Presented at the 1983 International Conference on
Acoustics, Speech and Signal Processing

RECONSTRUCTION FROM PROJECTIONS BASED ON DETECTION AND ESTIMATION OF OBJECTS

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ABSTRACT

This paper considers the problem of observing a 2D function via its 1D projections (Radon transform); it presents a framework for detecting, locating and describing objects contained within a 2D cross-section by using noisy measurements of the Radon transform *directly*, rather than post-processing a reconstructed image. This framework offers the potential for significant improvements in applications where (1) attempts to perform an initial inversion with insufficient measurement data result in severely degraded reconstructions, and (2) the ultimate goal of the process is to obtain several specific pieces of information about the cross-section. To illustrate this perspective, we focus our attention on the problem of obtaining maximum-likelihood (ML) estimates of the parameters characterizing a single random object situated within a deterministic background medium, and we investigate the performance, robustness, and computational structure of the ML estimation procedure.

Introduction

The problem of reconstructing a multi-dimensional function from its projections arises, typically in imaging applications, in a diversity of disciplines, including oceanography, medicine, and nondestructive testing [1,2]. In the two-dimensional version of this problem, a 2D function $f(\underline{x})$ is observed via noisy samples of its Radon transform

$$g(t,\theta) = \int_{\underline{x}'\underline{\theta}} f(\underline{x})ds \qquad (1)$$

where $\underline{\theta}$ is the unit vector $(\cos\theta \ \sin\theta)'$. The major emphasis of research and applications in this area has been on developing exact and approximate solutions to this integral equation in order to produce high-resolution cross-sectional imagery (this approach to the inverse problem requires a large number of high signal-to-noise ratio measurements taken over a wide viewing angle [3]). Perhaps the most popular example of reconstruction from projections is medical computerized axial tomography (CAT). Initial success with reconstruction from projections in CAT scanning, as well as in radio astronomy and electron microscopy, has led to suggestions to apply reconstruction techniques to a variety of novel and technologically demanding tasks, e.g. real-time monitoring of high production rate manufacturing processes, quality control nondestructive testing, "stop action" internal imaging of very rapidly changing media, and mesoscale imaging of oceanographic regions hundreds of kilometers on a side [1,2]. In such applications of reconstruction techniques, high resolution cross-sectional imaging may be neither possible nor the ultimate goal:

1. For a variety of reasons (e.g. time, economic, environmental or physical constraints that limit either the total measurement viewing angle or time, or limit the number or sensitivity of measurement transducers) it may be impossible to obtain a full set of low noise measurements over a wide viewing angle.
2. In many applications, abundent *a priori* information about the cross-section is available, and the *ultimate* goal is not necessarily to obtain an image, but rather to extract specific information about the cross-section. In oceanographic and nondestructive testing applications, for example, projection measurements are processed in order to determine the location of objects such as oceanographic cold-core rings [1], or to detect and locate cracks or flaws within a homogeneous material [2].

When projection measurements are incomplete or of low quality due to the factors in (1), reconstruction leads to images that have artifacts, poor resolution and/or high noise levels; attempted interpretation of such imagery may result in unreliable or inconsistent evaluation.

In this paper, we consider processing incomplete noisy projection measurements when the ultimate goal is to obtain very specific information about a cross-section. In particular, an approach to such problems is presented, along with the associated analysis, and is illustrated via a simple problem of locating an object contained within a known cross-sectional background.

Cross-section and Measurement Model

Consider a 2D cross-section

$$f(x) = f_b(x) + d \cdot f_o(x-c;\gamma) \qquad (2)$$

where $f_b(x)$ is a known background and $d \cdot f_o(x-c;\gamma)$ is a randomly located object having known *density* or contrast d [where $f_o(x;\gamma)=1$] and unknown location $c \in R^2$; γ is a known vector of parameters characterizing, for example, the size, shape and/or orientation of the object. By the linearity of (1), the Radon transform of $f(x)$ is the sum of two components,

$$g(t,\theta) = \int_{\underline{x}'\underline{\theta}=t} f_b(x)ds + d \cdot \int_{\underline{x}'\underline{\theta}=t} f_o(x-c;\gamma)ds$$

$$= g_b(t,\theta) + d \cdot g_o(t-\underline{c}'\underline{\theta},\theta;\gamma) \qquad (3)$$

where $g_o(t,\theta;\gamma)$ is the Radon transform of $f_o(x;\gamma)$, the unit-contrast object located at the origin. Note that because the location of the object $c \in R^2$ is random, the second term in the Radon transform (the component due to the object) is characterized by a random *sinusoidal* shift $\underline{c}'\underline{\theta}$ in the t variable.

Let the noisy projection measurements be given by[1]

$$y(t,\theta) = \left[d \cdot g_o(t-\underline{c}'\underline{\theta},\theta;\gamma) * h(t)\right] + w(t,\theta)$$

$$= s(t,\theta;c) + w(t,\theta) \qquad (4)$$

$$(t,\theta) \in S_y \subset S = \left\{(t,\theta) : -\infty < t < \infty \ , \ 0 \leq \theta < \pi\right\}$$

where $*$ denotes one-dimensional convolution in the t variable, $h(t)$ is a 1D measurement aperture function, and $w(t,\theta)$ is a zero-mean Gaussian noise process.[2] In terms of the present notation, the object localization problem may be stated as: *given* noisy, partial measurements of the Radon transform as shown in (4), *estimate* the location $c \in R^2$ of the object. Problems involving an unknown object density d and/or an unknown object geometry parameterized by γ are treated in the same way [4]. It should be noted that with the exception of the density parameter d, the parameters characterizing the object enter the problem nonlinearly, and lead to a nonlinear estimation problem of small dimensionality. This is in contrast to full image reconstruction, in which a linear estimation problem of high dimensionality is solved.

ML Object Localization

In this section, we consider the special case of *full-view* measurements (i.e., $S_y = S$), with $w(t,\theta)$ a 2D zero-mean Gaussian noise process with covariance $E[w(t,\theta)w(\tau,\phi)] = \frac{N_o}{2}\delta(t-\tau,\theta-\phi)$.[3] The *maximum-likelihood* (ML) location estimate $\hat{c}_{ML}$ is that value of the parameter c that maximizes the log likelihood function [6]

$$l(c;Y) = \frac{2}{N_o}\int_0^{\pi}\int_{-\infty}^{\infty} y(t,\theta)s(t,\theta;c)dtd\theta$$

$$- \frac{1}{N_o}\int_0^{\pi}\int_{-\infty}^{\infty} s^2(t,\theta;c)dtd\theta \qquad (5)$$

The first term in (5) corresponds to a matched filtering operation in Radon space (this operation maps the Radon-space measurements into a function on R^2); the second term in (5) involves the energy in the Radon space matched filtering template. Since $s(t,\theta;c)$ depends on c only via a shift in the t variable, the second term in (5) is c-independent and can be dropped, as can the $\frac{2}{N_o}$ scaling factor, to yield

$$L(c;Y) = \int_0^{\pi}\int_{-\infty}^{\infty} y(t,\theta)s(t-\underline{c}'\underline{\theta},\theta;\underline{0})dtd\theta \qquad (6)$$

The log likelihood function for this problem is seen to be obtained by a *convolution back-projection* (CBP) operation (such as that used in full image reconstruction [3]), where the generally θ-dependent and non-symmetric convolving kernel $s(t,\theta;\underline{0})$ has been specified in the solution to the optimal localization problem.

Performance Analysis - An Example

By substituting the measurements in (4) into (6), the log likelihood function may be written as the sum of a deterministic *ambiguity function* and a zero-mean 2D correlated random process. By examining the effects of both *local* and *global* types of errors [6] in the estimation procedure, the covariance of the estimation error may be analyzed [4,5]. As an example, consider a constant-density *disk object* of radius R,

$$f_o(x;R) = \begin{cases} 1 & \text{if } |x| < R \\ 0 & \text{otherwise} \end{cases} \qquad (7)$$

which is located within a circular region of radius T >> R. For this full-view circularly-symmetric case, the

1. By the stated assumptions, the Radon transform of the background, $g_b(t,\theta)$, is known and its effect has been subtracted from the measurements.

2. The problem where $y(t,\theta)$ is a counting process with rate that is a function of $g(t,\theta)$ may also be considered; such a model is appropriate for very low-dose x-ray problems.

3. Incomplete measurement cases, in which views are available over only a limited view angle or at a finite number of views, are treated similarly.

error covariance matrix is σ^2 times a 2×2 identity matrix. Figure 1 is an illustration of $(\sigma/T)^{-2}$, the inverse of the normalized error variance, versus the normalized object size R/T, for several values of the ratio of contrast squared to noise level. This Figure indicates a definite threshold behavior -- for given values of the object contrast d and measurement noise level N_o, there exists a smallest object size for reliable localization.

Summary

A framework has been presented for detecting, locating and characterizing objects in a cross-section by using noisy projection measurements *directly*, rather than post-processing a reconstructed image. Within this framework, the performance, robustness and computational structure of ML estimation procedures have been investigated for both object location and geometry parameters [4,5].

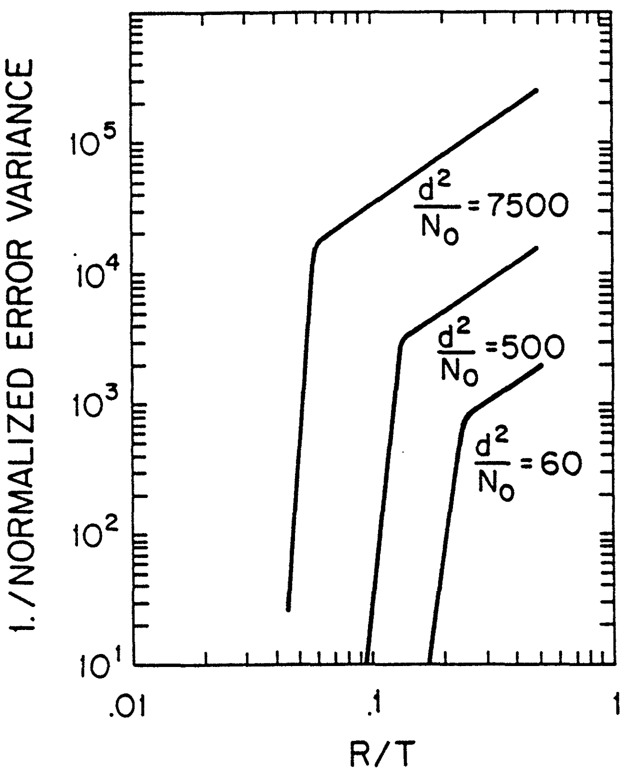

Figure 1. Localization performance versus disk object size.

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The work reported in this paper was performed at the M.I.T. Laboratory for Information and Decision Systems with partial support provided by the National Science Foundation under Grant ECS-8012668.