Examples of Solvable Quantum Groups
and Their Representations

by

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Abstract

In this paper we first define a q-deformation of the universal enveloping algebra of the Heisenberg Lie algebra. We study this algebra and its finite-dimensional irreducible representations when $q = \varepsilon$, where $\varepsilon$ is a primitive $\ell$th root of 1 with $\ell$ odd.

For each element of the Weyl group of a finite-dimensional simple Lie algebra, there is a corresponding solvable quantum group. We find generators and relations for each of these algebras in the case of the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$, and we also find the central elements. Setting $q = \varepsilon$, where $\varepsilon$ is a primitive $\ell$th root of 1 with $\ell$ odd, we then study the finite-dimensional irreducible representations of these algebras. It is shown that each representation has dimension either 1, $\ell$, or $\ell^2$, and that the dimension depends only on the central character.

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Chapter 1

The Quantum Heisenberg Algebra

In this chapter we define the quantum Heisenberg algebra $\mathcal{H}$, which is a $q$-deformation of the universal enveloping algebra of the Heisenberg Lie algebra. Setting $q = \varepsilon$, we obtain the algebra $\mathcal{H}_\varepsilon$. We examine the finite-dimensional irreducible representations of this algebra when $\varepsilon$ is a primitive $\ell$th root of 1, with $\ell > 2$.

1.1 Definition and Basic Properties

Definition 1.1 The quantum Heisenberg algebra $\mathcal{H}$ is the associative algebra over the ring $\mathcal{A} = \mathbb{C}[q, q^{-1}, (q - q^{-1})^{-1}]$ with generators $a, b, c$ and relations

\begin{align*}
ab - qba &= c \\
ac - q^{-1}ca &= 0 \\
bc - qcb &= 0
\end{align*}

We further define $\mathcal{H}_\varepsilon$, $\varepsilon \in \mathbb{C}$, $\varepsilon \neq 0, 1, -1$, as the algebra $\mathcal{H}/(q - \varepsilon)\mathcal{H}$. We observe that $\mathcal{H}_1$ is the universal enveloping algebra of the Heisenberg Lie algebra.

Proposition 1.2 (a) The elements $a^i b^j c^k, (i, j, k) \in \mathbb{Z}_+^3$, form a basis of $\mathcal{H}$ over $\mathcal{A}$ and of $\mathcal{H}_\varepsilon$ over $\mathbb{C}$.

(b) The algebras $\mathcal{H}$ and $\mathcal{H}_\varepsilon$ have no zero divisors.
Proof: (a) The elements $a^ib^jc^k$ clearly span. To prove they are a basis, it suffices to show that the element $c_{ba}$ reduces to the same element whether we begin by reducing $cb$ or $ba$ in the product. Checking, we have $(cb)a = (q^{-1}bc)a = q^{-1}b(ca) = b(ac) = (ba)c = q^{-1}(ab - c)c = q^{-1}abc - q^{-1}c^2$ and $c(ba) = q^{-1}c(ab - c) = q^{-1}(ca)b - q^{-1}c^2 = (ac)b - q^{-1}c^2 = a(cb) - q^{-1}c^2 = q^{-1}abc - q^{-1}c^2$.

(b) To see that there are no zero divisors, we note that $(a^ib^jc^k + \text{lower-degree terms})(a^ib^jc^k + \text{lower-degree terms}) = q^{k^r-j^r-k^s}a^{i+r+j+s+k+t} + \text{lower degree terms})$.

**Proposition 1.3** The element $(q - q^{-1})abc - q^{-1}c^2$ generates the center $Z$ of $H$.

Proof: It is easily checked that this element commutes with each of the generators $a$, $b$, and $c$. Let $z = (a^ib^jc^k + \text{lower-degree terms})$ be central. Then $a^ib^jc^k$ must commute, modulo lower-degree terms, with each of the generators $a$, $b$, and $c$. This gives the condition that $i = j = k$. Then $z = (a^mb^mc^m + \text{lower-degree terms}) - q^{(1/2)m(m-1)}[abc - q^{-1}(q - q^{-1})^{-1}c^2]^m$ is a central element of degree less than that of $z$. By induction on degree, the proof is complete.

**Lemma 1.4** (a) In $H$, for $m = 1, 2, 3, ...$

$$ab^m = q^m b^m a + (q^{-m+1} + q^{-m+3} + ... + q^{-m+1+q^{-m+1}b^{m-1}})$$

$$a^mb = q^m ba^m + (1 + q^2 + ... + q^{2(m-2)} + q^{2(m-1)})a^{m-1}c$$

(b) in $H_\epsilon$, for $m = 1, 2, 3, ...$

$$ab^m = \epsilon^m b^m a + \epsilon^{1-m} \left(\frac{1 - \epsilon^{2m}}{1 - \epsilon^2}\right) b^{m-1}c$$

$$a^mb = \epsilon^m ba^m + \left(\frac{1 - \epsilon^{2m}}{1 - \epsilon^2}\right) a^{m-1}c$$

Proof: (a) By induction on $m$. Part (b) follows from part (a), with $\epsilon \neq 0, 1, \text{or } -1$.

**Proposition 1.5** The center $Z_\epsilon$ of $H_\epsilon$, where $\epsilon$ is a primitive $\ell$th root of 1, is generated by $a^\ell, b^\ell, c^\ell$, and $(\epsilon - \epsilon^{-1})abc + \epsilon^{-1}c^2$. 

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Proof: $c^t$ clearly commutes with $a$ and $b$. $a^t$ commutes with $c$, and with $b$ by the preceding lemma. Likewise $b^t$ commutes with $a$ and $c$. The element $(q - q^{-1})abc - q^{-1}c^2$ lies in the center $Z$ of $H$, so $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2$ lies in $Z$. Let $z = (a^t b^t c^k + \text{lower-degree terms})$ be central. Then $a^t b^t c^k$ must commute, modulo lower-degree terms, with each of the generators $a$, $b$, and $c$. This gives the condition that $i = j = k \pmod{\ell}$. Then $z = (a^m + t r b^m + t s c^m + \ell t + \text{lower-degree terms}) - \varepsilon^{(1/2)m(m-1)}a^t b^t c^k\varepsilon'[abc - \varepsilon^{-1}(\varepsilon - \varepsilon^{-1})^{-1}c^2]^{m}$ is a central element of degree less than that of $z$. By induction on degree, the proof is complete.

1.2 Irreducible Representations of $H_\varepsilon$

We now consider the finite-dimensional irreducible representations of $H_\varepsilon$, where $\varepsilon$ is a primitive $\ell$th root of 1, with $\ell > 2$. Since $a^t, b^t, c^t$, and $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2$ are central elements of $H_\varepsilon$, by Schur's Lemma they act as scalars $a^t = x, b^t = y, c^t = z$, and $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2 = w$ in any finite-dimensional irreducible representation.

**Proposition 1.6** The finite-dimensional irreducible representations of $H_\varepsilon$, where $\varepsilon$ is a primitive $\ell$th root of 1, have the following dimensions:

1. $1$ if $z = 0$, and $x$ or $y$ is zero

2. $\ell/2$ if $z \neq 0, x = 0, y = 0$, and $\ell$ is even

3. $\ell$ if $z = 0, x \neq 0, y \neq 0$

   - if $z \neq 0, x \neq 0$ or $y \neq 0$
   - if $z \neq 0, x = 0, y = 0$, and $\ell$ is odd

Proof: Let $V$ be an irreducible $H_\varepsilon$-module.

Case 1: Suppose that $z = c^t = 0$ on $V$. Then, since $c$ q-commutes (see definition 2.1) with $a$ and $b$, it follows that $c = 0$ on $V$ (see the proof of lemma 2.2). Then $V$ is an irreducible module over the generators $a$ and $b$, which satisfy the relation $ab = \varepsilon ba$ on $V$.

   1a) If $x = a^t = 0$, then since $a$ and $b$ q-commute on $V$, it follows that $a = 0$ on $V$. Then $V$ is one dimensional, spanned by an eigenvector of $b$. Likewise, if $y = b^t = 0$, then $\dim V = 1$. 

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1b) If \( x = a^t \neq 0 \) and \( y = b^t \neq 0 \), let \( v \) be an eigenvector of \( a; av = \lambda v \), \( \lambda \neq 0 \). Then the vectors \( v, bv, ..., b^{t-1}v \) are eigenvectors (these vectors are nonzero since \( b^t \neq 0 \)) of \( a \) with distinct eigenvalues \( \lambda, \varepsilon \lambda, ..., \varepsilon^{t-1} \lambda \), respectively. The space \( \text{span}(v, bv, ..., b^{t-1}v) \) is invariant under \( a \) and \( b \) (and \( c \), since \( c = 0 \) on \( V \)), so by irreducibility is equal to \( V \). Thus \( \dim V = t \).

Case 2: \( z = c^t \neq 0 \), and \( x = a^t \neq 0 \) or \( y = b^t \neq 0 \). Suppose first that \( x = a^t \neq 0 \). \( V \) is also a module over the algebra with generators \( a \) and \( c \) and relation \( ca = \varepsilon ac \). Let \( U \) be an irreducible submodule of \( V \) over this algebra. By the same reasoning as in case 1b), we see that \( \dim U = t \). Then from \( (\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2 = w \), we can solve for \( b \), obtaining \( b = [xz(\varepsilon - \varepsilon^{-1})]^{-1}a^{t-1}[w - \varepsilon^{-1}c^2]c^{t-1} \). Thus \( U \) is invariant under \( b \), so \( V = U \) by irreducibility and \( \dim V = t \). Similarly, if \( y = b^t \neq 0 \) we have \( \dim V = t \).

Case 3: \( z = c^t \neq 0 \), \( x = a^t = 0 \), and \( y = b^t = 0 \). Let \( U \) be an irreducible submodule of \( V \) over the algebra with generators \( a \) and \( c \) and relation \( ca = \varepsilon ac \). Since \( a \) q-commutes with \( c \) and \( a^t = 0 \), it follows that \( a = 0 \) on \( U \) (see lemma 2.2). Thus \( U \) is one dimensional, spanned by an eigenvector \( u \) of \( c; cu = \lambda u \), with \( \lambda \neq 0 \) since \( c^t \neq 0 \). The space \( \text{span}(u, bu, ..., b^{t-1}u) \) is seen to be invariant under \( a, b, \) and \( c \), so \( V = \text{span}(u, bu, ..., b^{t-1}u) \). We note further that \( cb^m u = \varepsilon^{-m} \lambda u \), so the spaces \( U, bU, ... , b^{t-1}U \) are eigenspaces of \( c \) with distinct eigenvalues. Thus \( V = U \oplus bU \oplus ... \oplus b^{t-1}U \) (direct sum as vector spaces).

3a) Suppose \( \ell \) is odd. Let \( m \) be the least positive integer such that \( b^m u = 0 \). Applying equation 1.6 to \( u \), we obtain

\[
0 = \lambda \varepsilon^{1-m} \left( \frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2} \right) b^{m-1} u
\]

Thus \( \ell \) divides \( 2m \). Since \( \ell \) is odd, it follows that \( \ell = m \). Thus each of the spaces in the sum \( V = U \oplus bU \oplus ... \oplus b^{t-1}U \) is one dimensional, and \( \dim V = \ell \).

3b) Suppose \( \ell \) is even. Then, applying equation 1.6 to \( u \) with \( m = \ell/2 \), we obtain

\[
ab^{\ell/2} u = 0
\]
For $m = \ell/2 + 1, \ldots, \ell - 1$ we have

$$ab^m u = \lambda \varepsilon^{1-m} \left( \frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2} \right) b^{m-1} u \quad (1.10)$$

It follows that the space span$(b^{\ell/2}u, b^{\ell/2+1}u, \ldots, b^{\ell-1}u)$ is invariant under $a$. It is also invariant under $c$, and invariant under $b$ since $b^\ell = 0$. Thus this is an invariant subspace of $V$. But it does not contain the vector $u$, because $V = U \oplus bU \oplus \ldots \oplus b^{\ell-1}U$ and $u \in U$. Thus by irreducibility, since this space is not equal to $V$, it must be 0. Thus $V = U \oplus bU \oplus \ldots \oplus b^{\ell/2-1}U$. Now let $m$ be the least positive integer such that $b^m u = 0$. By the same reasoning as in case 3a, we see that $\ell$ divides $2m$. Since $m \leq \ell/2$, it follows that $m = \ell/2$. Thus each of the spaces in the sum $V = U \oplus bU \oplus \ldots \oplus b^{\ell/2-1}U$ has dimension one, so $\dim V = \ell/2$.

We now consider only the case where $\varepsilon$ is a primitive $\ell$th root of 1 with $\ell$ odd.

**Proposition 1.7** In any finite-dimensional irreducible representation of $\mathcal{H}_\varepsilon$, with $\varepsilon$ a primitive $\ell$th root of 1 with $\ell$ odd, we have the relation

$$w^\ell = (\varepsilon - \varepsilon^{-1})^\ell xyz + z^2 \quad (1.11)$$

where $a^\ell = x$, $b^\ell = y$, $c^\ell = z$, and $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2 = w$.

Proof: Case 1: If $z = 0$, then $c = 0$ so $w = 0$, and the relation is satisfied trivially.

Case 2: $z \neq 0$, and $x = 0$ or $y = 0$. Suppose first that $a^\ell = x = 0$. As in case 3 of Proposition 1.6, there is a vector $u$ such that $au = 0$ and $cu = \lambda u$, where $\lambda^\ell = z$.

We can rewrite the element $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2$ as $(\varepsilon^2 - 1)bac + c^2$. Applying this element to $u$, we get $wu = \lambda^2 u$. Thus $w = \lambda^2$, and raising this to the $\ell$th power gives $w^\ell = z^2 = (\varepsilon - \varepsilon^{-1})^\ell xyz + z^2$. If $b^\ell = 0$ the proof is similar.

Case 3: $z \neq 0$, $x \neq 0$, and $y \neq 0$. We have seen that in this case the representation is $\ell$ dimensional. Also in this case, $a$, $b$, and $c$ are diagonalizable. For example, letting $v$ be an eigenvector of $a$ with eigenvalue $\lambda$, the vectors $v, cv, \ldots, c^{\ell-1}v$ are a basis for the irreducible $\mathcal{H}_\varepsilon$-module $V$, and these are eigenvectors of $a$ with eigenvalues.
\(\lambda, \lambda^{-1}, \ldots, \lambda^{-(\ell - 1)}\). Thus the determinant of \(a\) is the product of these eigenvalues, which is \(x\). Similarly, the determinants of \(b\) and \(c\) are \(y\) and \(z\), respectively. We now take the determinant of the equation

\[
w - \varepsilon^{-1}c^2 = (\varepsilon - \varepsilon^{-1})abc
\]

(1.12)

The determinant of the right-hand side is \((\varepsilon - \varepsilon^{-1})^\ell xyz\). The determinant of the left-hand side is

\[
\prod_{j=0}^{\ell-1}(w - \varepsilon^{-1}\mu^2\varepsilon^{2j}),
\]

(1.13)

where \(\mu\) is an eigenvalue of \(c\), so \(\mu^\ell = z\). To compute this product we use the Gauss Binomial Formula

\[
\prod_{j=0}^{m-1}(\alpha + q^{2j}\beta) = \alpha^m + q^{m(m-1)}\beta^m + \sum_{j=1}^{m-1} \left( \frac{[m][m-j+1]}{[j][1]} \right) q^{j(m-1)}\alpha^{m-j}\beta^j
\]

(1.14)

where \([n] = (q^n - q^{-n})/(q - q^{-1})\), with \(m = \ell\), \(\alpha = w\), \(\beta = -\varepsilon^{-1}\mu^2\), and \(q = \varepsilon\). Noting that \([\ell] = 0\), this gives \(w^\ell + (-\mu^2)^\ell = w^\ell - z^2\).

**Proposition 1.8** In \(\mathcal{H}_\varepsilon\),

\(a)\)

\[
[(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2]^\ell = (\varepsilon - \varepsilon^{-1})^\ell a^\ell b^\ell c^\ell + (c^\ell)^2
\]

(1.15)

\(b)\)

\[
[(\varepsilon - \varepsilon^{-1})ab + \varepsilon^{-1}c]^\ell = (\varepsilon - \varepsilon^{-1})^\ell a^\ell b^\ell + c^\ell
\]

(1.16)

**Proof:**

\(a)\) \(Z_\varepsilon\) is a finitely generated commutative algebra. Thus, given any nonzero element \(z\) of \(Z_\varepsilon\), there is a finite-dimensional irreducible representation which maps \(z\) to a nonzero scalar. Also, since \(\mathcal{H}_\varepsilon\) is a finitely-generated module over \(Z_\varepsilon\) (as a \(Z_\varepsilon\)-module, \(\mathcal{H}_\varepsilon\) is generated by the monomials \(a^ib^jc^k\) with \(i, j, k < \ell\)). the canonical map Spec \(\mathcal{H}_\varepsilon \rightarrow\) Spec \(Z_\varepsilon\) is surjective. [4] Thus there is a finite-dimensional irreducible
representation of $\mathcal{H}_z$ which maps $z$ to a nonzero scalar. Since we have shown that
the element $[(e - e^{-1})abc + e^{-1}c^2]^\ell - (e - e^{-1})^\ell a^\ell b^\ell c^\ell - (c^\ell)^2$ is mapped to zero in any
finite-dimensional irreducible representation, it follows that this element must be zero
in $\mathcal{H}_z$.

b) Since $c$ commutes with $[(e - e^{-1})ab + e^{-1}c]$, we have $[(e - e^{-1})ab + e^{-1}c]^\ell c^\ell =
[(e - e^{-1})abc + e^{-1}c^2]^\ell = (e - e^{-1})^\ell a^\ell b^\ell c^\ell + (c^\ell)^2$. Now use the fact that $\mathcal{H}_z$ has no
zero divisors.

1.3 Another Quantum Heisenberg Algebra

Consider the algebra over the ring $\mathcal{A} = \mathbb{C}[q, q^{-1}, (q - q^{-1})^{-1}]$, with generators $a_i$, $b_i$,
($i = 1, 2$) and $c$ with relations

\begin{align*}
  b_1b_2 &= b_2b_1 & (1.17) \\
  a_1a_2 &= a_2a_1 & (1.18) \\
  ca_i &= qa_ic & (1.19) \\
  b_ic &= qcb_i & (1.20) \\
  b_ia_j &= qa_jb_i & \text{for } i \neq j & (1.21) \\
  a_ib_i - qb_ia_i &= c & (1.22)
\end{align*}

This is the algebra $U^{a_1a_2b_1b_2}$, which is examined in Chapter 2, with the relabeling
$E_1 \rightarrow b_1$, $E_3 \rightarrow b_2$, $E_{23} \rightarrow a_1$, $E_{12} \rightarrow a_2$, and $E_{123} \rightarrow q^{-1}c$. We find in Chapter 2
for this algebra that the element $[(q - q^{-1})b_1a_1 + c][(q - q^{-1})b_2a_2 + c]$ generates the
center. When $q = e$ where $e$ is a primitive $\ell$th root of 1 with $\ell$ odd, we find that the
finite-dimensional irreducible representations have the following dimensions:

1 if $c' = 0, a'_1 = 0, and a'_2 = 0$
1 if $c' = 0, b'_1 = 0, and b'_2 = 0$
$\ell^2$ if $c' \neq 0$ and $(e - e^{-1})^{\ell} a'_1b'_1 + c' \neq 0$
$\ell^2$ if $c' \neq 0$ and $(e - e^{-1})^{\ell} a'_2b'_2 + c' \neq 0$
\ell \text{ in all other cases.}
Chapter 2
Quantum Groups Associated With $\mathcal{U}_q^+(sl_n(C))$

Let $W$ be the Weyl group of the finite-dimensional simple Lie algebra $sl_n(C)$. For each $w \in W$, there is a corresponding solvable quantum group $\mathcal{U}^w$. Each of these quantum groups is a subalgebra of $\mathcal{U}_q^+(sl_n(C))$; when $w$ is the longest element of $W$, we obtain $\mathcal{U}_q^+(sl_n(C))$. In this chapter, we consider $sl_n(C)$ and give defining relations for $\mathcal{U}^w$ for each $w \in W$. Then, letting $q = \varepsilon$, a primitive $\ell$th root of 1 with $\ell$ odd, we obtain the algebras $\mathcal{U}^w$. We study the finite-dimensional irreducible representations of these algebras, showing that they all have dimensions which are powers of $\ell$. We also show that the dimensions depend only on the central character of the representation.

2.1 $\mathcal{U}_q(sl_n(C))$ and $\mathcal{U}^w$

Let $a_{ij}$ be the Cartan matrix of $sl_n(C)$. Quantum $sl_n(C)$, which we shall designate from this point on as $\mathcal{U}$, is the algebra over the ring $\mathcal{A} = \mathbb{C}[q, q^{-1}, (q - q^{-1})^{-1}]$ with generators $E_i, F_i, K_i, K_{-i}$ ($i = 1, \ldots, n - 1$) and relations

\begin{align*}
K_i K_j &= K_j K_i, \quad K_i K_{-i} = K_{-i} K_i = 1 \quad (2.1) \\
K_i E_j &= q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i \quad (2.2)
\end{align*}
\[ E_i F_j - F_j E_i = \delta_{ij} (K_i - K_i^{-1}) / (q - q^{-1}) \] (2.3)

\[ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if} \quad a_{ij} = -1 \] (2.4)

\[ E_i E_j - E_j E_i = 0 \quad \text{if} \quad a_{ij} = 0 \] (2.5)

\[ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad \text{if} \quad a_{ij} = -1 \] (2.6)

\[ F_i F_j - F_j F_i = 0 \quad \text{if} \quad a_{ij} = 0. \] (2.7)

We have the following automorphisms \( T_i \) \((i = 1, \ldots, n - 1)\) of the algebra \( \mathcal{U} \) [6]:

\[ T_i E_i = -F_i K_i \quad ; \quad T_i E_j = E_j \quad \text{if} \quad a_{ij} = 0 \] (2.8)

\[ T_i E_j = -E_i E_j + q^{-1} E_j E_i \quad \text{if} \quad a_{ij} = -1 \] (2.9)

\[ T_i F_i = -K_i^{-1} E_i \quad ; \quad T_i F_j = F_j \quad \text{if} \quad a_{ij} = 0 \] (2.10)

\[ T_i F_j = -F_j F_i + q F_i F_j \quad \text{if} \quad a_{ij} = -1 \] (2.11)

\[ T_i K_j = K_j K_i^{-a_{ij}} \] (2.12)

These automorphisms \( T_i \) satisfy the braid relations.

Let \( w \in W \), and let \( s_i, \ldots, s_{i_m} \) be a reduced expression for \( w \) in terms of simple reflections. Let \( \beta_1 = \alpha_{i_1}, \ldots, \beta_m = s_{i_1} \ldots s_{i_{m-1}}(\alpha_{i_m}) \). For \( l = 1, \ldots, m \), let \( E_{\beta_l} = T_{i_1} \ldots T_{i_{l-1}} E_{i_l} \) (these depend on the choice of reduced expression for \( w \)). For \( k = (k_1, \ldots, k_m) \in \mathbb{Z}^m_+ \), let \( E^k = E_{\beta_1}^{k_1} \ldots E_{\beta_m}^{k_m} \). These elements form a basis of \( \mathcal{U}^w \) over \( A \) [3].

And for \( i < j \) we have:

\[ E_{\beta_i} E_{\beta_j} - q^{(\beta_i \mid \beta_j)} E_{\beta_j} E_{\beta_i} = \sum_{k \in \mathbb{Z}_+^m} c_k E^k, \] (2.13)

where \( c_k \in \mathbb{C}[q, q^{-1}] \) and \( c_k \neq 0 \) only when \( k = (k_1, \ldots, k_m) \) is such that \( k_s = 0 \) for \( s \leq i \) and \( s \geq j \) [5]. The algebra \( \mathcal{U}^w \) is generated by the elements \( E_{\beta_1}, \ldots, E_{\beta_m} \) with defining relations 2.13. \( \mathcal{U}^w \) is independent of the choice of reduced expression for \( w \) [3].
Setting \( q = \varepsilon \), we obtain the algebra \( \mathcal{U}_\varepsilon^w \). The elements \( E^t_{\beta_i} \) \((i = 1, \ldots, m)\) are central in \( \mathcal{U}_\varepsilon^w \) [3].

### 2.2 Preliminary Results on Irreducible \( \mathcal{U}_\varepsilon^w \) Modules

**Definition 2.1** Let \( x \) and \( y \) be elements of \( \mathcal{U}^w \) (respectively \( \mathcal{U}_\varepsilon^w \)). We shall say that \( x \) and \( y \) \( q - \)commute if they satisfy \( xy - q^s yx = 0 \) (respectively \( xy - \varepsilon^s yx = 0 \)) for some \( s \in \mathbb{Z} \).

The elements \( E^t_{\beta_i} \) are central in the algebra \( \mathcal{U}_\varepsilon^w \). Thus, by Schur's lemma, they act as scalars in any finite-dimensional irreducible representation of \( \mathcal{U}_\varepsilon^w \).

**Lemma 2.2** Suppose \( E_{\beta_1} \) \( q \)-commutes with each of the generators \( E_{\beta_1} \) of \( \mathcal{U}_\varepsilon^w \). If \( E^t_{\beta_1} = 0 \) on a finite-dimensional irreducible \( \mathcal{U}_\varepsilon^w \)-module \( V \), then \( E_{\beta_1} = 0 \) on \( V \).

Proof: Let \( v \in V \) be an eigenvector of \( E_{\beta_1} \). Then \( E_{\beta_1} v = 0 \). By irreducibility, \( v \) generates \( V \) as a \( \mathcal{U}_\varepsilon^w \) module. Thus any element of \( V \) may be written as a linear combination of terms having the form \( E^{k_1}_{\beta_1} \cdots E^{k_m}_{\beta_m} v \). Then \( E_{\beta_1} E^{k_1}_{\beta_1} \cdots E^{k_m}_{\beta_m} v = \varepsilon^s E^{k_1}_{\beta_1} \cdots E^{k_m}_{\beta_m} E_{\beta_1} v = 0 \) (for some \( s \in \mathbb{Z} \)). Thus \( E_{\beta_1} = 0 \) on \( V \).

**Lemma 2.3** Let \( s_{i_1} \ldots s_{i_{m-1}} \) be a reduced expression for \( w \) in terms of simple reflections, and let \( s_{i_1} \ldots s_{i_m} \) be a reduced expression for \( \tilde{w} \). Let \( V \) be a finite-dimensional irreducible module over the algebra \( \mathcal{U}_\varepsilon^\tilde{w} \). \( V \) is also a module over the algebra \( \mathcal{U}_\varepsilon^w \). Let \( U \) be an irreducible submodule of \( V \) over the algebra \( \mathcal{U}_\varepsilon^w \). Then \( V = U \oplus E_{\beta_m} U \oplus \cdots \oplus E^k_{\beta_m} U \) (direct sum as vector spaces) for some \( 0 \leq k \leq \ell - 1 \), where \( \dim E^j_{\beta_m} U = \dim U \) for \( j = 0, \ldots, k \).

Proof: Let \( r \) be the smallest positive integer such that there exists \( u \in U \), \( u \neq 0 \), satisfying \( E^{r+1}_{\beta_m} \in U + E_{\beta_m} U + \ldots + E^r_{\beta_m} U \). We know that \( r \leq \ell - 1 \), because \( E^r_{\beta_m} \) acts as a scalar on \( V \). The sum \( U + E_{\beta_m} U + \ldots + E^r_{\beta_m} U \) is direct (by our choice of \( r \)).
From 2.13 it follows that for \( i < m \) we have

\[
E_{r_i} E_{r_m}^k = \varepsilon^{k(\beta_i|\beta_m)} E_{r_m}^k E_{r_i} + E_{r_m}^{k-1} f_{k-1}(E_{r_1}, ..., E_{r_{m-1}}) + ... + f_0(E_{r_1}, ..., E_{r_{m-1}}),
\]

(2.14)

where \( f_j(E_{r_1}, ..., E_{r_{m-1}}) \) is a polynomial in \( E_{r_1}, ..., E_{r_{m-1}} \). Applying 2.14 to a vector in \( U \), we see that \( U \oplus E_{r_m} U \oplus ... \oplus E_{r_m}^r U \) is invariant under \( E_{r_1}, ..., E_{r_{m-1}} \). We also have by 2.13

\[
E_{r_{m-1}}^{r+1}(E_{r_{m-1}}^{k-1}E_{r_1}^{k_1}) = \varepsilon^{s}(E_{r_{m-1}}^{k-1}E_{r_1}^{k_1}) E_{r_m}^{r+1} + E_{r_m} g_{k_{m-1}}(E_{r_1}, ..., E_{r_{m-1}}) + ...
\]

(2.15)

for some \( s \in \mathbb{Z} \). By irreducibility of \( U \) over \( U^w \), the element \( u \) generates \( U \) over \( U^w \), and any element of \( U \) may be written as a linear combination of terms each having the form \( E_{r_{m-1}}^{k-1}E_{r_1}^{k_1}u \). Applying (2.15) to \( u \), we see that the right-hand side of this equation lies in \( U \oplus E_{r_m} U \oplus ... \oplus E_{r_m}^r U \). Thus \( U \oplus E_{r_m} U \oplus ... \oplus E_{r_m}^r U \) is also invariant under \( E_{r_m} \), and by irreducibility over the algebra \( U^w \), we have \( V = U \oplus E_{r_m} U \oplus ... \oplus E_{r_m}^r U \).

Finally, if \( r = 0 \), the proof is complete. If \( r > 0 \), consider the maps \( E_{r_m} : E_{r_m}^{i-1}U \to E_{r_m}^i U \) (\( i = 1, ..., r \)). Suppose \( E_{r_m}^i \bar{u} = 0 \). Then by choice of \( r \), \( \bar{u} = 0 \) so \( E_{r_m}^{i-1} \bar{u} = 0 \). Thus the nullspace of each of these maps is 0, which implies \( \text{dim} E_{r_m}^{i-1}U \leq \text{dim} E_{r_m}^i U \), hence \( \text{dim} E_{r_m}^{i-1}U = \text{dim} E_{r_m}^i U \).

### 2.3 \( U^w \) for \( w = s_1, s_1s_2, s_1s_2s_1, s_1s_2s_1s_3, s_1s_2s_1s_3s_2, \) and \( s_1s_2s_1s_3s_2s_1 \)

For \( w = s_1s_2s_1s_3s_2s_1 \), we find that \( \beta_1 = \alpha_1, \beta_2 = \alpha_1 + \alpha_2, \beta_3 = \alpha_2, \beta_4 = \alpha_1 + \alpha_2 + \alpha_3, \beta_5 = \alpha_2 + \alpha_3, \) and \( \beta_6 = \alpha_3 \). We then find, using (2.8) through (2.12), that \( E_{\beta_1} = E_1, E_{\beta_2} = -E_1E_2 + q^{-1}E_2E_1, E_{\beta_3} = E_2, E_{\beta_4} = E_1E_2E_3 - q^{-1}E_2E_1E_3 - q^{-1}E_3E_1E_2 + q^{-2}E_3E_2E_1, E_{\beta_5} = -E_2E_3 + q^{-1}E_3E_2, \) and \( E_{\beta_6} = E_3 \). We shall write \( E_{\beta_2} = E_{\alpha_1 + \alpha_2} \)
as $E_{12}$, etc. With some computation, we find the relations (2.13) are as follows:

$$E_1 E_{12} = q E_{12} E_1$$  \hspace{1cm} (2.16)

$$E_1 E_2 = q^{-1} E_2 E_1 - E_{12}$$  \hspace{1cm} (2.17)

$$E_{12} E_2 = q E_2 E_{12}$$  \hspace{1cm} (2.18)

$$E_1 E_{123} = q E_{123} E_1$$  \hspace{1cm} (2.19)

$$E_{12} E_{123} = q E_{123} E_{12}$$  \hspace{1cm} (2.20)

$$E_2 E_{123} = E_{123} E_2$$  \hspace{1cm} (2.21)

$$E_1 E_{23} = q^{-1} E_{23} E_1 - E_{123}$$  \hspace{1cm} (2.22)

$$E_{12} E_{23} = E_{23} E_{12} + (q - q^{-1}) E_2 E_{123}$$  \hspace{1cm} (2.23)

$$E_2 E_{23} = q E_{23} E_2$$  \hspace{1cm} (2.24)

$$E_{123} E_{23} = q E_{23} E_{123}$$  \hspace{1cm} (2.25)

$$E_1 E_3 = E_3 E_1$$  \hspace{1cm} (2.26)

$$E_{12} E_3 = q^{-1} E_3 E_{12} - E_{123}$$  \hspace{1cm} (2.27)

$$E_2 E_3 = q^{-1} E_2 E_3 - E_{23}$$  \hspace{1cm} (2.28)

$$E_{123} E_3 = q E_3 E_{123}$$  \hspace{1cm} (2.29)

$$E_{23} E_3 = q E_3 E_{23}$$  \hspace{1cm} (2.30)

$U_{12}$ has the generator $E_1$ (and no relations). $U_{1213}$ has generators $E_1, E_{12}$ with relation (2.16).

$U_{1223}$ has generators $E_1, E_{12}, E_2$ with relations (2.16) through (2.18). In this algebra, we find that the element $E_{12}[(q - q^{-1}) E_1 E_2 + q E_{12}]$ is central (it is easily checked that it commutes with each of the generators).

$U_{123}$ has generators $E_1, E_{12}, E_2, E_{123}$ with relations (2.16) through (2.21).
\( U^{s_1 s_2 s_1 s_2} \) has generators \( E_1, E_{12}, E_2, E_{123}, E_{23} \) with relations (2.16) through (2.25). We find that the element \( E_{12}E_{23} - qE_2E_{123} \) is central in this algebra.

\( U^{s_1 s_2 s_1 s_2 s_2} \) (which is equal to \( U_q^+(s_4(C)) \)), because \( s_1 s_2 s_1 s_3 s_2 s_1 \) is the longest element of \( W \) for \( s_4(C) \) has generators \( E_1, E_{12}, E_2, E_{123}, E_{23}, E_3 \) with relations (2.16) through (2.30). The elements \( E_{12}E_{23} - qE_2E_{123} \) and \( (q - q^{-1})E_3([q - q^{-1}]E_1E_2 + qE_{12}]E_{123} + E_{123}([q - q^{-1}]E_1E_{23} + qE_{123}) \) are central in this algebra.

### 2.4 Irreducible Representations of \( U_\varepsilon^w \) for \( w = s_1, s_1s_2, s_1s_2s_1, s_1s_2s_1s_3, s_1s_2s_1s_3s_2, \) and \( s_1s_2s_1s_3s_2s_1 \)

We now let \( q = \varepsilon \), where \( \varepsilon \) is a primitive \( \ell \)th root of 1 (with \( \ell \) odd in most cases). All representations considered will be finite-dimensional. Recall that if \( s_{i_1}...s_{i_m} \) is a reduced expression for \( w \) in terms of simple reflections, then the elements \( E_{i_1}^\ell \) (\( i = 1, ..., m \)) are central in \( U_\varepsilon^w \), so they act as scalars in any finite-dimensional representation.

**Proposition 2.4** The finite-dimensional irreducible representations of \( U_\varepsilon^{s_1} \), where \( \varepsilon \) is a primitive \( \ell \)th root of unity, are one dimensional.

Proof: Let \( V \) be an irreducible \( U_\varepsilon^{s_1} \)-module. Since \( U_\varepsilon^{s_1} \) is generated by \( E_1 \), \( V \) is spanned by an eigenvector of \( E_1 \). So \( \dim V = 1 \).

**Proposition 2.5** The finite-dimensional irreducible representations of \( U_\varepsilon^{s_1 s_2} \), where \( \varepsilon \) is a primitive \( \ell \)th root of 1, have the following dimensions:

- \( \ell \) if \( E_1^\ell \neq 0 \) and \( E_{12}^\ell \neq 0 \)
- 1 in all other cases.

Proof: Let \( V \) be an irreducible \( U_\varepsilon^{s_1 s_2} \)-module. If \( E_1^\ell = 0 \) on \( V \), then \( E_1 = 0 \) on \( V \) by lemma 2.2. Then \( V \) is one-dimensional, spanned by an eigenvector of \( E_{12} \). Similarly if \( E_{12}^\ell = 0 \), then \( \dim V = 1 \). If \( E_1^\ell \neq 0 \) and \( E_{12}^\ell \neq 0 \), let \( v \) be an eigenvector of \( E_1 \), with eigenvalue \( \lambda (\lambda \neq 0) \). Then \( \text{span}(v, E_{12}v, ..., E_{12}^{\ell-1}v) \) is invariant under \( E_1 \) and \( E_{12} \), so this space is equal to \( V \). \( v, E_{12}v, ..., E_{12}^{\ell-1}v \) are eigenvectors of \( E_1 \) with
eigenvalues $\lambda, \varepsilon_\lambda, ..., \varepsilon^{\ell-1} \lambda$, respectively (each of these vectors is nonzero, because $E_{12}^\ell \neq 0$). Therefore these vectors are linearly independent and $\dim V = \ell$.

**Proposition 2.6** The finite-dimensional irreducible representations of $U_{q^{2_1}q^{2_2}}$, where $\varepsilon$ is a primitive $\ell$th root of unity with $\ell$ odd, have dimensions:

1. if $E_{12}^\ell = 0$, and $E_1^\ell$ or $E_2^\ell$ is zero
2. in all other cases

Proof: The algebra $U_{q^{2_1}q^{2_2}}$ is isomorphic to the quantum Heisenberg algebra $\mathcal{H}$ discussed in Chapter 1, with the identification $E_1 \to b$, $E_{12} \to q^{-1}c$, and $E_2 \to a$. The element $E_{12}[(q - q^{-1})E_1 E_2 + q E_{12}]$ is central in this algebra, corresponding to the element $q^{-2}c[(q - q^{-1})ab + q^{-1}c]$ in $\mathcal{H}$. We also note from applying Proposition 1.8 to this case that $[(\varepsilon - \varepsilon^{-1})E_1 E_2 + \varepsilon E_{12}]^\ell = (\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell$.

**Proposition 2.7** The finite-dimensional irreducible representations of $U_{q^{2_1}q^{2_2}}$, where $\varepsilon$ is a primitive $\ell$th root of unity with $\ell$ odd, have dimensions:

1. if $E_{12}^\ell = 0$, $E_{12}^\ell = 0$, and $E_2^\ell = 0$
2. if $E_{12}^\ell = 0$ and $E_1^\ell = 0$
3. if $E_{12}^\ell \neq 0$, $E_{12}^\ell \neq 0$, and $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell \neq 0$
4. in all other cases

Proof: Let $V$ be an irreducible $U_{q^{2_1}q^{2_2}}$-module.

Case 1: $E_{12}^\ell = 0$ on $V$. Since $E_{12}$ q-commutes with $E_1$, $E_{12}$, and $E_2$, $E_{12}^\ell = 0$ on $V$ implies that $E_{12} = 0$ on $V$. Thus, by lemma 2.3, $V = U$ where $U$ is an irreducible $U_{q^{2_1}q^{2_2}}$-module. So $\dim V = 1$ or $\ell$.

Case 2: $E_{12}^\ell \neq 0$, $E_{12}^\ell = 0$, and $E_1^\ell = 0$ on $V$. $E_{12}^\ell = 0$ implies $E_{12} = 0$. It then follows that $E_1$ q-commutes with $E_2$, so $E_1^\ell = 0$ implies $E_1 = 0$. We are thus left with the two generators $E_2$ and $E_{12}$, which commute. $V$ is spanned by a common eigenvector of these two generators, so $\dim V = 1$.

Case 3: $E_{12}^\ell \neq 0$, $E_{12}^\ell = 0$, and $E_1^\ell \neq 0$ on $V$. $E_{12}^\ell = 0$ implies $E_{12} = 0$. Let $v$ be a common eigenvector of $E_2$ and $E_{12}$, which commute. Then $E_{12} v = \lambda v$, with $\lambda \neq 0$. The space $\text{span}(v, E_1 v, ..., E_1^{\ell-1})$ is invariant under $E_1$, $E_2$, and $E_{12}$, so
is equal to \( V \). Furthermore, the vectors \( v, E_1 v, ..., E_1^{\ell-1} \) are all eigenvectors of \( E_{123} \) with distinct eigenvalues \( \lambda, \varepsilon^{-1}\lambda, ..., \varepsilon^{-(\ell-1)}\lambda \), respectively (the vectors are all nonzero because \( E_1 \neq 0 \)). Thus \( \dim V = \ell \).

Case 4: \( E_{123}^\ell \neq 0 \) and \( E_{12}^\ell \neq 0 \) on \( V \). \( V \) is a module over \( \mathcal{U}_e^{s_1 s_2 s_3} \); let \( U \) be an irreducible submodule of \( V \) over \( \mathcal{U}_e^{s_1 s_2 s_3} \). \( E_{123}[(\varepsilon - \varepsilon^{-1})E_1 E_2 + \varepsilon E_{12}] \) is a central element of \( \mathcal{U}_e^{s_1 s_2 s_3} \), so it acts as a scalar on \( U \). Let \( x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1 E_2 + \varepsilon E_{12}] \). We see by checking directly that \( xE_{123} = \varepsilon^2 E_{123}x \). We consider the following two subcases.

Case 4a: \( x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1 E_2 + \varepsilon E_{12}] \) acts as a nonzero scalar \( \alpha \) on \( U \) (thus \( (\varepsilon - \varepsilon^{-1})E_1^\ell E_2^\ell + E_{12}^\ell \neq 0 \)). We know that \( V = U + E_{123} U + ... + E_{123}^{\ell-1} U \). The spaces \( U, E_{123} U, ..., E_{123}^{\ell-1} U \) are eigenspaces of \( x \) with corresponding eigenvalues \( \alpha, \varepsilon^2 \alpha, ..., \varepsilon^{\ell-1} \alpha, \varepsilon^{\ell+1} \alpha, ..., \varepsilon^{2\ell-2} \alpha \); these eigenvalues are all distinct (because \( \ell \) is odd). Thus \( V = U \oplus E_{123} U \oplus ... \oplus E_{123}^{\ell-1} U \). Each of the spaces in the direct sum is nonzero and each has dimension equal to the dimension of \( U \), because \( E_{123} \neq 0 \). Thus \( \dim V = \ell \dim U \). From our previous results we know that \( E_{12}^\ell \neq 0 \) on \( U \) implies that \( \dim U = \ell \). Therefore \( \dim V = \ell^2 \).

Case 4b: \( x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1 E_2 + \varepsilon E_{12}] = 0 \) on \( U \) (thus \( (\varepsilon - \varepsilon^{-1})E_1^\ell E_2^\ell + E_{12}^\ell = 0 \)). Since \( V = U + E_{123} U + ... + E_{123}^{\ell-1} U \) and \( xE_{123} = \varepsilon^2 E_{123}x \), it follows that \( x = 0 \) on \( V \). Since \( E_{12}^\ell \neq 0 \), \( E_{12} \) is invertible so we have \( (\varepsilon - \varepsilon^{-1})E_1 E_2 + \varepsilon E_{12} = 0 \) on \( V \). Solving for \( E_{12} \), we find that \( E_{12} = (\varepsilon^{-2} - 1)E_1 E_2 \). Substituting this into the relation \( E_1 E_2 = \varepsilon^{-1} E_2 E_1 - E_{12} \), we find that \( E_1 E_2 = \varepsilon E_2 E_1 \) on \( V \). Thus we see that \( E_1 \) and \( E_2 \) q-commute with all generators. Thus \( E_1^\ell \neq 0 \) would imply that \( E_1 = 0 \), which would further imply that \( E_{12} = 0 \), contrary to assumption. Therefore \( E_{12}^\ell \neq 0 \), and likewise \( E_2^\ell \neq 0 \). Direct verification shows that the element \( E_{123} E_2^{\ell-1} \) commutes with each of the generators \( E_1, E_{12}, E_2, \) and \( E_{123} \). Thus this element is central and acts as a scalar \( \alpha \) on \( V \). \( E_2 \) acts as a scalar \( c \) (\( c \neq 0 \)) on \( V \). Multiplying both sides of the equation \( E_{123} E_2^{\ell-1} = \alpha \) on the right by \( E_2 \), we obtain \( E_{123} = (\alpha/c) E_2 \). Since we can express \( E_{12} \) and \( E_{123} \) in terms of \( E_1 \) and \( E_2 \), \( V \) must be irreducible over the generators \( E_1 \) and \( E_2 \), which satisfy \( E_1 E_2 = \varepsilon E_2 E_1 \). Since \( E_1^\ell \neq 0 \) and \( E_2^\ell \neq 0 \), we have \( \dim V = \ell \) (shown in the same way as when we let \( V \) be an irreducible \( \mathcal{U}_e^{s_1 s_2} \)-module).

**Proposition 2.8** The finite-dimensional irreducible representations of \( \mathcal{U}_e^{s_1 s_2 s_3 s_4} \).
where $\varepsilon$ is a primitive $\ell$th root of unity with $\ell$ odd, have dimensions:

1. if $E'_{123} = 0$, $E'_{12} = 0$, and any two or three of $E'_1, E'_2, E'_{23}$ are zero

2. if $E'_{123} \neq 0$ and $(\varepsilon - \varepsilon^{-1}) E'_1 E'_2 + E'_1 \neq 0$

$\ell$ in all other cases

Proof: Let $V$ be an irreducible $\mathcal{U}^{(1,2,3)}_{e}$-module. The element $E_{12} E_{23} - q E_{2} E_{123}$ commutes with the generators $E_1, E_{12}, E_2, E_{123}$, and $E_{23}$ of the algebra $\mathcal{U}^{(1,2,3)}$, so is central. Thus $E_{12} E_{23} - \varepsilon E_{2} E_{123}$ acts as a scalar $\alpha$ on $V$.

Case 1: $E'_{12} \neq 0$ on $V$. $E'_{12}$ acts as a scalar $b (b \neq 0)$ on $V$. From $E_{12} E_{23} - \varepsilon E_{2} E_{123} = \alpha$, we may solve for $E_{23}$: $E_{23} = (1/b)[\varepsilon E'_{12} E_{2} E_{123} + \alpha E'_{12}]$. Thus if $U$ is an irreducible $\mathcal{U}^{(1,2,3)}_{e}$-submodule of $V$, we see that $U$ is $E_{23}$-invariant, so $U = V$. From previous results, we know that $\dim U = \ell$ or $\ell^2$ when $E'_{12} \neq 0$ on $U$. Thus $\dim V = \ell$ or $\ell^2$.

Case 2: $E'_{12} = 0$ and $E'_{123} = 0$. $E_{123}$ q-commutes with all the generators, so $E'_{123} = 0$ implies $E_{123} = 0$ on $V$. It then follows that $E_{12}$ now q-commutes with all other generators in the representation, so $E'_{12} = 0$ implies $E_{12} = 0$ on $V$. We are left with the generators $E_1, E_2$, and $E_{23}$, which satisfy the relations $E_1 E_2 = \varepsilon^{-1} E_2 E_1$, $E_1 E_{23} = \varepsilon^{-1} E_{23} E_1$, and $E_2 E_{23} = \varepsilon E_{23} E_2$. We find that the elements $E_1 E_2 E_{23}^{-1}$ and $E_1^{-1} E_2 E_{23}^{-1}$ are central in the representation, so they act as scalars: $E_1 E_2 E_{23}^{-1} = \beta$ and $E_1^{-1} E_2 E_{23}^{-1} = \gamma$. Because $E_1, E_2$, and $E_{23}$ all q-commute, if the $\ell$th power of any of these generators is 0, then the generator itself is zero. Thus if any two (or all three) of these generators have $\ell$th powers equal to zero, then $V$ will be one-dimensional, spanned by an eigenvector of the third generator. Now suppose any two (or all three) of these generators have $\ell$th powers not equal to zero. Let $U$ be an irreducible submodule of $V$ over the algebra with those two generators and their relation. Then (as before) $\dim U = \ell$. But $U$ is invariant under the third generator, because we can solve for the third generator in terms of the first two from $E_1 E_2 E_{23}^{-1} = \beta$ or $E_1^{-1} E_2 E_{23}^{-1} = \gamma$. Thus in this situation $\dim V = \ell$. So in this case we then have $\dim V = 1$ or $\ell$.

Case 3: $E'_{12} = 0$, $E'_{123} \neq 0$, $E'_1 = 0$. Let $U$ be an irreducible $\mathcal{U}^{(1,2,3)}_{e}$-submodule of $V$. We have seen that in the case $E'_{12} = 0$, $E'_{123} \neq 0$, and $E'_1 = 0$ on $U$, that $U$ is
one-dimensional, spanned by a vector $u$ which satisfies $E_1u = 0$, $E_{12}u = 0$, $E_2u = \lambda u$, and $E_{13}u = \mu u$, where $\mu \neq 0$ since $E'_{13} \neq 0$. We have $V = U + E_{23}U + \ldots + E'_{23}^{-1}U$. $E_{13}(E_{23}^m u) = \mu \varepsilon^m (E_{23}^m u)$, and $\mu, \mu \varepsilon, \ldots, \mu \varepsilon^{t-1}$ are distinct, so $V = U \oplus E_{23}U \oplus \ldots \oplus E'_{23}^{-1}U$. It remains to show that each of these summands is nonzero. If $E'_{23} \neq 0$ this is clear. If $E'_{23} = 0$, we use the following formula, which is proven by induction on $m$ ($m = 1, 2, \ldots$):

$$E_1 E_{23}^m = \varepsilon^{-m} E_{23}^m E_1 - \varepsilon^{1-m} \left( \frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2} \right) E_{23}^{m-1} E_{123}. \tag{2.31}$$

Let $m$ be the least positive integer such that $E_{23}^m u = 0$. Then, applying (2.31) to $u$, we obtain

$$0 = -\varepsilon^{1-m} \left( \frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2} \right) \mu E_{23}^{m-1} u. \tag{2.32}$$

It follows that $\ell$ divides $2m$, which implies that $\ell$ divides $m$, since $\ell$ is odd. Thus $\ell = m$, and $\dim V = \ell$ in this case.

Case 4: $E'_{12} = 0$, $E'_{13} \neq 0$, $E'_{1} \neq 0$, $E'_2 = 0$ on $V$. Let $U$ be an irreducible $U^{a_1 a_3}_{a_2}$-submodule of $V$. From previous results we know that $E'_{12} = 0$ implies $E_{12} = 0$ on $U$. Now $E_2$ q-commutes with the other generators in the representation $U$, so $E'_2 = 0$ implies $E_2 = 0$ on $U$. Now a simple induction argument shows that $E_{12}(E_{23}^m U) = 0$, so $E_{12} = 0$ on $V$ (since $V = U + E_{23}U + \ldots + E'_{23}^{-1}U$). From the relation $E_{12}E_{23} = E_{23}E_{12} + (\varepsilon - \varepsilon^{-1}) E_2 E_{123}$, we now have $0 = (\varepsilon - \varepsilon^{-1}) E_2 E_{123}$ on $V$. Since $E'_{13} \neq 0$, this implies that $E_2 = 0$ on $V$. Then $V$ irreducible over the generators $E_1, E_{123}, E_{23}$, with relations $E_{123} = \varepsilon E_{123} E_1, E_{1} E_{23} = \varepsilon^{-1} E_{23} E_1 - E_{123}$, and $E_{123} E_{23} = \varepsilon E_{23} E_{123}$. Relabeling these generators $E_1 \rightarrow E_1, E_{123} \rightarrow E_{12}$, and $E_{23} \rightarrow E_2$, we see that we have the algebra $U^{a_1 a_2 a_3}_{a_3}$. Then from previous results (noting that $0 \neq E'_{13} \rightarrow E'_{12}$) we have $\dim V = \ell$.

Case 5: $E'_{12} = 0$, $E'_{13} \neq 0$, $E'_{1} \neq 0$, $E'_2 \neq 0$ on $V$. Let $U$ be an irreducible $U^{a_1 a_3}_{a_2}$-submodule of $V$. From previous results we know that $U$ has a basis given by $(u, E_1u, \ldots, E'^{-1}_1u)$, where $u$ is a common eigenvector of $E_{123}$ and $E_2$; $E_{123}u = \lambda u$ ($\lambda \neq 0$, since $E'_{13} \neq 0$), $E_2u = \mu u$ ($\mu \neq 0$, since $E'_2 \neq 0$). We also know that $E_{12} = 0$ on $U$. $E_2 E_{123}$ commutes with $E_1$, so we see that $E_2 E_{123} = \mu \lambda$ on $U$. 

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\( V = U + E_{23}U + \ldots + E_{23}^{\ell-1}U \), and \( E_2 E_{123}(E_{23}^m u) = \mu \lambda e^{2m}(E_{23}^m u) \) for any \( u \in U \). Thus \( U, E_{23}U, \ldots, E_{23}^{\ell-1}U \) are eigenspaces of \( E_2 E_{123} \) with respective eigenvalues \( \mu \lambda, \mu \lambda e^2, \ldots, \mu \lambda e^{2\ell-2} \) (distinct, since \( \ell \) is odd). Thus \( V = U \oplus E_{23}U \oplus \ldots \oplus E_{23}^{\ell-1}U \). If \( E_{23}^\ell \neq 0 \), then \( \dim V = \ell \dim U = \ell^2 \). If \( E_{23}^\ell = 0 \), we use the following formula, which is proven by induction on \( m \) (\( m = 1, 2, \ldots \)):

\[
E_{12}E_{23}^m = E_{23}^m E_{12} + \left( \frac{e^{2m} - 1}{e} \right) E_{23}^{m-1}E_2E_{123}. \tag{2.33}
\]

Let \( u \in U, u \neq 0 \), and let \( m \) be the least positive integer such that \( E_{23}^m u = 0 \). Applying (2.33) to \( u \), we have

\[
0 = \mu \lambda \left( \frac{e^{2m} - 1}{e} \right) E_{23}^{m-1}u. \tag{2.34}
\]

It follows that \( \ell \) divides \( 2m \), which implies that \( \ell \) divides \( m \), since \( \ell \) is odd. Thus \( \ell = m \). It then follows that each space in the sum \( V = U \oplus E_{23}U \oplus \ldots \oplus E_{23}^{\ell-1}U \) has dimension equal to the dimension of \( U \), and \( \dim V = \ell \dim U = \ell^2 \).

**Proposition 2.9** The finite-dimensional irreducible representations of \( \mathcal{U}_{e_1e_2e_3}^a \), where \( e \) is a primitive \( \ell \)th root of unity with \( \ell \) odd, have dimensions:

- \( 1 \) if \( E_{123}^\ell = 0, E_{12}^\ell = 0, E_{23}^\ell = 0, \) and \( E_2^\ell = 0 \)
- \( 1 \) if \( E_{123}^\ell = 0, E_{12}^\ell = 0, E_{23}^\ell = 0, E_1^\ell = 0, \) and \( E_3^\ell = 0 \)
- \( \ell^2 \) if \( E_{123}^\ell = 0, E_{12}^\ell = 0, E_1^\ell \neq 0, E_{23}^\ell \neq 0, \) and \( (e - e^{-1})^\ell E_2^\ell E_3^\ell + E_{23}^\ell \neq 0 \)
- \( \ell^2 \) if \( E_{123}^\ell = 0, E_{23}^\ell = 0, E_3^\ell \neq 0, E_{12}^\ell \neq 0, \) and \( (e - e^{-1})^\ell E_1^\ell E_3^\ell + E_{12}^\ell \neq 0 \)
- \( \ell^2 \) if \( E_{123}^\ell \neq 0, (e - e^{-1})^\ell E_1^\ell E_2^\ell E_{23}^\ell + E_{12}^\ell \neq 0 \)
- \( \ell^2 \) if \( E_{123}^\ell \neq 0, (e - e^{-1})^\ell E_2^\ell E_3^\ell + E_{12}^\ell \neq 0, \) and \( (e - e^{-1})^\ell E_1^\ell E_2^\ell E_{23}^\ell + E_{12}^\ell \neq 0 \)
- \( \ell^2 \) if \( E_{123}^\ell \neq 0, (e - e^{-1})^\ell E_1^\ell E_3^\ell E_{23}^\ell + E_{12}^\ell \neq 0, \) and \( (e - e^{-1})^\ell E_1^\ell E_2^\ell E_{23}^\ell + E_{12}^\ell \neq 0 \)
- \( \ell \) in all other cases

Proof: Let \( V \) be an irreducible \( \mathcal{U}_{e_1e_2e_3}^a \)-module.
Case 1: \( E_{123}^t = 0, E_{23}^t = 0, E_3^t = 0 \). It then follows that \( E_{123} = 0, E_{23} = 0, \) and \( E_3^t = 0 \) in the representation. We conclude then that \( V \) is an irreducible \( \mathcal{U}_{\varepsilon_1^{\varepsilon_2}} \) module, so \( \dim V = 1 \) or \( \ell \).

Case 2: \( E_{123}^t = 0, E_{23}^t = 0, E_3^t \neq 0, E_{12}^t = 0 \). Then \( E_{123} = 0, E_{23} = 0, \) and \( E_{12} = 0 \) on \( V \). Thus \( V \) is an irreducible module over the algebra with generators \( E_1, E_2, \) and \( E_3 \) with relations \( E_1E_2 = \varepsilon^{-1}E_2E_1, E_1E_3 = E_3E_1, \) and \( E_2E_3 = \varepsilon^{-1}E_3E_2 \). Let \( v \) be a common eigenvector of \( E_1 \) and \( E_3 \). If \( E_2^t = 0 \), then \( E_2 = 0 \) and \( \dim V = 1 \). If \( E_2^t \neq 0 \), then the vectors \((v, E_2v, ..., E_2^{t-1}v)\) form a basis for \( V \) (they are eigenvectors of \( E_1 \) and \( E_3 \), with distinct eigenvalues as eigenvectors of \( E_3 \)), so \( \dim V = \ell \).

Case 3: \( E_{123}^t = 0, E_{23}^t = 0, E_3^t \neq 0, E_{12}^t \neq 0 \). Then \( E_{123} = 0 \) and \( E_{23} = 0 \) on \( V \). Let \( U \) be an irreducible submodule of \( V \) over the algebra with generators \( E_1, E_{12}, E_2 \) and their relations, i.e. \( \mathcal{U}_{\varepsilon_1^{\varepsilon_2}} \). Since \( E_{12}^t \neq 0 \), we have \( \dim U = \ell \). We know that the element \( E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] \) is central and acts as a scalar \( \alpha \) on \( U \). Let \( x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] \). We find that \( xE_3 = \varepsilon^{-2}E_3x \), using \( E_{123} = 0 \) and \( E_{23} = 0 \). We consider the following two subcases.

Case 3a: \( x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] \) acts as a nonzero scalar \( \alpha \) on \( U \) (Thus \( (\varepsilon - \varepsilon^{-1})E_1E_2^t + E_{12}^t \neq 0 \)). Then \( V = U + E_3U + ... + E_3^{t-1}U \), and since \( xE_3 = \varepsilon^{-2}E_3x \), we see that \( U, E_3U, ... , E_3^{t-1}U \) are eigenspaces of \( x \) with eigenvalues \( \alpha, \alpha^2, ..., \alpha^{2t-2} \) (distinct, since \( \ell \) is odd). Furthermore, each of these spaces has dimension equal to the dimension of \( U \), since \( E_3^t \neq 0 \). So \( \dim V = \ell \dim U = \ell^2 \).

Case 3b: \( x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] = 0 \) on \( U \) (Thus \( (\varepsilon - \varepsilon^{-1})E_1E_2^t + E_{12}^t = 0 \)). Then since \( V = U + E_3U + ... + E_3^{t-1}U \) and \( xE_3 = \varepsilon^{-2}E_3E_3x \), it follows that \( x = 0 \) on \( V \). Since \( E_{12}^t \neq 0 \), this implies that \( [(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] = 0 \) on \( V \). As in case 4b of Proposition 2.7, we find that \( E_{12} = (\varepsilon^{-2} - 1)E_1E_2 \) and \( E_1E_2 = \varepsilon E_2E_1 \) on \( V \). Thus \( V \) is an irreducible module over the algebra with generators \( E_1, E_2, \) and \( E_3 \) with relations \( E_1E_2 = \varepsilon E_2E_1, E_1E_3 = E_3E_1, \) and \( E_2E_3 = \varepsilon^{-1}E_3E_2 \). We then find in the same manner as for case 2 of this proposition that \( \dim V = \ell \).

Case 4: \( E_{123}^t = 0, E_{12}^t = 0 \) on \( V \). It follows that \( E_{123} = 0, E_{12} = 0 \) on \( V \). Now relabel the generators as follows: \( E_1 \rightarrow E_3, E_2 \rightarrow E_2, E_3 \rightarrow E_1, \) and \( E_{23} \rightarrow -\varepsilon^{-1}E_{12} \). Now we find that we have the same generators and relations as we have for \( \mathcal{U}_{\varepsilon_1^{\varepsilon_2}} \).
When $E'_{123} = 0$, $E'_{23} = 0$ on $V$. $\epsilon^{-1}$ is also a primitive $\ell$th root of 1, so this is covered in cases 1 through 3 of this proposition. We conclude as in those cases that $\dim V = 1$, $\ell$, or $\ell^2$. Note that the condition (i.e. whether or not it is zero) on the element $(\epsilon^{-1} - \epsilon)^{\ell} E_1 E_2' + E_1' E_2$ of $U_{\epsilon^{-1} - \epsilon}$ becomes here the corresponding condition on the element $(\epsilon - \epsilon^{-1})^{\ell} E_1 E_3' + E_3' E_2'$. 

Case 5: $E'_{123} = 0$, $E'_{12} \neq 0$, $E'_{23} \neq 0$, and $E_1' \neq 0$ on $V$. $E'_{123} = 0$ implies $E'_{123} = 0$ on $V$. The element $E_1 E_2' - \epsilon E_2 E_{123} = E_1 E_{123}$ acts as a scalar $\alpha$ on $V$, so $E_{12} = (\alpha/b)E_{12}$ on $V$, where $b = E_{12}$. Let $U$ be an irreducible submodule of $V$ over the algebra with generators $E_1$, $E_{12}$, $E_{23}$, and $E_3$ with their relations. Let $u$ be a common eigenvector of $E_1$ (with eigenvalue $\lambda \neq 0$ for $E_1$) and $E_3$ (with eigenvalue $\mu$), which commute. Then the space $\text{span}(u, E_{12} u, ..., E_{12}^{\ell-1} u)$ is seen to be invariant under $E_1$, $E_{12}$, $E_{23}$, and $E_3$. Also, the vectors $u$, $E_{12} u$, ..., $E_{12}^{\ell-1} u$ are eigenvalues of $E_1$ with distinct eigenvalues, since $E_1 (E_{12}^m u) = \epsilon^m \lambda (E_{12}^m u)$. Thus $U = \text{span}(u, E_{12} u, ..., E_{12}^{\ell-1} u)$ and $\dim U = \ell$. Note also that $E_3 (E_{12}^m u) = \epsilon^m \mu (E_{12}^m u)$, so $E_3 = (\mu/\lambda)E_1$ on $U$. $E_{12}[(\epsilon - \epsilon^{-1}) E_1 E_2 + \epsilon E_{12}]$ commutes with $E_1$, $E_{12}$, and $E_2$, and thus must also commute with $E_{23}$ and $E_3$, since $E_{23} = (\alpha/b)E_{12}^{-1}$ and $E_3 = (\mu/\lambda)E_1$. Thus $E_{12}[(\epsilon - \epsilon^{-1}) E_1 E_2 + \epsilon E_{12}] = \beta$ ($\beta$ = constant) on $U$. Since $E_{12}' \neq 0$ and $E_1' \neq 0$, $E_{12}$ and $E_1$ are invertible and we can solve for $E_2$ in terms of $\beta$, $E_{12}$ and $E_1$. Thus $U$ is $E_2$-invariant, so $U = V$ and $\dim V = \ell$.

Case 6: $E'_{123} = 0$, $E'_{12} \neq 0$, $E'_{23} \neq 0$, and $E_1' \neq 0$ on $V$. Relabeling $E_1 \rightarrow E_3$, $E_2 \rightarrow E_2$, $E_3 \rightarrow E_1$, $E_{12} \rightarrow -\epsilon^{-1} E_{23}$ and $E_{23} \rightarrow -\epsilon^{-1} E_{12}$, we obtain the same algebra we had in case 5, with $\epsilon^{-1}$ in place of $\epsilon$. Since $\epsilon^{-1}$ is also a primitive $\ell$th root of 1, by case 5 we have $\dim V = \ell$.

Case 7: $E'_{123} = 0$, $E'_{12} \neq 0$, $E'_{23} \neq 0$, $E_1' = 0$, and $E_3' = 0$ on $V$. Then $E_{123} = 0$ on $V$. Let $U$ be an irreducible submodule of $V$ over the algebra with generators $E_1$, $E_{12}$, $E_{23}$, and $E_3$. These generators all $q$-commute (since $E_{123} = 0$ on $V$), so $E_1' = 0$ and $E_3' = 0$ on $U$ imply $E_1 = 0$ and $E_3 = 0$ on $U$. Let $u$ be an eigenvector of $E_{12}$; $E_{12} u = \lambda u$, $\lambda \neq 0$. As in case 5, we have $E_{23} = (\alpha/b)E_{12}^{\ell-1}$ on $V$, where $b = E_{12}$. Thus $u$ is also an eigenvector of $E_{23}$, so $\dim U = 1$. Now $V = U + E_2 U + ... + E_2^{\ell-1} U$, so $\dim V \leq \ell$. But $V$ is also a module over the algebra with generators $E_1$, $E_{12}$, and
and we have seen that an irreducible module over this algebra with $E_{12}^\ell \neq 0$ has dimension equal to $\ell$, thus $\dim V \geq \ell$. So $\dim V = \ell$.

We now consider the cases where $E_{123}^\ell \neq 0$. Direct verification shows that the element $(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}$ q-commutes with each of the generators $E_1$, $E_{12}$, $E_2$, $E_{123}$, and $E_{23}$ (but not with $E_3$). Let $x = (\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}$. Also recall that the element $(q-q^{-1})E_3[(q-q^{-1})E_1E_2 + qE_{12}]E_{123} + E_{123}[(q-q^{-1})E_1E_{23} + qE_{123}]$ is central in $U_{q,1}$, so $(\varepsilon - \varepsilon^{-1})E_3[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]E_{123} + E_{123}[(\varepsilon - \varepsilon^{-1})E_1E_{23} + \varepsilon E_{123}]$ acts as a scalar in a finite-dimensional irreducible representation of $U_{q,1}$. Let $y = [(\varepsilon - \varepsilon^{-1})E_1E_{23} + \varepsilon E_{123}]$ and $z = (\varepsilon - \varepsilon^{-1})E_3[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]E_{123} + E_{123}[(\varepsilon - \varepsilon^{-1})E_1E_{23} + \varepsilon E_{123}]$, so $z = (\varepsilon - \varepsilon^{-1})E_3 x E_{123} + E_{123} y$. $z = \alpha$ for some scalar $\alpha$ on $V$. For the following cases, let $U$ be an irreducible submodule of $V$ over the algebra with generators $E_1$, $E_{12}$, $E_2$, $E_{123}$, and $E_{23}$ with their relations. Since $x$ q-commutes with each of the generators, $x^\ell$ is central in this algebra. By the same proof as in lemma 2.2, either $x = 0$ on $U$ or $x^\ell \neq 0$ on $U$ (i.e. $x$ is invertible on $U$). Also we have $x^\ell = (\varepsilon - \varepsilon^{-1})E_1E_2 \ell + E_{12}^\ell$, and likewise $y^\ell = (\varepsilon - \varepsilon^{-1})E_1E_{23} \ell + E_{123}^\ell$.

Case 8: $E_{123}^\ell \neq 0$ on $V$, $x^\ell \neq 0$ on $U$. In this case we may solve for $E_3$ in terms of $E_1$, $E_{12}$, $E_2$, $E_{123}$, and $E_{23}$ from the equation $z = (\varepsilon - \varepsilon^{-1})E_3 x E_{123} + E_{123} y = \alpha$. Thus $U$ is $E_3$-invariant, so $V = U$ and $\dim U = \ell^2$ in this case.

Case 9: $E_{123}^\ell \neq 0$ on $V$, $x = 0$ on $U$ (thus $(\varepsilon - \varepsilon^{-1})E_1E_2 \ell + E_{12}^\ell = 0$), and $z = (\varepsilon - \varepsilon^{-1})E_3 x E_{123} + E_{123} y = \alpha \neq 0$ on $V$. It follows that $E_{123} y = \alpha$ on $V$. $V = U + E_3 U + \ldots + E_{123}^\ell U$, and we find that $(E_{123} y) E_3^m = \varepsilon^{2m} E_3^m (E_{123} y)$, so the spaces in the sum are eigenspaces of $E_{123} y$ with distinct (since $\ell$ is odd) eigenvalues $\alpha$, $\alpha \varepsilon^2$, $\ldots$, $\alpha \varepsilon^{2\ell-2}$. Thus $V = U \oplus E_3 U \oplus \ldots \oplus E_{123}^\ell U$. If $E_3^\ell \neq 0$, we conclude that $\dim V = \ell \dim U$. If $E_3^\ell = 0$, We use the following formula, which is proved by induction:

\[
(E_{123} x) E_3^m = E_3^m (E_{123} x) - \left( \frac{1 - \varepsilon^m}{1 - \varepsilon} \right) E_3^{m-1} (E_{123} y) \tag{2.35}
\]
Let \( u \in U, u \neq 0 \). Let \( m \) be the least positive integer such that \( E_3^m u = 0 \). Applying equation 2.35 to \( u \), we obtain

\[
0 = \alpha \left( \frac{1 - \varepsilon^m}{1 - \varepsilon} \right) E_3^{m-1} u,
\]

from which we conclude that \( m = \ell \). Thus we again have \( \dim V = \ell \dim U \). In this case we previously found that \( \dim U = \ell \), so \( \dim V = \ell^2 \).

Case 10: \( E_{123}^l \neq 0 \) on \( V \), \( x = 0 \) on \( U \) (thus \( (\varepsilon - \varepsilon^{-1}) E_1^l E_2^l + E_{12}^l = 0 \)), and \( z = (\varepsilon - \varepsilon^{-1}) E_3 x E_{123} + E_{123} y = 0 \) on \( V \). It follows that \( E_{123} y = 0 \) on \( U \), so \( y = 0 \) on \( U \). We find that \( y E_3 = \varepsilon E_3 y \), and since \( V = U + E_3 U + \ldots + E_3^{l-1} U \) we conclude that \( y = 0 \) on \( V \). Equation 2.35 now becomes \( (E_{123} x) E_3^m = E_3^m (E_{123} x) \), so we also see that \( E_{123} x = 0 \) on \( V \), hence \( x = 0 \) on \( V \). From \( x = 0 \) we find that \( E_{12} = (\varepsilon^{-2} - 1) E_1 E_2 \) and \( E_1 E_2 = \varepsilon E_2 E_1 \) on \( V \). From \( y = 0 \) we find that \( E_{123} = (\varepsilon^{-2} - 1) E_1 E_{23} \) and \( E_1 E_{23} = \varepsilon E_{23} E_1 \) on \( V \). Also, \( y^l = (\varepsilon - \varepsilon^{-1}) E_1^l E_2^l + E_{123}^l = 0 \); since we have assumed that \( E_{123}^l \neq 0 \), it follows that we must also have \( E_1^l \neq 0 \) and \( E_{23}^l \neq 0 \). \( V \) is thus an irreducible module over the generators \( E_1, E_2, E_{23} \), and \( E_3 \), which satisfy the relations:

\[
\begin{align*}
E_1 E_2 &= \varepsilon E_2 E_1 \quad (2.37) \\
E_1 E_{23} &= \varepsilon E_{23} E_1 \\
E_1 E_3 &= E_3 E_1 \\
E_2 E_{23} &= \varepsilon E_{23} E_2 \\
E_2 E_3 &= \varepsilon^{-1} E_3 E_2 - E_{23} \\
E_{23} E_3 &= \varepsilon E_3 E_{23} \\
\end{align*}
\]

Let \( W \) be an irreducible submodule of \( V \) over the algebra with generators \( E_2, E_{23}, \) and \( E_3 \). This algebra is obviously isomorphic to \( U_{\epsilon^2}^{e_1 e_2} \), which was considered earlier. The element \( E_{23}[(\varepsilon - \varepsilon^{-1}) E_2 E_3 + \varepsilon E_{23}] \) is central in this algebra, and so acts as a scalar \( \beta \) on \( W \). Let \( \tilde{x} = E_{23}[(\varepsilon - \varepsilon^{-1}) E_2 E_3 + \varepsilon E_{23}] \). We have \( V = W + E_1 W + \ldots + E_1^{l-1} W \).

We now consider the following subcases:
Case 10a: \( E'_1 \neq 0, E'_2 \neq 0 \), and \( \tilde{x} = E_{23}[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] \neq 0 \) on \( W \) (thus \( \varepsilon - \varepsilon^{-1} \neq 0 \)). We find that \( \tilde{x}E_1 = \varepsilon^{-2}E_1\tilde{x} \). Thus the spaces \( W, E_1W, ..., E_l^{-1}W \) are eigenspaces of \( \tilde{x} \) with distinct (since \( \ell \) is odd) eigenvalues \( \beta, \beta\varepsilon^{-2}, \ldots, \beta\varepsilon^{2l-2} \). Since \( E'_1 \neq 0 \), we have \( \dim V = \ell \dim W \). \( \dim W = \ell \), since \( E_2 \neq 0 \), so \( \dim V = 2 \).

Case 10b: \( E'_1 \neq 0, E'_2 \neq 0 \), and \( \tilde{x} = E_{23}[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] = 0 \) on \( W \). Then \( [(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] = 0 \) on \( W \). We find that \( [(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}]E_1 = \varepsilon^{-1}E_1[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] \). It then follows from \( V = W + E_1W + \ldots + E_l^{-1}W \) that \( [(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] = 0 \) on \( V \). We then find that \( E_{23} = (\varepsilon^{-2} - 1)E_2E_3 \) on \( V \), and that \( E_2E_3 = \varepsilon E_3E_2 \) on \( V \). \( V \) is then irreducible over the generators \( E_1, E_2, \) and \( E_3 \), which satisfy the relations \( E_1E_2 = \varepsilon E_2E_1, E_1E_3 = E_3E_1, \) and \( E_2E_3 = \varepsilon E_3E_2 \). Since \( (\varepsilon - \varepsilon^{-1})E'_2E_3 + E'_1 = 0 \) and \( E'_2 \neq 0 \), we have \( E'_1 \neq 0 \). \( V \) then has dimension \( \ell \).

2.5 \( U^w \) for \( w = s_1s_3, s_1s_3s_2, s_1s_3s_2s_1, \) and \( s_1s_3s_2s_1s_3 \)

For \( w = s_1s_3s_2s_1s_3 \), we find that \( \beta_1 = \alpha_1, \beta_2 = \alpha_3, \beta_3 = \alpha_1 + \alpha_2 + \alpha_3, \beta_4 = \alpha_2 + \alpha_3, \) and \( \beta_5 = \alpha_1 + \alpha_2 \). We then find, using (2.8) through (2.12), that \( E_{\beta_1} = E_1, E_{\beta_2} = E_3, E_{\beta_3} = E_{123} = E_3E_1E_2 - q^{-1}E_3E_2E_1 - q^{-1}E_1E_2E_3 + q^{-2}E_2E_1E_3. E_{\beta_4} = E_{23} = -E_3E_2 + q^{-1}E_2E_3, E_{\beta_5} = E_{12} = -E_1E_2 + q^{-1}E_2E_1 \). With some computation, we find the relations (2.13) are as follows:

\[
E_1E_3 = E_3E_1 \quad (2.43)
\]
\[
E_1E_{123} = qE_{123}E_1 \quad (2.44)
\]
\[
E_3E_{123} = qE_{123}E_3 \quad (2.45)
\]
\[
E_1E_{23} = q^{-1}E_{23}E_1 - E_{123} \quad (2.46)
\]
\[
E_3E_{23} = qE_{23}E_3 \quad (2.47)
\]
\[
E_{123}E_{23} = qE_{23}E_{123} \quad (2.48)
\]

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\[ E_1 E_{12} = q E_{12} E_1 \] (2.49)

\[ E_3 E_{12} = q^{-1} E_{12} E_3 - E_{123} \] (2.50)

\[ E_{123} E_{12} = q E_{12} E_{123} \] (2.51)

\[ E_{23} E_{12} = E_{12} E_{23} \] (2.52)

\[ U^{{s_1}{s_3}} \] has generators \( E_1, E_3 \) with relation (2.43). \( U^{{s_1}{s_3}{s_2}} \) has generators \( E_1, E_3, E_{123} \) with relations (2.43) through (2.45). \( U^{{s_1}{s_3}{s_2}{s_1}} \) has generators \( E_1, E_3, E_{123}, \) and \( E_{23} \) with relations (2.43) through (2.48).

\( U^{{s_1}{s_3}{s_2}{s_1}{s_2}} \) has generators \( E_1, E_3, E_{123}, E_{23}, \) and \( E_{12} \) with relations (2.43) through (2.52). We find that the element \([ (q - q^{-1}) E_{12} E_{23} + q E_{123} ] [ (q - q^{-1}) E_3 E_{12} + q E_{123} ] \) is central in this algebra.

### 2.6 Irreducible Representations of \( U_\varepsilon^w \) for \( w = s_1 s_3, s_1 s_3 s_2, s_1 s_3 s_2 s_1, \) and \( s_1 s_3 s_2 s_1 s_3 \)

**Proposition 2.10** The finite-dimensional irreducible representations of \( U_\varepsilon^{{s_1}{s_3}} \), where \( \varepsilon \) is a primitive \( \ell \)th root of unity, have dimension 1.

Proof: Let \( V \) be an irreducible \( U_\varepsilon^{{s_1}{s_3}} \)-module. \( E_1 \) and \( E_3 \) commute, so \( V \) is spanned by a common eigenvector of \( E_1 \) and \( E_3 \).

**Proposition 2.11** The finite-dimensional irreducible representations of \( U_\varepsilon^{{s_1}{s_3}{s_2}} \), where \( \varepsilon \) is a primitive \( \ell \)th root of unity, have dimensions:

1. if \( E_{123}^\ell = 0 \)
2. if \( E_1^\ell = 0 \) and \( E_3^\ell = 0 \)
3. \( \ell \) in all other cases

Proof: Let \( V \) be an irreducible \( U_\varepsilon^{{s_1}{s_3}{s_2}} \)-module. If \( E_{123}^\ell = 0 \), then \( E_{123} = 0 \) and \( V \) is spanned by a common eigenvector of \( E_1 \) and \( E_3 \). If \( E_1^\ell = 0 \) and \( E_3^\ell = 0 \), then \( E_1 = 0 \) and \( E_3 = 0 \) and \( V \) is spanned by an eigenvector of \( E_{123} \). If \( E_{123}^\ell \neq 0 \) and
$E_1^\ell \neq 0$, let $v$ be a common eigenvector of $E_1$ and $E_3$. Then span$(v, E_{123}v, ..., E_{123}^{\ell-1}v)$ is invariant under each of the three generators, and the vectors $v, E_{123}v, ..., E_{123}^{\ell-1}v$ are eigenvectors of $E_1$ with distinct eigenvalues. Thus dim$V = \ell$. Similarly, dim$V = \ell$ if $E_1^\ell \neq 0$ and $E_3^\ell \neq 0$.

**Proposition 2.12** The finite-dimensional irreducible representations of $U_{e_1^{a_1}e_2^{a_2}e_3^{a_3}}$, where $e$ is a primitive $\ell$th root of unity with $\ell$ odd, have dimensions:

1. if $E_{123}^\ell = 0$ and $E_2^\ell = 0$
2. if $E_{123}^\ell = 0$, $E_1^\ell = 0$, and $E_3^\ell = 0$
3. if $E_3^\ell \neq 0$, $E_{123}^\ell \neq 0$, and $(e - e^{-1})^\ell E_1^\ell E_{23}^\ell + E_{123}^\ell \neq 0$
4. in all other cases

**Proof:** If we relabel the generators $E_1 \to E_1$, $E_3 \to E_3$, $E_{123} \to E_{12}$, and $E_{23} \to E_2$, we find that we have the same generators and relations as we had for $U_{e_1^{a_1}e_2^{a_2}e_3^{a_3}}$ in the case where $E_{123}^\ell = 0$, $E_{23}^\ell = 0$. From these results we see that dim$V = 1$, $\ell$, or $\ell^2$.

**Proposition 2.13** The finite-dimensional irreducible representations of $U_{e_1^{a_1}e_2^{a_2}e_3^{a_3}}$, where $e$ is a primitive $\ell$th root of unity with $\ell$ odd, have dimensions:

1. if $E_{123}^\ell = 0$, $E_1^\ell = 0$, and $E_3^\ell = 0$
2. if $E_{123}^\ell = 0$, $E_{12}^\ell = 0$, and $E_{23}^\ell = 0$
3. if $E_{123}^\ell \neq 0$ and $(e - e^{-1})^\ell E_1^\ell E_{23}^\ell + E_{123}^\ell \neq 0$
4. if $E_{123}^\ell \neq 0$ and $(e - e^{-1})^\ell E_3^\ell E_{12}^\ell + E_{123}^\ell \neq 0$
5. in all other cases

**Proof:** Let $V$ be an irreducible $U_{e_1^{a_1}e_2^{a_2}e_3^{a_3}}$-module. The element $[(q - q^{-1})E_1 E_{23} + qE_{123}][(q - q^{-1})E_3 E_{12} + q E_{123}]$ is central in $U_{e_1^{a_1}e_2^{a_2}e_3^{a_3}}$, so letting $[(e - e^{-1})E_1 E_{23} + e E_{123}][(e - e^{-1})E_3 E_{12} + e E_{123}]$ acts as a scalar $\alpha$ on $V$. Let $x = [(e - e^{-1})E_1 E_{23} + e E_{123}]$, and let $y = [(e - e^{-1})E_3 E_{12} + e E_{123}]$, so $xy = \alpha$ on $V$. We also note that $x$ and $y$ each q-commute with each of the generators, so each is either 0 or invertible on $V$.

**Case 1:** $E_{123}^\ell = 0$ on $V$. Then $E_{123} = 0$ on $V$, and we are left with the generators
$E_1$, $E_3$, $E_{23}$, and $E_{12}$, which satisfy

\[
\begin{align*}
E_1E_3 &= E_3E_1 \\
E_1E_{23} &= \varepsilon^{-1}E_{23}E_1 \\
E_1E_{12} &= \varepsilon E_{12}E_1 \\
E_3E_{23} &= \varepsilon E_{23}E_3 \\
E_3E_{12} &= \varepsilon^{-1}E_{12}E_3 \\
E_{12}E_{23} &= E_{23}E_{12}
\end{align*}
\]

We find that $E_1E_3$ and $E_{23}E_{12}$ are central in the representation. If $E'_1 = 0$ and $E'_3 = 0$, then $E_1 = 0$ and $E_3 = 0$ on $V$, and $V$ is one dimensional, spanned by a common eigenvector of $E_{23}$ and $E_{12}$. Similarly, if $E'_{23} = 0$ and $E'_{12} = 0$, dim$V = 1$. In all other cases we find that dim$V = \ell$. For example, if $E'_1 \neq 0$ and $E'_{23} \neq 0$, let $v$ be a common eigenvector of $E_1$ and $E_3$. $E_{23}E_{12} = \beta$ on $V$, and since $E'_{23} \neq 0$ we can solve for $E_{12}$ in terms of $E_{23}$ from this equation. Thus span$(v, E_{23}v, \ldots, E^{(\ell-1)}_{23})$ is invariant under each of the generators. Also the vectors $v, E_{23}v, \ldots, E^{(\ell-1)}_{23}$ are eigenvectors of $E_1$ with distinct eigenvalues, so dim$V = \ell$. The other cases are similar.

**Case 2:** $E'_{123} \neq 0$, $x \neq 0$ on $V$, and $E'_3 \neq 0$. Let $U$ be an irreducible submodule of $V$ over the generators $E_1$, $E_3$, $E_{123}$, and $E_{23}$. From $x[\varepsilon - \varepsilon^{-1}]E_3E_{12} + \varepsilon E_{123}] = \alpha$, and using the fact that $x$ and $E_3$ are invertible, we can solve for $E_{12}$ in terms of the other generators. Thus $U$ in $E_{12}$-invariant, and $V = U$. By previous considerations we know then that dim$V = \ell$ or $\ell^2$.

**Case 3:** $E'_{123} \neq 0$, $x \neq 0$ on $V$, and $E'_1 \neq 0$. Relabeling $E_1 \to E_{23}$, $E_{23} \to E_1$, $E_{12} \to E_3$, and $E_{123} \to -\varepsilon^{-1}E_{123}$, we obtain the same algebra as in case 2, with $\varepsilon^{-1}$ in place of $\varepsilon$. Thus by case 2 we have dim$V = \ell$ or $\ell^2$.

**Case 4:** $E'_{123} \neq 0$, $x \neq 0$ on $V$, $E'_3 = 0$, and $E'_1 \neq 0$. Let $U$ be an irreducible submodule of $V$ over the algebra with generators $E_1$, $E_3$, $E_{123}$, and $E_{23}$. Since $E_3$ q-commutes with each of these generators and $E'_3 = 0$, $E_3 = 0$ on $U$. The remaining generators $E_1$, $E_{123}$, and $E_{23}$ on $U$ satisfy the same relations as the generators in $\mathcal{U}^{q}_{gr}$ (with $E_1 \to E_1$, $E_{123} \to E_{12}$, and $E_{23} \to E_2$). Thus dim$U = \ell$, since $E'_{123} \neq 0$.  31
Also, the element $E_{123}x$ acts as a nonzero scalar on $U$. We find that $(E_{123}x)E_{12} = \varepsilon^2 E_{12}(E_{123}x)$, so the spaces $U$, $E_{12}U$, ..., $E_{12}^{\ell-1}U$ are eigenspaces of $E_{123}x$ with distinct eigenvalues (since $\ell$ is odd). Thus $V = U \oplus E_{12}U \oplus \cdots \oplus E_{12}^{\ell-1}U$. We shall show that each of the spaces in the sum is nonzero. For that we need the following formula, which is proved by induction on $m$:

\[ E_3E_{12}^m = \varepsilon^{-m}E_{12}^mE_3 - \varepsilon^{1-m}\left(\frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2}\right)E_{12}^{m-1}E_{123} \] (2.53)

Let $u$ be an eigenvector of $E_{123}$ in $U$; $E_{123}u = \lambda u$ with $\lambda \neq 0$. Let $m$ be the least positive integer such that $E_{12}^m u = 0$. Applying 2.53 to $u$, we obtain:

\[ 0 = -\lambda \varepsilon^{1-m}\left(\frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2}\right)E_{12}^{m-1}u \] (2.54)

It follows that $m = \ell$, since $\ell$ is odd. Thus each of the spaces in the sum $V = U \oplus E_{12}U \oplus \cdots \oplus E_{12}^{\ell-1}U$ is nonzero, and by lemma 2.3 this gives $\text{dim} V = \ell \text{dim} U$. Thus $\text{dim} V = \ell^2$.

**Case 5:** $E_{123}' \neq 0$, $y \neq 0$ on $V$. If we relabel the generators $E_1 \rightarrow E_3$, $E_3 \rightarrow E_1$, $E_{23} \rightarrow E_{12}$, $E_{12} \rightarrow E_{23}$, and $E_{123} \rightarrow E_{123}$, we find that we have not changed the relations. Thus this case is covered by cases 2, 3, and 4.

**Case 6:** $E_{123}' \neq 0$, $x = 0$, and $y = 0$ on $V$. Then $E_{123} = (\varepsilon^{-2} - 1)E_1E_{23}$ and $E_{123} = (\varepsilon^{-2} - 1)E_3E_{12}$. We then find that $E_1E_{23} = \varepsilon E_{23}E_1$ and $E_3E_{12} = \varepsilon E_{12}E_3$. Thus $V$ is irreducible over the generators $E_1$, $E_3$, $E_{23}$, and $E_{12}$, which satisfy the relations

\[
\begin{align*}
E_1E_3 &= E_3E_1 \\
E_1E_{23} &= \varepsilon E_{23}E_1 \\
E_1E_{12} &= \varepsilon E_{12}E_1 \\
E_3E_{12} &= \varepsilon E_{12}E_3 \\
E_3E_{23} &= \varepsilon E_{23}E_3 \\
E_{12}E_{23} &= E_{23}E_{12}
\end{align*}
\]
Note that the \( \ell \)th power of each of these generators must be nonzero. For example, if \( E_1^\ell = 0 \), then \( E_1 = 0 \) on \( V \), which would imply that \( E_{123} = 0 \), a contradiction. We also find that \( E_{23} E_{12}^{\ell-1} \) commutes with each of the generators, so is equal to a scalar \( \beta \) on \( V \). Thus we can solve for \( E_{23} \) in terms of \( E_{12} \). Let \( v \) be a common eigenvector of \( E_1 \) and \( E_3 \). The space span\((v, E_{12}v, \ldots, E_{12}^{\ell-1}v)\) is invariant under each of the generators, and the vectors \( v, E_{12}v, \ldots, E_{12}^{\ell-1}v \) are eigenvectors of \( E_1 \) with distinct eigenvalues.

Thus \( \dim V = \ell \).

\[2.7 \quad \mathcal{U}^w \text{ for } w = s_2s_1s_3 \text{ and } s_2s_1s_3s_2\]

For \( w = s_2s_1s_3s_2 \), we find that \( \beta_1 = \alpha_2, \beta_2 = \alpha_1 + \alpha_2, \beta_3 = \alpha_2 + \alpha_3, \beta_4 = \alpha_1 + \alpha_2 + \alpha_3 \).

We then find, using (2.8) through (2.12), that \( E_{\beta_1} = E_2, E_{\beta_2} = E_{12} = -E_2 E_1 + q^{-1} E_1 E_2, E_{\beta_3} = E_{23} = -E_2 E_3 + q^{-1} E_3 E_2, E_{\beta_4} = E_{123} = E_2 E_3 E_1 - q^{-1} E_3 E_2 E_1 - q^{-1} E_1 E_2 E_3 + q^{-2} E_1 E_3 E_2 \). With some computation, we find the relations (2.13) are as follows:

\[
E_2 E_{12} = q E_{12} E_2 \tag{2.55}
\]

\[
E_2 E_{23} = q E_{23} E_2 \tag{2.56}
\]

\[
E_{12} E_{23} = E_{23} E_{12} \tag{2.57}
\]

\[
E_2 E_{123} = E_{123} E_2 + (q - q^{-1}) E_{12} E_{23} \tag{2.58}
\]

\[
E_{12} E_{123} = q E_{123} E_{12} \tag{2.59}
\]

\[
E_{23} E_{123} = q E_{123} E_{23} \tag{2.60}
\]

\( \mathcal{U}^{s_2s_1s_3} \) has generators \( E_2, E_{12}, \) and \( E_{23} \) with relations (2.55) through (2.57).

\( \mathcal{U}^{s_2s_1s_3s_2} \) has generators \( E_2, E_{12}, E_{23}, \) and \( E_{123} \) with relations (2.55) through (2.60). We find that the element \( E_2 E_{123} - q E_{12} E_{23} \) is central in this algebra.
2.8 Irreducible Representations of $\mathcal{U}_z^w$ for $w = s_2s_1s_3$ and $s_2s_1s_3s_2$

**Proposition 2.14** The finite-dimensional irreducible representations of $\mathcal{U}_z^{s_2s_1s_3}$, where $\varepsilon$ is a primitive $\ell$th root of unity, have dimensions:

1. if $E_2^\ell = 0$
2. if $E_{12}^\ell = 0$ and $E_{23}^\ell = 0$
3. $\ell$ in all other cases

**Proof:** Let $V$ be an irreducible $\mathcal{U}_z^{s_2s_1s_3}$-module. If $E_2^\ell = 0$, then $E_2 = 0$ and $V$ is spanned by a common eigenvector of $E_{12}$ and $E_{23}$. If $E_{12}^\ell = 0$ and $E_{23}^\ell = 0$, then $E_{12} = 0$ and $E_{23} = 0$, and $V$ is spanned by an eigenvector of $E_2$, so dim$V = 1$. If $E_2^\ell \neq 0$ and $E_{12}^\ell \neq 0$, then letting $v$ be a common eigenvector of $E_{12}$ and $E_{23}$, we find that span$(v, E_2v, ..., E_2^{\ell-1}v)$ is invariant under each of the generators, and the vectors $v, E_2v, ..., E_2^{\ell-1}v$ are eigenvectors of $E_{12}$ with distinct eigenvalues. Thus dim$V = \ell$. Similarly, if $E_2^\ell \neq 0$ and $E_{23}^\ell \neq 0$, then dim$V = \ell$.

**Proposition 2.15** The finite-dimensional irreducible representations of $\mathcal{U}_z^{s_2s_1s_3s_2}$, where $\varepsilon$ is a primitive $\ell$th root of unity with $\ell$ odd, have dimensions:

1. if $E_{12}^\ell = 0$ and $E_{23}^\ell = 0$
2. if $E_{12}^\ell = 0$, $E_2^\ell = 0$, and $E_{12}^\ell$ or $E_{23}^\ell$ is zero
3. $\ell$ in all other cases

**Proof:** Let $V$ be an irreducible $\mathcal{U}_z^{s_2s_1s_3s_2}$-module. The element $E_2E_{123} - qE_{12}E_{23}$ is central in $\mathcal{U}_z^{s_2s_1s_3s_2}$, so $E_2E_{123} - \varepsilon E_{12}E_{23} = \alpha$ for some scalar $\alpha$ on $V$.

Case 1: $E_2^\ell \neq 0$. Let $U$ be an irreducible submodule of $V$ over the generators $E_2$, $E_{12}$, and $E_{23}$. From previous results we know that dim$U = 1$ or $\ell$. From $E_2E_{123} - \varepsilon E_{12}E_{23} = \alpha$ we can solve for $E_{123}$ in terms of the other generators, so $U$ is $E_{123}$-invariant and $U = V$.

Case 2: $E_2^\ell = 0$, $E_{12}^\ell = 0$, and $E_{23}^\ell = 0$. Since $E_{12}$ and $E_{23}$ q-commute with the other generators, we have $E_{12} = 0$, and $E_{23} = 0$ on $V$. It then follows that $E_2$...
q-commutes with $E_{123}$, so $E_2 = 0$. Thus $V$ is spanned by an eigenvector of $E_{123}$, and $\dim V = 1$.

Case 3: $E_2' = 0$, $E_{12}' \neq 0$, and $E_{23}' = 0$. $E_{23}' = 0$ implies $E_{23} = 0$ on $V$. This implies that $E_2$ q-commutes with the other generators, so $E_2 = 0$. We are left with the generators $E_{12}$ and $E_{123}$, which satisfy $E_{123}E_{12} = \varepsilon^{-1}E_{12}E_{123}$. Since $E_{12}' \neq 0$, $\dim V = \ell$ if $E_{123}' \neq 0$ and $\dim V = 1$ if $E_{123}' = 0$.

Case 4: $E_2' = 0$, $E_{12}' = 0$, and $E_{23}' \neq 0$. By the same argument as in case 3, we have $\dim V = \ell$ if $E_{123}' \neq 0$ and $\dim V = 1$ if $E_{123}' = 0$.

Case 5: $E_2' = 0$, $E_{12}' \neq 0$, and $E_{23}' \neq 0$. Let $U$ be an irreducible submodule of $V$ over the generators $E_2$, $E_{12}$, and $E_{23}$. Since $E_2$ q-commutes with $E_{12}$ and $E_{23}$, $E_2 = 0$ on $U$. Then $U$ is spanned by a common eigenvector $u$ of $E_{12}$ and $E_{23}$: $E_{12}u = \lambda u$ and $E_{23}u = \mu u$, where $\lambda \neq 0$ and $\mu \neq 0$. $V = U + E_{123}U + \ldots + E_{123}^{l-1}U$, and the spaces in the sum are eigenspaces of $E_{12}$ with distinct eigenvalues, so $V = U \oplus E_{123}U \oplus \ldots \oplus E_{123}^{l-1}U$. If $E_{123}' \neq 0$, it follows immediately that each of the spaces in the sum is nonzero, and $\dim V = \ell \dim U = \ell$. If $E_{123}' = 0$, we use the following formula, which is proven by induction on $m$:

$$E_2E_{123}^m = E_{123}^mE_2 + (\varepsilon - \varepsilon^{-1}) \left( \frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2} \right) E_{123}^{m-1}E_{12}E_{23} \quad (2.61)$$

Let $m$ be the least positive integer such that $E_{123}^m u = 0$. Applying equation 2.61 to $u$, we obtain:

$$0 = \lambda \mu (\varepsilon - \varepsilon^{-1}) \left( \frac{1 - \varepsilon^{2m}}{1 - \varepsilon^2} \right) E_{123}^{m-1}u \quad (2.62)$$

We conclude that $\ell = m$, since $\ell$ is odd. Thus we again have $\dim V = \ell \dim U = \ell$.

### 2.9 $\mathcal{U}^w$ and Irreducible Representations of $\mathcal{U}_z^w$ for

$w = s_1 s_2 s_3$

For $w = s_1 s_2 s_3$, we find that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_1 + \alpha_2 + \alpha_3$. We then find, using (2.8) through (2.12), that $E_{\beta_1} = E_1$, $E_{\beta_2} = E_{12} = -E_1E_2 + q^{-1}E_2E_1$, $E_{\beta_3} = 

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\[ E_{123} = E_1 E_2 E_3 - q^{-1} E_2 E_1 E_3 - q^{-1} E_3 E_1 E_2 + q^{-2} E_3 E_2 E_1. \]

With some computation, we find the relations (2.13) for \( \mathcal{U}^{g_1 g_2} \) are as follows:

\[ E_1 E_{12} = q E_{12} E_1 \quad (2.63) \]
\[ E_1 E_{123} = q E_{123} E_1 \quad (2.64) \]
\[ E_{12} E_{123} = q E_{123} E_{12} \quad (2.65) \]

**Proposition 2.16** The finite-dimensional irreducible representations of \( \mathcal{U}_\varepsilon^{g_1 g_2} \), where \( \varepsilon \) is a primitive \( \ell \)th root of unity, have dimensions:

- 1 if any two or three of \( E_1', E_{12}' \), and \( E_{123}' \) are zero
- \( \ell \) in all other cases

Proof: Let \( V \) be an irreducible \( \mathcal{U}_\varepsilon^{g_1 g_2} \)-module. We find that \( E_1 E_{12}'^{-1} E_{123} \) and \( E_1^{-1} E_{12} E_{123}'^{-1} \) are central in \( \mathcal{U}_\varepsilon^{g_1 g_2} \), so we have \( E_1 E_{12}'^{-1} E_{123} = \alpha \) and \( E_1^{-1} E_{12} E_{123}'^{-1} = \beta \) on \( V \) for some scalars \( \alpha \) and \( \beta \).

Case 1: Any two (or all three) of the \( \ell \)th powers of the generators \( E_1, E_{12}, E_{123} \) are 0. Then those two generators are 0 on \( V \), and \( V \) is spanned by an eigenvector of the third generator. So \( \dim V = 1 \).

Case 2: Any two (or all three) of the \( \ell \)th powers of the generators \( E_1, E_{12}, E_{123} \) are nonzero. Suppose, for example, that \( E_1' \neq 0 \) and \( E_{12}' \neq 0 \). Let \( U \) be an irreducible submodule of \( V \) over the generators \( E_1 \) and \( E_{12} \). Then \( \dim U = \ell \), and from \( E_1 E_{12}'^{-1} E_{123} = \alpha \) we can solve for \( E_{123} \) in terms of \( E_1 \) and \( E_{12} \), so \( U \) is \( E_{123} \)-invariant and \( V = U \), so \( \dim V = \ell \). The other cases are similar.

### 2.10 \( \mathcal{U}^w \) for the Remaining Elements of the Weyl Group

The remaining (nonidentity) elements of the Weyl Group for \( sl_4(\mathbb{C}) \) are \( s_2, s_3, s_2 s_1, \)
\( s_2 s_3, s_3 s_2, s_3 s_2 s_3, s_3 s_2 s_1, s_3 s_1 s_2 s_3, \) and \( s_3 s_2 s_3 s_1 s_2 \). Each of their algebras \( \mathcal{U}^w \) have the same (with a change of indices) generators and relations as algebras
already considered. For example, with the change of indices $1 \to 3$, $2 \to 2$, and $3 \to 1$, $\mathcal{U}^{21222122}$ has the same generators and relations as the algebra $\mathcal{U}^{2122122}$.

## 2.11 A Final Word

In the paper [3], a conjecture is made regarding the dimensions of the irreducible representations of solvable quantum groups. Namely, this conjecture states that the dimension should be $\varepsilon^{(1/2)\dim O_x}$, where $O_x$ is the symplectic leaf containing the restriction of the central character of $\pi$ to $Z_0$. This conjecture has been shown by Kac to hold for the quantum Heisenberg algebra considered in Chapter 1. For the algebras of Chapter 2, this conjecture has not been checked but it does predict the possible dimensions of these representations correctly.
Bibliography


