ABSTRACT

We consider estimation of an nx1 vector noncausal stochastic process governed by the linear dynamics

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with two-point boundary condition

$$v = Vx(0) + V^Tx(T)$$

where \(u\) is an mx1 white input noise with covariance parameter \(Q(t)\) and \(v\) is an nx1 zero mean Gaussian boundary value with nonsingular covariance matrix \(N_v\), uncorrelated with \(u\) on \([0,T]\). Estimation of processes satisfying models of this type was first suggested by Krener [1].

1. INTRODUCTION

The now classical solutions to linear filtering and smoothing problems for one-dimensional (1-D), nonstationary, causal processes are discussed in the review paper by Kailath [2]. Notable in the derivations of these solutions is the use of the Markovian nature of the models for these 1-D processes. However, in as much as stochastic processes in higher dimensions (random fields) are typically noncausal and consequently are not Markovian in the usual sense, their estimators cannot be derived through a direct extension of these 1-D derivations. Thus, linear estimation problems for noncausal processes require new approaches, and a natural first step is to solve the estimation problem for 1-D noncausal processes.

With this as motivation, Krener [3] introduced a class of linear noncausal 1-D dynamic models. In his study of these models, Krener has developed results on controllability, observability and minimality and has solved a deterministic linear control problem. In addition, he has posed the fixed-interval linear smoothing problem for these systems and has derived integral equations for both the weighting pattern and the error covariance for the optimal smoother. Working directly with these equations he has had some success in obtaining a dynamic realization of the smoother for a special "stationary-cyclic" class of these processes [4].

In this paper we solve the estimation problem for Krener's 1-D noncausal models through a generalization of the complementary models approach to linear estimation which was first introduced by Weinert and Desai [5]. In particular, we present a procedure for finding the complementary model which is appropriate for the 1-D noncausal estimation problem and following the methodology introduced in [6], we augment the model of the process to be estimated with the complementary model and invert the combined system to obtain a differential realization of the smoother.

2. LINEAR STOCHASTIC TWO-POINT BOUNDARY VALUE PROCESS (TPBVP)

The General Solution

The 1-D noncausal process studied here is governed by an nxn order linear stochastic differential equation together with a specified two-point boundary condition. Accordingly, the process will be referred to as a linear stochastic two-point boundary value process or TPBVP. In describing the model we employ the white noise formalism for representing linear stochastic differential equations. Let \(u(t)\) be a nx1 white noise process with covariance parameter \(Q(t)\). Let \(v\) be a nx1 random vector, independent of \(u(t)\), with covariance matrix \(N_v\). The nx1 boundary value process \(x(t)\) is governed on the interval \([0,T]\) by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with boundary condition

$$v = Vx(0) + V^Tx(T) .$$

It will be assumed that \(A\) and \(B\) are continuous on \([0,T]\) and that all random variables are zero-mean.

It is instructive to derive one form of the general solution for (1) as the approach we take in this derivation will be used later. The form of the solution which we obtain differs from the usual Green's function solution (e.g. see [1]). Let \(\zeta(t,s)\) be the transition matrix associated with \(A(t)\). If \(x(0)\) were known, then \(x(t)\) could be represented in the variation-of-constants form

$$x(t) = \zeta(t,0)x(0) + x^0(t)$$

where \(x^0(t)\) is the solution of (1a) with \(x^0(0) = 0:

$$x^0(t) = \int_0^t \zeta(t,s)b(s)u(s)ds .$$

Substituting from (2a) at \(t = T\) into the boundary condition (lb), we can write

$$v = Vx^0(T) + V^\zeta(T,0)x(0) .$$

For a well-posed problem, there will be a unique \(x(0)\) for a given \(v\) and \(u\) on \([0,T]\). Thus
well-posedness requires that the \( n \times n \) matrix

\[
F = V^T + V^T \Phi(T,0)
\] (3b)

be nonsingular. With \( F \) invertible, we can solve for \( x(0) \) as

\[
x(0) = F^{-1} [v - V^T x(T)]
\] (3c)

Substituting \( x(0) \) into (2a) gives the general solution for (1a,b) as

\[
x(t) = \Phi(t,0) x(0) + \int_0^t \Phi(t,\tau) f(\tau) \, d\tau
\]

The noncausal nature of the TPBVF \( x(t) \) is clearly displayed if we correlate the value of \( x \) at \( t = 0 \) with future values of the input \( u \):

\[
E[x(0)u'(t)] = -F^{-1} V^T \Psi(T,t) B(t) Q(t) \, u \quad t \in [0, T].
\]

Thus, the \( n \)th order model in (1) is not Markovian, and consequently Kalman filtering and associated smoothing techniques are not directly applicable.

Below, as an alternative to (4), we present a second form for the general solution of (1) which leads to a numerically stable implementation. Consider the equivalent process obtained by transforming \( x \) as

\[
\begin{bmatrix}
x_f(t) \\
x_b(t)
\end{bmatrix} = T(t) x(t)
\] (6a)

where the transformation matrix \( T(t) \) is chosen so that 1) the dynamics of the system model in (1) become decoupled:

\[
\begin{bmatrix}
x_f(t) \\
x_b(t)
\end{bmatrix} = \begin{bmatrix}
A_f & 0 \\
0 & A_b
\end{bmatrix} \begin{bmatrix}
x_f(t) \\
x_b(t)
\end{bmatrix} + \begin{bmatrix}
B_f \\
B_b
\end{bmatrix} u
\]

and 2) \( A_f \) is exponentially stable in the forward direction and \( A_b \) is exponentially stable in the backward direction. For time-invariant systems this is always possible by assigning those modes associated with eigenvalues greater than or equal to zero to \( A_f \) and those less than zero to \( A_b \). For time-varying dynamics, it may be difficult to determine the dynamics and boundary conditions for a transformation \( T(t) \) which transforms the system dynamics into this form. However, for the systems of interest to us later in this paper we can overcome this difficulty by invoking results obtained previously for smoothing solutions for causal processes. The boundary condition for the transformed process will be written as

\[
\begin{bmatrix}
x_f(0) \\
x_b(0)
\end{bmatrix} = \begin{bmatrix}
0 & \Phi(T,0) \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_f(T) \\
x_b(T)
\end{bmatrix}
\]

The reason for our choice of subscripts \( f \) and \( b \), denoting forward and backward respectively, will become apparent below.

If \( x_f(0) \) and \( x_b(0) \) were known, then we could solve for \( x_f(t) \) and \( x_b(t) \) as

\[
x_f(t) = \Phi_f(t,0) x_f(0) + \int_0^t \Phi_f(t,\tau) f(\tau) \, d\tau
\] (7a)

and

\[
x_b(t) = \Phi_b(t,0) x_b(0) + \int_0^t \Phi_b(t,\tau) f(\tau) \, d\tau
\] (7b)

where \( x_f(0) \) is governed by (6b) with \( x_f(0) = 0 \) and \( x_b(0) \) is governed by (7b) with \( x_b(0) = 0 \).

Following a derivation similar to that used to obtain the general solution in (4), it can be shown that

\[
\begin{bmatrix}
x_f(t) \\
x_b(t)
\end{bmatrix} = \begin{bmatrix}
\Phi_f(t,0) & 0 \\
0 & \Phi_b(t,0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_f(t) \\
x_b(t)
\end{bmatrix} = \begin{bmatrix}
0 & \Phi_f(t,0) \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_f(t) \\
x_b(t)
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & \Phi_b(t,0)
\end{bmatrix}
\]

The covariance of the TPBVF \( x(t) \)

The covariance of \( x(t) \)

\[
P_x(t) = E[x(t)x'(t)]
\]

(11)

can be derived from (4) as

\[
P_x(t) = \begin{bmatrix}
P_x^0(t) + \Phi(t,0) F^{-1} [V^T + V^T \Phi(T,0) V'] \\
\end{bmatrix}
\]

\[
F^{-1} \Phi'(t,0) + \Phi(t,0) F^{-1} V P_x^0(t) - P_x^0(t) V^T F^{-1} \Phi'(t,0)
\]

(12a)
where \( P^0(t) \) is the covariance of \( x^0(t) \) satisfying
\[
\dot{P}^0 = AP^0 + P^0 A' + BQB' ; \quad P^0(0) = 0 .
\] (12b)

An additional expression for \( P \) can be derived from the two-filter form of the general solution (equation (8)). However, this expression is somewhat complex, and we will wait until a later publication to present it in the context of the study of the estimation error covariance [7].

**Green's Identity**

In establishing the solution to our estimation problem we will make use of Green's Identity for the differential operator:
\[
L: D(L) \rightarrow R(L) ; \quad (Lx)(t) = \dot{x}(t) - A(t)x(t) \quad (13)
\]
where \( D(L) \) is the space of once continuously differentiable \( n \times 1 \) vector functions on \([0,T]\) and \( R(L) \) is the Hilbert space of square integrable \( n \times 1 \) vector functions on \([0,T]\). Let \( E \) be a 2\( n \times 2n \) matrix partitioned into \( n \times n \) blocks with:
\[
E = \begin{bmatrix}
-I & 0 \\
0 & I
\end{bmatrix}
\] (14a)

and define the 2\( n \times 1 \) vector
\[
x_b = \begin{bmatrix}
x(0) \\
x(T)
\end{bmatrix} . \quad (14b)
\]
The formal adjoint of the operator \( L \) is
\[
(L^* \lambda)(t) = -\dot{\lambda}(t) - A'(t)\lambda(t) \quad (14c)
\]

Given these definitions, the Green's Identity for \( L \) on the interval \([0,T]\) is obtained directly by integration by parts, yielding
\[
\langle Lx, \lambda \rangle_{L^2[0,T]} = \langle x, L^* \lambda \rangle_{L^2[0,T]} + \langle x_b, E \lambda_b \rangle_{R^n} \quad (15)
\]
where \( \lambda_b \) takes a form similar to \( x_b \) in (14b).

3. **PROBLEM STATEMENT**

The fixed-interval smoothing problem for the noncausal process \( x(t) \) defined earlier is stated as follows. Let \( r(t) \) be a \( p \times 1 \) white noise process uncorrelated with \( v \) and \( u(t) \) and with continuous covariance parameter \( R(t) \). Let \( C(t) \) be a \( p \times n \) matrix whose elements are continuous on \([0,T]\). The observations of \( x(t) \) are given by the \( p \times 1 \) vector stochastic process:
\[
y(t) = C(t)x(t) + r(t) . \quad (16)
\]

In addition, we assume that there may be available a boundary observation \( y_b \) defined as follows. Let \( r_b \) be a \( q \times 1 \) random vector uncorrelated with \( r(t) \), \( u(t) \) and \( v \) with covariance matrix \( \Pi_b \). Define a \( q \times 2n \) matrix \( W \) partitioned into \( q \times n \) blocks as
\[
W = [W^0 ; W^T] . \quad (17a)
\]
The boundary observation is the \( q \times 1 \) random vector:
\[
y_b = Wx_b + r_b . \quad (17b)
\]
Define an \( n \times 2n \) matrix \( V \) as
\[
v = [V^0 ; V^T] . \quad (18a)
\]
so that the boundary condition in (1b) can be written as
\[
v = Vx_b \quad (18b)
\]
We will assume that the rows of \( W \) and the rows of \( V \) are linearly independent [8].

The fixed-interval smoothing problem is to find the linear minimum variance estimate of the noncausal TPBVP \( x(t), t \in [0,T] \), given the complete observation set \( Y \):
\[
Y = \{y(t) \mid 0 \leq t \leq T\} . \quad (19)
\]

4. **SOLUTION TO THE ESTIMATION PROBLEM**

Our approach to the solution of the estimation problem is based on an application of the method of complementary models introduced by Weinert and Desai [7] in deriving the smoother for causal stochastic processes. A brief discussion of the basis for this approach follows. Define the underlying process \( \xi \) as
\[
\xi = \begin{bmatrix}
u \\
v \\
\end{bmatrix} . \quad (20)
\]
The complementary process \( Z = \{Z, Z_b\} \) is orthogonal to the observations \( Y \{y, y_b\} \) and when combined with the observations, they span the space spanned by \( \xi \). In particular, we can express the observations \( Y \) and the process to be estimated \( x \) as linear mapping of the underlying process
\[
Y = \begin{bmatrix}y & y_b\end{bmatrix} = [H : I] \xi \quad (21a)
\]
\[
\Delta = M \xi
\]
and
\[
x = M \xi . \quad (21b)
\]
The complementary process
Given by
\[ Z = \begin{bmatrix} z \\ z_b \end{bmatrix} \] (22a)

is a linear mapping of \( \xi \)
\[ Z = M \xi \] (22b)

which satisfies: (1) the orthogonality condition
\[ Z \perp Y, \text{ i.e. } E \left\{ y(t) \begin{bmatrix} z'(t) \\ z_b'(t) \end{bmatrix} \right\} = 0 \] (23a)

and (2) the complementation condition: The augmented mapping \( M \) defined by
\[ \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} \mathcal{M}_Y \\ \mathcal{M}_Z \end{bmatrix} \] (23b)

is invertible so that \( \xi \) can be written as
\[ \xi = M^{-1} Y + N \xi 
\]
\[ \Delta \xi = \Delta Y + \xi \] (23c)

where the partitions of \( M^{-1} \) are
\[ M^{-1} = [N_Y: N_Z] \]

Given (22a) and (23c), the linear minimum variance estimate of \( \xi \) given \( Y \) is \( \hat{\xi} \), the projection of (23c) into \( \text{Span}(Y) \). Since \( \hat{x} \) is a linear mapping of \( \xi \), the linear minimum variance estimate of \( x \) is
\[ \hat{x} = M \hat{\xi} \]
\[ = M \hat{Y} \] (24a)

It is straight-forward to show that the estimation error is the minimum variance estimate given \( Z \):
\[ \hat{x} = x - \hat{x} 
\]
\[ = M \hat{\xi} - M \hat{Y} 
\]
\[ = M \hat{\xi} - M \hat{Y} 
\]
\[ = M \hat{\xi} \] (24b)

Here we have expressed the error as a linear mapping of the underlying process \( \xi \), and the estimation error variance is therefore a linear function of the variance of \( \xi \).

For the above construction to be useful we need to know the mapping \( M \) in (22b) and the inverse of the augmented map \( M \) in (23b). In [8] it is proved that the orthogonality and complementation conditions are satisfied by \( M \)

\[ M \xi = [-I : H^*] \begin{bmatrix} \Sigma_x^{-1} \xi \end{bmatrix} \] (25a)

where \( H^* \) is the Hilbert adjoint operator of \( H \) in (21a) and \( \Sigma_x \) is the integral operator whose kernel is the correlation function of the underlying process \( \xi \). From the definition of the underlying process for our problem
\[ Z = \begin{bmatrix} z \\ z_b \end{bmatrix} = H^* \begin{bmatrix} x^{-1} \xi \\ \Pi^{-1}_b \end{bmatrix} - \begin{bmatrix} \Pi^{-1}_v \end{bmatrix} \] (25b)

Given \( M \), the final step in determining the estimator is to augment \( M \) and \( M \) as in (23b) and invert. This step can be simplified if we first determine an internal differential realization for \( Z \) in a form similar to that for \( Y \). Then, following [5] the internal differential realizations for \( Y \) and \( Z \) can be augmented and inverted in a straightforward manner. From (25b) we see that an internal differential realization for \( Y \) implies an internal realization for \( H^* \). To obtain such a realization, start with the inner product identity for Hilbert adjoints
\[ \left< H \begin{bmatrix} u \\ v \end{bmatrix} \right| \begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \left< \begin{bmatrix} u \\ v \end{bmatrix} \right| \begin{bmatrix} H^* \beta \\ \Pi\nu \end{bmatrix} \] (26)

where \( \begin{bmatrix} \beta \\ \gamma \end{bmatrix} \) is an arbitrary element in the range of \( H, R(H) \), and \( \begin{bmatrix} u \\ v \end{bmatrix} \) is an arbitrary element in the domain of \( H, D(H) \). Employing
\[ \begin{bmatrix} u \\ v \end{bmatrix} = C x \]
and defining
\[ \begin{bmatrix} \lambda \\ \psi \end{bmatrix} = R \begin{bmatrix} u \\ v \end{bmatrix} \] (27a)

where
\[ H = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad (\text{i.e. } H^* = \begin{bmatrix} B^* & 0 \\ 0 & I \end{bmatrix} R^*) \] (27c)

(26) can be rewritten as
\[ \left< x, C^* u \right| + \left< x, H^* v \right| = \left< \beta \right| \left< \psi \right| \] (28a)
Applying Green's identity to replace \(<Lx, \lambda>\) in (28a) gives
\[ <x, [C^*u_\lambda - L^*\lambda]> = <\lambda, [E^*_b + V^*y_b - W^*V^\lambda]> \] (28b)

Now if \(W_c\) is a matrix such that
\[ \Gamma = \begin{bmatrix} W_c \\ -V \end{bmatrix} \] (29a)
is invertible, we can define
\[ \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = (\Gamma')^{-1} E. \] (29b)

Then it is shown in [8] that the map
\[ \psi = \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = H^* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \] (30a)
has an internal differential realization with internal process \(\lambda\) satisfying
\[ L^\lambda = C^*u_\lambda \] (30b)
with boundary condition
\[ \begin{bmatrix} v_\lambda \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_b \\ \lambda_c \end{bmatrix} \] (30c)
and output
\[ \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = B^\lambda \begin{bmatrix} \lambda \\ \lambda_b \end{bmatrix} \] (30d)
and that this choice of internal realization satisfies the inner product identity (28b) with both left and right hand sides identically zero.

Augmenting \(Z\) defined by (25b) and \(H^*\) given by (30) with the internal differential realization for \(Y\) and inverting as in [6] and projecting the solution onto the span of \(Y\) (see (23)) we find [8] that an internal differential realization of the estimator is given by the 2nth order system:
\[ \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\lambda}} \end{bmatrix} = H \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} + \begin{bmatrix} 0 \\ C^*R^{-1} \end{bmatrix} \] (31a)
with two-point boundary condition
\[
\begin{bmatrix} W^b \Pi^{-1} y_b \\ v_0^\lambda \hat{x}(0) \\ v_0^T \hat{x}(T) \end{bmatrix} = \begin{bmatrix} \lambda(0) \\ \lambda(T) \end{bmatrix} \] (31b)

where
\[ H = \begin{bmatrix} A & BQ^*B' \\ C^* & -A^* \end{bmatrix}, \] (31c)
\[ V^0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} V^T - W^T \Pi^{-1} W : I \] (31d)

and
\[ V^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} V^0 - W^T \Pi^{-1} W : 0 \] (31e)

Hamiltonian Diagonalization

The solution of the 2nth order boundary value process in (31) could be implemented by either of the two forms of the general solution derived earlier. However, by considering the "time"-invariant case we can anticipate that there may be numerical instabilities associated with the first of those methods. In the time-invariant case the eigenvalues of the 2nx2n Hamiltonian matrix \(H\) defined in (1a) are symmetric about the imaginary axis [9], i.e. there are n eigenvalues in each of the left and right half planes. Thus, for the time-invariant case, the right half plane eigenvalues will result in numerical instabilities for the unidirectional implementation suggested by (4). This stability problem can be avoided in general by transforming the smoother dynamics into the stable forward/backward form in (8). To achieve this second form, we need a transformation which diagonalizes the dynamics of \(H\) into two \(nxn\) blocks, one stable in the forward direction and the other backwards stable.

Since the dynamics of our smoother as represented by \(H\) are identical to those of the smoother for causal processes as originally derived by Bryson and Frazier [10], any transformation which results in a two-filter smoother for causal processes will also diagonalize our smoother. However, choosing a diagonalizing transformation for our problem requires special considerations not encountered in the causal case. First, because the two-point boundary condition provides incomplete information for both the initial and final values of the process, we will choose a transformation which corresponds to a two-filter solution for causal processes with both filters in information form. Second, it is important to choose the boundary conditions properly for the Riccati differential
equations which govern the time-varying elements of the diagonalizing transformation. Our choice of diagonalizing transformation and corresponding boundary conditions makes it possible to formulate a numerically stable two-filter form for our smoother which is remarkably similar to two-filter smoothers for causal processes.

Define the time-varying transformation \( T(t) \) as the \( 2n \times 2n \) matrix partitioned in \( n \times n \) blocks as

\[
T(t) = \begin{bmatrix}
\theta_f(t) & -I \\
-\theta_b(t) & I
\end{bmatrix}
\]  \hspace{1cm} (32a)

Let the transformed process be denoted by

\[
q(t) = \begin{bmatrix}
q_f(t) \\
q_b(t)
\end{bmatrix} = T(t) \begin{bmatrix}
\hat{x}(t) \\
\lambda(t)
\end{bmatrix}
\]  \hspace{1cm} (32b)

Also define

\[
\dot{\theta}^2_q = \frac{1}{2} T^{-1} + THT^{-1}
\]  \hspace{1cm} (33a)

and

\[
G_q = TG
\]  \hspace{1cm} (33b)

so that the dynamics of the transformed process can be written as

\[
\begin{bmatrix}
\dot{q}_f \\
\dot{q}_b
\end{bmatrix} = H_q \begin{bmatrix}
q_f \\
q_b
\end{bmatrix} + G_q y
\]  \hspace{1cm} (33c)

If we use the following form for the inverse of \( T \):

\[
T^{-1}(t) = \begin{bmatrix}
I & I \\
-\theta_b(t) & \theta_f(t)
\end{bmatrix}
\begin{bmatrix}
0 & P_s(t) \\
0 & P_s(t)
\end{bmatrix}
\]  \hspace{1cm} (34a)

where

\[
P_s(t) = [\theta_f(t) + \theta_b(t)]^{-1}
\]  \hspace{1cm} (34b)

and if we choose the dynamics for \( \theta_f \) and \( \theta_b \) as

\[
\begin{align*}
\dot{\theta}_f &= \theta_f A + \theta_f BQ B^T \theta_f - C^T R^{-1} C \\
\dot{\theta}_b &= \theta_b A + \theta_b BQ B^T \theta_b - C^T R^{-1} C
\end{align*}
\]  \hspace{1cm} (34c)

and

\[
\begin{align*}
\dot{\theta}_f &= \theta_f A + \theta_f BQ B^T \theta_f - C^T R^{-1} C \\
\dot{\theta}_b &= \theta_b A + \theta_b BQ B^T \theta_b + C^T R^{-1} C
\end{align*}
\]  \hspace{1cm} (34d)

then carrying out the calculation in (33a), it can be shown that \( H_q \) is diagonalized with diagonal blocks

\[
H_q = \begin{bmatrix}
A' + \theta_f BQ B^T \\
-\theta_b BQ B^T
\end{bmatrix}
\]  \hspace{1cm} (34e)

and

\[
H_b = \begin{bmatrix}
-A' - \theta_b BQ B^T \\
\theta_f BQ B^T
\end{bmatrix}
\]  \hspace{1cm} (34f)

Thus the dynamics of \( q_f \) and \( q_b \) are decoupled and are given by

\[
\begin{align*}
\dot{q}_f &= H_q q_f + C^T R^{-1} y \\
\dot{q}_b &= H_b q_b - C^T R^{-1} y
\end{align*}
\]  \hspace{1cm} (35a)

and

\[
\begin{align*}
\dot{q}_f &= H_f q_f + C^T R^{-1} y \\
\dot{q}_b &= H_b q_b - C^T R^{-1} y
\end{align*}
\]  \hspace{1cm} (35b)

If we assume for time-invariant dynamics that \( A, B \) is stabilizable and that \( A, C \) is detectable and for time-varying dynamics that \( A, B \) is uniformly completely controllable and \( A, C \) is uniformly completely reconstructable, then the invertibility of \( P \) in (34b) is guaranteed if both \( \theta_f(0) \) and \( \theta_b(0) \) are nonnegative definite [9]. Furthermore, these conditions guarantee that \( \theta_f \) and \( \theta_b \) and their derivatives are bounded and that \( H_f \) and \( H_b \) are forward and backward stable respectively.

Under the transformation (32a), the boundary condition (31b) becomes

\[
\begin{bmatrix}
W^0 T^{-1} y_b \\
W^T T^{-1} y_b
\end{bmatrix} = V^0 q_f(0) + V^T q_b(0) + V^0 q_f(T) + V^T q_b(T)
\]  \hspace{1cm} (36a)

where

\[
V_q^0 = V_q^0 T^{-1}(0)
\]  \hspace{1cm} (36b)

and

\[
V_q^T = V_q^T T^{-1}(T)
\]  \hspace{1cm} (36c)

To simplify the expressions for the boundary value coefficient matrices in (36b) and (36c), choose the following nonnegative definite initial and final conditions for the Riccati equations (34c) and (34e):

\[
\begin{align*}
\theta_f(0) &= W^0 \Pi^{-1}_v W^0' \\
\theta_b(0) &= W^T \Pi^{-1}_b W^T
\end{align*}
\]  \hspace{1cm} (37a)

and

\[
\begin{align*}
\theta_f(T) &= W^0 T^{-1} \Pi_v W^0' \\
\theta_b(T) &= W^T T^{-1} \Pi_b W^T
\end{align*}
\]  \hspace{1cm} (37b)

Then defining \( \theta_c \) as the following \( n \times n \) matrix:

\[
\theta_c = V_q^T \Pi_{v}^{-1} V_q^T + V_q^T \Pi_b^{-1} V_q^T
\]  \hspace{1cm} (38)
it can be shown that the boundary value coefficient matrices can be written as

\[
V_0^+ = \begin{bmatrix}
I & 0 \\
\theta_P(0) : \theta_P(0)
\end{bmatrix}
\]

and

\[
V_0^T = \begin{bmatrix}
\theta_P(T) : \theta_P(T) \\
0 : I
\end{bmatrix}
\]

Since the dynamics of \( q_f \) and \( q_b \) are uncoupled, the only coupling between the two enters through the boundary condition. By our choice of initial and final conditions for the Riccati equations, we have been able to display this coupling solely as a function of the matrix \( \theta_c \).

The smoothed estimate of \( x \) is recovered by inverting \( T(t) \) in (32b) so that we obtain

\[
\hat{x}(t) = P_s(t)[q_f(t) + q_b(t)].
\]

Following (8), an explicit expression for the two-filter solution for \( q_f \) and \( q_b \) is formulated as follows. Let \( q_f^0 \) and \( q_b^0 \) be governed by (35a) and (35b) respectively with boundary conditions: \( q_f^0(0) = 0 \) and \( q_b^0(T) = 0 \). Define \( F_{fb} \) and \( \phi_{fb} \) as the 2nx2n matrices

\[
F_{fb} = [V_f^0 + V_b^0, 0] \quad \phi_{fb}(T) = \begin{bmatrix}
\theta & 0 \\
0 & \theta
\end{bmatrix}
\]

Then the two-filter solution for \( q(t) \) is given by

\[
\begin{bmatrix}
q_f(t) \\
qu_b(t)
\end{bmatrix} = \phi_{fb}(T)F_{fb}^{-1} \begin{bmatrix}
v^0_f \\
v^0_b
\end{bmatrix} + \begin{bmatrix}
q_f(0) \\
qu_b(0)
\end{bmatrix}. \tag{41c}
\]

The computational complexity of the non-causal smoother implementation suggested by (41) is nearly the same as that of the two-filter smoothers for causal processes such as the Mayne-Fraser form [11,12]. We note, however, that before \( q_f \) and \( q_b \) can be evaluated for any \( t \in [0,T] \), both \( q_f^0 \) and \( q_b^0 \) must be computed and stored along with \( P_s \) and \( \phi_{fb} \) for the entire interval \([0,T] \). Thus, the required storage exceeds that of the smoother for causal processes. Indeed, the Mayne-Fraser solution and ours differ significantly in one aspect. That is, for our smoother the contribution of the forward filter to the smoothed estimate at some point \( t \) depends not only on past observations, as does the Mayne-Fraser solution, but also on future observations through the term \( \phi_c \hat{x} \) in (14). A similar statement applies for the backward filter contribution.

5. SMOOTHING ERROR

From (24b) we see that the smoothing error is the solution of the augmented and inverted system projected onto the span of the complementary process \( z \). By expressing the complementary process as a linear function of the underlying process and employing the internal differential realization of the inverted system it can be shown (8), that the smoothing error has the same Hamiltonian dynamics so the smoother (31a) and that the boundary condition for the smoothing error can be written in terms of \( \psi \) and \( \theta_c \) in (31d) and (31e). As a consequence, the same diagonalizing transformation (32a) can be used to write the error as a linear combination of a forward stable process and backward stable process. From this representation it can be shown (8) that the error covariance can be expressed in terms of \( \theta_{f}, \theta_{b} \) and \( \theta_{c} \) in (34c), (34d) and (38) plus the solution of one additional matrix Riccati equation.

6. CONCLUSIONS

An internal differential realization of the fixed-interval smoother for an \( n \)-th order non-causal two-point boundary value stochastic process (TPBVP) has been shown to have the same \( n \)-th order Hamiltonian dynamics as the fixed-interval smoother for causal processes. The simplicity of this two-filter form is achieved by employing an information form for the diagonalizing transformation with carefully chosen boundary conditions.
for the differential equations governing its elements. The significant difference between our two-filter implementation and that for causal processes is that in the noncausal case the smoothed estimate at a given point in the interval is a noncausal function of each of the forward and backward processes (see (40) and (41c)).

In closing we note that differential realizations for estimators of both discrete and continuous parameter multidimensional stochastic processes can be formulated as well by the method of complementary models. The solutions to these multidimensional estimators and problems associated with their implementation are addressed in [8].

REFERENCES


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