An ℓ-adic Fourier Transform over Local Fields

by

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Submitted to the
Department of Mathematics
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

January 1994

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January 15, 1994

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Acknowledgements

Time has come for me to discharge several long overdue acknowledgements.
In order of appearance: first comes Edoardo Ballico, who was an important point of reference and source of useful suggestions during my years at the Scuola Normale Superiore; he is also the person from whom I heard the word "Grothendieck" for the first time. Back then it sounded like a very mysterious word, and in many ways it still does.

Next, I should thank Fabrizio Catanese, who was the advisor for my undergraduate thesis. He had the not always easy task of dealing with me at a time when my mathematical maturity was no match for my ambition. Hopefully I have reached now a more balanced compromise (but then again...).

Of all the pleasant memories that I keep of my year in Rome at the Instituto di Alta Matematica, I like to recollect in particular the kindness of Ciro Ciliberto. Little or nothing of what I learnt from him transpires from this thesis, but has become an invaluable part of my background: it is the stuff against which I check my intuition of geometric phenomena.

The period that I spent at M.I.T. has been the most intense in my mathematical education. A great part of the reason is due to the seminars organized by Steven Kleiman: a most efficient, if demanding, way of learning mathematics. To him I am also grateful for all the help and advice that I received in these four years, in matters ranging from mathematics to life in the Institute.

Even greater is my debt to Marc Levine: through (uncountably) many conversations and generous suggestions, he has been, in the last three years, as much an advisor as anybody else. Although my thesis took eventually an unexpected turn, most of my research activity since it got started, is best characterized as joint work with him.

I thank Alexander Beilinson for having agreed to supervise this thesis, for suggesting the problem that originally motivated my work, for a lot of useful criticism and for a wonderful visit to Moscow in summer 1992.

Finally I'd like to thank Vladimir Berkovich for helpful e-mail assistance, Minhyong Kim for pointing out the work of Berkovich, and David Kazhdan for several interesting discussions and intriguing questions, most of which remain unanswered after this thesis.

Abstract

In the thesis I introduce an analogue of the Deligne-Fourier transform for vector spaces defined over a p-adic field. The construction makes use of techniques of rigid analytic geometry recently introduced by Berkovich, in combination with Lubin-Tate theory, reworked in a geometric guise. Most of the formal properties are derived as in Laumon, once the necessary technical preliminaries are dealt with. Numerous open questions arise from this work, and I list some of them in the last chapter of the thesis.

An ℓ-adic Fourier Transform over Local Fields

Lorenzo Ramero

Contents

The lord, who has the Delphi oracle,
does not speak nor conceals,
but gives signs.
- Heraclitus B 93 Diels

1 Introduction

The concept of the Fourier transform appears in many guises in geometry. In the context of intersection theory, it arises as a correspondence on the product of an abelian variety and its dual. For algebraic D-modules, it is given by an explicit formula on the algebra of differential operators of a vector space over a field of characteristic zero. Another example is the Fourier-Sato transform, defined on the category of conical sheaves over a fixed real vector bundle. Here a sheaf is said to be conical if it is invariant under the multiplicative action of $\mathbf{R}^+$ on the fibers of the vector bundle.

The cohomological Fourier transform for vector spaces over finite fields was defined by Deligne and studied extensively by Laumon in [Lau]. This is an anti-involution of the derived category of ℓ-adic sheaves on the vector space, whose construction makes essential use of the peculiar properties of the affine line over finite fields, and therefore it would seem to be hopelessly confined to positive characteristic.

In all known cases, Fourier transforms have yielded valuable insight on the fine structure of the categories in which they were introduced.

A common feature that accounts in part for the success of the Fourier transform (at least for the various sheaf-theoretic versions) is a certain capacity to translate local properties of an object into global properties of its transformed, and viceversa.

To be more specific, consider the case of the Fourier-Deligne transform $\mathcal{F} : \mathbf{D}_c^b(\mathbf{A}^n, \mathbf{Q}_\ell) \to \mathbf{D}_c^b(\mathbf{A}^n, \mathbf{Q}_\ell)$. The full significance of the theory is revealed by the following two facts. First, $\mathcal{F}$ preserves the subcategory of perverse sheaves: this result was obtained by Katz and Laumon and the proof amounts to a careful study of the ramification of the Artin-Schreier map over the point at infinity of $\mathbf{P}^1$. Second, by Deligne's solution of the Weil conjectures, $\mathcal{F}$ preserves the category of mixed complexes. These two facts together make of the Fourier-Deligne transform a powerful tool for the study of the monodromy of sheaves in positive characteristic.

As a demonstration, Laumon succeded in proving the conjecture on the local constant of the functional equation for varieties over finite fields. On the other hand, the Fourier-Deligne transform can be used to simplify significantly the proof of the Weil conjectures. More recently, the functor $\mathcal{F}$ has found important applications in representation theory, as witnessed by the work of Lusztig and Springer, among others.

Compared to the case of finite base field, algebraic geometry over p-adic fields is much less understood. Let V be an algebraic variety over a fixed p-adic field k with absolute Galois group $\mathcal{G}$. According to a folk conjecture, one expects that the ℓ-adic cohomology of V, considered as a $\mathcal{G}$-module, should carry a kind of "mixed Hodge" structure, in analogy with the case of complex algebraic varieties. Then a $\mathcal{G}$-module should be called pure if the monodromy filtration, coming from the action of the inertia subgroup, coincides with the weight filtration, coming from the action of any lifting of a Frobenius generator.

This conjecture is proved when char$k > 0$, because in this case it follows from the Weil conjectures. By contrast, the case char$k = 0$ is still very mysterious, and it is in fact the problem which originated the present thesis.

In this thesis I introduce an operation for sheaves over a vector space on a local field, that behaves in many respects like the Deligne-Fourier transform. The main ingredients for its construction are Lubin-Tate theory and the new etale cohomology theory for analytic varieties found by Berkovich.

Before passing to a more detailed description, I should add that my work does not achieve the goal of proving the conjecture, and probably does not even get anywhere close to it. It is my hope that the interest of this work lies more in the provocative questions that it prompts, than in the answers that it presently provides. In particular, this paper reflects the author's conviction that, even if we are only interested in the cohomology of "classical" algebraic varieties, the new analytic methods of Berkovich are inescapable, and provide the best framework for dealing with these problems.

Now, let k be a p-adic field of zero characteristic. The main obstacle in defining a cohomological Fourier transform over the base field k, is the well known fact that the algebraic fundamental group of $\mathbb{A}^1_k$ is trivial. Starting from here, it is perhaps natural to try to find a richer geometric category, in which the affine line may have non-trivial coverings.

Such a category does indeed exist, and it is embodied by the theory of rigid analytic varieties. This theory was discovered originally by Grothendieck and Tate; more recently Berkovich has found a different fundational approach, that leads basically to the same class of objects, but is more flexible and technically less demanding. I give a summary of Berkovich's theory in the second chapter of this thesis.

The affine line is also an analytic group. In our quest for the Fourier transform, the next question that we have to ask, is whether in this extended geometry the affine line admits non-trivial group coverings, i.e. if there exists an analytic group G with a surjective analytic group homomorphism $G \to \mathbb{A}^1_k$. A torsor over $\mathbb{A}^1_k$ would suffice to define an operator on $\mathbf{D}^b_c(\mathbb{A}^n, \mathbb{Q}_\ell)$, but the stronger property that $G \to \mathbb{A}^1_k$ be a map of groups is necessary if we want to show that the operator satisfies the usual properties, like for instance involutivity (see section 6.1).

It turns out that the most elaborate and complete answer to this question has been given almost thirty years ago in a paper of Lubin and Tate, now classical. In their paper they introduce a whole class of formal groups defined over any p-adic field, and they show how to recover a substantial portion of local class field theory in a very explicit fashion, by analysing the torsion in these groups. This theory is reviewed quickly in the first chapter of the thesis.

Although all the Lubin-Tate formal groups are analytically isomorphic, they contain different arithmetic information, and to pick up arbitrarily one of them would deprive us of this beautiful diversity. Therefore I decided to develop the entire thesis for general Lubin-Tate groups.

Lubin and Tate worked out their theory in a purely algebraic framework. We are left with the easy task of showing that in fact every Lubin-Tate formal group law gives rise to an analytic group G whose underlying analytic variety is a small disc in the plane. To produce the morphism $G \to \mathbb{A}^1_k$, we use the logarithm map, another useful gadget in the Lubin-Tate toolkit. This was originally defined as a formal isomorphism between G and the formal additive group, but we show that in fact it is a surjective etale morphism of analytic varieties. I am not aware of any other arithmetic geometrical problem for whose solution it has been found convenient to "geometrize" Lubin-Tate theory.

Let G_∞ be the kernel of the map $G \to \mathbb{A}^1_k$: now it is not hard to produce a G_∞-torsor out of G, and it seems only fair to call this the *Lubin-Tate torsor*. Chapter 5 gives the construction in full detail. Finally, the construction of the Fourier transform is accomplished by copying word by word from [Lau]. But to make sense of this, we still have to take care of some technical fine point: Berkovich develops his cohomology theory for finite coefficient rings only, and these are not enough for us, since the group G_∞ is p-divisible, and therefore does not possess any non-trivial homomorphism to a finite group. Thus we have to extend somewhat the theorems of Berkovich: this is the purpose of chapter 4. In the last section of the same chapter we also set up an ℓ-adic formalism, following the paper [Ek].

The definition of the Fourier transform is given in chapter 6, together with the proof of its main properties. Some examples of computations follow: these are all translated from the paper of Laumon, with the exception of the computation for the constant sheaf concentrated on a disc in the plane, which has no analogue over a finite field.

We conclude with a list of open questions, for future investigation.

2 Review of Lubin-Tate theory

We recall here some well known fact from Lubin-Tate theory. The paper [LT] is the original source, but a complete account can be found in Lang's book [La].

2.1 Formal groups over local fields

Let k be a local field with valuation $|\cdot|$; denote by $\mathcal{O}_k$ and resp. π the ring of integers of k and a uniformizing parameter in $\mathcal{O}_k$. Let q be the cardinality of the residue field $\tilde{k} = \mathcal{O}_k/m$, where of course $m = (\pi)$ is the maximal ideal. Set $p = \text{char } \tilde{k} > 0$. Let also $\overline{k}$ be the algebraic closure of k, with the unique valuation $|\cdot|$ that extends the valuation of k.

Following Lubin-Tate [LT] we let:

$$\begin{aligned} \mathcal{F}_\pi = \text{set of power series } f \in \mathcal{O}_k[[X]] \text{ such that:} \\ f(X) &\cong \pi X \text{ mod degree } 2 \\ f(X) &\cong X^q \text{ mod } \pi. \end{aligned}$$

The simplest example is just the polynomial $f(X) = \pi X + X^q$. Recall that a formal group F is a power series $F(X,Y) = \sum_{ij} a_{ij} X^i Y^j$ with coefficients $a_{ij} \in k$, satisfying the identities $F(F(X,Y),Z) = F(X,F(Y,Z))$, $F(X,Y) = F(Y,X)$ and $F(X,0) = 0$. A homomorphism of the formal group F into the formal group F' is a power series $f(X) \in k[[X]]$ such that $f(F(X,Y)) = F'(f(X),f(Y))$. In particular an endomorphism of F is a homomorphism of F into itself. We say that a formal group is defined over $\mathcal{O}_k$ if its coefficients a_{ij} are in $\mathcal{O}_k$.

The following theorem summarizes the main features of the Lubin-Tate construction:

Theorem 1 *a) For each $f \in \mathcal{F}_\pi$ there exists a unique formal group F_f, defined over $\mathcal{O}$ such that f is a (formal) endomorphism of F_f. Moreover, for any two power series $f, g \in \mathcal{F}_\pi$ and every $a \in \mathcal{O}_k$ there is a unique $[a]_{f,g} \in \mathcal{O}_k[[X]]$ such that $[a]_{f,g} \in \text{Hom}(F_f, F_g)$ and $[a]_{f,g} \cong aX$ mod degree 2.*

b) The map $a \mapsto [a]_{f,g}$ gives a group homomorphism $\mathcal{O}_k \to \text{Hom}(F_f, F_g)$ satisfying the composition rule

$$[a]_{g,h} \circ [a]_{f,g} = [ab]_{f,h}.$$

In particular, if $f = g$, this map is a ring homomorphism $\mathcal{O}_k \to \text{End}(F_f)$.

Proof: This is theorem 1.2, chapt. 8 of [La]. □

We will write $[a]_f$ in place of $[a]_{f,f}$; in particular notice that $[\pi]_f = f$.

Given $f \in \mathcal{F}_\pi$, the associated formal group F_f converges, as a power series, for all pairs (x,y) of elements of $\overline{k}$ such that $|x|, |y| < 1$. We introduce the notation $\Delta(a,\rho)$ for the set $\{x \in \overline{k} \text{ such that } |x - a| < \rho\}$. Here $a \in \overline{k}$ and ρ is a real number. Then it is clear that F induces a group structure on $\Delta(0,1)$. Any $a \in \mathcal{O}_k$ induces an endomorphism $[a]_f$ of this group.

Definition 1 *For any positive integer n we let $G_n \subset \overline{k}$ be the kernel of the iterated power $[\pi]_f^n$. Also we define $G_\infty = \cup_{n>0} G_n$.*

We collect here some well known results about G_n:

Theorem 2 *1) The action of $\mathcal{O}_k$ on $\Delta(0,1)$ induces an isomorphism of $\mathcal{O}_k-$modules between G_n and the additive group $\mathcal{O}_k/\pi^n\mathcal{O}_k$.*

2) The field $k(G_n)$ is a totally ramified abelian extension k with Galois group isomorphic to $(\mathcal{O}_k/\pi^n\mathcal{O}_k)^\times$.

Proof: See theorem 2.1, chapt. 8 of [La]. □

2.2 The logarithm

We specialize now to characteristic zero, that is $\mathrm{char}(k) = 0$. In this case it is known (see [La], section 8.6) that for any formal group F over k, there exists a formal isomorphism

$$\lambda : F \to \mathbf{G}_a$$

where $\mathbf{G}_a$ is the usual additive formal group over k, that is $\mathbf{G}_a(X,Y) = X+Y$. The isomorphism λ is called the logarithm of F, and it is uniquely determined by F.

Lemma 1 *Let F be a Lubin-Tate formal group, i.e. $F = F_f$ for some $f \in \mathcal{F}_\pi$. Then the logarithm $\lambda = \lambda_F$ can be written in the form:*

$$\lambda(X) = \sum_i g_i(X)\frac{X^{q^i}}{\pi^i}$$

with $g_i(X) \in \mathcal{O}[[X]]$.

Proof: This is lemma 6.3, chapt. 8 of [La]. □

It follows easily from the lemma that λ converges over $\Delta(0,1)$, therefore it induces a group homomorphism

$$\lambda : \Delta(0,1) \longrightarrow \mathbf{G}_a(\overline{k}).$$

The following theorem measures the extent to which λ fails to be an isomorphism of groups:

Theorem 3 *Let $e_F(Z)$ be the power series (with coefficient in k) which is the inverse of $\lambda_F(X)$. Then $e_F(Z)$ converges on the disc $\Delta(0,|\pi|^{1/(q-1)})$ and induces the inverse homomorphism to λ_F on the subgroups*

$$\Delta(0,|\pi|^{1/(q-1)}) \underset{\lambda_F}{\overset{e_F}{\rightleftarrows}} \mathbf{G}_a(0,|\pi|^{1/(q-1)}).$$

(the group on the right coincides set-theoretically with the group on the left, and we use the notation $\mathbf{G}_a$ to emphasize that it is endowed with additive group structure).
Proof: See lemma 6.4, chapt. 8 of [La]. □

Remark: *a*) It can be shown that λ is a homomorphism of $\mathcal{O}_k$-modules, i.e. for all $a \in \mathcal{O}_k$ there is an equality of powers series:

$$a \cdot \lambda = \lambda \circ [a]_f.$$

b) Using theorem 3 and (*a*) it's not hard to show that the kernel of λ is the subgroup G_∞.

In what follows we will reserve the symbol ρ_0 for the constant $|\pi|^{1/(q-1)}$.

3 Rigid analytic geometry

The purely algebraic Lubin-Tate theory is enriched with the adoption of the geometric language of Tate-Berkovich rigid analytic varieties. We give here in compendium the main foundational results of the theory of analytic varieties following Berkovich. Complete details can be found in [B4].

3.1 Banach algebras

Let A be a commutative ring with identity 1. A *seminorm* on A is a function $|| \ || : A \to \mathbf{R}$ with values in the set of non-negative real numbers such that:

1) $||0|| = 0, \quad ||1|| = 1,$
2) $||f - g|| \leq ||f|| + ||g||,$
3) $||fg|| \leq ||f|| \cdot ||g||$

for all elements $f, g \in A$. The seminorm is said to be *multiplicative* if in (3) the equality holds. Two seminorms $|| \ ||$ and $|| \ ||'$ on A are *equivalent* if there exist $C, C' > 0$ such that $C||f|| \leq ||f||' \leq C'||f||$ for all $f \in A$. A seminorm is a *norm* if the equality $||f|| = 0$ holds only for $f = 0$. A normed ring, that is complete with respect to the topology determined by its norm, is called a *Banach ring*. For instance, the base field k is a Banach ring.

A homomorphism $\phi : M \to N$ of Banach rings is said to be *bounded* if there exists $C > 0$ such that $||\phi(f)|| \leq C||f||$ for all $f \in M$. The *residue seminorm* on $M/\mathrm{Ker}(\phi)$ is defined as $|f| = \inf\{||g||, g \in f + \mathrm{Ker}(\phi)\}$. The homomorphism ϕ is *admissible* if the residue seminorm on $M/\mathrm{Ker}(\phi)$ is equivalent to the restriction to $\mathrm{Im}(\phi)$ of the seminorm of N.

Let A be a Banach ring with norm $|| \ ||$. A seminorm $| \ |$ on A is said to be *bounded* if there exists $C > 0$ such that $|f| \leq C||f||$ for all $f \in A$.

Definition 2 *The spectrum $\mathcal{M}(A)$ of a Banach ring A is the set of all bounded multiplicative seminorms on A provided with the weakest topology with respect to which all real valued functions on $\mathcal{M}(A)$ of the form $| \ | \mapsto |f|$, $f \in A$, are continuous.*

It's easy to see that the map $A \mapsto \mathcal{M}(A)$ defines a contravariant functor from the category of Banach algebras with bounded homomorphisms, to the category of topological spaces.

The following theorem is proved in [B4], section 1.2.

Theorem 4 *The spectrum $\mathcal{M}(A)$ is a non-empty, compact Hausdorff space.*

□

Remark: Let x be a point of $\mathcal{M}(A)$ and let $| \ |$ be the corresponding seminorm. The kernel $\mathfrak{p}_x$ of $| \ |$ is a closed prime ideal of A. The value $|f|$ depends only on the residue class of f in $A/\mathfrak{p}_x$. The resulting valuation on the integral domain $A/\mathfrak{p}_x$ extends to a valuation on its fraction field F. The closure of F with respect to the valuation is a valuation field denoted by $\mathcal{H}_x$. The image of an element $f \in A$ in $\mathcal{H}_x$ will be denoted by $f(x)$.

For the purposes of analytic geometry, we single out a special class of Banach rings.

Let A a Banach ring; for $r_1, ..., r_n > 0$, we set:

$$A\{r_1^{-1}T_1, ..., r_n^{-1}T_n\} = \left\{ f = \sum_{\nu=o}^{\infty} a_\nu T^\nu \,|a_\nu \in A \text{ and } |a_\nu| r^\nu \to 0 \text{ as } |\nu| \to 0 \right\}$$

(here $\nu = (\nu_1, ..., \nu_n)$, $|\nu| = \nu_1 + ... + \nu_n$, $T^\nu = T^{\nu_1} ... T^{\nu_n}$ and $r^\nu = r_1^{\nu_1} ... r_n^{\nu_n}$).

This is a Banach A-algebra with respect to the multiplicative norm $||f|| = \max_\nu |a_\nu| r^\nu$. For brevity we will also denote it by $A\{r^{-1}T\}$.

Definition 3 *A Banach k-algebra A is said to be k-affinoid if there exists an admissible epimorphism $k\{r^{-1}T\} \to A$.*

Note that if A is a k-affinoid algebra, then also $A\{r^{-1}T\}$ is k-affinoid.

3.2 Affinoid domains

The k-affinoid algebras provide the local model of a rigid analytic variety, in analogy with the affine schemes of usual algebraic geometry. The underlying topological space for this geometric construction will be the spectrum X of the k-affinoid algebra A.

Example: Let $A = k\{r^{-1}T\}$ be a power series ring in one variable $T = T_1$ as above. We want to describe $\mathcal{M}(A)$ explicitly. Let $\mathbf{C}_k$ be the completion of $\overline{k}$. First of all, any $a \in \mathbf{C}_k$ with $|a| \leq r$ determines a multiplicative seminorm by setting $f \mapsto |f(a)|$ for all $f \in A$. Two elements $a, b \in \mathbf{C}_k$ define the same point in $\mathcal{M}(A)$ if and only if they are conjugate by the Galois group of $\mathbf{C}_k$ over k. In Berkovich's terminology, these are the points of type (1).

Furthermore, given a as above and a real number ρ with $0 < \rho \leq r$, the closed disc $D = \overline{\Delta(a, \rho)}$ determines a multiplicative seminorm $f \mapsto |f|_D = \max_n |a_n| \rho^n$, where $\sum_{n=0}^{\infty} a_n (T-a)^n$ is the expansion of f with center a. This is a point of type (2). Notice that such a point depends only on the disc D and not on its center a.

Berkovich shows that all the points of $\mathcal{M}(A)$ are either of type (1) or type (2).

Definition 4 *Let A be a k-affinoid algebra. A closed subset $V \subset X = \mathcal{M}(A)$ is said to be an affinoid domain in X if there exists an affinoid algebra A_V and a bounded homomorphism of k-affinoid algebras $\phi : A \to A_V$ satisfying the following universal property. Given a bounded homomorphism of affinoid k-algebras $A \to B$ such that the image of $\mathcal{M}(B)$ in X lies in V, there exists a unique bounded homomorphism $A_V \to B$ making the diagram*

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr] & & A_V \arrow[dl] \\ & B & \end{tikzcd}$$

commutative.

Example: Let $f = (f_1, ..., f_n)$ and $g = (g_1, ..., g_m)$ be sets of elements of A, and let $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_m)$ be sets of positive numbers. Then the set

$$X(p^{-1}f, qg^{-1}) = \{x \in X | \, |f_i(x)| \leq p_i, |g_j(x)| \geq q_j, \; 1 \leq i \leq n, \; 1 \leq j \leq m\}$$

is an affinoid domain in X represented by the homomorphism:

$$A \to A\{p_1^{-1}T_1, ..., p_n^{-1}T_n, q_1S_1, ..., q_mS_m\}/(T_i - f_i, g_jS_j - 1).$$

Affinoid domains of the form $X(p^{-1}f)$ (resp. $X(p^{-1}f, qg^{-1})$) are called *Weierstrass* (resp. *Laurent*) *domains* in X.

Various properties of affinoid domains are proved in [B4]. For instance, if $\phi : \mathcal{M}(B) \to \mathcal{M}(A)$ is a map induced by a bounded homomorphism $A \to B$ is of k-affinoid algebras, then for any affinoid domain $V \in \mathcal{M}(A)$ the preimage $\phi^{-1}(V)$ is affinoid in $\mathcal{M}(B)$. Also, the intersection of two affinoid subdomain in $\mathcal{M}(A)$ is again affinoid. Most importantly, for any point $x \in X$, there is a basis of closed neighborhoods of x consisting of affinoid (in fact Laurent) domains.

A finite union of affinoid domain is called a *special* set. For a special set $V = \cup_i V_i$, we set $A_V = \mathrm{Ker}(\Pi_i A_{V_i} \to \Pi_{i,j} A_{V_i \cap V_j})$, where the maps $A_{V_i} \to A_{V_i \cap V_j})$ are as in the initial portion of the Cech complex.

Let X be the spectrum of a k-affinoid algebra A. We endow the space X with a presheaf of rings $\mathcal{O}_X$ as follows. For an open set $U \subset X$, we set

$$\Gamma(U, \mathcal{O}_X) = \varprojlim A_V,$$

where the limit is taken over all special subsets $V \subset U$. Using Tate's acyclicity theorem it can be shown that $\mathcal{O}_X$ is in fact a sheaf of rings and the stalk at any point $x \in X$ is a local ring.

The locally ringed space obtained in this way is called a *k-affinoid space.* A morphism of affinoid spaces $\mathcal{M}(A) \to \mathcal{M}(B)$ is a map of locally ringed spaces which comes from a bounded homomorphism $B \to A$.

A *k-quasiaffinoid space* is a pair $(\mathcal{U}, \phi)$ consisting of a locally ringed space $\mathcal{U}$ and an open immersion ϕ of $\mathcal{U}$ in a k-affinoid space X. A closed subset $V \subset \mathcal{U}$ is called an *affinoid domain* if $\phi(V)$ is an affinoid domain in X. A morphism of k-quasiaffinoid spaces $(\mathcal{U}, \phi) \to (\mathcal{V}, \psi)$ is defined as follows. This is a morphism of locally ringed spaces $\theta : \mathcal{U} \to \mathcal{V}$ such that, for any pair of affinoid domains $U \subset \mathcal{U}$ and $V \subset \mathcal{V}$ with $\theta(U) \subset \mathrm{Int}(V)$ (the topological interior of V in $\mathcal{V}$), the induced homomorphism of k-affinoid algebras $\mathcal{B}_V \to \mathcal{A}_U$ is bounded.

3.3 Analytic varieties

Now let X be a locally ringed space. A *k-analytic atlas* on X is a collection of pairs $(\mathcal{U}_i, \phi_i)$, $i \in I$, called *charts* of the atlas, satisfying the following conditions:

1) Each $\mathcal{U}_i$ is an open subset of X, and the $\mathcal{U}_i$ cover X.

2) Each pair $(\mathcal{U}_i, \phi_i)$ is a k-quasiaffinoid space.

3) The induced morphism of locally ringed spaces $\phi_j\phi_i^{-1} : \phi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \phi_j(\mathcal{U}_i \cap \mathcal{U}_j)$ is an isomorphism of k-quasiaffinoid spaces for each $i, j \in I$.

Definition 5 *A locally ringed space endowed with an atlas is called a k-analytic variety.*

In particular, any quasiaffinoid space is an analytic variety.

Let X, Y be two k-analytic varieties and let $f : X \to Y$ be a morphism of locally ringed spaces. Then f is called a *morphism of k-analytic varieties* if there exists an atlas $\{(\mathcal{U}_i, \phi_i)\}$ of X and an atlas $\{(\mathcal{V}_j, \psi_j)\}$ of Y such that for each i, j the morphism $\psi_j f \phi^{-1} : \phi_i(\mathcal{U}_i) \to \psi_j(\mathcal{V}_j)$ is a morphism of quasiaffinoid spaces.

We will show how to give a structure of analytic varieties to the spaces of main interest for us. First of all, we define the affine line $\mathbb{A}^1_k$ as follows. For any real number $r > 0$, let D_r be the affinoid variety determined by the k-affinoid ring $k\{r^{-1}T\}$. For $r < s$, there is a canonical isomorphism of D_r onto the Weierstrass domain $D_s(r^{-1}T)$. Then $\mathbb{A}^1_k$ is the inductive limit $\cup_{r>0} D_r$. The interior of D_r inside $\mathbb{A}^1_k$ is a quasiaffinoid space that we denote $\Delta(0,r)$. The collection of all $\Delta(0,r)$ will be an atlas for $\mathbb{A}^1_k$.

The set of $\overline{k}$-points of $\Delta(0,r)$ of type (1) is just the set of all $a \in \overline{k}$ such that $|a| < r$ and therefore this notation is consistent with our previous definition of $\Delta(0,r)$. More generally, for any $a \in k$, we can define the open disc $\Delta(a,\rho)$ as the interior of an appropriate affinoid domain inside $\mathbb{A}^1_k$.

Any formal group with coefficients in k, converging over $\Delta(0,\rho)$ gives rise to a *k-analytic group*. In particular, for any $f \in \mathcal{F}$ there exist an analytic Lubin-Tate group F_f. Of course also $\mathbb{G}_a$ with its additive group law is an example of analytic group. More generally, for any real number $\rho > 0$ we denote by $\mathbb{G}_a(\rho)$ the analytic group obtained by restricting the addition law of $\mathbb{G}_a$ to the open disc of radius ρ centered at the origin. Given any $f, g \in \mathcal{F}_\pi$ and $a \in \mathcal{O}_k$, the power series $[a]_{f,g}$ defines an analytic group homomorphism $F_f \to F_g$. Moreover, the logarithm λ_F associated to a Lubin-Tate group F, is an analytic homomorphism $\Delta(0,1) \to \mathbb{G}_a$.

The category of k-analytic varieties admits arbitrary fibre products and for an isometric imbedding of k inside another complete field K, there is a field extension functor $X \mapsto X\widehat{\otimes}K$ from k-analytic varieties to K-analytic varieties, that we will denote sometimes with a subscript. As an example, we can define the affine d-dimensional space $\mathbb{A}^d_k$ as the d-fold product $\mathbb{A}^1_k \times \ldots \times \mathbb{A}^1_k$.

We say that a morphism $X \to Y$ of k-analytic varieties is separated if the diagonal imbedding $X \to X \times_Y X$ is closed.

Definition 6 *The category $\mathcal{A}n_k$ of analytic spaces over k consists of all pairs (K, X) where K is a complete field in which k embeds isometrically, and X is a K-analytic variety. A morphism $(K, X) \to (L, Y)$ is a pair (j, ϕ) where j is an isometric imbedding $L \hookrightarrow K$ and ϕ is a morphism of K-analytic varieties $X \to Y_K$.*

Any scheme X locally of finite type over k gives rise to an analytic variety X^{an}. More precisely, let Φ be the functor from the category of k-analytic spaces to the category of sets which associates to every analytic space $\mathcal{X}$ the set of morphisms of k-ringed spaces $\mathrm{Hom}_k(\mathcal{X}, X)$. Berkovich proves the following;

Theorem 5 *The functor Φ is represented by a k-analytic variety X^{an} and a morphism $\pi : X^{an} \to X$. They have the following properties:*

(i) The map π is surjective and for any non-Archimedean field K over k, induces a bijection $X^{an}(K) \simeq X(K)$.

(ii) For any point $x \in X^{an}$, the homomorphism $\pi_x : \mathcal{O}_{X,\pi(x)} \to \mathcal{O}_{X^{an},x}$ is local and flat. Furthermore, if $x \in X^{an}(\overline{k})$, then π_x induces an isomorphism of completions $\hat{\pi}_x : \widehat{\mathcal{O}}_{X,\pi(x)} \to \widehat{\mathcal{O}}_{X^{an},x}$.

□

The map $X \mapsto X^{an}$ defines a functor from schemes locally of finite type over k to k-analytic varieties. We give a sample of properties that are preserved by this functor. Denote by $|\mathcal{X}|$ the underlying topological space of the analytic variety $\mathcal{X}$.

Theorem 6 *Let X be a scheme locally of finite type over k. Then:*

(i) X is separated $\Leftrightarrow$ $|X^{an}|$ is Hausdorff;

(ii) X is proper $\Leftrightarrow$ $|X^{an}|$ is Hausdorff and compact;

(iii) X is connected $\Leftrightarrow$ $|X^{an}|$ is connected.

Furthermore, let $\phi : X \to Y$ be a morphism of schemes locally of finite type over k. Then:

(iv) ϕ is injective $\Leftrightarrow$ ϕ^{an} is injective;

(v) ϕ is surjective $\Leftrightarrow$ ϕ^{an} is surjective;

(vi) ϕ^{an} is an open immersion $\Leftrightarrow$ ϕ^{an} is an open immersion.

□

From here on the theory proceeds parallel to algebraic geometry. For instance, there is a well behaved notion of proper morphism; there is a theory of coherent sheaves, with the relative finiteness results, and standard theorems like semicontinuity and proper base change hold in this new context. The same is true for Zariski's main theorem and Stein factorization, to cite just a few examples.

4 Berkovich's rigid etale cohomology

Berkovich has defined and studied a theory of etale cohomology for his analytic varieties. In the papers [B1], [B2], and [B3] he establishes the usual properties for his cohomology, like proper and smooth base change, Poincaré duality, and introduces also a notion of vanishing cycle.

4.1 The analytic etale topology

We start by complementing the prevoius section with the basic definitions needed to set up the etale topology for an analytic variety, following [B1].

Definition 7 *Let A be a Banach ring. An A-affinoid algebra B is called a finite Banach A-algebra and the map $A \to B$ is said to be finite, if B is finite as an A-module.*

A morphism of affinoid spaces $\mathcal{M}(B) \to \mathcal{M}(A)$ is said to be finite if it is induced by a finite map $A \to B$.

A morphism of analytic varieties $\phi : X \to Y$ is said to be finite if for any point $y \in Y$ there exists an affinoid neighborhood V of x such $\phi^{-1}(V) \to V$ is a finite morphism of affinoid spaces.

The morphism ϕ is said to be quasifinite if for any point $x \in X$ there exists a neighborhood V of x and U of $\phi(x)$ such that ϕ induces a finite morphism $V \to U$.

Recall that a morphism of schemes $\phi : \mathcal{X} \to \mathcal{Y}$ is called quasifinite if any point $y \in \mathcal{Y}$ is isolated in its fibre $\phi^{-1}(y)$. The following result is proved in [B1]:

Lemma 2 *(i) Quasifinite morphisms are preverved under composition, under any base change and under any ground field extension functor.*

(ii) A morphism $\phi : \mathcal{X} \to \mathcal{Y}$ between schemes of locally finite type over k is quasifinite if and only if the corresponding morphism $\phi^{an} : \mathcal{X}^{an} \to \mathcal{Y}^{an}$ is quasifinite.

□

Definition 8 *A morphism of analytic spaces $\phi : Y \to X$ is said to be flat at a point $y \in Y$ if $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,\phi(y)}$ algebra. ϕ is said to be flat if it is flat at all points.*

Unramified morphisms $\phi : Y \to X$ can be defined in analogy with the algebraic case, by looking at the sheaf of relative differentials $\Omega_{Y/X}$.

Let $\phi : M(B) \to M(A)$ be a morphism of k-affinoid spaces. The *module of differentials* $\Omega_{B/A}$ is the B-module J/J^2, where J is the kernel of the multiplication $B\widehat{\otimes}B \to B$. $\Omega_{B/A}$ determines a sheaf on $\mathcal{M}(B)$. As usual, this definition globalizes to general morphisms of analytic varieties $Y \to X$, and gives the sheaf $\Omega_{Y/X}$.

Definition 9 *Let $\phi : Y \to X$ be a quasifinite morphism. Then ϕ is said to be unramified if $\Omega_{Y/X} = 0$. ϕ is said to be etale if it is unramified and flat.*

Proposition 1 *(i) Etale morphisms are preverved under composition, under any base change and under any ground field extension functor.*

(ii) A morphism $\phi : \mathcal{X} \to \mathcal{Y}$ between schemes of locally finite type over k is etale if and only if the corresponding morphism $\phi^{an} : \mathcal{X}^{an} \to \mathcal{Y}^{an}$ is etale.

(iii) Let $\psi : Z \to Y$ and $\phi : Y \to X$ be quasifinite morphisms and suppose that $\phi\psi$ is etale and ϕ is unramified. Then ψ is etale.

□

For a k-analytic space X we set $\mathbb{A}^d_X = \mathbb{A}^d_k \times X$.

Definition 10 *A morphism of k-analytic spaces $\phi : Y \to X$ is said to be smooth at a point $y \in Y$ if there exists an open neighborhood V of y such that the induced morphism $V \to X$ factors as a composition of an etale morphism $V \to \mathbb{A}^d_X$ and the canonical projection $\mathbb{A}^d_X \to X$. ϕ is said to be smooth if it is smooth at all points $y \in y$.*

Proposition 2 *(i) Smooth morphisms are preserved under compositions, under any base change functor, and under extension of the ground field.*

(ii) A morphism $\phi : \mathcal{X} \to \mathcal{Y}$ between schemes of locally finite type over k is smooth if and only if the corresponding morphism $\phi^{an} : \mathcal{X}^{an} \to \mathcal{Y}^{an}$ is smooth.

□

We denote by $\mathrm{Et}(X)$ the category of etale analytic varieties over X. Objects of $\mathrm{Et}(X)$ are all the etale maps $U \to X$ and the morphisms between two objects $\psi : U \to X$ and $\phi : V \to X$ are all the maps $\beta : U \to V$ such that $\phi\beta = \psi$. Proposition 1.(iii) means that all the morphisms in $\mathrm{Et}(X)$ are etale.

The category $\mathrm{Et}(X)$ has an obvious Grothendieck topology for which the set of coverings of $(U \to X)$ consists of all the families of morphisms $\{f_i : U_i \to U\}_{i\in I}$ such that $U = \cup_{i\in I} f_i(U_i)$. This is the *etale site* of X that we denote by X_{et}. Any morphism $(X, K) \to (Y, L)$ of analytic spaces over k induces a morphism of sites $X_{et} \to Y_{et}$.

4.2 Higher direct images of torsion sheaves

For any ring Λ, the category of sheaves of Λ-modules on this topology is a topos, and the usual formalism of sheaf cohomology goes through, like in the algebraic case.

In his paper, Berkovich considers mainly finite rings of coefficients, of the form $\Lambda = \mathbb{Z}/n\mathbb{Z}$. For our purposes, these are not quite enough, since we have to consider characters of an infinite divisible group G_∞ into Λ^*.

Our next task is to extend the main results to more general torsion rings Λ. Instead of trying to reprove all the statements that we need beginning from scratch, we take a shortcut: we will show that in order to compute the effect of a cohomological functor on a sheaf F of Λ-modules, it suffices to regard F as a sheaf of abelian groups and compute the cohomological functor inside the category of sheaves of abelian groups. This will allow us to quickly derive our results from the theorems of Berkovich.

To start with, let Λ be any torsion ring and let $\mathbf{D}(X, \Lambda)$ (resp.($\mathbf{D}^+(X, \Lambda)$) be the derived category of complexes (resp. of complexes vanishing in large negative degrees) of sheaves $K^\cdot$ of Λ-modules and similarly define $\mathbf{D}^-(X, \Lambda)$; denote by Ψ_X the forgetful functor from $\mathbf{D}(X, \Lambda)$ to the derived category $\mathbf{S}(X)$ of complexes of abelian sheaves.

Let $f : X \to Y$ be a map of analytic spaces over k. First of all there is a direct image functor $Rf_* : \mathbf{D}^+(X, \Lambda) \to \mathbf{D}^+(Y, \Lambda)$.

Proposition 3 *The functor Rf_* commutes with the forgetful functor, i.e.*

$$Rf_* \circ \Psi_X = \Psi_Y \circ Rf_*.$$

Proof: For any sheaf F we will construct a resolution $I^{\cdot}$ by sheaves that are both injective as sheaves of Λ-modules and flabby as sheaves of abelian modules. One checks as in the algebraic case that flabby resolutions are f_*-acyclic : to do this one can look at [Mi] chapt. III sections 1,2,3 and convince oneself that all the arguments work without change in the present situation.

Then $I^{\cdot}$ computes at the same time Rf_* in the categories $D(Y, \Lambda)$ and $\mathbf{S}(X)$, and the proposition follows.

For each $x \in X$, choose a geometric point x' localized at x, i.e. an imbedding of the residue field $\mathcal{H}(x)$ of x in the completion of its algebraic closure. We form the locally ringed space $X' = \cup_{x \in X} x'$ that we endow with the discrete topology. This space is an inductive limit of analytic spaces and therefore carries a natural etale site X'_{et}. Let $\pi : X'_{et} \to X_{et}$ be the obvious map.

The sheaf $\pi^* F$ is the direct product over the stalks $F_{x'} = x'^* F$ at the points $x' \in X'$. For every $x' \in X'$ choose an imbedding into an injective Λ-module $F_{x'} \hookrightarrow I_{x'}$: we see $I_{x'}$ as an injective sheaf of Λ-modules over the point x'. The product $I^0 = \Pi_{x' \in X'} I_{x'}$ is an injective sheaf of Λ-modules on X' and clearly F imbeds into $\pi_* I$. Since π_* preserves injective sheaves, we have constructed the first step of an injective resolution of Λ-modules; if we iterate this construction we obtain a full Godement resolution $I^{\cdot}$ for F. On the other hand, I is also flabby as a sheaves of abelian groups (since every sheaf on X' is flabby) and π_* preserves flabby sheaves, therefore $I^{\cdot}$ is also a flabby resolution, as wanted. □

Next we turn to cohomology with support. We recall the relevant definitions and notation from [B1].

A family Φ of closed subsets of a topological space S is said to be a *family of supports* if it is preserved under finite unions and contains all closed subsets of any set from Φ. The family of supports Φ is said to be paracompactifying if any $A \in \Phi$ is paracompact and has a neighborhood $B \in \Phi$.

Example. Let S be a Hausdorff topological space. Then the family C_S of all compact subsets of S is a family of supports. If S is locally compact, then C_S is paracompactifying. More generally, suppose $\phi : T \to S$ is a continuous Hausdorff map of topological spaces (i.e. the diagonal imbedding $T \to T \times_S T$ is closed), and assume that each point of S has a compact neighborhood. Then the family C_ϕ of all closed subsets $A \subset T$ such that the induced map $A \to S$ is compact is a family of supports. We recall that a map of topological spaces is said to be compact if the preimage of any compact subset is compact.

Let Φ be a family of supports on the k-analytic variety X. We can define the following left exact functor $\Gamma_\Phi : \mathbf{S}(X) \to \mathcal{A}b - \mathcal{G}roups$;

$$\Gamma_\Phi(F) = \{s \in F(X) | \mathrm{Supp}(s) \in \Phi\}.$$

The value of its right derived functor are denoted by $H^n_\Phi(X, F)$, $n \geq 0$. For example, if Φ is the family of all closed subsets, these are just the groups $H^n(X, F)$. If $\Phi = C_X$ then we get the cohomology groups with compact support.

Let now $\phi : Y \to X$ be a morphism of k-analytic varieties. For any map $f : U \to V$ in $\mathrm{Et}(X)$ we will write U_ϕ (resp. V_ϕ) for the fiber product $U \times_X Y$ (resp. $V \times_X Y$) and f_ϕ for the induced map $U_\phi \to V_\phi$.

Definition 11 *(i) A ϕ-family of supports is a system Φ of families of supports $\Phi(f)$ in U_ϕ for all etale maps $(U \to X) \in \mathrm{Et}(X)$ such that it satisfies the following conditions:*

(1) for any morphism $g : V \to U$ in $\mathrm{Et}(X)$ one has $g_\phi^{-1}(\Phi(f)) \subset \Phi(fg)$;

(2) if for a closed subset $A \subset U$ there exists a covering $\{g_i : V_i \to U\}_{i \in I}$ in $\mathrm{Et}(X)$ such that $g_i^{-1}(A) \in \Phi(fg_i)$ for all $i \in I$, then $A \in \Phi(f)$.

(ii) The ϕ-family Φ is said to be paracompactifying if, for any $f : U \to X$ in $\mathrm{Et}(X)$, each point of U has a neighborhood $g : V \to U$ in $\mathrm{Et}(X)$ such that the family of supports $\Phi(fg)$ is paracompactifying.

A ϕ-family of supports Φ defines a left exact functor $\phi_\Phi : \mathbf{S}(Y) \to \mathbf{S}(X)$ as follows. If $F \in \mathbf{S}(Y)$ and $f : U \to X$ is etale, then

$$(\phi_\Phi F)(U) = \{s \in F(U_\phi) | \mathrm{Supp}(s) \in \Phi(f)\}.$$

For example, if Φ is the family of all closed subsets, then $\phi_\Phi F = \phi_*$. If the map $X \to Y$ is separated then $\Phi = C_\phi$ is a paracompactifying ϕ-family, and we get a left exact functor that is denoted by $\phi_!$.

We can derive the functor ϕ_Φ in the two categories $\mathbf{S}(X)$ and $\mathbf{D}^+(X, \Lambda)$, and in this way we obtain two functors that we denote both by $R\phi_\Phi$. The following proposition shows that in the cases of interest no ambiguity arises from this choice of notation.

Proposition 4 *Suppose that the family Φ is paracompactifying. Then the two functors defined above coincide, i.e.*

$$R\phi_\Phi \circ \Psi_X = \Psi_Y \circ R\phi_\Phi.$$

Proof: The proof of proposition 3 produces for any sheaf of Λ-modules a resolution that is injective in the category of sheaves of Λ-moodules and flabby in the category of sheaves of abelian groups.

To prove the theorem, it suffices to show that this resolution is acyclic for the functor ϕ_Φ defined on the category $\mathbf{S}(X)$, thus the proposition follows from lemma 3 below. □

Lemma 3 *Suppose that the family Φ is paracompactifying. Let F be a flabby sheaf of abelian groups. Then $R^n\phi_\Phi(F) = 0$ for all $n > 0$.*

Proof: It is shown in [B1], proposition 5.2.1, that $R^n\phi_\Phi(F)$ is the sheaf associated with the presheaf $(U \to X) \mapsto H^n_{\Phi(f)}(U_\phi, F)$. Therefore it suffices to show that under the stated hypothesis, $H^n_{\Phi(f)}(U_\phi, F) = 0$ for all etale morphism $U \to X$ and all $n > 0$. Since the restriction to U of a flabby sheaf of abelian groups on X, is a flabby sheaf, we have only to prove this for $U = X$.

Consider the morphism of sites $\pi : X_{et} \to |X|$, where $|X|$ is the space X with its underlying analytic topology. The morphism π induces a spectral sequence

$$H^p_\Phi(|X|, R^q\pi_* F) \Rightarrow H^{p+q}(X, F).$$

We will prove that $R^q\pi_*F = 0$ for all $q > 0$. Assuming this for the moment, we show how to conclude. It follows that $H^p_\Phi(|X|, \pi_*F) = H^p(X, F)$. Since F is flabby by hypothesis, we obtain from [B1], corollary 4.2.5, that π_*F is flabby in the analytic topology. Then π_*F is Γ_Φ-acyclic, by lemma 3.7.1 from [Gro] and the lemma is proved.

To see that $R^q\pi_*F = 0$, we can look at the stalks of this sheaf. For any point $x \in X$, let G_x be the Galois group of the algebraic closure of the residue field $\mathcal{H}(x)$. According to [B1], proposition 4.2.4, we have $(R^q\pi_*F)_x \simeq H^q(G_x, F_x)$, $q \geq 0$. Since F is flabby, it follows from [B1], corollary 4.2.5 that F_x is an acyclic G_x-module, as wanted. □

As a corollary, we get a proper base change theorem for sheaves of Λ-modules. From here on, we restrict to torsion rings Λ in which the prime $\mathrm{char}(\tilde{k})$ is invertible.

Theorem 7 *Let $\phi : Y \to X$ be a separated morphism of k-analytic spaces, and let $f : X' \to X$ be a morphism of analytic spaces over k, which gives rise to a cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ X' & \xrightarrow{f} & X \end{array}$$

Then for any complex $K^\cdot \in \mathbf{D}^+(Y, \Lambda)$ there is a canonical isomorphism in $\mathbf{D}^+(X', \Lambda)$

$$f^*(R\phi_! K^\cdot) \simeq R\phi'_!(f'^* K^\cdot).$$

Proof: The usual devissage reduces to the case where $K^\cdot$ is concentrated in degree 0. Then the theorem follows from proposition 4 and theorem 7.7.1 of [B1]. □

Let $\mathbf{D}^b(X, \Lambda)$ be the subcategory of $\mathbf{D}^+(X, \Lambda)$ consisting of cohomologically bounded complexes. Let $\phi : Y \to X$ be as in theorem 7 and suppose that the fibres of ϕ have bounded dimension. Then, by corollary 5.3.8 of [B1] and proposition 4 we deduce that $R\phi_!$ takes $\mathbf{D}^b(X, \Lambda)$ to $\mathbf{D}^b(Y, \Lambda)$ and extends to a functor $R\phi_! : \mathbf{D}^-(X, \Lambda) \to \mathbf{D}^-(Y, \Lambda)$.

The following projection formula is proved as in [B1], theorem 5.3.8.

Theorem 8 *Suppose that $F^\cdot \in \mathbf{D}^-(X, \Lambda)$ and $G^\cdot \in D^-(Y, \Lambda)$ or that $F^\cdot \in \mathbf{D}^b(X, \Lambda)$ has finite Tor-dimension and $G^\cdot \in \mathbf{D}(Y, \Lambda)$. Then there is a canonical isomorphism*

$$F^\cdot \overset{\mathbf{L}}{\otimes} R\phi_!(G^\cdot) \simeq R\phi_!(\phi^*(F^\cdot) \overset{\mathbf{L}}{\otimes} G^\cdot). \tag{1}$$

□

Remark: we point out that the isomorphism of the theorem is functorial in both $F^\cdot$ and $G^\cdot$. Explicitly, let $f : F^\cdot \to F'^\cdot$ and $g : G^\cdot \to G'^\cdot$ be maps complexes; then the isomorphism (1)

induces the following commutative diagram

$$\begin{tikzcd}
F^{\cdot} \overset{\mathbf{L}}{\otimes} R\phi_!(G^{\cdot}) \arrow[r, "\sim"] \arrow[d, "\overset{\mathbf{L}}{f \otimes R\phi_!(g)}"'] & R\phi_!(\phi^*(F^{\cdot}) \overset{\mathbf{L}}{\otimes} G^{\cdot}) \arrow[d, "R\phi_!(\phi^*(f) \overset{\mathbf{L}}{\otimes} g)"] \\
F^{\cdot} \overset{\mathbf{L}}{\otimes} R\phi_!(G^{\cdot}) \arrow[r, "\sim"] & R\phi_!(\phi^*(F^{\cdot}) \overset{\mathbf{L}}{\otimes} G^{\cdot}).
\end{tikzcd}$$

We conclude this section with the statement of a remarkable theorem, that allows us to compute cohomology in several important cases.

Let X be a scheme locally of finite type over k, and denote by X_{et} the (algebraic) etale site on X. It follows from proposition 1.(ii) that there is a natural morphism of sites $\pi : X_{et}^{an} \to X_{et}$. For any sheaf F of abelian groups on X_{et}, we denote by F^{an} the pullback $\pi^* F$. Given a morphism of schemes $f : X \to Y$ and a sheaf F on Y, it is clear that $(f^* F)^{an} = f^{an*} F^{an}$. The following Comparison Theorem is proved by Berkovich.

Theorem 9 *Let $f : X \to Y$ be a morphism of schemes locally of finite type over k and let F be an etale sheaf of abelian groups on Y. Then for all $q \geq 0$ there is a canonical isomorphism*

$$(R^q f_* F)^{an} \simeq R^q f_* F^{an}.$$

Proof: This is theorem 7.5(iii) in [B1]. □

4.3 Poincaré duality

Lastly, we want to establish Poincaré duality for sheaves of Λ-modules.

Let $\Lambda' \to \Lambda$ be a ring homomorphism and let F (resp. G) be a sheaf of Λ-modules (resp. of Λ'-modules) on the analytic space X. Then F becomes a sheaf of Λ'-modules by restriction of scalars, and we can form the tensor product $F \otimes_{\Lambda'} G$. The sheaf of Λ'-modules $F \otimes_{\Lambda'} G$ carries also a canonical structure of sheaf of Λ-modules. To describe this structure, recall that a sheaf of Λ-modules S is by definition a Λ-module object in the category of sheaves of abelian groups; in other words, the structure of S is determined by a collection of endomorphisms $\lambda_S^* : S \to S$ for all $\lambda \in \Lambda$, such that $\lambda_S^* \circ \lambda_S'^* = (\lambda\lambda')_S^*$ and $1_S^* = \mathrm{id}_S$. Then the structure of $F \otimes_{\Lambda'} G$ is given by the rule: $\lambda^*_{F \otimes_{\Lambda'} G} = \lambda_F^* \otimes_{\Lambda'} \mathrm{id}_G$.

The following propositions set up the formalism of trace mappings in our context.

Proposition 5 *One can assign to every separated flat quasifinite morphism $\phi : Y \to X$ and every sheaf of Λ-modules on X a trace mapping*

$$\mathrm{Tr}_\phi : \phi_! \phi^*(F) \to F.$$

These mappings are functorial on F and are compatible with base change and with composition. If ϕ is finite of constant rank d, then composition with the adjunction map

$$F \to \phi_* \phi^*(F) = \phi_! \phi^*(F) \xrightarrow{\mathrm{Tr}_\phi} F$$

gives the multiplication by d. These properties determine uniquely the trace mappings.

Proof: In theorem 5.4.1 from [B1] the mappings are constructed in the category of sheaves of abelian groups, but the construction shows that Tr_ϕ commutes with the homomorphisms λ^*, $\lambda \in \Lambda$, i.e. it preserves the structure of Λ-module. $\square$

Let X be an analytic variety over k. Denote by μ_n the sheaf of roots of unity of order n. We write μ_n^d for the d-th tensor power of μ_n with itself. Then we define the sheaves $\Lambda(d)_X = \mu_n^d \otimes_{\mathbf{Z}/n} \Lambda_X$. By the argument above, $\Lambda(d)_X$ is a sheaf of Λ-modules.

Berkovich shows that for all n prime to $\mathrm{char}(k)$ there is a Kummer short exact sequence, analogous to the usual one from the algebraic case:

$$0 \to \mu_n \to \mathbf{G}_m \to \mathbf{G}_m \to 0.$$

Taking $X = \mathbf{P}^1$, a well known argument produces an isomorphism $\mathrm{Pic}(\mathbf{P}^1)/n \simeq H^2(\mathbf{P}^1, \mu_n)$.

Let n be an integer prime to $\mathrm{char}(\tilde{k})$. Here we specialize further and assume that $n\Lambda = 0$.

Proposition 6 *Suppose that k is algebraically closed. Then one can assign to every smooth connected k-analytic curve X a trace mapping isomorphism*

$$\mathrm{Tr}_X : H_c^2(X, \Lambda(1)_X) \to \Lambda.$$

These mappings have the following properties and are uniquely determined by them:

a) for any flat quasi-finite morphism $\phi : Y \to X$ the following diagram is commutative

$$\begin{tikzcd} H_c^2(Y, \Lambda(1)_Y) \arrow[rr, "H_c^2(\mathrm{Tr}_\phi)"] \arrow[dr, "\mathrm{Tr}_Y"'] & & H_c^2(X, \Lambda(1)_X) \arrow[dl, "\mathrm{Tr}_X"] \\ & \Lambda & \end{tikzcd}$$

b) $\mathrm{Tr}_{\mathbf{P}^1}$ is the canonical mapping $H^2(\mathbf{P}^1, \Lambda(1)_{\mathbf{P}^1}) \simeq \Lambda$ induced by the degree isomorphism deg: $\mathrm{Pic}(X) \simeq \mathbf{Z}$.

Proof: Theorem 6.2.1 of [B1] constructs trace mappings

$$\mathrm{Tr}_X : H_c^2(X, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$$

with corresponding properties. By theorem 7, these mappings induce isomorphisms of abelian groups

$$\mathrm{Tr}_X : H_c^2(X, \Lambda(1)_X) \simeq H_c^2(X, \mu_n) \otimes_{\mathbf{Z}/n\mathbf{Z}} \Lambda \to \mathbf{Z}/n\mathbf{Z} \otimes_{\mathbf{Z}/n\mathbf{Z}} \Lambda \simeq \Lambda.$$

But the remark after the proof of theorem 7 implies that this isomorphism preserves the Λ-module structure. $\square$

Proposition 7 *One can assign to every separated smooth morphism $\phi : Y \to X$ of pure dimension d a trace mapping*

$$\mathrm{Tr}_\phi : R^{2d}\phi_!(\Lambda(d)_Y) \to \Lambda_X.$$

These mappings are compatible with base change and with composition. If $d = 0$, then Tr_ϕ is the trace mapping of proposition 5; if $d = 1$ and X is the spectrum of k, with k algebraically closed, then Tr_ϕ is the trace mapping of proposition 6. These properties determine the trace mappings uniquely.

Furthermore, if the fibres of ϕ are nonempty and connected, then Tr_ϕ is an isomorphism.

Proof: The proof is a formal argument, starting from the case of curves. The details can be found in the proof of theorem 7.2.1 of [B1]. □

Let $G^\cdot, G'^\cdot \in \mathbf{D}^b(Y, \Lambda)$; a general nonsense argument provides us with a canonical morphism in $\mathbf{D}^+(X, \Lambda)$

$$R\phi_*(\mathcal{H}om(G^\cdot, G'^\cdot)) \longrightarrow \mathcal{H}om(R\phi_! G^\cdot, R\phi_! G'^\cdot).$$

Applying this morphism to complexes of the form $\phi^* F^\cdot(d)[2d]$ and using the trace mapping $R\phi_!(\phi^* F^\cdot(d)[2d]) \to F^\cdot$ we obtain for any $G^\cdot \in \mathbf{D}^b(Y, \Lambda)$ and $F^\cdot \in \mathbf{D}^b(X, \Lambda)$ a duality morphism

$$R\phi_*(\mathcal{H}om(G^\cdot, \phi^F(d)[2d])) \longrightarrow \mathcal{H}om(R\phi_! G^\cdot, F^\cdot).$$

Theorem 10 *The duality morphism is an isomorphism.*

Proof: The proof is given in [B1], theorem 7.3.1, with $\Lambda = \mathbf{Z}/n\mathbf{Z}$. The reader can verify that the same proof goes through with no change for a general ring Λ such that $n\Lambda = 0$. □

4.4 ℓ-adic cohomology

To conclude, we want to set up an ℓ-adic formalism. To this purpose we will use the method of Ekedahl [Ek], appropriately downsized to fit our needs. The relevant proofs will be omitted, save for giving references to the paper [Ek].

Fix a prime ℓ different from $\mathrm{char}(\tilde{k})$. We will need a coefficient ring somewhat larger than the usual $\mathbf{Q}_\ell$; denote by $\mathbf{B}_\ell$ the extension of $\mathbf{Q}_\ell$ obtained by adding all the p^n-th roots of 1, for all integers n. Let $\mathbf{O}$ be the ring of integers of $\mathbf{B}_\ell$ and let m be its maximal ideal. Since the field $\mathbf{B}_\ell$ is an unramified extension of $\mathbf{Q}_\ell$, it's not hard to see that $\mathbf{O}$ is a discrete valuation ring, with residue field a certain infinite algebraic extension of $\mathbf{F}_\ell$.

Fix a k-analytic space X and let:

- $S = S_X$ be the topos of etale sheaves of sets on X;
- $(S_X, \mathbf{O})$ (resp. $(S_X, \mathfrak{Ab})$) be the category of $\mathbf{O}$-modules objects (resp. of abelian groups objects) in S;
- $\mathbf{D}(S, \mathfrak{Ab})$ (resp. $\mathbf{D}^+(S, \mathfrak{Ab}), \mathbf{D}^-(S, \mathfrak{Ab})$) be the derived category of complexes (resp. bounded from below, bounded from above) of objects of $(S_X, \mathfrak{Ab})$; same for $\mathbf{D}(S, \mathbf{O})$ and its variants;
- $S^{\mathbf{N}} = S_X^{\mathbf{N}}$ be the topos of inverse systems $\{M_n,\ n > 0,\ \pi_n : M_{n+1} \to M_n\}$ of elements of S, and we define $(S^{\mathbf{N}}, \mathfrak{Ab})$, $\mathbf{D}(S^{\mathbf{N}}, \mathfrak{Ab})$, $\mathbf{D}^+(S^{\mathbf{N}}, \mathfrak{Ab})$, $\mathbf{D}^-(S^{\mathbf{N}}, \mathfrak{Ab})$ in the obvious way.

Clearly $(S_X, \mathbf{O})$ becomes a ringed topos if we fix the ring object of S determined by the constant sheaf of sets with stalk $\mathbf{O}$ at every point.

We have a natural ring object in $S^{\mathbf{N}}$, namely the system of constant sheaves $\mathbf{O}_n = \mathbf{O}/m^n$, $n > 0$ with the natural projection maps $\mathrm{pr}_n : \mathbf{O}_{n+1} \to \mathbf{O}_n$. With this choice, $(S^{\mathbf{N}}, \{\mathbf{O}_n,\ n > 0,\ \mathrm{pr}_n\})$

becomes a ringed topos, and we have derived categories $\mathbf{D}(S^{\mathbf{N}}, \mathfrak{Ab})$, $\mathbf{D}(S^{\mathbf{N}}, \mathbf{O}.)$ with obvious notation.

There is a morphism of ringed topoi $\pi : S^{\mathbf{N}} \to S$ defined as follows:

$$\pi_* M. = \varprojlim \{M.\}, \quad (\pi^* M)_n = M \otimes_{\mathbf{O}} \mathbf{O}_n.$$

Definition 12 *i) An element $M.$ of $(S^N, \mathfrak{Ab})$ is said to be essentially zero if there is a covering $\{U_i\}_{i \in I}$ of X such that the following condition is satisfied. For the restriction of $M.$ to each U_i there is for all n an $m > n$ such that $\pi^{m-n} : M_m \to M_n$ is zero, where π^{m-n} is the composition of the appropriate π_i's.*

ii) An element $M^{\cdot}$ of $\mathbf{D}(S^{\mathbf{N}}, \mathfrak{Ab})$ will be said to be essentially zero if $H^i(M^{\cdot})$ is essentially zero for all i.

iii) The complex $M^{\cdot}$ is said to be essentially bounded (resp. from below, from above) if $H^m(M^{\cdot})$ is essentially zero for all m with $|m| >> o$ (resp. $-m >> 0$, $m >> 0$).

At this point Ekedahl introduces the following general notion. An $\mathbf{O}$-ringed topos T is said to satisfy the condition A if there is a class of generators T^{gen} of T and an integer N such that for all $M \in T^{gen}$ and all $\mathbf{O}_1$-module objects N in T, $H^i(M, N) = 0$ if $i > N$. Recall that in any topos, $H^i(M, N)$ is defined as $\mathrm{Ext}^i(M, N)$ for any two abelian group objects M, N of the topos.

Lemma 4 *The topos $(S_X, \mathbf{O})$ satisfies condition A.*

Proof: As generators we can take the family of all sheaves $f_* \mathbf{Z}_U$. where $\mathbf{Z}_U$ is the constant sheaf on U, and $f : U \to X$ ranges through all elements of $\mathrm{Et}(X)$. Then the lemma follows from theorem 4.2.6 of [B1]. □

Lemma 5 *i) The map $\pi_* : S^{\mathbf{N}} \to S$ has finite cohomological dimension and therefore extends to a morphism $R\pi_* : \mathbf{D}(S^N, \mathbf{O}.) \to \mathbf{D}(S, \mathbf{O})$.*

ii) If $M^{\cdot} \in \mathbf{D}(S^N, \mathbf{O}.)$ is essentially zero, then $R\pi_* M^{\cdot} = 0$.

iii) If $M^{\cdot} \in \mathbf{D}(S^N, \mathbf{O}.)$ is essentially bounded in some direction then $R\pi_* M^{\cdot}$ is bounded in the same direction.

Proof: Taking into account lemma 4, this is lemma 1.3 of [Ek]. □

We say that an element $M^{\cdot}$ in $\mathbf{D}(S^{\mathbf{N}}, \mathfrak{Ab})$ is *essentially constant* if there exists a complex $N^{\cdot}$ in $(S^{\mathbf{N}}, \mathfrak{Ab})$, a complex P in $(S, \mathfrak{Ab})$ and morphisms of complexes $M \xleftarrow{f} N \xrightarrow{g} \pi^* P$ such that the mapping cones of both f and g are essentially zero.

Definition 13 *Let $M^{\cdot}, N^{\cdot} \in \mathbf{D}(S^N, \mathbf{O}.)$.*

i) $M^{\cdot}$ is said to be an $\mathbf{O}$-complex if $\pi^*(\mathbf{O}_1) \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} M^{\cdot}$ is essentially constant;

ii) $M^{\cdot}$ is said to be negligible if $\pi^*(\mathbf{O}_1) \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} M^{\cdot}$ is essentially zero;

iii) a morphism $M^{\cdot} \to N^{\cdot}$ is said to be essentially an isomorphism if it has a negligible mapping cone.

Next, one notes that the negligible complexes form a thick subcategory of $\mathbf{D}(S^N, \mathbf{O}.)$. This allows to make the following definition.

Definition 14 *The derived category of complexes* $\mathbf{O}$*-adic modules is the category* $\mathbf{D}(X, \mathbf{O})$ *whose objects are* $\mathbf{O}$*-complexes and morphisms are the morphisms* $M^{\cdot} \to N^{\cdot}$ *with essential isomorphisms inverted. If* $* = b, +$ *or* $-$ *we denote by* $\mathbf{D}^*(X, \mathbf{O})$ *the essential image of the* $\mathbf{O}$*-complexes which are essentially bounded (resp. from below, from above).*

Proposition 8 *i) The functor*

$$\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} R\pi_*(-) : \mathbf{D}(S^{\mathbf{N}}, \mathbf{O}.) \to \mathbf{D}(X, \mathbf{O}_n)$$

factors to give a functor $\mathbf{D}(X, \mathbf{O}) \to \mathbf{D}(X, \mathbf{O}_n)$ *which we will denote* $\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (-)$.

ii) The functor $\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (-) : \mathbf{D}(X, \mathbf{O}) \to \mathbf{D}(X, \mathbf{O}_n)$ *is conservative, i.e.:*

$$M^{\cdot} = 0 \Longleftrightarrow \mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (M^{\cdot}) = 0$$

for all $M^{\cdot} \in \mathbf{D}(X, \mathbf{O})$.

Proof: Part (i) is lemma 2.6 of [Ek] and part (ii) is proposition 2.7 also from [Ek]. □

We show now how to deal with tensor products, Hom functors, direct and inverse images in our category $\mathbf{D}(X, \mathbf{O})$.

Definition 15 *i) Let* $M^{\cdot} \in \mathbf{D}^-(S^{\mathbf{N}}, \mathbf{O}.)$ *and* $N^{\cdot} \in \mathbf{D}(S^{\mathbf{N}}, \mathbf{O}.)$ *(or viceversa). Put*

$$M^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} N^{\cdot} = L\pi^*(R\pi_* M^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} R\pi_* N^{\cdot}).$$

ii) Let $M^{\cdot} \in \mathbf{D}(S^{\mathbf{N}}, \mathbf{O}.)$ *and* $N^{\cdot} \in \mathbf{D}^+(S^{\mathbf{N}}, \mathbf{O}.)$. *Put*

$$R\mathrm{Hom}_{\mathbf{O}}(M^{\cdot}, N^{\cdot}) = L\pi^* R\mathrm{Hom}_{\mathbf{O}}(R\pi_* M^{\cdot}, R\pi_* N^{\cdot}).$$

Clearly $(-) \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (-)$ and $R\mathrm{Hom}_{\mathbf{O}}(-,-)$ factor to give bifunctors on $\mathbf{D}(X, \mathbf{O})$.

Proposition 9 *i) Let* $M^{\cdot} \in \mathbf{D}^-(S^{\mathbf{N}}, \mathbf{O}.)$ *and* $N^{\cdot} \in \mathbf{D}(S^{\mathbf{N}}, \mathbf{O}.)$ *(or viceversa). The natural morphism*

$$\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (M^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} N^{\cdot}) \to (\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} M^{\cdot}) \overset{\mathbf{L}}{\otimes}_{\mathbf{O}_n} (\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} N^{\cdot})$$

is an isomorphism.

ii) Let $M^{\cdot} \in \mathbf{D}(S^{\mathbf{N}}, \mathbf{O}.)$ *and* $N^{\cdot} \in \mathbf{D}^+(S^{\mathbf{N}}, \mathbf{O}.)$. *The natural morphism*

$$\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (R\mathrm{Hom}_{\mathbf{O}}(M^{\cdot}, N^{\cdot})) \to R\mathrm{Hom}_{\mathbf{O}_n}((\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} M^{\cdot}), (\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} N^{\cdot}))$$

is an isomorphism.

Proof: This is proposition 4.2 of [Ek]. □

Proposition 10 *i) Let $M^{\cdot} \in \mathbf{D}(X, \mathbf{O})$, $N^{\cdot} \in \mathbf{D}^{+}(X, \mathbf{O})$ and denote by $\mathbf{O}$ the image of the inverse system $\mathbf{O}$. in $\mathbf{D}(X, \mathbf{O})$. Then*

$$\mathrm{Hom}_{\mathbf{D}(X,\mathbf{O})}(\mathbf{O}, R\mathrm{Hom}_{\mathbf{O}}(M^{\cdot}, N^{\cdot}) = \mathrm{Hom}_{\mathbf{D}(X,\mathbf{O})}(M^{\cdot}, N^{\cdot}).$$

ii) Let $M^{\cdot} \in \mathbf{D}(X, \mathbf{O})$, $N^{\cdot} \in \mathbf{D}^{-}(X, \mathbf{O})$ and $P^{\cdot} \in \mathbf{D}^{+}(X, \mathbf{O})$. Then

$$R\mathrm{Hom}_{\mathbf{O}}(M^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} N, P) = R\mathrm{Hom}_{\mathbf{O}}(M^{\cdot}, R\mathrm{Hom}_{\mathbf{O}}(N, P)).$$

In particular $(-) \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} N$ and $R\mathrm{Hom}_{\mathbf{O}}(N, -)$ are adjoint functors on $\mathbf{D}(X, \mathbf{O})$.

Proof: This is proposition 4.4 of [Ek]. □

Suppose Y is another k-analytic space, and let $f : X \to Y$ be a morphism. There is an induced map of topoi $f : S_X \to S_Y$, from which we derive morphisms of triangulated categories $Rf_* : \mathbf{D}^{+}(S_X^N, \mathbf{O}.) \to \mathbf{D}(S_Y^N, \mathbf{O}.)$ and $Lf^* : \mathbf{D}^{-}(S_Y^N, \mathbf{O}.) \to \mathbf{D}(S_X^N, \mathbf{O}.)$. It is clear that Rf_* and Lf^* preserve $\mathbf{O}$-complexes as well as essential isomorphisms and therefore induce maps $Rf_*\mathbf{D}(X, \mathbf{O}) \to \mathbf{D}(Y, \mathbf{O})$ and $Lf^* : \mathbf{D}(Y, \mathbf{O}) \to \mathbf{D}(X, \mathbf{O})$.

The following proposition is not explicitly stated in [Ek], but it is used implicitly in that paper. This is my justification for including it here, even though I cannot confidently point to an adequate reference.

Proposition 11 *i) Let $M^{\cdot} \in \mathbf{D}^{+}(S_X^{\mathbf{N}}, \mathbf{O}.)$. Then the natural morphism*

$$\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (Rf_* M^{\cdot}) \to Rf_*(\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} M^{\cdot})$$

is an isomorphism.

ii) Let $N^{\cdot} \in \mathbf{D}^{-}(S_Y^{\mathbf{N}}, \mathbf{O}.)$. Then the natural morphism

$$\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} (Lf^* N^{\cdot}) \to Lf^*(\mathbf{O}_n \overset{\mathbf{L}}{\otimes}_{\mathbf{O}} N^{\cdot})$$

is an isomorphism. □

This completes the set up of our ℓ-adic theory. Without giving details, we mention that all the main results (proper base change, Poincaré duality, and so on), stated for the sheaves of Λ-modules, translate into corresponding theorems in $\mathbf{D}(X, \mathcal{O})$. For the proof one uses repeatedly the conservativeness property of proposition 8(ii).

Finally, we get the derived category $\mathbf{D}(X, \mathbf{B}_\ell)$ of $\mathbf{B}_\ell$-adic sheaves by inverting the morphisms in $\mathbf{D}(X, \mathcal{O})$ whose cone is a torsion complex.

In the same spirit, there should be a theory of perverse sheaves, with the usual properties. It seems that the main obstacle in completing this program is the definition of a good category of constructible sheaves, including the related finiteness theorems.

Let Λ be a torsion ring. Tentatively, a constructible sheaf of Λ-modules on the analytic variety X, should be a sheaf $\mathcal{F}$ for which there exists a filtration $X_0 \subset X_1 \subset \ldots \subset X_n = X$ of X by closed analytic subvarieties such that the restriction of $\mathcal{F}$ to each open stratum $X_{i+1} \backslash X_i$, is locally constant and of finite rank.

See [BGR] section 9.5, for the definition of (closed) analytic subvariety, where it is shown that there is a bijection between coherent analytic sheaves of ideals and analytic subvarieties.

We record here the precise statement of the conjecture:

Conjecture 1 *Berkovich's etale cohomology extends to a theory of the derived category of constructible $\mathbf{B}_\ell$-adic sheaves by the method of Ekedahl, in such a way that theorem 6.3 of [Ek] holds for this derived category, i.e. there is a formalism of the six operations, satisfying Poincaré-Verdier duality.*

5 The Lubin-Tate torsor

In this chapter we introduce and study the sheaf that plays the role covered by the Lang torsor in positive characteristic. I believe the name "Lubin-Tate torsor" is appropriate enough for this object. Let F be a fixed Lubin-Tate group.

5.1 Construction of the torsor

Lemma 6 *The logarithm $\lambda_F : \Delta(0,1) \to \mathbb{A}^1_k$ is an etale covering of $\mathbb{A}^1_k$.*

Proof: Let $\mathbb{A}^1_k = \cup_{r>0} D_r$ be the covering of the affine line by affinoid domains described in section 3. Denote by E_r the connected component of $\lambda^{-1}(D_r)$ containing 0.

From remark (a) following theorem 3 we get an equality of formal power series: $\lambda \circ [\pi]^n_f = \pi^n \cdot \lambda$. By analytic continuation, this formal identity gives rise to a commutative diagram of analytic maps:

$$\begin{tikzcd}
\Delta(0,1) \arrow[r, "{[\pi]^n_f}"] \arrow[d, "\lambda"'] & \Delta(0,1) \arrow[d, "\lambda"'] & \Delta(0,\rho_0) \arrow[l] \arrow[d, "\lambda"'] \\
\mathbb{A}^1 \arrow[r, "\pi^n"] & \mathbb{A}^1 & \mathbb{G}_a(\rho_0). \arrow[l]
\end{tikzcd}$$

Looking at the diagram above, we see that the restriction of λ to E_r is a finite map, hence E_r is an affinoid domain in $\Delta(0,1)$ for all r and $\Delta(0,1) = \cup_{r>0} E_r$. Note that for $r < s$, E_s is a closed neighborhood of E_r. It follows easily that λ is etale and surjective if and only if the induced maps $E_r \to D_r$ are etale and surjective for all r.

Given $r > 0$, choose an integer n_r large enough such that $[\pi]^{n_r}_f(E_r) \subset \Delta(0,\rho_0)$. By theorem 3, the power series e_F converges on $\Delta(0,\rho_0)$. This means that e_F defines a morphism on the quasiaffinoid space $\Delta(0,\rho_0)$, and therefore the restriction of λ to $\Delta(0,\rho_0)$ is an isomorphism of quasiffinoid spaces. It follows that $\lambda : E_r \to D_r$ is an etale covering if and only if $[\pi]^{n_r}_f : E_r \to \pi^{n_r} \cdot D_r$ is an etale covering. Let $g \in \mathcal{F}_\pi$ be any other power series; the homomorphism $[1]_{f,g} : \Delta(0,1) \to \Delta(0,1)$ of quasiaffinoid spaces has an inverse $[1]_{g,f}$ and therefore it is an isomorphism. $[1]_{g,f}$. From theorem 1.(b) we see that $[1]_{f,g} \circ [\pi]_f \circ [1]_{g,f} = [\pi]_g$. Therefore it suffices to prove that for some $g \in \mathcal{F}$ the morphism $[\pi]_g$ is an etale covering. Then we select $g(Z) = \pi Z + Z^q$. Now consider the map of schemes $\mathbb{A}^1_k \to \mathbb{A}^1_k$ defined by the polynomial $g(Z)$: this map ramifies over a finite set of points $x_1, ..., x_m \in \mathbb{A}^1_k(\overline{k}) = \overline{k}$, and using the jacobian criterion one checks easily that $|x_i| \geq 1$ for all i. On the complement of $x_1, ..., x_n$, g restricts to an etale covering $U \to \mathbb{A}^1_k - \{x_1, ..., x_n\}$. By proposition 1.(ii), it follows that the map $g^{an} : U^{an} \to \mathbb{A}^{1\,an}_k - \{x_1, ..., x_n\}$ is also etale, and by theorem 6.(v), it is surjective. But clearly $[\pi]_g$ is obtained from g^{an} by base change to $\Delta(0,1) \subset \mathbb{A}^{1\,an}_k$, and the lemma follows from proposition 1.(i). $\square$

Remark: the proof of the lemma shows in particular that the restriction of the analytic covering $\lambda : \Delta(0,1) \to \mathbb{A}^1_{\overline{k}}$ to any bounded disc $\Delta(0,\rho) \hookrightarrow \mathbb{A}^1_{\overline{k}}$ factors as a trivial (split) covering followed by an algebraic covering of finite degree.

For any positive integer n, let $k_n = k(G_n)$, and let $k_\infty = \cup_{n>0} k_n$.

It has already been remarked that $G_\infty = \mathrm{Ker}(\lambda : \Delta(0,1) \to \mathbb{A}^1_{\bar{k}})$. In particular, this kernel is contained in k_∞.

We define a sheaf of sets (in the rigid etale topology) over $\mathbb{A}^1_k$ as follows: for any U etale over $\mathbb{A}^1_k$, let $\phi(U)$ be the set of analytic maps $f : U \to \Delta(0,1)$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & \Delta(0,1) \arrow[dl, "\lambda"] \\ \mathbb{A}^1_k. & \end{tikzcd}$$

For any given extension field E of k, there is a base change map $p : \mathbb{A}^1_E \to \mathbb{A}^1_k$ and we can form the pull back $\phi_E = p^*\phi$. For our purposes, the really useful sheaf is ϕ_{k_∞}; for brevity we will denote it simply by ϕ_∞.

Lemma 7 *The sheaf ϕ_∞ is a torsor for the group G_∞.*

Proof: Of course the group G_∞ acts on the local sections of ϕ_∞ by translation and this action is free and transitive. From lemma 6 we see that the sheaf ϕ_∞ trivializes over the etale covering $\Delta(0,1)_{k_\infty}$. □

Definition 16 *The sheaf ϕ_∞ is called the Lubin-Tate torsor.*

Let $\chi : G_\infty \to \mathbb{B}_\ell^\times$ be a character of G_∞. The following push out diagram:

$$\begin{tikzcd} 0 \arrow[r] & G_\infty \arrow[r] \arrow[d, "\chi"] & \Delta \arrow[r] \arrow[d] & \mathbb{G}_a \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & \mathbb{B}_\ell^\times \arrow[r] & \mathcal{L}_\chi \arrow[r] & \mathbb{G}_a \arrow[r] & 0 \end{tikzcd}$$

defines a torsor for the group $\mathbb{B}_\ell^\times$, or equivalently, a locally free rank 1 $\mathbb{B}_\ell$-adic sheaf over $\mathbb{G}_a$ that we denote also by $\mathcal{L}_\chi$. In concrete terms $\mathcal{L}_\chi$ is a projective system $\{\mathcal{L}_\chi^{(n)}\}_{n\in\mathbb{N}}$ of $\mathcal{O}/m^n$-modules. For this kind of generalities, [Ek] is a good reference.

A note about notation: for a map $f : X \to \mathbb{G}_a$ sometime we will write $\mathcal{L}(f)$ in place of $f^*\mathcal{L}$.

We list here some elementary properties of $\mathcal{L}_\chi$, that follow from the general yoga of torsors. Let $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ be the addition map, and $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ the projection maps on the first and second factor. Then $\mathcal{L}_\chi$ comes with:

LT1) a rigidification at the origin:

$$\mathcal{L}_{\chi,\{0\}} \simeq \mathbb{B}_{\ell,\{0\}}$$

LT2) a trivialization:

$$m^*\mathcal{L}_\chi \otimes \mathrm{pr}_1^*\mathcal{L}_\chi^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}_\chi^{-1} \simeq \mathbb{B}_{\ell,\mathbb{G}_a\times\mathbb{G}_a}$$

compatible with the rigidification at the origin $\{0,0\}$ induced by **LT1**.

LT3) In particular:

$$\mathcal{L}_{\chi^{-1}} \simeq \mathcal{L}_{\chi}^{-1}.$$

We will denote by $\rho(\chi)$ the supremum of all real numbers ρ such that $\mathcal{L}_\chi$ trivializes on $\mathbf{G}_a(\rho)$. Notice that $\rho(\chi) \geq \rho_0$ and we have equality if and only if χ is injective. Moreover $\rho(\chi) = \infty$ if and only if χ is trivial.

The proof of the following proposition is taken from [SGA4$\frac{1}{2}$], Sommes trig. We reproduce it here to be on the safe side.

Proposition 12 *Let $\chi : G_\infty \to \mathbf{B}_\ell^\times$ be a non-trivial character. Then:*

$$H_c^*(\mathbf{G}_a(\rho)_{\overline{k}}, \mathcal{L}_\chi) = 0$$

for all $\rho > \rho(\chi)$.

Proof: Let Δ_ρ be the connected component of $\lambda^{-1}(\mathbf{G}_a(\rho))$ containing 0. For a $\overline{k}$-rational point x of Δ_ρ, let t_x be the translation $t_x(g) = g[+_f]x$ on Δ_ρ, where $[+_f]$ is Lubin-Tate group law. Also, let t'_y be the translation by $y \in \mathbf{G}_a$, with respect to usual addition law on $\mathbf{G}_a$. The formula $\lambda \circ t_x = t'_{\lambda(x)} \circ \lambda$ states that the pair $(t_x, t'_{\lambda(x)})$ is an automorphism of the diagram $\Delta_\rho \to \mathbf{G}_a(\rho)$.

Let $\psi(x)$ be the induced automorphism of $(\mathbf{G}_a(\rho), \mathcal{L}_\chi)$. For $x \in G_\infty$ this automorphism gives the identity on $\mathbf{G}_a(\rho)$, and multiplication by $\chi(x)^{-1}$ on $\mathcal{L}_\chi$.

Let $\psi_H(x)$ be the automorphism of $H_c^*(\mathbf{G}_a(\rho)_{\overline{k}}, \mathcal{L}_\chi)$ induced by $\psi(x)$. Then $\psi_H(x)$ is multiplication by $\chi(x)^{-1}$. On the other hand, the following "homotopy" lemma shows that $\psi_H(x) = \psi_H(0)$. Since by hypothesis $\rho > \rho(\chi)$, we can find $x \in G(\rho) \cap G_\infty$ such that $\chi(x) \neq 1$ and we see that multiplication by $(1 - \chi(x)^{-1}) \neq 0$ is the zero map, therefore the claim follows.

Lemma 8 ("Homotopy" lemma) *Let X and Y be two rigid analytic varieties over $\overline{k}$, with Y connected. Let $\mathcal{G}$ be a sheaf on X and (ψ, ϵ) a family of endomorphisms of $(X, \mathcal{G})$ parametrized by Y, i.e.:*

$$\begin{array}{ll} \psi : Y \times X \longrightarrow Y \times X & \textit{is a } Y\textit{-morphism and} \\ \epsilon : \psi^* \mathrm{pr}_2^* \mathcal{G} \longrightarrow \mathrm{pr}_2^* \mathcal{G} & \textit{a morphism of sheaves.} \end{array}$$

Assume ψ is proper. For $y \in Y(\overline{k})$, let $\psi_H(y)^*$ the endomorphism of $H_c^*(X, \mathcal{G})$ induced by $\psi_y : X \longrightarrow X$ and $\epsilon_y : \psi_y^* \mathcal{G} \longrightarrow \mathcal{G}$. Then $\psi_H(y)^*$ is independent of y.

Proof: In fact, $R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathcal{G}$ is the constant sheaf on Y with stalk $H_c^p(X, \mathcal{G})$, and $\psi_H(y)^*$ is the fiber at y of the endomorphism :

$$R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathcal{G} \xrightarrow{\psi^*} R^p \mathrm{pr}_{1!} \psi^* \mathrm{pr}_2^* \mathcal{G} \xleftarrow{\epsilon} R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathcal{G}.$$

□

To apply the homotopy lemma to the present situation, we take $\psi : \Delta_\rho \times \mathbf{G}_a(\rho) \longrightarrow \Delta_\rho \times \mathbf{G}_a(\rho)$ defined by $\psi(x, y) = (x, y + \lambda(x))$. □

5.2 The character induced by Galois action

We conclude this chapter with some observations about the Galois action on $\mathcal{L}_\chi$.

Let $\overline{\phi}$ be the pull back of ϕ_∞ to $\mathbb{A}^1_{\overline{k}}$; by transport of structure we get a natural action of $\mathrm{Gal}(\overline{k}/k_\infty)$ on $\overline{\phi}$, covering the action on $\mathbb{A}^1_{\overline{k}}$. This action is inherited by $\mathcal{L}_\chi$. In particular, if p is a k_∞-rational point of $\mathbb{A}^1_{\overline{k}}$, then the stalk $\mathcal{L}_{\chi,p}$ becomes a one-dimensional representation of $\mathrm{Gal}(\overline{k}/k_\infty)$.

For any $n \leq \infty$, let k_n^{ab} denote the maximal abelian extension of k_n. It is clear that the action on $\mathcal{L}_{\chi,p}$ factors through $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$.

I don't know the complete structure of $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$; in particular I don't know whether there is a canonical generator that takes the place of the Frobenius element as in the finite field case. Instead we make the following:

Definition 17 *Let k^{ur} be the maximal unramified extension of k. Clearly $k^{ur} \subset k_\infty^{ab}$ and $k^{ur} \cap k_\infty = k$. We say that an element $\sigma \in \mathrm{Gal}(k_\infty^{ab}/k_\infty)$ is a Frobenius element if the image of σ in $\mathrm{Gal}(k^{ur}/k)$ is the canonical Frobenius generator.*

Our aim is to give an explicit formula for the trace $\mathrm{Tr}(\sigma, \mathcal{L}_{\chi,p})$ of the endomorphism induced by the Frobenius element σ on the stalk of $\mathcal{L}_\chi$ at the point p. We start with two elementary lemmas:

Lemma 9 *The map $p \mapsto \mathrm{Tr}(\sigma, \mathcal{L}_{\chi,p})$ is a continuous group homomorphism $\mathrm{Tr}_\sigma : k_\infty \to \mathbb{B}_\ell^\times$.*

Proof: It follows easily from **LT1** and **LT2** that the map Tr_σ is a group homomorphism. Moreover, it follows from lemma 3 that the restriction of ϕ_∞ to $\Delta(0, \rho_0)$ is the trivial G_∞-torsor; therefore the restriction of $\mathcal{L}_\chi$ to the same disc is a trivial line bundle, and we conclude that the kernel of Tr_σ contains this entire disc, i.e. the map is continuous. □

Lemma 10 $k_\infty^{ab} = \cup_{n<\infty} k_n^{ab}$.

Proof: It is clear that $k_n^{ab} \subset k_\infty^{ab}$. On the other hand, let $x \in k_\infty^{ab}$ and let $x_1, ..., x_m$ be the orbit of x for the action of the full Galois group $\mathrm{Gal}(\overline{k}/k)$; take n big enough such that $[k_n(x_1, ..., x_m) : k_n] = [k_\infty(x_1, ..., x_m) : k_\infty]$. Then there is a natural isomorphism $\mathrm{Gal}(k_n(x_1, ..., x_m)/k_n) \simeq \mathrm{Gal}(k_\infty(x_1, ..., x_m)/k_\infty)$, and this last group is abelian, being a quotient of $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$. □

It follows from the lemma that the choice of a Frobenius element σ in $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$ is equivalent to the choice of a sequence $\sigma_0, \sigma_1, ...$ of liftings of Frobenius $\sigma_n \in \mathrm{Gal}(k_n^{ab}/k_n)$ such that the restriction of σ_{n+1} to k_n^{ab} acts as σ_n, and such that σ_0 acts trivially on k_∞. Let $\beta_n \in k_n$ such that the Artin symbol $(\beta_n, k_n^{ab}/k_n)$ acts on k_n^{ab} as σ_n. Then by local class field theory, it follows $\mathrm{Nm}_{k_{n+1}/k_n}(\beta_{n+1}) = \beta_n$. Also, by Lubin-Tate theory it follows $\beta_0 = \pi$.

Viceversa, the choice of a compatible system of elements $\beta_n \in k_n$ as before is equivalent to the choice of a Frobenius element σ.

For the next result we need to fix some notation. First of all we select for each positive integer n:

1) a generator v_n of G_n as an $\mathcal{O}_k$-module, such that $[\pi^{m-n}]_f(v_m) = v_n$;

2) an element $\beta_n \in k_n$ such that the sequence of these elements satisfies the compatibility condition above, and corresponds to the choice of a Frobenius element σ_β;

3) a power series $b_n(z) = z \cdot r_n(z)$, where $r(z) \in \mathcal{O}_k[[z]]$ satisfies $r(0) \neq 0$ and such that $b_n(v_n) = \beta_n$.

Finally, let T_n be the trace map from k_n to k.

Theorem 11 *Let p be a point in $\mathbb{A}^1_{\bar{k}}(k_\infty) = k_\infty$, and choose an integer n such that:*

(a) $|\pi^n p| < \rho_0$;

(b) $[k(p) : k] \leq n$.

Let m be any integer $\geq 2n+1$. Then, with reference to the notation above:

$$\mathrm{Tr}(\sigma_\beta, \mathcal{L}_{\chi,p}) = \chi\left(\left[\frac{1}{\pi^{m-n}} T_m\left(\frac{p}{\lambda'(v_m)} \frac{db_m}{dz}\Big|_{z=v_m}\right)\right]_f (v_n)\right).$$

Proof: First of all, notice that the group $\mathrm{Gal}(\bar{k}/k_\infty)$ acts also on $\Delta(0,1)_{\bar{k}}$ in such a way that the logarithm becomes an equivariant morphism. Let $q \in \lambda^{-1}(p)$. Let $\tilde{\sigma}$ be any lifting of σ_β to $\mathrm{Gal}(\bar{k}/k_\infty)$; then essentially by definition we have:

$$\mathrm{Tr}(\sigma_\beta, \mathcal{L}_{\chi,p}) = \chi(\tilde{\sigma}(q)[-_f]q) \tag{2}$$

(where $[-]_f$ denotes subtraction in the formal group). Obviously this formula is independent of the choices involved. Take n such that (a) is satisfied; by inspecting the proof of lemma 6 and the remark that follows it, we obtain:

$$\lambda^{-1}(p) = [\pi^n]_f^{-1}(e(\pi^n p))[+_f]G_\infty.$$

In particular we can take $q \in [\pi^n]_f^{-1}(e(\pi^n p))$ in equation (2). We recall now the definition of the generalized Kummer pairing, introduced by Frolich in [Fr]: let $F(k_n)$ be the subgroup of $\Delta(0,1)(k_\infty)$ consisting of the elements rational over k_n; then there is a bilinear map:

$$(\ ,\ )_n^F : F(k_n) \times k_n^\times \longrightarrow G_n$$

defined as follows. If $\beta \in k_n^\times$, let τ_β be the element of the $\mathrm{Gal}(k_n^{ab}/k_n)$ which is attached to β by the Artin symbol. If $\alpha \in F(k_n)$, choose γ in $\Delta(0,1)(\bar{k})$ such that $[\pi^n]_f(\gamma) = \alpha$. Then $(\alpha, \beta)_n^F = \tau_\beta(\gamma)[-_f]\gamma$. Clearly, if we take n such that both (a) and (b) are satisfied, the right side in formula (2) translates as $\chi((e(\pi^n p), \beta_n)_n^F)$.

Then the formula of the theorem follows immediately from theorem 1 of [Wi]. □

Remark: It would be interesting to determine the maximal pro-ℓ-quotient of $\mathrm{Gal}(\bar{k}/k_\infty)$. In any case, recall that the maximal pro-ℓ-quotient of $\mathrm{Gal}(\bar{k}/k)$ sits in a short exact sequence:

$$0 \longrightarrow \mathbf{Z}_\ell \longrightarrow G_\ell \longrightarrow \mathbf{Z}_\ell(1) \longrightarrow 0.$$

Let H be the cokernel of the natural map $\mathrm{Gal}(\bar{k}/k_\infty) \longrightarrow G_\ell$. We obtain a surjective map $\mathcal{O}_k \simeq \mathrm{Gal}(k_\infty/k) \to H$. But the group $\mathcal{O}_k$ decomposes as $\mu_{q-1} \times (1 + \pi\mathcal{O})$, that is a product of a pro-p-group and a finite group. Therefore the image of this map is a finite group, and thus H itself is finite.

We could interpret this as saying that no essential information on the ℓ-part of the Galois action is lost, in passing from k to the infinite extension k_∞.

6 Fourier Transform

We are now ready to define the Fourier transform. With the set-up of the previous chapters, we only have to mimic the construction of the Deligne-Fourier transform. The proofs of the main properties reduce to routine verifications, carried out by applying projection formulas, proper base change theorem and Poincaré duality, exactly as in Laumon's paper.

6.1 Definition and main properties

We introduce the following notation:

- $V \simeq \mathbb{A}^n_k$ is a linear variety of dimension n over k,
- V' is the dual of V,
- $\sigma : \{0\} \hookrightarrow V'$ is the imbedding of the origin,
- $\pi : V \to \mathrm{Spec}\, k$ is the structural map of V,
- $s : V \times V \to V$ is the addition law in the vector space V,
- $\langle , \rangle : V \times_k V' \longrightarrow \mathbb{G}_a$ is the canonical dual pairing.

Fix a character $\chi : G_\infty \longrightarrow \mathbb{B}_\ell{}^\times$. We define an operator:

$$\mathcal{F}_\chi : \mathbf{D}^b(V, \mathbb{B}_\ell) \longrightarrow \mathbf{D}^b(V', \mathbb{B}_\ell)$$

by the formula:

$$\mathcal{F}_\chi(K^\cdot) = R\mathrm{pr}_{2!}(\mathcal{L}_\chi(\langle , \rangle) \otimes \mathrm{pr}_1^* K^\cdot))[n]$$

where $\mathrm{pr}_1 : V \times V' \longrightarrow V$ and $\mathrm{pr}_2 : V \times V' \longrightarrow V'$ are the projections and $K^\cdot \in \mathbf{D}^b(V, \mathbb{B}_\ell)$.

Definition 18 *$\mathcal{F}_\chi$ is called the Fourier Transform associated to the character χ.*

Next we would like to show that $\mathcal{F}$ shares some interesting properties with the Fourier transform defined over finite fields.

Involutivity is easily established: denote by V'' the double dual of V. The previous construction applies to V' and its dual V'' to give a Fourier transform $\mathcal{F}'$. We consider the composition:

$$\mathbf{D}^b(V, \mathbb{B}_\ell) \xrightarrow{\mathcal{F}} \mathbf{D}^b(V', \mathbb{B}_\ell) \xrightarrow{\mathcal{F}'} \mathbf{D}^b(V'', \mathbb{B}_\ell).$$

Denote by $a : V \xrightarrow{\sim} V''$ the isomorphism defined by $a(v) = -\langle v, \cdot \rangle$.

Theorem 12 *There is a functorial isomorphism:*

$$\mathcal{F}' \circ \mathcal{F}(K^\cdot) \simeq a_*(K^\cdot)(-n)$$

for $K^\cdot \in \mathbf{D}^b(V, \mathbb{B}_\ell)$ (The brackets denoting Tate twist, as usual).

Proof: (Cp. [Lau], theorem (1.2.2.1)). We fix some notation: let $\alpha : V \times V' \times V'' \longrightarrow V' \times V''$ be defined as $\alpha(v, v', v'') = (v', v'' - a(v))$ and $\beta : V \times V'' \longrightarrow V''$ as $\beta(v, v'') = v'' - a(v)$.

Consider the commutative diagram:

$$\begin{tikzcd}
 & V \times V'' \arrow[rr, "\beta"] \arrow[ddl, "\mathrm{pr}"'] \arrow[ddrr, "\mathrm{pr}''"] & & V'' \\
 & V \times V' \times V'' \arrow[u, "\mathrm{pr}_{13}"] \arrow[rr, "\alpha"] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{23}"] & & V' \times V'' \arrow[u, "\mathrm{pr}''"'] \\
V & V \times V' \arrow[l, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}'"] & V^* \times V'' \arrow[dl, "\mathrm{pr}'"'] \arrow[dr, "\mathrm{pr}''"] & \\
 & V' & & V''
\end{tikzcd}$$

where the two squares are fiber diagrams.

It follows easily from property **LT2** that

$$\mathrm{pr}_{12}^*\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}_{23}^*\mathcal{L}(\langle,\rangle) = \alpha^*\mathcal{L}(\langle,\rangle). \tag{3}$$

Then we have:

$$\begin{aligned}
\mathcal{F}'\circ\mathcal{F}(K^\cdot) &\simeq \mathcal{F}'(R\mathrm{pr}'_!(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^*K^\cdot)[n]) \\
&\simeq R\mathrm{pr}''_!(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}'^*(R\mathrm{pr}'_!(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^*K^\cdot)))[2n] \\
&\simeq R\mathrm{pr}''_!(\mathcal{L}(\langle,\rangle) \otimes R\mathrm{pr}_{23!}\mathrm{pr}_{12}^*(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^*K^\cdot))[2n] && \text{(proper base change)} \\
&\simeq R\mathrm{pr}''_!(\mathcal{L}(\langle,\rangle) \otimes R\mathrm{pr}_{23!}(\mathrm{pr}_{12}^*\mathrm{pr}^*K^\cdot))[2n] \\
&\simeq R\mathrm{pr}''_!(\mathcal{L}(\langle,\rangle) \otimes \mathcal{L}(\langle,\rangle) \otimes R\mathrm{pr}_{23!}(\alpha^*\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}_{23}^*\mathcal{L}(\langle,\rangle)^{-1} \otimes \mathrm{pr}_{12}^*\mathrm{pr}^*K^\cdot))[2n] \\
& && \text{(by formula (3))} \\
&\simeq R\mathrm{pr}''_!(\mathcal{L}(\langle,\rangle) \otimes \mathcal{L}(\langle,\rangle)^{-1} \otimes R\mathrm{pr}_{23!}(\alpha^*\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}_{12}^*\mathrm{pr}^*K^\cdot))[2n] && \text{(proj. formula)} \\
&\simeq R\mathrm{pr}''_!R\mathrm{pr}_{23!}(\alpha^*\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}_{12}^*\mathrm{pr}^*K^\cdot) \\
&\simeq R\mathrm{pr}''_!R\mathrm{pr}_{13!}(\alpha^*\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}_{13}^*\mathrm{pr}^*K^\cdot) && \text{(functoriality)} \\
&\simeq R\mathrm{pr}''_!(\mathrm{pr}^*K^\cdot \otimes R\mathrm{pr}_{13!}\alpha^*\mathcal{L}(\langle,\rangle)) && \text{(projection formula)} \\
&\simeq R\mathrm{pr}''_!(\mathrm{pr}^*K^\cdot \otimes \beta^*\mathrm{pr}''_!\mathcal{L}(\langle,\rangle)). && \text{(proper base change)}
\end{aligned}$$

The proof is concluded with an application of the lemma that follows. □

Lemma 11 *For any $L^\cdot \in \mathbf{D}^b(\mathrm{Spec}k, \mathbf{B}_\ell)$ we have:*

$$\mathcal{F}(\pi^*L[n]) \simeq \sigma_*L^\cdot(-n).$$

Proof: By the projection formula:

$$\mathcal{F}(\pi^* L[n]) = L^{\cdot} \otimes R\mathrm{pr}'_! \mathcal{L}^{\cdot}(\langle , \rangle)[2n].$$

On the other hand, using proper base change, property **LT1** and proposition 12, we get:

$$\sigma^* R\mathrm{pr}'_! \mathcal{L}^{\cdot}(\langle , \rangle) = R\pi_! \mathbf{B}_\ell = \mathbf{B}_\ell(-n)[-2n]$$
$$R\mathrm{pr}'_! \mathcal{L}^{\cdot}(\langle , \rangle)_{|V^* - \{0\}} = 0.$$

□

Corollary 1 *$\mathcal{F}$ is an equivalence of triangulated categories of $\mathbf{D}^b(V, \mathbf{B}_\ell)$ onto $\mathbf{D}^b(V', \mathbf{B}_\ell)$, with inverse $a^* \mathcal{F}'(-)(n)$.*

□

In the case of the Fourier transform over a finite field, it is known moreover that $\mathcal{F}$ preserves the t-structure coming from middle perversity. Even in absence of a theory of perverse sheaves for analytic varieties, we can formulate a suitable conjecture:

Conjecture 2 *There is an isomorphism of functors:*

$$\mathcal{F}(-) \simeq R\mathrm{pr}_{2*}(\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_1^*(-)).$$

In the next few theorems we derive the other main formal properties of the Fourier transform. The proofs have the same flavour as the previous proof of involutivity, therefore we leave the details to the reader.

Theorem 13 *Let $V_1 \to V_2$ a linear morphism of vector spaces of dimension r_1 and r_2 respectively, and let $f' : V_2' \to V_1'$ be the transpose of f. Then there is a canonical isomorphism*

$$\mathcal{F}_2(Rf_! K_1^{\cdot}) \simeq f'^* \mathcal{F}_1(K_1^{\cdot})[r_2 - r_1]$$

for all $K_1^{\cdot} \in \mathbf{D}^b(V_1, \mathbf{B}_\ell)$.

Proof: (Cp. [Lau], theorem (1.2.2.4)). Since f and f' are adjoint maps, we have

$$(f \times 1_{V_2})^* \mathcal{L}(\langle , \rangle_2) = (1_{V_1} \times f')^* \mathcal{L}(\langle , \rangle_1).$$

Then the proof proceeds by repeated application of projection formulas and proper base change by chasing the following commutative diagram with cartesian squares

$$\begin{tikzcd}
 & & V_1 \times V_2' \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "f \times 1"] \arrow[drr, "\mathrm{pr}_1'"] & & \\
V_1 \arrow[dr, "f"'] & V_1 \times V_2' \arrow[l, "\mathrm{pr}_1"'] \arrow[ur, "1 \times f'"] \arrow[dr, "f \times 1"] & & V_2 \times V_1' \arrow[r, "\mathrm{pr}_1'"] & V_1' \\
 & V_2 & V_2 \times V_2' \arrow[l, "\mathrm{pr}_2"'] \arrow[r, "\mathrm{pr}_2'"] \arrow[ur, "1 \times f'"] & V_2' \arrow[ur, "f'"'] &
\end{tikzcd}$$

□

Corollary 2 *There is a canonical isomorphism*

$$R\pi_!'\mathcal{F}(K^\cdot) \simeq \sigma^* K^\cdot(-r)[-r]$$

for all $K^\cdot \in \mathbf{D}^b(V_1, \mathbf{B}_\ell)$.

Proof: We apply theorem 13 to the case $V_1 = V'$, $V_2 = \mathrm{Spec}\,k$, $f = \pi'$ and $K_1^\cdot = \mathcal{F}(K^\cdot)$ and then we use theorem 12. □

Definition 19 *The convolution product on* V *is the operation*

$$* : \mathbf{D}^b(V, \mathbf{B}_\ell) \times \mathbf{D}^b(V, \mathbf{B}_\ell) \to \mathbf{D}^b(V, \mathbf{B}_\ell)$$

defined as

$$K_1^\cdot * K_2^\cdot = Rs_!(K_1^\cdot \boxtimes K_2^\cdot).$$

Proposition 13 *There is a canonical isomorphism*

$$\mathcal{F}(K_1^\cdot * K_2^\cdot) \simeq \mathcal{F}(K_1^\cdot) \odot \mathcal{F}(K_2^\cdot)[-r]$$

for all $K_1^\cdot, K_2^\cdot \in \mathbf{D}^b(V, \mathbf{B}_\ell)$.

Proof: (Cp. [Lau], theorem (1.2.2.7)). If we denote again by $\mathcal{F}$ the Fourier transform for the vector space $V \times V$, then by Kunneth formula we have

$$\mathcal{F}(K_1 \boxtimes K_2^\cdot) = \mathcal{F}(K_1^\cdot) \boxtimes \mathcal{F}(K_2^\cdot).$$

Then it suffices to apply theorem 13 to the case $V_1 = V \times V$, $V_2 = V$, $f = s$ with K_1 replaced by $K_1 \boxtimes K_2$. □

Proposition 14 *There is a canonical "Plancherel" isomorphism*

$$R\pi_!'(\mathcal{F}(K_1) \otimes \mathcal{F}(K_2^\cdot)) \simeq R\pi_!(K_1^\cdot \odot [-1]^* K_2^\cdot)(-r)$$

for all $K_1, K_2^\cdot \in \mathbf{D}^b(V, \mathbf{B}_\ell)$.

Proof: (Cp. [Lau], theorem (1.2.2.8)). One applies in sequence proposition 13, corollary 2, and proper base change for the cartesian square diagram

$$\begin{tikzcd} V \arrow[r, "(1_V, -1_V)"] \arrow[d, "\pi"'] & V \times V \arrow[d, "s"] \\ \mathrm{Spec}\,k \arrow[r, "\sigma"] & E. \end{tikzcd}$$

□

6.2 Computation of some Fourier transforms

The following examples of calculation of Fourier trasforms are taken from [Lau], with the exception of proposition 18, that has no analogue in positive characteristic.

Proposition 15 *Let $W \overset{i}{\hookrightarrow} V$ be a linear subspace of dimension s. Denote by $W^{\perp} \overset{i^{\perp}}{\hookrightarrow} V'$ the orthogonal of W in V'. Then there is a canonical isomorphism*

$$\mathcal{F}(i_*\mathbf{B}_{\ell,F}[s]) \simeq i_*^{\perp}\mathbf{B}_{\ell,F^{\perp}}(-s)[r-s].$$

Proof: It follows from theorem 13 and lemma 11. □

Proposition 16 *Let $v \in V(k)$. Denote by $\tau_v : V \to V$ the translation by v. Then there is a canonical isomorphism*

$$\mathcal{F}(\tau_{v*}K^{\cdot}) \simeq \mathcal{F}(K) \otimes \mathcal{L}(\langle v, \ \rangle)$$

for all $K^{\cdot} \in \mathbf{D}^b(V, \mathbf{B}_\ell)$.

Proof: We have $\tau_{v*}K^{\cdot} = (v_*\mathbf{B}_\ell) * K^{\cdot}$ and one applies proposition 13. □

Proposition 17 *Let $\alpha : V \xrightarrow{\sim} V'$ be a symmetric isomorphism. Denote by $q : V \to \mathbf{G}_a$ and $q' : V' \to \mathbf{G}_a$ the quadratic forms associated to α (i.e. $q(v) = \langle e, \alpha(v)\rangle$ and $q'(v') = \langle \alpha^{-1}(v'), v'\rangle$). Let $[2] : E' \to V'$ be multiplication by 2 on the vector space V'. Then there is a canonical isomorphism*

$$[2]^*\mathcal{F}(\mathcal{L}(q)) \simeq \mathcal{L}(-q') \otimes \pi'^* R\pi_!\mathcal{L}(q)[r].$$

Proof: It follows from the formula

$$q(v) + \langle v, 2v'\rangle = q(v + \alpha^{-1}(v')) - q'(v').$$

□

For the next result, we suppose V has dimension one for simplicity, and we identify both V and V' with $\mathbf{G}_a$, in such a way that $\langle , \rangle$ becomes multiplication in $\mathbf{G}_a$. For any positive real number ρ, let j_ρ (resp. i_ρ) be the imbedding of $\mathbf{G}_a(\rho)$ (resp. D_ρ) in $\mathbf{G}_a$.

Proposition 18 *Fix a real number $\alpha > 0$ and let β such that $\alpha\beta = \rho(\chi)$. Then*
i) $\mathcal{F}(i_{\alpha*}\mathbf{B}_{\ell,D_\alpha}) = j_{\beta!}\mathbf{B}_{\ell,\mathbf{G}_a(\beta)}[1]$,
ii) $\mathcal{F}(j_{\alpha!}\mathbf{B}_{\ell,\mathbf{G}_a(\alpha)}) = i_{\beta*}\mathbf{B}_{\ell,D_\beta}(-1)[-1]$.

Proof: By theorem 12 we see that (i) and (ii) are equivalent. We will prove (ii). Set $T = \mathbf{G}_a(\alpha) \times D_\beta$. Note that the condition $\alpha\beta = \rho(\chi)$ implies that $\mathcal{L}(\langle , \rangle)$ trivializes on T. It follows that the restriction of $\mathcal{F}(j_{\alpha!}\mathbf{B}_{\ell,\mathbf{G}_a(\alpha)})$ to D_β coincides with $\mathbf{B}_\ell[-1]$.Therefore it suffices to show that $\mathcal{F}(j_{\alpha!}\mathbf{B}_{\ell,\mathbf{G}_a(\alpha)})$ vanishes outside D_β. To this purpose we can check on the stalks, and then the claim follows from proposition 12. □

Your soul is waiting for you!
-Swami Niranjan

7 Open Questions

Beside the two conjectures stated in the previous chapters, several questions are naturally prompted by our work. Some of them must still remain vague for lack of an adequate language, but some others seem to be already accessible to the present means, and hopefully they will provide research material for the near future.

Collected in this final chapter are some of these open questions.

1.*Analytic Class Field Theory.* As it has been seen in chapter 5, the closed one-dimensional disc D of radius 1 possesses many connected analytic etale coverings. By mimicking algebraic geometry, we can easily define the fundamental group of an analytic variety, and therefore the observation above can be paraphrased by saying that the fundamental group of D is non-trivial.

Suppose $X = \mathcal{X}^{an}$ arises by "analytification" of the scheme $\mathcal{X}$. Then both the fundamental groups of X and $\mathcal{X}$ are defined, and the first classifies analytic coverings of X, while the second classifies algebraic coverings of $\mathcal{X}$. It should be noticed that these two groups do *not* coincide. Such a behaviour is already manifest in the case of the affine line $\mathbb{A}^1$: this is simply connected as an algebraic variety, but its fundamental group as an analytic variety is very large.

Given any analytic variety X we will denote $\pi_1^{an}(X)$ this new invariant, for which we propose the name "*analytic fundamental group*". Of course it would be interesting to compute $\pi_1^{an}(X)$, at least for some simple varieties. In first approximation we could try to determine its abelianization. In analogy with the case of algebraic geometry, this should be called the problem of Analytic Class Field Theory. One would expect that there should be an adelic description of $\pi_1^{an}(X)$, on the model of Kato-Saito theory. Already the case of curves presents interesting aspects. Kato and Saito produce a map from a certain adelic group to the abelianized fundamental group of any non-singular algebraic curve. This adelic group is built as a restricted product of local factors associated via K-theory to the local rings of the closed points. What are the local factors for analytic class field theory?

As far as I know, this is an almost totally unexplored territory, but here are the basic known facts.

a) Analytic etale cohomology computes the abelianization of $\pi_1^{an}(X)$: we have the formula

$$\pi_1^{an,ab}(X) = \mathrm{Hom}(H^1(X, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}).$$

b) For any analytic variety X defined over the field k it is possible to find a formal scheme $\mathcal{X}$ locally of finite type over the ring of integers of k, such that the "generic fibre" of $\mathcal{X}$ is X and the special fibre is a scheme of finite type $\mathcal{X}_s$ over the residue field of k (see [Bo-Lü]). If X is proper, the fundamental group of $\mathcal{X}$ is isomorphic to the fundamental group of $\mathcal{X}_s$, and there is a surjective group homomorphism

$$\pi_1^{an}(X) \longrightarrow \pi_1(\mathcal{X}).$$

For $X = D$ the closed disc, (b) gives a surjection

$$\pi_1^{an}(D) \longrightarrow \pi_1(\mathbb{A}^1_{\mathbf{F}_q})$$

where $\mathbf{F}_q$ is the residue field of k. The kernel of this map should be considered as some sort of "analytic inertia", and it is this part of $\pi_1^{an}(D)$ to be responsible for the covering maps $[\pi]_f : D \to D$ that arise in the thesis.

2. *Harmonic Analysis.* Let X be a scheme defined over a finite field $\mathbf{F}_q$ and let $K^{\cdot}$ be any complex of ℓ-adic sheaves on X. Grothendieck showed how to associate to $K^{\cdot}$ a function $f(K^{\cdot}) : X(\mathbf{F}_q) \to \mathbf{Q}_\ell$. By means of this correspondence, common operations on functions can be given cohomological interpretations. Here is a sample:

$$f(K^{\cdot} \oplus L^{\cdot}) = f(K^{\cdot}) + f(L^{\cdot})$$
$$f(K^{\cdot} \overset{\mathbf{L}}{\otimes} L^{\cdot}) = f(K^{\cdot}) \cdot f(L^{\cdot})$$
$$f(R\phi_!(K^{\cdot}))(y) = \textstyle\sum_{x \in \phi^{-1}(y) \cap X(\mathbf{F}_q)} f(K^{\cdot})(x)$$

for any map of schemes $\phi : X \to Y$ and any $y \in Y(\mathbf{F}_q)$.

To define $f(K^{\cdot})(x)$, we consider the action of the local Frobenius generator Fr_x on the stalks of the homology of the complex $H^i(K^{\cdot})$. Then we let

$$f(K^{\cdot})(x) = \sum_i (-1)^i \mathrm{Trace}(\mathrm{Fr}_x, H^i(K^{\cdot})_x).$$

The first two formulas above follow immediately, and the third is a reformulation of the Grothendieck-Nielsen fixed point formula for the action of Frobenius.

Given any function f on $\mathbf{F}_q^n = \mathbb{A}^n(\mathbf{F}_q)$ with values in $\mathbf{Q}_\ell$, we denote by $\mathcal{F}_\chi(f)$ the discrete Fourier transform of f, relative to the character $\chi : \mathbf{F}_q \to \mathbf{Q}_\ell^\times$. Recall that this is the function $\mathcal{F}_\chi(f) : \mathbf{F}_q^n \to \mathbf{Q}_\ell$ defined by the formula

$$\mathcal{F}_\chi(f)(y) = \sum_{x \in \mathbf{F}_q^n} \chi(\langle x, y \rangle) \cdot f(x)$$

where $\langle , \rangle$ is the scalar product in $\mathbf{F}_q^n$.

On the other hand, let $\mathcal{F}_\chi(K^{\cdot})$ be the Deligne-Fourier transform of the complex $K^{\cdot}$ of ℓ-adic sheaves on $\mathbb{A}^n(\mathbf{F}_q)$. Then it is not hard to check the equality

$$f(\mathcal{F}_\chi(K^{\cdot})) = \mathcal{F}_\chi(f(K^{\cdot})).$$

Notice in particular that if $\mathcal{L}'_\chi$ is the Lang torsor associated to the character χ, then

$$f(\mathcal{L}'_\chi) = \chi. \tag{4}$$

We could try to repeat these constructions in the p-adic case. In this case our Fourier transform should correspond to the Fourier transform defined on the space of L^2-functions on a vector space V over k. Recall that V is a locally compact topological group, therefore it is endowed with an invariant measure, unique up to multiplication by a constant. The L^2-Fourier

transform is determined by assigning a character $k \to \mathbb{C}^\times$, in the same way as our ℓ-adic Fourier transform is determined by a character $G_\infty \to \mathbf{B}_\ell{}^\times$.

Then the calculations of section 5.2 can be interpreted as establishing the analogue of equation 4. But the analogy falls short of the expectations: to complete the dictionary we should be able to express integration of L^2-functions associated to complexes, in terms of cohomological operations on the complexes themselves. In the finite field case, this is achieved by means of the Grothendieck-Nielsen formula, but in the ℓ-adic case no analogue of this formula is known.

In fact, in my opinion it is doubtful whether the formalism of etale sheaves is suited for such a purpose. Possibly some new fundational ideas will be required to clarify the situation.

But perhaps a first step could be already at the reach of our means. In Deligne's words, 'where a Fourier transform is, there should be an action of the metaplectic group'. The metaplectic representation is an action of a double covering of $SL(2,k)$ on the space of L^2-functions over $\mathbb{A}^1_k$, defined in vast generality by Weil for any locally compact Hausdorff topological field k. In a letter to Kazhdan (see [Del], Deligne shows how to construct a complex of constructible ℓ-adic complexes $K^\cdot$ on the variety $\mathcal{W}_{\mathbf{F}_q} = SL(2,\mathbf{F}_q) \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q}$, that works as the cohomological counterpart of the Weil representation for finite fields.

By this we mean the following. First of all, notice that in the case of a finite field, the metaplectic representation descends to an action ρ of the group $SL(2,\mathbf{F}_q)$. Let

$$p_1, p_2 : SL(2,\mathbf{F}_q) \times SL(2,\mathbf{F}_q) \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q} \to \mathcal{W}_{\mathbf{F}_q}$$

be the projections defined as

$$\begin{aligned} p_1(g_1, g_2, x_1, x_2, x_3) &= (g_2, x_2, x_3) \\ p_2(g_1, g_2, x_1, x_2, x_3) &= (g_1, x_1, x_2) \end{aligned}$$

and define

$$p_3 : SL(2,\mathbf{F}_q) \times SL(2,\mathbf{F}_q) \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q} \to SL(2,\mathbf{F}_q) \times SL(2,\mathbf{F}_q) \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q}$$

by the formula

$$p_3(g_1, g_2, x_1, x_2, x_3) = (g_1, g_2, x_1, x_3).$$

Lastly, let

$$\mu : SL(2,\mathbf{F}_q) \times SL(2,\mathbf{F}_q) \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q} \to \mathcal{W}_{\mathbf{F}_q}$$

be given by

$$\mu(g_1, g_2, x_1, x_2) = (g_1 \cdot g_2, x_1, x_2).$$

Then we have the equality

$$Rp_{3!}(p_1^* K^\cdot \otimes p_2^* K^\cdot) = \mu^* K^\cdot. \tag{5}$$

Moreover, for any $\mathbf{F}_q$-rational point g of $SL(2,\mathbf{F}_q)$, let $K^\cdot_g$ be the restriction of $K^\cdot$ to $\{g\} \times \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q}$. Then $f(K^\cdot_g)$ is the kernel of the operator $\rho(g) : L^2(\mathbb{A}^1_{\mathbf{F}_q}(\mathbf{F}_q)) \to L^2(\mathbb{A}^1_{\mathbf{F}_q}(\mathbf{F}_q))$. In particular, for $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the sheaf $K^\cdot_g$ is the pull-back of $\mathcal{L}'_\chi$ by means of the multiplication map $m : \mathbb{A}^1_{\mathbf{F}_q} \times \mathbb{A}^1_{\mathbf{F}_q} \to \mathbb{A}^1_{\mathbf{F}_q}$.

In case k is a p-adic field, the Weil representation does not descend any longer to $SL(2,k)$, however the metaplectic group M is still an algebraic group defined over k, and we can form

the variety $\mathcal{W}_k = M \times \mathbf{A}^1_k \times \mathbf{A}^1_k$. Let $g \in M$ be a lifting of the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,k)$. Then it is natural to ask the following

Question: Is it possible to find a complex $K^{\cdot}$ of ℓ-adic sheaves on $\mathcal{W}_k$ for which the obvious analogue of equation 5 holds, and such that the restriction of $K^{\cdot}$ to $\{g\} \times \mathbf{A}^1_k \times \mathbf{A}^1_k$ is the sheaf $m^*\mathcal{L}_\chi$?

3.*Subanalytic Stratifications.* According to one of our conjectures, we expect that there should be a theory of perverse sheaves for analytic varieties, and the Fourier transform should preserve perversity. There is little doubt that for any reasonable definition of perverse sheaf, the computations of section 6.2 will all serve to corroborate this conjecture. That is, all except for proposition 18: if we model our notion of perverse sheaf according to the examples of algebraic geometry or complex analytic geometry, the kind of sheaves considered in proposition 18 simply have no right of membership in this category.

More precisely: to say what a perverse sheaf is, we must first decide what a constructible sheaf is, and in turn this amounts to select, among all the possible stratifications of a variety, a set of 'admissible' ones. In the complex analytic category, a stratification is admissible if all the strata are analytic subsets; in the algebraic category, we restrict to stratifications by algebraic subsets.

Now, in defining admissible stratifications for rigid analytic varieties, one is tempted to follow the model of complex analytic varieties: in this case the strata would have to be analytic subvarieties. That this choice should give a reasonable theory is the content of Conjecture 1 in the thesis. Yet, according to this definition, a lot of interesting sheaves would be declared not-constructible, and among them also some which seem to exhibit a rather 'controlled' behavior. An example is the extension by zero of a constant sheaf on a bounded disc, that appears in proposition 18.

Question: Is it possible to find a class of admissible stratifications that is

1) general enough so that most interesting sheaves (like the one in proposition 18) become constructible, and at the same time

2) sufficiently restrictive so that there is a good theory of perverse sheaves based on this class of stratifications?

Here, the closest known analogue seems to be the theory of *subanalytic sets.* I refer to [Ka-Sch] for a complete exposition of a theory of perverse sheaves in the real analytic category, based on the notion of subanalytic stratification. Note that a (closed or open) bounded disc in the real plane *is* a subanalytic set.

Perhaps rigid analytic varieties are closer to real analytic manifolds than to complex varieties. Notice in particular that if the analogy holds, we would expect two distinct middle perversities, and the Fourier transform should preserve both of them.

References

[B1] V. Berkovich, Etale cohomology for rigid analytic varieties. *To appear in Publ. Math. IHES*

[B2] V. Berkovich, Vanishing cycles for rigid analytic varieties. *To appear in J. Am. Math. Soc.*

[B3] V. Berkovich, Vanishing cycles for formal schemes. *To appear in Inventiones Math.*

[B4] V. Berkovich, Spectral Theory and Analytic Geometry over non-Archimedean Fields *Math. Surveys and Monographs* 33 (1990)

[Bo-Lü] S. Bosch and W. Lütkebohmert, Formal and Rigid Geometry. I. Rigid Spaces. *Mathematische Annalen* 295 (1993) pp. 291-318

[BGR] S. Bosch, U. Guntzer,R. Remmert, Non-Archimedean Analysis. *Grundlehren der Mathematischen Wissenschaften* 261 (1984).

[Del] P.Deligne, Letter to Kazhdan, *29-11-1976.*

[SGA4$\frac{1}{2}$] P. Deligne et al., Seminaire de Geometrie Algebrique; Cohomologie Etale. *Lecture Notes in Mathematics* 569 (1977).

[Ek] T. Ekedahl, On the adic formalism. *The Grothendieck Festschrift* vol.II (1988)

[Fr] A. Frolich, Formal Groups. *Lecture Notes in Mathematics* 74 (1968).

[Gro] A. Grothendieck, Sur quelques points d'algebre homologique. *Tohoku Math. J.* 9 (1956)

[Ka-Sch] Kashiwara and Schapira, Sheaves on Manifolds. *Grundlehren der Mathematischen Wissenschaften* 292 (1990).

[La] S.Lang, Cyclotomic Fields I and II. *Springer GTM* 121 (1990)

[Lau] G.Laumon, Transformation de Fourier, Constantes d'equations fonctionelles et Conjecture de Weil. *Publ. Math. IHES* 68 (1987)

[LT] J. Lubin and J. Tate, Formal complex multiplication in local fields. *Ann. of Math.* 81 (1965) pp.380-387

[Mi] J.S. Milne, Etale Cohomology, *Princeton Mathematical Series* 33 (1980).

[Wi] A. Wiles, Higher Explicit Reciprocity Laws. *Ann. of Math.* 107 (1978) pp. 235-254