappa_r^{\rho,\delta} = \kappa_r^\rho.$$

So $x \in L(\kappa_r^\rho)$ and $x \leq_E \rho, \delta$. Hence

$$\kappa_0^{x,\rho} \leq \kappa_0^{\rho,\delta} \leq \kappa_r^{\delta,\rho} = \kappa_r^\rho.$$

So $\kappa_r^{x,\rho} \leq \kappa_r^\rho$ and therefore $\kappa_r^{x,\rho} = \kappa_r^\rho$.

♦

Definition 14: A partial order P is locally proper iff

$$\forall p \in P \, \forall a \in E(\rho) \, \exists \delta \;\; p, a \in H_{\delta+1} \text{ and } \exists q \leq p \; (q \text{ is } (H_{\delta+1}, P)\text{-generic.})$$

Definition 15: q is $(H_{\delta+1}, P)$-generic if and only if for all D dense in P and definable by an (r.e. $\vee$ co-re) formula from parameters in $H_{\delta+1}$ we have $D \bigcap H_{\delta+1}$ predense below q.

3.4 Local Properness

We will now begin the proof of the theorem stated at the beginning of the chapter. We assume that $P \in H_0$. We will suppress the parameter P in the discussion below. We introduce the concepts of *low* and *high*.

Definition 16: $p \Vdash t \; low \iff \exists \, \gamma < \kappa_r \; p \Vdash |t| = \gamma$.

Definition 17: $p \Vdash^* t \; high \iff \forall \, q \leq p \; \neg q \Vdash t \; low$.

Claim 28: $p \Vdash t \; low \Longrightarrow p \Vdash |t| = \gamma$ where $\gamma \leq_E p, t, P$.

Proof: The relation $p \Vdash |t| = \gamma$ is recursive. Looking at our hypothesis we have, $p \Vdash |t| = \gamma < \kappa_r \le \kappa_r^{p,t}$. So $L(\kappa_r^{p,t})$ sees a convergent computation witnessing the truth of this $\Sigma_1(p,t)$ fact. By reflection, $L(\kappa_0^{p,t})$ sees such a computation, that is $\gamma < \delta \le_E p, t$. Now we can search through δ for a γ such that $p \Vdash |t| = \gamma$ and so $\gamma \le_E p, t$.

♦

Corollary 2: "$p \Vdash$ t *low*" is r.e. in parameter $P \in H_0$.

Proof: The relation "$\gamma < \kappa_r$" is recursive. By Gandy selection we can find an index n_0 such that

1. $\{n_0\}(p,t,P) \downarrow \iff p \Vdash |t| = \gamma$ for some γ where $\gamma < \kappa_r$ and $\gamma \le_E p, t, P$.
2. If $p \Vdash |t| = \gamma$ and $\gamma \le_E p, t, P$ where $\gamma < \kappa_r$ then $\{n_0\}(p,t,P) = \gamma$.

♦

Claim 29: "$p \Vdash^* t$ *high*" is co-re in $P \in H_0$

Proof: We show the negation is r.e.

$$\neg p \Vdash^* t \textit{ high} \iff \exists q \le p \; q \Vdash t \textit{ low}.$$

We will show that if such a q exists, then some such q is E-recursive in p, t and P. Let $q \le p$ and $q \Vdash t = \gamma$ where $\gamma < \kappa_r$. As in Claim 1, $L(\kappa_r^{p,t})$ sees a computation witnessing the $\Sigma_1(p,t)$ assertion that such a q exists. So by reflection again there exists $q \in L(\kappa_0^{p,t})$ such that $q \le p$ and $q \Vdash t$ *low*. That is, $q \Vdash |t| = \gamma$ where we now know that $q, \gamma < \kappa_0^{p,t}$. Now we can find a $\delta \leq_E p, t$ such that $q, \gamma < \delta$ and search through $\delta \times \delta$ for a (q, γ) such that $q \Vdash |t| = \gamma$ and $\gamma < \kappa_r$. This shows

$$\exists\, q \leq p\; q \Vdash t\; low \Longrightarrow \exists\, q \leq p\;\; (\, q \leq_E p, t, P \text{ and } q \Vdash t\; low).$$

Now by Gandy selection we can find an index n_1 such that

1. $\{n_1\}(p, t, P) \downarrow \Longleftrightarrow\; \exists q \leq p\; q \Vdash t\; low.$

2. If $\exists q \leq p\; q \Vdash t\; low$ then $\{n_1\}(p, t, P) = q$ for some such q.

♦

Claim 30: $\forall\; a, e\; (\{e\}(a, G) \uparrow$ or $|\{e\}(a, G)| < \kappa) \iff L(\kappa, G)$ is $E-$ closed.

Proof: For the forward direction, let $x_i \in L(\kappa, G)$ then $x_i = \{e_i\}(a_i, G)$. If $\{f\}(\vec{x}) \downarrow$ then $\{f\}(\{e_1\}(a_1, G) \ldots \{e_n\}(a_n, G)) \downarrow$ in less than κ steps. So it will converge to an element of $L(\kappa, G)$.

Conversely, suppose $L(\kappa, G)$ is E-closed and $|\{e\}(a, G)| \geq \kappa$. Now

$$|\{e\}(a, G)| \leq_E a, G \in L(\kappa, G).$$

But $|\{e\}(a, G)| \in L(\kappa, G)$ as $L(\kappa, G)$ E-closed. Contradiction.

♦

Hence, to prove Theorem 1, it is sufficient to show

$$(*) \qquad \forall\, a, e\; (\{e\}(a, G) \uparrow \text{ or } |\{e\}(a, G)| < \kappa).$$

holds.

Fix $t \equiv \{e\}(a, \dot{G})$. It is enough to show the set

$$A = \{ \, r \mid r \Vdash |t| < \kappa \text{ or } r \Vdash^{*} t \uparrow \, \}$$

is dense. Fix p. We know by definition of local properness that $p, t \in H_{\delta+1}$ for some δ and there is $q \leq p$ where q is $(H_{\delta+1}, P)$-generic. For simplicity let $\delta + 1 = 0$.

If there is $r \leq p$ such that $r \Vdash t$ *low*, then $r \Vdash |t| < \kappa$ and so $r \in A$. Otherwise, $p \Vdash^{*} t$ *high*. We will show $q \Vdash^{*} t \uparrow$ by building a tree and showing that for any G P-generic and $q \in G$, we can find a Moschovakis Witness to the divergence of t. We want to build the tree to have top node $\langle p, t\rangle$ and for for each node a set of daughters such that given an arbitrary node $\langle r, u\rangle$ where $u = \{2^m, 3^n\}(a, G)$ and $u_0 = \{m\}(a, G)$

1. $r \in P \bigcap H$.

2. u is a term in H.

3. $r \Vdash^{*} u$ *high*.

4. $r \parallel q$.

These four conditions keep the construction going.

The daughters (immediate successors) of $\langle r, u\rangle$ are a set of nodes $\langle s, v\rangle$ such that

1. $s \leq r$.

2. $s \Vdash \dot{v}$ is an immediate subcomputation of $\dot{u}$.

3. The set $\{ \, s \mid \exists v \langle s, v\rangle$ is a daughter of $\langle r, u\rangle \, \}$ is predense below $q \wedge r$.

Lemma 6: Suppose we have built a tree with the properties above. Suppose $q \in G$. Then we can read off a Moschovakis Witness that $t^G \uparrow$ from the tree.

Proof: As $q \leq p$, $p \in G$. We claim that for some daughter node (p_1, u_1), we have $p_1 \in G$. This is true because the set of such p_1 is predense below $p \wedge q$ and $p \wedge q \in G$. Now $p_1 \Vdash u_1$ is a subcomputation of u, where $p_1 \in G$. So u_1^G really is a subcomputation of u_0^G. Repeating, we can build a sequence $\lambda n\ u_n^G$ which is a Moschovakis witness that $t^G \uparrow$.

◆

Define

$$D = \{ s \leq r \mid s \Vdash u_0\ \mathit{low} \text{ or } s \Vdash^* \ u_0\ \mathit{high} \}.$$

D is r.e. $\vee$ co-re in parameters from H.

Lemma 7: Suppose D is dense and r.e. $\vee$ co-re in parameters from H, $p \parallel q$ and q is (H, P)-generic. Then $D \cap H$ is predense below $p \wedge q$

Proof: Let

$$D^* = \{ r \mid r \in D \text{ or } r \text{ incompatible with } p \}.$$

D^* is dense. To see this let $x \in P$. If x is incompatible with p, then $x \in D^*$. If not, $\exists\, y \leq x, p$. As $y \leq p$ and D is dense below p, $\exists z \ \leq y\ z \in D$.

D^* is r.e. $\vee$ co-re in parameters from H. So $D^* \cap H$ is predense below q.

Let $w \leq p \wedge q$. Then there is $y \in D^* \cap H$ such that $w \ \parallel\ y$ as $w \leq q$, $D^* \cap H$ is predense below q. $p \| y$ because $w \leq p$, so $y \in D$.

◆

By the above lemma, $D \cap H$ is predense below $r \wedge q$. Let

$$E = \{ s \in D \bigcap H \mid s \parallel q \}.$$

then E is predense below $r \wedge q$.

Let

$$E_H = \{ s \in E \mid s \Vdash^* u_0\ \mathit{high} \},$$

and

$$E_L = \{\ s\ inE \mid s \Vdash^* u_0\ low\ \}.$$

$E_H \cup E_L$ is predense below $r \wedge q$. For each $s \in E_H$, make $\langle s, u_0 \rangle$ a daughter of $\langle r, u \rangle$ where

1. $s, u_0 \in H_0$
2. $s \Vdash^* u_0\ high$.
3. $s \parallel q$.
4. $s \leq r$.
5. $s \Vdash u_0$ an immediate subcomputation of u.

In the case of $s \in E_L$, the situation is harder. We will throw in a lot of daughter nodes. (So many that the set of first coordinates is predense below $q \wedge s$.)

Claim 31: Under the above assumptions, the set of first coordinates of daughters from E_L and E_H is predense below $q \wedge r$.

Proof: Let $x \leq q \wedge r$. E is predense below $q \wedge r$, so $\exists\ y \in E\ x||y$. If $y \in E_H$ then y is the first coordinate of a daughter node. If $y \in E_L$, let $z \leq x, y$. $z \leq x \leq q$ so $z \leq y, q$ so $z \leq y \wedge q$. The set of first coordinates of daughters added when we considered y is predense below $y \wedge q$ so there exists a daughter $\langle s, r \rangle$ such that $z \parallel s$, so $x \parallel s$.

♦

Given this fact, we need to find a set of daughter nodes whose first coordinates are predense below $s \wedge q$, for each $s \in E_L$. We know the following facts.

1. $s \in H, u_0 \in H_0$.

2. $s \Vdash u_0$ *low.*

3. $s \Vdash |u_0| = \gamma$ where $\gamma \leq_E s, u_0$ and $\gamma \in H_0$.

Let $M = T(s, u_0, \gamma)$ be the set of terms for potential members of u_0. Let F be the dense set of conditions r_1 where $r_1 \leq s$ and either

1. $\exists\, x \in M(r_1 \Vdash x \in u_0$ and $r_1 \Vdash^* \{n\}(x)$ *high*$)$ or

2. $\forall\, x \in M\ \forall w \leq r_1\ (w \Vdash x \in u_0 \Longrightarrow \exists\, z \leq w\ z \Vdash \{n\}(x)$ *low*$)$.

Lemma 8: F is r.e. $\vee$ co-re in parameters from H.

Proof: Let $\pi(r_1)$ be an abbreviation for equation (2) above. We claim that $\pi(r_1)$ is re. By a reflection argument exactly as in Claim 1,

$$\exists\, z \leq w\ z \Vdash \{n\}(x)\ low \Longrightarrow \exists\, z \leq_E w, x, P.$$

So by Gandy selection we can find an e_0 such that

1. $\exists\, z \leq w\ z \Vdash \{n\}(x) low \iff \{e_0\}(w, x)\downarrow$.

2. $\exists\, z \leq w\ a \Vdash \{n\}(x) low \Longrightarrow \{e_0\}(w, x) \downarrow = z$ for some such z.

Now define e_1 such that $\{e_1\}(r_1)$ first computes

$$Z = \{\ (w, x) \in P \times M \mid w \Vdash x \in u_0, w \leq r_1\ \},$$

and then computes

$$\bigcup_{(w,x)\in Z} \{e_0\}(w, x).$$

It is clear that

$$\{e_1\}(r_1)\downarrow \iff \pi(r_1).$$

Let the formula $\sigma(r_1)$ abbreviate equation (1) above. We claim that $\sigma(r_1)$ is co-re in parameters from H.

$$\neg\sigma(r_1) \iff \forall\, x \in M\ (\neg r_1 \Vdash x \in u_0 \text{ or } \exists\, w \le r_1\ u \Vdash \{n\}(x)\ low).$$

The argument is essentially the same as in the previous case.

♦

F is dense and re $\vee$ co-re in parameters from H and hence $F \bigcap H$ is predense below $s \bigcap q$.

Claim 32: If $r_1 \in F \bigcap H$, $r \,||\, q$, then r_1 gets into F by way of clause 1 of the definition of F.

Proof: Let $r_1 \in F \cap H$, and $\sigma(r_1)$. By argument that $\sigma(r_1)$ is co-re , we can find $\eta \le_E \nu, P$ where $\eta \in H$ such that

$$\forall x \in M\ \forall w \le q\ (w \Vdash x \in u_0 \Longrightarrow \exists z\ \le w\ z \Vdash^* \{n\}(x) \le \eta).$$

But now we claim $r_1 \Vdash^* t\ low$. (This is a contradiction as $q \Vdash^* u\ high$) Well, $r_1 \le s$ and $s \Vdash |u_0| = \gamma$. By a density argument r_1 forces all subcomputations of the form $\{n\}(b)$, $b \in u_0$ to converge in less than η steps. Then

$$r_1 \Vdash |t| \le sup(\gamma, \eta)$$

where $\gamma, \eta \in H$.

♦

From this claim, we are done by making $\langle r_1, \{n\}(x)\rangle$ a daughter of $\langle r, u\rangle$ where x is a witness to Clause 1, for each $r_1 \in F \bigcap H$ s.t. $r_1 || q$.

3.5 Is Local Properness Necessary?

Sacks showed (see [8]) that if $E(\rho)$ is an inadmissible model of $< \rho$-selection and $\mathbb{P} \in E(\rho)$ satisfies the ρ-chain condition, then forcing with $\mathbb{P}$ preserves E-closure. The next logical question to address is that of whether, in general, forcing notions that preserve E-closure must be locally proper.

In [10] Sacks has addressed this question. He formulates a weaker notion of local properness for $E(\omega_1)$, in which it is only demanded that a "generic" condition should handle those r.e.$\vee$co-r.e. dense sets which are used in the proof that E-closure is preserved. He then shows that this form of local properness is actually a necessary and sufficient condition for the preservation of the E-closure of $E(\omega_1)$.

Sacks' work can be generalized to the context of $E(\rho)$, where $E(\rho)$ believes that the greatest cardinal is regular. This will be discussed in the author's forthcoming paper [6].

3.6 Properness

Definition 18: P is a proper partial order if and only if there exists a club C of co-re substructures, $C \leq_E L(\kappa)$ such that

$$\forall\, J \in C, \forall p \in P \cap J\ \exists q\ q \leq p\ q \text{ generic over } J$$

Recall the definition of "p is $J-$generic" where J is a limit co-re substructure.

Definition 19: p is J-generic if and only iff for all dense sets D $\Sigma_1 \vee \Pi_1$ definable in parameters over J then $D \cap I$ is predense below p.

Theorem 16: Let P be a proper partial order where $P \in L(\kappa)$, and $L(\kappa)$ is E-closed and inadmissible. Let S be a recursive-on-$L(\kappa)$ set of countable co-re substructures, stationary in the following sense:

$S \cap C \neq \emptyset$ for every C where C is a recursive-on-$L(\kappa)$ club of countable co-re substructures.

Then P preserves the stationarity of S in the sense described above.

Proof: Let D be a new club recursive on $L(\kappa, G)$. It is sufficient to show that

$$D \cap S \neq \emptyset.$$

For a contradiction, suppose that

$$\emptyset \Vdash D \text{ is a club and } D \cap S = \emptyset$$

We want to build a club of countable co-re substructures J with the property

$$(*) \quad \forall a \in J \cap P\, \forall r \in J\, \exists s \in J \cap P\, \exists k \in J\, (s \leq r,\ a \subseteq k \text{ and } s \Vdash k \in D).$$

The point of $(*)$ is that if q is generic for J which has $(*)$ and $J \in S$, then $q \Vdash J \in S \cap D$. We shall see this later.

We first want to prove that there exists a recursive club of structures J with $(*)$ [so that there exists a $J \in S$ with $(*)$]. To prove closure, let $J_0 \subseteq J_1 \subseteq \ldots J_n \ldots.$ where each J_n has $(*)$, then $\bigcup J_n$ has property $(*)$.

To prove unboundedness, we use

$$\emptyset \Vdash D \text{ is unbounded.}$$

Given J an arbitrary countable object, we want to show that sets with property $(*)$ are unbounded by finding an expansion of J with $(*)$. Given $J = J_0$, we define J_1 as follows: For each $a \in J$, $r \in J \cap P$,

$$r \Vdash \exists\, k \in D\ a \subseteq k,$$

as D is forced to be unbounded. So there exist $s \leq r$ and $k \in D$ such that

$$s \Vdash a \subseteq k.$$

We build J_1 by adding in appropriate s, k for all $a, r \in J_0$. Repeat the process to get $J_2, J_3, \ldots$ and $J_\omega = \bigcup_n J_n$. Then J_ω has $(*)$ and $J \subseteq J_\omega$. We have just proved that countable sets with $(*)$ form a recursive club; we then intersect them with the recursive club of co-re substructures to see that there is a recursive club of co-re substructures with $(*)$. We have proved that we can find a $J \in S$ with $(*)$.

To complete the proof, we need to show the following:

$$\text{If } q \text{ is generic over } J \text{ then } q \Vdash J \in D.$$

Define the dense set

$$D_k = \{\, s \mid \exists\, L \in J\ k \subseteq L\ s \Vdash L \in D \text{ or } s \Vdash \forall\, L \in J \neg(k \subseteq L \text{ and } L \in D) \,\}$$

Now D_k is $\Sigma_1 \vee \Pi_1$ definable in parameters over J.

Elements of $D_k \cap J$ only get in via the positive clause of D_k. To see this, let $s \in D_k \cap J$. Now $k \in J$ and $s \in J \cap P$, so for some $r \leq s$ and some $L \supseteq k$ we have $r, L \in J$ and that $s \Vdash L \in D$. This cannot be if s obeys the negative clause.

We build a tree to show that

$$q \Vdash^* J \in D.$$

The proof is exactly analogous to the Moschovakis Witness construction proof showing that local properness implies the preservation of E-closure. Fix an enumeration of J as $\langle a_0, a_1, \ldots \rangle$. Elements of the tree are pairs (p, k) where

1. $p \in J \cap P$.

2. $k \in J$.

3. $p \Vdash k \in D$.

4. $p||q$.

5. If (p, k) is a node on level m, then the first entries of its daughter nodes are all in $D_{k\cup\{a_{m+1}\}} \cap J$ and the set of these first entries is predense below $p \wedge q$.

6. If (p, k) is on level m, $\{a_0 \dots a_m\} \subseteq k$.

How do we build the daughters of (p, k) on level m? Fix $r \leq p, q$. $k \cup \{a_{m+1}\} \in J$, so $D_{k\cup\{a_{m+1}\}} \cap J$ is predense below r. Pick out elements comparable to q all obeying the positive definition of $D_{k\cup\{a_{m+1}\}}$ together with the witnesses that they are in $D_{k\cup\{a_{m+1}\}}$ Just as in the "local properness implies preservation of E-closure" proof, we can show that if $q \in G$ then G picks out a branch of the tree which will write J as a union of a ω-sequence of sets in D. That is, $q \Vdash J \in D$.

♦

A natural question at this stage would be the question of whether locally proper implies proper. The answer is no. The motivation for the solution is from Baumgartner,Harrington, and Kleinbergs' [1] forcing for killing a stationary subset of ω_1 while preserving ω_1. Sacks and the author have constructed in $E(\omega_1)$ forcing which preserves E-closure while destroying the stationarity of some subset of ω_1: this forcing is an example of some forcing which is locally proper, but not proper. See [10] for details of the construction. There now arises two questions arising from an analogy of Shelah's work in ZFC. Shelah showed that in the ZFC context, there is an iteration theory for proper forcing and that properness is equivalent to preservation of all stationary sets of

$\mathcal{P}_{\omega_1}(X)$. At the moment, the effective version is not so clear. However, we have partial results worth mentioning. Below is a sketch of a proof with a gap which shows that preservation of stationary sets is equivalent to proper. Given this result, then the finite iteration theory for E-closed structures is complete.

Theorem 17: If P preserves stationary sets, then P is proper.

Proof: Suppose not. Let P preserve stationary sets, but be improper. Define

$$S = \{ I \mid \exists p \in P \cap I \, \forall \, q \leq p \; q \text{ is not } I - generic. \}$$

By Fodor's lemma, there is a stationary set $T \subseteq S$ and a fixed $p \in P$ such that $\forall I \in T p \in I$ and $\forall q \leq p \; q$ is not $I - generic.$. We force a generic G with $p \in G$. T is stationary in $L(\kappa, G)$. We now define a set C, consisting of those $I \ni p$ such that

$$\forall r \in I \cap G \, \forall a \in I \, \forall D \; \Sigma_1 \vee \Pi_1 \text{ in } a \text{ and dense } \exists s \leq r \; s \in D \cap I \cap G.$$

Note that we need a regularity assumption of ω_1 to know that C is club. Also, it is unclear whether C is a recursive club. Assuming that this is the case then we can find

$$I \in T \cap C$$

Choose $q \leq p$ such that $q \in G$ and $q \Vdash I \in \dot{C}$. The following claim gives a contradiction.

Claim 33: q is I-generic.

Proof: Let D be $\Sigma_1 \vee \Pi_1(a)$ where $a \in I$. Let $r \leq q$. Force a generic H with $r \in H$. Then $q \in H$, so $I \in \dot{C}^H$. Now $p \in I \cap H$, as $I \in \dot{C}^H$, we can find $s \leq p$ such that $s \in D \cap I \cap H$. As $r, s \in H$, $r \parallel s$. We have shown that $D \cap I$ is predense below q.

♦

3.7 Finite Step Iteration

This is a proof with one gap making a more direct attack on the problem of two step iteration of proper forcing.

Theorem 18: Let P be a proper partial order such that $P \in L(\kappa)$, where $L(\kappa)$ is E-closed and inadmissible . Let $\dot{Q}$ be a name for a proper partial ordering in $L(\kappa, G)$. Then $P * \dot{Q}$ is proper.

Proof: Let

$$\emptyset \Vdash \dot{D} \text{ is a recursive-on-}L(\kappa, G) \text{ club of countable co-re substructures,}$$

and

$$\emptyset \Vdash \dot{Q} \text{ is proper as witnessed by } \dot{D}.$$

We want to find a club $E \subseteq C$ of structures such that

$$\forall J \in E\ \forall (p, \dot{q}) \in P * \dot{Q} \cap J\ \exists (r, \dot{s}) \leq (p, \dot{q})\ (r, \dot{s}) \text{ is generic over } J.$$

Let $D = \dot{D}^G$,and let

$$\bar{D} = \{\ H \mid H \text{ is a canonical substructure of } L(\kappa) \text{ such that } H[G] \in D\ \}$$

Claim 34: $\bar{D}$ is club.

Proof: The canonical co-re substructures of $L(\kappa, G)$ are club in the set of all co-re substructures. If I is a canonical co-re substructure of $L(\kappa, G)$, then

$I = H[G]$ for some H a canonical co-re substructure of $L(\kappa)$.

♦

As in Chapter s, we can find a club of J, recursive-on-$L(\kappa)$, such that if q is generic over J then

$$q \Vdash J \in \bar{D}.$$

That is,

$$q \Vdash J[\dot{G}] \in \dot{D}.$$

Let E be the club of those $J \in C$ with the property just described, and suppose that we have $J \in E$ and $(p, \dot{q}) \in P * \dot{Q} \cap J$. Now, $J \in C$ and $p \in J \cap P$ so there exists $r \leq p$ with r generic over J.

By the construction of E, and the fact that $\dot{q} \in J$

$$r \Vdash J[G] \in D \text{ and } \dot{q} \in J[\dot{G}]$$

$\dot{D}$ names a club witnessing properness of $\dot{Q}$, so

$$r \Vdash \exists \dot{s} \leq \dot{q}\ \dot{s} \text{ is generic over } J[G]$$

So there exists $\dot{s}$ such that

$$r \Vdash \dot{s} \leq \dot{q} \text{ and } \dot{s} \text{ is generic over } J[\dot{G}]$$

At this point, the gap in the proof occurs. It remains to show that $(r, \dot{s})$ is generic over J for $P * \dot{Q}$. but this seems to be difficult and Sacks and the author are skeptical that it is actually true.

♦

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