A PARAMETRIZATION OF ALL STABLE STABILIZING
COMPENSATORS FOR SINGLE-INPUT-OUTPUT SYSTEMS*

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ABSTRACT

A characterization is given of all stable compensators that stabilize a given scalar plant. In the process, a simple proof is given of an interpolation lemma due to Youla et al.

Key words: strong stabilization, interpolation by units

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1. INTRODUCTION

In this paper, we study the following problem: Given a lumped linear scalar plant described by its transfer function $p(\cdot)$, find all stable compensators $c(\cdot)$ (if any) such that the transfer function $p/(1+cp)$ is stable. Previously, Youla et al. [1] have given necessary and sufficient conditions for the existence of a stable stabilizing compensator for a given plant (if such a compensator can be found, the plant is said to be strong stabilizable [1]). However, at present no characterization is available of all stable stabilizing compensators. This is given in the present paper. In the process, we also obtain an exceedingly simple proof of an interpolation lemma first proved in [1]. Finally, the results given here can also be used to characterize the set of all compensators that simultaneously stabilize each of two given plants $p_1$ and $p_2$. 
2. LOGARITHMS IN THE DISC ALGEBRA

Throughout this paper, the main technique used is that of embedding the problem in a Banach algebra and then taking logarithms. Hence it is appropriate to begin with a discussion of this topic.

The disc algebra \( A \) consists of functions that are analytic on the open unit disc and continuous on the closed unit disc [2, p. 77]. If we define the norm of a function \( f \) in \( A \) by

\[
||f|| = \max_{|z| < 1} |f(z)|
\]

and the product of two functions in \( A \) to be their point-wise product, then \( A \) becomes a commutative Banach algebra with identity. We let \( A_s \) denote the subalgebra of \( A \) consisting of symmetric functions; i.e.

\[
A_s = \{ f \in A : \overline{f(z)} = f(z) \quad \forall z \in D \}
\]

where \( D \) is the closed unit disc and the bar denotes conjugation. It is important to note that \( A \) is viewed as an algebra over the field of complex numbers, whereas \( A_s \) is an algebra over the reals.

Suppose a linear scalar time-invariant plant has the transfer function \( p(s) \). We say that the plant is **stable** if the function \( z \to p(z) = \tilde{p}[(1+z)/(1-z)] \) is in \( A_s \). The bilinear transformation is used to map the closed right half-plane plus the point at infinity (which is usually considered to be the unstable region for continuous-time systems) into the closed unit disc \( D \). In the case of discrete-time systems, a plant with transfer function \( p(z) \) is stable if \( p \in A_s \). The plant is **lumped** if \( p \) or \( \tilde{p} \), as appropriate, is a rational function.
A unit of $A_s$ is an element that has an inverse in $A_s$. It is easy to see that $f \in A_s$ is a unit if and only if $f(z) \neq 0$ for all $z \in D$.

An element of $f \in A_s$ is said to have a logarithm in $A_s$ if there exists a $g \in A_s$ such that

$$f = \exp(g) = \sum_{i=0}^{\infty} \frac{g^i}{i!}$$  \hspace{1cm} (3)

If $f$ has a logarithm $f$, then $f \exp(-g) = 1$, so that $f$ must be a unit. The converse question, namely whether every unit in $A_s$ has a logarithm, is answered next.

**Proposition 1** [3, p.14, Theorem (1.4.10)] Suppose $B$ is a Banach algebra with identity, and let $L$ denote the set of units in $B$ that have a logarithm. Then the group (under multiplication) generated by $L$ coincides with the connected component of the set of units containing the identity. In particular, if $B$ is commutative, then $L$ is a group, so that $L$ coincides with this component.

**Proposition 2.** A unit $f$ in $A_s$ has a logarithm in $A_s$ if and only if $f(z) > 0$ for all $z \in [-1,1]$.

**Proof.** "only if" Suppose $f = \exp(g)$, where $g \in A_s$. Then $g(z)$ is real for all $z \in [-1,1]$, so that $f(z) = \exp[g(z)] > 0$ for all $z \in [1,1]$.

"if" Suppose a unit $f$ satisfies $f(z) > 0$ for all $z \in [-1,1]$. Then, in particular, we have that $f(0) > 0$. Now define

$$h(r,z) = f(rz) \forall r \in [0,1], \forall z \in D.$$  \hspace{1cm} (4)

Then $h$ is a continuous map from $[0,1] \times A_s$ into $A_s$. Moreover, since
f(z) \neq 0 \ \forall z \in D, \text{ it is easy to see that } h(r,z) \neq 0 \ \forall z \in D, \ \forall r \in [0,1].

Hence \( h(r,.\) is a unit for every \( r \) in \([0,1]\). Finally, \( h(0,.\) = f(0), \( h(1,.\) = f(.), so that \( f \) is homotopic to the constant function \( f(0) \).

Since \( f(0) > 0 \), it is homotopic to the identity via the map

\[
    w(r,z) = rf(0) + (1-r), \ \forall z \in D, \ \forall r \in [0,1]
\]

This shows that \( f \) is homotopic to the identity. By Proposition 1, it now follows that \( f \) has a logarithm in \( A_s \).

Though it is not needed for this paper, we state another proposition in the interests of completeness.

Proposition 3. Every unit in \( A \) has a logarithm in \( A \).

Proof. If \( f \) is a unit of \( A \), then \( f(z) \neq 0 \ \forall z \in D \). Now define

\[
    h(r,z) = \begin{cases}
    f[(1-2r)z], & r \in [0,1/2] \\
    \exp[1-2r]a, & r \in [1/2,1]
    \end{cases} \ \forall z \in D
\]

where \( \exp(a) = f(0) \neq 0 \). Then \( h \) provides a homotopy from \( f \) to 1 within the set of units of \( A \), so that \( f \) has a logarithm in \( A \), by Proposition 1.

In particular, every unit of \( A_s \) (which is also a unit of \( A \)) has a logarithm in \( A \), but not necessarily in \( A_s \). The difference arises because the set of nonzero complex numbers is connected, while the set of nonzero real numbers is not.

Proposition 4. Suppose a unit \( f \) in \( A_s \) has a logarithm in \( A_s \). Then the logarithm is unique.
Proof. Let $g_1$, $g_2$ be logarithms of $f$ in $A_s$. Then we have successively

\[ \exp(g_1(z)) = \exp(g_2(z)) \quad \forall \, z \in D \]  
\[ g_1(z) = g_2(z) + j2\pi m(z) \quad \forall \, z \in D \]

where $m(z)$ is an integer for all $z \in D$. Since $m(z) = (g_1(z) - g_2(z))/j2\pi$, it is analytic in $D$ and therefore continuous in $D$. Now a continuous function that assumes only discrete values must be constant. Hence there is an integer $m$ such that

\[ g_1(z) = g_2(z) + j2\pi m, \quad \forall z \in D \]

Finally, since $g_1, g_2 \in A_s$, the quantity $g_1(z) - g_2(z)$ is real whenever $z \in [1,1]$. This shows that $m = 0$.

The same argument shows that logarithms in $A$ are unique to within integer multiples of the constant function $j2\pi$. 
3. A SIMPLE PROOF OF AN INTERPOLATION LEMMA

In this section, we give a simple proof of an interpolation lemma from [1] which forms the basis of all the results on strong stabilizability given in [1].

Suppose we are given a finite collection \( \{z_1, \ldots, z_l\} \) of points in \( \mathbb{D} \), together with nonnegative integers \( m_1, \ldots, m_l \) and complex numbers \( c_{ij} \), \( 0 \leq j \leq m_i, 1 \leq i \leq l \). The objective is to determine whether there exists a unit \( u \in A_s \) such that \( u^{(j)}(z_i) = c_{ij} \) for all \( i, j \), where as usual the zeroth order derivative is the function itself. Since \( u \) is required to be in \( A_s \), all the required interpolation conditions must occur in complex conjugate pairs; that is, if \( z_i \) is real, then \( c_{ij} \) must be real for all \( j \), and if \( z_i = \overline{z}_k \), then \( m_i = m_k \) and \( c_{ij} = \overline{c}_{kj} \) for all \( j \). Let us assume without loss of generality that these elementary consistency conditions are met, and that the \( z_i \) are so numbered that \( z_1, \ldots, z_r \) are real \( z_{r+1}, \ldots, z_l \) have nonreal.

Proposition 5. With the above notation, there exists a unit \( u \in A_s \) such that \( u^{(j)}(z_i) = c_{ij} \) for all \( i, j \) if and only if \( c_{10}, \ldots, c_{r0} \) all have the same sign.

Proof "only if" If such a \( u \) exists, then \( u(z) \) is real and does not change sign as \( z \) varies over \([-1, 1]\). In particular, \( u(z_1) = c_{10}, \ldots, u(z_r) = c_{r0} \) are all of the same sign.

"if" we may assume without loss of generality that \( c_{10}, \ldots, c_{r0} \) are all positive; if not, we construct a unit \( v \in A_s \) such that \( v^{(j)}(z_i) = -c_{ij} \), \( \forall i, j \), and let \( u = -v \). The problem is solved if we can construct a
function $f \in A$ such that

$$\frac{d^j}{dz^j} \exp(f(z)) \bigg|_{z=z_i} = c_{ij}, \quad \forall i,j$$

(8)

Let us translate the conditions in (8) into conditions on the values of $f$ and its derivatives. First, we get

$$f(z_i) = \log c_{i0}, \quad i=1,\ldots,\ell$$

(9)

The point to note is that when $z_i$ is real, $c_{i0}$ is real and positive, so that $\log c_{i0}$ is (more precisely, can be taken to be) real. When $z_i$ is nonreal, $\log c_{i0}$ may be nonreal, but this is no problem. For higher derivatives, we get

$$f'(z_i) = \frac{c_{i1}}{c_{i0}}$$

(10)

$$f''(z_i) = \frac{c_{i2} - [f'(z_i)]^2}{c_{i0}}$$

et cetera. Since $c_{i0} \neq 0$ for all $i$, the expressions in (10) are all well-defined. Moreover, the quantities $f^{(j)}(z_i)$ are all real whenever $z_i$ is real, and occur in complex conjugate pairs. Thus the original interpolation problem is reduced to one of constructing a symmetric analytic function meeting prespecified interpolation conditions. This is trivial to solve: in fact, $f$ can always be chosen to be a polynomial.

**Example 1.** Suppose it is required to construct a unit in $A$ satisfying

$$u(1) = 2, \quad u(0) = 1, \quad u'(0) = -1, \quad u(j) = 1 + 2j$$

Since $u(1)$ and $u(0)$ are both positive, such a unit exists. To solve the problem, let $u = \exp(f)$. Then $f$ must satisfy
\[ f(l) = \log 2 \approx 0.7 \]
\[ f(0) = 0 \]
\[ f'(0) = -1 \]
\[ f(j) = \log(1+2j) \approx 0.8 + j \cdot 1.1 \]

Using Lagrange interpolation, one can readily find a polynomial satisfying the above conditions, namely:

\[
\begin{align*}
    f(z) &= 0.7 \cdot \frac{z^2}{2} + \frac{z(z-1)(z^2+1)}{2} + (0.8 + j \cdot 1.1) \cdot \frac{z^2}{2} \\
    &\quad + (0.8 - j \cdot 1.1) \cdot \frac{z^2}{2} \\
    &= 2.3 \cdot z^4 - 2.1 \cdot z^3 + 1.5 \cdot z^2 - z
\end{align*}
\]

It should be noted that the procedure described above does not result in a rational unit function, in contrast with the procedure in [1]. The existence of a rational unit is a consequence of the next result.

**Proposition 6.** Suppose \( d, n \) are rational functions in \( A_S \) with no common zeros in \( D \). Then there exists a rational \( c \) in \( A_S \) such that \( d+cn \) is a unit if and only if the values of \( d \) at the real zeros of \( n \) in \([-1,1]\) are all of the same sign (or equivalently, if the number of zeros of \( d \) (counting multiplicity) between any pair of real zeros of \( n \) in \([-1,1]\) is even).

**Proof.** Let \( z_1, \ldots, z_\ell \) be the zeros of \( n \) in \( D \), with multiplicities \( m_1, \ldots, m_\ell \). Then a function \( g \) in \( A_S \) is of the form \( d+cn \) for some \( c \) in \( A_S \) if and only if

\[
g^{(j)}(z_i) = d^{(j)}(z_i), \quad 0 \leq j \leq m_i - 1; \quad 1 \leq i \leq \ell
\] (11)
"only if" Suppose there exists a rational \( c \) such that \( d+cn \) is a unit. Let \( u \) denote \( d+cn \); then \( u \) satisfies (11). By Proposition 5, this implies that \( d(z_1), \ldots, d(z_r) \) are all of the same sign, where \( z_1, \ldots, z_r \) are the real zeros of \( n \) in \( D \).

"if" Suppose \( d(z_1), \ldots, d(z_r) \) are all of the same sign. Then by Proposition 4 there exists a (not necessarily rational) unit \( u_1 \) in \( A_S \) such that \( u_1 = d+c_1n \) for some \( c_1 \) in \( A_S \). If \( c_1 \) is not rational, find a rational function \( c \) in \( A_S \) (e.g. a polynomial) such that \( ||c-c_1|| < 1/||u_1^{-1}|| \cdot ||n|| \). Let \( u = d+cn \). Then \( ||u-u_1|| \leq ||c-c_1|| \cdot ||n|| < 1/||u_1^{-1}|| \), so that \( u \) is also a unit.

Now we return to Proposition 5 and examine the existence of a rational unit \( u \) satisfying \( u^{(j)}(z_i) = c_{ij} \) for all \( i,j \). Let \( d \) be any polynomial in \( A_S \) such that \( d^{(j)}(z_i) = c_{ij} \) \( \forall i,j \), and let

\[
    n(z) = \prod_{i=1}^{l} \frac{1}{m_i} (z-z_i)^{m_i}. \tag{12}
\]

The condition \( c_{i0} > 0 \) for \( i=1, \ldots, r \) insures that \( d \) and \( n \) satisfy the hypotheses of Proposition 6. Thus there exists a rational \( c \in A_S \) such that \( u = d+cn \) is a unit. Clearly \( u \) is also rational and satisfies \( u^{(j)}(z_i) = c_{ij} \) \( \forall i,j \).
4. STABLE STABILIZING COMPENSATORS

Given rational functions $d, n$ in $A_s$, the problem of finding all $c \in A_s$ such that $d + cn$ is a unit is equivalent to finding all units $u \in A_s$ such that $d-u$ is a multiple of $n$. Let $z_1, \ldots, z_p$ be the zeros of $n$ in $D$, with multiplicities $m_1, \ldots, m_p$. Then the problem is one of finding all units $u \in A_s$ that satisfy

$$u^{(j)}(z_i) = d^{(j)}(z_i), \quad 0 \leq j \leq m_i - 1; \quad 1 \leq i \leq p \quad (13)$$

Suppose $u_1$ and $u_2$ are two units that each satisfy (13), and let $v$ denote the unit $u_2 u_1^{-1}$. Then a routine calculation shows that

$$v(z_i) = 1, \quad 1 \leq i \leq p \quad (14)$$

$$v^{(j)}(z_i) = 0, \quad 0 \leq j \leq m_i - 1; \quad 1 \leq i \leq p \quad (15)$$

Equivalently, $v$ is of the form $1 + fn$ for some $f \in A_s$, since $v$ interpolates the function "1" and its derivatives at the zeros of $n$. This leads to the next result.

**Proposition 7.** Given $d, n \in A_s$, let $u$ be any unit in $A_s$ satisfying (13). Then the set of all units in $A_s$ satisfying (13) is given by

$$\{uv : v \in A_s \text{ is a unit of the form } 1 + fn \text{ for some } f \in A_s\}.$$
finite number of zeros in $D$. Let $z_1, \ldots, z_s, \bar{z}_1, \ldots, \bar{z}_s, \sigma_1, \ldots, \sigma_r$ denote
the distinct zeros of $g$ in $D$, where $z_1, \ldots, z_s$ are nonreal and $\sigma_1, \ldots, \sigma_r$ are real. For convenience, let $z_1, \ldots, z_{2s+r}$ denote the same sequence,
and let $\mu_i$ denote the multiplicity of $z_i$ as a zero of $g$. Select
polynomials $p_1(z), \ldots, p_{2s}(z)$ such that

\begin{align*}
p_i(z_i) &= 1, \quad p_i^{(j)}(z_i) = 0 \quad \text{for } j = 1, \ldots, \mu_i - 1 \quad (16) \\
p_i^{(j)}(z_k) &= 0 \quad \text{for } j = 0, \ldots, \mu_k - 1 \text{ if } k \neq i
\end{align*}

Since the zeros of $g$ occur in complex conjugate pairs, we may suppose
that $p_i(z) = p_{i+s}^\ast(z)$ for $i = 1, \ldots, s$. Finally, define the polynomials

\begin{align*}
\phi_i(z) &= j[p_i(z) - p_{i+s}(z)], \quad i = 1, \ldots, s \quad (17)
\end{align*}

and observe that $\phi_i \in \mathbb{A}_s$ for all $i$.

**Proposition 8.** Suppose $g \in \mathbb{A}_s$ has only a finite number of zeros
in $D$, and that $g$ has at least one real zero in $D$. Then every unit in
$\mathbb{A}_s$ of the form $1 + fg$, $f \in \mathbb{A}_s$, can be expressed as $\exp(v)$, where
$v \in \mathbb{A}_s$ has the form

\begin{align*}
v &= hg + \sum_{i=1}^{s} 2\pi m_i \phi_i \\
&= \sum_{i=1}^{s} 2\pi m_i \phi_i \\
&= \sum_{i=1}^{s} 2\pi m_i \phi_i
\end{align*}

(18)

where $h \in \mathbb{A}_s$, $m_1, \ldots, m_s$ are arbitrary integers, and $\phi_i$ are defined in
(17). Conversely every unit $\exp(v)$ where $v$ is of the form (18) can
also be written as $1 + fg$ for some $f \in \mathbb{A}_s$. In summary,

$$U(g) = \{\exp(v) : v \text{ is of the form (18)}\} \quad (19)$$

**Proof.** Suppose $u = 1 + fg$ is a unit. Since $g$ has at least one real
zero in $D$, $u(z)$ equals one for some real $z$ in $D$. Hence $u(z) > 0$ for
z ∈ [-1,1], and by Proposition 2, has a logarithm v in A_s. Since

\[ g(z_i) = 0 \text{ for } i = 1, \ldots, 2s+r, \]

it follows that \( u(z_i) = 1 \) \( \forall i \), which in turn implies that

\[ v(z_i) = \frac{j2\pi m_i}{\phi}, \text{ } m_i \text{ an integer, for } i = 1, \ldots, 2s+r \]  

(20)

Since \( v(z) = \overline{v(z)} \), it is immediate from (20) that \( m_i = -m_{i+s} \) for \( i = 1, \ldots, s \), and that \( m_{2s+1} = \ldots = m_{2s+r} = 0 \). At multiple zeros of g, successive higher derivatives of g vanish, which implies that the corresponding derivatives of u and v also vanish. Thus

\[ v^{(j)}(z_i) = 0 \text{ for } j = 1, \ldots, \nu_i-1; \text{ } i = 1, \ldots, 2s+r \]  

(21)

Now (20) and (21) lead to the conclusion that g divides the function

\[ v = \sum_{i=1}^{2s} \frac{j2\pi m_i}{\phi} p_i, \text{ which equals } v = \sum_{i=1}^{s} \frac{2\pi m_i}{\phi} \phi_i. \]  

This is precisely (18).

Conversely, suppose v is of the form (18). Then v satisfies (20) and (21), which in turn implies that \( \exp(v) - 1 \) is divisible by g. Hence \( \exp(v) \) is of the form \( 1+fg \) for some \( f \in A_s \).

In practice, the assumption that g has at least one real zero in D does not pose a major restriction, because if g is the "numerator" of a strictly proper plant, then \( g(1) = 0 \). Also, note that g is not required to be rational - merely to have only a finite number of zeros in D.

We state the next result in the interests of completeness. The proof is omitted, as it closely follows that of Proposition 8.

**Proposition 9.** Suppose \( g \in A_s \) has only a finite number of zeros in D, all of them nonreal. Then every unit \( u \in A_s \) of the form \( 1+fg, f \in A_s \)
can either be written as \( \exp(v) \) where

\[
v = h + \sum_{i=1}^{s} 2m_i \phi_i; \quad m_1, \ldots, m_s \text{ arbitrary integers, } h \in A_s
\]

or as \(-\exp(v)\) where \(v \in A_s\) has the form

\[
v = h + \sum_{i=1}^{s} (2m_i + 1) \phi_i; \quad m_1, \ldots, m_s \text{ arbitrary integers, } h \in A_s
\]

Conversely, every unit \(\exp(v)\) where \(v\) is of the form (22) can be written as \(1 + fg\) for some \(f \in A_s\); every unit \(-\exp(v)\) where \(v\) is of the form (23) can be written as \(1 + fg\) for some \(f \in A_s\).

**Example.** Consider the problem of determining all stable stabilizing compensators for the plant whose transfer function is \((s^2 + 1)/[(s+2)(s-3)]\).

After substituting \(s = (1+z)/(1-z)\), this becomes

\[
n(z) (z + 1) (1-z) \quad \frac{d(z)}{d(z)} (z-3)^2 (2z-1)
\]

where \(n\) and \(d\) denote the numerator and denominator polynomials, respectively.

Thus the problem is one of finding all \(c \in A_s\) such that \(d+cn\) is a unit of \(A_s\). This can be solved using Propositions 7 and 8. First, we construct a unit \(u_0\) such that \(d-u_0\) is a multiple of \(n\). Since the zeros of \(n\) inside the unit disc are at 1, \(j\), \(-j\), we require \(u_0(z)\) to equal \(d(z)\) at these points. Thus we must have

\[
u_0(1) = 4, \quad u_0(j) = 4 + 22j
\]

If \(u_0 = \exp f_0\), then \(f_0\) must satisfy

\[
f_0(1) = \log 4 \approx 1.4, \quad f_0(j) = \log (4 + 22j) \approx 3.1 + j1.4
\]
Using Lagrange interpolation, one can find

\[ f_0(z) = -1.55 z^2 + 1.4z + 1.55 \]

Therefore, one stabilizing compensator is given by

\[ c_0(z) = [d(z) - \exp f_0(z)] / n(z) \]

To find all stabilizing compensators, we compute the set \( U(n) \). Again using Lagrange interpolation, one can find polynomials to satisfy (16), namely

\[ p_1(z) = \frac{(z-1)(z+1)}{2+2i}, \quad p_2(z) = p_1(z), \quad p_3(z) = \frac{z^2 + 1}{2} \]

\[ \phi_1(z) = j[p_1(z) - p_2(z)] = -\frac{\pi}{2} (z-1)^2 \]

Thus

\[ U(n) = \{ \exp u, \text{ where } v = hn + \Im(m(z-1)^2), \text{ m an integer, } h \in \mathbb{A}_s \} \]

Thus the set of all stable stabilizing compensators is given by

\[ \mathcal{C} = \{ [d(z) - \exp f(z)] / n(z), \text{ where } \]

\[ f(z) = f_0(z) + h(z)n(z) + \Im(m(z-1)^2), \text{ m an integer, } h \in \mathbb{A}_s \} \]
5. MULTIVARIABLE PLANTS

Suppose \( D \in A_{s}^{p \times p} \), \( N \in A_{s}^{q \times p} \) are rational and right-coprime. Then necessary and sufficient conditions for the existence of a rational \( C \in A_{s}^{p \times q} \) such that \( D+CN \) is a unit of \( A_{s}^{p \times p} \) are given in [1]. Now the problem of finding all \( C \in A_{s}^{p \times q} \) that stabilize \( ND^{-1} \) is equivalent to the problem of finding all units \( U \) in \( A_{s}^{p \times p} \) such that \( D-U \) is a left multiple of \( N \). It turns out that Proposition 7 can be readily extended to the multivariable case. Define

\[
U(N,D) = \{ U \in A_{s}^{p \times p} : U \text{ is a unit and } U = D+CN \text{ for some } C \in A_{s}^{p \times q} \} \tag{24}
\]

Let \( U(N) \) be a shorthand for \( U(N,I) \). Thus

\[
U(N) = \{ U \in A_{s}^{p \times p} : U \text{ is a unit and } U = I+FN \text{ for some } F \in A_{s}^{p \times q} \} \tag{25}
\]

Proposition 10: Let \( U \) be any element of \( U(N,D) \). Then

\[
U(N,D) = \{ UV : V \in U(N) \} \tag{26}
\]

Proof. Suppose \( D-U = C_0 N \). We show first that \( UV \in U(N,D) \) for all \( V \in U(N) \), and then we show that every \( W \in U(N,D) \) is of the form \( UV \) for some \( V \in U(N) \). So suppose first that \( V \in U(N) \), and let \( V = I + FN \). Then \( D - UV = (U + C_0 N) - U(I+FN) = (C_0 - F)N \) is a left multiple of \( N \), so that \( UV \in U(N,D) \). Conversely, suppose \( W \in U(N,D) \), and let \( D-W = CN \) for some \( C \in A_{s}^{p \times q} \). Define \( F = U^{-1}(C-C_0) \) and \( V = I+FN \). Then \( W = D+CN = D+C_0 N + (C-C_0)N = U + UFN = UV \). Since \( W \) and \( U \) are units, so is \( V \); i.e., \( V \in U(N) \).
The main difficulty in extending Propositions 8 and 9 to the multi-variable case is that the set of units that have logarithms is not necessarily a group under multiplication. Hence, even if a unit is homotopic to the identity, it need not have a logarithm. As a result, it is not possible to characterize all stable stabilizing compensators, since it is not possible to characterize all units of the form $I + FN$ for a given $N$. However, it is possible to generate an infinite family of such compensators.

Suppose $N \in A^{pxq}_s$. Then every function $\exp(HN), H \in A^{qxp}_s$ is a unit of $A^{qxp}_s$ of the form $I + FN$; in fact

$$\exp(HN) = I + \left[ \sum_{i=0}^{\infty} \frac{(HN)^i}{(i+1)!} H \right] N$$

Thus, for any $H \in A^{qxp}_s$, the function

$$F = \sum_{i=0}^{\infty} \frac{(HN)^i}{(i+1)!} H$$

belongs to $U(N)$. In this way it is possible to generate infinitely many $C$ such that $D + CN$ is a unit, using Proposition 10.
6. CONCLUDING REMARKS

In this paper, we have given a characterization of all stable compensators that stabilize a given scalar plant. In the process, we have also given a simple proof of an interpolation lemma due to Youla et al. [1]. In the case of multivariable plants, we have given a procedure for generating infinitely many (but not necessarily all) stable stabilizing compensators.

It is shown in [4] that the problem of simultaneously stabilizing two plants is equivalent to stabilizing an auxiliary system using a stable compensator. Thus the results of this paper can also be used to determine all compensators that simultaneously stabilize each of two given scalar plants, and to determine an infinite number of compensators that simultaneously stabilize each of two given multivariable plants.
REFERENCES


