Private Risk

by

Mahdi H. Mattar

M.S. Civil and Environmental Engineering, M.I.T., August 1998
B.E. Civil and Environmental Engineering, American University of Beirut, July 1997

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Signature of the Author: ______________________
August 26, 2002

Certified by: ______________________
Professor Gordon M. Kaufman
Morris A. Adelman Professor of Management

Certified by: ______________________
Professor Fred Moavenzadeh
Professor of Engineering Systems and Civil and Environmental Engineering
Director, Center for Technology, Policy, and Industrial Development
Director, Technology and Development Program

Accepted by: ______________________
Professor Oral Buyukozturk
Chairman, Departmental Committee on Graduate Studies

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Abstract

In the first essay of this thesis, we extend the traditional decision analysis theory of buying price and selling price of a lottery. We allow the decision maker to rebalance his financial portfolio in the course of determination of a lottery's buying (selling) price. We build on the classical portfolio allocation problem in complete markets, generalizing to include both traded and non-traded unique risks. Our principal focus is on *private* risks—risks that are not tradable in financial markets. The first essay:

- Generalizes the treatment of the buying price and the selling price of a private risk lottery by allowing portfolio rebalancing in the course of determining these prices and
- Outlines the implications of this generalization for distributive bargaining.

The second essay is a study of methods for pricing unique risks in real options problems. This essay is a critical evaluation of how methods currently in vogue for pricing private risks affect real option value. We build a framework for valuing investments under uncertainty in the presence of private risks and demonstrate by example that different methods for pricing private risk can lead to decisively different real option values. To this end we use the classical oil and gas exploration and development example pioneered by Paddock, Siegel and Smith(1978). We show how, when private risks are present in this setting, alternative methods for valuation can lead to large differences in choice of a development policy and in associated valuations.

Thesis Supervisors:

Gordon M. Kaufman
Morris A. Adleman Professor of Management

Fred Moavenzadeh
Professor of Engineering Systems and Civil and Environmental Engineering
Director, Center for Technology, Policy, and Industrial Development
Director, Technology and Development Program

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OUTLINE

Introduction

In part I of this thesis, we extend the traditional decision analysis theory of buying price and selling price of a lottery by allowing the decision maker to rebalance his financial portfolio in the course of determination of b lottery's buying (selling) price. We build on the classical portfolio allocation problem in complete markets, generalizing to include both traded and non-traded unique risks. Our principal focus is on *private* risks—risks that are not tradable or traded in financial markets. The first essay:

- Generalizes the treatment of the buying price and the selling price of a private risk lottery by allowing portfolio rebalancing in the course of determining these prices and
- Outlines the implications of this generalization for distributive bargaining.

Part II is a study of methods for pricing unique risks in real options problems. This essay is a critical evaluation of how methods currently in vogue for pricing private risks affect real option value. We build a framework for valuing investments under uncertainty in the presence of private risks and demonstrate by example that different methods for pricing private risk can lead to decisively different real option values. To this end we use the classical oil and gas exploration and development example pioneered by Paddock, Siegel and Smith(1978). We show how, when private risks are present in this setting, alternative methods for valuation can lead to large differences in choice of a development policy and in associated valuations.

Unique, Private and Market Risks

Among approaches to valuing investment under uncertainty, contingent claim analysis plays a central role. It is the bridge that ties financial markets to valuation of investment projects available to managers: the Law of One Price says that if it is possible to construct a portfolio of financial market securities whose probability distribution over time perfectly mimics the probability distribution of cash flows of a project over time, then the value of the project is the value of the portfolio. In a simpler vernacular, we shall call a portfolio of traded market securities whose returns perfectly mimic the outcome of a lottery, a *market lottery.* If an investment project can be represented as a market lottery and financial markets are in equilibrium, the price of the lottery—hence the value of the project-- is uniquely determined by the Law of One Price. Said differently, if a spanning portfolio exists, *no arbitrage* dictates that the price of the lottery equals the current value of the spanning portfolio.

A *unique or unsystematic risk* is a risk that is uncorrelated with market risk. A consequence is that the market prices a unique risk lottery as bearing no risk premium. Unique risks are, from the market's perspective, *diversifiable* in the sense that such a risk can, in principle, be "atomized" by partitioning it into smaller and smaller components that can be sold individually in the market at closer and closer to a generic individual investor's zero risk level as the magnitude of each atom decreases.

We define a *private risk* to be a risk that may either be correlated with the market or be unique, but has the following additional characteristics:

- It represents a substantial portion of the investor's current wealth,
- It is either not tradable in securities markets or is inhibited from trading by large agency costs.

If a private risk is not tradable, it cannot be diversified away. Then subjective expected utility is an indispensable tool for rational pricing of that risk.

Our aim here is to explain how the owner of a private risk lottery should go about determination of his selling price to a (single) investor who has expressed interest in buying it. There is one buyer and one seller negotiating a selling (buying price) for this lottery and neither buyer nor seller can observe or deduce the price of the lottery from market prices by building a spanning portfolio from market securities. However, both seller and buyer are allowed to rebalance their market security portfolios in the course of determination of their respective buying and selling prices for this private lottery. We shall explain how the Zone of Possible Agreement (ZOPA) is affected by enlarging the choice sets of buyer and seller in this fashion.

Literature Review

Lessard and Miller (2001) classify types of risks faced in large engineering projects. They define *residual risks* to be those risks that remain after strategizing to reduce, shift, transform and diversify away identifiable risks.[1] Sponsors of a project who possess a *comparative advantage* in bearing residual risks often embrace them. Their comparative advantage may

> "...arise for any one of three reasons: some parties may have more information about particular risks and their impacts than others; some parties or stakeholders may have different degrees of influence over outcomes; or some investors differ in their ability to diversify risks".[2]

Residual risks are, in the terminology adopted here, *private risks*. Lessard and Miller give several examples where local partners load-up on these private or residual risks in recognition of their ability to influence outcomes. While possessing a competitive advantage relative to private risks, local partners may "...have little ability to diversify risks and little knowledge about commercial prospects worldwide". For example, a Chilean firm Endesa is planning on buying a power-generating plant in Argentina.

[1] Lessard and Miller (2000) Ch. 3 Mapping and Facing the Landscape of Risks pp. 87-88.
[2] Lessard and Miller (2000) Ch. 3 Mapping and Facing the Landscape of Risks pp. 89

Endesa has prior experience in privatization and knows more about the future of the Argentine power sector than the local government. "Based on its experience as an operator, Endesa has a clear information and influence advantage when it comes to operating risk". [3]

Private risks are often private because the investor *chooses* to hold them to exploit a comparative advantage, despite the fact that they may be diversifiable in a market context. To sell successfully such a private risk in the market, he would have to find a mechanism that compensates market participants that do not possess his comparative advantage for agency costs.

If the consequences of a lottery are uncorrelated with the market (a *unique risk lottery)* then, from the market's perspective this lottery is *diversifiable* and is priced in the market with no risk premium--just take the expected value of lottery consequences at each point in time and discount at the risk-free rate. This is the standard nostrum adopted by financial engineers to price diversifiable risk. Luenberger (1998) defines this valuation procedure as ***zero-level pricing***:

> "One way to assign a value to such a project is to make believe that the project value is a price, and then set the price so that you would be indifferent between either purchasing a small portion of the project or not. This is called zero-level pricing since you will purchase the project at zero level."
>
> "If there is only private uncertainty the zero-level price is just the discounted expected value of the project (using actual probabilities)." [4]

He generalizes zero-level pricing to lotteries (or projects) that have both market and unique risk components. Suppose that the consequences of a lottery *Y* depend on both the state s of the market and a state e of a unique lottery. Define y_{se} to be the consequence of the joint event $s \cap e$. The marginal probability that a market event $s \in S$ occurs is p_s, and

[3] Example From Lessard and Miller (2000) pp90.
[4] Luenberger (1998) p 458

the marginal probability that a unique lottery event $e \in E$ occurs is q_e. Assuming that market events and unique lottery events are probabilistically independent, the lottery Y is representable as $L_Y = \{(y_{se}; p_s q_e) \mid s \in S, e \in E\}$.

By definition, if there exists a set of market securities that spans all market states then there exists a unique risk-neutral probability π_s for each market state $s \in S$. Zero-level pricing requires that the risk neutral probability π_{se} of state $s \cap e$ satisfy $\pi_{se} = \pi_s q_e$. As we show in Section__, for an individual investor-decision maker, π_{se} depends on the investor's utility function and is in general, not representable as a product $\pi_{se} = \pi_s q_e$ of probabilities.

In a similar vein, Neely (1998)[5] argues that:

> "Simply put, endogenous project uncertainties are not correlated with the external market events. Therefore, the beta of cash-flows that are functions of endogenous uncertainties is zero, and the proper discount rate for evaluating these cash-flows is the risk-free rate".

As does Luenberger, he applies zero-level pricing to value contingent claims contracts on real assets and real option problems.

According to Trigeorgis (1998)[6],

> "...no premium would be required for the part of an asset's risk (i.e., the unique or firm specific risk) that can be diversified away".

However Trigeorgis does not go further in pricing unique risks and limits his treatment of real options to market risks.

The previous discussion on pricing unique risk is a reasonable representation of financial economists' approach to valuing unique risk. The key assumption driving this valuation procedure is that project specific risks are uncorrelated with the market

[5] Neely (1998) pp79
[6] Trigeorgis (1998) pp 41

portfolio and can be diversified away, so investors do not require a risk premium in pricing these risks. When an investor owns a private risk with consequences that represent a substantial proportion his wealth, we expect his subjective beliefs *and* preferences for risk bearing to come into play. If he is risk averse, we intuitively expect that he would assign a positive risk premium to such a private risk.

Once risks are outside the realm of financial markets, subjective expected utility is the sensible alternative for measuring the value of risk to an individual investor. Any other defined (statistical, mathematical) measure of risk can be justified as an approximation to subjective expected utility evaluation. For the owner of a private risk, subjective expected utility is the analytical glue that binds financial market valuation to private risk valuation.

Luenberger expands his treatment of unique risk to the case in which unique risk cannot be diversified away, in particular when "...the cash outlay required may represent a significant portion of one's investment capital". This is similar to our definition of private risk where the risk is not traded and the investor cannot diversify it away. He proposes a ***buying price analysis*** for valuing a cash flow lottery of this type. The buying price b of a private risk lottery is the price at which the investor is indifferent between owning the lottery or not. If b is the buying price, the investor's expected utility without the lottery equals his expected utility with the lottery purchased at price b. For an investor who is not risk neutral, this cash flow buying price clearly depends on the investor's risk preferences and probability beliefs about the unique risk component of such a private cash flow lottery.

Luenberger does a buying price analysis of cash flow lotteries with both market and private risks that differs from a zero-level pricing approach: he assumes that the investor is constantly risk averse; i.e. the investor's utility function for wealth is exponential at each of a discrete set of future time points. He first calculates the certainty equivalent for uncertain cash flow at each discrete point in time and then computes the discounted value of cash flow certainty equivalents. Exponential utility for terminal wealth has the advantage of mathematical tractability: the certainty equivalent for a single stage lottery is functionally independent of initial wealth prior to observation of the outcome of the lottery. A consequence is that one does not need to address the problem of portfolio rebalancing when the private lottery is added. The price paid is lack of flexibility in capturing the shape of preferences for investors who may be decreasingly or increasingly risk averse as their level of wealth changes.

Similarly, Smith and Nau (1995) propose an integrated valuation procedure for pricing projects under uncertainty with private risks. The investor's subjective probabilities and utility function are used to compute the certainty equivalent for the private risk component of a cash flow at each of a discrete set of points in time. Market risks are priced using complete market risk-neutral probabilities. As they employ exponential utility, they do not need to address the issue of portfolio rebalancing in the course of calculating the buying and the selling price for an uncertain cash flow.

When the investor is faced with a private lottery that is *perfectly correlated* with the market the decision variable is not the price of the lottery – since it is observed in the market – but it is how much of the risky asset to hold. The market prices this lottery. If the market is in equilibrium, there are no arbitrage opportunities and the equilibrium investor has to adjust his holdings of other risky and non-risky assets so as to align his own risk neutral probabilities with the market risk neutral probabilities. The investor can use his personal risk neutral probabilities to price the private lottery and arrive at a higher or lower price than the market quoted price. If so, he will engage in buying or selling of other risky assets in order to create a hedge strategy for his private risk. The investor will continue to buy or sell risky assets until his subjective expected utility prices are in line with market prices.

David Mayers (1973, 1976) derives a pricing model for investors who hold "two kinds of assets, perfectly liquid (marketable) or perfectly non-liquid (nonmarketable)". Labor income is an example of a perfectly non-liquid asset. Mayers constructs and solves this special portfolio problem. He builds a single period "extended model" of capital asset pricing. He also shows that the composition of the market portfolio varies widely across investors:

> "Each investor holds a portfolio of marketable assets that solves his personal and possibly unique portfolio problem". [7]

A principal difference between Mayers' analysis and ours is that in Mayers' analysis non-marketable assets are correlated with market securities and so are not unique risks. Truly private risks, unlike human capital, may be uncorrelated with all market securities. When there is only one market security Mayers' analysis implies that adding a private risk to the investor's portfolio will not affect the composition of his market portfolio (it will only affect the proportion of his total wealth invested in the market portfolio as we show in the next chapter). As is the case with "modern portfolio theory", all investors will hold the same market portfolio. When a private risk asset is uncorrelated with market securities, capital asset pricing models can not be applied to price it. We will show later in the thesis how to use subjective expected utility to price such a private risk asset.

[7] Mayers (1973) pp 259-260

I Private risk and the Investor's Portfolio Problem

The focus of our development of an expanded theory of buying and selling price for a private lottery is this: if an investor can simultaneously rebalance his market portfolio and buy or sell a private lottery, then both buying and selling prices must account for rebalancing opportunities. This broader perspective leads to new insights.

If rebalancing is ignored or not allowed the calculating buying and selling price of a private lottery is well understood (Raiffa 1968). However, when rebalancing is allowed:

- How does the buying (or selling) price of a private lottery change?
- What happens of the ZOPA for buyer and seller?

I.1 The Generic Investor's Problem

We adopt a formulation of the investor portfolio problem similar to that of Huang and Litzenberger (1988) and Leroy and Werner (2001). These authors provide an exhaustive review of the literature which we do not reproduce here. An investor endowed with wealth w_0 at t =0 must decide to allocate w_0 among N market securities. With the exception of one risk free security, values (market price per share) of the remaining N-1 securities are uncertain. If the investor wishes to behave rationally – maximize his subjective expected utility for the wealth w_1 at t = 1, how should he behave?

The following standard formulation of his problem will serve as a benchmark for our treatment of private risk. Let $P_i^{(0)}$ be the price of security i at t = 0, and set x_{is} equal to the change in value $P_i^{(1)}$-$P_i^{(0)}$ plus any cash flow or dividend from security i if market state

$s \in S = \{s_1, \ldots, s_N\}$ obtains at time t = 1. If he buys (sells short) α_i shares of security i at t = 0, his wealth at t = 1 given that states s obtains is

$$w_{1,s} = w_0 + \sum_{i=1}^{N} \alpha_i x_{is} \tag{1.1}$$

Define $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ and

$$\underset{N\times N}{\mathbf{X}} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{bmatrix} = \left[\mathbf{x}^{(1)}, \quad \cdots \quad , \mathbf{x}^{(N)}\right] \tag{1.2}$$

Namely, $\boldsymbol{x}_i$ is the vector composed of the elements of the i^th^ row of the (NxN) matrix $\mathbf{X}$ with x_{ij} the payoff or change in value of security i from t = 0 to t = 1 if the market state $s=j$ obtains. (Also $\mathbf{x}^{(s)}$ is the s^th^ column vector) This terminology allows us to write $w_1(s, \boldsymbol{\alpha})$ at t = 1 given (s, $\boldsymbol{\alpha}$) as

$$w_1(s; \boldsymbol{\alpha}) = w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} \tag{1.3}$$

If our investor assigns probability p_s to the state s for each $s \in$ S and possess a monotone increasing concave utility function U for terminal wealth at t = 1, then there exits a unique solution to his investment problem:

$$\max_{\alpha} E_{\mathbf{X}}\left[U\left(w_0 + \alpha_1 \mathbf{x}_1 + \cdots + \alpha_N \mathbf{x}_N\right)\right] \tag{1.4}$$

subject to $\boldsymbol{\alpha}\mathbf{P}^{(0)} = w_0$; if no short selling is allowed, $\boldsymbol{\alpha} \geq 0$. Here $\mathbf{P}^{(0)} = (P_1^{(0)}, \ldots, P_N^{(0)})^t$. Because we have chosen $\mathbf{X}$ to be nonsingular, a unique solution $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$ exists and satisfies the LaGrangian conditions:

$$\frac{\partial L_i}{\partial \alpha_i} = \sum_{s \in S} p_s U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)}) x_{is} = \lambda P_i^{(0)} \ , \ i = 1, 2, \ \ldots, \ \mathrm{N} \tag{1.5}$$

and $$w_0 = \boldsymbol{\alpha}\mathbf{P}^{(0)} \, .$$

These conditions can be recast in terms of risk neutral probabilities as follows:

define $$\pi_s \equiv \frac{1}{r} \frac{p_s U'\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)}\right)}{\sum_{s \in S} p_s U'\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)}\right)} \ , \qquad s = 1, \ldots, N \, . \tag{1.6}$$

Summing over states,

$$\sum_{s=1}^{N} \pi_s = \frac{1}{1 + r_f} \equiv \frac{1}{r} \tag{1.7}$$

with $\pi^{(s)} \equiv (\pi_1, \ \ldots \ , \ \pi_s, \ \ldots, \ \pi_N)^t$ and at $\alpha = \alpha^*$:

$$\mathbf{X}\boldsymbol{\pi}^{(s)} = \mathbf{P}^{(o)} \qquad \text{or} \qquad \boldsymbol{\pi}^{(s)} = \mathbf{X}^{-1}\mathbf{P}^{(0)} \, , \tag{1.8}$$

provided that **X** is nonsingular (complete markets).

I.2 Private Risk

Suppose that this investor possess the opportunity to buy a *private risk* that will unfold on (a) a domain of mutually exclusive and collectively exhaustive events $E \equiv \{e_1, \ldots, e_M\}$ and possibly on both E and S at once. Which particular event will obtain in E is uncertain, so our investor assigns marginal probabilities q_j to the event "e_j obtains at the time t = 1", j = 1, 2, …, M., and a joint probability q_{sj} to each (s,e) ∈ S x E. Only our investor observes e ∈ E, at time t = 1. If we label a rv "private" we mean that this rv or uncertain quantity is *probabilistically independent* of the uncertain event "state of the market at t + 1". We suppose that the investor receives a payoff from ownership of the private risk lottery *Y*. The rv *Y* has domain $\{s \cap e \mid s \in S, e \in E\} \equiv S{\times}E$ and range $\mathbb{R} = (-\infty, \infty)$ or some measurable subset of $\mathbb{R}$. When market and private risk events are independent, *Y* is a lottery $\{(y_{se}; p_s q_e) \mid s \in S, e \in E\}$.

In traditional discussions of portfolio optimization, Ingersoll (1987) Huang and Litzenberger (1993) LeRoy and Werner (2000) assumes that all risks are market risks or that market alternatively are totally incomplete (where all risks are private risks). we construct a portfolio problem in which the investor can buy or sell private risk and simultaneously rebalance his market portfolio.

I.2.1 Hoff's Analysis

Hoff(1997) develops a valuation approach for uncertain payoffs when markets are incomplete. The basic idea of his research is derived from the field of financial economics especially the application of portfolio optimization and valuation using state contingent securities. Hoff assumes the investor's utility function is of the form: $U(c_0, w_1) = U(c_0) + U(w_1)$ where c_0 is the consumption is year 0, and w_1 is wealth in year one. $U(w_1)$ is a utility vector over states *se*.

In order to render the relation between our analysis and Hoff's, we rederive that of Hoff's in our notation. The main distinction between our analysis and Hoff's is that we consider only year one wealth without consumption in the previous year. This difference does not change the character of results in the context of valuation of uncertain investments. From now on, we will assume the investor's utility is of the form $U(w_1)$.

An investor, endowed with wealth w_0 at t =0, wishes to determine the buying price b of a private risk lottery $Y \equiv \{(y_e;q_e)|\ e \in E\}$.He must decide how to allocate w_0 among N market securities given that Y is purchased at b. Hoff argues that the buying price b of Y is determined as follows,

$$b = \boldsymbol{\psi}.\mathbf{y} \tag{1.9}$$

Here ψ is a vector of risk neutral probabilities defined in the following way:

$$\psi = \frac{1}{r}\frac{\nabla \mathbf{u}}{\nabla \mathbf{u} \bullet \mathbf{1}} \tag{1.10}$$

where

$r \equiv 1 + r_f$, with r_f the risk free rate,

and

$$\nabla \mathbf{u} = [\nabla u_1, \nabla u_2, \ldots, \nabla u_e]$$

$$\nabla u_e = \begin{cases} \dfrac{E_{X,Y}[U(\hat{w}_{se})] - E_{X,Y}[U(w_{se})]}{\Delta w_e + y_e - b}, & if(\Delta w_{se} + y_e - b) \neq 0 \\ E[U'(w_{se})], & if(\Delta w_{se} + y_e - b) = 0 \end{cases} \tag{1.11}$$

Here w_{se} $(= w_s)$ is the period 1 wealth in state *se* before purchase the private risk, and $\hat{w}_{se}$ is period 1 wealth after the buying (selling) of the lottery and re-optimizing the market portfolio. Δw_{se} is the change in market wealth (proportion of the investor wealth allocation to market securities) in period one due to re-optimization. y_e is the payoff of the private lottery in state *e*.

Hoff proves that (1.9)- (1.11) hold for both complete and incomplete markets. However his definition of an incomplete market is based on two assumptions: (1) The private risk Y is not spanned by market securities and (2) the investor can only trade in the risk free asset – he can not buy the private risk lottery and trade in market securities at the same time. This second assumption is a severe practical limitation as investors can always trade a current market portfolio.

The following is the proof of the above results modified to our notation and assumptions about U.

Complete Markets: In the compete market case Y is spanned by market securities. Then Y can be represented as a linear combination of market securities. In this case $\hat{w}_{se} = w_{se}$ and the investor doesn't need to rebalance his portfolio, i.e. $(\Delta w + y_e - b) = 0$. Then

$$\psi = \frac{1}{r}\frac{\nabla \mathbf{u}}{\nabla \mathbf{u} \cdot \mathbf{1}} = \frac{1}{r}\frac{E[U'(\mathbf{w})]}{E[U'(\mathbf{w})] \cdot \mathbf{1}} \tag{1.12}$$

which is exactly the market's risk neutral probability (assuming no arbitrage). Then the price of Y becomes $b = \boldsymbol{\psi}.\mathbf{y}$.

Incomplete Markets: According to Hoff's second assumption, in incomplete markets the investor can trade exclusively in the risk free asset. When markets are incomplete the proof goes as follow: By definition the buying price is the price that makes the investor indifferent between the status quo and buying the lottery:

$$E_{X,Y}\left[U(\hat{w})\right] = E_{X,Y}\left[U(w)\right] \tag{1.13}$$

where

$$\hat{w} = w + \Delta w + Y - b \tag{1.14}$$

and since the investor can only modify his investment in the risk free asset,

$$E_{X,Y}\left[U(\hat{\mathbf{w}})\right]=E_Y\left[U((w-b)r\cdot\mathbf{1}+\mathbf{y})\right] \tag{1.15}$$

and $$E_{X,Y}\left[U(\mathbf{w})\right]=U(wr) \tag{1.16}$$

then

$$\begin{aligned}
&E_Y\left[U((w-b)r\cdot\mathbf{1}+\mathbf{y})\right]=U(wr)\\
&\frac{E_Y\left[U((w-b)r\cdot\mathbf{1}+\mathbf{y})\right]-U(wr)}{(\mathbf{y}-br\cdot\mathbf{1})}(\mathbf{y}-br\cdot\mathbf{1})=0\\
&(r.\nabla\mathbf{u}.\mathbf{1})b=\nabla\mathbf{u}.\mathbf{y}\\
&b=\frac{1}{r}\frac{\nabla\mathbf{u}}{\nabla\mathbf{u}.\mathbf{1}}.\mathbf{y}
\end{aligned} \tag{1.17}$$

as in the second part of (1.11).

The main disadvantage with this approach to evaluate the buying price of a private risk is that in order to calculate risk neutral probabilities one must first calculate $\nabla\mathbf{u}$. However one needs to know b in order to calculate $\nabla\mathbf{u}$. If we approximate $\nabla\mathbf{u}_i$ by $E\left[U'(w_i)\right]$ then we are *ignoring* the portfolio rebalancing effect. Hoff seems to ignore the portfolio rebalancing in his thesis. By limiting rebalancing to the risk free security alone, he argues that the portfolio rebalancing effect on buying and selling price is minimal and can be ignored. Had he allowed trading in general when evaluating buying and selling price of private risk, the above analysis may fail as we will show by example in sections I.3 and I.4.

In the next section we will explore pricing of private risk. Our definition of "incomplete market" differs from that of Hoff's: a private risk lottery is not spanned by existing market securities and the investor will rebalance his existing market portfolio (not just the risk free asset) as he or she adjusts for the addition or sale of a private lottery. This freedom to rebalance expands the domain of choice and yields buying and selling prices that are decisively different from buying and selling prices in the absence of rebalancing.

I.3 Buying and Selling Price

The buying price of a private lottery is the maximum amount the investor is willing to pay given his current wealth allocation. In other word, it is the value that makes him indifferent between buying the lottery and the status quo Let b_{max} be the buying price for the lottery *Y*:

At the status quo, the investor's expected utility is,

$$\bar{U}_0 \equiv E_X\left[U\left(w_0 + \boldsymbol{\alpha}^* \mathbf{x}^{(s)}\right)\right] \tag{1.18}$$

For a fixed known buying price b of a private lottery *Y*, the investor will reallocate his portfolio to conform to that price. Namely, he will maximize his portfolio selection $\boldsymbol{\alpha}$ for given *Y* and b:

$$\begin{aligned} &\max_{\alpha} E_{X,Y}\left[U\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + Y - b\right)\right] \\ &s.t. \quad \boldsymbol{\alpha}\mathbf{P}^{(0)} + b = w_0 \end{aligned} \tag{1.19}$$

At the optimum ($\boldsymbol{\alpha}^b$) expected utility is

$$\bar{U}(b) \equiv E_{X,Y}\left[U\left(w_0 + \boldsymbol{\alpha}^b \mathbf{x}^{(s)} + Y - b\right)\right] \tag{1.20}$$

The **buying price** b_{max} is then defined to be

$$\begin{aligned} &\max \ b \\ &s.t. \quad \bar{U}(b) = \bar{U}_0 \quad and \quad \boldsymbol{\alpha}\mathbf{P}^{(0)} + b = w_0 \end{aligned} \tag{1.21}$$

Similarly the selling price of a lottery is the minimum price the investor is willing to accept in exchange for his lottery. Let s_{min} be the selling price of the lottery *Y*. Note in this case, the status quo, the investor owns the private lottery *Y*.

At the status quo, if the investor owns *Y*, his expected utility is,

$$\begin{aligned} &\max_{\alpha} E_{X,Y}\left[U\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + Y\right)\right] \\ &s.t. \quad \boldsymbol{\alpha}\mathbf{P}^{(0)} = w_0 \end{aligned} \tag{1.22}$$

Define

$$\bar{U}_Y \equiv E_{X,Y}\left[U\left(w_0 + \boldsymbol{\alpha}^M\mathbf{x}^{(s)} + Y\right)\right] \tag{1.23}$$

For a fixed known selling price s, the investor will reallocate his portfolio to conform to that price.

$$\begin{aligned} &\max_{\alpha} E_{X}\left[U\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + s\right)\right] \\ &s.t. \quad \boldsymbol{\alpha}\mathbf{P}^{(0)} - s = w_0 \end{aligned} \tag{1.24}$$

At the optimum portfolio rebalance $\boldsymbol{\alpha}^s$, given selling price *s*, the investor's expected utility is

$$\bar{U}(s) \equiv E_X\left[U\left(w_0 + \boldsymbol{\alpha}^s\mathbf{x}^{(s)} + s\right)\right] \tag{1.25}$$

The **selling price** s_{min} for *Y* is defined to be

$$\begin{aligned} &\min \quad s \\ &s.t. \quad \bar{U}(s) = \bar{U}_Y \quad and \quad \boldsymbol{\alpha}\mathbf{P}^{(0)} - s = w_0 \end{aligned} \tag{1.26}$$

I.3.1 Effect of B&S Price on Expected Utility

If our investor added to his holdings Y at cost b, then his expected utility is

$$E_{X,Y}\left[U\left(w_0+\boldsymbol{\alpha}\mathbf{x}^{(s)}+Y-b\right)\right] \tag{1.27}$$

Where $\boldsymbol{\alpha}$ satisfies $w_0 = \boldsymbol{\alpha}\, \boldsymbol{P}^{(0)} + b$. That is, his wealth allocate to market securities is diminished by b. If he buys Y, he will re-allocate $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}^*(b)$ to satisfy

$$\frac{\partial L_i}{\partial \alpha_i}=\sum_{s\in S}\sum_{e\in E} p_s q_e U'(w_0+\boldsymbol{\alpha}\mathbf{x}^{(s)}+y_{se}-b)x_{is}=\lambda P_i^{(0)}, \quad \mathrm{i}=1,2,\ \ldots,\ \mathrm{N} \tag{1.28}$$

and $\quad w_0 = \boldsymbol{\alpha}\boldsymbol{P}^{(0)} + b.$

The investor's expected utility at the optimal portfolio becomes:

$$E_{X,Y}\left[U\left(w_0+\boldsymbol{\alpha}^*(b)\mathbf{x}^{(s)}+Y-b\right)\right] \tag{1.29}$$

Proposition 1: Equation (1.29) is monotonic decreasing in b. That is, if portfolio rebalancing is allowed and the investor rebalancing optimally for each possible buying price b, expected utility is monotone decreasing with increasing b.

Proof: Assume that u(y) is continuous in y, monotone increasing in y and (at least) twice differentiable.

Let $\quad f_\alpha(b)=E_{X,Y}\left[u\left(w_o+\boldsymbol{\alpha}\mathbf{x}^{(s)}+Y-b\right)\right]$ $\forall$ fixed value of $\boldsymbol{\alpha}$.

Then $$\frac{d}{db} f_{\alpha}(b) = -E_{X,Y}\left[U'\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + Y - b\right)\right]$$

As $U' > 0$, $\frac{d}{db} f_{\alpha}(b) < 0$ i.e. $\forall \boldsymbol{\alpha}$ fixed $f_{\alpha}(b)$ is monotone decreasing as a function of b.

Suppose $b_1 > b_2$. Then $\forall \alpha$, $f_{\alpha}(b_1) < f_{\alpha}(b_2)$

Let $\boldsymbol{\alpha}_1$ = the value of $\boldsymbol{\alpha}$ that maximizes $f_{\alpha}(b_1)$ at fixed $b = b_1$.

Let $\boldsymbol{\alpha}_2$ = the value of $\boldsymbol{\alpha}$ that maximizes $f_{\alpha}(b_2)$ at fixed $b = b_2$.

We know $f_{\alpha_1}(b_1) < f_{\alpha_1}(b_2)$ and $f_{\alpha_1}(b_2) \le f_{\alpha_2}(b_2)$ since α_2 is the value of α that maximizes $f_{\alpha}(b_2)$. Consequently $f_{\alpha_1}(b_1) \le f_{\alpha_2}(b_2)$ and equation (1.29) is monotonic decreasing in b. □

If our investor **shorts** his holdings *Y* at 'price' *b*, then his expected utility is

$$E_{X,Y} = \left[U\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} - Y + b\right)\right] \tag{1.30}$$

with $\boldsymbol{\alpha}$ satisfying $w_0 + b = \boldsymbol{\alpha} \boldsymbol{P}^{(0)}$. That is, his wealth allocation to market securities is increased by *b* in return for exposure to *-Y*. If he shorts *Y*, he will re-allocate $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}^*(b)$ to satisfy

$$\frac{\partial L_i}{\partial \alpha_i} = \sum_{s\in S}\sum_{e\in E} p_s q_e U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} - y_{se} + b)x_{is} = \lambda P_i^{(0)}, \quad i = 1, 2, \ldots, N \tag{1.31}$$

and $w_0 = \boldsymbol{\alpha}\boldsymbol{P}^{(0)} - b$.

The investor's expected utility at the optimal portfolio becomes:

$$E_{X,Y}\left[U\left(w_0+\boldsymbol{\alpha}^*(b)\ \ \mathbf{x}^{(s)}-Y+b\right)\right] \tag{1.32}$$

Proposition 2: Equation (1.32) is monotonic increasing in b. That is, if portfolio rebalancing is allowed and the investor rebalancing optimally for each possible selling price b, expected utility is monotone increasing with increasing b.

Proof: Let $f_\alpha(b)=E_{X,Y}\left[u\left(w_o+\boldsymbol{\alpha}\mathbf{x}^{(s)}-Y+b\right)\right]$ $\forall$ fixed value of $\boldsymbol{\alpha}$. Then

$$\frac{d}{db}f_\alpha(b)=E_{X,Y}\left[U'\left(w_0+\boldsymbol{\alpha}\ \ \mathbf{x}^{(s)}-Y+b\right)\right].$$

As $U'>0$, $\frac{d}{db}f_\alpha(b)>0$ i.e. $\forall\boldsymbol{\alpha}$ fixed $f_\alpha(b)$ is monotone increasing as a function of b.

Suppose $b_1<b_2$. Then $\forall\boldsymbol{\alpha}$, $f_\alpha(b_1)<f_\alpha(b_2)$

Let $\boldsymbol{\alpha}_1$ = the value of $\boldsymbol{\alpha}$ that maximizes $f_\alpha(b_1)$ at fixed $b=b_1$.

Let $\boldsymbol{\alpha}_2$ = the value of $\boldsymbol{\alpha}$ that maximizes $f_\alpha(b_2)$ at fixed $b=b_2$.

We know $f_{\alpha_1}(b_1)<f_{\alpha_1}(b_2)$ and $f_{\alpha_1}(b_2)\le f_{\alpha_2}(b_2)$ since α_1 is the value of α that maximizes $f_\alpha(b_1)$. Consequently $f_{\alpha_2}(b_2)\ge f_{\alpha_1}(b_1)$ and equation (1.32) is monotonic increasing in b. □

Similarly, If our investor **sold** his holdings *Y* at the 'price' *b*, then his expected utility is

$$E_{X,Y}=\left[U\left(w_0+\boldsymbol{\alpha}^*\mathbf{x}^{(s)}+b\right)\right] \tag{1.33}$$

and we can prove in the same fashion that the above equation is monotone increasing in b.

I.3.2 Illustration

An investor considers investing in market securities for one time period.. There are two traded securities, a risk-free security and a risky security. Possible outcomes of returns are described in the table below:

	Probability	***Payoff***	***Return***
	0.3	3	200%
Risky Security	0.4	1	0
	0.3	0.5	-50%
Risk free	1	1.05	5%

Best portfolio allocation in the status quo: The investor problem is to select an optimal portfolio allocation between the risky security α_1 and the risk free α_2. Assume the investor's utility, for market returns, is U(x) = ln(x), and no short selling is allowed. If the investor's initial wealth is $w_0 = 10.0$, he wishes to choose α_1 and α_2 to achieve

$$\max_{\alpha_1 \alpha_2} E[\ln(w_0 + \alpha_1 x_1 + \alpha_2 x_2)]$$

$$\text{s.t.} \qquad \alpha_1 P_1 + \alpha_2 P_2 = w_0$$

Given the numerical specifications of returns in the table above, his portfolio allocation problem is

$$\max_{\alpha_1,\alpha_2}\left[0.3\ln(10+2\alpha_1+0.05\alpha_2)+0.4\ln(10+0.05\alpha_2)+0.3\ln(10-0.5\alpha_1+0.05\alpha_2)\right]$$

$$\text{s.t. } \alpha_1+\alpha_2=10$$

The optimal allocation $\boldsymbol{\alpha}^*$ is $\alpha_1 = 6.364$, $\alpha_2 = 3.636$. In other words, the investor should invest 63.64% of his wealth in the risky security. At the optimal solution expected utility is

$$\bar{U}_0 \equiv E_X\left[U\left(w_0+\boldsymbol{\alpha}^*\mathbf{x}^{(s)}\right)\right]=E\left[\ln\left(w_0+\ \alpha_1^*x_1+\alpha_2^*x_2\right)\right]$$

The certainty equivalent of the optimal portfolio is $\bar{U}_0 = 2.4515$

Buying price: Assume the investor is considering investing in a venture whose payoffs are probabilistically independent of market returns – a private (unique) risk. This venture payoffs are 4 with probability 0.5 and 1 with probability 0.5.

For a fixed known cost b (investment in the venture) the investor wishes to choose $\boldsymbol{\alpha}^*$ to satisfy

$$\max_{\alpha_1,\alpha_2} E_{X,\,Y}\left[U\left(w_0+\alpha_1x_1+\alpha_2x_2+Y-b\right)\right]$$
$$s.t.\quad \alpha_1P_1+\alpha_2P_2+b=w_0$$

At the optimum ($\alpha_1^b+\alpha_2^b$) expected utility is

$$\bar{U}(b)\equiv E_{X,Y}\left[\ln\left(w_0+\ \alpha_1^bx_1+\alpha_2^bx_2+Y-b\right)\right]=$$

$$0.5*\left[0.3\ln(10+2\alpha_1^b+0.05\alpha_2^b+4-b)+0.4\ln(10+0.05\alpha_2^b+4-b)+0.3\ln(10-0.5\alpha_1^b+0.05\alpha_2^b+4-b)\right]$$
$$+0.5*\left[0.3\ln(10+2\alpha_1^b+0.05\alpha_2^b+1-b)+0.4\ln(10+0.05\alpha_2^b+1-b)+0.3\ln(10-0.5\alpha_1^b+0.05\alpha_2^b+1-b)\right]$$

In order to find the buying price of this private lottery - the maximum price (b_{max}) that the investor is willing to pay for it if portfolio rebalancing is allowed - is the value of b for which the investor is indifferent between investing in the venture and the status quo.

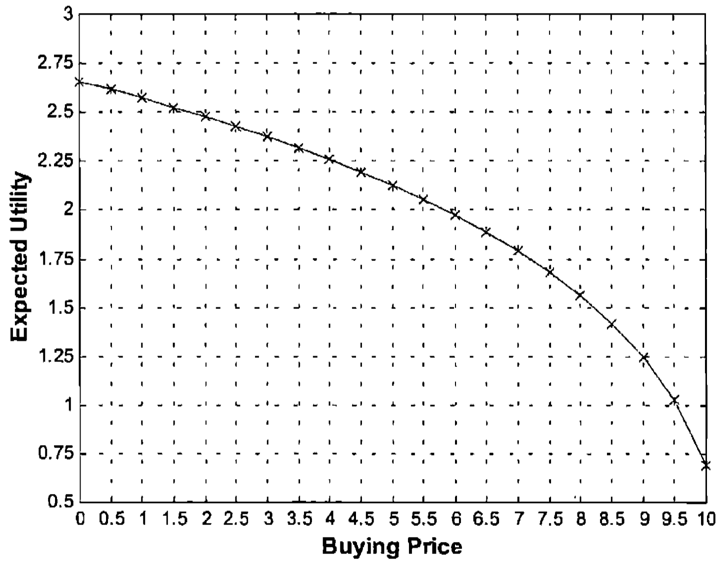

Figure I-1 Expected utility is a decreasing function of the venture buying price

Here b_{max} is defined as the solution to

$$\begin{aligned} &\max \quad b \\ &s.t. \quad \bar{U}(b) = \bar{U}_0 = 2.4515 \\ &\qquad \alpha_1 + \alpha_2 + b = 10 \end{aligned}$$

The buying price of the private lottery Y is $\mathbf{b_{max}}$= **2.2636** and the optimal portfolio rebalancing is α_1= 6.1950, α_2= 1.5414.

Best portfolio allocation if the investor owns the lottery: suppose that the investor owns this venture and is considering selling it. His expected utility of rebalancing is allowed is

$$\max_{\alpha_1,\alpha_2} \; E_{X,Y}\left[\ln\left(w_0 + \; \alpha_1 x_1 + \alpha_2 x_2 + Y\right)\right]$$

$$\text{s.t.} \qquad \alpha_1 P_1 + \alpha_2 P_2 = w_0$$

We find that are α_1= 7.6785, α_2= 2.3215. At optimal rebalancing , expected utility is

$$\bar{U}_M \equiv E_{X,Y}\left[U\left(w_0 + \boldsymbol{\alpha}^M \mathbf{x}^{(s)} + Y\right)\right] = E\left[\ln\left(w_0 + \; \alpha_1^M x_1 + \alpha_2^M x_2 + Y\right)\right]$$

$$\bar{U}_M = 2.6573$$

Selling price: for a fixed known selling price s the problem is

$$\max_{\alpha_1,\alpha_2} E_X\left[U\left(w_0 + \alpha_1 x_1 + \alpha_2 x_2 + s\right)\right]$$

$$s.t. \quad \alpha_1 P_1 + \alpha_2 P_2 - s = w_0$$

At the optimum ($\alpha_1^s + \alpha_2^s$), expected utility is

$$\bar{U}(s) \equiv E\left[\ln\left(w_0 + \; \alpha_1^b x_1 + \alpha_2^b x_2 + s\right)\right]$$

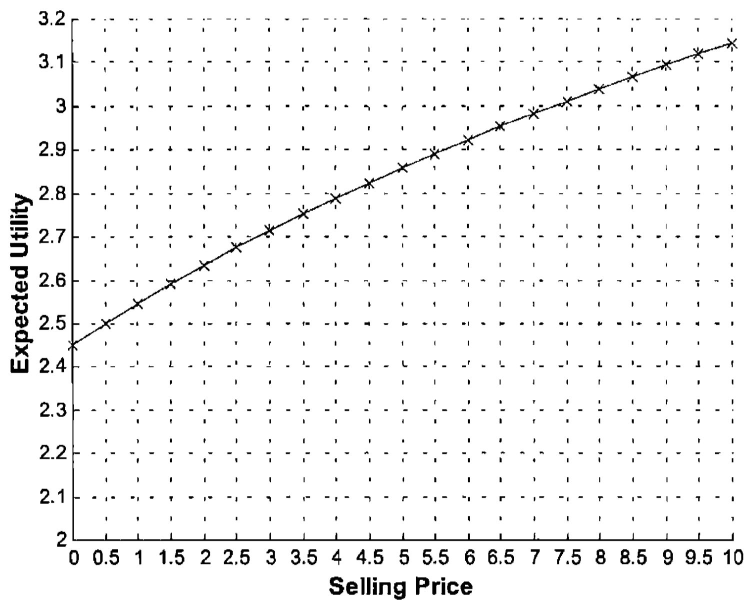

Figure I-2 Expected utility is an increasing function of the selling price

The selling price, the minimum he should accept with portfolio rebalancing is

$$\begin{aligned} &\min \quad s \\ &s.t. \quad \bar{U}(s) = \bar{U}_M = 2.6573 \\ &\qquad \alpha_1 + \alpha_2 - s = 10 \end{aligned}$$

The results are α_1= 7.8177, α_2= 4.4673 and $\mathbf{s_{min}}$=**2.2849** .

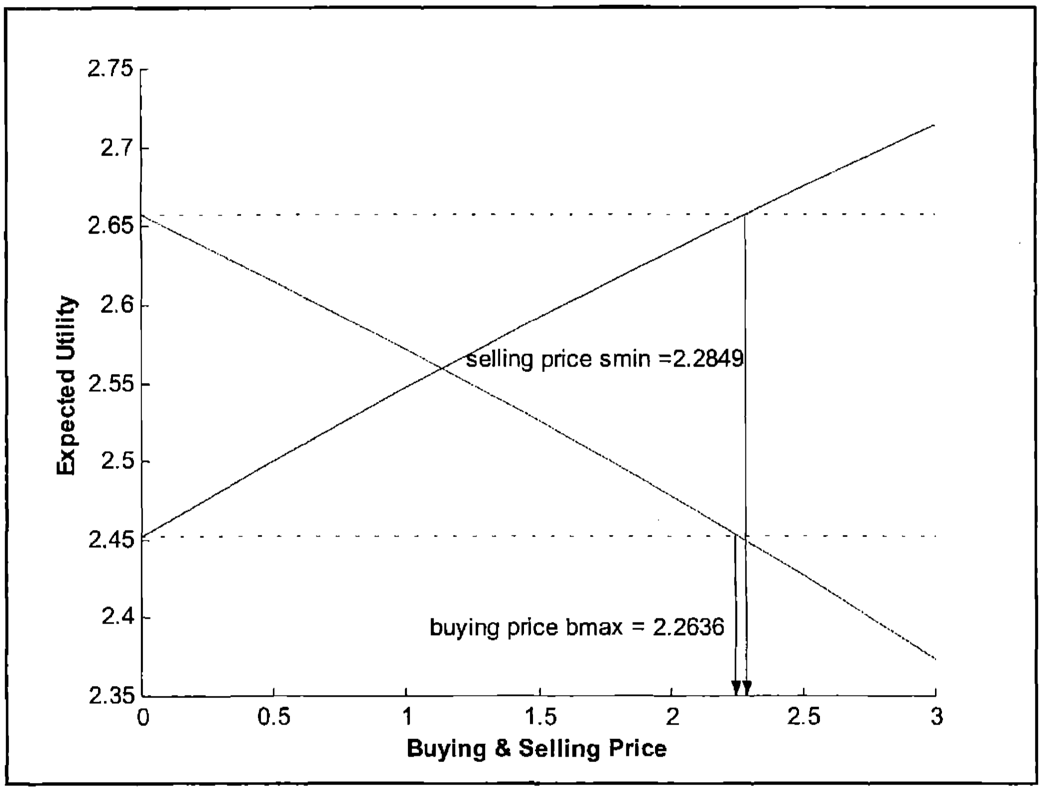

Figure I-3 Selling and buying price

I.3.3 Risk Neutral Probability Approach

If our investor adds Y to his holdings at the cost b, then his expected utility is

$$E_{X,Y}=\left[U\left(w_0+\sum_{i=1}^{N}\alpha_i\mathbf{x}_i+Y-b\right)\right] \tag{1.34}$$

where $\boldsymbol{\alpha}$ satisfies $w_0 = \boldsymbol{\alpha}\boldsymbol{P}^{(0)} + b$. That is, his market security wealth allocation is diminished by b. If he buys Y, he will re-allocate $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}^*$ to satisfy

$$\frac{\partial L_i}{\partial \alpha_i} = \sum_{s\in S}\sum_{e\in E} p_s q_e U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + y_{se} - b)x_{is} = \lambda P_i^{(0)}, \quad i = 1, 2, \ldots, N \tag{1.35}$$

and $$w_0 = \boldsymbol{\alpha}\boldsymbol{P}^{(0)} + b. \tag{1.36}$$

The above system has a unique solution $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{**}$ (N+1 unknowns and N+1 equations).

Because we have introduced a risk free market asset

$$\lambda = \sum_{s\in S}\sum_{e\in E} p_s q_e U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + y_{se} - b) \tag{1.37}$$

we can calculate, at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{**}$, risk neutral probabilities

$$\pi_{es} = \frac{1}{r}\,\frac{p_s q_e U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + y_{se} - b)}{\sum_{s\in S}\sum_{e\in E} p_s q_e U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + y_{se} - b)} \tag{1.38}$$

In terms of the NM risk neutral probabilities (π_{es})[8], we have

$$\sum_{s\in S}\sum_{e\in E} \pi_{es} x_{is} = P_i^{(o)}, \quad i = 1, 2, \ldots, N \tag{1.39}$$

and

$$\sum_{s\in S}\sum_{e\in E} \pi_{es} y_{es} = b$$

(1.40)

[8] If we relax the assumption that events in E and S are *probabilistically independent.* Equation (1.40) still hold with π_{se} defined as follows:

$$\pi_{es} = \frac{1}{r}\,\frac{p_{se} U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + y_{se} - b)}{\sum_{s\in S}\sum_{e\in E} p_{se} U'(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + y_{se} - b)}$$

Where p_{se} is the probability of the joint event $s \cap e$.

As x_{is} and e are functionally independent, upon setting $\pi_s = \sum_{e \in E} \pi_{es}$,

$$\sum_{s \in S} \pi_s x_{is} = P_i^{(o)} \tag{1.41}$$

and we recover the conditions of the complete market case.

Now decompose π_{es} into π_s and $\pi_{e|s}$ ≡ the risk neutral probability that e obtains conditional upon market state s. Define

$$\sum_{e \in E} \pi_{e|s} y_{es} = \bar{y}_s \tag{1.42}$$

a *risk neutral conditional expectation of payoff* from Y given market state s. then (1.40) is representable as

$$\sum_{s \in S} \pi_s \bar{y}_s = b \tag{1.43}$$

Upon setting $\bar{\mathbf{y}} = (\bar{y}_1, \ldots, \bar{y}_N)$, optimality conditions become

$$\begin{bmatrix} \mathbf{X} \\ \bar{\mathbf{y}} \end{bmatrix} \boldsymbol{\pi}^{(s)} = \begin{pmatrix} \mathbf{P}^{(0)} \\ b \end{pmatrix}. \tag{1.44}$$

Because we have chose $\boldsymbol{X}$ to be (N×N) and non-singular $\boldsymbol{\pi}^{(s)} = \mathbf{X}^{-1}\mathbf{P}^{(0)}$ as in the absence of Y – the market is complete – but we have something new:

Proposition 3: If the investor can purchase the private risk lottery Y at cost b, then

(1) There exist NM risk-neutral probabilities

$$\underset{(N \times M)}{\mathbf{\Pi}} \equiv \left[\left(\pi_{se} \right) \right]$$

for which $\sum_{e \in E} \pi_{es} = \pi_s$, s ∈ S and $\sum_{s \in S} \pi_{es} = \pi_e$, e ∈ E,

and satisfies:

$$\sum_{s \in S} \sum_{e \in E} \pi_{es} y_{es} = b$$

(2) At the optimal choice α^{**} for *this* investor, Y is equivalent to a unique market portfolio represented by θ (N×1) satisfying:

$$\boldsymbol{\theta} \ \mathbf{X} = \overline{\mathbf{y}} \quad \text{and} \quad \boldsymbol{\theta} \mathbf{P}^{(0)} = b$$

where $\overline{\mathbf{y}}$ is a *risk neutral conditional expectation of payoff* from *Y*.

<u>Proof</u>: Because X (N ×N) is non-singular, there exists a unique solution $\boldsymbol{\theta} = \overline{\mathbf{y}}\mathbf{X}^{-1}$ for any given $\overline{\mathbf{y}}$.

I.4 Effect of Portfolio Rebalancing.

I.4.1 Valuation of Private Risk

As discussed earlier, the value of a private risk lottery is determined by its buying price or selling price, depending whether one is buying or selling the lottery. The maximum buying price, for example, should be such that the investor is indifferent between the status quo and adding the lottery at the maximum buying price. Of course, any price bellow the maximum buying price would make the investor better off and its expected utility would be higher. (Refer to proposition 1, the expected utility is decreasing with the buying price).

Adding the lottery at any given price should be followed by rebalancing of the existing portfolio of market securities. Falling to do so would lead to suboptimal results.

Proposition 4: The maximum buying price with portfolio rebalancing is higher than the maximum buying price without rebalancing.

Proof: At the status quo the investor expected utility is $E_X\left[U\left(w_0+\boldsymbol{\alpha}_0\mathbf{x}^{(s)}\right)\right]=\bar{U}_0$ where α_0 is the optimal portfolio allocation. If the investor calculates the maximum buying price without rebalancing the market portfolio then the buying price ($b_{\max}^{unb}$) is defined to be

$$\begin{aligned}&\max \quad b\\ &s.t. \quad \bar{U}_{unb}(b)=E_{X,Y}\left[U\left(w_0+\boldsymbol{\alpha}_0\mathbf{x}^{(s)}+Y-b\right)\right]=\bar{U}_0\end{aligned}$$

or ,

$$\bar{U}_{unb}\left(\boldsymbol{\alpha}_0,b_{\max}^{unb}\right)=\bar{U}_0 \ .$$

We know that

$$\max_{\boldsymbol{\alpha}} E_{X,Y}\left[U\left(w_0+\boldsymbol{\alpha}\mathbf{x}^{(s)}+Y-b\right)\right]\geq E_{X,Y}\left[U\left(w_0+\boldsymbol{\alpha}_0\mathbf{x}^{(s)}+Y-b\right)\right], \qquad \forall b$$

as a consequence for $b = b_{\max}^{unb}$ we have $\bar{U}\left(b_{\max}^{unb}\right) \geq \bar{U}_{unb}\left(b_{\max}^{unb}\right)$, which implies $\bar{U}\left(b_{\max}^{unb}\right) \geq \bar{U}_0$. On the other hand, we have by definition that $\bar{U}\left(b_{\max}\right) = \bar{U}_0$, so then $\bar{U}\left(b_{\max}^{unb}\right) \geq \bar{U}\left(b_{\max}\right)$. Since $\bar{U}(b)$ decreasing in b (proposition 1) then $b_{\max}^{unb} \leq b_{\max}$. □

Proposition 5: The minimum selling price with portfolio rebalancing is lower than the minimum selling price without rebalancing.

Proof: At the status quo the investor own the utility and his expected utility is equal to $E_{X,Y}\left[U\left(w_0 + \boldsymbol{\alpha}_M \mathbf{x}^{(s)} + Y\right)\right] = \bar{U}_M$ where $\boldsymbol{\alpha}_M$ is the optimal portfolio allocation. If the investor calculates the minimum selling price without rebalancing the market portfolio then the selling price ($s_{\min}^{unb}$) is defined to be

$$\begin{aligned} &\min \quad s \\ &s.t. \quad \bar{U}_{unb}(s) = E_X\left[U\left(w_0 + \boldsymbol{\alpha}_M \mathbf{x}^{(s)} + s\right)\right] = \bar{U}_M \end{aligned}$$

or,

$$\bar{U}_{unb}\left(\boldsymbol{\alpha}_M, s_{\min}^{unb}\right) = \bar{U}_M .$$

We know that

$$\max_{\boldsymbol{\alpha}} E_X\left[U\left(w_0 + \boldsymbol{\alpha}\mathbf{x}^{(s)} + s\right)\right] \geq E_X\left[U\left(w_0 + \boldsymbol{\alpha}_M \mathbf{x}^{(s)} + s\right)\right], \qquad \forall s$$

as a consequence for s = $s_{\min}^{unb}$ we have $\bar{U}\left(s_{\min}^{unb}\right) \geq \bar{U}_{unb}\left(s_{\min}^{unb}\right)$, which implies $\bar{U}\left(s_{\min}^{unb}\right) \geq \bar{U}_M$. On the other hand, we have by definition that $\bar{U}\left(s_{\min}\right) = \bar{U}_M$, then $\bar{U}\left(s_{\min}^{unb}\right) \geq \bar{U}\left(s_{\min}\right)$. Since $\bar{U}(s)$ is an increasing function in s (proposition 2) then $s_{\min} \leq s_{\min}^{unb}$. □

I.4.2 Illustration

An investor is considering investing in the market for one period.. There exist only two traded securities, the risk-free security and a risky security. The possible outcome of these securities is described in the shown table

	Probability	*Payoff*	*Return*
	0.3	3	200%
Risky Security	0.4	1	0
	0.3	0.5	-50%
Risk free	1	1.05	5%

Best portfolio allocation in the status quo: The investor problem is to select the optimal portfolio allocation between the risky and the risk-free security. The investor's utility is U(x) = ln(x), and no short selling is allowed. Assume $w_0 = 10$. Let α be the proportion invested the risky security (with return r).

$$\max_{\alpha} E\left(U\left(w_1\right)\right)$$

or

$$\max_{\alpha} \; E\left[\ln\left(w_0\left(r_f+\alpha\left(r-r_f\right)\right)\right)\right]$$

or

$$\max_{\alpha}\left[0.3\ln\left(10\left(1.05+\alpha(1.95)\right)\right)+0.4\ln\left(10\left(1.05+\alpha(-0.05)\right)\right)+0.3\ln\left(10\left(1.05+\alpha(-0.55)\right)\right)\right]$$

The result is $\alpha_0 = 0.6364$. In other words, the investor should invest 63.64% of his wealth in the risky security. At the optimal solution the expected utility is equal $\overline{U}_0 = 2.4515$

Buying price: Let's assume that the investor is considering investing in a venture. The risk entailed in this investment is uncorrelated with the market and hence is a unique risk. The payoff on this assets is 4 with probability 0.5, and 1 with probability 0.5.

For a fixed known cost b the problem becomes

$$\bar{U}(b) = \max_{\alpha} \; E\left[\ln\left(\left(w_0 - b\right)\left(r_f + \alpha\left(r - r_f\right)\right)\right) + Y\right]$$

In order to find the buying price of the this private lottery, we need the find the maximum price (b_{max}) that the investor is willing to pay. In other word, b_{max} is the value makes the investor indifferent between investing in the venture and the status quo.

The buying price b_{max} is defined as

$$\begin{aligned} &\max \quad b \\ &s.t. \quad \bar{U}(b) = \bar{U}_0 = 2.4515 \end{aligned}$$

The result is $\alpha_b = 0.8008$ and $\mathbf{b_{max}}$= **2.2636**

Best portfolio allocation if the investor owns the lottery: If the investor already owns this venture and is considering selling it. So his initial state is

$$\bar{U}_M = \max_{\alpha} \; E\left[\ln\left(\left(w_0\right)\left(r_f + \alpha\left(r - r_f\right)\right)\right) + Y\right]$$

The result is $\alpha_M = 0.7679$. In other words, the investor should invest 76.85% of his wealth in the risky security. At the optimal, expected utility is $\bar{U}_M = 2.6573$

Selling price: For a fixed knows cost s the problem becomes

$$\bar{U}(s)=\max_{\alpha}\ E\left[\ln\left(\left(w_0+s\right)\left(r_f+\alpha\left(r-r_f\right)\right)\right)\right]$$

If the investor wants to sell this venture. The minimum selling price he should accept is

$$\begin{aligned}&\min\quad s\\&s.t.\quad \bar{U}(s)=\bar{U}_M=2.6573\end{aligned}$$

The result is α_s = 0.6364 **s_{min}=2.2849 .**

Without rebalancing: If the investor does not rebalance his portfolio while adding the private lottery. The buying price is

$$\begin{aligned}&\max\quad b\\&s.t.\quad \bar{U}_{unb}(b)=\bar{U}_0=2.4515\end{aligned}$$

where

$$\bar{U}_{unb}(b)=E\left[\ln\left(\left(w_0-b\right)\left(r_f+\alpha_0\left(r-r_f\right)\right)\right)+Y\right]$$

In this case the result is $b_{\max}^{unb}$ = **2.2294**. Similarly the selling price is

$$\begin{aligned}&\min\quad s\\&s.t.\quad \bar{U}_{unb}(s)=\bar{U}_M=2.6573\end{aligned}$$

where

$$\bar{U}_{unb}(s) = \max_{\alpha} \; E\left[\ln\left(\left(w_0 + s\right)\left(r_f + \alpha_M\left(r - r_f\right)\right)\right)\right]$$

and $s_{\min}^{unb} = \mathbf{2.3278}$.

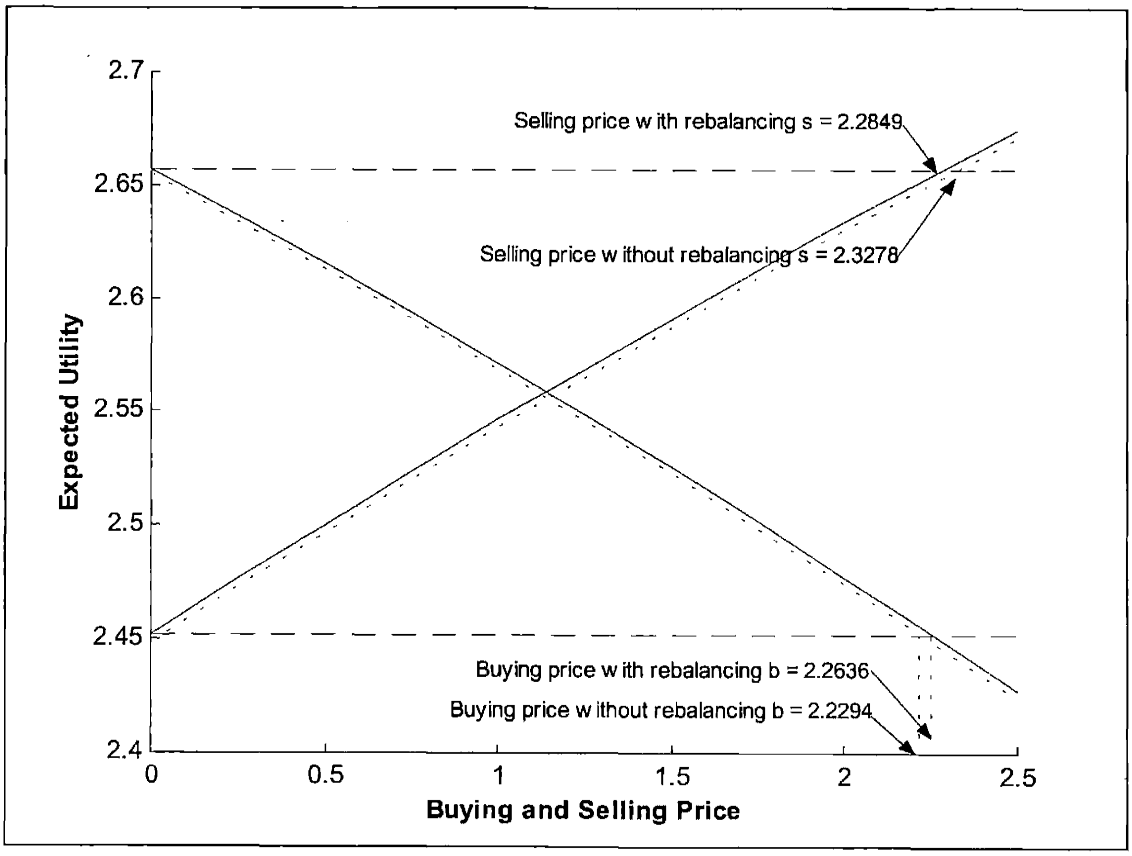

Figure I-4 Effect of Rebalancing on Buying and Selling Prices. The Difference between buying an selling prices with and without portfolio rebalancing is economically significant.

I.4.3 Distributive Bargaining

The buyer's reservation price is the maximum he will pay for *Y*; the seller's reservation price is the minimum he will accept for *Y*. In a distributive bargain between an owner of a private risk lottery Y and a potential buyer, the zone of possible agreement (ZOPA) is the intersection of the buyer's bargaining range and the seller's bargaining range. The buyer's bargaining zone is the region between the buyer's target point (lower bound) and reservation price (upper bound). Similarly the seller bargaining zone is the region between its reservation price (lower point) and its target point (upper bound).

Proposition 6: the ZOPA is larger if portfolio rebalancing is allowed by buyer, by seller, or by both.

Proof: By definition the buyer's reservation price is the price that makes him indifferent between reaching an agreement and walking away from the negotiation. In other words, at this level of buying price the utility of the buyer is equal to his utility at status quo ≡ $\bar{U}_0$. If the buyer does not rebalance his portfolio in the course of calculating the maximum buying price, he calculates a reservation price b_R^{unb}. Let b_R be his reservation price if he rebalances his portfolio to an optimal asset allocation for each possible buying price. Proposition 4 tells us that $b_R^{unb} \leq b_R$.

Similarly the seller reservation price is the selling price that makes the seller indifferent between the sale transaction and the status quo. At the reservation price the seller expected utility is equal to the his utility in the status quo. If the seller does not rebalance his portfolio in the course of calculating the minimum selling price, he calculates a reservation price s_R^{unb}. Let s_R be his reservation price if he rebalances his portfolio to an optimal asset allocation for each possible selling price. Proposition 5 tells us that $s_R \leq s_R^{unb}$.

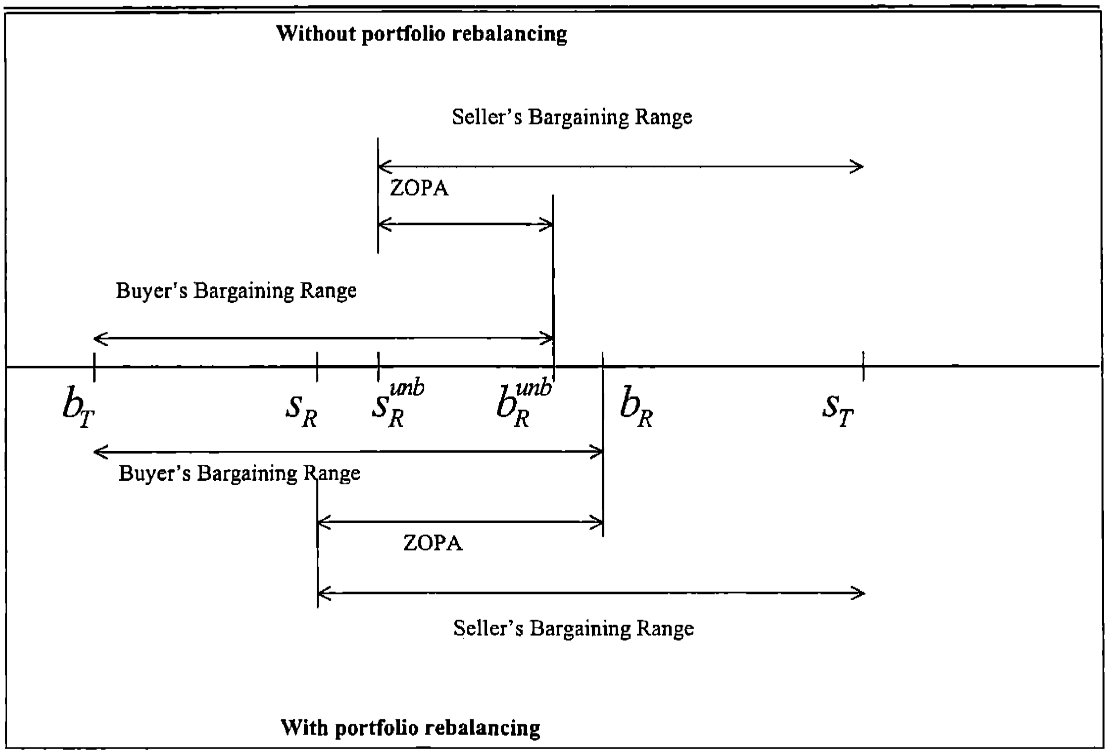

Figure I-5 Effect of simultaneous portfolio rebalancing on ZOPA

II Valuing Investments Under Uncertainty – Private Risk Effect

II.1 Introduction

Firms continuously make decisions whether or not to invest in risky projects, many of which require investment of an initial fixed amount today in exchange for an uncertain stream of future payoffs. The decision becomes more interesting when theses projects have a great deal of uncertainty ex-ante regarding the value of the project. Firms typically learn more about a project's value as they invest overtime and as uncertainties are resolved, and they often have managerial flexibility (also called operating flexibility) to respond actively to this new information. Operating flexibility allows the decision maker to take intermediate sequence of decisions, as new information is presented, to eliminate outcomes that are unfavorable. Some examples include: oil companies investing in exploration projects, pharmaceutical firms investing in R&D projects....

This chapters begins with a review of the origin of the real option approach. We review the market completeness assumption since it is the fundamental assumption of the real option approach that allow us to use financial economics tools such as no arbitrage and risk neutral pricing. Many real option problems have project specific risk that can not be hedged in the market. Under such circumstances market completeness does not hold. In what follows, we discuss several approaches to valuing real option in incomplete markets.

II.1.1 Discounted Cash Flow and its Limitations

The most common tool for asset valuation in capital budgeting problems has been the Discounted Cash-Flow (DCF) analysis. DCF is mainly suitable for valuing projects which cash flows have no uncertainty and where no future decisions are made. However DCF does not properly value investments under uncertainty and/or investments where management has the choice to take future decisions. This weakness of DCF analysis suggested that financial evaluation be replaced by strategic analysis. However the problem is not in the financial evaluation but the DCF analysis. DCF analysis has several weaknesses especially when valuing investments under uncertainty (calculating the ENPV):

- DCF, by using a constant discount rate, assumes that the risk in the project is same at different stages. This creates a bias towards long-term projects or strategic decision-making projects by discounting the future excessively.
- The analysis mainly depends on the choice of the discount rate (or the hurdle rate). Although there might be some guidelines for the choice of the discount rate, such as the CAPM, this doesn't take into account the change in the risk profile of the project as information is revealed with time.
- DCF analysis is an analysis of the "now or never" alternatives. It does not take into account other alternatives that are the result of managerial flexibility and ability to respond to future events.
- DCF deals with risk in an ad-hoc way. Risk is accounted for by a combination of discount rate adjustments and a measure of the spread in the valuation results across several scenarios.

II.1.2 Decision Tree Analysis

One attempt to value risky projects with managerial flexibility is decision-tree analysis (DTA)[9]. DTA is particularly useful in mapping up the available future decisions contingent on the various states of the world. It is extremely useful in analyzing sequential investments under uncertainties. It highlights the optimal managerial strategy in sequential and interdependent decisions. Combining DCF and DTA would fairly evaluate managerial flexibility. The net present value calculated by optimally folding the tree is called "expanded NPV" (Trigeorgis and Mason 1987, Trigeorgis 1996). This value is equal to the usual expected NPV plus any value added by optimally managing the project:

Expanded NPV = expected NPV + value of managerial flexibility

DTA is particularly useful for discrete time problems, where information and decisions present themselves at few discrete points. A continuous version of DTA is available for problems with continuous flow of information and decision making. This is nothing more than valuing the project using the dynamic programming approach (Dixit and Pindyck 1994).

The main serious problem in the DTA (and dynamic programming approach) is the choice of the discount rate. Determining the appropriate discount rate is one problem, another serious problem is in the use of a constant discount rate throughout the life of the project. This assumes that the project's risk is constant in all periods and that risks are resolved continuously at constant rate over time. These assumptions do not hold even in the simplest projects. For example a single-stage project will have a different risk profile if one takes into account managerial flexibility, which requires a different discount rate for different decisions.

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[9] It is important to note that decision-tree analysis is different than decision analysis. Decision analysis, in valuing a project, uses risk preference (utility function), time preference (discount rate), subjective probability, and a decision-tree.

II.1.3 Decision Analysis

Another important tool for valuing investments under uncertainty is decision analysis (DA). DA is often confused with DTA. DA uses a decision tree (or dynamic programming), however in rolling back the tree, the optimal strategy is determined by the maximum certainty equivalent instead of the maximum expected value. By using a utility function, which capture the investor risk preference, no adjustments to the discount rate is needed. Hence the choice of the discount rate problem is avoided. (For further discussion of DA and its applications, please refer to Keeney and Raiffa 1993). The benefits of decision analysis is that it can be applied to all kinds of projects from to oil exploration to medical decision. The major criticism of DA is that it focuses only on the decision maker, ignoring market information and its effect on the optimal strategy. Smith and Nau (1995) show that in complete markets (where all project risks can be hedged in the market) DA if applied correctly will yield the same results as any other market valuation such as real option valuation. We will further discuss the contributions of this paper in later chapters.

II.1.4 Real Options

In order to overcome the limitations of discounted cash flow techniques in capital budgeting valuation with managerial flexibility, Myers (1977) suggested the use of financial option methods (or contingent claim analysis). Real options are the natural extension of financial options valuation theory to real life capital expenditure projects that have the option-like characteristics, mainly due to managerial flexibility (also called operating flexibility).The financial option pricing which started with Black and Scholes (1973) and Merton (1973) is used to price "real"[10] projects instead of financial derivatives. Real option valuation (ROV) method is the use of financial option theories to capital budgeting project. Real option valuation started with Myers (1977) and Ross (1978), and got popularized by Myers (1984).

[10] The term "real options" first appeared in Myers (1977).

The attractiveness of ROV is that one does not need to estimate subjective probabilities, a choice of discount rate, or a utility function. ROV seeks a portfolio of securities that perfectly mimics the cash flows of the project, in all state of the world. By the "law of one price", the value of project is equal to the current value of the portfolio. ROV will provide a market-based valuation of the project as well as the optimal decision choices that maximize this value.

The application of this valuation technique to capital budgeting problems has gained a lot of popularity in the last two decades. The most common real option problems in the literature can be categorized in the following manner: growth option, exit option, option to wait, flexibility option, switch option. Trigeorgis (1996) and Dixit and Pindyck (1994) describe each of the following options with an intensive literature review of the main references.

II.2 Real Option Valuation

II.2.1 Valuation By Arbitrage

The main idea behind ROV is the no-arbitrage concept, also called "law of one price". No-arbitrage states that two different assets with identical cash-flow must have the same price. In order to price an asset with a complex cash flow, one need to replicate its cash flow by a portfolio of traded securities. Once this portfolio is determined, the value of the asset is equal to the value of the portfolio. The value of the portfolio is simply the sum of the value of its parts.

We will illustrate the concept of ROV and the use of contingent claim analysis and no-arbitrage by a simple example. A more elaborate example can be found in Trigeorgis and Mason (1987), Copeland, Koller and Murin (1990), Nau and McCardle (1991), Smith and Nau (1995). Assume a firm has the opportunity to invest in a project that requires an initial investment of $100 now and yields an uncertain payoff one year from now. In the "up" state the payoff is $130 and in the "down" state it is $60. The firm believes that the two states are equally likely. The firm may either accept or reject this lottery now or delay the investment for one year. If the firms delays, after the uncertainty about the payoff is resolved one year from now, it can invest $110 and receive either $130 or $60 for certain.

Suppose that there exist two securities, a risk-free security and a "twin security". The risk free rate is 5%. The "twin security" value in one year, similarly to the project, depends on the state uncertainty. The current value of the security is S = $20, and it will be worth S_u = $30 in the up state, and S_d = $15 in the down state.

Obviously, the defer option is preferred. Let's calculate the value of this defer option. We need to construct the replicating portfolio for this alternative. Let β_1 and β_2 denote the number of shares of the risk-free security and the twin security in the replicating portfolio. We know that the payoffs of the portfolio are equal to the payoffs of the project, which leads to the following equations

Up state: $\beta_1(1.05) + \beta_2(30) = 20$

Down state: $\beta_1(1.05) + \beta_2(15) = 0$

Solving these equations would yield $\beta_1 = -19.05$ and $\beta_2 = 1.33$. The replicating portfolio consists of buying 1.33 shares of the twin security and borrowing \$19.05. The defer alternative is valued at $-19.05(1) + 1.33(20) = \$7.62$.

The risk-neutral approach is another tool for valuing options using no-arbitrage. Cox and Ross (1975) observed that in finding the riskless hedge, no assumption about the risk preference was made. This suggests that a solution for the problem, assuming a particular risk preference structure, must be the solution for the problem for any preference structure. They choose risk-neutral preference structure for it mathematical tractability.

To value the project we need first to determine the probabilities that satisfy the valuation of the twin security under risk-neutral assumption. Let π denote the risk-neutral probability of an up state.

$$20 = \frac{\pi(30) + (1-\pi)(15)}{1.05}$$

This implies that $\pi = 0.4$. Using these probability we can now value the defer alternative. The value of the defer strategy is equal to $[0.4(20) + 0.6(0)]/1.05 = 7.62$. Knowing the risk-neutral probabilities we can determine the state prices which are equal to the risk-neutral probabilities times the discount factor.

A more elaborate example of no arbitrage pricing along with a comparison with DTA and its limitation can be found in Smith and Nau (1995).

II.2.2 Real Option Analysis Using An Equilibrium Derivation

McDonald and Siegel (1985) in their valuation of a project with an option to shut-down production temporarily with no additional cost, did not use the no-arbitrage argument to determine the option value. Their approach originates from Samuelson (1965), and is similar to Constantinides (1978). They compute the expected option price and then discount it using a pre-determined discount rate. They also replace the underlying asset growth rate by an equilibrium rate of return (assuming an equilibrium model for asset pricing such as Merton (1973) ICAPM). Their analysis yields the Black-Scholes-Merton formula for option value if they assume risk-neutrality and use the risk free rate for discounting the option's expected value). This treatment pricing real options is not the main-stream approach. If one does not assume risk neutrality, several problems arises: such as the choice and the use of a constant discount rate, and the assumption about the investors preferences. This is why McDonald and Siegel's approach is viewed as a generalization of the DCF and DTA approaches and not a rigorous method.

II.2.3 Complete Market Assumption

The ability to apply contingent claim analysis, by constructing a replicating portfolio or calculating risk neutral probabilities, requires the critical assumption of "complete markets" with respect to the project risks. This assumptions states that capital markets are sufficiently complete so that a replicating portfolio can be constructed whose value are perfectly correlated with the value of the project. In other words, stochastic changes in the project's value are spanned by the existing market securities. Only when the complete market requirement is met, we can apply financial option pricing to capital budgeting problems. Most of academic literature on real options uses the complete market assumption. Trigeorgis (1996) and Dixit and Pindyck (1994) analyze projects under the complete market assumption. When the project's underlying asset is a traded security, or its equilibrium value is know, then a straightforward analysis similar to the Black and Scholes (1973) and Merton (1973) analysis can be applied.

Mason and Merton (1985) in their discussion of valuing sequential investments, present a solution for problems where the underlying asset is not traded, and hence its value is not observed. They suggest the price of this non-traded asset be adjusted by deducting the no-arbitrage value of a corresponding traded asset. Their reasoning is similar to the DCF analysis where the market required rate of return of a traded asset is used as a discount rate to price an "equivalent in risk" non-traded asset.

Majd and Pindyck (1987) also tackled the problem of non-traded underlying assets. They calculate the value of the underlying asset as if the asset was traded in the market. Using the equilibrium required rate of return of an existing identical traded asset. Similar to Mason and Merton (1985), this analysis requires that the portfolio of existing traded securities spans the risks of the project's identical asset, and hence spans the project's risks.

II.2.4 Project-Specific Risks

The essence of real option's valuation is the pricing and analysis of investments under uncertainty taking into account managerial flexibility and using the tools of financial economics for valuation. No arbitrage and its consequence for contingent claim analysis are the main tools of real option valuation. The usual assumption is that stochastic changes in a project's cash flow can always be spanned by existing market securities. This is a strong assumption about market completeness with respect to the project risks.
If the market is complete with respect to a project's risks, these risks can be perfectly hedged by trading securities. In fact, most real project risks can be only partly replicated by financial markets. This is the biggest shortcoming of the ROV. The market for the underlying asset is complete only if the project's risks are market risks. In most capital investment projects there exist project-specific risks that are not contained in the set of traded securities. These risks are also called diversifiable risks, unsystematic risks, unique risks. Since these risk are not spanned by the market securities, they are not priced in the financial market. This is why they are also called zero-priced risks or unpriced risks.
Valuing an undeveloped oil reserve is an example of capital budgeting investment where ROV is extremely useful and fairly easy to apply (assuming development cost and the size of the reserve are know variables). The owner of the project has the option to acquire developed reserve by exercising his option and paying the development price. The underlying asset in this example are developed oil reserves whose value are know by observing traded securities like oil future and bonds. In this example, the market is complete with respect of the project risks, a replicating portfolio that perfectly mimics the

cash flow of the project can be constructed and the exact value of the project can be determined. However if we consider an oil exploration project. We can still determine by ROV the value of the undeveloped oil reserves contingent on oil discovery. On the other hand, the risk of not finding oil or the uncertainty about deposit size are *unique* risks that can not be spanned by market securities and are not priced by the market. Other examples of unique risks in real options problem included research and development projects, new market entries, technological performance, and project cost risks associated with the quality of performance.….

II.3 Pricing Projects With Project-Specific Risks

II.3.1 Financial Economist Approach

Financial economists argue that project specific risks are not correlated with any traded security and thus its correlation with the market is zero. This would result in a beta(s) of zero for these risks, requiring that cash flows of private risks be discounted at the risk-free rate (Neely 1998, Luenberger 1998) . The theory behind this idea is that a shareholder, who can select any security while constructing its market portfolio, requires a premium from all systematic (non diversifiable) risks, since these risks contribute to the overall riskness of its market portfolio. On the other hand, unique risks, which are firm specific risk, have no covariance with the market portfolio and can be diversified away, thus require no risk premium (Trigeorgis 1998). Applying this approach to real option problems results in using the replicating portfolio and no arbitrage tools when possible, and when faced with project-specific risks one takes the expected value of these risks and discount them at the risk free rate.

II.3.2 Approximate Approach

Dixit and Pindyck (1994) propose an alternative approach for valuing real options when market are incomplete. They propose the use of dynamic programming approach as in the complete market case, however when spanning does not hold one should use an "assumed discount rate". This discount rate is an arbitrary discount rate since there is "no theory for determining the correct value of the discount rate". This approach proves extremely useful in solving for real option values in incomplete markets. Although it is based on a theoretical inaccuracy when it comes to choosing a discount rate, it is still a practical method for valuing a project under uncertainty taking into account the value of flexibility. This explain the popularity of this method in the literature of real option valuation. Trigeorgis uses a similar method for valuing real option when spanning does not hold. He called this method 'the expanded NPV'.

Both these methods are well know techniques in the decision science field. It is nothing more than the decision tree analysis. Dixit and Pindyck presented the continuous time form and Trigeorgis the discrete time form

II.3.3 Integrated Valuation Procedure

If the project-specific risks are private risks, as defined in the introduction, in the sense the investor chooses to hold them, despite the fact that they are potentially diversifiable in market context. These risks cannot be traded or whose trade is inhibited by large agency costs, and possess consequences that, with non-negligible probability, represent a substantial portion of the owner's current wealth.

The most appropriate approach to evaluate these private risks is the integrated valuation procedure (IVP) developed by Smith and Nau (1995). This approach, as the previous approach, uses market information and prices to evaluate market risks. However project-specific risks (risks that can not be hedged in the market) are priced using the firm's subjective believes and preferences. In other words, IVP uses the market's risk-neutral probabilities to price market risks and subjective probabilities and utility functions in order to calculate certainty equivalents of unique risks. IVP, as noted by its name, integrates option pricing procedure and decision analysis using the former when markets are complete and the later when markets are incomplete. One would use market valuation and inferred prices as much as possible, and managerial risk preference when market prices are absent. This method captured the firm's risk attitude while pricing private risk.

II.3.4 Valuation Framework

As we have discussed, pricing market risk is straightforward. One can use the replicating portfolio and law of one price, or risk neutral valuation. It is worth mentioning that option pricing and decision analyses (if applied properly) give the same price for market risks. If one includes market opportunities to borrow and trade then decision analysis yield the same results as the real option approach (Nau and Smith 1995).

Pricing of project-specific risks is somewhat unclear. Financial economists argue that unique risks are not correlated with any traded security and thus its correlation with the market is zero. However this approach to price unique risks could be challenged easily, especially when these risks are a considerable proportion of the investor's portfolio and the investor can not or choose not to diversify them away . If the arguments of diversification were applicable in practice, then we would argue that managers should not be worried about risk, and should not involve in any active risk management practice. Managers should then maximize shareholders values by selecting investments with the highest expected net present value. However the above is not true due to several reasons. First there is the managerial self-interest. Managers and employees are generally not well diversified with most their capital investment and all their human capital invested in their company. Added to that managers try to seek stability in their position and wealth, which would be reflected in their decision to stabilize the performance of the company by avoiding risky investment even if they were strategic investment. Hacket (1985) noted that it is unrealistic to assume that managers are merely agents for the shareholders, instead managers attempt to reconcile the interests of all stakeholders including themselves, employees, suppliers, customers.... Second, firms are faced with the possibility of financial distress especially when considering large risky investments. Due to capital market imperfections, the possibility of bankruptcy makes the cost of external financing extremely high, if not impossible, especially when the firm is in most need for these funds. All the above reasons foster risk aversion among managers and lead them to actively manage both market and unique risks.

The argument that firms behave in a risk averse fashion seems quite reasonable. For example, Dyers and Walls (1996) studied the performance and strategies of the top 25 firms in the petroleum exploration industry. They concluded that all these firms have a significant risk aversion behavior. Howard (1988) argued that firms of a particular industry tend to have a similar risk attitude, he also found that there might be a relation between a corporation's risk tolerance and its financial measures (such as net income). Jensen and Ruback (1983) maintain that managers exhibit risk aversion especially in the short term, since management has to report that its company is viable in each period if it wants to keep its job, thus making risky projects an inferior good. Finally Greenwald and Stiglitz (1990) showed that managers act in risk averse manner due to asymmetric information between providers of capital and the firm management.

It seems logical at this point to differentiate between two types of unique risks. First there are unique risks that can be diversified away. For example when similar identical risks exists, then the investor can trade these risks with other investor or he can hedge his current position. We will refer to these risk from now on as simply unique risks. However if the investor can not trade (or chooses not to trade them due to his comparative advantage as discussed in the introduction) project-specific risks and they constitutes a considerable part of his portfolio, we call these project specific risks "private risks".

Pricing of project-specific risks depends whether it can be traded or not. In the case of unique risks, the investor has to price it at zero premium since this is the price that the market prices it and he has to conform with market prices or else arbitrage opportunity is created. (we are assuming that adding the project to the existing traded securities does not effect the market equilibrium prices). In this case the financial economist approach is the valuation tool that should be used for capital budgeting problem.

In the case of private risks, the investor is the only holder of the private risks, he can not trade it and no similar risks are price in the markets or by other investors. He will price the risks using his own subjective probabilities and preferences. Similarly to the decision analysis problems pricing of a private risk lottery will be priced using the subjective expected utility theory. And in the case of pricing projects under uncertainty with private risks, Smith and Nau's integrated valuation procedure is the appropriate valuation approach.

Market risks

Project-Specific Risks

	Low	**High**
Low	NPV	Real Option
High	Decision Analysis	IVP (for private risks) / Financial economist approach (for unique risks)

Table II-1 Valuation Framework

II.4 Oil development project – the value of discovered, undeveloped reserves

This section describes the application of real option to valuing managerial flexibility in the timing of the exploration and development of an oil prospect, the classical example pioneered by Paddock, Siegel, and Smith (1988). Our treatment is similar to that Paddock, Siegel, and Smith (1988) and Pickles and Smith (1993). We introduce project specific risks in both the exploration and the development phases. In particular, we allow the exercise price of the real option to be uncertain, as it would be if development costs are uncertain ex ante. On the other hand, we will adopt the simplest assumption about price uncertainties (as in the above papers) and productions profiles in order not to distract the focus of the reader from the treatment of private risks.

We analysis a development project where the exploration phase has revealed the discovery of reserves and the size of the reserves is well known. The firm at this stage has the option to pay development cost (install production facilities) to transform undeveloped reserves into developed reserves. We will assume that the firm will not postpone extraction - it will immediately extract developed reserves.

The value of a development project can be represented as

$$OP_{dev} = QX\left(V, T-t, \tilde{D}\right) \tag{2.1}$$

where

- Q = Size of the undeveloped reserves.
- $\tilde{D}$ = Development cost. We will assume that D is uncertain (for example D could be function of Q, and Q could be uncertain). In our case we are more interested in the value of X rather than V_{dev}, So we will assume Q to be constant and D to be uncertain.
- T = Expiration time. Time that the firm has before relinquishing the project if it doesn't develop it.

t = Current time.

V = Current value of a barrel of developed reserve.

X = Option value of a barrel of undeveloped reserve.

We will assume that V follows a geometric brownian motion (GBM), and has the following diffusion process:

$$\frac{dV}{V} = (\alpha - \delta)dt + \sigma dw_t \tag{2.2}$$

α is the expected rate of return on developed reserves (or a stock with risk σdw_t). δ is the payout rate of developed reserves. It is equal to the convenience yield earned by holders of the commodity minus any storage costs. σ is the volatility (standard deviation of reserve prices), and dw_t is the one period increment of a Wiener process. We will assume that α,δ, and σ are constant over the life time of the project.

Time to expiry of the project is 3 years. Current price of developed reserve is \$12/bbl, the price is assume to follow a GBM with σ = 25% and the payout rate of 7%. The risk free rate is r = 5%. The project have a variable cost equal 20% of the oil price. The fixed cost is uncertain and could be \$400, \$600, or \$1000 million equally likely. This uncertainty about the cost is assumed to uncover once development has started and the process is irreversible.

The difference between our analysis and other analysis in the literature is that we allow the exercise price to be stochastic. This creates a unique project risk. In our example the exercise price per bbl of oil,\$4, 6\$ or 10\$, are equally likely.

Figure II-1 displays all the project data. Figure II-2 shows the net revenues from extracting one barrel of developed reserve, taking into account the production profile. Extracting one barrel at time τ yields NPV (at time τ) of 0.6829, for V = \$1.00/bbl. This number can be used for other developed reserve prices, for example if V = \$12.00/bbl then the NPV would be equal \$8.1954.

Input Data

Time to expiry of the lease			T = 3
Real risk free rate			r = 5.00%
Payout rate			delta = 7.00%
Present price of developed reserve, per barrel			P = $12.00
Price volatility (annualized)			sigma = 25%
proven reserve			Q = 100,000,000
variable operating cost per barrel			20%
Development cost	500,000,000	600,000,000	1,000,000,000

Calculated Parameters

length of one time period (years)	dt = 0.125
Upward price change in one time period	
=exp{sigma*sq.root(dt)}-1	= 9.24%
Downward price change in one time period	
=1- exp{-sigma*sq.root(dt)}	= 8.46%
Change in expected price in one time period	
=exp{(rho-delta)*dt}-1	= -0.25%

Risk neutral probabilities

p*upward price + (1-p)*downward price = expected price

Probability of upward change	p = 0.4638
Probability of downward change	(1-p) = 0.5362
risk neutral SDE of oil price	dS=(-0.02).S.dt+(0.25).S.dWt

Figure II-1 Project Data

Year	Expected Price	Production Profile	Discount Factor	Expected NPV $/BBL
1	1.0113	0.15	0.952	0.14447
2	1.0228	0.15	0.907	0.13915
3	1.0343	0.15	0.864	0.13402
4	1.0460	0.15	0.823	0.12909
5	1.0579	0.11	0.784	0.09117
6	1.0698	0.08	0.746	0.06387
7	1.0819	0.06	0.711	0.04613
8	1.0942	0.05	0.677	0.03703
9	1.1066	0.04	0.645	0.02853
10	1.1191	0.03	0.614	0.02061
11	1.1317	0.02	0.585	0.01323
12	1.1445	0.01	0.557	0.00637
				0.85369

Assuming a 20% operating cost: Net present value of the revenue of a bbl of oil is = *0.6829*

Figure II-2 Production profile and net revenue per bbl for V =1

We have assumed that oil prices follow a geometric Brownian motion. Using the binomial lattice method developed by Cox, Ross and Rubenstein (1979), we generate the option prices. Given the price V of the developed oil reserve, next period price can move up by a factor of 1.092 ($e^{\sigma\sqrt{dt}}$) or down by a factor of 0.915 ($e^{-\sigma\sqrt{dt}}$). Next period risk neutral expected price of developed reserve can be calculated from the risk-free rate and the payout rate, and it is equal $Ve^{(r-\delta)dt} = 0.995\,V$. Now we can calculate the risk neutral probability π that the price increases by the factor of 1.092 and decreases by the factor of 0.915. Solving the equation $0.9975V = p1.092V + (1-p)0.915V$, yields p = 0.464 and (1-p) = 0.536.

Period	0	1	2	3	4	5	6	7	←→	23	24
	12.00	13.11	14.32	15.64	17.09	18.67	20.39	22.28		91.64	100.11
		10.98	12.00	13.11	14.32	15.64	17.09	18.67		76.79	83.89
			10.06	10.98	12.00	13.11	14.32	15.64		64.35	70.29
				9.20	10.06	10.98	12.00	13.11		53.92	58.90
					8.43	9.20	10.06	10.98		45.18	49.36
						7.71	8.43	9.20		37.86	41.36
							7.06	7.71		31.73	34.66
								6.46		26.59	29.04
										22.28	24.34
										18.67	20.39
										15.64	17.09
										13.11	14.32
										10.98	12.00
										9.20	10.06
										7.71	8.43
										6.46	7.06
										5.42	5.92
										4.54	4.96
										3.80	4.15
										3.19	3.48
										2.67	2.92
										2.24	2.44
										1.88	2.05
										1.57	1.72
											1.44

Figure II-3 Oil Price Movement

The development cost is the exercise price of the development option. The uncertainty of the exercise price is a project specific risk. As discussed in the previous chapter the treatment of this risk depend on the ability or willingness of diversifying this risk and the risk attitude of the investor (or the firm). We will now show both treatment of this risk.

If the investor is risk neutral toward project unique risks (or has the ability to diversify away these risks) he will require no premium for holding this risk and will price it using the tradition financial economist approach.

Under these assumptions the price of the development option will be as follow. The firm has the choice at each period t (t<T) to either exercise the option and receive the value $V_t - E(\tilde{D})$ per barrel, or wait one period. The later strategy has the value of $e^{r.dt}E^*(X_{t+1})$, where $E^*(.)$ denotes the risk neutral expected value (using risk neutral probability). More formally the value of the option at time t is:

$$X_t = \max\left(e^{r.dt}E^*(X_{t+1}), V_t - E(\tilde{D})\right) \qquad (2.3)$$

at t=T, the investor can either exercise the option or kill it,

$$X_T = \max\left(0, V_T - E(\tilde{D})\right) \qquad (2.4)$$

Using the above formulation, we can solve for X_0 using dynamic programming or rolling back the binomial lattice tree. We find $X_0 = \$1.63 / bbl$, and the valuation of the project $OP_{dev} = 163.2$ millions. Figure II-4 shows part of the binomial tree. At each step we have the price of the developed reserve, we then calculate the value of the exercise strategy and the value of the wait or hold strategy. Then the option value is the maximum of both strategies. In addition to that we have a cell that indicates when the optimal strategy is to exercise or to wait ("Yes" for exercise).

						← 24 periods →			
Period	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Price	12.00	13.11	14.32	15.64	17.09	18.67	20.39	22.28	24.34
Hold value	1.63	2.17	2.84	3.66	4.61	5.68	6.85	8.13	9.52
Exc. Value	1.20	1.95	2.78	3.68	4.67	5.75	6.93	8.22	9.62
Option Value	1.632	2.17	2.84	3.68	4.67	5.75	6.93	8.22	9.62
Exercise?	No	No	No	Yes	Yes	Yes	Yes	Yes	Yes
Price		10.98	12.00	13.11	14.32	15.64	17.09	18.67	20.39
Hold value		1.19	1.61	2.15	2.83	3.66	4.61	5.68	6.85
Exc. Value		0.50	1.20	1.95	2.78	3.68	4.67	5.75	6.93
Option Value		1.19	1.61	2.15	2.83	3.68	4.67	5.75	6.93
Exercise?		No	No	No	No	Yes	Yes	Yes	Yes
Price			10.06	10.98	12.00	13.11	14.32	15.64	17.09
Hold value			0.84	1.16	1.58	2.13	2.82	3.65	4.61
Exc. Value			-0.13	0.50	1.20	1.95	2.78	3.68	4.67
Option Value			0.84	1.16	1.58	2.13	2.82	3.68	4.67
Exercise?			No	No	No	No	No	Yes	Yes
Price				9.20	10.06	10.98	12.00	13.11	14.32
Hold value				0.57	0.81	1.13	1.56	2.11	2.81
Exc. Value				-0.71	-0.13	0.50	1.20	1.95	2.78
Option Value				0.57	0.81	1.13	1.56	2.11	2.81
Exercise?				No	No	No	No	No	No
Price					8.43	9.20	10.06	10.98	12.00
Hold value					0.37	0.54	0.77	1.10	1.53
Exc. Value					-1.25	-0.71	-0.13	0.50	1.20
Option Value					0.37	0.54	0.77	1.10	1.53
Exercise?					No	No	No	No	No
Price						7.71	8.43	9.20	10.06
Hold value						0.22	0.34	0.51	0.74
Exc. Value						-1.73	-1.25	-0.71	-0.13
Option Value						0.22	0.34	0.51	0.74
Exercise?						No	No	No	No
Price							7.06	7.71	8.43
Hold value							0.13	0.20	0.31
Exc. Value							-2.18	-1.73	-1.25
Option Value							0.13	0.20	0.31
Exercise?							No	No	No
Price								6.46	7.06
Hold value								0.07	0.11
Exc. Value								-2.59	-2.18
Option Value								0.07	0.11
Exercise?								No	No
Price									5.92
Hold value									0.03
Exc. Value									-2.96
Option Value									0.03
Exercise?									No

Figure II-4 Binomial Tree for development option (using tradition valuations approach). Value per bbl.

The cost of development risk is a specific project specific risk. If the investor can not trade and can not diversify away this risk, as is the case in most large project, in this case the investor is risk averse towards this project specific risk. He will require a risk premium to hold this risk. This premium depends on the investor believes and preferences.

The investor in this case values his choices using the integrated valuation approach.

We will assume that the investor has utility of the form

$$U(w) = \ln\left(\tilde{w}/\rho\right)$$

where ρ is the investor's risk tolerance. The value of ρ is assumed to be 200 millions. In the previous analysis we made all the calculation on a per barrel basis. In this analysis we have to use the investor's utility function, then calculations can not be performed on a per barrel basis, since this will not capture the risk aversion of the investor.

At each period the value of the exercise now strategy is $CE\{QV_t - D\}$, where CE denotes the certainty equivalent using the investor's utility function. The delay strategy has value $e^{r.dt}E^*(X_{t+1})$. The investor at each period will choose the strategy that has the higher certainty equivalent. The value of the option at t<T is

$$X_t = \max\left(e^{r.dt}E^*(X_{t+1}), CE\left(QV_t - \tilde{D}\right)\right) \qquad (2.5)$$

and at t=T

$$X_T = \max\left(0, CE\left(QV_T - \tilde{D}\right)\right) \qquad (2.6)$$

In this case $X_0 = OP_{dev}$ since we are calculating option value for the project as a whole and not on a per barrel basis. Figure II-5 shows the binomial tree for both valuation procedures. The difference in the value of the development option between both valuation procedure is $1.63 - 1.09 = \$0.54$, or \$54 million. This difference is the premium that the investor (or the firm) requires to hold the private risk. We define this difference to be a **private risk premium**. Figure II-6 shows the value of the project for both valuation methods. The difference is the private risk premium.

Period	0	1	2	3	4	5	6
Price	12.00	13.11	14.32	15.64	17.09	18.67	20.39
Hold value	1.63	2.17	2.84	3.66	4.61	5.68	6.85
Exc. Value	1.20	1.95	2.78	3.68	4.67	5.75	6.93
Option Value	1.63	2.17	2.84	3.68	4.67	5.75	6.93
Exercise?	No	No	No	Yes	Yes	Yes	Yes
Hold value	*1.09*	*1.49*	*2.01*	*2.67*	*3.51*	*4.51*	*5.66*
Exc. Value	*0.00*	*0.76*	*1.58*	*2.49*	*3.47*	*4.55*	*5.73*
Option Value	*1.09*	*1.49*	*2.01*	*2.67*	*3.51*	*4.55*	*5.73*
Exercise?	*No*	*No*	*No*	*No*	*No*	*Yes*	*Yes*
Price		10.98	12.00	13.11	14.32	15.64	17.09
Hold value		1.19	1.61	2.15	2.83	3.66	4.61
Exc. Value		0.50	1.20	1.95	2.78	3.68	4.67
Option Value		1.19	1.61	2.15	2.83	3.68	4.67
Exercise?		No	No	No	No	Yes	Yes
Hold value		*0.75*	*1.05*	*1.45*	*1.98*	*2.65*	*3.50*
Exc. Value		*-0.69*	*0.00*	*0.76*	*1.58*	*2.49*	*3.47*
Option Value		*0.75*	*1.05*	*1.45*	*1.98*	*2.65*	*3.50*
Exercise?		*No*	*No*	*No*	*No*	*No*	*No*
Price			10.06	10.98	12.00	13.11	14.32
Hold value			0.84	1.16	1.58	2.13	2.82
Exc. Value			-0.13	0.50	1.20	1.95	2.78
Option Value			0.84	1.16	1.58	2.13	2.82
Exercise?			No	No	No	No	No
Hold value			*0.50*	*0.72*	*1.02*	*1.42*	*1.95*
Exc. Value			*-1.33*	*-0.69*	*0.00*	*0.76*	*1.58*
Option Value			*0.50*	*0.72*	*1.02*	*1.42*	*1.95*
Exercise?			*No*	*No*	*No*	*No*	*No*
Price				9.20	10.06	10.98	12.00
Hold value				0.57	0.81	1.13	1.56
Exc. Value				-0.71	-0.13	0.50	1.20
Option Value				0.57	0.81	1.13	1.56
Exercise?				No	No	No	No
Hold value				*0.32*	*0.47*	*0.68*	*0.98*
Exc. Value				*-1.91*	*-1.33*	*-0.69*	*0.00*
Option Value				*0.32*	*0.47*	*0.68*	*0.98*
Exercise?				*No*	*No*	*No*	*No*
Price					8.43	9.20	10.06
Hold value					0.37	0.54	0.77
Exc. Value					-1.25	-0.71	-0.13
Option Value					0.37	0.54	0.77
Exercise?					No	No	No
Hold value					*0.19*	*0.29*	*0.43*
Exc. Value					*-2.44*	*-1.91*	*-1.33*
Option Value					*0.19*	*0.29*	*0.43*
Exercise?					*No*	*No*	*No*
Price						7.71	8.43
Hold value						0.22	0.34
Exc. Value						-1.73	-1.25
Option Value						0.22	0.34
Exercise?						No	No
Hold value						*0.10*	*0.16*
Exc. Value						*-2.93*	*-2.44*
Option Value						*0.10*	*0.16*
Exercise?						*No*	*No*
Price							7.06
Hold value							0.13
Exc. Value							-2.18
Option Value							0.13
Exercise?							No
Hold value							*0.05*
Exc. Value							*-3.37*
Option Value							*0.05*
Exercise?							*No*

Figure II-5 Value of the development option (6 first period only), in hundred millions. In each period the first 5 rows are for the traditional valuation method. The next 4 columns shows the calculations for the certainty equivalent valuation

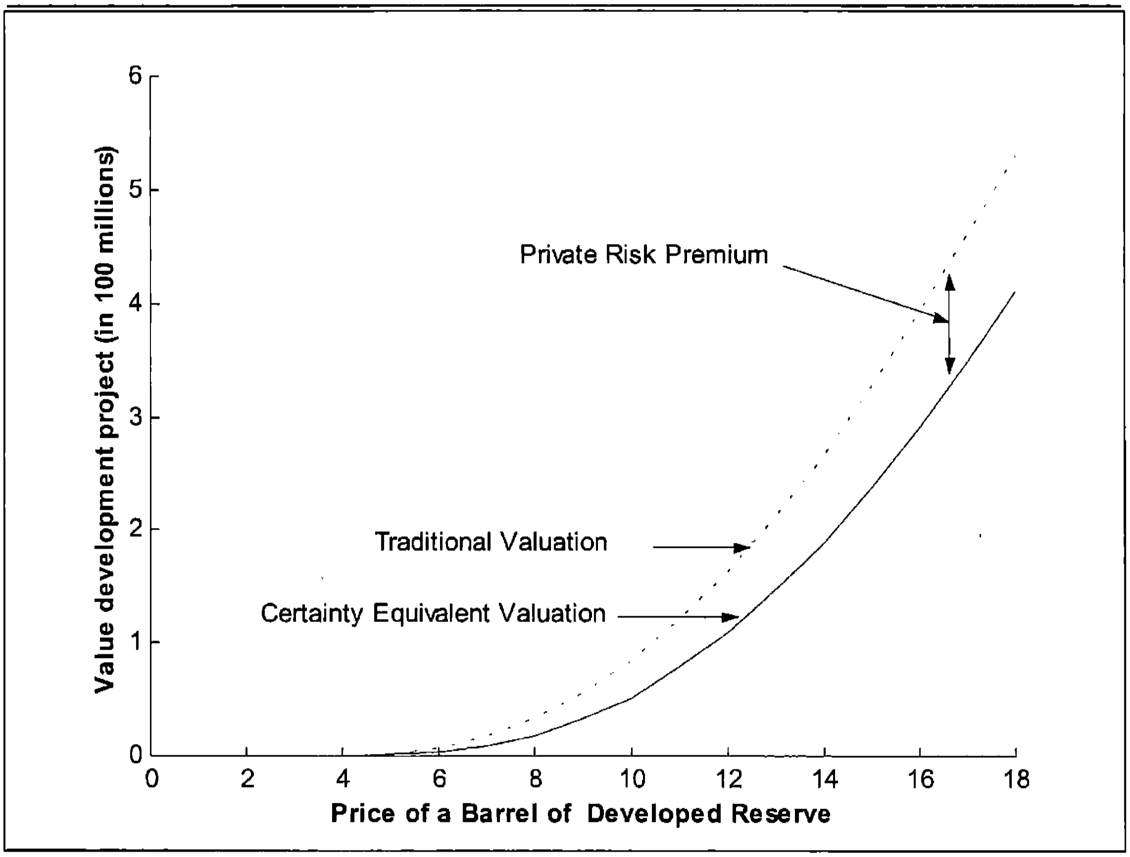

Figure II-6 Development option value for both valuation method

In addition the difference in the value of the project, each valuation approach yields a difference in the development strategy. For example in figure II-5 in period six, if the reserve price reaches 17.09, the traditional approach values the exercise strategy higher than the hold strategy. Where as in the certainty equivalent approach the optimal strategy is to wait and not to exercise.

Each valuation approach result in a different a different exercise strategy. The difference is not clear in the binomial tree due to the size of the time step. The difference is clear in figure II-7, it shows the minimum price at each period to exercise the option. the dashed line is the results using the traditional valuation approach. The solid line is the strategy for the certainty equivalent approach.

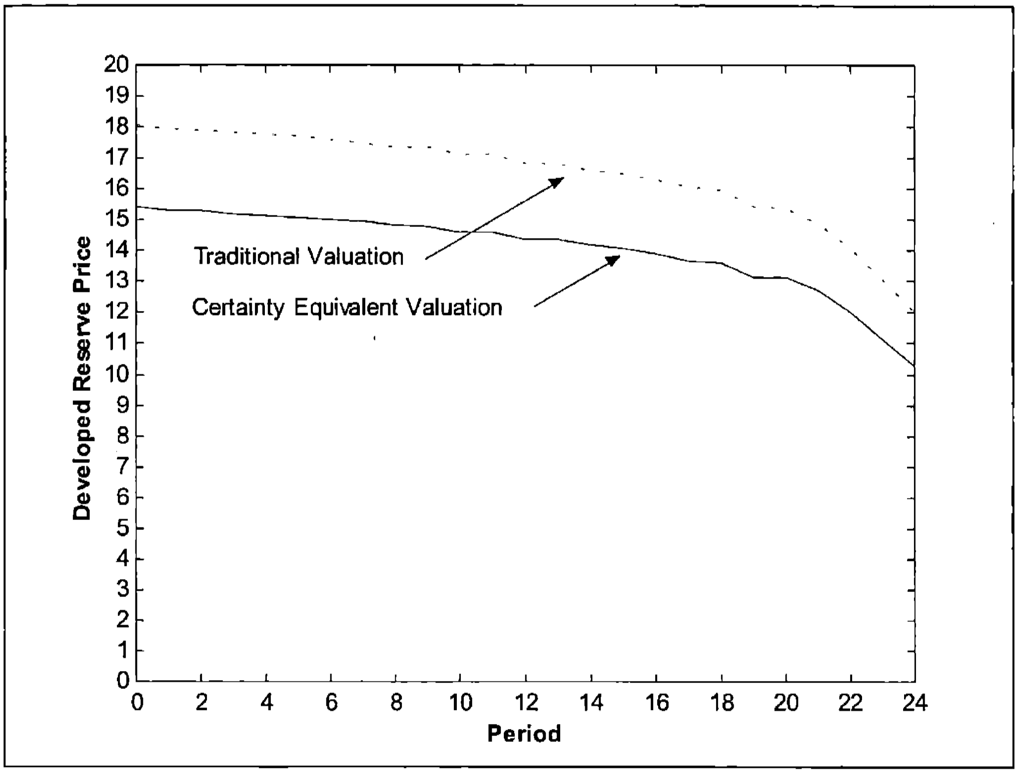

Figure II-7 Exercise Strategy - Minimum required price for exercising the option

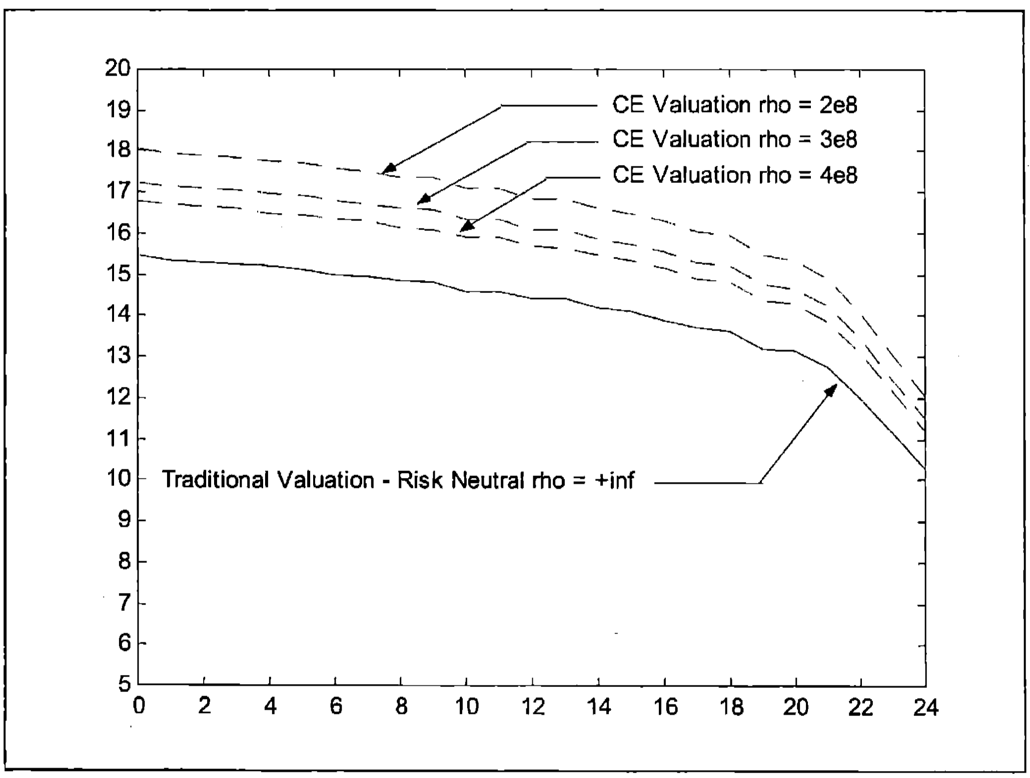

Figure II-8 Minimum required price for exercising the option for different risk tolerence levels

Conclusion

In the second essay, we build a framework for valuing investments under uncertainty in the presence of private risks. We demonstrated by example that different methods for pricing private risk can lead to decisively different real option values. This difference is mainly due to *private risk premium* – the premium the investor (or the firm) requires to hold the private risk. We also showed how, when private risks are present, alternative methods for valuation can lead to large differences in choice of exercise strategy of the real option. This difference in exercise strategy contributes to the difference in real option values. We also showed that that the exercise strategy depends on the investor's (or firm's) risk aversion.

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