LOAD-BEARING ROLE OF THE HUMAN KNEE MENISCUS

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ABSTRACT

Pathology of the meniscus is a significant orthopaedic problem. The surgical
treatment of meniscal tears is partial or total meniscectomy and these are some of the
most frequent orthopaedic procedures. Unfortunately, meniscectomies are not benign
procedures; clinical studies have linked meniscectomies to subsequent degenerative
osteoarthritic changes in the knee joint. Attempts at meniscal repair have been
unsuccessful and the need for meniscectomies to alleviate joint pain and disability
continues, albeit under more conservative protocols.

In spite of the strong clinical link between meniscal pathology and subsequent
osteoarthritic degeneration of the knee joint, mechanical modeling of the effects of a
meniscectomy on subchondral bone stresses has not been previously attempted. The
objectives of this work are to quantitatively model the biomechanics of the meniscus
and to elucidate its role in normal and pathological load-bearing in the knee. Even
though experimental studies have substantiated the load-bearing role of the meniscus,
mathematical modeling of this function is necessary to estimate physical variables of
the system that cannot be measured experimentally.

Model development began with a lumped-parameter knee model incorporating
axial compressive stiffness and circumferential tensile stiffness of the meniscus and
contact stiffness of the articular cartilage. This model provides a qualitative under-
standing of the mechanics of load transmission in the knee joint and the important
gEometric and material parameters. A two-dimensional axisymmetric finite element
model of the knee was developed. Important contributions in the model include
incorporation of articular cartilage layers and anisotropic material properties. The
load-bearing role of the meniscus was addressed through analysis of the overall joint
stiffness, contact pressure profiles, and subchondral bone stress distribution. The
model results indicate that the meniscus does transmit a significant percentage of
the total joint load and distribute the load to reduce subchondral bone stresses. The
model predictions indicate subchondral bone stresses approach the subchondral bone
strength following partial and total meniscectomy. This result is consistent with the-
tories proposing subchondral microfractures as a precipitating event for osteoarthritic
degeneration.

Model verification involved measuring meniscal displacements and strains in human cadaver knees. An X-ray stereophotogrammetry system and a knee loading apparatus were designed to perform the measurements. Measured strains are consistent with the model predictions.

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Chapter 1

Introduction

The knee joint is the largest synovial joint in the human body and arguably the most complex joint. The knee is required to move through a large range of motion (145 degrees in flexion/extension) and to transmit significant forces (3–5 times body weight). These requirements are not complementary. Transmission of large body forces requires conforming surfaces to reduce the contact stresses. However, rigid conforming surfaces restrict the kinematic freedom of the joint. The menisci achieve these competing requirements by acting as mobile conforming surfaces and play a primary role in knee load-bearing.

1.1 Significance of Meniscal Pathology

Unfortunately, the incidence of meniscal injuries is significant. Meniscal injuries are frequently encountered in orthopaedic practice and some authors report only fractures are more common. Appel [10] reports “ruptures of the semilunar cartilage (meniscus) of the knee are the most common intra-articular injuries and consequently, their removal is the most usual intra-articular procedure.” Fox et al [52] reviewed 6000 knee surgical procedures and found 38 percent of the procedures involved medial meniscectomy. In the most recent statistics, the 1988 National Patient Profile reports 51,488 meniscectomies were performed. This figure may underestimate the actual number of procedures performed.
Cadaver studies corroborate the prevalence of meniscal lesions. In a study of 100 autopsies, Noble and Turner [110] report that 60 percent of the individuals had at least one significant meniscal lesion. Casscells [31] undertook a larger study of 369 cadaver knees and found 116 knees (32%) with some pathological change either in the meniscus or the adjacent femoral and/or tibial articular surfaces. Even in a study of a younger population (less than 55 years of age), Noble found that 18.6 percent had at least one tear of the meniscus [108]. These autopsy studies further quantify the significance of meniscal pathology.

1.2 Meniscectomy Linked to Osteoarthritis

The clinical treatment for meniscal tears is partial or total meniscectomy. However, this form of treatment is not a benign procedure. In 1936, King [81] demonstrated partial and total meniscectomy resulted in articular cartilage degeneration in dogs and hypothesized that the menisci protect the articular cartilage. Fairbank [47] (1948) was the first to propose the menisci protected the articular cartilage by sharing in the weight-bearing. He reported three radiographic changes in the knee following meniscectomy: "(1) formation of and antero-posterior ridge projecting downwards from the margin of the femoral condyle over the old meniscus site; (2) generalized flattening of the marginal half of the femoral articular surface . . . ; (3) narrowing of the joint space on the side of the operation . . . ." Since Fairbank, several follow-up studies have linked meniscectomies to subsequent osteoarthritic changes in the knee joint (Jackson [71] 1968, Appel [10] 1970, Johnson et al [74] 1974, Cox et al [40] 1975, Noble and Erat [109] 1980, Allen et al [8] 1984, Jorgensen et al [77] 1987). Attempts to repair the meniscus have been unsuccessful so the need for meniscectomies to alleviate joint problems continues, albeit under more conservative protocols.
1.3 Research Objectives

In spite of the strong clinical link between meniscectomy and subsequent osteoarthritic degeneration of the knee joint, mechanical modeling of the meniscus to quantify its role in load-bearing under normal and pathological conditions is insufficient. The importance of modeling the meniscus to elucidate its mechanical role in normal and pathological processes was highlighted in the recommendations by a seminar conducted by the American Academy of Orthopaedic Surgeons and the National Institute of Arthritis and Musculoskeletal and Skin Diseases:

Although important functions have been ascribed to the meniscus, little detailed mechanical information is available on how the meniscus carries out these functions or how the tissue responds to various stresses. Studies characterizing the mechanical functions of the meniscus should be directed at establishing models to relate mechanical functions to the composition and structure of the meniscus and the interactions among the meniscus, the synovial fluid, and the adjacent articular cartilage. ... The development of these models to describe meniscal function will lead to an understanding of the mechanical response of menisci in normal and abnormal environments. These models will provide the means to evaluate plausible causes of meniscal injury and degeneration, and the effects of different surgical procedures on meniscal repair, including partial and total meniscectomy. [11]

Development of these models is crucial to understanding the mechanical role of the meniscus and possible mechanisms in the pathogenesis of osteoarthritis because the models provide a means of estimating variables that cannot be measured experimentally (e.g. subchondral bone stresses).

Several meniscal functions have been proposed [75], including: (1) joint stability, (2) shock absorption, (3) increased congruity, (4) lubrication, (5) limitation of
extremes of flexion and extension, (6) load transmission, and (7) contact stress reduction. The particular functions addressed in this research are functions related to the meniscal load-bearing role: increased congruity, load transmission, and contact stress reduction. The quantitative measures used to study the load-bearing role of the meniscus are the contact pressure distribution, the tibial subchondral bone stress distribution, axial load-displacement curves of the joint, and load-displacement curves of the meniscal radial displacement. The role of the meniscus is examined in the intact knee and pathologic cases following partial and total meniscectomy.

In order to conduct this research, the following research objectives were adopted:

1. Determine an anisotropic stiffness tensor incorporating meniscal material properties to describe the macroscopic elastic behavior of the meniscus.

2. Develop a lumped-parameter model of the knee incorporating coupled axial and circumferential stiffnesses of the meniscus to qualitatively explain the meniscal role in load transmission in the knee joint.

3. Develop axisymmetric finite element model of the knee including an anisotropic meniscus and articular cartilage to examine the load-bearing role of the meniscus and the stress distribution in the tibial subchondral bone.

4. Examine mechanical mechanisms for the development of osteoarthritis following meniscectomy.

5. Conduct X-ray stereophotogrammetry studies to measure the displacements and strains of human cadaver knee menisci. These experiments are intended to verify the model predictions.
Chapter 2

Anatomic and Clinical Background

A thorough understanding of the relevant anatomy is essential to the development of accurate mathematical models. Model geometry and boundary conditions are determined by the anatomic structures and attachments, respectively. The material symmetries and properties derive from the tissue microanatomy (histology and biochemistry). Once models are developed, knowledge of surgical treatments are necessary to simulate procedures and predict outcomes. This chapter is not intended to be a comprehensive review of the anatomic and clinical aspects of the meniscus, but is intended to provide a brief background on normal knee anatomy and surgical removal of the meniscus (meniscectomy).

2.1 Anatomy

The knee joint is a compound synovial joint consisting of two condylar joints between the corresponding condyles of the femur and tibia and a sellar joint between the patella and the patellar surface of the femur [156]. Interposed between the femoral condyles and the tibial plateau are two menisci, semilunar fibrocartilages that are regarded as extensions of the tibia. The menisci serve to deepen the surfaces of the articular fossae of the head of the tibia for reception of the condyles of the femur. The peripheral border of the meniscus is attached to the joint capsule and is thick
and convex; the inner border tapers to a thin, free edge. The proximal (superior) surface of the meniscus articulating with the femoral condyles is concave while the distal (inferior) surface is flat as it rests on the tibial plateau [11] (see Figure 2.1).

The medial meniscus is approximately semicircular in the transverse plane, but it is narrower in cross-section anteriorly than posteriorly (see Figure 2.2). The anterior horn of the medial meniscus is attached anteromedially to the insertion of the anterior cruciate ligament in the intercondylar area of the tibial plateau. Peripheral fibers of the anterior horn are continuous with the transverse ligament (see Figure 2.2). The posterior horn of the medial meniscus is attached anteromedially to the posterior cruciate ligament insertion in the intercondylar area. The peripheral edge of the medial meniscus is attached to the tibia through the coronary ligament, which is part of the capsule. The coronary ligament is relatively lax, but short, so that it does not allow excessive motion of the medial meniscus relative to the tibia [88, 151]. The superficial and deep layers of the medial collateral ligament (tibial collateral ligament) are attached to the medial meniscus (see Figures 2.3a and 2.3b). The deep layer of the medial collateral ligament may be referred to as the medial capsular ligament [75].
Figure 2.2: Superior aspect of the right tibia, showing the menisci and the tibial attachments of the cruciate ligaments with menisci and cruciate ligaments in situ [152].

There is a discrepancy between Last [88] and Warren [151] regarding whether or not the deep portion of the medial collateral ligament connects to the tibia. Last states that the periphery of the medial meniscus is more firmly attached to the femur than to the tibia by both the superficial and deep layers of the medial collateral ligament [88]. The short femoral fibers move with the femur; the long tibial fibers are free from the tibia over the semimembranosus expansion and the medial inferior genicular vessels. In rotation of the knee, therefore, the medial meniscus moves with the femur on the tibia.

The lateral meniscus forms approximately four-fifths of a complete circle and is of constant breadth throughout its extent [156] (see Figure 2.2). The anterior and posterior attachments of the lateral meniscal horns are quite centrally located in the intercondylar area. The anterior horn of the lateral meniscus is attached anterior to the lateral intercondylar tubercle, lateral and posterior to the anterior
Figure 2.3: (a) The superficial layer of the medial collateral ligament is attached posteriorly to the medial meniscus. (b) The deep layer of the medial meniscus is attached to the medial meniscus. [151]
cruciate ligament with which it partly blends [156]. The posterior horn is attached between the medial and lateral intercondylar tubercles. Additionally, the posterior horn of the lateral meniscus sends off the anterior meniscofemoral ligament (ligament of Humphry) and the posterior meniscofemoral ligament (ligament of Wrisberg) that pass anteriorly and posteriorly to the posterior cruciate ligament, respectively, and attach to the medial femoral condyle (see Figure 2.4). The meniscofemoral ligaments are often the only attachments of the posterior horn of the lateral meniscus [156]. The transverse ligament connects the anterior convex border of the lateral meniscus to the anterior horn of the medial meniscus (see Figure 2.2). The thickness of the transverse ligament varies between subjects and is absent in about 40 percent of the subjects [156]. The lateral meniscus is tethered to the tibia by the coronary ligament.

The remaining attachments of the lateral meniscus involve the popliteus muscle, the arcuate ligament, and the lateral collateral ligament (fibular collateral ligament). The literature is quite confusing concerning this region of the lateral meniscus, with several authors providing seemingly different descriptions of the region [38, 75, 87, 88, 156]. However, two points are quite clear: (1) the lateral collateral ligament is not connected to the meniscus; (2) the medial or upper half of the popliteus muscle is attached to the posterior arch (circumference) of the lateral meniscus through a quadrilateral aponeurosis. The remainder of the popliteus muscle is attached to the lateral condyle of the femur (see Figure 2.4). As the popliteus muscle contracts, the posterior arch of the meniscus is retracted from the joint space. The following description attempts to provide a consistent interpretation of the region and clarify the differing naming conventions. Gray's Anatomy describes the arcuate popliteal ligament as a Y-shaped system of capsular fibers whose stem arises from the fibular head. The anterior (or lateral) branch of the arcuate ligament extends to the lateral epicondyle of the femur and fuses with the femoral tendon of the popliteus muscle. This branch also attaches to the lateral meniscus. The names for this anterior bundle of the arcuate ligament are numerous: (1) short lateral ligament [156], (2) short
Figure 2.4: The upper half of the muscle is inserted into the posterior arch of the lateral meniscus and the meniscus is attached to the femur by the ligaments of Humphry and Wrisberg [88].
Figure 2.5: Detailed drawing of the region of the popliteal hiatus. The superior fascicle is posterior to the tendon, and the inferior fascicle is anterior. The exposed area of the tibial condyle is actually within the popliteal bursa, and is separated from the tendon by a thin sheet of capsule and synovium [38].

external lateral ligament [88], (3) lateral capsular ligament [75], and (4) meniscofibular ligament as described by Živanović in 80 percent of knee joints [156]. The posterior (or medial) branch of the arcuate ligament arches over the tendon of the popliteus to the posterior joint capsule with some fibers blending with the popliteal attachment of the lateral meniscus. The posterior branch of the arcuate ligament arching over the popliteus tendon creates the popliteal hiatus [38] (see Figure 2.5).

2.2 Histology

In a classic study of the histologic structure of the meniscus, Bullough et al [27] described the collagen fiber orientation of the meniscus (see Figure 2.6). The predominant collagen fiber orientation is circumferential. This is to withstand the hoop
Figure 2.6: Schema of the meniscus shows collagen fiber orientation. [27]
Figure 2.7: Longitudinal section of a medial meniscus viewed under polarized light demonstrates the orientation of the coarse, deep, circumferentially oriented collagen fibers. [11]

stresses induced in the meniscus under axial loading of the knee. The circumferential fibers are visualized using polarized light microscopy (see Figure 2.7). Although the principal meniscus fiber orientation is circumferential, a few radially oriented fibers arise from the femoral and tibial surfaces to act as "ties" to provide structural rigidity and resist longitudinal splitting of the circumferential fibers under load.

The core region of the meniscus is enveloped by a 100 μm thick surface layer [11]. The collagen fibers in the surface layer are randomly oriented in the planes parallel to the articulating surfaces. This layer can be seen in Figure 2.6.

2.3 Biochemistry

As with most biological materials, the major constituent of the meniscus is water (73.6% ± 2.1% [112]). In addition to the water, the extracellular matrix consists of collagen, proteoglycans, elastin, and noncollagenous proteins. Collagen is the largest
Table 2.1: Biochemical comparison of connective tissues. Collagen, proteoglycan, and elastin components are presented as a percentage of dry tissue weight.

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>70–78</td>
<td>65–80</td>
<td>78</td>
<td>65–70</td>
<td>65–70</td>
</tr>
<tr>
<td>Collagen</td>
<td>60–90</td>
<td>65–75</td>
<td>66</td>
<td>75–80</td>
<td>75–80</td>
</tr>
<tr>
<td>Type I</td>
<td>1.3–3.4</td>
<td>20</td>
<td>15–25</td>
<td>50</td>
<td>1–1.5</td>
</tr>
<tr>
<td>Type II</td>
<td></td>
<td>25</td>
<td>Type I &amp; II</td>
<td></td>
<td>Type 1</td>
</tr>
<tr>
<td>Type II</td>
<td></td>
<td></td>
<td>Type II</td>
<td></td>
<td>Type 1</td>
</tr>
<tr>
<td>Type I</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type I</td>
<td></td>
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<td>Type I</td>
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<tr>
<td>Type I</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Type I</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elastin</td>
<td>&lt;1</td>
<td></td>
<td></td>
<td>&lt;3</td>
<td>&lt;5</td>
</tr>
</tbody>
</table>

component averaging 78% of the tissue dry weight. Meniscal collagen content is slightly greater than that of articular cartilage. The meniscus contains at least four types of collagen [11]: type I is the most abundant, comprising approximately 90% of the total collagen; types II and V are both 1% to 2%; and type VI accounts for 1%. Type III collagen was tentatively identified by electrophoresis, but the molecule has not been identified with certainty [11].

Considerable variation in the glycosaminoglycan content exists in the human meniscus depending on site and age. Adams and Ho [2] found chondroitin sulfate, dermatan sulfate, keratan sulfate, and hyaluronate present in the human meniscus, with chondroitan sulfate being the primary constituent. The overall amount of glycosaminoglycan is approximately 10% of that present in articular cartilage (2.4% of dry weight for chondroitin sulfate [112]). Peters and Smillie [112] report the mean elastin content is 0.6% of dry weight.

Also of interest is the biochemical similarity of various connective tissues. Assuming that form follows function, one can speculate that similar biochemical compositions have similar functions. Table 2.1 compares the meniscus to articular cartilage, annulus and nucleus of the intervertebral disk, tendon, and ligament. Although both articular cartilage and menisci distribute stress in the knee, the proteoglycan content of articular cartilage is an order of magnitude greater than that of menisci. This is indicative of a fundamental difference in the way the two tissues transmit load.
The relatively high proteoglycan content of articular cartilage uses electrostatic repulsive forces to balance compressive loads [46] while the menisci use circumferentially oriented collagen fibers to balance hoop tensile forces. The biochemical similarities between menisci, tendon, and ligament further support the theory that the meniscus is subjected to large circumferential tensile forces.

2.4 Meniscal Lesions, Treatment, and Regeneration

The mechanical causes of meniscal injuries are poorly understood. Current hypotheses suggest that external torques are applied to the knee during weight-bearing and result in abnormally high shear and compressive stresses in the meniscus [119, 134]. Tears of the medial meniscus are five to seven times more common than those of the lateral meniscus [134]. This is attributed to the decreased mobility of the medial meniscus due to its medial collateral ligament attachment. The most common type of lesion is a longitudinal tear of the posterior segment of the medial meniscus. Figure 2.8 provides classifications for typical meniscal tears.

Following meniscal injury, partial or total meniscectomy is the standard surgical treatment. There is considerable controversy regarding which procedure is most effective. Central to this debate is the issue of meniscal regeneration. It is generally accepted that in order for the meniscus to regenerate, the excision must extend to the peripheral vascularized region of the meniscus [134]. Therefore, total meniscectomy permits regeneration of the meniscus [119]. The controversy arises because many investigators believe that the quality of the regenerated meniscus is inferior to the original meniscus [134]. They prefer conservative partial meniscectomy procedures that minimize removal of the original tissue. The results of the long-term follow-up clinical studies (see Section 1.2) linking meniscectomy to degenerative changes in the knee seem to provide empirical evidence that the regenerated menisci do not function as the original menisci. Consequently, meniscal-sparing procedures are currently the
Figure 2.8: Typical meniscus tears: (a) and (b) partial and subtotal longitudinal tears; (c) and (d) total longitudinal tears (bucket handle tears with displacement of the torn meniscus fragment into the interior of the joint); (e) and (f) tear of the anterior or posterior horn; (g) tongue-shaped tear with peripheral displacement; (h) transverse tear. [119]
procedures of choice [134].
Chapter 3

Meniscal Material Properties

The meniscus is a collagen reinforced composite material. Its microstructure (see Section 2.2) makes the tissue inherently anisotropic. This chapter reviews the meniscal material properties necessary to model the meniscus as an anisotropic material. The review concentrates on the moduli: (1) tensile elastic modulus, (2) compressive elastic modulus, and (3) shear modulus. A review of the constitutive equations for isotropic and anisotropic elastic materials is presented in Appendix A. The theoretical relationships between the moduli for various conditions of material symmetry and the number of independent elastic parameters associated with each condition of material symmetry are included in the appendix.

Bimodularity is an important property of composite materials and is fundamental to an understanding of relevant material properties. Bimodular materials have different elastic moduli in tension and compression (see Figure 3.1). This behavior is a result of different material components transmitting tensile and compressive loads (e.g. in steel reinforced concrete, the concrete carries the compressive load and the steel carries the tensile load). In the meniscus, the collagen fibers transmit the tensile stresses and the entire extracellular matrix transmits the compressive stresses. Therefore, selection of appropriate elastic moduli for a composite material depends on the presumed stress state. The presumed meniscal stress state is compressive in the axial and radial directions and tensile in the circumferential direction.
Figure 3.1: Stress-strain constitutive relationship of a bimodular material ($E_{\text{tension}} > E_{\text{compression}}$).
3.1 Tensile Elastic Modulus

The tensile elastic modulus of the meniscus was first measured by Kleckner [82, 150]. Four fresh normal adult specimens were tested. Circumferentially oriented specimens were cut into dumbbell-shaped samples and rigidly clamped at each end. A least-squares quadratic regression of the experimental results yielded the following relationship:

\[ E = 62(1 + 11\varepsilon + 108\varepsilon^2) \text{ MPa} \]  \hspace{1cm} (3.1)

The Young's modulus presented in Table 3.1 is the modulus evaluated at 5% strain which is the maximum strain applied in the experimental protocol.

Uezaki et al [145] examined the viscoelastic nature of porcine menisci. The specimens were held with a Vibron-II type chuck with rasp-like tongues and immersed in a silicon oil bath for testing. The Young's modulus was measured for a range of strain rates. The results are presented in Table 3.1. The measured magnitudes of Young's modulus are roughly an order of magnitude lower than results reported by other investigators. The discrepancy in results may be due to problems associated with initial gauge length measurement, specimen chuck slippage, or the silicon oil bath (the authors did report control tests to validate the use of the silicon oil bath). It is unlikely that the differences can be attributed to species variablity.

The most extensive measurement of meniscal elastic moduli to date is from Proctor et al in 1989 [115]. Some of these results were reported earlier by Whipple et al [155, 154]. Specimens were taken from skeletally mature bovine medial menisci. The specimens were cut parallel to the femoral surface (Figure 3.2a) and oriented in radial and circumferential directions (Figure 3.2b). Squares of waterproof polishing paper were affixed to the specimen ends with cyanoacrylate and clamped with spring-loaded jaws. The results are presented in Table 3.1. The tissue is relatively isotropic on the surface, but becomes much stiffer in the circumferential direction than the radial direction in the core of the meniscus (see Figure 3.3). This correlates with the microstructure of the meniscus which has a random collagen distribution on
Table 3.1: Meniscal elastic moduli in tension.

<table>
<thead>
<tr>
<th>Investigator</th>
<th>Species</th>
<th>Specimen Orientation</th>
<th>Young's Modulus (MPa)</th>
<th>Strain Rate (%/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uesaki et al (1979) [145]</td>
<td>Porcine</td>
<td>Circumferential</td>
<td>17.2</td>
<td>0.13</td>
</tr>
<tr>
<td>Porcine</td>
<td>Circumferential</td>
<td>17.8</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>Porcine</td>
<td>Circumferential</td>
<td>17.5</td>
<td>1.7</td>
<td></td>
</tr>
<tr>
<td>Porcine</td>
<td>Circumferential</td>
<td>19.4</td>
<td>16.0</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. circum. middle</td>
<td>198.4 ± 87.5</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. circum. deep</td>
<td>139.0 ± 79.2</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. radial surface</td>
<td>71.4 ± 41.6</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. radial middle</td>
<td>2.8 ± 1.2</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. radial deep</td>
<td>4.6 ± 2.1</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. circum. middle post.</td>
<td>259.0 ± 69.2</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. circum. deep post.</td>
<td>194.9 ± 59.0</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. circum. middle ant.</td>
<td>117.9 ± 23.1</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Med. circum. deep ant.</td>
<td>61.5 ± 25.6</td>
<td>0.167</td>
<td></td>
</tr>
<tr>
<td>Schmidt et al (1986) [123] and Arnoczky et al (1988) [12]</td>
<td>Canine</td>
<td>Medial circumferential (50%-60% of radius)</td>
<td>158.0 ± 53.3</td>
<td>0.556</td>
</tr>
<tr>
<td>Canine</td>
<td>Medial circumferential (70%-80% of radius)</td>
<td>99.5 ± 38.1</td>
<td>0.556</td>
<td></td>
</tr>
<tr>
<td>Krause et al (1989) [83]</td>
<td>Canine</td>
<td>Medial circumferential</td>
<td>362. ± 238.</td>
<td>2.0</td>
</tr>
<tr>
<td>Canine</td>
<td>Medial circumferential</td>
<td>362. ± 176.</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Canine</td>
<td>Medial circumferential</td>
<td>337. ± 135.</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Canine</td>
<td>Medial circumferential</td>
<td>246. ± 101.</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Pithian et al (1989) [50]</td>
<td>Human</td>
<td>Med. circum. anterior</td>
<td>159.6 ± 26.2</td>
<td>0.005</td>
</tr>
<tr>
<td>Human</td>
<td>Med. circum. central</td>
<td>93.2 ± 52.4</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Human</td>
<td>Med. circum. posterior</td>
<td>110.2 ± 40.7</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Human</td>
<td>Lat. circum. anterior</td>
<td>159.1 ± 47.4</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Human</td>
<td>Lat. circum. central</td>
<td>228.8 ± 51.4</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Human</td>
<td>Lat. circum. posterior</td>
<td>294.1 ± 90.4</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Skaggs and Mow (1990) [135]</td>
<td>Bovine</td>
<td>Radial anterior</td>
<td>10.8</td>
<td>1.0</td>
</tr>
<tr>
<td>Bovine</td>
<td>Radial central</td>
<td>32.7</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Radial posterior</td>
<td>42.3</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Radial, complete bundle</td>
<td>74.2</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Radial, partial bundle</td>
<td>30.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>Radial, no bundle</td>
<td>12.0</td>
<td>1.0</td>
<td></td>
</tr>
</tbody>
</table>
Figure 3.2: (a) After removing the apex, serial slices were obtained from the femoral surface. (b) Dumbbell-shaped specimens were obtained in either the circumferential or radial direction. [115]

the surface and a circumferential orientation deeper in the meniscus as described by Bullough [27]. Also measured was the variability of the tensile moduli from specimens taken from the anterior and posterior segments of the medial menisci.

Arnoczky et al (1988) conducted a study on the effects of cryopreservation on the mechanical properties of the meniscus [12]. Some of the results had been reported previously by Schmidt et al (1986) [123]. Circumferentially oriented specimens were cut at varying radial distances (see Figure 3.4). The specimens were mounted using a protocol similar to Proctor [115]. The results presented in Table 3.1 are for the control specimens and show the tensile modulus decreases as the radius increases.

Krause et al (1989) examined the modulus of repaired canine menisci [83]. The results presented in Table 3.1 are the results of the control experiments.

Fithian et al (1989) measured the tensile modulus of normal human knees from subjects aged 24-40 years [50]. The specimens were prepared using the same protocol as Whipple [154]. Additionally, the authors report that a tare load 0.01 N and 5 preconditioning cycles to 0.05 N were applied. The results are presented in Table 3.1. The authors hypothesize that the lower tensile moduli of the posterior two-thirds of the medial meniscus may be due to the disruption of the circumferential
Figure 3.3: Typical stress-strain curves demonstrating the isotropy of the surface and the anisotropy of the deep tissue. The variation between the circumferential and radial direction for all surface specimens tested is not statistically significant. [155, 115]
Figure 3.4: (a) Schematic diagram of specimen location relative to meniscal geometry. The longitudinal axis of the dumbbell-shaped specimen is parallel to the predominantly circumferential collagen fiber orientation. (b) The cross-sectional view of the meniscus. The relative radial location of each test specimen was expressed as a percentage of the total radial width of the meniscus, measured from the internal free edge to the peripheral rim at the anterior-posterior midline. The average total width for all canine menisci tested was 5.89 ± 0.84 mm. [12]
collagen fiber orientation by the capsular attachments.

Skaggs and Mow (1990) studied the function of radial tie fibers in the bovine meniscus [135]. The protocol was similar to that used by Fithian with 10 preconditioning cycles. The specimens were divided into three groups according to the occurrence of large radial collagen bundles which were visible to the naked eye: I. radial bundles through the entire specimen; II. radial bundles through part of the specimen; III. no radial bundles. The mean values for the three groups and for the anterior, central, and posterior regions are presented in Table 3.1.

For completeness, the work by Ghosh and Taylor (1987) is included [58]. They examined the tensile modulus of fresh and preserved meniscal samples at very low strains and found an empirical relationship

\[ E = \frac{d\sigma}{d\varepsilon} = B\sigma + C \]  

(3.2)

Integrating this equation results in the stress-strain constitutive relationship

\[ \sigma = \frac{C}{B}(e^{Be} - 1) \]  

(3.3)

Substituting the constitutive relationship into the expression for the elastic modulus yields

\[ E = Ce^{Be} \]  

(3.4)

Using a Taylor series expansion,

\[ E = C \left(1 + B\varepsilon + \frac{B^2\varepsilon^2}{2} + \cdots\right) \]  

(3.5)

which is analogous to Equation 3.1. Unfortunately, the constants \( B \) and \( C \) cannot be evaluated from the reported results, but the graphical results are included (see Figure 3.5).

In summary, the reported meniscal tensile moduli have a range spanning two orders of magnitude. The variation in the tensile modulus is dependent on the orientation (radial or circumferential), region (anterior, central, or posterior), and location (depth from surface or radius) of the specimen. These variations are demonstrated in Figure 3.6.
Figure 3.5: Comparison of Young's modulus for fresh and preserved menisci. Exposure of sections to neutral formaldehyde for 16 hours (B-D) increased the stiffness relative to the nonfixed tissues (A). [58]
Figure 3.6: (a) Variations of the tensile modulus with respect to depth for radial and circumferentially oriented specimens [115]. (b) Variations of the tensile moduli of circumferential specimens from the anterior and posterior segments of the medial meniscus [115]. (c) Relationship between the tensile modulus and the relative radial location of fresh and cryopreserved meniscal test specimens. Modulus decreased as the location of the specimen approached the peripheral border of the meniscus [12].
3.2 Compressive Elastic Modulus

The work on the meniscal compressive elastic moduli is not as extensive as the work on the meniscal tensile elastic moduli. The only reported measurements are for the aggregate modulus in confined compression for a creep experiment. This work was first reported by Favenesi et al in 1983 [48]. A more detailed reporting of the results are presented by Proctor et al in 1989 [115] and reproduced in Table 3.2 for reference. Although the aggregate modulus and permeability are important parameters for modeling the meniscus as a poroelastic solid, they are measured over relatively long time scales (20,000 s). The aggregate modulus is measured after fluid has exuded from the tissue and the tissue is in equilibrium. Human gait frequencies are in the range 1–10 Hz (0.1–1.0 s). Extracellular fluid is retained by the tissue because there is not sufficient time for flow to develop. Therefore, the dynamic stiffness is significantly greater than the aggregate modulus (e.g. for cartilage, Sah et al [121] report the dynamic stiffness of cartilage is $\sim 3 \text{MPa}$ at 0.0001 Hz and reaches a plateau of $\sim 37 \text{MPa}$ at 0.1–1.0 Hz). It is inappropriate to use the aggregate modulus as the dynamic elastic compression modulus for either meniscus or cartilage when considering phenomena that occur at physiologic gait frequencies.

One option for estimating the dynamic compressive stiffness of the meniscus is to use poroelastic models developed for cartilage. Kim developed a poroelastic model for unconfined compression with axisymmetric boundary conditions [80]. The meniscal values for the aggregate modulus and permeability in addition to Poisson’s ratio of the solid phase are used in the model. Kim found that the maximum change in modulus with respect to frequency occurred when Poisson’s ratio was zero or $\mu^* = \frac{1}{2}$. This is not an unreasonable assumption to adopt given that the solid phase primarily consists of a network of collagen fibers that should exhibit very little lateral strain under compression of just the solid phase. Unfortunately, the poroelastic model has not been experimentally verified so it would be premature to use the model.

There being no substitute for actual measurement of Young’s modulus for meniscal
Table 3.2: Comparison of water content, permeability, and aggregate modulus in confined compression of bovine medial meniscus and bovine and human articular cartilage. (Values are ±σ) [48, 115]

<table>
<thead>
<tr>
<th>Articular cartilage</th>
<th>Water content (%$\pm$σ)</th>
<th>Permeability ($10^{-15}$ m$^4$/N·s$\pm$σ)</th>
<th>Aggregate modulus (MPa$\pm$σ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meniscus</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Superficial</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Posterior</td>
<td>76.8 ± 3.0</td>
<td>0.76 ± 0.47</td>
<td>0.393 ± 0.109</td>
</tr>
<tr>
<td>Central-posterior</td>
<td>73.8 ± 3.7</td>
<td>0.83 ± 0.39</td>
<td>0.421 ± 0.074</td>
</tr>
<tr>
<td>Central-anterior</td>
<td>73.2 ± 1.8</td>
<td>0.78 ± 0.38</td>
<td>0.377 ± 0.076</td>
</tr>
<tr>
<td>Anterior</td>
<td>73.4 ± 2.0</td>
<td>0.63 ± 0.47</td>
<td>0.440 ± 0.108</td>
</tr>
<tr>
<td>Deep</td>
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</tr>
<tr>
<td>Posterior</td>
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<td>0.91 ± 0.52</td>
<td>0.491 ± 0.042</td>
</tr>
<tr>
<td>Central-posterior</td>
<td>72.3 ± 3.3</td>
<td>0.86 ± 0.51</td>
<td>0.408 ± 0.047</td>
</tr>
<tr>
<td>Central-anterior</td>
<td>72.8 ± 1.1</td>
<td>1.03 ± 0.58</td>
<td>0.384 ± 0.092</td>
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<tr>
<td>Anterior</td>
<td>73.0 ± 2.3</td>
<td>0.74 ± 0.14</td>
<td>0.375 ± 0.042</td>
</tr>
<tr>
<td>Mean</td>
<td>73.8 ± 3.1</td>
<td>0.81 ± 0.45</td>
<td>0.410 ± 0.088</td>
</tr>
<tr>
<td>Articular cartilage</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bovine</td>
<td>78.6 ± 3.9</td>
<td>4.70 ± 3.6</td>
<td>0.790 ± 0.360</td>
</tr>
<tr>
<td>Human</td>
<td></td>
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<td></td>
</tr>
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</table>
fibrocartilage in compression, the the stress-strain curve and dynamic stiffness of the meniscus in unconfined compression were measured. Fresh human cadaver menisci were harvested and frozen until testing. The meniscal specimens were cut into 1 mm slices parallel to the tibial surface using a microtome with a freezing stage. Cylindrical plugs of 3 mm diameter were cut from the 1 mm slice using a dermal punch. An unconfined compression chamber was constructed of plexiglas. The chamber was capable of testing a single plug and provided a buffered isotonic saline bath for the specimen. The compression chamber was mounted in an mechanical spectrometer (Dynastat, IMASS, Hingham, MA). Load-displacement curves were measured over a range of strain rates (0.1–10 %/s) and the dynamic modulus was measured over a frequency range of 0.001–10 Hz.

The stress-strain curves and the modulus-strain curves for one specimen are given (see Figure 3.7). The stress-strain curves intersect at the two higher strain rates suggesting that the compressive stiffness of the meniscus has reached a plateau and the poroelastic material is acting as a purely elastic material. The moduli-strain curves for the two higher strain rates intersect at $\sim 15 \, MPa$ ($\sim 10 \, \%$ strain).

### 3.3 Shear Modulus

Chern et al (1989,1990) measured the equilibrium and dynamic shear moduli for the meniscus [34, 35]. Bovine specimens were prepared as disks. The axial and radial specimens have the circumferentially oriented collagen fibers running parallel to the specimen face and the circumferential specimens have the collagen fibers oriented perpendicular to the its face (see Figure 3.8). The equilibrium shear moduli as determined from the equilibrium stress-strain relationship are $29.8 \pm 9.7 \, kPa$, $21.4 \pm 6.5 \, kPa$ and $36.8 \pm 11.6 \, kPa$ for the axial, radial, and circumferential specimens, respectively. The dynamic shear moduli were measured through the frequency range of 1-100 rad/s with superimposed compressive strains of 7%, 10%, and 13%. The dynamic shear moduli measured are in the range of 25–125 kPa. The results are
Figure 3.7: (a) Stress-strain curves for meniscal plugs in unconfined compression. (b) Modulus-strain curves for meniscal plugs in unconfined compression at various strain rates. The strain rates are: 0.1 %/s (solid line), 1.0 %/s (dashed line), 10.0 %/s (dotted line).
Figure 3.8: Orientation of meniscal shear test specimens used by Chern et al [35].

presented in Figures 3.9 and 3.10a.

Anderson et al also measured the dynamic shear modulus of the meniscus [9]. Equine menisci were cut into $3.4 \times 3.4 \times 0.28$ mm$^3$ specimens. The specimens were tested with and across the predominant fiber orientation over a frequency range of 100–800 Hz. The measured dynamic shear moduli are in the range of 100–400 kPa, slightly higher than the values reported by Chern (see Figure 3.10b).

### 3.4 Tensile Strength

The first measurements of meniscal tensile strength were conducted by Mathur et al at the Mayo Clinic in 1949 [97]. Fresh menisci removed from amputated limbs were used for the study. The menisci were held between two metal clamps and suspended from a frame. Incremental weights were added to the lower clamp until fracture occurred. The initial length of the meniscus and the stretching induced by the incremental weights were measured. The results for sixty-five menisci are presented in Table 3.3. Only gross macroscopic rupture loads are reported. Unfortunately, neither the percent elongation at fracture nor the force-displacement data were reported. Due to the lack of information, intrinsic material properties such as an average failure strain or stress cannot be estimated.

Bullough et al conducted a comprehensive investigation into the strength of the meniscus as it relates to tissue microstructure [27]. Human specimens were obtained at autopsy or from amputated limbs. Thin sections 9–20 μm in thickness were cut on
Figure 3.9: Experimental measurements and theoretical predictions of the dynamic shear modulus of an axial specimen [34].

Figure 3.10: Dynamic shear modulus of meniscus as related to specimen orientation. (a) Results from Chern et al with superimposed compressive strain [35]. (b) Results from Anderson et al [9].
Table 3.3: Failure loads of menisci from Mathur [97].

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Age and Sex</th>
<th>Joint</th>
<th>Volume of meniscus (cm$^3$)</th>
<th>Initial length between clamps (mm)</th>
<th>Weight needed to produce fracture (lbs)</th>
<th>Type of fracture</th>
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<tbody>
<tr>
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<td></td>
<td></td>
<td>Med</td>
<td>Lat</td>
<td>Med</td>
<td>Lat</td>
</tr>
<tr>
<td>1</td>
<td>32 M</td>
<td>Left</td>
<td></td>
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</tr>
<tr>
<td>2</td>
<td>58 M</td>
<td>Left</td>
<td></td>
<td></td>
<td>53</td>
<td>60</td>
</tr>
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<td></td>
<td></td>
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<td></td>
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<tr>
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<td>38 M</td>
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<td></td>
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<td>45</td>
</tr>
<tr>
<td>4</td>
<td>54 M</td>
<td>Right</td>
<td></td>
<td></td>
<td>46</td>
<td>37</td>
</tr>
<tr>
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<td>68 F</td>
<td>Left</td>
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<td>3.4</td>
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<td></td>
</tr>
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<td>Left</td>
<td>3.3</td>
<td>2.5</td>
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<tr>
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<td>61 M</td>
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<td>4.6</td>
<td>4.7</td>
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<tr>
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<td>2.8</td>
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<td>2.9</td>
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<td>44</td>
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<td>61 F</td>
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<td>4.8</td>
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<td>19 F</td>
<td>Right</td>
<td>1.7</td>
<td>1.9</td>
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<td>73 F</td>
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<tr>
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<td>17 F</td>
<td>Right</td>
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<td>2.1</td>
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<td>10 F</td>
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<td>47 M</td>
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<td>4.7</td>
<td>4.5</td>
<td>52</td>
<td>50</td>
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a cryostat. All reported results were performed on formalin-fixed tissue (preliminary control tests revealed no difference in strength between fresh and fixed tissue). Collagen fiber orientation of each specimen was determined by polarized light microscopy. The results demonstrate that the strength of the specimen is dependent on the orientation of the specimen (as would be expected for a composite material). The average strength of the specimens oriented perpendicularly and parallel to the collagen fibers is 0.5–1.0 $MPa$ and 6.3–12.5 $MPa$, respectively for the lateral meniscus. For the medial meniscus, the corresponding strengths are 0.5–2.5 $MPa$ and 3.4–8.3 $MPa$ (see Tables 3.4 and 3.5).

Arnoczky et al measured the tensile strength of circumferentially oriented specimens from canine menisci [123, 12]. The experimental protocol was reported in Section 3.1. The results demonstrate that the tissue strength is dependent on the radial location of the specimen. The mean failure stress for the control specimens harvested at 50–60% of the radial width was $31.3 \pm 8.5 \, MPa$ and for the specimens harvested from the 70–80% region was $11.9 \pm 9.3 \, MPa$. These values are higher than the values reported by Bullough.

Krause et al also measured the tensile strength in conjunction with the tensile modulus [83]. The meniscal tensile strength results for four control groups are $14.8 \pm 6.7 \, MPa$, $15.6 \pm 9.0 \, MPa$, $14.5 \pm 8.0 \, MPa$, and $10.8 \pm 4.6 \, MPa$. 
Table 3.4: Lateral meniscus tensile strength properties from Bullough [27].

<table>
<thead>
<tr>
<th>Spec. no.</th>
<th>Age</th>
<th>Sex</th>
<th>Adjacent bone surface</th>
<th>Location of specimen in meniscus</th>
<th>Principal orientation of the fibers in specimen in relation to the tensile axis</th>
<th>Tensile strength Maximum (kg/cm²)</th>
<th>Tensile strength Minimum (kg/cm²)</th>
<th>Tensile strength Mean (kg/cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>Male</td>
<td>Femoral</td>
<td>Central</td>
<td>Perpendicular with many cross fibers</td>
<td>9.70</td>
<td>6.12</td>
<td>7.73</td>
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<tr>
<td>2</td>
<td>14</td>
<td>Male</td>
<td>Femoral</td>
<td>Central</td>
<td>Oblique (approx. 30 degrees) with some cross fibers</td>
<td>20.4</td>
<td>12.02</td>
<td>14.62</td>
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<td>3</td>
<td>29</td>
<td>Male</td>
<td>Tibial</td>
<td>Anterior</td>
<td>Perpendicular with many cross fibers</td>
<td>11.53</td>
<td>8.79</td>
<td>9.98</td>
</tr>
<tr>
<td>4</td>
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<td>Tibial</td>
<td>Anterior</td>
<td>Mainly parallel with many cross fibers</td>
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<td>Male</td>
<td>Femoral</td>
<td>Central</td>
<td>Perpendicular</td>
<td>10.55</td>
<td>9.70</td>
<td>10.12</td>
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<td>Parallel</td>
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<td>Perpendicular</td>
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<td>Perpendicular</td>
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Table 3.5: Medial meniscus tensile strength properties from Bullough [27].

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<th>Sex</th>
<th>Adjacent bone surface</th>
<th>Location of specimen in meniscus</th>
<th>Principal orientation of the fibers in specimen in relation to the tensile axis</th>
<th>Tensile strength</th>
</tr>
</thead>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Maximum (kg/cm²)</td>
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<td>Posterior</td>
<td>Perpendicular</td>
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<tr>
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<td>Posterior</td>
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<td>Posterior</td>
<td>Oblique with cross fibers</td>
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<td>Tibial</td>
<td>Posterior</td>
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<tr>
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<td>Femoral</td>
<td>Central</td>
<td>Perpendicular</td>
<td>13.79</td>
</tr>
<tr>
<td>30</td>
<td>53</td>
<td>Male</td>
<td>Femoral</td>
<td>Central</td>
<td>Parallel with some oblique</td>
<td>47.45</td>
</tr>
<tr>
<td>31</td>
<td>60</td>
<td>Female</td>
<td>Femoral</td>
<td>Central</td>
<td>Perpendicular</td>
<td>13.43</td>
</tr>
<tr>
<td>32</td>
<td>60</td>
<td>Female</td>
<td>Femoral</td>
<td>Central</td>
<td>Oblique with some cross fibers</td>
<td>31.28</td>
</tr>
</tbody>
</table>
Chapter 4

Lumped Parameter Modeling of the Meniscus

Fairbank was the first investigator to experimentally verify the load-bearing role of the knee meniscus in 1948. His description of the meniscal function and simple experimental confirmation is concise and elegant.

As compression [of the knee joint] increases, ... the circumference of the meniscus must be forced centrifugally. But in so far as the two ends of the meniscus are firmly attached to bone this force is resisted by rising tension in the stretched and elastic fibrocartilage. The greater the degree of joint compression, the greater the circumferential tension in the meniscus. It is submitted that this tension resists extrusive forces and enables the meniscus to share in weight-bearing. That the tension is real was confirmed by stripping a knee joint of the outer soft tissues, while leaving the cruciate ligaments intact and the menisci in position but attached only by their central ends. In full extension the menisci remained slightly mobile until compression was applied, when the periphery of the menisci at once became hard and tense. [47]

Although this description of the meniscus may be difficult to improve, a mathematical model describing the biomechanics of the meniscus can provide engineering insight into the contributions of the meniscal geometry and material properties.
Figure 4.1: Superimposed tracings of sagittal sections of the tibial and femoral condyles to show the effect of joint compression in producing centrifugal displacement of the meniscus (see quotation for explanation). [47]

The earliest mechanical models of the meniscus are lumped-parameters models. The term *lumped-parameter* means that the elastic properties and geometry of the system are “lumped” into discrete mechanical elements such as idealized massless springs. The distributed mass of the system is modeled as discrete masses. The dissipative properties of the system are modeled as dashpots. Then the state of the system can be described by the force and displacement or velocity of the lumped elements.\(^1\) This is in contrast to *continuum* models that describe the state of each point throughout a distributed system (e.g. stress and pressure distributions). This brief contrast should help the reader understand the complexity involved in dealing with continuum models and the appeal of considering simplified lumped-parameter models.

\(^{1}\text{The author is fully aware that the power state variables for capacitances, inertances, and resistances are force and velocity. However, the analyses in this section are quasistatic in nature and as such concern only the elastic elements for which the energy state variables will be used.}\)
Figure 4.2: (a) A simple two-spring model of the knee. (b) Load-displacement curve demonstrating fraction of load transmitted by meniscus.

4.1 Two-Spring Knee Models

The simplest lumped-parameter knee model consists of two springs. This model was presented by Seedhom [125, 126] (see Figure 4.2). In this model, $k_m$ is the meniscal stiffness ascribed to the circumferential stiffness of the tissue. $k_c$ is the stiffness of the direct cartilage-to-cartilage contact. The load is distributed between the two springs by the ratio of the springs stiffnesses.

$$\frac{F_m}{F_c} = \frac{k_m x}{k_c x} = \frac{k_m}{k_c} \tag{4.1}$$

The fraction of the load carried by the meniscus is

$$\frac{F_m}{F_{total}} = \frac{k_m x}{k_m x + k_c x} = \frac{k_m}{k_m + k_c} \tag{4.2}$$
This relationship assumes the springs are linear. If the system is nonlinear, an alternative expression is necessary.

\[
\frac{F_m}{F_{total}} = \frac{F_{total} - F_c}{F_{total}}
\]  \hspace{1cm} (4.3)

This second relationship can be applied to experimental load-displacement curves to calculate the percentage of the load carried by the meniscus [126]. The experimental protocol involves measuring the load-displacement curve for the intact joint. The menisci are removed and the load displacement curve is remeasured. For a given displacement, the curves provide \( F_{total} \) and \( F_c \) so the percentage of load transmitted by the meniscus is calculated (see Figure 4.2).

Shrive used such an analysis to calculate the percentage of the transmitted by the meniscus for in vitro load-displacement curves of the knee [133]. Shrive presented an axisymmetric model in the article (see Figure 4.3), but no analysis of the model was discussed.

Newman [106] modified Seedhom's two-spring model to incorporate a so-called meniscal spacer effect. The meniscal spacer effect results from the meniscus being thick enough to prevent direct cartilage-to-cartilage contact and a finite load is necessary to deform the meniscus and initiate direct contact. The spacer effect has been reported by a number of investigators [133, 148, 129], but not all investigators have reported the phenomenon (Shrive [133] reports a gap in 9 of 16 human specimens). The nonlinear spacer effect is incorporated into the two-spring model by using springs of unequal length (Figure 4.4a). This model produces the piecewise linear load-displacement shown in Figure 4.4b.

4.2 Four-Spring Knee Models

Seedhom [126] also proposed a more complex four-spring model of the knee joint. The model was based on several assumptions:

1. The model contains only elastic elements.
Figure 4.3: Axisymmetric model of schematic knee joint. The relative displacement of the indenter is depicted for (a) circumferentially stiff annulus, and (b) circumferentially flexible annulus. [133]
Figure 4.4: (a) Modification of Seedhom's two-spring model incorporating springs of unequal length and stiffness, which takes into account the meniscal spacer effect. (b) Resultant curve for the intact joint, obtained by summing the forces produced by each load-bearing element at each displacement. [106]
2. The subchondral bone is assumed to be rigid.

3. The springs are nonlinear.

4. The springs can have unequal length.

5. The forces acting on each spring are assumed to pass through the centroid of the respective areas on which the forces are transmitted.

The model is presented in Figure 4.5, but the reader is referred to the original article for the corresponding analysis. Newman [106] also proposed a four-spring model (the slope labels for Figure 4.4b include four-spring constants), but no analysis of the model was included. Seedhom's four-spring knee model was employed for analysis of experimental data.

4.3 Rheological Knee Models

Rheological models add dissipative elements to the linear models. The dashpot elements are added to model to account for the creep behavior of the joint. Seedhom [128] presented the first rheological model of the meniscus shown in Figure 4.6 by substituting Maxwell fluid elements (a spring and dashpot in series) for the linear springs in the four-spring model. The use of a Maxwell fluid element does not incorporate the necessary creep response observed experimentally. The more appropriate element would be a Kelvin solid (a spring in series with parallel combination of a spring and dashpot). model does not accurately reproduce creep behavior because a Kelvin solid is needed to incorporate a necessary decay response.

Jaspers et al [72, 41] developed a rheological model of the knee, but only included the Kelvin solid elements in the cartilage and not the meniscus (see Figure 4.7). The model assumes that the state of loading is uniaxial and the midsagittal plane is one of symmetry. The mechanical effect of the meniscus is separated into two effects: a

\footnote{The reader is referred to Y.C. Fung, *Biomechanics*, Springer-Verlag, New York, 1981, for a brief explanation of linear modeling of viscoelastic behavior.}
Figure 4.5: (a) The four-spring model of the knee with menisci intact. Top represents the four load-bearing areas. Below the distribution of load $F$ between the four springs. (b) The model representing the knee without menisci. Top, the two load-bearing areas where direct contact takes place between femur and tibia. Below, the load $F$ distributed on these two springs. [126]
nonlinear time-independent elastic effect from the meniscal circumferential stretching and a nonlinear viscoelastic effect related to the load-carrying area and the articular cartilage material properties. The modeling develops constitutive relationships for each element. Experimental measurement and literature data were used for estimating some of the model parameters. The remaining parameters were estimated by fitting the model to the experimental data. Unfortunately, the physical significance of the fitted parameters is obscured with this modeling approach.

4.4 Walker's Meniscus Model

Although not exactly a lumped-parameter model, Walker [150] presents a derivation of a load-displacement relationship for the meniscus. The meniscus is modeled as a semicircular annulus with a wedge-shaped cross-section (see Figure 4.8). Assuming the meniscal surfaces are frictionless, the resultant force must be perpendicular to the meniscal surface. Assuming the axial load is distributed uniformly, the resultant
Figure 4.7: Rheological model of the knee with elastic menisci and viscoelastic articular cartilage. [72, 41]

Figure 4.8: Semicircular model of meniscus used by Walker. [150]
force on a differential element can be separated into its component forces.

\[
\delta F_x = F \frac{\delta s}{\pi r} \quad (4.4)
\]

\[
\delta F_n = F \frac{\delta s}{\cos \alpha \pi r} \quad (4.5)
\]

\[
\delta F_r = F \tan \alpha \frac{\delta s}{\pi r} \quad (4.6)
\]

where \( r \) is the mean radius and the subscripts \( z, n, \) and \( r \) denote the axial, normal, and radial directions, respectively. Assuming that the meniscus is unattached so the circumferential tension \( T \) is equal throughout the body of the meniscus, a radial force balance results in

\[
2T \sin \frac{1}{2} \delta \theta = F \tan \alpha \frac{\delta s}{\pi r} \quad (4.7)
\]

Using the small angle approximation for the differential angle \( \delta \theta \), \( \sin \frac{1}{2} \delta \theta \approx \frac{1}{2} \delta \theta \) and substituting \( r \delta \theta = \delta s \) reduces the equation to

\[
T = \frac{F \tan \alpha}{\pi} \quad (4.8)
\]

The circumferential stretching of the meniscus is equal to the circumferential strain of the meniscus multiplied by the circumferential length.

\[
\Delta L = \varepsilon_{\theta \theta} L \quad (4.9)
\]

Substituting Hooke's law \( \sigma = E \varepsilon \) into the equation yields

\[
\Delta L = \frac{\sigma}{E} L = \frac{TL}{AE} \quad (4.10)
\]

The circumferential strain is equal to

\[
\frac{\Delta L}{L} = \frac{\pi \Delta r}{\pi r} = \frac{\Delta r}{r} \quad (4.11)
\]

and the radial expansion is equal to

\[
\Delta r = \Delta L \frac{r}{L} = \Delta L \frac{r}{\pi r} = \frac{\Delta L}{\pi} = \frac{TL}{\pi AE} \quad (4.12)
\]

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The kinematics of the model are such that the axial displacement is equal to the \( \Delta z = \tan \alpha \Delta r \). Substituting the previously derived expressions for the radial expansion and the meniscal tension yields

\[
\Delta z = \frac{TL \tan^2 \alpha}{\pi^2 AE}
\]  
(4.13)

or rewriting the equation in the form of an equivalent spring \( F = k_{eq} \Delta z \),

\[
F = \frac{\pi^2 AE}{L \tan^2 \alpha} \Delta z
\]  
(4.14)

where the equivalent spring stiffness is

\[
k_{eq} = \frac{\pi^2 AE}{L \tan^2 \alpha}
\]  
(4.15)

The stiffness relationship was used to quantitatively predict the deflection of the femur for a given load.

\section*{4.5 Three-Spring Knee Model}

The modeling approach taken in this thesis began with the development of a lumped-parameter elastic knee model. Once the three-spring model was developed, the spring constants were related to the appropriate material properties and joint geometry. An attempt has been made to incorporate as much of the knee anatomy and material properties as possible so a qualitative understanding of the relevant parameters can be gained.

Whereas the previous models used one spring to model the meniscal stiffness and attributed the stiffness to the circumferential stiffness of the meniscus, this model uses two coupled, but distinct springs to model the compressive and circumferential stiffnesses of the meniscus. The fact that the meniscal compressive modulus is roughly the same magnitude as the compressive modulus of articular cartilage means that the compressive stiffness of the meniscus is important. The third spring in the model is the stiffness related to the direct contact of the articular cartilage surfaces of the femur and tibia. A schematic of the model is shown in Figure 4.9.
Figure 4.9: Three-spring model of the knee with meniscal compressive stiffness $k_1$, meniscal circumferential stiffness $k_2$, and cartilage direct contact stiffness $k_3$.

The first step is determining the kinematics and kinetics of a meniscus compressed between an oblique femur and a flat tibial plateau as shown in Figure 4.10. Initially, the meniscus is assumed to be incompressible in the axial direction. The femoral surface is tilted by an angle $\alpha$ with respect to the tibial plateau. As the femur progresses a distance $\Delta z$, the meniscus is displaced radially a distance $\Delta r$. By focusing on a specific point of the meniscus such as the top left corner, it is apparent that the axial and radial displacements are related by the simple geometry of a right triangle.

$$\tan \alpha = \frac{\Delta z}{\Delta r}$$ (4.16)

Once the kinematics have been determined, an energy balance can be used to solve the kinetics. It is assumed that the surfaces are frictionless so the resultant contact forces act normal to each surface. The displacement of the femur is constrained to move in an axial direction. In the given configuration, the meniscus is acting as an idealized force transformer and as such neither stores nor dissipates energy. Therefore the sum of the work put into the system by the femur must balance the work done
Figure 4.10: Kinematics of an axially incompressible meniscus compressed between an oblique femur and a flat tibial plateau.

by the meniscus.

\[ W_{\text{femur}} + W_{\text{meniscus}} = 0 \]  
(4.17)

\[ F_{\text{femur}} \cos \alpha (\Delta z) + F_{\text{meniscus}} (-\Delta r) = 0 \]  
(4.18)

\[ F_{\text{meniscus}} = \cos \alpha \frac{\Delta z}{\Delta r} F_{\text{femur}} \]  
(4.19)

\[ = \cos \alpha \tan \alpha F_{\text{femur}} \]  
(4.20)

\[ = \sin \alpha F_{\text{femur}} \]  
(4.21)

Alternatively, a free-body diagram of the meniscus could be used to determine the relationships between the contact forces. A summation of forces in the radial direction yields

\[ \sum F_r = F_{\text{femur}} \sin \alpha - F_{\text{meniscus}} = 0 \]  
(4.22)

\[ F_{\text{femur}} \sin \alpha = F_{\text{meniscus}} \]  
(4.23)
Summing the forces in the axial direction yields

\[ \sum F_z = F_{\text{femur}} \cos \alpha - F_{\text{tibia}} = 0 \]  \hspace{1cm} (4.24)

\[ F_{\text{femur}} \cos \alpha = F_{\text{tibia}} \]  \hspace{1cm} (4.25)

Once the simple case of an axially incompressible meniscus is understood, axial compression can be added and the kinematics and kinetics modified appropriately. With the added degree of freedom for the meniscus, the \( \Delta z \) displacement in Equation 4.16 is replaced by the relative displacement of the superior meniscal surface. Also, the differential displacements are replaced by the absolute displacement minus the initial position and the initial positions are defined to be zero at the uncompressed spring lengths (i.e. \( \Delta z = z - z_0 = z \)).

\[ \tan \alpha = \frac{z - z_m}{r_m} \]  \hspace{1cm} (4.26)
Figure 4.12: (a) Model of an axially compressible meniscus. (b) Free-body diagram of an axially compressible meniscus.

Using the same assumption regarding frictionless surfaces, the resultant of the two spring forces must be equal and opposite to the femoral force (see Figure 4.12b). A transformed coordinate system with the \( r' \) coordinate parallel to the femoral surface is used to perform a force balance. A summation of forces in the transformed radial direction yields

$$
\sum F_{r'} = k_1 z_m \sin \alpha - k_2 r_m \cos \alpha = 0
$$

(4.27)

Solving for each meniscal displacement as a function of the other results in

$$
r_m = \left( \frac{k_1}{k_2} \right) \tan \alpha z_m
$$

(4.28)

and

$$
z_m = \left( \frac{k_2}{k_1} \right) \left( \frac{1}{\tan \alpha} \right) z_m
$$

(4.29)

Summing the forces in the transformed axial direction yields

$$
\sum F_{z'} = F_{femur} - k_1 z_m \cos \alpha - k_2 r_m \sin \alpha = 0
$$

(4.30)
Substitution of Equation 4.29 into Equation 4.30 or substitution of Equation 4.28, respectively, simplify to the following equations:

\[ k_2 r_m = F_{femur} \sin \alpha \]  
\[ k_1 z_m = F_{femur} \cos \alpha \]  

(4.31)  
(4.32)

These equations would have resulted from a force balance using the original coordinate system and Equations 4.28 and 4.29 result by eliminating \( F_{femur} \) from the previous set of equations.

The derived meniscal kinematics can be verified by determining \( r_m \) and \( z_m \) as functions of the axial displacement of the femur and examining the meniscal displacements at the appropriate stiffness limits. These functions can be derived by substituting Equation 4.26 into Equations 4.28 and 4.29.

\[ r_m = \left( \frac{1}{\tan \alpha + \left( \frac{F_{femur}}{k_1} \right) \frac{1}{\tan \alpha}} \right) z \]  
\[ z_m = \left( \frac{1}{1 + \left( \frac{k_1}{k_2} \right) \tan^2 \alpha} \right) z \]  

(4.33)  
(4.34)

As \( k_1 \) becomes infinitely stiff, \( z_m \to 0 \) and \( r_m \to z/\tan \alpha \) which is the same as Equation 4.16. As \( k_2 \) becomes infinitely stiff, \( r_m \to 0 \) and \( z_m \to z \). The model is consistent in both limits.

The next step in the analysis is to determine the percentage of the load transmitted by the meniscus. In the previous analysis, only the meniscal stiffnesses were considered so the stiffness of the direct cartilage contact must now be considered. Consider the free-body diagram of the femur shown in Figure 4.13. Summing the forces in the axial direction, the axial component of the femoral force is the total transmitted force and is equal to the tibial force and the axial compressive force of the meniscus.

\[ \sum F_x = F_{femur,y} - F_{tibia} - F_{meniscus,y} = 0 \]  
\[ F_{total} \]

(4.35)
Figure 4.13: Free-body diagram of the femur with the applied tibial and meniscal forces and the resultant femoral force.

\[ F_{\text{total}} = F_{\text{tibia}} + F_{\text{meniscus,y}} \]  
\[ = k_3 z + k_1 z_m \]  

Substituting Equation 4.34 for \( z_m \) yields

\[ F_{\text{total}} = k_3 z + k_1 \left( \frac{1}{1 + \left( \frac{k_1}{k_2} \right) \tan^2 \alpha} \right) z \]  
\[ = \left\{ k_3 + \frac{k_1}{1 + \left( \frac{k_1}{k_2} \right) \tan^2 \alpha} \right\} z \]  

where \( k_{eq} \) is the equivalent stiffness of the total knee joint (\( F_{\text{total}} = k_{eq} z \)). The percentage of the load transmitted by the meniscus is the meniscal axial force divided by the total force.

\[ \% \text{Load} = \frac{k_1 z_m}{F_{\text{total}}} \]  

69
\[
\begin{align*}
\frac{k_1 \left( \frac{1}{1 + \left( \frac{k_1}{k_2} \right) \tan^2 \alpha} \right) z}{k_3 + \left( \frac{k_1}{1 + \left( \frac{k_1}{k_2} \right) \tan^2 \alpha} \right) z} &= \frac{\left( \frac{k_1}{k_3} \right)}{1 + \left( \frac{k_1}{k_3} \right) \tan^2 \alpha + \left( \frac{k_1}{k_3} \right)} \\
(4.41) & \\
\text{To verify the result, consider the limit as } k_1 \to \infty \text{ and } k_2 \to \infty. \text{ The intuitive result is that the meniscus would transmit all the load if the meniscus is infinitely stiff.}
\]

\[
\%\text{Load} = \frac{\left( \frac{1}{k_3} \right)}{1 + \left( \frac{k_1}{k_3} \right) \tan^2 \alpha + \left( \frac{k_1}{k_3} \right)} \\
(4.43) \\
= \frac{\left( \frac{1}{k_3} \right)}{\left( \frac{1}{k_1} \right) + \left( \frac{1}{k_3} \right) \tan^2 \alpha + \left( \frac{1}{k_3} \right)} \\
(4.44) \\
= \frac{\left( \frac{1}{k_3} \right)}{\left( \frac{1}{k_3} \right)} \\
(4.45) \\
= 1 \\
(4.46)
\]

The opposite limiting case is when \( k_3 \to \infty \). In this case the entire load should be transmitted directly between the femur and tibia and the meniscus should transmit no load.

\[
\%\text{Load} = \frac{\left( \frac{k_1}{k_3} \right)}{1 + \left( \frac{k_1}{k_3} \right) \tan^2 \alpha + \left( \frac{k_1}{k_3} \right)} \\
0 \\
(4.47) \\
= \frac{0}{1 + \left( \frac{k_1}{k_3} \right) \tan^2 \alpha} \\
(4.48) \\
= 0 \\
(4.49)
\]

Having developed the three-spring model, it is important to relate the spring constants to the relevant physical parameters. Beginning with \( k_1 \), the axial compressive stiffness of the meniscus, a first-order model can be motivated by considering Hooke's law for uniaxial compression.

\[
\sigma_{ss} = E_{ss} \varepsilon_{ss} \\
(4.50)
\]
The stress is equal to the force divided by the contact area

\[ \sigma_{zz} = \frac{F}{A_{\text{contact}} A_{\text{meniscus}}} \]  

(4.51)

and the strain is equal to the change in length divided by the original length, which is the average thickness of the meniscus divided by the axial displacement of the superior surface of the meniscus.

\[ \varepsilon_{zz} = \frac{z_m}{t_{\text{avg}}} \]  

(4.52)

Substituting these results into Hooke's law yields

\[ \frac{F}{A_{\text{contact}} A_{\text{meniscus}}} = E_{zz} \frac{z_m}{t_{\text{avg}}} \]  

(4.53)

This equation can be rewritten in the form of the constitutive equation for a spring.

\[ F = \left( \frac{E_{zz} A_{\text{contact}} A_{\text{meniscus}}}{t_{\text{avg}}} \right) z_m \]  

(4.54)

The model stiffness \( k_1 \) is a function of the meniscal compressive modulus, the meniscal contact area, and the thickness of the meniscus.

\[ k_1 = \left( \frac{E_{zz} A_{\text{contact}} A_{\text{meniscus}}}{t_{\text{avg}}} \right) \]  

(4.55)

The second stiffness in the model, \( k_2 \), can be determined in a similar manner. By analogy to \( k_1 \), Hooke's law for the meniscus in circumferential tension is

\[ \frac{T}{A_{\text{cross-section}}} = E_{\theta\theta} \frac{\Delta L}{L} \]  

(4.56)

where \( T \) is the circumferential tension on the meniscus and \( L \) is the circumferential length of the meniscus. This relationship cannot be used to directly obtain an expression for circumferential stiffness because \( k_2 \) relates the radial force to the radial displacement. Therefore, relationships between the radial force and the circumferential tension and the radial displacement and the circumferential lengthening must first be developed. Assuming the radial load on the meniscus is distributed uniformly,
the distributed load is equal to the radial load divided by the angular extent of the meniscus \((\Theta_m = \frac{L}{r_{avg}})\) as shown in Figure 4.14a. In order to relate the radial and tensile forces, consider the free-body diagram shown in Figure 4.14b. By summing the forces in the \(x\)-direction, the two tensile components are balanced by the integral of the \(x\)-component of the distributed radial load.

\[
\sum F_x = \int_{-\frac{\Theta_m}{2}}^{\frac{\Theta_m}{2}} \frac{F_r}{\Theta_m} \cos \theta d\theta - 2T = 0 \tag{4.57}
\]

Performing the integration leads to

\[
2T = \int_{-\frac{\Theta_m}{2}}^{\frac{\Theta_m}{2}} \frac{F_r}{\Theta_m} \cos \theta d\theta \tag{4.58}
\]

\[
= \frac{F_r}{\Theta_m} \sin \theta \bigg|_{-\frac{\Theta_m}{2}}^{\frac{\Theta_m}{2}} \tag{4.59}
\]

\[
= \frac{F_r}{\Theta_m} [1 - (-1)] \tag{4.60}
\]

\[
T = \frac{F_r}{\Theta_m} \tag{4.61}
\]
Relating the circumferential lengthening to the radial displacement is identical to the analysis presented by Walker [150] (see Equation 4.12).

\[ \Delta L = \Theta_m r_m \]  \hspace{1cm} (4.62)

Substituting Equations 4.61 and 4.62 into Equation 4.56 results in

\[ F_r = \left( \frac{E \theta A_{\text{meniscus}}}{L} \frac{\Theta_m^2}{k_2} \right) r_m \] \hspace{1cm} (4.63)

The circumferential stiffness of the meniscus is a function of the circumferential tensile modulus, the cross-sectional area, the angular extent, and the circumferential length of the meniscus.

\[ k_2 = \left( \frac{E \theta A_{\text{meniscus}}}{L} \frac{\Theta_m^2}{k_2} \right) \] \hspace{1cm} (4.64)

As a check on the derivation of \( k_2 \), compare the knee stiffness relationship developed by Walker [150] to the three-spring model. To make this comparison, the axial stiffness of the meniscus must be infinite \( (k_1 \rightarrow \infty) \) and the direct contact stiffness of the cartilage must be zero \( (k_3 \rightarrow 0) \). Rearranging Equation 4.39, the knee stiffness becomes

\[ F_{\text{total}} = \left\{ k_3 + \frac{1}{\left( \frac{1}{k_1} + \left( \frac{1}{k_2} \right) \tan^2 \alpha \right)} \right\} z \] \hspace{1cm} (4.65)

\[ = \left\{ \frac{k_2}{\tan^2 \alpha} \right\} z \] \hspace{1cm} (4.66)

\[ = \left\{ \frac{E \theta A_{\text{meniscus}}}{L \tan^2 \alpha} \right\} z \] \hspace{1cm} (4.67)

This is identical to Equation 4.14 when \( \Theta_m = \pi \).

The third stiffness of the knee model is related to the direct contact of the femoral and tibial articular cartilage surfaces. The direct contact stiffness can be written by inspection from the axial compressive stiffness of the meniscus \( k_1 \).

\[ k_3 = \left( \frac{E_{\text{contact}}}{t_{\text{avg}}} \right) \] \hspace{1cm} (4.68)
The direct contact stiffness is a function of the compressive modulus of articular cartilage, the contact area, and the combined average thickness of the femoral and tibial cartilage layers.

Although the spring stiffnesses $k_1$, $k_2$, and $k_3$ have been presented as linear springs, the nonlinear strain effects of the material properties, the nonlinear load effects on the contact area, and the nonlinear spacer effect of the meniscus could be incorporated into the three-spring model. All of the elastic effects of the previous models are incorporated into the three-spring model (except the secondary elastic effect of the peripheral cartilage compression between the meniscus and the subchondral bone that was included by Jaspers et al. [72, 41] plus the addition of the axial compressive stiffness of the meniscus and its coupling to the circumferential stiffness. The dissipative effects are neglected because the complicated flow patterns of a fluid passing through a porous media are not easily related to a lumped-parameter dashpot.

A parameter analysis of the three-spring knee model is presented in Figure 4.15. The plot shows the percentage of load transmitted by the meniscus increases as the circumferential stiffness $k_2$ increases relative to the meniscal compressive stiffness because the increased circumferential stiffness prevents the meniscus from expanding radially and the thus the meniscus transmits a greater share of the load. As the meniscal compressive stiffness increases relative to the direct contact stiffness, the meniscus again transmits an increasing percentage of the load.
Figure 4.15: Parameter analysis of the three-spring knee model. The percentage of the load transmitted by the meniscus is plotted as a function of the stiffness ratios $k_1/k_2$ and $k_1/k_3$ for a meniscal contact angle of $30^\circ$. 
Chapter 5
Finite Element Modeling of the Meniscus

Due to the complicated geometry of the knee and the anisotropic material properties of the meniscus, analytical solutions of the knee contact problem are not feasible. Consequently, an alternative method must be used, the finite element method (FEM). Although there are numerous different implementations, the following description is for the displacement-based FEM, which is the standard formulation for solid mechanics.¹ The finite element method entails dividing a continuum into a group of smaller elements which are defined by a set of nodal points. The elastic properties of the elements are distributed to the element nodes in the form of element stiffness matrices which are assembled into the stiffness matrix for the entire model. The applied loads and boundary conditions (boundary conditions are related to the physical boundary conditions of the problem, symmetry assumptions, and contact boundary conditions) are added to the model to generate a system of linear equations of the form:²

\[ K u = F \]  

¹The ADINA finite element program, ADINA Engineering Inc., Watertown, MA was used for all analyses. A general reference for the finite element problem formulation and solution techniques is Klaus-Jürgen Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Inc., Englewood Cliffs, 1982.

²This representation is not strictly correct because the contact boundary conditions are nonlinear in nature and must be solved iteratively, but the general concept of solving a large set of linear equations is sufficient for understanding the general solution method.
The solution of Equation 5.1 is the displacement vector $\mathbf{u}$, which contains the displacements of each node. The strains are subsequently calculated from the displacements. The stresses are calculated from the strains using the appropriate stress-strain constitutive relationships. It is important to understand that the strains and stresses are derived quantities and even though the elemental displacements are continuous across element boundaries, the displacement derivatives (strains) and stresses are not in general continuous. These potential discontinuities will be employed later to check the validity of the finite element models.

Before the specifics of the finite element modeling are presented, it is necessary to discuss the author's philosophy of the finite element method. FEM is a powerful analytical tool because it permits the solution of otherwise unsolvable problems. FEM is also exceedingly seductive for the very same reason. Assuming the model converges numerically, FEM always provides a solution. But the solution can be no better than the assumptions that went into the development of the finite element model and mesh. The results must be verified with physical intuition. Therefore, all FEM results should be reviewed with a healthy dose of skepticism. FEM users who blindly accept numerical results have led to the witticism, "Garbage in, gospel out." On a final note, if an analytical solution is possible, it is generally preferred.

Given the author's philosophical perspective of FEM, the reductionist approach to modeling is obvious. Begin with the simplest knee model that still retains the physical effects of interest. Glean as much information as possible regarding the load-bearing role of the meniscus from a simple model. It is intuitively easier to understand the effect of varying model parameters in an axisymmetric two-dimensional knee model than in a three-dimensional knee model with anatomically complex geometry. Studying a two-dimensional knee model provides insight into the finite element mesh requirements for examining meniscal mechanics. This physical intuition is a prerequisite for three-dimensional model development.

The model for investigating the load-bearing role of the meniscus assumes the knee
is axisymmetric. This assumption retains the important aspect of the circumferential stiffness of the meniscus while eliminating the computational disadvantages of a three-dimensional analysis. The femoral condyle is modeled as a sphere, the tibial plateau is modeled as a plane, and the meniscus is modeled as a wedge-shaped toroid. All contact surfaces are assumed to be frictionless.

5.1 Hertzian contact

Removal of the meniscus from the model described above, reduces the model to a class of contact problems known as Hertzian contact. Analytical solutions for Hertzian contact can be used to assess various finite element mesh refinement approaches and verify the finite element model without the meniscus.

Before employing the Hertz theory of contact, it is important to understand the implicit assumptions [73]. Denoting the radius of the contact area by $a$, the relative radius of curvature by $R$, the radii of each body by $R_1$ and $R_2$, and the major dimensions of the bodies both laterally and in depth by $l$, the assumptions are summarized as follows:

1. The surfaces are continuous and non-conforming, $a \ll R$; 

2. The strains are small, $a \ll R$; 

3. Each solid can be considered an elastic half-space, $a \ll R_{1,2}$ and $a \ll l$; 

4. The surfaces are frictionless.

These assumptions must be kept in mind when comparing the analytical Hertzian solution to the FEM solution.

5.1.1 Analytical solution

The analytical solution of two elastic bodies in frictionless contact was first solved by H. Hertz in 1881. The equations presented are taken from Timoshenko and Goodier
[142] and Johnson [73]. The following is not intended to be a derivation of the Hertzian contact solution, but merely a presentation of the relevant results.

For the case of axisymmetric bodies, the pressure is distributed over a circular region of radius \( a \). The pressure distribution is hemispherical in its form and is equal to the applied compressive load \( F \).

\[
\frac{P_0}{a} \left( \frac{2}{3} \pi a^3 \right) = F \tag{5.2}
\]

The peak pressure is then equal to

\[
P_0 = \frac{3}{2} \pi a^2 \frac{F}{R_1 R_2} = \frac{3}{2} p_{\text{uniform}} \tag{5.3}
\]

or the peak pressure is equal to three halves the uniform pressure. The contact radius \( a \) is determined by

\[
a = \sqrt[4]{\frac{3 \pi}{4} \frac{F(k_1 + k_2) R_1 R_2}{R_1 + R_2}} \tag{5.4}
\]

where \( k_1 \) and \( k_2 \) are the elastic constants of the materials and \( R_1 \) and \( R_2 \) are defined as the radii of curvature of the contact bodies. In the general formulation, the elastic properties of the two materials need not be the same. The elastic constants \( k_1 \) and \( k_2 \) are defined as

\[
k_1 = \frac{1 - \nu_1^2}{\pi E_1} \tag{5.5}
\]

and

\[
k_2 = \frac{1 - \nu_2^2}{\pi E_2} \tag{5.6}
\]

where \( E \) is Young’s modulus and \( \nu \) is Poisson’s ratio. Consider the case when one of the bodies is planar \((R_2 \to \infty)\) and the material properties of both bodies are identical \((k = k_1 = k_2)\). The expression for \( a \) simplifies to

\[
a = \sqrt[4]{\frac{3(1 - \nu^2)}{2E} F R_1} \tag{5.7}
\]

The contact pressure distribution is then

\[
p(r) = P_0 \sqrt{1 - \left( \frac{r}{a} \right)^2} = \frac{3F}{2\pi a^3} \sqrt{1 - \left( \frac{r}{a} \right)^2} \tag{5.8}
\]
This is the pressure distribution to which the finite element model results are compared.

The overall stiffness of two bodies in contact can be determined. Two points on the bodies on the axis of symmetry at large distances from the initial point of contact approach each other by an amount \( \alpha \).

\[
\alpha = \sqrt[3]{\frac{9\pi^2 F^2(k_1 + k_2)^2(R_1 + R_2)}{16 R_1 R_2}}
\]  
(5.9)

Reconsidering the case of one planar body with identical material properties, the distance \( \alpha \) simplifies to

\[
\alpha = \sqrt[3]{\frac{9(1 - \nu^2)^2 F^2}{4 E^2 R_1}}
\]  
(5.10)

The overall displacement \( \alpha \) will be compared to the load-displacement curve of the finite element model.

\[
F = \frac{2ER_1^{\frac{1}{3}}}{3(1 - \nu^2)^{\frac{1}{3}} \alpha^{\frac{4}{3}}}
\]  
(5.11)

Johnson [73] presents the equations for the surface displacements and stresses along the surface and axis of symmetry for Hertzian contact of an elastic half-space. The surface displacement \( u_s \) is

\[
u_s = \left( \frac{1 - \nu^2}{E} \right) \left( \frac{\pi p_0}{4a} \right) \left( 2a^2 - r^2 \right), r \leq a
\]  
(5.12)

\[
u_s = \left( \frac{1 - \nu^2}{E} \right) \left( \frac{p_0}{2a} \right) \left[ (2a^2 - r^2) \sin^{-1} \left( \frac{a}{r} \right) + a\sqrt{r^2 - a^2} \right], r > a
\]  
(5.13)

The stresses along the axis of symmetry are:

\[
\sigma_{rr} = \sigma_{\theta \theta} = -p_0(1 + \nu) \left[ 1 - \left( \frac{z}{a} \right) \tan^{-1} \left( \frac{a}{z} \right) \right] + \frac{p_0}{2} \left( 1 + \frac{z^2}{a^2} \right)^{-1}
\]  
(5.14)

\[
\sigma_{ss} = -p_0 \left( 1 + \frac{z^2}{a^2} \right)^{-1}
\]  
(5.15)

The three stresses are all principal stresses so the maximum shear stress is one half the difference between the stresses.

\[
\tau_{\text{max}} = \frac{1}{2} |\sigma_{ss} - \sigma_{rr}|
\]  
(5.16)
Figure 5.1: Stress distributions at the surface and along the axis of symmetry caused by uniform pressure (left) and Hertzian pressure (right) acting on a circular area of radius $a$. [73]
The stresses for Hertzian contact are shown in Figure 5.1 and compared to the stresses from a uniform pressure distribution. An important feature of the maximum shear stress profile along the axis of symmetry is that the peak shear stress occurs beneath the surface of the body.

### 5.1.2 Hertzian contact model

An elastic sphere of radius 30 mm is considered to come into contact with a planar elastic surface. The elastic sphere is chosen to be the target body and the elastic plane is chosen to be the contactor surface because the ADINA program only outputs results for the contactor surface. The model geometry was chosen to match that of existing knee finite element models [122, 68].

The loading boundary conditions of the model apply a compressive load at the axis of symmetry on the superior surface of the sphere. All nodes of the superior surface are constrained to have equal displacements in the axial direction (z-direction), but are free to move in the radial direction. This type of boundary condition distributes the compressive load along the surface without requiring an assumption about the load distribution function. The underlying assumption is that the superior surface remains planar. A similar constraint is applied to the inferior surface of the elastic plane. The axial nodal positions are fixed, but the nodes are free to move in the radial direction. The contact boundary condition specifies the contact surfaces to be frictionless \((\mu = 0)\) so only normal contact stresses are transmitted. The model axisymmetry restricts the motions of nodes on the axis to be axial.

A soft truss is used to fix the sphere to ground to insure numerical stability when the model is not in contact (see Figure 5.2). Otherwise, rigid body displacements of the sphere are possible. Only a negligible amount of the load is carried by the truss and the results from the model are unaffected.

A total of eighteen different finite element meshes were generated in an effort to optimize the mesh and validate models. The information on these different models...
Table 5.1: Summary of information on finite element models.

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is contained in Table 5.1. Initially, three different meshing approaches were used to study the effects of changing element type and element size.

Each subgroup of FEM model meshes represents a variation on a basic mesh so only one model from each subgroup is presented. Group I includes cases 1–7 and 17 and represents the simplest mesh pattern of subdividing each body into a roughly rectilinear pattern. Meshes 1 and 5 are shown in Figures 5.2 and 5.3, respectively. Mesh 5 demonstrates the effect of the element size ratio. The size ratio is a scaling ratio for specifying the relative element sizes. Notice that the size of the peripheral elements are considerably larger than the elements near the point of contact.

Group II includes cases 8–12 and is represented by mesh 10 (see Figure 5.4). This mesh group was an attempt to refine the mesh in the region of greatest stress. The element orientation in the refined region is similar to the pressure band plots discussed
Figure 5.2: Mesh for Hertzian contact Case 1. A soft truss is present to fix the sphere to ground.
Figure 5.3: Mesh for Hertzian contact Case 5. Notice the relative element sizes of the peripheral and central elements.
in the next section.

Group III also refines the mesh in the region of contact, but the elements are in a rectilinear pattern as opposed to the polar pattern of Group II. Cases 13 and 14 are included in this group and are represented by Case 13 (see Figure 5.5). This approach avoids the degenerated elements present at the corner of the Group II refined region where 4-node and 8-node elements are collapsed to 3-node and 6-node elements, respectively.

The remaining Hertzian contact cases were used for verification of the actual meshes used for studying the load-bearing role of the meniscus. Case 16 is the mesh (minus the meniscus) used for verification of Sauren’s model [122] and case 18 is the verification of the anisotropic axisymmetric knee model (minus the meniscus and cartilage). Case 16 is presented in Figure 5.6 and a detail of the contact region is presented in Figure 5.7. The mesh for Case 18 is shown in Figure 5.8.

5.1.3 Results

Given that the primary focus of the thesis is the load-bearing role of the meniscus, the variables of interest in the finite element models are the contact pressures and the resultant stresses. Therefore, the pressure profile and the stress distribution were chosen as the critical indicators of the FEM model accuracy.

The analytical pressure profile for Hertzian contact presented in Section 5.1.1 is compared to the model pressure profile. Three different errors measures are used for the pressure profile comparison:

1. the pressure at the axis of symmetry,

2. the peak pressure (the peak pressure does not necessarily occur at the axis of symmetry in the FEM model),

3. the integrated pressure profile (applied force).

The results and errors are tabulated for each case in Tables 5.2, 5.3, and 5.4.
Figure 5.4: Mesh for Hertzian contact Case 10. Notice the degenerated elements at the point of contact.
Figure 5.5: Mesh for Hertzian contact Case 13.
Figure 5.6: Mesh for Hertzian contact Case 16.
Figure 5.7: Detailed mesh of contact region for Case 16.
Figure 5.8: Mesh for Hertzian contact Case 18.
The general conclusion after reviewing the pressure errors is that the mesh should be finest where the greatest pressure gradients occur. Thus, the mesh should be most refined at the contact boundary edge. Unfortunately, the contact boundary edge moves with increasing load so the mesh should be equally refined throughout the area of contact.

A subtler point involves the choice of elements along the contact surface. The contact elements in ADINA employ linear interpolation functions. 4-node and 8-node solid elements use linear and quadratic interpolation functions, respectively. Consequently, 8-node solid elements require the use of two linear contact elements to interpolate the contact stresses along the contact surface of the quadratic solid element. A problem arises because the surface midnode of an 8-node element is stiffer than the equivalent node of two contiguous 4-node elements. This effect can be seen by examining the pressure profiles for Cases 13 and 14 (see Figure 5.9. Even though Case 14 has more nodes (the number of elements are equal), the pressure profile has greater error.

A final note on the pressure profile relates to a probable artifact of the FEM contact algorithm. Frequently, the last or second to last contact element showed a small pressure spike (see Figure 5.9 for Case 13). Numerical simulations to replicate the effect were unsuccessful, but the inconsistency of the effect supports the idea that it is an artifact.³

³For a detailed explanation of the ADINA contact solution algorithm, the reader is referred to a paper by Bathe and Chaudhary [19].
Table 5.2: Contact pressure at axis of symmetry.

<table>
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Analytical 4.34 7.73 13.7 24.4
Table 5.3: Peak pressure along contact surface.

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<td>7.73</td>
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</tbody>
</table>
Table 5.4: Integrated pressure (force) along contact surface.

<table>
<thead>
<tr>
<th>Case</th>
<th>Force, N</th>
<th>Percentage error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P1</td>
<td>P2</td>
</tr>
<tr>
<td>1</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>6.10</td>
<td>34.5</td>
</tr>
<tr>
<td>3</td>
<td>6.10</td>
<td>34.5</td>
</tr>
<tr>
<td>4</td>
<td>7.25</td>
<td>41.0</td>
</tr>
<tr>
<td>5</td>
<td>6.27</td>
<td>33.0</td>
</tr>
<tr>
<td>6</td>
<td>6.73</td>
<td>33.8</td>
</tr>
<tr>
<td>7</td>
<td>6.74</td>
<td>33.7</td>
</tr>
<tr>
<td>8</td>
<td>5.82</td>
<td>32.2</td>
</tr>
<tr>
<td>9</td>
<td>6.13</td>
<td>33.3</td>
</tr>
<tr>
<td>10</td>
<td>6.19</td>
<td>32.3</td>
</tr>
<tr>
<td>11</td>
<td>4.32</td>
<td>32.6</td>
</tr>
<tr>
<td>12</td>
<td>5.71</td>
<td>32.0</td>
</tr>
<tr>
<td>13</td>
<td>6.05</td>
<td>32.2</td>
</tr>
<tr>
<td>14</td>
<td>5.69</td>
<td>32.0</td>
</tr>
<tr>
<td>15</td>
<td>5.95</td>
<td>32.3</td>
</tr>
<tr>
<td>16</td>
<td>5.71</td>
<td>32.0</td>
</tr>
<tr>
<td>17</td>
<td>58.1</td>
<td>114.</td>
</tr>
<tr>
<td>18</td>
<td>5.96</td>
<td>32.3</td>
</tr>
</tbody>
</table>

The continuity of the stress distribution is an indicator of the accuracy of the predicted stresses. As mentioned previously, the stress is derived from the nodal displacements and strains so the stress need not be continuous from element to element. Sussman and Bathe [139] present a technique for examining the stress continuity through the use of band plots. Examining each of the stress components individually would be tedious, so representative stress measures are defined. The hydrostatic pressure is defined as

\[ p = \frac{-(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz})}{3} \]  

and is related to the first invariant of the stress tensor. This is a measure of the

95
Figure 5.9: Comparison of pressure profile for 4-node and 8-node surface solid elements. Numerically, the 4-node surface solid elements result in a more accurate pressure profile.
normal stresses. The shear stresses are measured using the effective von Mises stress.

\[
\sigma_{\text{eff}} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}
\] (5.18)

where \(\sigma_1, \sigma_2,\) and \(\sigma_3\) are the principal stresses. It is often inconvenient to calculate the principal stresses, so an alternative expression for the effective stress for an axisymmetric formulation \((\tau_{r\theta} = \tau_{\theta\theta} = 0)\) is

\[
\sigma_{\text{eff}} = \sqrt{\frac{(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{ss})^2 + (\sigma_{ss} - \sigma_{rr})^2}{2}} + 3\tau_{rs}
\] (5.19)

The effective stress is related to the second invariant of the stress tensor.

A band plot of the hydrostatic pressure for Case 1 is shown in Figure 5.10. Notice the visible discontinuities and the bands are not distinguishable. This indicates that the mesh is not sufficiently refined. For comparison, the respective pressure and effective stress band plots for Case 18 are presented in Figures 5.11 and 5.12. The bands are distinguishable with negligible discontinuities.

Given an appropriate selection of the band widths (5% to 10% of maximum stress), Sussman [139] states that “if all of the stress bands are continuous between elements, then the mesh is probably overrefined.” Judging by this criteria, the final mesh could be overrefined in regards to determining the stress distributions. However, the pressure profile does require a finer mesh so the mesh requirements for the stress distribution are less restrictive than those for accurate determination of the pressure profile.

The conclusions of the Hertzian contact study are:

1. Use 7-node solid elements along the contact surface. These elements have linear interpolation functions along the contact surface of the element and quadratic interpolation functions along the other surfaces for more accurate stress calculations.

2. Use 8-node elements solid elements throughout the rest of the mesh for accurate stress calculation.
Figure 5.10: Isoband pressure plot for Case 1. Stress bands are not distinguishable.
Figure 5.11: Isoband pressure plot for Case 18. Notice the distinguishable bands.
Figure 5.12: Isoband effective stress plot for Case 18. Notice the distinguishable bands.
3. Prevent overrefinement of the mesh in locations of low stress. The stress gradients are high near the contact regions.

4. Space contact elements equally spaced to predict consistent pressure profiles for various applied loads.

The verification of the final mesh includes comparison of the analytical Hertzian solutions for stiffness, displacements, pressures, and stresses for an applied load of 1000 N. The model stiffness is the slope of the load-displacement curve and is shown in Figure 5.13 along with the axial displacement of the planar contact surface. The FEM model predictions match the basic shape of the analytical solutions, but have a small offset. This is due to the fact that the analytical solution assumes the contact bodies are semi-infinite. The model pressure and stress predictions reproduce the analytical solution with a reasonable degree of accuracy. Predicted stresses diverge from the analytical solution at the rigid boundary condition (see Figure 5.14). The deformed model is presented in Figure 5.15 so the contact region can be visualized. Before concluding this analysis, it is important to check the initial Hertzian assumptions. The contact radius is not much less than the dimensions of the elastic bodies \((R_3/a \approx 7\) and \(l/a \approx 3.5\)) so some divergence from the analytical solutions is expected.

5.2 Previous Finite Element Models of the Meniscus

Sauren et al [122] were the first to model the meniscus using finite element methods. The axisymmetric model geometry and mesh are shown in Figure 5.16. The model was used for a parametric analysis of bone and meniscal material properties. The conclusions of the analysis are:

- The meniscus transmits a significant portion of the load in the knee joint.
- The meniscus considerably reduces the femorotibial contact stresses.
Figure 5.13: Comparison of FEM model predictions and analytical solutions for load-displacement curve and axial displacement of planar contact surface (Case 18).

- The inner half of the meniscus accounts for the major part of the meniscal load transmission.

- The axial compression of the joint and the radial displacement of the meniscus are nonlinear functions of the axial joint load.

- The combination of bone and meniscus material properties is more important for meniscal load transmission than the meniscal dimensions.

The authors conclude by stating that extrapolation of the results to reality is unwarranted because of the gross simplifications regarding joint geometry and material properties, most notably the exclusion of articular cartilage and anisotropic meniscal material properties. The actual numerical results of Sauren’s model are presented in the next section.

Hefzy et al [68, 67] attempted to replicate Sauren’s work (see Figure 5.17). The femoral elements in the region of tibiofemoral contact are highly distorted and any
Figure 5.14: Comparison of FEM model predictions and analytical solutions for contact pressure and principal stresses along axis of symmetry of planar body (Case 18).
<table>
<thead>
<tr>
<th>LOAD STEP</th>
<th>TIME (s)</th>
<th>ORIG. DEFORM.</th>
<th>XMIN</th>
<th>XMAX</th>
<th>YMIN</th>
<th>YMAX</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8.000</td>
<td></td>
<td>1.257</td>
<td>1.257</td>
<td>-15.00</td>
<td>15.00</td>
</tr>
</tbody>
</table>

Figure 5.15: Deformation of Hertzian FEM model (Case 18).
Figure 5.16: Finite element model of the knee with meniscus used by Sauren et al [122].

Figure 5.17: Finite element model of the knee with meniscus used by Hefsy et al [68]. Notice the highly distorted femoral elements in the tibiofemoral contact region.
stress distributions predicted by the model should be viewed with skepticism. Band plots of the pressure and effective stress would verify this criticism. Additionally, the applied load appears to be applied as a point load on the femur at the axis of symmetry. This conclusion is supported by the displacement plot of the top femoral surface (see Figure 5.18). Such an applied load is not physiologic. The load is distributed throughout the subchondral bone. If a point load is used, it is more reasonable to apply the load at the peripheral edge of the top femoral surface where the load is transmitted by the metaphyseal cortex of the femur. Hefzy claims agreement with Sauren's model predictions of the percentage of the load transmitted by the meniscus. Given that Hefzy reports differences of up to 50 percent for predicted displacements, agreement between predicted load transmission percentages is insufficient to validate his model. Sauren's model appears to be more accurate.

Aspden [15] examined the failure modes of the meniscus using an axisymmetric meniscus model. The meniscus rests on a compliant base to model articular cartilage (see Figures 5.19a and 5.19b). In the absence of available material properties, a parametric analysis was performed. Cross-sectional meniscal properties varied from 1-10 MPa for Young's modulus, 0.5-2.5 MPa for shear modulus, and 0.2-0.4 for Poisson's ratio. The fact that the meniscus is considered anisotropic allows all three of the elastic parameters to be specified independently. In the circumferential direction, the meniscal constitutive stress-strain relationship is assumed to be the same as for tendon and the magnitude of the elastic constant is scaled by the relative collagen
Figure 5.19: (a) Axisymmetric model of the meniscus resting on a compliant base. 
(b) Cross-section of the meniscal model showing FEM mesh. [15]
content.

\[ \sigma = (470 \ MPa) \times \varepsilon^3 \]  \hspace{1cm} (5.20)

The articular cartilage is given a modulus of 5 MPa. Regarding the applied load, Figure 5.19a suggests that the applied load is uniformly distributed normal to the superior meniscal surface.

The model predictions are:

- A circumferential tensile strain is developed throughout the cross-section of the meniscus.

- A region of radial tensile strain developed in the cross-section (see Figure 5.20b). The extent of the tensile region was dependent on the meniscus geometry and Poisson’s ratio.

- A regions of positive and negative shear strain are developed (see Figure 5.20c).
Figure 5.21: Biphasic anisotropic finite element model of the meniscus used by Spilker et al. [137]

Aspden is reluctant to present numerical results because of the lack of accurate elastic parameters and failure strains. However, he does point out the correlation between the radial strain and shear strain interface locations and the longitudinal and horizontal meniscal tears, respectively. The positive radial strain result is important if it can be validated.

Spilker et al [137] developed a biphasic (poroelastic) anisotropic model of the meniscus. The meniscus is axisymmetric and triangular in cross-section (see Figure 5.21). The meniscus is free to expand radially on the frictionless, rigid tibial surface. A prescribed parabolic displacement loads the femoral surface and is ramped up to a maximum value over a time period of 100 s.

Results of the model are given in Figure 5.22. The magnitude of the hoop stresses is seemingly small. Spilker reports the isotropic model has a maximum tensile hoop stress at the peripheral surface and the hoop stress becomes compressive as the radius decreases. No physical explanation is given to describe how an axisymmetric body under axial and radial load can develop a compressive hoop stress and this result is in conflict with all experimental and finite element modeling studies to date. Spilker reports that the anisotropic model has a maximum tensile hoop stress at the peripheral edge and remains tensile throughout the body of the meniscus. This result is also suspect because Aspden [15] reports the maximum hoop strain occurs
Figure 5.22: Hoop stresses developed in the meniscus as applied displacement ramp reaches maximum ($t=100 \text{ s}$). The results are in units of MPa. (a) isotropic model; (b) anisotropic model. Notice the irregular pattern of the stress contours at the inner edge of the meniscus. [137]

at the inner edge of the meniscus because that is where the greatest loading occurs. The anomalous hoop stress results may occur because of the prescribed displacement loading condition. The physiologic loading state is an applied set of contact and muscle forces resulting in a distributed pressure load on the meniscus, not prescribed meniscal displacements.

Regarding the poroelastic response of the meniscus, the author agrees with Spilker et al that poroelastic phenomenon must be included in meniscal models to account for their time dependent behavior, if 100 s is the appropriate physiologic time scale. However, daily activities of walking and running are in the frequency range of 1 Hz or greater and these are the activities in which meniscal distribution of rapidly applied loads would seem to be most crucial. At higher strain rates, the meniscus can be modeled as a purely elastic (albeit anisotropic) material.
Tissakht et al [144, 143] developed a three-dimensional model of the knee including the menisci, articular cartilage, and tibia (see Figure 5.23a). The menisci are modeled as anisotropic materials with an isotropic membrane over the surface. The isotropic membrane is incorporated to model the thin layer of collagen covering the meniscus (see Figure 5.23b). A distributed pressure load of 1300 $N$, according to the results of Ahmed [5], is applied to the model with/without $20^\circ$ of external axial rotation.

The first goal of the FEM study was to determine the effect of ligamentous attachments on meniscal displacements. The conclusion was that the effect of the transverse ligament was negligible, but the medial collateral ligament (MCL) attachment to the medial meniscus significantly reduced meniscal motion (see Figure 5.24). The second goal was to confirm if torsional motions are necessary to produce meniscal failure. Axial loading alone did not produce sufficient radial or circumferential stresses to cause meniscal failure, but the addition of external rotation caused significant stress increases in the posterior horn of the medial meniscus (see Figure 5.25). Given the coarseness of the mesh, the meniscal displacement results may be reasonable, assuming the loading conditions are accurate. However, the stresses should be examined further. The results do suggest the importance of axial rotation in predicting meniscal failures. Torsional effects cannot be studied in axisymmetric meniscal models.

5.3 Preliminary Knee Model

Of the finite element models of the knee, only Sauren et al [122] and Hefzy et al [68, 67] studied the load-bearing role of the meniscus. In order to verify previous results, a preliminary knee model was developed using Sauren's geometry. The preliminary model will be used to compare to later models which incorporate anisotropic material properties and articular cartilage.

The anatomic basis for the model is an axially loaded knee joint in full extension. The model is axisymmetric and consists of a planar tibial plateau, a spherical femoral condyle and an annular meniscus interposed between the femur and tibia. Under no
Figure 5.23: (a) Three-dimensional finite element model of the knee including meniscus. [144] (b) Cross-section of meniscus demonstrating FEM mesh and isotropic membrane surrounding meniscus. [143]
Figure 5.24: Three-dimensional finite element model predictions of meniscal displacements for (a) intact knee with axial load; (b) medial collateral ligament detached with axial load; (c) 10° external rotation. [144]

Figure 5.25: Three-dimensional finite element model predictions of meniscal stresses for axial load with/without external rotation: (a) proximal meniscal layer; (b) distal meniscal layer. [143]
load conditions, the meniscofemoral and meniscotibial surfaces are conforming. The tibiofemoral surfaces are in point contact. All contact surfaces are assumed frictionless due to the negligible friction present in normal synovial joints.

Sauren's finite element model and geometry were shown in Figure 5.16. The model developed for comparison is shown in Figure 5.26. The radius of the femur is 30 mm. The radial width of the tibia and femur is 20 mm and the axial lengths of the tibia and femur are 10 mm and 15 mm, respectively. The inner and outer surfaces of the meniscus are cylindrical with respective radii of 8 mm and 18 mm. A soft truss was added to fix the femur to ground. The same mesh is used for Cases 1–3. The finite element models for Cases 4 and 6 remove the outer and inner halves of the meniscus, respectively. The model for Case 5 was presented as Case 16 of the Hertzian contact modeling (see Figure 5.6).

As mentioned previously, Sauren's study was a parameter analysis of material properties. Both the bone and meniscus meniscus are modeled as isotropic materials. The material properties for each case are listed in Table 5.5. Sauren included a case which removed the outer half of the meniscus. Though this may demonstrate that most of the load transmitted by the meniscus is through the inner half of the meniscus, partial meniscectomy is simulated by removal of the inner half of the meniscus. Consequently, Case 6 has been added to the this study.

The preliminary model confirms Sauren's earlier results. The overall joint stiffness and meniscal radial stiffness demonstrate similar trends and values. Case 3 is the stiffest joint and Case 5 without the meniscus is the most compliant. The load-displacement curves for the meniscus demonstrate: (1) a stiffer meniscus undergoes less radial expansion; (2) a stiffer bone results in reduced radial displacements of the meniscus; (3) a narrower meniscus has diminished radial stiffness; (4) a meniscus with a larger inner radius is subjected to lower axial (and radial) loads. Graphical comparison of the results are shown in Figures 5.27 and 5.28.

Comparison of the pressure profiles for Case 1 shows small differences (see Fig-
Figure 5.26: Finite element model for Cases 1-3 used for comparison with Sauren’s results.
Figure 5.27: Overall joint stiffness comparison with Sauren. (a) Sauren [122], (b) preliminary knee model.
Figure 5.28: Meniscal radial stiffness comparison with Sauren. (a) Sauren [122], (b) preliminary knee model.
Table 5.5: Material properties for various case studies.

<table>
<thead>
<tr>
<th>Case</th>
<th>Sauren Case</th>
<th>Bone $E$ (MPa)</th>
<th>$\nu$</th>
<th>Meniscus $E$ (MPa)</th>
<th>$\nu$</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>500</td>
<td>0.2</td>
<td>20</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5000</td>
<td>0.2</td>
<td>20</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5000</td>
<td>0.2</td>
<td>200</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>500</td>
<td>0.2</td>
<td>20</td>
<td>0.3</td>
<td>Outer 1/2 of meniscus removed</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>500</td>
<td>0.2</td>
<td></td>
<td></td>
<td>Meniscus removed (Hertz contact)</td>
</tr>
<tr>
<td>6</td>
<td>–</td>
<td>500</td>
<td>0.2</td>
<td>20</td>
<td>0.3</td>
<td>Inner 1/2 of meniscus removed</td>
</tr>
</tbody>
</table>

The preliminary knee model uses more contact elements which results in a more refined pressure profile and is sufficient to explain the differences. Sauren’s model seems to overestimate the tibiofemoral contact pressure and underestimate the meniscotibial contact pressure. These differences are manifested in the calculation of the load transmitted by the meniscus. This is particularly evident for Case 3 (see Figure 5.30). Trapezoidal integration in polar coordinates was used to calculate the forces for both the tibiofemoral and meniscotibial contact regions.

\[
F = \int \int p(r)rdrd\theta
\]

\[
= 2\pi \sum_{i=0}^{N-1} \int_{r_i}^{r_{i+1}} \left\{ \left( p_{i+1} - p_i \right) \frac{r}{r_{i+1} - r_i} + \left( p_i r_{i+1} - p_{i+1} r_i \right) \frac{r}{r_{i+1} - r_i} \right\} rdr
\]

\[
= \frac{\pi}{3} \sum_{i=0}^{N-1} \left\{ \left( 2p_{i+1} + p_i \right) r_{i+1}^2 - \left( p_{i+1} - p_i \right) r_{i+1} r_i - \left( p_{i+1} + 2p_i \right) r_i^2 \right\}
\]

The sum of the two forces should be equal to the applied load and provides an estimate of the error. The maximum error was 1-2%. Sauren did not describe the method he used to calculate the percentage of load transmitted by the meniscus.

Contact areas were also calculated. Several early studies of the meniscal load-bearing used contact area as an indirect measure of load transmission. The meniscal percentage of the total contact area is compared to the percentage of load transmitted.
Figure 5.29: Contact pressure profile comparison with Sauren. (a) Sauren [122], (b) preliminary knee model.
Figure 5.30: Meniscal load transmission percentage comparison with Sauren. (a) Sauren [122], (b) preliminary knee model.
Figure 5.31: Percentage of contact area versus applied load for preliminary knee model geometry.

to check the assumptions of these earlier experimental studies. Admittedly, contact area calculations are extremely sensitive to joint geometry, but the gross differences between the meniscal percentage of transmitted load and the meniscal percentage of total contact area (see Figures 5.30 and 5.31) suggest caution should be exercised when making claims about meniscal load bearing from contact area measurements.

As a final measure of the model validity, isoband plots for pressure and effective stress were generated (see Figures 5.32 and 5.33). The bands are distinguishable with minor discontinuities. Of particular interest is the stress concentration caused by the corners of the inner radius of the meniscus. These stress concentrations have no physiologic basis because the normal meniscus has a tapered inner radius.
Figure 5.32: Isoband pressure plot of preliminary knee model for Case 1.
Figure 5.33: Isoband effective stress plot of preliminary knee model for Case 1.
5.4 Anisotropic Axisymmetric Knee Model

Sauren's work was the first FEM attempt to model meniscal force transmission in the knee. His model demonstrated that the meniscus does transmit a significant percentage of the joint load. Unfortunately, major deficiencies in the model detract from the quantitative results. The primary objectives of the anisotropic axisymmetric knee model were to obtain results representative of actual physiologic and pathologic states of the knee joint. In accordance with these objectives, three major additions to Sauren's model are incorporated:

1. Articular cartilage is added to the tibial and femoral contact surfaces;

2. The meniscus is modeled as an anisotropic material;

3. The meniscal geometry is roughly triangular with a concave proximal surface and a tapered inner edge.

While developing a model, it is important to justify major assumptions and parameter choices. Although disagreement may exist regarding a specific assumption, full disclosure provides the only environment for the discussion of results.

The major geometric assumption is modeling the knee in full extension as an axisymmetric joint. The femoral condyle is spherical, the tibial plateau is planar, and the meniscus is a triangular-shaped annulus. The meniscus is assumed to be axisymmetric so a two-dimensional finite element analysis could be used without eliminating the important circumferential stiffness of the meniscus. Given the meniscal attachments are sufficiently stiff to prevent meniscal splaying (greater radial expansion at the attachments than the central region), the axisymmetric assumption is reasonable.

The respective inner and outer radii of the meniscus are 6 mm and 18 mm, resulting in a meniscal width of 12 mm. The thickness of the meniscus is chosen to provide initial contact between the meniscus and femur and is equal to 6 mm. These dimensions are in agreement with published anatomic data. Noble [108] reported
dimensions on eighteen pairs of normal menisci. The average width and thickness of
the medial meniscus are approximately 13 mm and 7 mm, respectively.

The dimensions of the femur and tibia were taken from Sauren, except that the
tibial length was increased from 10 mm to 15 mm. The tibial length was increased to
insure that the distal tibial boundary condition did not affect the tibial subchondral
bone stress distribution. The femur and tibia have equal radial widths of 20 mm. The
model is axisymmetric so the anterior/posterior and medial/lateral dimensions are 40
mm diameters. Mensch and Amstutz [100] presented dimensions from 30 cadaveric
knees. The respective average widths (M/L) of the medial tibial plateau and the
medial femoral condyle are 30.9 mm and 26.6 mm and the respective average depths
(A/P) of the medial tibial plateau and medial femoral condyle are 48.9 mm and 64.4
mm. The chosen tibial and femoral dimensions are rough averages of the medial
compartment widths and depths.

Kurosawa et al [86] measured the radii of curvature of the posterior region of
the femoral condyles in the frontal and sagittal planes and found that it could be
modeled as spheres. However, the anterior region of the femoral condyles is the region
of interest for full extension. Kurosawa [86] reports the average frontal plane radius
of the medial condyle is 21.4 mm for 0° flexion and Mensch [100] reports the average
anterior medial condylar radius in the sagittal plane is 37.5 mm. These results
are consistent with the ranges reported by Walker and Hajek [149] (see Table 5.6).
Modeling the medial femoral condyle as a sphere with a 30 mm radius is reasonable.
The tibial values reported by Walker and Hajek indicate slight tibial concavity and
convexity, but these values appear negligible relative to the femoral curvature. The
tibial plateau is modeled as planar.

As mentioned above, femoral and tibial articular cartilage layers are incorporated
in the model. The cartilage thickness is important because the cartilage distributes
the contact stresses. Walker and Hajek [149] report the cartilage thickness is 2.3 mm
for both the medial femoral condyle and medial tibial plateau (measured at the center
Table 5.6: Radii of curvature of femoral condyles and tibial plateaus, measured at contact area, for various flexion angles of one specimen. Positive implies convex, negative concave. [149]

<table>
<thead>
<tr>
<th></th>
<th>Sagittal plane (A/P) (mm)</th>
<th>Frontal plane (M/L) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medial femoral condyle</td>
<td>17– 24</td>
<td>18– 33</td>
</tr>
<tr>
<td>Medial tibial plateau</td>
<td>-400– -50</td>
<td>-200– 20</td>
</tr>
<tr>
<td>Lateral femoral condyle</td>
<td>14– 29</td>
<td>16– 29</td>
</tr>
<tr>
<td>Lateral tibial plateau</td>
<td>-50– 200</td>
<td>-250– 20</td>
</tr>
</tbody>
</table>

of the contact area for full extension, one subject). Hall and Wyshak [62] performed a radiographic study of knee cartilage thickness. The mean maximum thickness of the medial femoral condyle is 4.0 mm, but divergence of the X-ray beam magnifies the actual cartilage thickness so the reported values are overestimates. Articular cartilage layer thicknesses of 2 mm were chosen. This cartilage thickness is identical to the values used by Askew and Mow [14] and Galbraith and Bryant [57].

The material properties of the knee model are taken from the literature. Goldstein [60] provides a review of trabecular bone properties and their dependence on anatomic location. In particular, elastic moduli of the distal femur and proximal tibia are reported. The range of values is extremely broad, but in general the subchondral bone of the distal femur is stiffer than that of the proximal tibia. The particular elastic modulus of the femur was chosen from a range of 60 MPa to 3000 MPa reported by Ducheyne et al [44]. These values were selected because the experiments were conducted using a high loading rate of 440 cm/min. The femoral subchondral bone is modeled as an isotropic elastic material with an elastic modulus of 1000 MPa and a Poisson’s ratio of 0.3. Ashman et al [13] provides an extensive review of the elastic moduli of the proximal tibia, including orthotropic moduli. However, the tibial subchondral bone is assumed to be isotropic with an elastic modulus of 500 MPa and
Poisson's ratio of 0.3.

The elastic parameters of articular cartilage are the subject of considerable controversy in the literature. Articular cartilage is a poroelastic material with complex physicochemical interactions [46, 53, 54], but at loading rates of interest (walking and running), the fluid flow in the tissue is not significant and cartilage can be modeled as a purely elastic solid. This assumption is discussed by Brown [26]. The articular cartilage is modeled as an isotropic elastic material with an elastic modulus of 12 MPa and a Poisson's ratio of 0.45 (at rapid loading rates, cartilage is nearly incompressible). These values are taken from Galbraith and Bryant [57] and Askew and Mow [14] and are originally reported by Hayes et al. [66]. Kempson [78] presents similar values for the elastic modulus of cartilage (range equals 8.4 MPa to 15.3 MPa). These values may actually underestimate the dynamic elastic modulus of articular cartilage because Sah et al. [121] reports a modulus of 37 MPa at strain rates of 5 %/s.

The material properties of the meniscus are extensively reviewed in Chapter 3. Therefore, only a brief summary of the pertinent parameters is presented. The meniscus is modeled as a transversely isotropic elastic material. As such, five independent elastic parameters determine the stiffness tensor. The meniscus is assumed to be isotropic in the plane orthogonal to the circumferential direction. The elastic compressive modulus in the axial and radial directions is taken from preliminary measurements and is 15 MPa. The circumferential tensile modulus is 120 MPa. This value is obtained by averaging the data presented by Fithian [50] for different regions of the medial meniscus. The shear moduli are assumed to be identical and equal to 0.2 MPa. The values are taken from Anderson [9]. The Poisson's ratios must satisfy the constraints in Appendix A, but no experimental measurements are available. Values are simply specified as \( \nu_{rz} = 1/2 \) and \( \nu_{r\theta} = \nu_{z\theta} = 1 \) and the thermodynamic constraints are satisfied.

\[
-2.83 < \nu_{r\theta} < 2.83 \tag{5.24}
\]
\[ -1 < \nu_{zz} < 0.75 \]  

Arguably, the most difficult assumption to justify is the applied loading condition. A knee can be dissected and geometry measured to within reasonable accuracy. Tissue specimens can be sectioned and tested to determine material properties. Not that geometric measurements and material characterization are trivial, but physiologic loading must be measured in vivo. The only available data are estimates of the forces transmitted by the knee joint. Two fundamentally different approaches to modeling diverge at this junction. One approach uses the knee force estimates to load cadaveric knees and measure the contact pressures in the knee joint (e.g. the femoral contact pressures). Then only the tibia and menisci need be modeled and the femoral contact pressures can be applied as distributed pressure boundary conditions. This is the approach taken by Tissakht et al in their three-dimensional knee model [144, 143]. The alternative approach models the entire knee (femur, tibia, and meniscus) and applies force loading conditions. Although computationally more expensive, this approach circumvents errors inherent in the experimental pressure measurements and presumably results in more “natural” contact pressures and results. The latter approach is the one selected for the two-dimensional axisymmetric knee model.

The range of applied knee forces is taken from estimates in the literature. Morrison [103, 104, 105] reports the maximum knee joint force during walking to be 3 times body weight (average value of 12 subjects) and the greater portion of the load was transmitted by the medial condyles. Harrington [65] calculated the average maximum bearing force transmitted at the knee was 3.5 times body weight and the center of joint pressure for all normal subjects was located in the medial compartment throughout most of the stance phase. Cheng [33]\(^4\) estimated the maximum knee contact force to be 4.3 times body weight (\(BW\)) with 3.0 \(BW\) on the medial condyle. Fijian [49] presents a similar value of 4.5 \(BW\) for the maximum joint contact force with 2.5 \(BW\) on the medial condyle. These predictions probably underestimate the actual

\(^4\)As reported by Fijian [49].
maximum contact force because it is difficult to include co-contraction. A maximum applied load of 3.0 \( BW \) was used. Assuming an average male of 70 \( kg \) mass, 3 \( BW \) is roughly equal to 2100 \( N \).

The actual finite element knee model is shown in Figure 5.34. The FEM model consists of the femur with articular cartilage, the tibia with articular cartilage, and the meniscus. The boundary conditions are the same as for the preliminary knee model described in the previous section. The meniscal surfaces are frictionless and a soft truss fixes the meniscus to ground to prevent rigid body dislocations. The truss carries a negligible percentage of the load.

An additional truss is placed in parallel with the articular cartilage layers along the axis of symmetry (see Figure 5.35 with the cartilage removed). The truss is added to provide stability at initial contact. Contact solutions are iterative and use an initial contact stiffness to estimate an initial femoral displacement. At the first time step, tibial and femoral articular surfaces are engaged in point contact with a relatively low stiffness. The initial displacement results in gross overlap of the contact elements and the solution is aborted. The truss increases the effective contact stiffness and reduces the initial displacements so the solution converges. Once additional contact elements were engaged, the truss was removed and the applied load increased. The truss was removed at a load of 50 \( N \) so reported results do not include the truss stiffness.

The described anisotropic finite element knee model represents a normal knee (Case 1) and forms the basis with which to compare subsequent models. Cases 2 and 3 represent partial and total meniscectomies, respectively. Cases 4 (intact meniscus) and 5 (total meniscectomy) simulate degenerative changes in the knee with articular cartilage thinning. Case 6 is a control to demonstrate the effects of an isotropic meniscus. Cases 7 (Figure 5.36) and 8 (Figure 5.37) provide different meniscal geometries. The different cases are summarized in Table 5.7.

Finite element models provide the investigator with copious amounts of data, so only results directly related to the meniscal load-bearing role are presented. The eight
Figure 5.34: Anisotropic axisymmetric finite element model of normal knee (Case 1).
Figure 5.35: Model of normal knee with cartilage removed to demonstrate stabilizing truss (Case 1).
Figure 5.36: Anisotropic axisymmetric finite element model of normal knee with initial line contact between femur and meniscus (Case 7).
Figure 5.37: Anisotropic axisymmetric finite element model of normal knee with 1 mm initial gap between femur and tibia (Case 8).
Table 5.7: Case descriptions for two-dimensional axisymmetric knee models.

<table>
<thead>
<tr>
<th>Case</th>
<th>Cartilage thickness (mm)</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>Partial meniscectomy</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>Total meniscectomy</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>Total meniscectomy</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>Isotropic meniscus</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>Meniscofemoral line contact</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>1 mm gap between femur and tibia</td>
</tr>
</tbody>
</table>

cases described above have been classified into three groups:

1. Cases 1-5 are used to examine the load-bearing role of the meniscus in normal and pathological knees.

2. Cases 1 and 6 demonstrate the effects of including anisotropic meniscal material properties.

3. Cases 1, 7, and 8 provide a sensitivity analysis of the meniscal geometry.

The overall joint stiffness (load-displacement curve), meniscal radial stiffness, tibial contact pressure distribution, and tibial subchondral bone von Mises stresses (below the cartilage surface and along the axis of symmetry) are compared for each group at a load of 1.5 \( BW \) (1050 \( N \)). The results are presented graphically in the next several pages. A discussion of the results is included in Chapter 7 so only a few major points are noted here.

The normal and pathologic models demonstrate increases in the tibiofemoral contact pressures and subchondral bone stresses as the meniscus is removed and as the cartilage thins. The magnitude of the contact pressures and stresses are significantly
lower than the results obtained with the preliminary model, in which no articular cartilage was present.

The differences between the isotropic and anisotropic meniscus are most obvious in the meniscal radial stiffness plot (Figure 5.44) and percentage of the load transmitted by the meniscus.

The third group of cases examines the effects of meniscal geometry, or more specifically, the initial meniscofemoral contact and tibiofemoral spacing. A 1 mm tibiofemoral gap was originally reported by Walker [148] and has been observed by others [133, 126, 55, 106]. This spacer effect significantly shifts the load-displacement curve to the left (see Figure 5.48).
Figure 5.39: Meniscal radial stiffness for normal and pathological knees at 1.5 $BW$.

Figure 5.40: Tibial contact pressure distribution for normal and pathological knees at 1.5 $BW$.  

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Figure 5.41: Subchondral bone von Mises stresses immediately below articular cartilage surface for normal and pathological knees at 1.5 BW.

Figure 5.42: Subchondral bone von Mises stresses along axis of symmetry for normal and pathological knees at 1.5 BW.
Figure 5.43: Overall joint stiffness for knees with anisotropic and isotropic menisci at 1.5 BW.

Figure 5.44: Meniscal radial stiffness for knees with anisotropic and isotropic menisci at 1.5 BW.
Figure 5.45: Tibial contact pressure distribution for knees with anisotropic and isotropic menisci at 1.5 \( BW \).

Figure 5.46: Subchondral bone von Mises stresses immediately below articular cartilage surface for knees with anisotropic and isotropic menisci at 1.5 \( BW \).
Figure 5.47: Subchondral bone von Mises stresses along axis of symmetry for knees with anisotropic and isotropic menisci at 1.5 \( BW \).
Figure 5.48: Overall joint stiffness for knees with various meniscal geometries at 1.5 BW.

Figure 5.49: Meniscal radial stiffness for knees with various meniscal geometries at 1.5 BW.
Figure 5.50: Tibial contact pressure distribution knees with various meniscal geometries at 1.5 BW.

Figure 5.51: Subchondral bone von Mises stresses immediately below articular cartilage surface for knees with various meniscal geometries at 1.5 BW.
Figure 5.52: Subchondral bone von Mises stresses along axis of symmetry for knees with various meniscal geometries at 1.5 BW.
Chapter 6

Experimental Methods and Results

Mathematical models can be excellent analytical tools, but to assume these models reproduce physiologic reality is unwarranted without experimental verification. Experimental reproduction of mathematical predictions validates the model; the confidence in the model is related to the number of independent experimental verification results. Therefore, it was important to develop an experimental method capable of validating the axisymmetric finite element knee model.

Review of the knee literature revealed numerous experimental techniques for examining knee load transmission and the load-bearing role of the meniscus. These techniques can be classified into the following general categories:

1. Contact area measurements provide an indirect estimate of the relative load distribution. Average contact stresses can be calculated by dividing the applied load by the measured contact area. Measurement techniques include forming joint space casts with polymethylmethacrylate (Walker et al [149, 148] or silicone rubber (Seedhom and Tsubuku [130], Fukubayashi and Kurosawa [55], and Kurosawa et al [85]), radiographic contrast studies (Kettelkamp et al [79] and Maquet et al [94, 95]), staining and dye exclusion studies (Krause et al [84] and Seedhom and Hargreaves [129]), and pressure sensitive film (Baratz et al [18], Rieger et al [120], and Newman et al [106]).
2. Contact pressure measurements provide direct measurements of meniscal load-bearing. Pressure sensitive film (Fukubayashi and Kurosawa [55], Baratz et al [18], and Riegger et al [120]) and a micro-indentation transducer (Ahmed and Burke [5, 3, 4]) were used for static measurements. Pressure transducers (Walker and Erkman [148, 150] and Brown and Shaw [25]) were used for dynamic measurements.

3. Differences in load-displacement curves with and without the meniscus quantify the load transmitted by the meniscus (Seedhom et al [127, 128, 124, 125, 126, 129, 64], Shrive et al [132, 133], Walker and Erkman [148], Krause et al [84], Kurosawa et al [85], and Newman et al [106]).

4. Photoelastic stress studies of planar knee models depict femoral and tibial stress patterns (Chand et al [32], Maquet [95], Nishizaki et al [107], and Radin et al [118]).

5. Meniscal load-displacement measurements demonstrate the meniscal stiffness. Specially-designed displacement transducers (Krause et al [84]), radiographic studies with radiopaque markers (Bylski et al [28, 29, 30, 37]), and magnetic resonance (MR) imaging techniques (Thompson et al [141]) have all been used.

6. Meniscal strain measurements provide estimates of meniscal stresses. Strains can be calculated from circumferential displacements (Krause et al [84]) or from direct observation of the meniscal surface viewed through an acrylic femur (Bylski et al [28, 29, 30, 37]).

7. Tibial strain measurements provide an estimate of the tibial stress state (Bourne et al [22]).

Each of these techniques has merits and drawbacks, but the overwhelming characteristic that struck the author upon reviewing these studies was the invasive and disruptive nature of the approaches. The majority of the methods disarticulated the
knee joint and/or placed transducers and markers into the joint space. Disarticulating the joint removes the natural ligamentous and capsular constraints and inevitably alters the knee kinematics. Placement of objects within the joint space destroys the physiologic joint lubrication mechanisms because the thickness of the transducers and markers is orders of magnitude greater than the lubricating film thickness. Additionally, the foreign objects can impinge on the meniscus and articular cartilage and prevent normal motion of the structures. The non-invasive methods also have problems. Radiographic contact area measurements are relatively inaccurate because surrounding bone attenuates the X-ray beam and blurs the contact margin. Model predictions of contact area are very sensitive to model geometry, resulting in a wide range of model predictions. Total joint stiffness measurements can be used to calculate the percentage of load transmitted by the meniscus (see Seedhom [126] or Chapter 4), but the method assumes that the knee can be loaded in the same manner with and without the meniscus. Fixing the knee so the loading kinematics are repeatable is counter to the concept of passive equilibrium during loading, so this method is rejected.

These concerns led to the development of a less disruptive (but still invasive) experimental verification technique. Biplanar X-ray stereophotogrammetry was used to measure the motions of small radiopaque markers in the body of the meniscus while the knee was loaded. Because the markers were placed within the body of the meniscus, the natural lubrication of the articulating surfaces was preserved. Only one of the major ligaments was transected (anterior cruciate ligament) and the majority of the joint capsule contiguous with the menisci was intact. Consequently, the meniscal displacement and strain measurement method minimized disruption of the joint.

6.1 Loading Apparatus

Numerous investigators have designed knee loading apparatuses with a wide range of complexity. The specifics of each design are related to the application and constraints,
but some a review of the different designs defines important concepts. The loading apparatuses can be classified into two groups: (1) loading fixtures adapted to Instron (Walker et al [149, 148] and Jaspers et al [72, 41]) or MTS (Markolf et al [96] and Sullivan et al [138]) material testing equipment; or (2) self-contained loading devices that do not rely on external load-generating equipment (Ahmed et al [6, 5], Seedhom [126], Bylski et al [29, 30], and Blankevoort et al [20]). This is not an exhaustive set of references, but provides a spectrum of the different designs.

6.1.1 Design Criteria

The approach of designing a loading fixture for an MTS machine was chosen because of its reduced cost and complexity compared to designing a self-contained device. The loading case of interest was axial compression of the fully-extended knee and did not warrant designing a special apparatus. In normal gait, the peak knee loads occur during stance phase while the knee is near full extension. Because the purpose of the experimental study was to verify the finite element model predictions, the apparatus design was limited to knee loading in full extension.

A 5000 N design was selected. According to Humanscale [43], the 95th percentile of the U.S. male population has a mass of 100 kg or approximately 1000 N. Assuming an upper limit of the peak knee contact force to be 5 times body weight (BW) [105, 49], the design load is 5000 N.

The most critical design features are the kinematic degrees of freedom. Current assumptions suggest the knee attains a state of passive equilibrium in vivo. In vivo, the knee is constrained by the contact surfaces and soft tissues (ligaments, menisci, joint capsule, etc.) and when the knee is loaded by the muscles forces, the knee seeks an equilibrium position subject to these passive constraints [70, 20]. Consequently, under in vivo loading conditions, it is important that the knee be allowed to attain a passive equilibrium position and that the knee not be fixed in rigid fixtures which confine the knee to a nonphysiologic configuration. Of the six degrees of freedom (DOF)
of the knee, only two DOF should be constrained under a pure axial load. Axial
displacement of the joint is determined by the applied axial load because causality
is such that both the load and displacement can not be independent parameters.
Additionally, the flexion/extension angle must be fixed so the knee is stable. The
remaining DOF should remain unconstrained and include:

1. internal/external rotation;
2. varus/valgus rotation;
3. anterior/posterior (A/P) displacement;
4. medial/lateral (M/L) displacement.

The following geometric constraints were imposed:

1. The apparatus is to be used in conjunction with a MTS materials testing ma-
   chine equipped with 1"-14 female-threaded holders to attach the apparatus to
   the machine.

2. The femur and tibia are mounted by cementing (grouting) 0.375 in threaded
   rods into each intramedullary canal. The femur and tibia are to be sectioned 15
   cm above and below the joint space, respectively. The overall specimen length
   is approximately 30 cm plus 2–3 cm of threaded rod extending from each end.
   The femoral fixture should accommodate the 10° average angle of obliquity of the
   femur [113].

3. The X-ray photogrammetry system will use two oblique projections and the
   loading apparatus must not mask the view of the reference points or the mark-
   ers. The X-ray photogrammetry reference frame is mounted on the loading
   apparatus.
6.1.2 Loading Apparatus Design

The loading apparatus was designed with modular components. This approach was taken to minimize the drawbacks of unforeseen problems. In the event one of the components failed, that component could be redesigned and manufactured without requiring a redesign of the entire system. Also, if a particular kinematic DOF resulted in knee instability, that component could be removed. The final design consisted of four components: the ball bearing thrust plate assembly, the tibial fixture, the femoral fixture, and the clevis joint. Figure 6.1 shows the components mounted in the MTS machine.

The ball bearing thrust plate assembly consists of a ball bearing thrust plate with two hardened race plates with holders. Slightly oversized hemispherical holes were countersunk in the two brass plates. Ball bearings were placed in the countersunk holes and the brass plates were fastened together with machine screws. The brass plates act as a bearing cages and the ball bearings protrude from both plate surfaces and are free to rotate in the oversized holes. Sliding friction exists between the ball bearings and the race, but the friction is negligible under load. The bearing races are constructed of hardened steel with a ground finish. The bearing race holders are machined from aluminum stock. The upper race diameter is 4 in and the lower race diameter is 6 in, so the assembly has 1 in of travel in the A/P and M/L directions. The lower race holder has four symmetrically placed thumb screws that limit the travel of the upper race if desired. The position of the upper race can also be fixed if desired. A superior view of the assembly with the upper race and holder removed is presented in Figure 6.2. The ball-bearing thrust plate assembly allows three kinematic DOF: internal/external rotation, A/P translation, and M/L translation.

The tibial fixture is a rod with 1\(^{\text{\textquoteleft\text{--}}}^{14}\) male threads on the inferior segment to engage the superior race holder or the MTS holder, and 3/8\(^{\text{\textquoteleft\text{--}}}^{16}\) UNC female threads in the superior segment to holder the cemented tibial rod. A 1\(^{\text{\textquoteleft\text{--}}}\) shoulder is machined in the middle segment to mate with a corresponding hole in the X-ray photogrammetry
Figure 6.1: Loading apparatus with four components and knee mounted in MTS machine.
Figure 6.2: Superior view of thrust plate assembly with upper race and holder removed.
reference frame and clamp the frame between the tibial fixture and the upper race holder.

The femoral fixture consists of a block and channeled brace. The block has a 3/8"-16 UNC threaded hole to hold the femoral threaded rod. The axis of the hole is tilted 10° from vertical in the coronal plane to accommodate the femoral obliquity. The block can be adjusted in the M/L direction to initially align the joint and then the block is fixed in place with cap screws.

The clevis joint is simply a hinge. Roller bearings are mounted in both sides of the inferior clevis bracket to reduce friction in the joint. The shaft axis is oriented in the A/P direction to allow varus/valgus rotations. As long as the line of action of the axial load remains between the femoral condyles, the knee should remain stable.

6.2 X-Ray Stereophotogrammetry

X-ray stereophotogrammetry is an established research tool in the field of biomechanics in general (McNeil [98], Selvik [131], Takamoto [140], and Brown et al [24]) and in the knee specifically (van Dijk et al [146], de Lange et al [42], Huiskes et al [69], and Meijer et al [99]). X-ray photogrammetry was selected as the experimental method for measuring meniscal marker motions because of its known accuracy (RMS errors of 0.1 mm or less for relative motions [90]). Derivation and explanation of the X-ray stereophotogrammetry methods are presented in Appendix B. This section is concerned with the specific design and application of these methods for the measurement of meniscal displacements and strains.

6.2.1 Imaging Constraints and System Design

The imaging constraints of the design are primarily concerned with the projections necessary for the reconstruction of the three-dimensional marker positions. Standard X-ray stereophotogrammetry methods project two images onto the same film. This would require superior/inferior projections through the transverse plane of the knee.
joint. Superior/inferior projections are impractical because the X-ray film would need to be located in line with the loading apparatus and the metal components of the loading apparatus and MTS machine would shield much of the view. Consequently, biplanar X-ray stereophotogrammetry methods were considered. Biplanar methods use two projections onto two separate X-ray films. Because the meniscal markers are in the transverse plane of the knee, anterior/posterior and medial/lateral projections cannot easily differentiate the different markers, so oblique projections were chosen. Both projections are in a sagittal or coronal plane with the principal direction of the projection tilted 45° with respect to the transverse plane. Therefore, if the meniscal markers are placed in a roughly elliptical pattern, the projected markers will be in a distorted elliptical pattern that allows easy differentiation of the individual markers.

In addition to the projection constraints, the object space of the markers had to be calibrated. Calibration was carried out by imaging a frame of control points. A direct linear transform (DLT) method was selected for coordinate point reconstruction. The control points had to fill the entire viewing volume because the DLT method is sensitive to extrapolation errors [158].

The X-ray stereophotogrammetry frame design consists of an aluminum frame to hold the X-ray cassettes and four acrylic windows embedded with 1 mm steel balls and 1/32 in metal wires as control points. The view of the frame through these windows is shown in Figure 6.3. The frame is mounted on top of the bearing thrust plate to minimize the viewing volume and increase the relative accuracy of the system. The control points allow each projection to be calibrated.

The X-ray films are digitized by hand. The films are placed on the digitizing tablet (Scriptel RDT Transparent Glass Digitizer, Columbus, OH) supported by a light table. Each image was digitized 5 times and the average coordinates used for further processing.
Figure 6.3: View of X-ray photogrammetry frame through projection windows. The intersections of the wires serve as control points for calibration of each film.
Table 6.1: Results of X-ray stereophotogrammetry calibration study.

<table>
<thead>
<tr>
<th>Ref. Dist. (mm)</th>
<th>No. of Meas.</th>
<th>Mean Error (mm)</th>
<th>Std. Dev. Error (mm)</th>
<th>Std. Error Mean (mm)</th>
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</thead>
<tbody>
<tr>
<td>7.83</td>
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<td>-0.016</td>
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<td>0.013</td>
</tr>
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<td>10.00</td>
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</tr>
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<td>0.028</td>
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<tr>
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<td>9</td>
<td>-0.146</td>
<td>0.054</td>
<td>0.018</td>
</tr>
</tbody>
</table>

6.2.2 System Calibration and Resolution

The accuracy of the system was tested with a precisely machined calibration array. The array pattern consisted of two pairs of concentric circular arcs with 1 mm steel balls placed at 15 degree intervals along the arcs and one ball was placed at the center of the arcs. The radii of the inner and outer arcs are 30 mm and 40 mm, respectively. The calibration array was imaged within the X-ray stereophotogrammetry frame, the X-ray films were digitized, and the three-dimensional coordinates of each marker were calculated. The array was designed to provide multiple measurements of specific relative distance. The results of the calibration tests are given in Table 6.1. For measurements of the relative distance between markers, the results show that the system has a resolution of approximately 0.1 mm (the markers are spaced approximately 10 mm apart).

Error propagation techniques can be used to calculate the resolution of the circumferential strain measurements. The equation for calculating the circumferential strain is

$$\varepsilon = \frac{l - l_0}{l_0}$$

(6.1)

Using the error propagation methods described by Mischke [101], the error in the
strain measurements is equal to

\[
\sigma_z^2 = \left( \frac{\partial \epsilon}{\partial l} \right)^2 \sigma_l^2 + \left( \frac{\partial \epsilon}{\partial l_0} \right)^2 \sigma_{l_0}^2 + \frac{1}{2} \left( \frac{\partial^2 \epsilon}{\partial^2 l_0} \right)^2 \sigma_{l_0}^4
\]  

(6.2)

which simplifies to

\[
\sigma_z^2 = \left( \frac{\sigma_l}{l_0} \right)^2 + \left( \frac{\sigma_{l_0}}{l_0} \right)^1 + 2 \left( \frac{\sigma_{l_0}}{l_0} \right)^4
\]  

(6.3)

for small strains \((l \approx l_0)\). The standard deviations of \(l\) and \(l_0\) are approximately 0.1 m and \(l\) and \(l_0\) are approximately 10 m, so the standard deviation of the circumferential strain measurements is approximately 1.4\% strain.

### 6.3 Specimen Preparation

Two knee specimens were tested (bilateral knees from a 75 y/o female, 66 kg). The specimens were frozen immediately after being harvested and thawed for experimental preparation and testing. All soft tissue was removed from the knees to the joint capsule layer. The tibial and femoral medullary canals were reamed and 3/8 in threaded stainless steel rod was cemented into each canal with polymethylmethacrylate (PMMA). The joint was opened by cutting the joint capsule on the medial and lateral sides of the patella and the patella was removed. The anterior cruciate ligament was cut to provide sufficient space within the joint to place 5–6 1 m钢 balls around the peripheral margin of each meniscus with roughly 1 cm spacing between the balls. Care was taken to avoid damaging the anterior attachments of the menisci. Additionally, three straight pins were inserted into the tibia to measure tibial displacements. The knee was mounted in the loading apparatus by threading the tibial and femoral cemented rods into their respective holders. Figure 6.1 shows the experimental set-up with a mounted knee.

### 6.4 Results

The objective of the experimental study was to verify the anisotropic axisymmetric knee model predictions of meniscal displacements and circumferential strains. Al-
though two knees were tested, the results of the right knee are subject to qualification due to mounting problems.

6.4.1 Load-Displacement Measurements

The marker positions were calculated first by digitizing the films and reconstructing the marker coordinates using the DLT methods in Appendix B. The meniscal markers roughly form a plane on the tibial plateau so the marker coordinates were projected onto a “best-fit” plane (see Appendix C). The displacements of the markers are plotted for the initial preload up to 2 BW and 3 BW for the left and right knees, respectively (see Figures 6.4 and 6.5). The magnitudes of the marker displacements are approximately 5 mm and 10 mm for the left and right knees, respectively.

The displacements of the meniscal markers relative to the tibia were calculated. The centroid of the tibial markers was determined for each load case. Subtracting the centroidal position from each of the meniscal markers positions yielded the relative displacements. The relative displacements were much smaller than the absolute displacements. Relative displacements of the left medial meniscus markers were all less than 1 mm (see Figure 6.6). Relative displacements of the right medial meniscus markers were slightly greater (see Figure 6.7).

6.4.2 Load-Strain Measurements

The meniscal circumferential strains were calculated by two different methods. The first method calculated the spatial average circumferential strain by dividing the change in distance between two markers by the original distance. Using this method, the meniscal circumferential strains were 1.4 ± 0.6% strain and 2.1 ± 0.5% strain for loads of 1 BW and 2 BW, respectively.

The second method involved fitting a circle to the markers and dividing the change in radius by the original radius. Figure 6.8 shows the “best-fit” circle to the left medial meniscus markers. The radius of the circle for the initial preload is 15.6 mm. The
Figure 6.4: Meniscal and tibial marker displacements of left knee under load. (+ preload, o 2 BW)
Figure 6.5: Meniscal and tibial marker displacements of right knee under load. (+ preload, o 3 BW)
Figure 6.6: Relative meniscal marker displacements of left knee under load. (+ preload, ○ 2 BW)
Figure 6.7: Relative meniscal marker displacements of right knee under load. (+ preload, o 3 BW)
calculated circumferential strains are 3.8% strain and 4.5% strain for loads of 1 BW and 2 BW, respectively. The previous strain estimates are probably more reliable because there is no reason the meniscal markers should form a circle.

The circumferential strains of the right knee are not discussed because of the mounting difficulty mentioned previously.
Figure 6.8: "Best-fit" circle to left medial meniscus markers for calculating circumferential strain.
Chapter 7
Discussion and Conclusions

7.1 Importance of Modeling Contributions

Modeling is a never ending process by its very nature. Modifications are always imaginable even if they can not be feasibly incorporated into the model. Consequently, it is important to determine whether or not model modifications significantly affect the model predictions and all first-order effects should be incorporated into the model before higher order effects are added (usually at considerable expense in terms of complexity). Although models can always be improved, it is important to use the simplest model that explains the observed phenomena and does not obscure the results with unnecessary complexity. The modifications incorporated into the anisotropic axisymmetric finite element knee model are significant and provide insight into load transmission in the knee.

7.1.1 Articular Cartilage Layer

Weightman and Kempson [153] suggest one role of articular cartilage is to distribute contact stresses, protecting the subchondral bone. Articular cartilage provides a compliant layer that deforms and increases the contact area to distribute the contact forces. The end result is a significant reduction in the peak contact pressures experienced by the joint and a concomitant reduction in subchondral bone stresses.

Sauren's finite element knee model did not include articular cartilage so the
tibiofemoral contact was direct bone-to-bone contact. The peak bone contact pressure is approximately 25 MPa for a 1000 N load. The anisotropic axisymmetric finite element knee model includes 2 mm articular cartilage layers on the femur and tibia which reduce the peak contact stress to 3.3 MPa for the same axial load. The articular cartilage produces an order of magnitude reduction in the peak contact pressure (see Figure 7.1a and 7.1b). Also of note is the order of magnitude difference between the tibiofemoral peak contact pressure and the meniscotibial peak contact pressure in Sauren's model while inclusion of articular cartilage layers reduce the difference to factor of two. The cartilage concomitantly reduces the peak von Mises stress in the subchondral bone. The peak von Mises stresses for Sauren's model and the anisotropic knee model are 17.4 MPa and 2.8 MPa, respectively.

7.1.2 Anisotropic Material Properties

The meniscus is a fiber-reinforced composite material. A 100 μm randomly oriented collagen layer covers the meniscus and the body of the meniscus is primarily composed of circumferentially oriented collagen fibers. The material properties of the meniscus have been measured (see Chapter 3) and are notably anisotropic. The circumferential tensile modulus and axial/radial compressive elastic moduli differ by an order of magnitude and assuming isotropic meniscal material properties is overly simplistic.\(^1\) The meniscus transmits load because the circumferential stiffness prevents the meniscus from being expelled from the joint space as the knee is loaded.

Results of incorporating anisotropic meniscal material properties into the knee model are most evident in the radial displacement of the meniscus and the percentage of the load transmitted by the meniscus. Case 6 of the knee model was added to elucidate the effects of an isotropic meniscus. The predicted meniscal radial displacements differ by an order of magnitude with the isotropic meniscus being extruded from the knee (see Figure 7.2). Calculations of the percentage of load transmitted by

\(^1\)In fairness to Sauren et al [122], experimental measurements of the meniscal material properties were not available when their model was developed.
Figure 7.1: Comparison of contact pressure profile for (a) Sauren's model without cartilage (Case 1); and (b) anisotropic knee model with cartilage (Case 1). 1000 N axial load.
Figure 7.2: Comparison of meniscal radial displacements for isotropic (Case 6) and anisotropic (Case 1) meniscal material properties.

The meniscus show that the anisotropic meniscus transmits approximately 30% more of the joint load than the isotropic meniscus on average (see Figure 7.3).

7.1.3 Meniscal Geometry

The meniscal geometry used by Sauren et al [122] was necessary to insure contact between the femur and meniscus. The femur would not engage the meniscus during loading because the relatively stiff bone-on-bone contact reduces the axial displacement of the femur so the meniscofemoral surfaces had to be conforming to develop significant contact pressures on the mensical surfaces.

While solving the conformity problem in one place, Sauren's model meniscal geometry induces significant nonphysiologic effects elsewhere. The inner edge of the
Figure 7.3: Comparison of percentage of load transmitted by meniscus for isotropic (Case 6) and anisotropic (Case 1) meniscal material properties.
meniscus is blunt with square corners. These corners result in nonphysiologic stress concentrations in the femur and tibial at the inner edge of the meniscus (see Figure 7.4). The blunt edge also prevents a portion of the exposed articular surface from contacting its opposing surface because the surfaces do not conform under load. By tapering the inner edge of the meniscus, the stress concentrations are eliminated (see Figure 7.5) and the surfaces of the meniscus, tibia, and femur can conform under load to distribute the contact force (see Figure 7.6).

7.2 Model Verification

To establish confidence in the anisotropic axisymmetric knee model, the model must be verified. Verification can be accomplished in a number of different ways.

One method is comparison with analytical solutions. The model was verified for the degenerate case of Hertzian contact by comparing the model predictions to the analytical Hertzian solution. Eberhardt et al [45] recently published an abstract describing an analytical solution for joint contact problems with articular cartilage included. The details of the solution are not yet available, but would provide a second analytical method to verify the model.

Unfortunately, the previously mentioned types of verification are circular in their logic because models are used to validate models and the basic geometric and material property assumptions are never tested. In vitro (or preferably in vivo) experimental studies of the knee must be used to independently verify the anisotropic axisymmetric knee model. The experimental component of this thesis involved measuring meniscal displacements and circumferential strains of a fully-extended cadaver knee under axial load to verify the model predictions. Additionally, experimental measurements of contact pressure and the percentage of the joint load transmitted by the meniscus were taken from the literature for verification.

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Figure 7.4: Von Mises stress band plot for Sauren's model (Case 1) for a 1000 N load.
Figure 7.5: Von Mises stress band plot for anisotropic knee model (Case 1) for a 2100 N load (tibia only).
Figure 7.6: Deformed mesh for Case 1 of anisotropic knee model.
7.2.1 Circumferential Strains

The experimental verification conducted as part of this thesis involved measuring meniscal displacements and circumferential strains during axial loading of the knee. Calculation of the radial displacement component of the markers was complicated by the actual knee kinematics, but the total displacements of the meniscal markers relative to the tibia for the first knee specimen were all less than 1 mm for loads up to 2 body weight (BW). The menisci are not extruded from the joint space and therefore transmit load.

Calculation of average circumferential strains was possible. The relative changes in length between markers divided by the original length provides an estimate of the spatial average of the circumferential strain between two markers. The spatial averages of several marker pairs were averaged to calculate an overall average circumferential strain of the medial meniscus. The calculated average circumferential strains of the medial meniscus are 1.4 ± 0.6 % strain and 2.1 ± 0.5 % strain for 1 BW and 2 BW, respectively. These results are consistent with the anisotropic knee model predictions for the outer region of the meniscus, where the meniscal markers were placed.

These findings are also consistent with measurements reported by Krause et al [84] for circumferential displacements of the meniscus (see Figure 7.8). Krause inserted a humpback displacement transducer into the peripheral margin of the meniscus and measured the circumferential displacement of the meniscus as the knee was loaded. The initial spacing of the displacement transducer was 0.5 in (1.27 cm) so 0.1 mm of circumferential displacement corresponds to 0.8 % circumferential strain. For curve A of Figure 7.8, the circumferential strain of the meniscus is approximately 1 % strain at 1 BW, which is the strain predicted by the knee model at the outer radius of the meniscus.
Figure 7.7: Predicted meniscal circumferential strains for Case 1 of anisotropic knee model. The experimental values are superimposed on the model predictions.

### 7.2.2 Contact Pressures

Previous models that examined tibial bone stresses assumed tibial contact pressure distributions and avoided solving the two body (or three body if the meniscus is included) contact problem (except Sauren). The loading assumptions have a significant effect on tibial stress predictions. It is preferable to assume the femoral loading and predict the contact pressure distribution by considering the three body contact problem. However, this approach is only valid if the predicted contact pressures are observed experimentally.

Brown and Shaw [25] measured knee pressure distributions using piezoresistive pressure transducers recessed in the femoral cartilage. The knee specimens were subjected to cyclic loads of 3-4 BW at 1 Hz. Series-averaged peak contact pressures for various flexion angles and for meniscectomies were reported (see Figure 7.9). The anisotropic knee model predictions of the peak contact pressure for Case 1 are superimposed on the figure and are consistent with the experimental results. However,
Figure 7.8: Circumferential displacements of the human medial meniscus. The average peripheral displacement of the menisci of three human knees tested at: A, zero flexion and 5 degrees of external rotation; B, 45 degrees of flexion and zero external rotation; C, zero flexion and zero external rotation. [84]
the model predictions following meniscectomy are inconsistent with Brown's findings. Brown did not find a significant difference in the peak contact pressure following meniscectomy and concluded the contact pressure increase reported by others (Fukubayashi and Kurosawa [55], Ahmed and Burke [5]) resulted from poroelastic behavior of the materials under static loading. Brown also hastened to point out that the experimental data should be interpreted as a "lower bound" of meniscectomy effects because of the resolution limitations for the small number of transducers. Brown's results probably do underestimate the contact changes following meniscectomy. Seedhom measured the percentage of load transmitted by the meniscus using similar loading rates (1.5–2.0 Hz) and found that the meniscus transmitted the majority of the knee load [129]. Seedhom's work and the entire body of experimental literature on the meniscus imply that the contact pressure does increase following meniscectomy and the anisotropic knee model predictions agree with that suggestion.

7.2.3 Percentage of Meniscal Load Transmission

The percentage of the load transmitted by the meniscus is a succinct indicator of the load-bearing role of the meniscus. It can be calculated by several different methods. Integration of the contact pressure profiles yields the force transmitted by the meniscus and the total joint force. The percentage of force transmitted by the meniscus is a ratio of the two forces. Alternatively, load-displacement curves can be used. For a given displacement, the force difference between the load-displacement curves with and without the meniscus is the force transmitted by the meniscus. At the same displacement, the total force is taken from the load-displacement curve with the meniscus intact. The percentage of load transmitted by the meniscus is a ratio of the meniscal force and total force.

Both methods described above were used to estimate the percentage of the load transmitted by the meniscus. For the model cases with an intact meniscus, the meniscus transmitted between 70–100% of the total load (see Figure 7.10). Following partial
Figure 7.9: Series-average peak contact pressure reported by Brown and Shaw [25]. The anisotropic axisymmetric knee model peak contact pressure predictions for Case 1 are superimposed on the figure (knee model predictions •).
Figure 7.10: Percentage of load transmitted by meniscus as a function of the applied load. Discrete points were calculated from integrated pressure profiles. Lines were calculated from differences in load-displacement curves.

meniscectomy, the percentage of the load transmitted by the meniscus decreased to 50% at higher loads. Seedhom and Hargreaves (1979) measured similar percentages experimentally [129] (see Figure 7.11), which are higher than previously reported values. Seedhom (1976) [125] reported the medial and lateral menisci transmit 40–50% and 65–70% of the total load, respectively, and Shrive et al (1978) [133] reported the menisci carried less than 60% of the total load. The more recent work by Seedhom entailed a more detailed analysis and consequently the higher percentages of load transmission are probably more reliable.
Figure 7.11: Medial and lateral meniscal loads as fractions of the loads acting on their respective sides. (a) Male, Age 63. (b) Male, Age 73 - the thickness of the medial meniscus was such that direct contact between femur and tibia did not occur even at high loads. (● medial meniscus, × lateral meniscus) [129]
7.3 Tibial Subchondral Bone Stresses

Models are useful for estimating information that is difficult to measure experimentally, such as tibial subchondral bone stresses. Radin [117, 116] has proposed that subchondral bone stiffening is an initiating mechanical factor in the pathogenesis of osteoarthritis. The anisotropic knee model provides a means of testing Radin’s hypothesis for the degenerative knee changes that follow meniscectomy.

The following is a brief description of the proposed mechanism by which meniscectomies cause osteoarthritis. Meniscectomies increase subchondral bone stresses. The subchondral bone adapts to the increased stresses by increasing its stiffness (Wolff’s law). The stiffened subchondral bone results in increased cartilage stresses. Through mechanisms that are not yet unknown, the increased cartilage stresses cause cartilage erosion and thinning. Reducing the cartilage thickness produces even greater subchondral bone stresses and positive feedback in the system results in continuing degeneration of the joint (i.e. an unstable system).

The modeling efforts in this thesis support at least two important causative links in Radin’s hypothesis. Increased subchondral bone stresses following meniscectomy are predicted. Figure 7.12 presents the peak von Mises stress versus applied load for the first five cases of the anisotropic knee model. The two cases with total meniscectomy (Cases 3 and 5) are subject to maximum von Mises stresses of 10 MPa at 3 BW and 2 BW, respectively. A survey of trabecular bone material properties by Goldstein [60] demonstrates that the strength of tibial subchondral bone is on the order of 10 MPa, so meniscectomy could result in microfracture of the subchondral bone. Microfractures can lead to bone sclerosis, which is seen radiographically as increased bone density. Many investigators have observed bone density increases radiographically following meniscectomy. Most recently, Odgaard et al [111] measured bone density changes at the proximal tibia after medial meniscectomy using quantitative computed tomography. They report that the bone density in the region of tibiofemoral contact is significantly higher in the meniscectomized knees as opposed
Figure 7.12: Peak von Mises stress versus applied load for Cases 1-5 of the anisotropic knee model.

to the contralateral intact knees for 5- and 10-year follow-up groups. The density difference is greater for total meniscectomies than for partial meniscectomies. The anisotropic knee model predictions are consistent with these experimental findings.

The second causal link supported by the knee model is that cartilage thinning increases subchondral bone stress. Cases 1 and 4 have cartilage layer thicknesses of 2 mm and 1 mm on the femoral and tibial contact surfaces, respectively, and the loss of 1 mm of cartilage from each surface results in a 1–2 MPa increase in the peak subchondral bone stress (see Figure 7.12).
7.4 Conclusions

The purpose of this research was to quantitatively examine the load bearing role of the human knee meniscus. A lumped-parameter three-spring model and an anisotropic axisymmetric finite element knee model were developed to accomplish this purpose. The three-spring lumped parameter knee model is the first such model to couple meniscal compressive and circumferential stiffnesses. Explicit relationships for the model stiffnesses were developed to understand the relevant material properties and geometries. A parametric study of the three-spring lumped parameter model predicted that the meniscus transmitted a significant percentage of the total joint load for physiologic stiffness ratios.

The anisotropic axisymmetric finite element knee model predicts the meniscus transmits 70–100% of axial knee loads and, in conjunction with the articular cartilage, distributes the load to reduce contact pressures and subchondral bone stresses. A degenerate form of the model is consistent with analytical Hertzian contact solutions and the model demonstrates strong correlation with experimental results (meniscal circumferential strains, contact pressures, meniscal percentage of load transmission).

The anisotropic axisymmetric finite element knee model is a useful tool for quantifying mechanical etiologic mechanisms involved in the development of post-meniscectomy osteoarthritis. The model predicts partial and total meniscectomies increase contact pressures and subchondral bone stresses. The increased subchondral bone stresses may result in microfractures and bone stiffening. This is consistent with clinical radiographic findings following meniscectomy and with theories which propose subchondral bone stiffening initiates degenerative changes. This does NOT suggest that other factors are not important.

Experimental measurements of meniscal circumferential strains confirm that the meniscus transmits load by developing hoop tension in the meniscus. The meniscal circumferential strains are of the order of 1–2 percent strain so the circumferential collagen fibers resist the load and prevent extrusion of the meniscus from the joint.
space.

In summary, the anisotropic axisymmetric finite element knee model is predicts contact pressures and cartilage and subchondral bone stresses consist with experimental findings in the knee. The major improvements of the model are incorporation of articular cartilage, anisotropic meniscal material properties, and anatomic meniscal geometry. Each of these modifications have demonstrated important first-order effects in knee load transmission and must be included in future modeling efforts of the knee.

7.5 Future Work

One obvious use of the anisotropic axisymmetric knee model is further exploration of the link between meniscectomies and osteoarthritis. Attempts to quantify Wolff's law could be utilized to estimate the increases in subchondral bone density. Odgaard's [111] measurement of tibial bone density changes following meniscectomy provide a means of verifying predicted bone density increases. The effects of increased bone stiffness on articular cartilage stresses can also be examined.

As was stated at the being of this chapter, modeling is never finished. The excerpt from the joint symposium of the American Academy of Orthopaedic Surgeons and the National Institute of Arthritis and Musculoskeletal and Skin Diseases given in Chapter 1 stated meniscal models are necessary to understand the mechanical functions of the meniscus and to evaluate causes of meniscal injury and degeneration. Only the load-bearing role of the meniscus and the link between meniscectomies have been addressed. A three-dimensional finite element knee model incorporating articular cartilage and anisotropic material properties is necessary to further examine meniscal mechanical functions.

The proposed mechanism of posterior horn tears of the meniscus combines axial loading with external rotation of the tibia. This cannot be simulated with an axisymmetric model. A three-dimensional knee model with medial collateral ligament
attachments is necessary. Predicted stresses could be compared to meniscal tissue strength to test hypothesized failure mechanisms.

Other mechanical roles of the meniscus can be examined with a three-dimensional knee model (e.g. joint stability). Fijan [49] recently developed a three-dimensional mathematical model of the knee joint (without the meniscus) to predict muscle forces and joint contact forces from gait data. He found that anterior/posterior force equilibrium could not established. This could be related to tibial surface geometry which can be planar or even slightly convex. If menisci are added to the knee model, the tibial surfaces become more concave. Tibial concavity permits the contact surfaces to transmit anterior/posterior shear forces and enhance the mechanical stability of the knee.

Distinct from the total joint is the meniscus itself. Further work needs to be done to characterize the material properties of the meniscus. Attempting to map material properties from macroscopic samples from the meniscus has limitations and restrictions. Nondestructive testing methods may provide a method of mapping material properties. Transmission ultrasound tomography produces a map of wave propagation velocities which are related to the local tissue density and modulus. Estimation of elastic moduli may be possible using this method.

Material properties may also be estimated from the microstructure of the tissue. Ault [16] used micromechanical models of composite materials to predict the material properties of fibrous soft tissues. Using histologic techniques to quantify the collagen distribution of the meniscus could potentially provide information on meniscal material properties.

In summary, our understanding of meniscal functions is minimal and many questions remain to be answered regarding the meniscus and its role in the knee.
Bibliography


American Society of Mechanical Engineers. 1989 Winter Annual Meeting in San Francisco, CA.


Appendix A

Stiffness Tensor

The following appendix is intended to familiarize the reader with the stiffness tensor and the number of independent elements in the stiffness tensor for materials exhibiting various forms of symmetry. Relationships between the stiffness tensor elements and material parameters such as Young's modulus, shear modulus, and Poisson's ratio are developed. Finally, the thermodynamic restrictions on the material parameters are presented. For an indepth treatment of elasticity and solid mechanics, the reader is directed to references by Timoshenko [142], Sokolnikoff [136], Fung [56], and Malvern [92].

A.1 Strain Energy Density Function

The strain energy density function, $W$, was first introduced by Green in 1839\(^1\) (see Love for a brief proof of the existence of the strain energy density function [91]). Assuming the strain energy density function exists, the function is a potential energy function of the form

$$W = W(\varepsilon_{ij}) \quad (A.1)$$

\(^1\)Attributed to G. Green by I.S. Sokolnikoff in 1946 [136].
A Taylor series expansion of $W$ about $\varepsilon_{ij} = 0$ [36] results in

$$W = W_0 + C_{ij} \varepsilon_{ij} + \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{3!} C_{ijklmn} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{mn} + \cdots \quad (A.2)$$

In the unstrained state, the strain energy $W \to W_0$ as $\varepsilon_{ij} \to 0$. By requiring the strain energy to vanish in the unstrained state, the reference strain energy must be identically zero ($W_0 = 0$).

The incremental change in strain energy density is equal to the work done by the stress acting through an incremental strain.

$$\delta W = \sigma_{ij} \delta \varepsilon_{ij} \quad (A.3)$$

Additionally, the strain energy density function is a potential energy function that is solely a function of strain, so the incremental change in strain energy density must be

$$\delta W = \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} \quad (A.4)$$

Comparing coefficients of Equations A.3 and A.4,

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \quad (A.5)$$

Since the stress is the partial derivative of the strain energy density with respect to the strain, applying this to Equation A.2 yields

$$\sigma_{ij} = C_{ij} + C_{ijkl} \varepsilon_{kl} + \frac{1}{2} C_{ijklmn} \varepsilon_{kl} \varepsilon_{mn} + \cdots \quad (A.6)$$

As with the reference strain energy, requiring the residual stresses to vanish in the unstrained state, $C_{ij} = 0$ and

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} + \frac{1}{2} C_{ijklmn} \varepsilon_{kl} \varepsilon_{mn} + \cdots \quad (A.7)$$

Equations A.5 and A.7 are general forms of the constitutive equations for a three-dimensional elastic solid. However, for linear elasticity, the nonlinear terms are neglected and Equation A.7 becomes

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad (A.8)$$
and the strain energy density function is
\[ W = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \]  \hspace{1cm} (A.9)

Equation A.8 is the generalized Hooke's law for a linearly elastic solid under the conditions of three-dimensional small strain.

An alternative method for deriving the strain energy density function would be to begin with Equation A.8 as the definition of a linearly elastic solid and integrate Equation A.3:\(^2\)
\[ \int_0^W dW' = \int_0^{\varepsilon_{ij}} \sigma_{ij} \varepsilon_{ij} \]  \hspace{1cm} (A.10)
Substituting Hooke's law into the integrand yields\(^3\)
\[ W = \int_0^{\varepsilon_{ij}} C_{ijkl} \varepsilon_{kl} \varepsilon_{ij} = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \]  \hspace{1cm} (A.11)

This result is analogous to the uniaxial stress case. Hooke's law for uniaxial stress is
\[ \sigma = E \varepsilon \]  \hspace{1cm} (A.12)
so the strain energy density is
\[ W = \int_0^{\varepsilon_0} \sigma \varepsilon = \int_0^{\varepsilon_0} E \varepsilon \varepsilon = \frac{1}{2} E \varepsilon^2 \]  \hspace{1cm} (A.13)
which has the same form as Equation A.11.

\(^2\)The prime, \(\prime\), denotes a substitute variable of integration.

\(^3\)The integral of Equation A.3 is a path integral in \(n\)-dimensional strain space. Consequently, the path must be parameterized to perform the integration. Since the strain energy density function is a potential energy state function of the strain only, the integration from one strain state to another strain state is independent of the path. Thus, the simplest possible path of a straight line can be used to parameterize the strain path:
\[ \varepsilon_{ij}' = \varepsilon_{ij} t \]
\[ \varepsilon_{kl}' = \varepsilon_{kl} t \]
\[ d\varepsilon_{ij}' = \varepsilon_{ij} dt \]
so
\[ \int_0^{\varepsilon_{ij}} C_{ijkl} \varepsilon_{kl} \varepsilon_{ij}' d\varepsilon_{ij}' = \int_0^{1} C_{ijkl} (\varepsilon_{kl} t)(\varepsilon_{ij} dt) = C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \int_0^{1} t dt \]
\[ = C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \frac{t^2}{2} \bigg|_0^1 = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \]
A.2 Symmetry Relationships

In general, an \( n^{th} \)-order tensor has \( 3^n \) elements. The stiffness tensor, \( C_{ijkl} \), is a fourth-order tensor and has \( 3^4 = 81 \) elements. However, the symmetry of the stiffness tensor reduces the number of independent elements. Combining Equations A.5 and A.8 results in the expression

\[
\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = C_{ijkl} \varepsilon_{kl} \tag{A.14}
\]

Differentiating Equation A.14 with respect to \( \varepsilon_{kl} \) yields

\[
C_{ijkl} = \frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \tag{A.15}
\]

Since the order of differentiation may be reversed, \( \frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 W}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} \) and

\[
C_{ijkl} = C_{klji} \tag{A.16}
\]

Further symmetry relationships can be developed due to the symmetry of the strain tensor\(^4\), \( \varepsilon_{ij} = \varepsilon_{ji} \).

\[
\frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 W}{\partial \varepsilon_{ji} \partial \varepsilon_{kl}} = \frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{lk}} = \frac{\partial^2 W}{\partial \varepsilon_{ji} \partial \varepsilon_{lk}} \tag{A.17}
\]

\(^4\)Under small strain conditions, the strain, as defined by Sokolnikoff [136], is

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

For normal strains \( (i = j) \),

\[
\varepsilon_{ii} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} \right) = \frac{\partial u_i}{\partial x_i}
\]

For shear stress \( (i \neq j) \),

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = \varepsilon_{ji}
\]

Consequently, the off-diagonal terms are symmetric and the strain tensor is symmetric:

\[
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{pmatrix}
= 
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}
\end{pmatrix}
\]

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Table A.1: Tabular representation of the stiffness tensor. Elements in boxes are equal.

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<tr>
<td>$\sigma_{32}$</td>
<td>$C_{3211}$</td>
<td>$C_{3212}$</td>
<td>$C_{3213}$</td>
<td>$C_{3221}$</td>
<td>$C_{3222}$</td>
<td>$C_{3223}$</td>
<td>$C_{3231}$</td>
<td>$C_{3232}$</td>
</tr>
<tr>
<td>$\sigma_{33}$</td>
<td>$C_{3311}$</td>
<td>$C_{3312}$</td>
<td>$C_{3313}$</td>
<td>$C_{3321}$</td>
<td>$C_{3322}$</td>
<td>$C_{3323}$</td>
<td>$C_{3331}$</td>
<td>$C_{3332}$</td>
</tr>
</tbody>
</table>

so

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$  \hspace{1cm} (A.18)

By analogy,

$$C_{klji} = C_{lkij} = C_{klji} = C_{lkji}$$  \hspace{1cm} (A.19)

so

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} = C_{klji} = C_{lkji}$$  \hspace{1cm} (A.20)

Table A.1 shows the equivalent elements for $i = 1$, $j = 2$, $k = 2$, and $l = 3$. The results also demonstrate the symmetry of the stress tensor\(^5\), $\sigma_{ij} = \sigma_{ji}$.

Equation A.20 can be used to deduce the 21 independent elements of the stiffness tensor:

$$C_{1111}$$

$$C_{1112} = C_{1121} = C_{1211} = C_{2111}$$

$$C_{1113} = C_{1131} = C_{1311} = C_{3111}$$

$$C_{1122} = C_{2211}$$

\(^5\)The symmetry of the stress tensor is a result of the equilibrium equations for an infinitesimal element (see Sokolnikoff for a proof of this stress tensor property [136]). The symmetry of the stress tensor can also be proven using Hooke’s law and the symmetry of the strain and stiffness tensors.
\[
\begin{align*}
C_{1123} &= C_{1132} = C_{2311} = C_{3211} \\
C_{1133} &= C_{3311} \\
C_{1212} &= C_{1221} = C_{2112} = C_{2121} \\
C_{1213} &= C_{1231} = C_{2113} = C_{2131} = C_{1312} = C_{1331} = C_{3112} = C_{3121} \\
C_{1222} &= C_{2122} = C_{2212} = C_{2221} \\
C_{1223} &= C_{1232} = C_{2123} = C_{2132} = C_{2312} = C_{2321} = C_{3212} = C_{3221} \\
C_{1233} &= C_{2133} = C_{3312} = C_{3321} \\
C_{1313} &= C_{1331} = C_{3113} = C_{3131} \\
C_{1322} &= C_{3122} = C_{3213} = C_{3231} \\
C_{1323} &= C_{1332} = C_{3123} = C_{3132} = C_{3231} = C_{3312} = C_{3321} = C_{3331} \\
C_{1333} &= C_{3133} = C_{3313} = C_{3331} \\
C_{2222} &= C_{2232} = C_{3222} = C_{3222} \\
C_{2233} &= C_{3322} \\
C_{2323} &= C_{2332} = C_{3223} = C_{3232} \\
C_{2333} &= C_{3233} = C_{3323} = C_{3332} \\
C_{3333} &
\end{align*}
\]

Hence, there are 21 independent elements in the stiffness tensor.

Due to the fact that both the stress tensor and the strain tensor are symmetric, i.e. \( \sigma_{ij} = \sigma_{ji} \) and \( \varepsilon_{ij} = \varepsilon_{ji} \) respectively, and the symmetry relationships of the stiffness tensor, the following shorthand notation can be employed:

\[
\begin{align*}
\sigma_1 &= \sigma_{11} \\
\sigma_2 &= \sigma_{22} \\
\sigma_3 &= \sigma_{33} \\
\sigma_4 &= \sigma_{23} = \sigma_{32} \\
\sigma_5 &= \sigma_{31} = \sigma_{13} \\
\sigma_6 &= \sigma_{12} = \sigma_{21} \\
\end{align*}
\] (A.21)
and

\[ \begin{align*}
\varepsilon_1 &= \varepsilon_{11} \\
\varepsilon_2 &= \varepsilon_{22} \\
\varepsilon_3 &= \varepsilon_{33} \\
\varepsilon_4 &= \varepsilon_{23} + \varepsilon_{32} = 2\varepsilon_{23} \\
\varepsilon_5 &= \varepsilon_{31} + \varepsilon_{13} = 2\varepsilon_{31} \\
\varepsilon_6 &= \varepsilon_{12} + \varepsilon_{21} = 2\varepsilon_{12}
\end{align*} \]  \hspace{1cm} (A.22)

and the symmetric stiffness matrix

\[
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{pmatrix}
\quad C_{ij} = C_{ji} \hspace{1cm} (A.23)
\]

so

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{pmatrix} \hspace{1cm} (A.24)
\]

However, it must be noted that the shorthand notations for \(\sigma_i\), \(\varepsilon_j\), and \(C_{ij}\) are not tensor quantities and therefore can not be transformed as a tensor [147]. The shorthand matrix notation is simply for ease of representation of fourth-order tensors.

## A.3 Monotropic Materials

In general, a stiffness tensor can be transformed according to the law

\[
C'_{ijkl} = a_{ir} a_{js} a_{kl} a_{tu} C_{rstu} \hspace{1cm} (A.25)
\]

where \(a_{ij}\) is the cosine of the angle between the \(x'_i\) and \(x_j\) axes\(^6\). A material that exhibits one plane of symmetry (mirror image symmetry) is known as a monotropic

\(^6\)A vector is transformed according to the law

\[ x'_i = a_{ij} x_j \]
material. For a monotropic material exhibiting symmetry about the \( x_i \) plane, the transformed stiffness tensor must be identically equal to the stiffness tensor in the original coordinate frame when \( x_i' = -x_i \).

\[
C'_{ijkl} = C_{ijkl}
\]  \hspace{1cm} (A.26)

For the case of \( x_3 \) plane symmetry, the transformation direction cosines are

\[
a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]  \hspace{1cm} (A.27)

Using Equation A.25 to transform \( C_{ijkl} \), for \( i = j = k = l = 1 \), this becomes

\[
C'_{1111} = a_{11}a_{11}a_{11}a_{11}C_{1111} + a_{11}a_{11}a_{11}a_{12}C_{1112} + a_{11}a_{11}a_{11}a_{13}C_{1113} + a_{11}a_{12}a_{12}a_{11}C_{1121} + a_{11}a_{12}a_{12}a_{12}C_{1122} + a_{11}a_{12}a_{12}a_{13}C_{1123} + a_{11}a_{13}a_{13}a_{11}C_{1131} + a_{11}a_{13}a_{13}a_{12}C_{1132} + a_{11}a_{13}a_{13}a_{13}C_{1133} + a_{11}a_{12}a_{12}a_{11}a_{11}C_{1211} + a_{11}a_{12}a_{12}a_{11}a_{12}C_{1212} + a_{11}a_{12}a_{12}a_{11}a_{13}C_{1213} + a_{11}a_{12}a_{12}a_{12}a_{11}C_{1221} + a_{11}a_{12}a_{12}a_{12}a_{12}C_{1222} + a_{11}a_{12}a_{12}a_{12}a_{13}C_{1223} + a_{11}a_{12}a_{12}a_{13}a_{11}C_{1231} + a_{11}a_{12}a_{12}a_{13}a_{12}C_{1232} + a_{11}a_{12}a_{12}a_{13}a_{13}C_{1233} + a_{11}a_{13}a_{13}a_{11}a_{11}C_{1311} + a_{11}a_{13}a_{13}a_{11}a_{12}C_{1312} + a_{11}a_{13}a_{13}a_{11}a_{13}C_{1313} + a_{11}a_{13}a_{13}a_{12}a_{11}C_{1321} + a_{11}a_{13}a_{13}a_{12}a_{12}C_{1322} + a_{11}a_{13}a_{13}a_{12}a_{13}C_{1323} + a_{11}a_{13}a_{13}a_{13}a_{11}C_{1331} + a_{11}a_{13}a_{13}a_{13}a_{12}C_{1332} + a_{11}a_{13}a_{13}a_{13}a_{13}C_{1333} + a_{12}a_{11}a_{11}a_{11}a_{11}C_{2111} + a_{12}a_{11}a_{11}a_{11}a_{12}C_{2112} + a_{12}a_{11}a_{11}a_{11}a_{13}C_{2113} + a_{12}a_{11}a_{12}a_{12}a_{11}C_{2121} + a_{12}a_{11}a_{12}a_{12}a_{12}C_{2122} + a_{12}a_{11}a_{12}a_{12}a_{13}C_{2123} + a_{12}a_{11}a_{13}a_{13}a_{11}C_{2131} + a_{12}a_{11}a_{13}a_{13}a_{12}C_{2132} + a_{12}a_{11}a_{13}a_{13}a_{13}C_{2133} + a_{12}a_{12}a_{11}a_{11}a_{11}C_{2211} + a_{12}a_{12}a_{11}a_{11}a_{12}C_{2212} + a_{12}a_{12}a_{11}a_{11}a_{13}C_{2213} + a_{12}a_{12}a_{12}a_{12}a_{11}C_{2221} + a_{12}a_{12}a_{12}a_{12}a_{12}C_{2222} + a_{12}a_{12}a_{12}a_{12}a_{13}C_{2223} + a_{12}a_{12}a_{13}a_{13}a_{11}C_{2311} + a_{12}a_{12}a_{13}a_{13}a_{12}C_{2312} + a_{12}a_{12}a_{13}a_{13}a_{13}C_{2313} + a_{12}a_{13}a_{11}a_{11}a_{11}C_{2311} + a_{12}a_{13}a_{11}a_{11}a_{12}C_{2312} + a_{12}a_{13}a_{11}a_{11}a_{13}C_{2313} + a_{12}a_{13}a_{12}a_{11}a_{11}C_{2321} + a_{12}a_{13}a_{12}a_{12}a_{12}C_{2322} + a_{12}a_{13}a_{12}a_{12}a_{13}C_{2323} + a_{12}a_{13}a_{13}a_{13}a_{11}C_{2331} + a_{12}a_{13}a_{13}a_{13}a_{12}C_{2332} + a_{12}a_{13}a_{13}a_{13}a_{13}C_{2333} + a_{12}a_{13}a_{12}a_{11}a_{11}a_{11}C_{2331} + a_{12}a_{13}a_{12}a_{12}a_{11}a_{12}C_{2332} + a_{12}a_{13}a_{12}a_{12}a_{11}a_{13}C_{2333}
+ a_{12}a_{13}a_{13}a_{11}C_{2331} + a_{12}a_{13}a_{13}a_{12}C_{2332} + a_{12}a_{13}a_{13}a_{13}C_{3333} \\
+ a_{13}a_{11}a_{11}a_{11}C_{3111} + a_{13}a_{11}a_{11}a_{12}C_{3112} + a_{13}a_{11}a_{11}a_{13}C_{3113} \\
+ a_{13}a_{11}a_{12}a_{11}C_{3121} + a_{13}a_{11}a_{12}a_{12}C_{3122} + a_{13}a_{11}a_{12}a_{13}C_{3123} \\
+ a_{13}a_{11}a_{13}a_{11}C_{3131} + a_{13}a_{11}a_{13}a_{12}C_{3132} + a_{13}a_{11}a_{13}a_{13}C_{3133} \\
+ a_{13}a_{12}a_{11}a_{11}C_{3211} + a_{13}a_{12}a_{11}a_{12}C_{3212} + a_{13}a_{12}a_{11}a_{13}C_{3213} \\
+ a_{13}a_{12}a_{12}a_{11}C_{3221} + a_{13}a_{12}a_{12}a_{12}C_{3222} + a_{13}a_{12}a_{12}a_{13}C_{3223} \\
+ a_{13}a_{12}a_{13}a_{11}C_{3321} + a_{13}a_{12}a_{13}a_{12}C_{3322} + a_{13}a_{12}a_{13}a_{13}C_{3323} \\
+ a_{13}a_{13}a_{11}a_{11}C_{3331} + a_{13}a_{13}a_{11}a_{12}C_{3332} + a_{13}a_{13}a_{11}a_{13}C_{3333} \\
+ a_{13}a_{13}a_{12}a_{11}C_{3331} + a_{13}a_{13}a_{12}a_{12}C_{3332} + a_{13}a_{13}a_{12}a_{13}C_{3333} \phantom{C_{33}} \quad (A.28)

But because all of the off-diagonal direction cosines are equal to zero, this reduces to

\[
C'_{1111} = a_{11}a_{11}a_{11}a_{11}C_{1111} = C_{1111} \quad (A.29)
\]

Similar expressions can be developed for the remaining 20 independent elements. However, any elements with an odd power of a_{33} result in C'_{ijkl} = -C_{ijkl} (i.e. C'_{1113} = a_{11}a_{11}a_{11}a_{33}C_{1113} = -C_{1113}), so the only way to satisfy Equation A.26 is for C'_{ijkl} = C_{ijkl} = 0. Therefore,

\[
C_{1113} = C_{1123} = C_{1213} = C_{1223} = C_{1323} = C_{1333} = C_{2223} = C_{2333} = 0 \quad (A.30)
\]

In shorthand matrix notation,

\[
[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{33} & 0 & 0 & C_{36} \\
C_{44} & C_{45} & 0 & 0 & C_{56} \\
C_{55} & 0 & C_{66} \\
C_{66} & & & & \\
\end{bmatrix} \quad (A.31)
\]

Similarly, for the case of z_1 plane symmetry

\[
C_{1112} = C_{1113} = C_{1212} = C_{1222} = C_{1322} = C_{1332} = C_{2323} = C_{1333} = 0 \quad (A.32)
\]

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and

\[
[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\
C_{22} & C_{23} & C_{24} & 0 & 0 \\
C_{33} & C_{34} & 0 & 0 \\
C_{44} & 0 & 0 \\
C_{55} & C_{56} \\
C_{66}
\end{bmatrix}
\]  \hspace{1cm} (A.33)

And for the case of \(x_2\) plane symmetry

\[
C_{1111} = C_{1123} = C_{1213} = C_{1222} = C_{1233} = C_{1323} = C_{2222} = C_{2323} = C_{3333} = 0
\]  \hspace{1cm} (A.34)

and

\[
[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\
C_{22} & C_{23} & C_{24} & 0 & C_{25} & 0 \\
C_{33} & 0 & C_{35} & 0 \\
C_{44} & 0 & C_{46} \\
C_{55} & 0 \\
C_{66}
\end{bmatrix}
\]  \hspace{1cm} (A.35)

There are 13 independent stiffness tensor elements for a monotropic material.

### A.4 Orthotropic Materials

A material that exhibits two planes of symmetry is called an orthotropic material.

In addition to meeting the transformation requirements of a monotropic material, an orthotropic material must meet a second transformation requirement. For example, \(x_1\) and \(x_2\) plane symmetry require that

\[
C'_{ijkl} = C_{ijkl}
\]  \hspace{1cm} (A.36)

\[
C''_{ijkl} = C_{ijkl}
\]  \hspace{1cm} (A.37)

where the respective direction cosines are

\[
a_{ij} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  \hspace{1cm} (A.38)

and

\[
a_{ij} = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  \hspace{1cm} (A.39)
The restrictions imposed by Equations A.36 and A.37 result in all stiffness tensor elements with subscripts "1" and/or "2" to and odd power are equal to zero. From Equation A.32,

$$ C_{1112} = C_{1113} = C_{1222} = C_{1223} = C_{1333} = C_{1322} = C_{1333} = C_{1333} = 0 $$  \hspace{1cm} (A.40) \\

and from Equation A.34

$$ C_{1123} = C_{1213} = C_{2223} = C_{2333} = 0 $$  \hspace{1cm} (A.41) \\

In shorthand matrix notation

$$ [C] = \begin{bmatrix} 
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} 
\end{bmatrix} $$  \hspace{1cm} (A.42) \\

$z_1$ and $z_3$ plane symmetry or $z_2$ and $z_3$ plane symmetry yield the same result. There are nine independent stiffness tensor elements for an orthotropic material.

In the principal directions of an orthotropic material, the shear strains are coupled to their respective shear stress components and the normal stresses and strains are decoupled from the shear stresses and strains.

### A.5 Transversely Isotropic Materials

A transversely isotropic material is an orthotropic material whose properties are constant in a given plane. For example, if the $z_3$ plane is the isotropic plane, then

$$ C'_{ijkl} = C_{ijkl} $$  \hspace{1cm} (A.43) \\

for any arbitrary rotation about the $z_3$ axis, for which

$$ a_{ij} = \begin{bmatrix} 
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 
\end{bmatrix} $$  \hspace{1cm} (A.44) \\

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Using these transformation requirements, a system of equations can be derived to reduce the number of independent stiffness tensor elements.

\[
C'_{1111} = C_{1111} = a_{11}a_{11}a_{11}a_{11}C_{1111} + a_{11}a_{11}a_{12}a_{12}C_{1112} + a_{11}a_{12}a_{12}a_{11}C_{1112} + a_{11}a_{12}a_{12}a_{12}C_{1121} \\
+ a_{11}a_{12}a_{12}a_{12}C_{1221} + a_{11}a_{12}a_{12}a_{12}C_{1222} + a_{11}a_{12}a_{12}a_{12}C_{2111} \\
+ a_{12}a_{11}a_{11}a_{12}C_{2112} + a_{12}a_{11}a_{12}a_{11}C_{2121} + a_{12}a_{11}a_{12}a_{12}C_{2122} \\
+ a_{12}a_{12}a_{11}a_{11}C_{2211} + a_{12}a_{12}a_{12}a_{11}C_{2212} + a_{12}a_{12}a_{12}a_{12}C_{2221} \\
+ a_{12}a_{12}a_{12}a_{12}C_{2222} \quad \text{(A.45)}
\]

All other terms have an \(a_{13} = 0\) term. Substituting the direction cosines yields

\[
C_{1111} = \cos^4 \theta C_{1111} + 2 \cos^2 \theta \sin^2 \theta C_{1122} + 4 \cos^2 \theta \sin^2 \theta C_{1212} + \sin^4 \theta C_{2222} \quad \text{(A.46)}
\]

Similarly for \(C_{2222}\),

\[
C'_{2222} = C_{2222} = a_{21}a_{21}a_{21}a_{21}C_{1111} + a_{21}a_{21}a_{22}a_{22}C_{1122} + a_{21}a_{22}a_{21}a_{22}C_{1212} \\
+ a_{21}a_{22}a_{22}a_{21}C_{2121} + a_{22}a_{21}a_{21}a_{22}C_{2112} + a_{22}a_{21}a_{22}a_{21}C_{2122} \\
+ a_{22}a_{22}a_{21}a_{21}C_{2211} + a_{22}a_{22}a_{22}a_{22}C_{2222} \quad \text{(A.47)}
\]

so

\[
C_{2222} = \sin^4 \theta C_{1111} + 2 \cos^2 \theta \sin^2 \theta C_{1122} + 4 \cos^2 \theta \sin^2 \theta C_{1212} + \cos^4 \theta C_{2222} \quad \text{(A.48)}
\]

Subtracting Equation A.48 from A.46 yields

\[
C_{1111} - C_{2222} = \cos^4 \theta (C_{1111} - C_{2222}) - \sin^4 \theta (C_{1111} - C_{2222}) \\
= (\cos^4 \theta - \sin^4 \theta)(C_{1111} - C_{2222}) \\
= \frac{(\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)}{\cos 2\theta} (C_{1111} - C_{2222}) \quad \text{(A.49)}
\]

so

\[
(C_{1111} - C_{2222})(\cos 2\theta - 1) = 0 \quad \text{(A.50)}
\]
Since \( \cos 2\theta - 1 \) is only equal to zero for \( \theta = 0, \pm \frac{\pi}{2}, \pm \pi, \ldots \) and Equation A.50 must be true for arbitrary \( \theta \), then

\[
C_{1111} = C_{2222}
\]  

(A.51)

Substituting this result back into Equation A.46 yields

\[
(1 - \cos^4 \theta)C_{1111} = 2 \cos^2 \theta \sin^2 \theta C_{1122} + 4 \cos^2 \theta \sin^2 \theta C_{1212} + \sin^4 \theta C_{1111}
\]

\[
(1 - \frac{\cos^2 \theta}{\sin^2 \theta})(1 + \cos^2 \theta)C_{1111} = 2 \cos^2 \theta \sin^2 \theta C_{1122} + 4 \cos^2 \theta \sin^2 \theta C_{1212} + \sin^4 \theta C_{1111}
\]

\[
(1 + \cos^2 \theta)C_{1111} = 2 \cos^2 \theta C_{1122} + 4 \cos^2 \theta C_{1212} + \sin^2 \theta C_{1111}
\]

\[
(\frac{1 - \sin^2 \theta}{\cos^2 \theta} + \cos^2 \theta)C_{1111} = 2 \cos^2 \theta C_{1122} + 4 \cos^2 \theta C_{1212}
\]

\[
2 \cos^2 \theta C_{1111} = 2 \cos^2 \theta C_{1122} + 4 \cos^2 \theta C_{1212}
\]

\[
C_{1111} = C_{1122} + 2C_{1212}
\]

\[
C_{1212} = \frac{C_{1111} - C_{1122}}{2}
\]

(A.52)

For \( C_{1133}, C_{1133} = a_1a_1a_{33}a_{33}C_{r_1r_3} \) because \( a_1 = a_{31} = a_{23} = a_{32} = 0 \). Therefore,

\[
C'_{1133} = C_{1133} = a_{11}a_{11}a_{33}a_{33}C_{1133} + a_{11}a_{12}a_{33}a_{33}C_{1233} + a_{12}a_{11}a_{33}a_{33}C_{2133} + a_{12}a_{12}a_{33}a_{33}C_{2233}
\]

(A.53)

So

\[
C_{1133} = \cos^2 \theta C_{1133} + \sin^2 \theta C_{2233}
\]

\[
(1 - \cos^2 \theta)C_{1133} = \sin^2 \theta C_{2233}
\]

\[
C_{1133} = C_{2233}
\]

(A.54)

Similarly for \( C_{2333},
\]

\[
C'_{2333} = C_{2333} = a_{21}a_{33}a_{21}a_{33}C_{1313} + a_{21}a_{33}a_{22}a_{33}C_{1323} + a_{22}a_{33}a_{21}a_{33}C_{2133} + a_{22}a_{33}a_{22}a_{33}C_{2233}
\]

(A.55)
\[
C_{2323} = \sin^2 \theta C_{1313} + \cos^2 \theta C_{2323} \\
(1 - \cos^2 \theta) C_{2323} = \sin^2 \theta C_{1313} \\
C_{2323} = C_{1313} 
\tag{A.56}
\]

In shorthand notation,

\[
[C] = \begin{bmatrix}
  C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
  C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
  C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & C_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & C_{44} & 0 \\
  C_{44} & 0 & 0 & 0 & 0 & \frac{C_{11} - C_{13}}{2}
\end{bmatrix} 
\tag{A.57}
\]

So a transversely isotropic material has 5 independent stiffness tensor elements.

By analogy, the transverse isotropy relationships for a material with \( x_2 \) isotropy are

\[
C_{1111} = C_{3333} 
\tag{A.58}
\]

\[
C_{1313} = \frac{C_{1111} - C_{1133}}{2} 
\tag{A.59}
\]

\[
C_{1122} = C_{2233} 
\tag{A.60}
\]

\[
C_{1212} = C_{2323} 
\tag{A.61}
\]

and

\[
[C] = \begin{bmatrix}
  C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
  C_{12} & C_{22} & C_{12} & 0 & 0 & 0 \\
  C_{13} & C_{12} & C_{11} & 0 & 0 & 0 \\
  0 & 0 & 0 & C_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & C_{44} & 0 \\
  C_{44} & 0 & 0 & 0 & 0 & \frac{C_{11} - C_{13}}{2}
\end{bmatrix} 
\tag{A.62}
\]

and the transverse isotropy relationships for a material with \( x_1 \) isotropy are

\[
C_{2222} = C_{3333} 
\tag{A.63}
\]

\[
C_{3323} = \frac{C_{2222} - C_{2233}}{2} 
\tag{A.64}
\]

\[
C_{1122} = C_{1133} 
\tag{A.65}
\]

\[
C_{1212} = C_{1313} 
\tag{A.66}
\]

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and

\[
[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{22} & C_{23} & C_{23} & 0 & 0 & 0 \\
C_{33} & C_{33} & C_{33} & 0 & 0 & 0 \\
\frac{C_{22} - C_{33}}{2} & 0 & 0 & C_{55} & 0 \\
0 & C_{55} & C_{55} & 0 & C_{55}
\end{bmatrix}
\] (A.67)

### A.6 Isotropic Materials

An isotropic material is an orthotropic material whose material properties are independent of the orientation of the material (i.e. the elastic properties are the same in all directions). Then in addition to meeting the transformation requirements for rotations about the \(x_3\) axis, the stiffness tensor must also meet the same transformation requirements for rotations about the \(x_1\) axis where

\[
a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}
\] (A.68)

and for rotations about the \(x_2\) axis where

\[
a_{ij} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}.
\] (A.69)

Considering arbitrary rotations about the \(x_2\) axis,

\[
C'_{1111} = C_{1111} = a_{11}a_{11}a_{11}a_{11}C_{1111} + a_{11}a_{11}a_{13}a_{13}C_{1133} + a_{11}a_{13}a_{11}a_{13}C_{1313}
\]

\[
+ a_{11}a_{13}a_{13}a_{11}C_{1331} + a_{13}a_{11}a_{11}a_{13}C_{3113} + a_{13}a_{11}a_{13}a_{11}C_{3131}
\]

\[
+ a_{13}a_{13}a_{11}a_{11}C_{3311} + a_{13}a_{13}a_{13}a_{13}C_{3333}
\] (A.70)

so

\[
C_{1111} = \cos^4 \theta C_{1111} + 2 \cos^2 \theta \sin^2 \theta C_{1133} + 4 \cos^2 \theta \sin^2 \theta C_{1313} + \sin^4 \theta C_{3333}
\] (A.71)

and similarly

\[
C_{3333} = \sin^4 \theta C_{1111} + 2 \cos^2 \theta \sin^2 \theta C_{1133} + 4 \cos^2 \theta \sin^2 \theta C_{1313} + \cos^4 \theta C_{3333}
\] (A.72)
By analogy to Equations A.46 and A.48, Equations A.71 and A.72 reduce to

\[ C_{1111} = C_{3333} \]  \hspace{1cm} (A.73)

and by analogy to Equation A.52,

\[ C_{1313} = \frac{C_{1111} - C_{1133}}{2} \]  \hspace{1cm} (A.74)

To derive the relationship for \( C_{1122} \),

\[ C'_{1122} = C_{1122} = a_{11}a_{11}a_{22}a_{22}C_{1122} + a_{11}a_{13}a_{22}a_{22}C_{1322} + a_{13}a_{11}a_{22}a_{22}C_{3122} + a_{13}a_{13}a_{22}a_{22}C_{3322} \]  \hspace{1cm} (A.75)

so

\[ C_{1122} = \cos^2 \theta C_{1122} + \sin^2 \theta C_{3322} \]

\[ \frac{(1 - \cos^2 \theta)}{\sin^2 \theta} C_{1122} = \sin^2 \theta C_{3322} \]

\[ C_{1122} = C_{3322} \]  \hspace{1cm} (A.76)

By combining the constraints required for a transversely isotropic material and the additional constraints for an isotropic material, the relationships for an isotropic material can be derived. Equations A.51 and A.73 yield

\[ C_{1111} = C_{2222} = C_{3333} \]  \hspace{1cm} (A.77)

Combining Equations A.54 and A.76 yields

\[ C_{1122} = C_{1133} = C_{2233} \]  \hspace{1cm} (A.78)

And substituting Equation A.78 into Equation A.74 and combining this result with Equations A.52 and A.56 yields

\[ C_{2333} = C_{3131} = C_{1212} = \frac{C_{1111} - C_{1122}}{2} \]  \hspace{1cm} (A.79)

Consequently, there are only two independent elastic parameters for an isotropic material. These stiffness tensor restrictions for an isotropic material could be derived
using any two of the axial rotation transformations \((x_1 \text{ and } x_2, x_1 \text{ and } x_3, \text{ or } x_2 \text{ and } x_3)\). The third transformation provides no additional independent restrictions.

In shorthand notation,

\[
[C] = \begin{bmatrix}
  C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
  C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
  C_{11} & 0 & 0 & 0 & 0 & 0 \\
  C_{11} & 0 & 0 & 0 & 0 & 0 \\
  C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
  C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
\end{bmatrix}
\]  

(A.80)

In 1852, Lamé introduced the two elastic constants \(\lambda\) and \(\mu\), where

\[
\lambda = C_{1122} = C_{1133} = C_{2233} \quad \text{(A.81)}
\]

\[
\mu = C_{2332} = C_{3131} = C_{1212} = \frac{C_{1111} - C_{1122}}{2} \quad \text{(A.82)}
\]

\[
\lambda + 2\mu = C_{1111} = C_{2222} = C_{3333} \quad \text{(A.83)}
\]

These constants allow the stiffness tensor of an isotropic solid to be succinctly expressed in the form

\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj}) \quad \text{(A.84)}
\]

or in matrix form

\[
[C] = \begin{bmatrix}
  \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
  \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
  \lambda + 2\mu & 0 & 0 & 0 & 0 & 0 \\
  \lambda + 2\mu & 0 & 0 & 0 & 0 & 0 \\
  \mu & 0 & 0 & 0 & 0 & 0 \\
  \mu & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]  

(A.85)

Hooke's law can then be succinctly written in tensor form as

\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \quad \text{(A.86)}
\]

or

\[
\varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{2\mu(2\mu + 3\lambda)} \sigma_{kk} \delta_{ij} \quad \text{(A.87)}
\]

\(^7\text{Attributed to Lamé by Malvern in 1969 [92].}\)
A.7 Engineering Constants Relationships

The number of independent material properties for an isotropic material is two, as derived in Section A.6. These two independent parameters were represented as the Lamé coefficients \( \lambda \) and \( \mu \). However, these are not the parameters typically reported in the engineering literature. The more commonly specified parameters are Young’s modulus or modulus of elasticity \( - E \), shear modulus or modulus of rigidity \( - G \), Poisson’s ratio \( - \nu \), and bulk modulus \( \kappa \). An additional parameter reported in cartilage research is the confined compression modulus or bulk longitudinal modulus \( - H \). The relationships between the Lamé constants and \( E \), \( G \), \( \nu \), \( \kappa \), and \( H \) can be derived by examining the appropriate stress states.

To derive expressions for Young’s modulus and Poisson’s ratio, consider the uniaxial stress state exemplified by the uniaxial tension test \( \sigma_{11} = E \varepsilon_{11}, \sigma_{22} = \sigma_{33} = 0 \).

\[
\varepsilon_{11} = \frac{1}{E} \sigma_{11} = \frac{1}{2\mu} \sigma_{11} - \frac{\lambda}{2\mu(2\mu + 3\lambda)} \sigma_{11} \tag{A.88}
\]

\[
\frac{1}{E} = \frac{1}{2\mu - \frac{\lambda}{2\mu(2\mu + 3\lambda)}}
= \frac{2\mu + 3\lambda - \lambda}{2\mu(2\mu + 3\lambda)}
= \frac{\mu + \lambda}{\mu(2\mu + 3\lambda)}
E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \tag{A.89}
\]

Poisson’s ratio is defined as the ratio of the transverse strain to the longitudinal strain when stressed in the longitudinal direction.

\[
\nu_{ij} = -\frac{\varepsilon_{ij}}{\varepsilon_{ii}} \tag{A.90}
\]

where \( \varepsilon_{ii} \) and \( \varepsilon_{jj} \) are normal strain components and do not imply the summation convention. For the case of uniaxial tension,

\[
\nu = \frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{\varepsilon_{33}}{\varepsilon_{11}} = -\frac{\frac{\lambda}{2\mu(2\mu + 3\lambda)} \sigma_{11}}{\frac{1}{2\mu} \sigma_{11} - \frac{\lambda}{2\mu(2\mu + 3\lambda)} \sigma_{11}}
\]

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\[ \nu = \frac{\lambda}{2(\mu + \lambda)} \quad (A.91) \]

In order to derive a relationship for the shear modulus, consider the case of pure shear loading. As mentioned previously, for an isotropic material, the normal stresses and strains are decoupled from the shear stresses and strains and \( \sigma_{12} = G(e_{12} + e_{21}) \) \( (\tau = G \gamma \) in more common engineering notation). Comparing this with Equation A.86

\[ \sigma_{12} = G(e_{12} + e_{21}) = 2\mu e_{12} \quad (A.92) \]

Since the strain components are symmetric \( (e_{12} = e_{21}) \), \( \mu \) is identically equal to \( G \).

\[ G = \mu \quad (A.93) \]

Now that the relationships between the Lamé coefficients and the engineering constants are known, the tensor form of Hooke's law can be written as

\[ \sigma_{ij} = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \varepsilon_{kk} \delta_{ij} + \frac{E}{1 + \nu} e_{ij} \quad (A.94) \]

or

\[ e_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \quad (A.95) \]

The bulk modulus relates the change in pressure to the volumetric strain according to the relationship

\[ dp = -\kappa \frac{dV}{V} \quad (A.96) \]

For small strains, the volumetric changes due to shear strains are negligible so the volumetric strain is calculated by

\[ \frac{dV}{V} = \frac{V_0(1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33}) - V_0}{V_0} = 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} + \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} - 1 \quad (A.97) \]

Neglecting higher order terms for small strains,

\[ \frac{dV}{V} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{kk} \quad (A.98) \]
For hydrostatic pressure, \( dp = p \) \((p_0 = 0)\) and the pressure is related to the stresses by \( \sigma_{11} = \sigma_{22} = \sigma_{33} = -p \). The pressure can be related to the volumetric strain by summing the normal stress components.

\[
\sigma_{11} = -p = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \epsilon_{kk} + \frac{E}{1 + \nu} \epsilon_{11} \quad (A.99)
\]

\[
\sigma_{22} = -p = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \epsilon_{kk} + \frac{E}{1 + \nu} \epsilon_{22} \quad (A.100)
\]

\[
\sigma_{33} = -p = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \epsilon_{kk} + \frac{E}{1 + \nu} \epsilon_{33} \quad (A.101)
\]

and

\[
-3p = \frac{3\nu E}{(1 + \nu)(1 - 2\nu)} \epsilon_{kk} + \frac{E}{1 + \nu} \epsilon_{kk}
\]

\[
= \frac{E}{(1 - 2\nu)} \epsilon_{kk}
\]

\[
= \frac{E}{(1 - 2\nu)} \frac{dV}{V}
\]

\[
p = -\frac{E}{3(1 - 2\nu)} \frac{dV}{V} \quad (A.102)
\]

Comparing terms with Equation A.96

\[
\kappa = \frac{E}{3(1 - 2\nu)} \quad (A.103)
\]

Confined compression involves placing a material sample in a rigid cylindrical fixture and compressing the sample along the longitudinal axis. The constitutive relationship is simply \( \sigma_{11} = H \epsilon_{11} \) with \( \epsilon_{22} = \epsilon_{33} = 0 \) because of the fixture constraints. Using Equation A.86,

\[
\sigma_{11} = H \epsilon_{11} = (\lambda + 2\mu)\epsilon_{11} \quad (A.104)
\]

so

\[
H = \lambda + 2\mu \quad (A.105)
\]

or in an alternative form

\[
H = \kappa + \frac{4}{3} G = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \quad (A.106)
\]
A more complete list of relationships between engineering constants was compiled by Malvern [92] and is provided here for reference.

\[
\lambda = \frac{2G\nu}{1-2\nu} = \frac{G(E-2G)}{3G-E} = \kappa - \frac{2G}{3} = \frac{E\nu}{(1+\nu)(1-2\nu)}
\]

\[
= \frac{3\kappa\nu}{1+\nu} = \frac{3\kappa(3\kappa-E)}{9\kappa-E}
\]

\[
\mu \equiv G = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(\kappa-\lambda) = \frac{E}{2(1+\nu)} = \frac{3\kappa(1-2\nu)}{2(1+\nu)}
\]

\[
= \frac{3\kappa E}{9\kappa-E}
\]

\[
\nu = \frac{\lambda}{2(\lambda+G)} = \frac{\lambda}{(3\kappa-\lambda)} = \frac{E}{2G} - 1 = \frac{3\kappa-2G}{2(3\kappa+G)} = \frac{3\kappa-E}{6\kappa}
\]

\[
E = \frac{G(3\lambda+2G)}{\lambda+G} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9\kappa(\kappa-\lambda)}{3\kappa-\lambda} = 2G(1+\nu)
\]

\[
= \frac{9\kappa G}{3\kappa+G} = 3\kappa(1-2\nu)
\]

\[
\kappa = \lambda + \frac{2}{3}G = \frac{\lambda(1+\nu)}{3\nu} = \frac{2G(1+\nu)}{3(1-2\nu)} = \frac{GE}{3(3G-E)} = \frac{E}{3(1-2\nu)}
\]

A.8 Compliance Tensor

Depending upon the stress-strain state, it is often convenient to have Hooke's law in the form \(\varepsilon_{ij} = S_{ijkl}\sigma_{kl}\) as opposed to \(\sigma_{ij} = C_{ijkl}\varepsilon_{kl}\) where \(S_{ijkl}\) is the compliance tensor\(^8\) and is equal to the inverse of the stiffness tensor. When using matrix notation,

\[
[S] = [C]^{-1}
\]

The compliance and stiffness matrices for orthotropic, transversely isotropic, and isotropic materials are listed for reference.

---

\(^{8}\)The author is aware of the confusion caused by using the nomenclature of \(C_{ijkl}\) for stiffness tensor and \(S_{ijkl}\) for compliance tensor, but the nomenclature is the standard convention.
A.8.1 Orthotropic Materials

Orthotropic materials have nine independent parameters. Symmetry relations can be used to relate the off-diagonal terms of the compliance matrix.

\[
\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad (A.113)
\]

Therefore, three of the independent Poisson's ratios must be known to determine the compliance matrix. The compliance matrix is

\[
[S] = \begin{bmatrix}
\frac{1}{E_1} & \frac{-\nu_{12}}{E_2} & \frac{-\nu_{13}}{E_3} & 0 & 0 & 0 \\
\frac{\nu_{12}}{E_2} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_3} & 0 & 0 & 0 \\
\frac{\nu_{13}}{E_3} & \frac{\nu_{23}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sigma_{22}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sigma_{33}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sigma_{12}}
\end{bmatrix} \quad (A.114)
\]

The stiffness matrix is the inverse of the compliance matrix and the nine independent components are:

\[
C_{11} = \frac{1}{E_1 E_2 E_3} \\
C_{12} = \frac{-\nu_{12}}{E_2 E_3} = \frac{\nu_{12} + \nu_{21} \nu_{13}}{E_1 E_2 E_3} \\
C_{13} = \frac{\nu_{13}}{E_2 E_3} = \frac{\nu_{13} + \nu_{21} \nu_{23}}{E_1 E_2 E_3} \\
C_{22} = \frac{1}{E_2 E_3} \\
C_{23} = \frac{-\nu_{23}}{E_2 E_3} = \frac{\nu_{23} + \nu_{12} \nu_{23}}{E_1 E_2 E_3} \\
C_{33} = \frac{1}{E_3} \\
C_{44} = G_{23} \\
C_{55} = G_{31} \\
C_{66} = G_{12}
\] (A.115)

where

\[
\Delta = \frac{1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{31} \nu_{13} - 2 \nu_{21} \nu_{32} \nu_{13}}{E_1 E_2 E_3}
\] (A.116)

A.8.2 Transversely Isotropic Materials

For a transversely isotropic material, there are five independent parameters. By choosing the 1-2 plane to be the plane of isotropy, the following substitutions can be
used:

\[
\begin{align*}
\nu_{TT} &= \nu_{12} = \nu_{21} \\
\nu_{TL} &= \nu_{13} = \nu_{23} \\
\nu_{LT} &= \nu_{31} = \nu_{32} \\
E_T &= E_1 = E_2 \\
E_L &= E_3 \\
G_{TL} &= G_{LT} = G_{23} = G_{31} \\
G_{TT} &= G_{12},
\end{align*}
\] (A.117)

where the subscripts \( T \) and \( L \) denote transverse and longitudinal respectively.

The compliance matrix is

\[
[S] = \begin{bmatrix}
\frac{1}{E_T} & \frac{-\nu_{TT}}{E_T} & \frac{-\nu_{TL}}{E_L} & 0 & 0 & 0 \\
\frac{-\nu_{TT}}{E_T} & \frac{1}{E_T} & \frac{-\nu_{LT}}{E_L} & 0 & 0 & 0 \\
\frac{-\nu_{TL}}{E_T} & \frac{-\nu_{LT}}{E_L} & \frac{1}{E_L} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{TL}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{TL}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_T}
\end{bmatrix}
\] (A.118)

and the independent stiffness matrix components are

\[
\begin{align*}
C_{11} = C_{22} &= \frac{1 - \nu_{TT} \nu_{LT}}{E_T E_L \Delta} \\
C_{12} &= \frac{\nu_{TT} + \nu_{LT} \nu_{TL}}{E_T E_L \Delta} \\
C_{13} = C_{23} &= \frac{\nu_{TL} + \nu_{TT} \nu_{LT}}{E_T E_L \Delta} = \frac{\nu_{TL} + \nu_{TT} \nu_{LT}}{E_T \Delta} \\
C_{33} &= \frac{1 - \nu_{TT}^2}{E_T \Delta} \\
C_{44} = C_{55} &= G_{TL} \\
C_{66} &= G_{TT} = C_{11} - C_{12}
\end{align*}
\] (A.119)

where

\[
\Delta = \frac{1 - \nu_{TT}^2 - 2 \nu_{TL} \nu_{LT} - 2 \nu_{TT} \nu_{LT} \nu_{TL}}{E_T^2 E_L}
\] (A.120)

### A.8.3 Isotropic Materials

As demonstrated in Section A.6, there are two independent parameters for an isotropic material. From the conditions of isotropy,

\[
\begin{align*}
\nu &= \nu_{12} = \nu_{21} = \nu_{13} = \nu_{23} = \nu_{31} = \nu_{32} \\
E &= E_1 = E_2 = E_3 \\
G &= G_{23} = G_{31} = G_{12}
\end{align*}
\] (A.121)

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Therefore, the compliance matrix simplifies to

\[
[S] = \begin{bmatrix}
\frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\]

(A.122)

where the shear modulus can be written as a function of Young’s modulus and Poisson’s ratio (see Equation A.108).

The stiffness matrix is then

\[
[C] = \begin{bmatrix}
\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\
\frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{(1+\nu)(1-2\nu)}{E\nu} & \frac{(1+\nu)(1-2\nu)}{E\nu} & 0 & 0 & 0 \\
\frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{(1+\nu)(1-2\nu)}{E\nu} & \frac{(1+\nu)(1-2\nu)}{E\nu} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\]

(A.123)

A.9 Thermodynamic Restrictions on Elastic Constants

In order to avoid violating the fundamental laws of thermodynamics, the strain energy of a material must increase when the material is stressed. Otherwise, a stressed material could be relieved to extract energy while the strain energy increased, thus creating energy. Formally, this means that the stiffness and compliance matrices must be positive-definite (i.e. the matrices must have positive principal values or invariants). Although the thermodynamic restrictions on orthotropical materials are much more complex than on isotropic materials, it is illustrative to begin with orthotropic materials and view isotropic materials as a subset of orthotropic materials.

A.9.1 Orthotropic Materials

The thermodynamic restrictions for orthotropic materials were first determined by Lempriere [89]. Cowin and Van Buskirk [39] applied these restrictions to bone (an bio-
logical orthotropic material) to verify that the measured Poisson's ratios met the thermodynamic restrictions. Applying the condition of positive-definiteness, Lempriere\(^9\) demonstrated that

\[ E_1, E_2, E_3, G_{23}, G_{31}, G_{12} > 0 \]  
(A.124)

\[ (1 - \nu_{23}\nu_{32}), (1 - \nu_{13}\nu_{31}), (1 - \nu_{12}\nu_{21}) > 0 \]  
(A.125)

and

\[ 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13} > 0 \]  
(A.126)

Using the symmetry of the compliance matrix

\[ \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \]  
(A.127)

where the summation convention is not implied, conditions A.125 reduce to the more useful form

\[ |\nu_{31}| < \left( \frac{E_2}{E_1} \right)^\frac{1}{2}, \quad |\nu_{12}| < \left( \frac{E_3}{E_2} \right)^\frac{1}{2} \]
\[ |\nu_{32}| < \left( \frac{E_2}{E_3} \right)^\frac{1}{2}, \quad |\nu_{23}| < \left( \frac{E_1}{E_2} \right)^\frac{1}{2} \]
\[ |\nu_{13}| < \left( \frac{E_1}{E_3} \right)^\frac{1}{2}, \quad |\nu_{31}| < \left( \frac{E_1}{E_2} \right)^\frac{1}{2} \]  
(A.128)

Equation A.126 can be rearranged so that

\[ \nu_{21}\nu_{32}\nu_{13} < \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13}}{2} \]
\[ \nu_{21}\nu_{32}\nu_{13} < \frac{1 - \nu_{21}^2 \left( \frac{E_2}{E_1} \right) - \nu_{32}^2 \left( \frac{E_3}{E_2} \right) - \nu_{13}^2 \left( \frac{E_1}{E_3} \right)}{2} \]  
(A.129)

Since all the terms on the right hand side must be positive, the right hand side must be less than one half.

\[ \nu_{21}\nu_{32}\nu_{13} < \frac{1 - \nu_{21}^2 \left( \frac{E_2}{E_1} \right) - \nu_{32}^2 \left( \frac{E_3}{E_2} \right) - \nu_{13}^2 \left( \frac{E_1}{E_3} \right)}{2} < \frac{1}{2} \]  
(A.130)

\(^9\)The meaning of \(i\) and \(j\) in the definition of Poisson's ratio used by Lempriere is the reversed form of that used in this appendix which was adopted from Jones [76].
A.9.2 Transversely Isotropic Materials

For the case of transverse isotropy in the 1-2 plane using the previous substitution of variables, the conditions of A.128 become

\[
|\nu_{TT}| < 1 \\
|\nu_{LT}| < \left(\frac{E_L}{E_T}\right)^{\frac{1}{2}} \\
|\nu_{TL}| < \left(\frac{E_T}{E_L}\right)^{\frac{1}{2}}
\]  

(A.131)

or

\[-1 < \nu_{TT} < 1 \]  

(A.132)

\[-\left(\frac{E_L}{E_T}\right)^{\frac{1}{2}} < \nu_{LT} < \left(\frac{E_L}{E_T}\right)^{\frac{1}{2}}\]  

(A.133)

and A.126 becomes

\[1 - \nu_{TT}^2 - 2\nu_{TT}\nu_{LT} - 2\nu_{TT}\nu_{LT}\nu_{TL} > 0\]  

\[(1 - \nu_{TT})(1 + \nu_{TT}) - 2\nu_{TT}\nu_{LT}(1 + \nu_{TT}) > 0\]  

\[(1 - \nu_{TT} - 2\nu_{LT}\nu_{LT})(1 + \nu_{TT}) > 0\]  

\[(1 - \nu_{TT} - 2\nu_{LT}^2 \left(\frac{E_T}{E_L}\right))(1 + \nu_{TT}) > 0\]  

(A.134)

Since the restriction \(\nu_{TT} > -1\) has already been imposed, the expression reduces to

\[\nu_{TT} < 1 - 2\nu_{LT}^2 \left(\frac{E_T}{E_L}\right)\]  

(A.135)

This restriction is more stringent than the upper limit of condition A.132 so this can be rewritten as

\[-1 < \nu_{TT} < 1 - 2\nu_{LT}^2 \left(\frac{E_T}{E_L}\right)\]  

(A.136)

A.9.3 Isotropic Materials

For the condition of isotropy, the restrictions on Poisson's ratio are the same as the restrictions on \(\nu_{TT}\) for a transversely isotropic material.

\[-1 < \nu < 1\]  

(A.137)
Equation A.126 becomes

\[ 1 - 3\nu^2 - 2\nu^3 > 0 \]  \hspace{1cm} (A.138)

which factors into

\[ (\nu + 1)^2(-2\nu + 1) > 0 \]  \hspace{1cm} (A.139)

Since \((\nu + 1)^2\) must be positive, the condition becomes

\[ (-2\nu + 1) > 0 \]  \hspace{1cm} (A.140)

or

\[ \nu < \frac{1}{2} \]  \hspace{1cm} (A.141)

which is more stringent than the upper limit of Equation A.137, so Poisson's ratio for a isotropic material must be in the range

\[ -1 < \nu < \frac{1}{2} \]  \hspace{1cm} (A.142)

The upper limit of \(\nu = 1/2\) is the value of Poisson's ratio for an incompressible isotropic material. This can be seen by rewriting Equation A.96 in the form

\[ \frac{dV}{V} = -\frac{1}{\kappa} dp \]  \hspace{1cm} (A.143)

For an incompressible material, \(dV\) must be zero for a nonzero pressure change. Therefore,

\[ \frac{1}{\kappa} = \frac{3(1 - 2\nu)}{E} = 0 \]  \hspace{1cm} (A.144)

or

\[ \nu = \frac{1}{2} \]  \hspace{1cm} (A.145)
Appendix B

Biplanar X-Ray Stereophotogrammetry

The primary function of photogrammetric methods is to determine three-dimensional coordinates of object points from reconstructed projection rays. Fortunately, the geometric relationships of X-ray photogrammetry are less complex than typical photogrammetric techniques because lens optics do not complicate the projection ray reconstruction. The following appendix provides derivations of the analytical X-ray photogrammetric methods used in this thesis. For a more detailed treatment of X-ray photogrammetry, the reader is referred to the standard text by Hallert [63] and a doctoral thesis by Takamoto [140]. Ghosh [59] and Moffitt and Mikhail [102] provide an excellent coverage of general analytical photogrammetry.

B.1 Basic Definitions

The various coordinate systems are considered first. The global coordinate system (object space) is the absolute coordinate system in which the three-dimensional object coordinates are represented. The focus coordinate system has its origin fixed at the X-ray tube focus with its xy-plane oriented parallel to the plane of the image coordinate system (the z-axis is parallel to the u-axis and the y-axis is parallel to the v-axis, see Figure B.1). The image coordinate system is embedded in the X-ray film plane.

The relative orientation of the focus and image coordinate systems is determined
Figure B.1: Projection schematic of an X-ray photograph (radiograph) with coordinate system definitions.
Figure B.2: Schematic diagram of an X-ray tube. The electrons from the cathode strike the anode, thus causing the emission of X-rays. The focus of the X-ray tube is modeled as a distinct mathematical point [63].

by their definition. However, the image coordinate system is displaced relative to the focus coordinate system. This displacement is characterized by another set of definitions. The focus of the X-ray tube is modeled as a single point and is assumed to be the perspective center or projection center (see Figure B.2). The position of the focus relative to the image is defined by the principal axis, the ray passing through the focus perpendicular to the image plane. The principal point, \((u_P, v_P)\), is defined in the image coordinates and represents the intersection of the principal axis and the image plane. The principal distance, \(d\), is the distance between the focus, \(F\), and the principal point, \(P\), along the principal axis (see Figure B.1).

The transformation between the focus and image coordinate systems is simply a translational displacement. However, the relative orientation transformation between the global and focus coordinate systems is not as simple. In general, the coordinate
systems are related by rotational and translational transformations:

\[ X_F = AX + B \]  \hspace{1cm} (B.1)

or

\[
\begin{pmatrix}
  x_F \\
  y_F \\
  z_F
\end{pmatrix} =
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} +
\begin{pmatrix}
  b_x \\
  b_y \\
  b_z
\end{pmatrix}
\]  \hspace{1cm} (B.2)

where \( A \) is an orthonormal rotation matrix and \( B \) is a translation vector.

\section*{B.2 Collinearity Condition}

With the definitions given above, it is possible to derive an algebraic expression relating an object point's image coordinates to its global coordinates. This is accomplished through the use of the \textit{collinearity condition}. The collinearity condition simply states that the perspective center (focus), object point, and image point must be collinear. Applying the geometric relationships of similar triangles to the projection geometry shown in Figure B.1, the collinearity equations can be derived:

\[ \frac{u - u_P}{d} = \frac{x_F}{z_F} \]  \hspace{1cm} (B.3)

\[ \frac{v - v_P}{d} = \frac{y_F}{z_F} \]  \hspace{1cm} (B.4)

To relate the image coordinates to the global coordinates, the focus coordinates must be transformed into global coordinates using Equation B.2 and substituted into the collinearity equations.

\[ \frac{u - u_P}{d} = \frac{a_{11}x + a_{12}y + a_{13}z + b_x}{a_{31}x + a_{32}y + a_{33}z + b_z} \]  \hspace{1cm} (B.5)

\[ \frac{v - v_P}{d} = \frac{a_{21}x + a_{22}y + a_{23}z + b_y}{a_{31}x + a_{32}y + a_{33}z + b_z} \]  \hspace{1cm} (B.6)

Rearranging and solving for \( u \) and \( v \) explicitly,

\[ u = \frac{a_{11}x + a_{12}y + a_{13}z + b_x}{a_{31}x + a_{32}y + a_{33}z + b_z}d + \frac{a_{31}x + a_{32}y + a_{33}z + b_z}{a_{31}x + a_{32}y + a_{33}z + b_z}u_P \]  \hspace{1cm} (B.7)

\[ v = \frac{a_{21}x + a_{22}y + a_{23}z + b_y}{a_{31}x + a_{32}y + a_{33}z + b_z}d + \frac{a_{31}x + a_{32}y + a_{33}z + b_z}{a_{31}x + a_{32}y + a_{33}z + b_z}v_P \]  \hspace{1cm} (B.8)
or

\[
\begin{align*}
    u &= \frac{(a_{11}d + a_{31}u_P)x + (a_{12}d + a_{32}u_P)y + (a_{13}d + a_{33}u_P)z + (b_xd + b_xu_P)}{a_{31}x + a_{32}y + a_{33}z + b_x} \\
    v &= \frac{(a_{21}d + a_{31}u_P)x + (a_{22}d + a_{32}u_P)y + (a_{23}d + a_{33}u_P)z + (b_yd + b_yu_P)}{a_{31}x + a_{32}y + a_{33}z + b_x}
\end{align*}
\] (B.9) (B.10)

The focus coordinates have been eliminated and the collinearity equations directly relate the image coordinates to the object coordinates.

In order to use the collinearity equations for calculating object point coordinates, the system must first be calibrated. The principal point, principal distance, rotation matrix, and translation vector are explicitly determined in the system calibration. Traditionally, calibration is accomplished by imaging a reference frame of known geometry and is executed in two steps. The interior calibration involves determination of the principal point and principal distance. The exterior calibration determines the six degrees of freedom (3 rotational and 3 translational) of the transformation between the global and focus coordinate systems. Complete descriptions of these explicit analytical calibration methods are presented by Hallert [63] and Takamoto [140].

### B.3 Direct Linear Transform

Abdel-Aziz and Karara [1] developed a method for implicitly calibrating a photogrammetric system without explicitly determining the internal and external calibration parameters. This method is known as the *direct linear transform* (DLT). By dividing the numerator and denominator of both Equations B.9 and B.10 by \( b_x \), eleven DLT coefficients can be defined.

\[
\begin{align*}
    L_1 &= \frac{(a_{11}d + a_{31}u_P)}{b_x}, & L_5 &= \frac{(a_{21}d + a_{31}u_P)}{b_x} & L_9 &= \frac{a_{31}}{b_x} \\
    L_2 &= \frac{(a_{12}d + a_{32}u_P)}{b_x}, & L_6 &= \frac{(a_{22}d + a_{32}u_P)}{b_x} & L_{10} &= \frac{a_{32}}{b_x} \\
    L_3 &= \frac{(a_{13}d + a_{33}u_P)}{b_x}, & L_7 &= \frac{(a_{23}d + a_{33}u_P)}{b_x} & L_{11} &= \frac{a_{33}}{b_x} \\
    L_4 &= \frac{(b_xd + b_xu_P)}{b_x}, & L_8 &= \frac{(b_yd + b_yu_P)}{b_x}
\end{align*}
\]

Substituting the DLT coefficients into the collinearity equations results in

\[
\begin{align*}
    u &= \frac{L_1x + L_2y + L_3z + L_4}{L_9x + L_{10}y + L_{11}z + 1}
\end{align*}
\] (B.11)
\[
v = \frac{L_5 x + L_6 y + L_7 z + L_8}{L_9 x + L_{10} y + L_{11} z + 1}
\]  

(B.12)

The DLT coefficients are determined by imaging a reference frame of control points having a known three-dimensional distribution. Since each control point specifies two independent collinearity equations, a minimum of six control points are required to generate a linear system of equations that overdetermine the eleven DLT coefficients, \(L_1 \ldots L_{11}\). Rewriting the collinearity equations as linear equations in terms of the DLT coefficients yields

\[
L_1 x + L_2 y + L_3 z + L_4 - L_9 u x - L_{10} u y - L_{11} u z = u
\]  

(B.13)

\[
L_5 x + L_6 y + L_7 z + L_8 - L_9 v x - L_{10} v y - L_{11} v z = v
\]  

(B.14)

For \(n\) points \((n \geq 6)\), the linear system of equations in matrix form is

\[
\begin{bmatrix}
x_1 & y_1 & z_1 & 1 & 0 & 0 & 0 & 0 & -u_1 x_1 & -u_1 y_1 & -u_1 z_1 \\
0 & 0 & 0 & 0 & x_1 & y_1 & z_1 & 1 & -u_2 x_1 & -u_2 y_1 & -u_2 z_1 \\
x_2 & y_2 & z_2 & 1 & 0 & 0 & 0 & 0 & -u_3 x_2 & -u_3 y_2 & -u_3 z_2 \\
0 & 0 & 0 & 0 & x_2 & y_2 & z_2 & 1 & -u_4 x_2 & -u_4 y_2 & -u_4 z_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_n & y_n & z_n & 1 & 0 & 0 & 0 & 0 & -u_n x_n & -u_n y_n & -u_n z_n \\
0 & 0 & 0 & 0 & x_n & y_n & z_n & 1 & -u_n x_n & -u_n y_n & -u_n z_n
\end{bmatrix}
\begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4 \\
L_5 \\
L_6 \\
L_7 \\
L_8 \\
L_9 \\
L_{10} \\
L_{11}
\end{bmatrix} =
\begin{bmatrix}
u_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8 \\
v_9 \\
v_{10} \\
v_{11}
\end{bmatrix}
\]  

(B.15)

or in matrix notation

\[
C_{2n \times 11} L_{11 \times 1} = D_{2n \times 1}
\]  

(B.16)

This system of equations is similar to those presented by Alem et al [7].

Because the system of equations is overdetermined, a least-squares solution is necessary. The pseudo-inverse method is a type of least squares solution and requires premultiplying both sides of the equation by the transpose of the non-square matrix \(C\).

\[
C^T CL = C^T D
\]  

(B.17)
This system of equations is known as the normal equations in statistics. The inverse of $C^T C$ is known as the covariance matrix $V$ so the DLT coefficients equal

$$L = (C^T C)^{-1} C^T D = V C^T D$$ (B.18)

where $(C^T C)^{-1} C^T$ is known as the pseudo-inverse matrix. Numerically, Gauss-Jordan elimination [114] is used to solve Equation B.17 and generate the covariance matrix $V$, which is used for error estimation.

Once the direct linear transform coefficients have been determined for a given projection, the object point coordinates can be determined by rearranging Equations B.13 and B.14 with $x$, $y$, and $z$ as the unknowns.

$$(L_{1,i} - L_{9,i} u_{i,j}) x_j + (L_{2,i} - L_{10,i} u_{i,j}) y_j + (L_{3,i} - L_{11,i} u_{i,j}) z_j = u_{i,j} - L_{4,i} \quad (B.19)$$

$$(L_{5,i} - L_{9,i} v_{i,j}) x_j + (L_{6,i} - L_{10,i} v_{i,j}) y_j + (L_{7,i} - L_{11,i} v_{i,j}) z_j = v_{i,j} - L_{8,i} \quad (B.20)$$

where the subscripts in the equations are for projection $i$ and point $j$. For two projections, there are four equations with only three unknowns.

$$\begin{bmatrix}
L_{1,1} - L_{9,1} u_{1,j} \\
L_{5,1} - L_{9,1} v_{1,j} \\
L_{1,2} - L_{9,2} u_{2,j} \\
L_{5,2} - L_{9,2} v_{2,j}
\end{bmatrix} \begin{bmatrix}
L_{2,1} - L_{10,1} u_{1,j} \\
L_{6,1} - L_{10,1} v_{1,j} \\
L_{2,2} - L_{10,2} u_{2,j} \\
L_{6,2} - L_{10,2} v_{2,j}
\end{bmatrix} \begin{bmatrix}
L_{3,1} - L_{11,1} u_{1,j} \\
L_{7,1} - L_{11,1} v_{1,j} \\
L_{3,2} - L_{11,2} u_{2,j} \\
L_{7,2} - L_{11,2} v_{2,j}
\end{bmatrix} \begin{bmatrix}
x_j \\
y_j \\
z_j
\end{bmatrix} = \begin{bmatrix}
u_{1,j} - L_{4,1} \\
u_{1,j} - L_{8,1} \\
u_{2,j} - L_{4,2} \\
u_{2,j} - L_{8,2}
\end{bmatrix}$$ \quad (B.21)

Once again a least-squares solution is necessary. Using the same matrix notation as the least-squares DLT coefficient problem,

$$C_{4 \times 3} X_{G,3 \times 1} = D_{4 \times 1}$$ (B.22)

and the solution is

$$X_G = V C^T D$$ (B.23)

Additionally, an error estimate of the coordinates can be carried out as described by Takamoto [140]. The residual vector $R$ is calculated by substituting the solution into the initial equation

$$R = D - CX_G$$ (B.24)
The variance of an observation of unit weight is

$$\sigma_0^2 = \frac{\mathbf{R}^T \mathbf{R}}{DOF} \quad (B.25)$$

where the degrees of freedom (DOF) are the number of equations minus the number of unknowns (DOF = 1 for the given case). The standard error of each coordinate is then

$$\sigma_x = \sigma_0 \sqrt{V_{11}} \quad (B.26)$$

$$\sigma_y = \sigma_0 \sqrt{V_{22}} \quad (B.27)$$

$$\sigma_z = \sigma_0 \sqrt{V_{33}} \quad (B.28)$$

### B.4 Reduced Direct Linear Transform

Explicit determination of the rotation matrix is difficult. Therefore, a priori knowledge of the rotation matrices of both projections simplifies the calibration procedure for determining the remaining calibration parameters. Given a biplanar imaging configuration with orthogonal imaging planes, the global coordinate system is defined relative to the reference frame so the orientations of the global coordinate system and the first projection focus coordinate system are identical. For the configuration shown in Fig B.3, the rotation matrices\(^1\) are:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Substituting the rotation matrices into the collinearity equations for each projection (Equations B.9 and B.10) results in the simplified collinearity equations.

$$u_1 = \frac{d_1 x + u_{P,1} z + (b_{x,1} d_1 + b_{z,1} u_{P,1})}{x + b_{z,1}} \quad (B.29)$$

\(^1\)The first rotation matrix is simply the identity matrix. The second rotation matrix is for rotations about the \(Y_G\) axis.

$$A_2 = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

For the given configuration, \(\theta = \pi/2\).
Figure B.3: Definition of the global coordinate system and the two focus coordinate systems.

\[ v_1 = \frac{d_1 y + v_{P,1} z + (b_{y,1} d_1 + b_{z,1} v_{P,1})}{x + b_{z,1}} \]  \hspace{1cm} (B.30)

and

\[ u_2 = \frac{u_{P,2} x - d_2 z + (b_{z,2} d_2 + b_{z,2} u_{P,2})}{x + b_{z,2}} \]
\[ v_2 = \frac{v_{P,2} x + d_2 y + (b_{y,2} d_2 + b_{z,2} v_{P,2})}{x + b_{z,2}} \]  \hspace{1cm} (B.31)

(B.32)

Reduced sets of DLT coefficients with only six independent coefficients can be defined.

\[ L_{1,1} = \frac{d_1}{b_{z,1}} \quad , \quad L_{7,1} = \frac{v_{P,1}}{b_{z,1}} \]
\[ L_{2,1} = 0 \quad , \quad L_{8,1} = \frac{(b_{y,1} d_1 + b_{z,1} v_{P,1})}{b_{z,1}} \]
\[ L_{3,1} = \frac{u_{P,1}}{b_{z,1}} \quad , \quad L_{9,1} = 0 \]
\[ L_{4,1} = \frac{(b_{z,1} d_1 + b_{z,1} u_{P,1})}{b_{z,1}} \quad , \quad L_{10,1} = 0 \]
\[ L_{5,1} = 0 \quad , \quad L_{11,1} = \frac{1}{b_{z,1}} \]
\[ L_{6,1} = \frac{d_1}{b_{z,1}} = L_{1,1} \]
and

\[ L_{1,2} = \frac{u_{p,2}}{b_{x,2}} \]
\[ L_{2,2} = 0 \]
\[ L_{3,2} = -\frac{d_2}{b_{x,2}} \]
\[ L_{4,2} = \frac{b_{x,2}d_2 + b_{x,2}u_{p,2}}{b_{x,2}} \]
\[ L_{5,2} = \frac{v_{p,2}}{b_{x,2}} \]
\[ L_{6,2} = \frac{d_2}{b_{x,2}} = -L_{3,2} \]

the collinearity equations become

\[ u_1 = \frac{L_{1,1}x + L_{3,1}z + L_{4,1}}{L_{11,1}z + 1} \]  
(B.33)

\[ v_1 = \frac{L_{1,1}y + L_{7,1}z + L_{8,1}}{L_{11,1}z + 1} \]  
(B.34)

and

\[ u_2 = \frac{L_{1,2}x + L_{3,2}z + L_{4,2}}{L_{9,2}x + 1} \]  
(B.35)

\[ v_2 = \frac{L_{5,2}x - L_{3,2}y + L_{8,2}}{L_{9,2}x + 1} \]  
(B.36)

Equations for solving for the DLT coefficients are:

\[ L_{1,1}x + L_{3,1}z + L_{4,1} - L_{11,1}u_1z = u_1 \]  
(B.37)

\[ L_{1,1}y + L_{7,1}z + L_{8,1} - L_{11,1}v_1z = v_1 \]  
(B.38)

and

\[ L_{1,2}x + L_{3,2}z + L_{4,2} - L_{9,2}u_2x = u_2 \]  
(B.39)

\[ L_{5,2}x - L_{3,2}y + L_{8,2} - L_{9,2}v_2x = v_2 \]  
(B.40)

As with the full DLT method, a reference frame of control points is necessary to determine the reduced DLT coefficients. Since each point specifies two equations, a minimum of three control points are required to determine the six independent DLT parameters for each projection. In practice, a volume must be calibrated so a minimum of four points is needed. The systems of equations for determining the
reduced DLT coefficients for each projection are

\[
\begin{bmatrix}
x_1 & x_1 & 1 & 0 & 0 & -u_{1,1}z_1 \\
y_1 & 0 & 0 & z_1 & 1 & -v_{1,1}z_1 \\
x_2 & x_2 & 0 & 0 & 1 & -u_{1,2}z_2 \\
y_2 & 0 & 0 & z_2 & 1 & -v_{1,2}z_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_n & x_n & 0 & 0 & 1 & -u_{1,n}z_n \\
y_n & 0 & 0 & z_n & 1 & -v_{1,n}z_n
\end{bmatrix}
\begin{bmatrix}
L_{1,1} \\
L_{3,1} \\
L_{4,1} \\
L_{7,1} \\
L_{8,1} \\
L_{11,1}
\end{bmatrix}
= 
\begin{bmatrix}
u_{1,1} \\
v_{1,1} \\
u_{1,2} \\
v_{1,2} \\
u_{1,n} \\
v_{1,n}
\end{bmatrix}
\]

(B.41)

and

\[
\begin{bmatrix}
x_1 & x_1 & 1 & 0 & 0 & -u_{2,1}x_1 \\
0 & -y_1 & 0 & x_1 & 1 & -v_{2,1}x_1 \\
x_2 & x_2 & 0 & 0 & 1 & -u_{2,2}x_2 \\
0 & -y_2 & 0 & x_2 & 1 & -v_{2,2}x_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_n & x_n & 0 & 0 & 1 & -u_{2,n}x_n \\
0 & -y_n & 0 & x_n & 1 & -v_{2,n}x_n
\end{bmatrix}
\begin{bmatrix}
L_{1,2} \\
L_{3,2} \\
L_{4,2} \\
L_{5,2} \\
L_{8,2} \\
L_{9,2}
\end{bmatrix}
= 
\begin{bmatrix}
u_{2,1} \\
v_{2,1} \\
u_{2,2} \\
v_{2,2} \\
u_{2,n} \\
v_{2,n}
\end{bmatrix}
\]

(B.42)

The least squares pseudo-inverse method described in the previous section can be used to determine the reduced set of DLT coefficients for each projection. Once the object space has been calibrated by determining the reduced set of DLT coefficients, the collinearity equations can be rewritten to determine the coordinates of the object points.

\[
L_{1,1}x_j + (L_{3,1} - L_{11,1}u_{1,j})z_j = u_{1,j} - L_{4,1}
\]

(B.43)

\[
L_{1,1}y_j + (L_{7,1} - L_{11,1}v_{1,j})z_j = v_{1,j} - L_{8,1}
\]

(B.44)

\[
(L_{1,2} - L_{9,2}u_{2,j})x_j + L_{3,2}z_j = u_{2,j} - L_{4,2}
\]

(B.45)

\[
(L_{5,2} - L_{9,2}v_{2,j})x_j - L_{3,2}y_j = v_{2,j} - L_{8,2}
\]

(B.46)

By writing the equations in matrix form, the least squares pseudo-inverse method can be used to determine the object coordinates.

\[
\begin{bmatrix}
L_{1,1} & 0 & (L_{3,1} - L_{11,1}u_{1,j}) \\
0 & L_{1,1} & (L_{7,1} - L_{11,1}v_{1,j}) \\
(L_{1,2} - L_{9,2}u_{2,j}) & 0 & L_{3,2} \\
(L_{5,2} - L_{9,2}v_{2,j}) & -L_{3,2} & 0
\end{bmatrix}
\begin{bmatrix}
x_j \\
y_j \\
z_j
\end{bmatrix}
= 
\begin{bmatrix}
u_{1,j} - L_{4,1} \\
v_{1,j} - L_{8,1} \\
u_{2,j} - L_{4,2} \\
u_{2,j} - L_{8,2}
\end{bmatrix}
\]

(B.47)

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To estimate the error in the coordinates, the residuals can be calculated and the standard errors determined as in the previous section (Equations B.26, B.27, and B.28).

### B.5 Skew Ray Errors

In addition to calculating the standard errors of the object coordinates, there is a second measure of the accuracy of the coordinate reconstruction. As discussed previously, each of the two collinearity equations represent an equation of a plane. The combination of the pair of collinearity equations represents the intersection of these two planes to form a ray in object space. Two such reconstructed rays from two different projections are necessary to calculate the position of an object point in global coordinates. However, due to errors in the reconstructed projection rays, the rays do not intersect exactly. The distance between the rays is defined as the skew ray error (SRE). The distance between two rays is defined as the length of the vector that is mutually perpendicular to both rays and connects points on each ray.

In order to calculate the skew ray error, it is important to understand the physical basis for the solution of Equation B.21. The normal form for the equation of a plane is

\[
 n_x x + n_y y + n_z z = p
\]

where \( n^T = (n_x, n_y, n_z) \) is the plane's unit normal vector and \( p \) is the signed distance between the origin and the plane. The solution of Equation B.21 can be interpreted as determining the intersection of four planes. As such, Equation B.21 can be rewritten as

\[
 \begin{bmatrix}
 n_{11} & n_{12} & n_{13} \\
 n_{21} & n_{22} & n_{23} \\
 n_{31} & n_{32} & n_{33} \\
 n_{41} & n_{42} & n_{43}
\end{bmatrix}
\begin{bmatrix}
 x \\
 y \\
 z
\end{bmatrix}
= 
\begin{bmatrix}
 p_1 \\
 p_2 \\
 p_3 \\
 p_4
\end{bmatrix}
\]

where the matrix coefficients are normalized. In general, four planes will not intersect at a point, so a point that minimizes the distance to each plane must be selected.
To reconstruct the two projection rays, the first and second equations and the third and fourth equations are combined. Flanders and Price [51] describe a method for vector construction of the parametric form of the projection rays.

\[ x = ms + q \] (B.50)

where \( m \) is the slope vector of the ray and \( q \) is a point on the ray. Because the slope of the ray must be perpendicular to both normal vectors of the planes, \( m \) is equal to the cross product of the two normal vectors.

\[ m_1 = n_1 \times n_2 \] (B.51)
\[ m_2 = n_3 \times n_4 \] (B.52)

The point \( q \) can be determined as a linear combination of the normal vectors.

\[ q_1 = \alpha_1 n_1 + \beta_1 n_2 \] (B.53)
\[ q_2 = \alpha_2 n_3 + \beta_2 n_4 \] (B.54)

Since \( q_1 \) must lie in both the first and second plane, \( n_1 \cdot q_1 = p_1 \) and \( n_2 \cdot q_1 = p_2 \).

These equations can be rewritten in terms of \( \alpha_1 \) and \( \beta_1 \) to yield

\[
\begin{bmatrix}
  n_1 \cdot n_1 & n_1 \cdot n_2 \\
  n_2 \cdot n_1 & n_2 \cdot n_2
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \beta_1
\end{bmatrix}
= 
\begin{bmatrix}
  p_1 \\
  p_2
\end{bmatrix}
\] (B.55)

Similarly for the second ray,

\[
\begin{bmatrix}
  n_3 \cdot n_3 & n_3 \cdot n_4 \\
  n_4 \cdot n_3 & n_4 \cdot n_4
\end{bmatrix}
\begin{bmatrix}
  \alpha_2 \\
  \beta_2
\end{bmatrix}
= 
\begin{bmatrix}
  p_3 \\
  p_4
\end{bmatrix}
\] (B.56)

\( q_1 \) and \( q_2 \) are determined, and the four planar equations reduce to two parametric equations for projection rays.

\[ x = m_1 s_1 + q_1 \] (B.57)
\[ x = m_2 s_2 + q_2 \] (B.58)

To calculate the skew ray error, the first step is to determine a unit normal vector perpendicular to both rays. Once again, the vector cross product is useful. The vector
cross product of \( m_1 \) and \( m_2 \) divided by the magnitude of the cross product will result in the desired unit normal vector.

\[
\mathbf{n} = \frac{\mathbf{m}_1 \times \mathbf{m}_2}{|\mathbf{m}_1 \times \mathbf{m}_2|} \quad (B.59)
\]

Then the skew ray error is the projection of a vector connecting a point on each ray onto the unit normal vector or

\[
SRE = \mathbf{n} \cdot (\mathbf{q}_1 - \mathbf{q}_2) \quad (B.60)
\]

### B.6 TRACK Algorithms for Calculating Object Coordinates and Skew Ray Errors

Explicit determination of the calibration parameters provides the opportunity to use alternative photogrammetric processing methods. Mansfield [93] developed a unique method for processing stereophotogrammetric data in conjunction with a large volume motion analysis system (TRACK). These processing methods are general in their development and are applicable to any photogrammetric system that explicitly determines the calibration parameters. The primary reason for developing the reduced DLT method was to calculate the calibration parameters in a simple manner. Given the previous definition of the rotation matrices, the principal point, \((u_P, v_P)\), principal distance, \(d\), and coordinate translation vector, \(B\), can be determined directly from the DLT coefficients. The calibration parameters for both projections are presented as functions of the reduced set of DLT coefficients.

\[
\begin{align*}
\mathbf{u}_{P,1} &= \frac{L_{3,1}}{L_{11,1}}, \quad b_{x,1} = \frac{L_{4,1} L_{11,1} - L_{3,1}}{L_{11,1}}, \\
\mathbf{v}_{P,1} &= \frac{L_{7,1}}{L_{11,1}}, \quad b_{y,1} = \frac{L_{8,1} L_{11,1} - L_{7,1}}{L_{11,1}}, \\
\mathbf{d}_1 &= \frac{L_{11,1}}{L_{11,1}}, \quad b_{z,1} = \frac{1}{L_{11,1}}, \\
\mathbf{u}_{P,2} &= \frac{L_{1,2}}{L_{9,2}}, \quad b_{x,2} = \frac{L_{1,2} - L_{4,2} L_{9,2}}{L_{3,2} L_{9,2}},
\end{align*}
\] (B.61)
\[ v_{P,2} = \frac{L_{5,2}}{L_{9,2}} \quad b_{y,2} = \frac{L_{5,2} - L_{8,2}L_{9,2}}{L_{3,2}L_{9,2}} \]  \\
\[ d_2 = -\frac{L_{3,2}}{L_{9,2}} \quad b_{z,2} = \frac{1}{L_{9,2}} \]  

(B.62)

The TRACK processing of stereophotogrammetric data begins with the same collinearity equations. Normalized image plane coordinates \( u'_i \) and \( v'_i \) are introduced for ease of algebraic and numerical manipulation.

\[
u'_i = \frac{u_i - u_{F_i}}{d_i} = \frac{x_{F_i}}{z_{F_i}} \tag{B.63}
\]
\[
u'_i = \frac{v_i - v_{F_i}}{d_i} = \frac{y_{F_i}}{z_{F_i}} \tag{B.64}
\]

The collinearity equations can be used to rewrite the focus coordinates as a function of \( z_{F_i} \).

\[
X_{F_i} = \begin{pmatrix} x_{F_i} \\ y_{F_i} \\ z_{F_i} \end{pmatrix} = \begin{pmatrix} u'_iz_{F_i} \\ v'_iz_{F_i} \\ z_{F_i} \end{pmatrix} = z_{F_i} \begin{pmatrix} u'_i \\ v'_i \\ 1 \end{pmatrix}
\tag{B.65}
\]

Substituting this result into Equation B.1 relating focus to global coordinates eliminates the focus coordinates \( x_{F_i} \) and \( y_{F_i} \).

\[
z_{F_i} \begin{pmatrix} u'_i \\ v'_i \\ 1 \end{pmatrix} = A_iX_G + B_i
\tag{B.66}
\]

Rearranging the equation to solve for the global coordinates yields \(^2\)

\[
X_G = A_i^{-1} \left( z_{F_i} \begin{pmatrix} u'_i \\ v'_i \\ 1 \end{pmatrix} - B_i \right) = A_i^T \left( z_{F_i} \begin{pmatrix} u'_i \\ v'_i \\ 1 \end{pmatrix} - B_i \right)
\tag{B.67}
\]

For a given point, the global coordinates must be identical for both projections. Therefore, the RHS of Equation B.67 can be equated for the two projections. Therefore,

\[
A_i^T \left( z_{F_1} \begin{pmatrix} u'_1 \\ v'_1 \\ 1 \end{pmatrix} - B_1 \right) = A_2^T \left( z_{F_2} \begin{pmatrix} u'_2 \\ v'_2 \\ 1 \end{pmatrix} - B_2 \right)
\tag{B.68}
\]

\(^2\)Both rotation matrices \( A_i \) are orthonormal matrices and possess the property that \( A_i^{-1} = A_i^T \).
\[ A_1^T \left\{ \begin{array}{c} u_1' \\ v_1' \\ 1 \end{array} \right\} z_{F_1} \quad - A_2^T \left\{ \begin{array}{c} u_2' \\ v_2' \\ 1 \end{array} \right\} z_{F_2} = A_1^T B_1 - A_2^T B_2 \]  

(B.69)

\[
\left[ A_1^T \left\{ \begin{array}{c} u_1' \\ v_1' \\ 1 \end{array} \right\} \quad - A_2^T \left\{ \begin{array}{c} u_2' \\ v_2' \\ 1 \end{array} \right\} \right] \left\{ \begin{array}{c} z_{F_1} \\ z_{F_2} \end{array} \right\} = A_1^T B_1 - A_2^T B_2
\]  

(B.70)

Substituting for \( A_1^T \) and \( A_2^T \) and expanding terms results in a set of equations with only two unknowns.

\[
\begin{bmatrix}
 u_1' & -1 \\
 v_1' & -v_2' \\
 1 & u_2' \\
\end{bmatrix}
\begin{bmatrix}
 z_{F_1} \\
 z_{F_2} \\
\end{bmatrix}
= \begin{bmatrix}
 b_{z_1} - b_{z_2} \\
 b_{v_1} - b_{v_2} \\
 b_{x_1} + b_{x_2} \\
\end{bmatrix}
\]  

(B.71)

The variables \( z_{F_1} \) and \( z_{F_2} \) are overdetermined and a least squares solution is necessary. However, the solution involves inverting a \( 2 \times 2 \) matrix as opposed to a \( 3 \times 3 \) matrix for the previous methods which results in faster solution times.

Having calculated the distance of the object point along the reconstructed projection ray in the focus coordinate system, Equation B.67 is used to calculate the global coordinates of the projection ray points that define the mutually perpendicular vector between the projection rays. The least squares solution of the object point coordinates is the average of the two projection points.

\[
X_G = \frac{1}{2} X_{G_1} + \frac{1}{2} X_{G_2}
\]  

(B.72)

The skew ray error is simply the perpendicular distance between the two points.

\[
SRE = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]  

(B.73)

Although the main advantage of the TRACK algorithm is the speed of computation, the physical insight gained by understanding the solution method is important for pedagogical reasons.

### B.7 Sensitivity Analysis of X-Ray Stereophotogrammetry

The processing of X-ray photogrammetry data involves the digitization of marker locations. These digitizing procedures introduce errors into the calculation of the
Figure B.4: Basic configurations for convergent X-ray stereophotogrammetry. [98]

object point coordinates and it is important to understand the sensitivity of the measurements to such input errors. Since the intent of this analysis is to compare the relative sensitivities of convergent X-ray stereophotogrammetry with one image plane and biplanar X-ray stereophotogrammetry with two imaging planes, simplified two-dimensional models will be employed.

B.7.1 Convergent X-Ray Stereophotogrammetry

Convergent X-ray stereophotogrammetry projects two images onto the same film. Figure B.4 shows four different configurations with the same effective projection geometry. A schematic of the projection geometry is presented in Figure B.5.

In order to perform a sensitivity analysis, it is necessary to develop functional relationships between the object point coordinates and the image coordinates. This can be accomplished by writing the collinearity equation for each projection. Because this is a planar analysis, there is only one collinearity equation for each projection. Using the standard equation for a line passing through a known point with a given
Figure B.5: Schematized projection geometry for convergent X-ray stereophotogrammetry.

slope,
\[
\frac{z-d}{x-d} = \frac{d}{u_{P_1} - u_1}
\]  \hspace{1cm} (B.74)

and
\[
\frac{z-d}{x} = \frac{d}{u_2 - u_{P_2}}
\]  \hspace{1cm} (B.75)

Substituting \( u_{P_1} = \delta \) and \( u_{P_2} = 0 \) and solving the collinearity equations for \( x \) and \( z \) yields

\[
x = \frac{\delta u_2}{\delta + u_2 - u_1}
\]  \hspace{1cm} (B.76)

\[
z = \frac{d(u_2 - u_1)}{\delta + u_2 - u_1}
\]  \hspace{1cm} (B.77)

The partial derivatives of \( x \) and \( z \) with respect to \( u_1 \) and \( u_2 \) determine the sensitivities.

\[
\frac{\partial x}{\partial u_1} = \frac{\delta u_2}{(\delta + u_2 - u_1)^2}
\]  \hspace{1cm} (B.78)

\[
\frac{\partial x}{\partial u_2} = \frac{\delta^2 - \delta u_1}{(\delta + u_2 - u_1)^2}
\]  \hspace{1cm} (B.79)

\[
\frac{\partial z}{\partial u_1} = -\delta d
\]  \hspace{1cm} (B.80)

\[
\frac{\partial z}{\partial u_2} = \frac{\delta d}{(\delta + u_2 - u_1)^2}
\]  \hspace{1cm} (B.81)

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For purposes of comparison, a reference point in the object space must be used. Figure B.5 shows the object point to be halfway between the projection centers at a height $a$ from the film plane. The values of $u_1$ and $u_2$ for the reference point can be determined using similar triangles.

\[
\tan \theta_1 = \frac{\delta - u_1}{d} = \frac{\delta}{2} - \frac{u_1}{a} \tag{B.82}
\]

\[
\tan \theta_2 = \frac{u_2}{d} = \frac{u_2 - \frac{\delta}{2}}{a} \tag{B.83}
\]

Solving for $u_1$ and $u_2$,

\[
u_1 = \frac{\frac{\delta d}{2} - \delta a}{d - a} \tag{B.84}
\]

\[
u_2 = \frac{\frac{\delta d}{2}}{d - a} \tag{B.85}
\]

Substituting these results into the partial derivatives yields

\[
\frac{\partial x}{\partial u_1} = \frac{1}{2} \left( \frac{d - a}{d} \right) \tag{B.86}
\]

\[
\frac{\partial x}{\partial u_2} = \frac{1}{2} \left( \frac{d - a}{d} \right) \tag{B.87}
\]

\[
\frac{\partial z}{\partial u_1} = -\frac{d}{\delta} \left( \frac{d - a}{d} \right)^2 \tag{B.88}
\]

\[
\frac{\partial z}{\partial u_2} = \frac{d}{\delta} \left( \frac{d - a}{d} \right)^2 \tag{B.89}
\]

### B.7.2 Biplanar X-Ray Stereophotogrammetry

Biplanar X-ray stereophotogrammetry uses two X-ray sources projecting onto their two respective films. A schematic of the projection geometry is presented in Figure B.6.

Repeating the steps in the previous section, the collinearity equations are written.

\[
\frac{z - d}{x - a} = \frac{d}{u_{R_1} - u_1} \tag{B.90}
\]

and

\[
\frac{z - a}{x - d} = \frac{u_2 - u_{R_2}}{d} \tag{B.91}
\]
Substituting $u_{R1} = u_{R2} = 0$ and solving the collinearity equations for $x$ and $z$ yields

$$x = \frac{du_1 u_2 - adu_1 + d^2 u_1 + ad^2}{d^2 + u_1 u_2} \quad (B.92)$$

$$z = \frac{du_1 u_2 + adu_2 - d^2 u_2 + ad^2}{d^2 + u_1 u_2} \quad (B.93)$$

The partial derivatives of $x$ and $z$ with respect to $u_1$ and $u_2$ are

$$\frac{\partial x}{\partial u_1} = \frac{d^3 u_2 - ad^3 + d^4 - ad^2 u_2}{(d^2 + u_1 u_2)^2} \quad (B.94)$$

$$\frac{\partial x}{\partial u_2} = \frac{d^3 u_1 + adu_1^2 - d^2 u_1 - ad^2 u_1}{(d^2 + u_1 u_2)^2} \quad (B.95)$$

$$\frac{\partial z}{\partial u_1} = \frac{d^3 u_2 + adu_2^2 - d^2 u_2 - ad^2 u_2}{(d^2 + u_1 u_2)^2} \quad (B.96)$$

$$\frac{\partial z}{\partial u_2} = \frac{d^3 u_1 + ad^3 - d^4 - ad^2 u_1}{(d^2 + u_1 u_2)^2} \quad (B.97)$$

Using an analogous reference point, the partial derivatives are evaluated at $u_1 = u_2 = 0$.

$$\frac{\partial x}{\partial u_1} = \left( \frac{d - a}{d} \right) \quad (B.98)$$
\[ \frac{\partial x}{\partial u_2} = 0 \]  
\[ \frac{\partial z}{\partial u_1} = 0 \]  
\[ \frac{\partial z}{\partial u_2} = -\left( \frac{d - a}{d} \right) \]  

(B.99)  
(B.100)  
(B.101)

Comparing the sensitivities of the convergent and bi-plane stereophotogrammetric methods, the biplanar method is less sensitive to image coordinate errors. This is particularly important for reconstruction of the \( z \) coordinate. The sensitivities differ by a factor of \( d/\delta \), which is the height/base ratio. Takamoto [140] advised using a height/base ratio of 14, so the convergent photogrammetry method is an order of magnitude more sensitive to image coordinate errors than the biplanar method.
Appendix C

"Best Fit Plane" Approaches

The "best fit plane" for a given set of points has no inherent meaning unless an error criterion is specified. For example, the error function can be defined as the sum of the absolute values of the perpendicular distance between each individual point and the plane. Dealing with absolute value operations can be difficult, so the most common error function is the least-squared-error function. In general, the least-squared-error approach squares each error term so each term in the sum is positive without using the absolute value operator. However, the error must still be specified to implement a least-squared-error formulation. For example, if the error is defined as the perpendicular distance between each point and the plane, the error function to be minimized is the sum of the squared perpendicular distances between each point and the plane. Three different approaches for determining the "best fit plane" will be presented.

C.1 Linear Regression of a Plane

The linear regression approach minimizes the sum of the squares of the distances between each point and the plane in the z-direction (as opposed to the perpendicular distance between each point and the plane). Assume that the equation of the plane takes the form

\[ z = g(x, y) = ax + by + c \]  \hspace{1cm} (C.1)
Then the error for each point as defined by the distance between each point and the plane in the z-direction is

\[ e_i = z_i - g(x_i, y_i) = z_i - ax_i - by_i - c \]  \hspace{1cm} (C.2)

and the sum of the squared errors is

\[ E^2 = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} (z_i - ax_i - by_i - c)^2 \]  \hspace{1cm} (C.3)

The linear regression coefficients \( a, b, \) and \( c \) are determined by minimizing Equation C.3 with respect to each coefficient separately. To find the minimum, take the partial derivative of \( E^2 \) with respect to \( a, b, \) and \( c \) and set each partial derivative equal to zero. This yields the following system of linear equations:

\[
\frac{\partial E^2}{\partial a} = 0 = \sum_{i=1}^{N} 2(z_i - ax_i - by_i - c)(-x_i)
\]

\[
\frac{\partial E^2}{\partial b} = 0 = \sum_{i=1}^{N} 2(z_i - ax_i - by_i - c)(-y_i)
\]

\[
\frac{\partial E^2}{\partial c} = 0 = \sum_{i=1}^{N} 2(z_i - ax_i - by_i - c)(-1)
\]

Eliminating the factor of two and summing each term results in

\[
a \sum x_i^2 + b \sum x_i y_i + c \sum z_i = \sum z_i x_i
\]

\[
a \sum y_i x_i + b \sum y_i^2 + c \sum y_i = \sum z_i y_i
\]

\[
a \sum x_i + b \sum y_i + c N = \sum z_i
\]

or in matrix form

\[
\begin{bmatrix}
\sum x_i^2 & \sum x_i y_i & \sum x_i \\
\sum y_i x_i & \sum y_i^2 & \sum y_i \\
\sum x_i & \sum y_i & N
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
\sum z_i x_i \\
\sum z_i y_i \\
\sum z_i
\end{bmatrix}
\]  \hspace{1cm} (C.5)

This set of equations can be solved by Gaussian elimination.

Alternatively, the pseudo-inverse method can be used. This approach uses Equation C.1 to write \( N \) equations with three unknowns \( (N > 3) \).

\[
\begin{bmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
\vdots & \vdots & \vdots \\
x_N & y_N & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_N
\end{bmatrix}
\]  \hspace{1cm} (C.7)

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Using matrix notation,

\[ xa = z \]  \hspace{1cm} (C.8)

Multiplying both sides of the equation by \( x^T \) yields

\[ x^T xa = x^T z \]  \hspace{1cm} (C.9)

which is identical to Equation C.6. To solve for \( a \), multiply both sides of the equation by the inverse of \( x^T x \).

\[ a = \left( x^T x \right)^{-1} x^T z \]  \hspace{1cm} (C.10)

where \( \left( x^T x \right)^{-1} x^T \) is known as the pseudo-inverse matrix.

### C.2 Principal Axis Method

The principal axes and moments determine the orientation and dispersion of a swarm of points in space [17]. A familiar example of this concept is the principal axes and principal moments of inertia of the inertia tensor, where the moment of inertia is the sum of the moments of the points about an axis. Alternatively, the moments of the points for three orthogonal planes can be defined. Then the minimum principal moment and the corresponding principal axis define a "best fit plane."

If a swarm of points is translated so that its center of mass is at the origin, a symmetric "scatter matrix" \( M \) can be defined.

\[
M = \sum_{i=1}^{N} x_i x_i^T = \begin{bmatrix}
    \sum x_i^2 & \sum x_i y_i & \sum x_i z_i \\
    \sum y_i x_i & \sum y_i^2 & \sum y_i z_i \\
    \sum z_i x_i & \sum z_i y_i & \sum z_i^2 
\end{bmatrix}
\]  \hspace{1cm} (C.11)

where

\[
x_i = \begin{bmatrix}
    x_i \\
    y_i \\
    z_i
\end{bmatrix}
\]  \hspace{1cm} (C.12)

To determine the "best fit plane," define the error as the perpendicular distance between each point and the plane. For a plane that passes through the origin (the centroid of the points), the perpendicular distance between each point and the plane
is the dot product of the point coordinate vector and the unit normal vector of the plane.

\[ e_i = x_i^T n \]  
(C.13)

The sum of the squared errors is

\[ E^2 = \sum_{i=1}^{N} (x_i^T n)^2 \]  
(C.14)

\[ = \sum_{i=1}^{N} (n^T x_i) (x_i^T n) \]  
(C.15)

\[ = n^T \left( \sum_{i=1}^{N} x_i x_i^T \right) n \]  
(C.16)

\[ = n^T M n \]  
(C.17)

This equation can be shown to be equivalent to the standard form of an eigenvalue problem [23]

\[ M n = \lambda n \]  
(C.18)

where the eigenvalues \( \lambda_i \) are the principal values of the scatter matrix or the principal sums of squared errors \( E^2 \). Left-multiplying both sides of the equation by the transpose of the unit normal vector yields

\[ n^T M n = n^T \lambda n \]  
(C.19)

\( \lambda \) is a scalar and can be commuted with \( n \) and \( n^T n = 1 \) because \( n \) is a unit vector, so

\[ n^T M n = \lambda n^T n = \lambda \]  
(C.20)

which is identical to Equation C.17. To solve for the eigenvalues of the scatter matrix, consider Equation C.19 in the form

\[ n^T M n - n^T \lambda n = 0 \]  
(C.21)

\[ n^T (M - \lambda I) n = 0 \]  
(C.22)

Since \( n \) is a unit normal vector, the trivial solution that \( n = 0 \) is not possible. Therefore, the eigenvalues must meet the condition

\[ \det (M - \lambda I) = 0 \]  
(C.23)
In general, the eigenvalues are determined by calculating the roots of the characteristic equation (Equation C.23) which would be a third-order polynomial for this scatter matrix. The eigenvectors are then determined by solving

\[(M - \lambda I) \mathbf{n} = 0\]  

(C.24)

The solution to the "best fit plane" problem is the minimum eigenvalue. As stated above, the eigenvalues are the principal values of the sums of squared errors. Therefore, the minimum eigenvalue is the minimum sum of the squared perpendicular distances between each point and the "best fit plane." The eigenvector corresponding to the minimum eigenvalue is the unit normal vector for the plane. The plane is specified by its unit normal vector and a point on the plane which happens to be the origin (centroid of the points) in this case and the equation of the plane has the normal form

\[n_xx + n_yy + n_zz = 0\]  

(C.25)

The scatter matrix \(M\) as defined above is related to the inertia tensor of a swarm of points. The relationship between the scatter matrix and the inertial tensor will be derived to assist the reader in further understanding the principal axis formulation. The scatter matrix is defined as

\[M = \begin{bmatrix}
\sum x_i^2 & \sum x_i y_i & \sum x_i z_i \\
\sum y_i x_i & \sum y_i^2 & \sum y_i z_i \\
\sum z_i x_i & \sum z_i y_i & \sum z_i^2
\end{bmatrix}\]  

(C.26)

By adding and subtracting terms to the diagonal elements,

\[M = \begin{bmatrix}
\sum(x_i^2 + y_i^2 + z_i^2) & 0 & 0 \\
0 & \sum(x_i^2 + y_i^2 + z_i^2) & 0 \\
0 & 0 & \sum(x_i^2 + y_i^2 + z_i^2)
\end{bmatrix} - \begin{bmatrix}
\sum(y_i^2 + z_i^2) & -\sum x_i y_i & -\sum x_i z_i \\
-\sum y_i x_i & \sum(x_i^2 + z_i^2) & -\sum y_i z_i \\
-\sum z_i x_i & -\sum z_i y_i & \sum(x_i^2 + y_i^2)
\end{bmatrix}\]  

(C.27)

\[= \begin{bmatrix}
\sum d_i^2 & 0 & 0 \\
0 & \sum d_i^2 & 0 \\
0 & 0 & \sum d_i^2
\end{bmatrix} - \begin{bmatrix}
I_{xx} & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} & I_{yz} \\
I_{zx} & I_{zy} & I_{zz}
\end{bmatrix}\]  

(C.28)
\[ (\sum d_i^2) I - \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \]  

where \( d_i \) is the distance of each point to the origin. Consequently, the scatter matrix and inertia tensor are closely related to each other. The eigenvalues of the scatter matrix are simply shifted eigenvalues of the inertia tensor

\[ \lambda_{\text{scatter},j} = (\sum d_i^2) - \lambda_{\text{inertia},4-j} \]  

where eigenvalue subscripts \( j = 1 - 3 \) are the minimum to maximum eigenvalues and the eigenvectors of the eigenvalue pairs are equal.

A less formal explanation for this duality is that the diagonal terms of the scatter matrix are the sums of the distances from the plane perpendicular to the respective axis while the moments of inertia are the sums of the distances to the respective axis. Therefore, the sum of the diagonal terms of the scatter matrix and inertia tensor is the sum of the distances from each point to the origin.

### C.3 Lagrangian Multiplier Method

An alternative approach to the principal axis method is to use Lagrangian multipliers to minimize the sum of the squared perpendicular distances between each point and the “best fit plane.”\(^1\) The equation for the signed perpendicular distance (error) between a point and an arbitrary plane is

\[ e_i = n_z x_i + n_y y_i + n_x z_i - p \]  

where \( n \) is the plane unit normal vector and \( p \) is the signed perpendicular distance between the plane and the origin\(^2\) and the positive direction for \( e_i \) and \( p \) is defined by

\[ p = \vec{x}^T \vec{n} = \vec{x} n_x + \vec{y} n_y + \vec{z} n_z \]  

\(^1\) A brief derivation of this method was written by Dawei Qi in the Newman Laboratory for Biomechanics and Human Rehabilitation at MIT, Cambridge, MA.

\(^2\) This distance \( p \) is equal to the dot product of the position vector of the centroid of the swarm of points and the plane unit normal vector.

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the unit normal vector \( \mathbf{n} \). The sum of the errors is the function \( f \) to be minimized.

\[
f(n_x, n_y, n_z, p) = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} (n_x x_i + n_y y_i + n_z z_i - p)^2
\]

(C.34)

However, only three independent parameters are required to specify a plane, so the constraint that the normal vector be a unit vector must be included.

\[
\phi(n_x, n_y, n_z) = n_x^2 + n_y^2 + n_z^2 = 1
\]

(C.35)

The method of Lagrangian multipliers minimizes a function \( f \) subject to the constraint \( \phi \) by constructing a new function \( F \) such that

\[
F(n_x, n_y, n_z, p) = f(n_x, n_y, n_z, p) + \lambda \phi(n_x, n_y, n_z)
\]

(C.36)

\[
= \sum_{i=1}^{N} (n_x x_i + n_y y_i + n_z z_i - p)^2 + \lambda \left( n_x^2 + n_y^2 + n_z^2 \right)
\]

(C.37)

where \( \lambda \) is an undetermined Lagrangian multiplier [21]. This new function can now be minimized with respect to the variables \( n_x, n_y, n_z \), and \( p \). Differentiating \( F \) with respect to each of the variables and setting each partial derivative equal to zero results in the set of equations

\[
\frac{\partial F}{\partial n_x} = 0 = \sum_{i=1}^{N} 2(n_x x_i + n_y y_i + n_z z_i - p)x_i + \lambda (2n_x)
\]

(C.38)

\[
\frac{\partial F}{\partial n_y} = 0 = \sum_{i=1}^{N} 2(n_x x_i + n_y y_i + n_z z_i - p)y_i + \lambda (2n_y)
\]

\[
\frac{\partial F}{\partial n_z} = 0 = \sum_{i=1}^{N} 2(n_x x_i + n_y y_i + n_z z_i - p)z_i + \lambda (2n_z)
\]

\[
\frac{\partial F}{\partial p} = 0 = \sum_{i=1}^{N} 2(n_x x_i + n_y y_i + n_z z_i - p)(-1)
\]

Eliminating a factor of two from the equations and summing each individual term yields

\[
\begin{align*}
 n_x \sum x_i^2 + n_y \sum y_i + n_z \sum z_i = & \frac{1}{p} \sum x_i + n_x \lambda = 0 \\
 n_x \sum y_i x_i + n_y \sum y_i^2 + n_z \sum z_i = & \frac{1}{p} \sum y_i + n_y \lambda = 0 \\
 n_x \sum z_i x_i + n_y \sum z_i y_i + n_z \sum z_i^2 = & \frac{1}{p} \sum z_i + n_z \lambda = 0 \\
 n_x \sum x_i + n_y \sum y_i + n_z \sum z_i = & -pN = 0
\end{align*}
\]

(C.39)

When the origin of the coordinate system is translated to the centroid of the swarm of points, \( p \to 0 \). Then the error distance becomes

\[
e_i = x_i^T \mathbf{n}
\]

(C.33)

which is identical to Equation C.13.
The last equation can be rearranged to explicitly solve for \( p \).

\[
p = \frac{1}{N} \left( n_x \sum x_i + n_y \sum y_i + n_z \sum z_i \right)
\]

(\ref{C.40})

\[
= n_x \bar{x} + n_y \bar{y} + n_z \bar{z}
\]

(\ref{C.41})

Substituting this result into the first three equations of Equation \( \ref{C.39} \) to eliminate \( p \) and rewriting the equations in matrix form gives

\[
\begin{bmatrix}
\sum x_i^2 - \bar{x} \sum x_i + \lambda & \sum x_i y_i - \bar{y} \sum x_i & \sum x_i z_i - \bar{z} \sum x_i \\
\sum y_i x_i - \bar{x} \sum y_i & \sum y_i^2 - \bar{y} \sum y_i + \lambda & \sum y_i z_i - \bar{z} \sum y_i \\
\sum z_i x_i - \bar{x} \sum z_i & \sum z_i y_i - \bar{y} \sum z_i & \sum z_i^2 - \bar{z} \sum z_i + \lambda
\end{bmatrix}
\begin{bmatrix}
n_x \\
n_y \\
n_z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(\ref{C.42})

Using the definition of the centroid (\( \sum x_i = N \bar{x}, \ldots \)), the matrix can be rewritten as

\[
\begin{bmatrix}
\sum x_i^2 - N \bar{x}^2 + \lambda & \sum x_i y_i - N \bar{x} \bar{y} & \sum x_i z_i - N \bar{x} \bar{z} \\
\sum y_i x_i - N \bar{x} \bar{y} & \sum y_i^2 - N \bar{y}^2 + \lambda & \sum y_i z_i - N \bar{y} \bar{z} \\
\sum z_i x_i - N \bar{x} \bar{z} & \sum z_i y_i - N \bar{y} \bar{z} & \sum z_i^2 - N \bar{z}^2 + \lambda
\end{bmatrix}
\begin{bmatrix}
n_x \\
n_y \\
n_z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(\ref{C.43})

The terms with the centroid coordinates can be separated and shown to be equal to \( \bar{x} N \bar{x}^T \).

\[
\begin{bmatrix}
\sum x_i^2 & \sum x_i y_i & \sum x_i z_i \\
\sum y_i x_i & \sum y_i^2 & \sum y_i z_i \\
\sum z_i x_i & \sum z_i y_i & \sum z_i^2
\end{bmatrix}
\begin{bmatrix}
\bar{x} N \bar{x}^T + \lambda I
\end{bmatrix}
\begin{bmatrix}
n_x \\
n_y \\
n_z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(\ref{C.44})

\[
(M_{\text{origin}} - \bar{x} N \bar{x}^T + \lambda I) \mathbf{n} = 0
\]

(\ref{C.45})

The first two terms of the equation are equal to the scatter matrix for a coordinate system located at the centroid. The transformation for the scatter matrix due to translation is analogous to the parallel axis theorem for the inertia tensor, \( M_{\text{origin}} = M_{\text{centroid}} + \bar{x} N \bar{x}^T \).

\[
(M_{\text{centroid}} + \lambda I) \mathbf{n} = 0
\]

(\ref{C.46})

Since the trivial solution that \( \mathbf{n} = 0 \) is not possible because the normal vector is constrained to be a unit vector, the determinant of the matrix must be zero.

\[
det (M_{\text{centroid}} + \lambda I) = 0
\]

(\ref{C.47})

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This is equivalent to the eigenvalue solution of the principal axis formulation with the opposite sign on the Lagrangian multiplier (see Equation C.23). So by analogy to the principal axis method, one can calculate the three Lagrangian multipliers and choose the one with the minimum absolute value. The normal vector can then be determined and normalized to a unit normal vector. Finally, $p$ can be calculated from Equation C.41. It should not be a surprise that the principal axis method and the Lagrangian multiplier method give the same results because both methods are formulated to minimize the sum of the squared perpendicular distance between each point and the "best fit plane."