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RECURSIVE ESTIMATION FOR 2-D ISOTROPIC RANDOM FIELDS

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Abstract

In this paper we develop efficient recursive smoothing algorithms for isotropic random fields described by non-causal internal differential models . The 2-D estimation problem is shown to be equivalent to a countably infinite set of 1-D separable two-point boundary value smoothing problems. The 1-D smoothing problems are solved using either a Markovianization approach followed by a standard 1-D smoothing algorithm, or by using a recently developed smoothing technique for two-point boundary value problems. The desired field estimate is then obtained as a properly weighted sum of the 1-D smoothed estimates.

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1 INTRODUCTION

Problems involving spatially-distributed data and phenomena arise in various fields including image processing, meteorology, geophysical signal processing, oceanography and optical processing. A major challenge in any such problem is to develop algorithms capable of dealing effectively with the increased computational complexity of multidimensional problems and which can be implemented in a recursive fashion. In one dimension the ways in which data can be organized for efficient processing are extremely limited and causality typically provides a natural choice. Furthermore, in one dimension, internal differential realizations of random processes were exploited to develop an efficient estimation algorithm,namely the Kalman filtering technique. This has led researchers in estimation theory to investigate the extension of 1-D Kalman filtering and smoothing methods to non-causal 2-D random fields. The work of Woods and Radewan [27], Habibi [9], Attassi [3], Jain and Angel [11], Wong [25], Ogier and Wong [18] to name a few, has shown that such extensions do exist. However, the methods developed by these researchers are either approximate or can be applied only to a limited class of 2-D fields, namely to fields that can be described by hyperbolic partial differential equations, and which therefore are causal in some sense.

The objective of this paper is to study the smoothing problem for a class of random fields which have non-causal internal differential realizations but which also have enough structure to allow the development of efficient recursive smoothing algorithms. Note that, unlike one dimension, the most natural estimation problem in higher dimensions is the smoothing problem, rather than the causal filtering problem. This is due to the fact that in higher dimensions, the filtering problem requires an artificial partition of the data between past and future, whereas the smoothing problem does not assume any causal ordering of the data. Specifically, in this paper we investigate efficient recursive smoothing techniques for *isotropic* random fields which can be represented as the output of rational 2-D filters driven by white noise, and which admit therefore simple internal differential models. An isotropic field is characterized by the fact that its mean value is a constant independent of position and its autocovariance function is invariant under all rigid body motions, i. e. under translations and rotations. In some sense, isotropy is the natural extension of the notion of stationarity in one dimension. Furthermore, isotropic random fields arise in a number of practical problems such as the black body radiation problem [17], the study of underwater ambient noise in horizontal planes parallel to the surface of the ocean [6], and the investigation of temperature and pressure distributions at constant altitude in the atmosphere [12].

The smoothing problem for 2-D random fields has been studied from an ***input-output*** point of view by Ramm [20] and by Levy and Tsitsiklis [15] among others. In particular, Ramm studied the integral equation governing the optimal linear filter for estimating a general random field given some observations, while Levy and Tsitsiklis developed efficient Levinson-like recursions for computing the optimal smoothing filter for the case where both the field of interest and the observations are isotropic. Here, in contrast, we develop a ***recursive, differential-model-based*** estimation technique for the smoothing problem for 2-D isotropic random fields. The difference existing between our approach and that of [15] and [20] is therefore the same as that existing between Kalman and Wiener filtering methods in 1-D estimation theory. Specifically, we consider the smoothing problem for isotropic random fields $z(\vec{r})$[1] having an internal differential realization involving the Laplacian operator. The motivation for studying a model of this form is that any isotropic process that can be obtained by passing 2-D white noise through a rational linear filter has an internal realization of this type (see Section 2). Another motivation for considering such a model is that it can be used to describe a large class of physical phenomena such as the variation of the electric potential created by a random charge distribution.

An important property of 2-D isotropic fields is that when they are expanded in a Fourier series in terms of the polar coordinate angle θ, the Fourier coefficient processes of different orders are uncorrelated [26]. Given noisy observations of

[1]Throughout this paper we use $\vec{r}$ to denote a point in 2-D Cartesian space. The polar coordinates of this point are denoted by r and θ.

the isotropic random field $z(\vec{r})$ over a finite disk of radius R , our approach is to reduce the 2-D smoothing problem to a countable set of decoupled 1-D smoothing problems for the uncorrelated Fourier coefficient processes $z_k(r)$ corresponding to the process $z(\vec{r})$. Using the internal model of the process $z(\vec{r})$, 1-D state space two-point boundary value (TPBV) models are constructed for the Fourier coefficient processes . The resulting 1-D TPBV smoothing problems are then solved using either a Markovianization technique which transforms the non-causal state-space model to a causal one to which standard 1-D smoothing techniques can be applied, or directly by using the method of Adams et al. [2]. Finally, the best linear least squares estimate of $z(\vec{r})$ given the observations is obtained as a properly weighted combination of the 1-D smoothed estimates of all the Fourier coefficient processes $z_k(r)$. Observe that by properly exploiting the structure of isotropic random fields, a *recursive* solution to the smoothing problem for a non-causal isotropic process has thus been constructed . The recursions here are with respect to the radius r in a polar coordinate representation of the fields.

This paper is organized as follows. In Section 2, we introduce an internal differential model for the class of 2-D isotropic fields to be studied and show that this class includes isotropic random fields which can be represented as the output of rational 2-D filters driven by white noise. In Section 3, the smoothing problem for the isotropic random field $z(\vec{r})$ given noisy measurements over a disk of radius R is defined and is reduced to a countably infinite set of decoupled 1-D estimation problem. Two-point boundary value models are then developed to describe the 1-D estimation problems. Section 4 outlines two solutions to the 1-D two-point boundary value smoothing problems of Section 3. In Section 5 we briefly discuss some implementation issues. Finally, in Section 6, we study the asymptotic behavior of the differential models introduced in Section 3 as the radius R of the disk of observations tends to infinity. In particular, we show that the filters that we use to solve the 1-D estimation problems are asymptotically stable.

2 INTERNAL MODEL

A Differential Model

The class of random fields considered in this paper is described over the plane $\mathbf{R}^2$ by the differential model

$$(I_n\nabla^2 - A^2)x(\vec{r}) = Bu(\vec{r}) \quad (2.1)$$
$$z(\vec{r}) = Cx(\vec{r}) \quad (2.2)$$

with the asymptotic condition

$$E[x(\vec{r})x^T(\vec{s})] \to 0 \quad as \quad |\vec{r}-\vec{s}| \to \infty. \quad (2.3)$$

Here, $x(\vec{r}) \in \mathbf{R}^n$, $u(\vec{r}) \in \mathbf{R}^m$, $z(\vec{r}) \in \mathbf{R}^p$, and A , B, and C are real matrices of appropriate dimensions. The eigenvalues of the matrix A are assumed to have strictly positive real parts. This assumption insures that there exists a solution $x(\vec{r})$ to (2.1) that obeys the asymptotic condition (2.3). In equation (2.1) $u(\vec{r})$ is a random zero-mean two-dimensional white Gaussian noise process with

$$E[u(\vec{r})u^T(\vec{s})] = I_m\delta(\vec{r}-\vec{s}), \quad (2.4)$$

where I_m is the $m \times m$ identity matrix.

From an input-output point of view equations (2.1)-(2.2) together with the asymptotic condition (2.3) are equivalent to the representation

$$x(\vec{r}) = -\frac{1}{2\pi}\int_{\mathbf{R}^2} K_0(A|\vec{r}-\vec{r'}|)Bu(\vec{r'})\,d\vec{r'}, \quad \vec{r} \in \mathbf{R}^2 \quad (2.5)$$
$$z(\vec{r}) = Cx(\vec{r}) \quad (2.6)$$

where $d\vec{r'} = dx'\,dy'$ denotes an element of area. Here, $K_0(Ar)$ denotes a matrix modified Bessel function of the second kind and of order zero [4]. In fact, $G(\vec{r},\vec{s}) = \frac{1}{2\pi}K_0(A|\vec{r}-\vec{s}|)$ is the Green's function associated to the differential equation (2.1), i. e. $G(\vec{r},\vec{s})$ satisfies the equation

$$(I_n\nabla^2 - A^2)G(\vec{r},\vec{s}) = -I_n\delta(\vec{r}-\vec{s}) \quad (2.7)$$

for $\vec{r}, \vec{s} \in \mathbf{R}^2$, with the asymptotic condition

$$G(\vec{r}, \vec{s}) \to 0 \quad as \quad |\vec{r} - \vec{s}| \to \infty. \tag{2.8}$$

Matrix modified Bessel functions of the first and second kinds arise naturally in the study of rational isotropic random fields. A brief discussion of some of their properties appears in Appendix A. (For more details see [4] and the references therein).

The main property of the process $x(\vec{r})$ defined by (2.1) and (2.3) is that it is a 2-D rational isotropic random field as is shown below.

Theorem 2.1 *The process $x(\vec{r})$ defined by equation (2.1) together with the asymptotic condition (2.3)is an isotropic random field, i.e. its autocorrelation function $R_x(\vec{r}, \vec{s}) = E[x(\vec{r})x^T(\vec{s})]$ is invariant under translations and rotations.*

Proof

We will first show that $R_x(\vec{r}, \vec{s})$ is invariant under translation. From (2.5) we have

$$\begin{aligned} R_x(\vec{r}, \vec{s}) &= E[x(\vec{r})x^T(\vec{s})] & (2.9) \\ &= \frac{1}{4\pi^2} \int_{\mathbf{R}^2} K_0(A|\vec{r} - \vec{u}|)BB^T K_0(A|\vec{s} - \vec{u}|)\, d\vec{u} & (2.10) \end{aligned}$$

Now perform the transformation

$$\vec{v} = \vec{u} + \vec{h} \tag{2.11}$$

to obtain

$$R_x(\vec{r}, \vec{s}) = \frac{1}{4\pi^2} \int_{\mathbf{R}^2} K_0(A|\vec{r} + \vec{h} - \vec{v}|)BB^T K_0(A|\vec{s} + \vec{h} - \vec{v}|)\, d\vec{v}. \tag{2.12}$$

This shows that $R_x(\vec{r}, \vec{s})$ is invariant under translation. Using this fact, we can write

$$R_x(\vec{r}, \vec{s}) = R_x(\vec{v}, 0) \tag{2.13}$$

where $\vec{v} = \vec{r} - \vec{s}$. Hence,

$$\begin{aligned} R_x(\vec{r}, \vec{s}) &= \frac{1}{4\pi^2} \int_{\mathbf{R}^2} K_0(A|\vec{v} - \vec{u}|)BB^T K_0(A|\vec{u}|)\, d\vec{u} & (2.14) \\ &= \frac{1}{4\pi^2} \int_{\mathbf{R}^2} K_0(A(v^2 + u^2 - 2uv\cos(\theta - \phi))^{\frac{1}{2}})BB^T K_0(Au)\, d\vec{u} & (2.15) \end{aligned}$$

where $\vec{v} = (v, \phi)$ and $\vec{u} = (u, \theta)$. Letting $\alpha = \phi - \theta$, we conclude from the above equation and the periodicity of $\cos\alpha$ that

$$R_x(\vec{r}, \vec{s}) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} K_0(A(v^2 + u^2 - 2uv\cos\alpha)^{\frac{1}{2}}) BB^T K_0(Au) u\, du\, d\alpha \quad (2.16)$$

□□

Theorem (2.1) implies also that the output process $z(\vec{r})$ is isotropic with autocorrelation function

$$R_z(\vec{r}, \vec{s}) = C R_x(\vec{r}, \vec{s}) C^T. \quad (2.17)$$

Since $R_x(\cdot)$ is translation-invariant we can define its spectral density matrix $S_x(\vec{\lambda})$, which is the 2-D Fourier transform of $R_x(\vec{r})$:

$$\begin{aligned} S_x(\vec{\lambda}) &= \int_{\mathbf{R}^2} R_x(\vec{r}) e^{-j\vec{\lambda}\cdot\vec{r}}\, d\vec{r} && (2.18) \\ &= 2\pi \int_0^{\infty} R_x(r) J_0(\lambda r) r\, dr && (2.19) \\ &= (\lambda^2 I_n + M)^{-1} BB^T (\lambda^2 I_n + M^T)^{-1} && (2.20) \\ &= S_x(\lambda), && (2.21) \end{aligned}$$

where we have taken advantage of the circular symmetry of $R_x(\vec{r})$,and where $M = A^2$. Here M^T denotes the transpose of M. Observe that $S_x(\lambda)$ is *rational* in λ, the magnitude of $\vec{\lambda}$. Furthermore, the poles of the spectrum $S_x(\lambda)$, obtained by setting $p = j\lambda$ in (2.20), have a quadrantal symmetry property when plotted in the complex p-plane. Another important property of the process $x(\vec{r})$ that follows from equation (2.20) is that $x(\vec{r})$ is pseudo-Markovian of order 1 [19], i.e. the value of $x(\vec{r})$ inside a closed curve Γ is independent of the value of $x(\vec{r})$ outside Γ given the value of $x(\vec{r})$ and of its normal derivative along Γ. In the sequel, we shall make extensive use of the isotropic and pseudo-Markovian nature of the process $x(\vec{r})$.

B Motivation

The motivation for considering model (2.1)-(2.3) is that it can be used to describe a large class of physical phenomena such as the variation of the electric potential created by a uniformly distributed random sources in a lossy medium, where the loss is described here by A^2. Another important motivation for considering such a model is given in the following theorem.

Theorem 2.2 *Any isotropic process that is obtained by passing 2-D white noise through a rational and proper 2-D circularly symmetric linear filter has an internal realization of the form (2.1)-(2.2).*

Proof

Consider the scalar 2-D random field $z(\vec{r})$ described by the partial differential equation

$$P(\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2})z(\vec{r}) = Q(\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2})u(\vec{r}) \tag{2.22}$$

where $u(\vec{r})$ is a 2-D white noise process of intensity I_m. Here, $P(s_1, s_2)$ and $Q(s_1, s_2)$ are 2-D polynomials in the variables s_1 and s_2. Equation (2.22) implies that $z(\vec{r})$ is the output of a rational 2-D filter $H(\vec{\lambda})$ driven by the noise process $u(\vec{r})$, where

$$H(\vec{\lambda}) = \frac{Q(j\lambda_1, j\lambda_2)}{P(j\lambda_1, j\lambda_2)}. \tag{2.23}$$

The spectrum of $z(\vec{r})$ is given by

$$S_z(\vec{\lambda}) = |H(\vec{\lambda})|^2. \tag{2.24}$$

In [28], Yadrenko shows that the process $z(\vec{r})$ is isotropic if and only if the 2-D polynomials $P(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ are functions of $\lambda = (\lambda_1^2 + \lambda_2^2)^{1/2}$ only, i.e. if $P(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ are of the form

$$\begin{aligned} P(j\lambda_1, j\lambda_2) &= \sum_{k=0}^{n} p_k(-\lambda^2)^k && (2.25)\\ &= P(-\lambda^2) && (2.26)\\ Q(j\lambda_1, j\lambda_2) &= \sum_{k=0}^{q} q_k(-\lambda^2)^k && (2.27)\\ &= Q(-\lambda^2). && (2.28) \end{aligned}$$

In this case, the model (2.22) reduces to

$$P(\nabla^2)z(\vec{r}) = Q(\nabla^2)u(\vec{r}). \tag{2.29}$$

Furthermore, if $n > q$, by writing

$$H(-\lambda^2) = \frac{Q(-\lambda^2)}{P(-\lambda^2)} \quad (2.30)$$

$$= C(-\lambda^2 I_n - M)^{-1} B \quad (2.31)$$

and using any of the standard 1-D state-space realization techniques with the variable s replaced by λ^2 and the operator $\frac{d}{dt}$ by the operator ∇^2, we can obtain a state space realization of $z(\vec{r})$ in the form (2.1)-(2.2).

□□

We see therefore that the class of random fields with an internal differential realization of the form (2.1)-(2.2) is quite large. It is, in fact, the analog of the class of 1- D stationary processes which are obtained by passing white-noise through a finite dimensional, linear time-invariant filter.

Finally, note that in [20], Ramm has investigated the solution of a set of integral equations that arise in the study of input-output estimation problems for a class of random fields that includes all 2-D isotropic random fields with an internal differential realization of the form (2.1)-(2.2). In particular, Ramm gives a procedure for constructing the optimal linear smoothing filter for estimating random fields in the class that he studies given noisy observations of such fields. As mentioned earlier, we consider here a different problem. Specifically, we are interested in developing a ***recursive differential model based*** procedure for computing the linear least squares estimate of the field $z(\vec{r})$ over the disk $r \leq R$ given noisy observations of $z(\cdot)$ over that same disk. In the next section, we compute estimates of $z(\cdot)$ directly (i.e. without ever writing the estimate in integral form), by properly exploiting the internal realization (2.1)-(2.2) of the field $z(\cdot)$.

C Model over a Finite Disk

Over the finite disk $D_R = \{\vec{r} : r \leq R\}$, the field $z(\vec{r})$ defined by (2.1)-(2.3) or alternatively by the integral representation (2.5) can be modeled by

(2.1)-(2.2) together with a suitable boundary condition on the edge Γ of D_R. In addition to well-posedness, we would like to specify this boundary condition so that it is independent of the noise $u(\vec{r})$ inside the disk D_R. This will allow us later to directly apply the results of Adams et al. [2] for the estimation of boundary value processes. We shall call a boundary condition that satisfies the above conditions an *admissible* boundary condition. An admissible boundary condition can be specified as follows.

Theorem 2.3 *An admissible boundary condition for the process $x(\vec{r})$ over the disk D_R is given by*

$$\int_\Gamma [G(\vec{R},\vec{s})\frac{\partial x}{\partial n}(\vec{s}) - (\frac{\partial G}{\partial n}(\vec{R},\vec{s}))x(\vec{s})]\, dl = \beta(R,\theta), \quad 0 \le \theta < 2\pi \tag{2.32}$$

where Γ is the circle of radius R, $G(\vec{r},\vec{s}) = \frac{1}{2\pi}K_0(A|\vec{r}-\vec{s}|)$, and

$$E[\beta(R,\theta)] = 0 \tag{2.33}$$

$$\begin{aligned} E[\beta(R,\theta)\beta^T(R,\phi)] &= \Pi_\beta(R;\theta-\phi) \\ &= \sum_{k=-\infty}^{\infty} I_k(AR)\Pi_{\eta_k}(R)I_k^T(AR)e^{jk(\theta-\phi)} \end{aligned} \tag{2.34}$$

with

$$\Pi_{\eta_k}(R) = \frac{1}{2\pi}\int_R^\infty K_k(As)BB^TK_k(As)s\, ds. \tag{2.35}$$

Here, $\frac{\partial}{\partial n}$ and dl denote respectively the normal derivative with respect to Γ and an infinitesimal element of arc length along Γ. The functions $I_k(Ar)$ and $K_k(Ar)$ are matrix modified Bessel functions of the first and second kind respectively, and of order k (see Appendix A and [4]).

Theorem (2.3) is proved in Appendix B, where by repeatedly applying Green's identity it is shown that the boundary condition (2.32) leads to a well-posed problem and that the process $x(\vec{r})$ given by (2.5) is the unique solution to eq. (2.1) with the boundary condition (2.32). It is further shown that the boundary process $\beta(R,\theta)$ is independent of the noise $u(\vec{r})$ for $r < R$. Since we are primarily interested in the smoothing problem for the field $z(\vec{r})$ over the finite disk D_R, we shall assume throughout the remainder of this paper that $z(\vec{r})$ is described by the model (2.1)-(2.2) together with the boundary condition (2.32) or equivalently by equations (2.5)-(2.6).

3 THE SMOOTHING PROBLEM

A Problem Statement

Let

$$y(\vec{r}) = z(\vec{r}) + v(\vec{r}), \quad \vec{r} \in D_R \tag{3.1}$$

with $D_R = \{\vec{r} : r \leq R\}$, be noisy observations of the isotropic field $z(\vec{r})$ defined by the internal model (2.1)-(2.2) together with the boundary condition (2.32). Here, $v(\vec{r})$ is a two-dimensional white Gaussian noise field of dimension p uncorrelated with $u(\vec{r})$ and $\beta(R,\theta)$, and with intensity V, where V is a positive definite matrix. Thus,

$$E[v(\vec{r})u^T(\vec{s})] = 0 \tag{3.2}$$

$$E[v(\vec{r})\beta^T(R,\theta)] = 0 \tag{3.3}$$

$$E[v(\vec{r})v^T(\vec{s})] = V\delta(\vec{r}-\vec{s}) \tag{3.4}$$

where $\delta(\vec{r})$ denotes a two-dimensional delta function. The estimation problem that we consider here consists in computing the conditional mean

$$\hat{z}(\vec{r}|R) = E[z(\vec{r})|\, y(\vec{s}) \,:\, 0 \leq s \leq R] \tag{3.5}$$

for all $\vec{r} \in D_R$.

B Solution via Fourier Series Expansions

Following [15], our estimation procedure relies on the Fourier series expansions of the observation, signal and observation and process noise fields, e. g.

$$f(r,\theta) = \sum_{k=-\infty}^{\infty} f_k(r)\, e^{jk\theta}, \tag{3.6}$$

$$f_k(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)e^{-jk\theta}\, d\theta \tag{3.7}$$

where $f(\cdot)$ stands for $y(\cdot)$, $z(\cdot)$, $x(\cdot)$, $u(\cdot)$ or $v(\cdot)$. Note that the Fourier coefficient processes $y_k(r)$, $z_k(r)$, $u_k(r)$ and $v_k(r)$ are one dimensional processes. Substituting the Fourier series expansions of $y(\cdot)$, $z(\cdot)$, and $v(\cdot)$ into (3.1) yields

$$y_k(r) = z_k(r) + v_k(r), \quad 0 \le r \le R. \tag{3.8}$$

The main feature of this approach is that the Fourier coefficient processes of different orders are *uncorrelated*, i.e.

$$E[\alpha_k(r)\gamma_l^H(s)] = 0, \quad \text{for } k \neq l \tag{3.9}$$

where $\alpha(\cdot)$ and $\gamma(\cdot)$ stand for $y(\cdot)$, $z(\cdot)$, $x(\cdot)$, $u(\cdot)$ or $v(\cdot)$. Consequently, our original two-dimensional estimation problem requires the solution of a countable set of decoupled 1-D smoothing problems for the Fourier coefficient process $z_k(r)$ given the observations $y_k(s)$ over the interval $0 \le s \le R$. Once the smoothed estimates $\hat{z}_k(r|R) = E[z_k(r)|y_k(s) \ : \ 0 \le s \le R]$ are found, $\hat{z}(\vec{r}|R)$ may be computed as

$$\hat{z}(\vec{r}|R) = \sum_{k=-\infty}^{\infty} \hat{z}_k(r|R)\, e^{jk\theta} \tag{3.10}$$

where the equality in (3.10) is to be understood in the mean-square sense. In practice, of course, one would consider only a finite number N of the above one dimensional estimation problems. We shall have more to say about this point in Section 5.

C State-Space Models For The Fourier Processes

Using the internal model (2.1)-(2.2) and (3.1) for the process $z(\vec{r})$ and the observations $y(\vec{r})$, 1-D state-space two-point boundary value models can be constructed for the Fourier coefficient processes $z_k(r)$ and $y_k(r)$ as follows.

Theorem 3.1 *A two-point boundary value (TPBV) model describing $z_k(r)$ and $y_k(r)$ over the interval [0,R] is given by*

$$\frac{d}{dr}\begin{bmatrix} \xi_k(r) \\ \eta_k(r) \end{bmatrix} = \begin{bmatrix} -rI_k(Ar)B \\ rK_k(Ar)B \end{bmatrix} u_k(r) \tag{3.11}$$

$$z_k(r) = [CK_k(Ar) \ \ CI_k(Ar)] \begin{bmatrix} \xi_k(r) \\ \eta_k(r) \end{bmatrix} \tag{3.12}$$

$$y_k(r) = z_k(r) + v_k(r), \tag{3.13}$$

with the boundary conditions

$$\xi_k(0) = 0 \qquad \textit{with probability 1} \tag{3.14}$$

and

$$\eta_k(R) \sim N(0, \Pi_{\eta_k}(R)) \tag{3.15}$$

where $\Pi_{\eta_k}(R)$ *is given by equation (2.35). Here,* $u_k(r)$ *and* $v_k(r)$ *are two one-dimensional zero-mean white Gaussian noise processes with covariance*

$$E\left[\begin{bmatrix} u_k(r) \\ v_k(r) \end{bmatrix} \begin{bmatrix} u_k^T(s) & v_k^T(s) \end{bmatrix}\right] = \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \frac{\delta(r-s)}{2\pi r}. \tag{3.16}$$

Note that the TPBV model dynamics (3.11) are extremely simple, consisting of a gain matrix multiplying the input noise process $u_k(r)$. This is to be contrasted with the more complicated dynamics of an equivalent *Markovian* model for $z_k(r)$ that we shall develop in the next section.

Proof

To derive equations (3.11)-(3.13), we shall use the following identity [4]

$$K_0(A|\vec{r} - \vec{s}|) = \sum_k I_k(Ar_<)K_k(Ar_>)\cos(k(\theta - \phi)) \tag{3.17}$$

where $\vec{r} = (r, \theta)$, $\vec{s} = (s, \phi)$, $r_< = \min(r, s)$ and $r_> = \max(r, s)$. Upon multiplying both sides of (2.1)-(2.2) and (3.1) by $e^{-jk\theta}/2\pi$ and integrating from 0 to 2π, we obtain

$$\begin{aligned} x_k(r) &= -K_k(Ar)\int_0^r I_k(As)Bu_k(s)s\,ds \\ &\quad - I_k(Ar)\int_r^\infty K_k(As)Bu_k(s)s\,ds & (3.18) \\ z_k(r) &= Cx_k(r) & (3.19) \end{aligned}$$

and

$$y_k(r) = z_k(r) + v_k(r). \tag{3.20}$$

Define the state variables $\xi_k(r)$ and $\eta_k(r)$ by

$$\xi_k(r) = -\int_0^r I_k(As)Bu_k(s)s\,ds \qquad (3.21)$$

and

$$\eta_k(r) = -\int_r^\infty K_k(As)Bu_k(s)s\,ds. \qquad (3.22)$$

Then, it follows from (3.18)-(3.22) that a TPBV model describing $y_k(r)$ over the interval [0,R] is given by the system (3.11)-(3.13).

□□

Note that the boundary condition for the process $\eta_k(r)$ follows directly from the boundary condition (2.32) for the process $x(\vec{r})$ upon recognizing from identity (B.8) that

$$\beta(R,\theta) = \sum_{k=-\infty}^{\infty} I_k(Ar)\eta_k(R)\,e^{jk\theta}. \qquad (3.23)$$

Note also that the TPBV model (3.11)-(3.15) is well-posed, since $z_k(r)$ can be expressed uniquely in terms of $u_k(r)$ and $\eta_k(R)$ as

$$\begin{aligned} z_k(r) &= -C(K_k(Ar)\int_0^r I_k(As)Bu_k(s)s\,ds \\ &\quad + I_k(Ar)\int_r^R K_k(As)Bu_k(s)s\,ds + I_k(Ar)\eta_k(R)) \end{aligned} \qquad (3.24)$$

Furthermore, observe that $\eta_k(R)$ is independent of $u_k(r)$ for $r \leq R$.

4 1-D SMOOTHERS

In this section we discuss two solutions to the 1-D TPBV smoothing problems for the Fourier coefficient processes. The first solution is based on a Markovianization procedure followed by standard 1-D smoothing techniques, while the second solution is a direct application of the method proposed by Adams et al. [2]. Conceptually, the difference between the two approaches lies in the way they deal with the boundary conditions for the smoother. In the method of Adams et al. the boundary conditions are replaced initially by zero boundary conditions and a two-filter smoothing formula with simple dynamics is used. Once all the measurements $y_k(r)$ have been processed, a second step is required to take the true boundary conditions into account. On the other hand, the Markovianization approach deals with the boundary conditions directly as the measurements are processed. It does so by properly incorporating the boundary conditions into the dynamics of the estimator, a step that results in a more complicated smoother implementation.

A The Markovianization Approach

As mentioned earlier, the main feature of the TPBV model (3.11)-(3.15) describing the k^{th} order Fourier coefficient is that it is separable, i.e. the boundary conditions $\xi_k(0)$ and $\eta_k(R)$ are decoupled (cf. [13]). Hence, a Markovian model of the same order as the model (3.11)-(3.15) can be constructed for $x_k(r)$ by reversing the direction of propagation of $\eta_k(r)$ using a technique introduced by Verghese and Kailath [24] for constructing backwards Markovian models. Let

$$\mathcal{F}_r^k = \sigma\{\eta_k(s)\,,\ 0 \leq s \leq r\} \tag{4.1}$$

be the sigma field generated by the process $\eta_k(s)$ over the interval $[0, r]$. Then

$$\begin{aligned} \hat{u}_k(r) &= E[u_k(r)|\mathcal{F}_r^k] \\ &= E[u_k(r)\eta_k^H(r)]E[\eta_k(r)\eta_k^H(r)]^{-1}\eta_k(r) \\ &= -\frac{1}{2\pi}B^T K_k(Ar)\Pi_{\eta_k}^{-1}(r)\eta_k(r), \end{aligned} \tag{4.2}$$

where

$$\Pi_{\eta_k}(r) = \frac{1}{2\pi}\int_r^\infty K_k(As)BB^T K_k^T(As)s\,ds, \tag{4.3}$$

and where we have assumed that $\Pi_{\eta_k}(r)$ is non-singular. The process $\tilde{u}_k(r)$ defined by

$$\tilde{u}_k(r) = u_k(r) - \hat{u}_k(r) \tag{4.4}$$

is then an $\mathcal{F}_r^k$ - martingale with the same intensity $I_m/2\pi r$ as $u_k(r)$. Substituting (4.2) and (4.4) into (3.11)-(3.15) yields the forward-propagating model

$$\frac{d}{dr}\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} = \begin{bmatrix}0 & G_k(r)\\ 0 & F_k(r)\end{bmatrix}\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} + \begin{bmatrix}-rI_k(Ar)B\\ rK_k(Ar)B\end{bmatrix}\tilde{u}_k(r) \tag{4.5}$$

$$y_k = [CK_k(Ar) \;\; CI_k(Ar)]\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} + v_k(r), \tag{4.6}$$

with

$$E[\tilde{u}_k(r)] = 0 \tag{4.7}$$

$$E[v_k(r)] = 0 \tag{4.8}$$

$$E\left[\begin{bmatrix}\tilde{u}_k(r)\\ v_k(r)\end{bmatrix}[\tilde{u}_k^H(s) \;\; v_k^H(s)]\right] = \begin{bmatrix}I & 0\\ 0 & V\end{bmatrix}\frac{\delta(r-s)}{2\pi r}, \tag{4.9}$$

and where

$$G_k(r) = \frac{r}{2\pi}I_k(Ar)BB^TK_k^T(Ar)\Pi_{\eta_k}^{-1}(r) \tag{4.10}$$

and

$$F_k(r) = -\frac{r}{2\pi}K_k(Ar)BB^TK_k^T(Ar)\Pi_{\eta_k}^{-1}(r). \tag{4.11}$$

The initial conditions for the state-space model (4.5) at r=0 are given by

$$\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} \sim N(0, \Pi_k(0)) \tag{4.12}$$

with

$$\Pi_k(0) = \begin{bmatrix}0 & 0\\ 0 & \Pi_{\eta_k}(0)\end{bmatrix}, \tag{4.13}$$

where we have used the fact that

$$E[\xi_k(0)\eta_k^H(0)] = 0. \tag{4.14}$$

Here, $\eta_k^H(r)$ denotes the complex conjugate transpose of $\eta_k(r)$. The smoothing problem associated with the system (4.5)-(4.6) over [0,R] is a standard causal smoothing problem and can be solved using any of the 1-D smoothing techniques such as the Mayne-Fraser two-filter formula [16], [8], or the Rauch-Tung-Striebel formula [21], among others.

B The TPBV Smoother Formulation

Directly applying the results of [2] to the TPBV model (3.11)-(3.15), we find that the smoothed estimates of $\xi_k(r)$ and $\eta_k(r)$, $\hat{\xi}_k(r)$ and $\hat{\eta}_k(r)$ respectively, satisfy the following Hamiltonian TPBV system

$$\frac{d}{dr}\begin{bmatrix} \hat{\xi}_k(r) \\ \hat{\eta}_k(r) \\ \hat{\gamma}_k(r) \\ \hat{\delta}_k(r) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2\pi r}\mathcal{B}\mathcal{B}^T \\ 2\pi r \mathcal{C}^T V^{-1}\mathcal{C} & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}_k(r) \\ \hat{\eta}_k(r) \\ \hat{\gamma}_k(r) \\ \hat{\delta}_k(r) \end{bmatrix} + \begin{bmatrix} 0 \\ -2\pi r \mathcal{C}^T V^{-1} y_k(r) \end{bmatrix}, \tag{4.15}$$

where

$$\mathcal{B}^T = [-rB^T I_k^T(Ar) \;\; rB^T K_k(Ar)] \tag{4.16}$$

$$\mathcal{C} = [CK_k(Ar) \;\; CI_k(Ar)], \tag{4.17}$$

and with the boundary conditions,

$$\hat{\xi}_k(0) = 0, \tag{4.18}$$

$$\hat{\delta}_k(0) = 0, \tag{4.19}$$

$$\hat{\gamma}_k(R) = 0, \tag{4.20}$$

$$\hat{\delta}_k(R) = -\Pi_{\eta_k}^{-1}(R)\hat{\eta}_k(R). \tag{4.21}$$

An alternative way of deriving the Hamiltonian system (4.15) is to note that, since $x(\vec{r})$ is described by the model (2.1)-(2.2) with the boundary condition (2.32), then according to Adams et al. [2], the 2-D smoothed estimate of $x(\vec{r})$, $\hat{x}(\vec{r}|R)$, satisfies the Hamiltonian system

$$(I_n\nabla^2 - M)\hat{x}(\vec{r}|R) = BB^T\hat{\Theta}(\vec{r}|R) \tag{4.22}$$

$$(I_n\nabla^2 - M^T)\hat{\Theta}(\vec{r}|R) = C^T V^{-1}(y(\vec{r}) - C\hat{x}(\vec{r}|R)) \tag{4.23}$$

with the boundary condition [2]

$$\begin{bmatrix} -\frac{\partial\hat{\Theta}}{\partial n}(\vec{R}|R) \\ \hat{\Theta}(\vec{R}|R) \end{bmatrix} = V^*\,\Pi_\beta^{-1}\,V \begin{bmatrix} \hat{x}(\vec{R}'|R) \\ \frac{\partial}{\partial n}\hat{x}(\vec{R}'|R) \end{bmatrix} \tag{4.24}$$

where $\vec{R} = (R,\theta)$ and $\vec{R'} = (R,\phi)$. In the above identity, if $L_2^k(\Gamma)$ denotes the space of k vector functions which are square-integrable over Γ, the operator $V : L_2^{2n}(\Gamma) \rightarrow L_2^n(\Gamma)$ is such that for

$$f(\vec{R}) = [f_1^T(\vec{R}) \;\; f_2^T(\vec{R})]^T \tag{4.25}$$

we have

$$(Vf)(\vec{R}) = \int_\Gamma [-(\frac{\partial G}{\partial n}(\vec{R},\vec{R'}))f_1(\vec{R'}) + G(\vec{R},\vec{R'})f_2(\vec{R'})]\, dl \tag{4.26}$$

where $G(\vec{R},\vec{R'}) = \frac{1}{2\pi}K_0(A|\vec{R}-\vec{R'}|)$ and where dl denotes an element of arc length along Γ. In (4.24) the operators V^* and Π_β^{-1} denote respectively the Hilbert adjoint of the operator V and the inverse of the correlation operator associated to the kernel $\Pi_\beta(R;\theta-\phi)$ defined in eq. (2.35). If we introduce the variable

$$\Psi(\vec{r}|R) = 2\pi r\hat{\Theta}(\vec{r}|R) \tag{4.27}$$

and substitute the Fourier expansions of $\hat{x}(\vec{r}|R)$ and $\hat{\Psi}(\vec{r}|R)$ into (4.22)- (4.23) we obtain the following Hamiltonian system for the k^{th} order Fourier coefficient process

$$\begin{aligned} (I_n(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k^2}{r^2}) - M)\hat{x}_k(r|R) &= \frac{1}{2\pi r}BB^T\hat{\psi}_k(r|R) \qquad (4.28)\\ (I_n(\frac{d^2}{dr^2} - \frac{d}{dr}[\frac{1}{r}\cdot] - \frac{k^2}{r^2}) - M^T)\hat{\psi}_k(r|R) &= 2\pi rC^TV^{-1} \\ & \quad (y_k(r) - C\hat{x}_k(r|R)). \qquad (4.29) \end{aligned}$$

By properly selecting the state variables

$$\hat{\xi}_k(r) = r\frac{d}{dr}(I_k(Ar))\hat{x}_k(r|R) - rI_k(Ar)\frac{d}{dr}\hat{x}_k(r|R), \tag{4.30}$$

$$\hat{\xi}_k(r) = -r\frac{d}{dr}(K_k(Ar))\hat{x}_k(r|R) + rK_k(Ar)\frac{d}{dr}\hat{x}_k(r|R), \tag{4.31}$$

$$\hat{\gamma}_k(r) = (\frac{(k+1)}{r}A^{-T}K_k^T(Ar) - K_{k+1}^T(Ar))\hat{\psi}_k(r|R) - A^{-T}K_k^T(Ar)\frac{d}{dr}\hat{\psi}_k(r|R) \tag{4.32}$$

$$\hat{\delta}_k(r) = (\frac{(k+1)}{r}A^{-T}I_k^T(Ar) + I_{k+1}^T(Ar))\hat{\psi}_k(r|R) - A^{-T}I_k^T(Ar)\frac{d}{dr}\hat{\psi}_k(r|R) \tag{4.33}$$

it can be shown that the second order Hamiltonian system (4.28)-(4.29) has the realization (4.15).

The Hamiltonian TPBV system (4.15) for the smoother with the boundary conditions (4.18)-(4.21), can be solved efficiently by using the procedure proposed by Adams et al. (see [2] for details).

Once the smoothed estimates $\hat{\xi}_k(r)$ and $\hat{\eta}_k(r)$ have been computed for all k, the smoothed estimate $\hat{z}(\vec{r}|R)$ of $z(\vec{r})$ can be found as

$$\hat{z}(\vec{r}|R) = \sum_{k=-\infty}^{\infty} C(K_k(Ar)\hat{\xi}_k(r) + I_k(Ar)\hat{\eta}_k(r))e^{jk\theta}. \tag{4.34}$$

5 IMPLEMENTATION ISSUES

In this section we briefly discuss some implementation issues. Specifically, we examine the problem of truncating the series (3.10) and the problem of implementing the 1-D smoothers of Section 4.

A Truncation of the Series Representation of the Smoothed Estimate

The smoothed estimate $\hat{z}(\vec{r}|R)$ is given by equation (3.10) as an infinite sum of the 1-D Fourier coefficient processes smoothed estimates $\hat{z}_k(r|R)$. In practice, of course, one would consider a finite set of 1-D smoothing problems and one would approximate the series (3.10) by the finite series

$$\hat{z}_N(\vec{r}) = \sum_{|k|\leq N} \hat{z}_k(r|R)\, e^{jk\theta}. \tag{5.1}$$

Note that, with $\tilde{z}_N(\vec{r}) = z(\vec{r}) - \hat{z}_N(\vec{r})$ and with $\tilde{z}_k(r) = z_k(r) - \hat{z}_k(r|R)$, we have

$$E[\tilde{z}_N(\vec{r})\tilde{z}_N^T(\vec{r})] = \sum_{|k|\leq N} E[\tilde{z}_k(r)\tilde{z}_k^H(r)] + \sum_{|k|>N} E[z_k(r)z_k^H(r)]. \tag{5.2}$$

As r tends to zero, the matrix $E[z_k(r)z_k^H(r)]$ tends to zero as r^2 for $k \neq 0$. Furthermore, as r tends to infinity, the matrix $E[z_k(r)z_k^H(r)]$ tends to zero as r^{-1} for all values of k. Hence, in order to keep the variance of the estimator error small, the number 2 N + 1 of terms to be used in (5.1) should increase with the distance between the origin and the point where $z(\vec{r})$ is to be estimated. If r is small, one can use very few terms in (5.1) and still obtain a good estimate of $z(\vec{r})$. In fact, for $r = 0$ one needs only the zeroth order Fourier coefficient process smoothed estimate, $\hat{z}_0(0)$, in order to compute $\hat{z}(0|R)$ exactly.

B 1-D Smoother Implementation

At first glance the implementation of the 1-D smoothers of Section 4 poses some problems since the models (3.11)-(3.13) and (4.5)-(4.6) are not well behaved in the vicinity of $r = 0$ for $k \neq 0$. This can be seen from the singularity of $K_k(Ar)$ at $r = 0$, and is not surprising since the Fourier series decomposition degenerates at the origin. We now show that this is of no practical consequence. In practice, to compute $\hat{z}(\vec{r}|R)$ we divide the intervals $[0, R]$ and $[0, 2\pi]$ into M and N subintervals of length $\Delta_1 = R/M$ and $\Delta_2 = 2\pi/N$ respectively. As a result, the Fourier coefficient processes $y_k(r)$ are available at the positions $r = m\Delta_1$, $0 \leq m \leq M$. The 1-D smoothed estimates $\hat{z}_k(r|R)$ are then found by discretizing the smoother equations corresponding to models (3.11)-(3.13) and (4.5)-(4.6). In particular, for $k \neq 0$ we consider the 1-D discretized smoother implementations for $1 \leq m \leq M$. Note that, since $z_k(0) = 0$ and $y_k(0) = 0$ with probability one for $k \neq 0$, then

$$\hat{z}_k(\Delta_1|0) = 0, \quad k \neq 0. \tag{5.3}$$

Thus,

$$E[z_k(m\Delta_1)|y_k(l\Delta_1)\ :\ 0 \leq l \leq M] = E[z_k(m\Delta_1)|y_k(l\Delta_1)\ :\ 1 \leq l \leq M] \tag{5.4}$$

for $k \neq 0$. For $k = 0$, the models (3.11)-(3.13) and (4.5)-(4.6) are well behaved at $r = 0$. Hence, we solve the 1-D discretized smoothing problem for the zeroth order Fourier coefficient for $0 \leq m \leq M$. Observe that the zeroth order Fourier coefficient process is the only process needed to compute $\hat{z}(0|R)$. Consequently, in practice, the smoothed estimate of $\hat{z}(m\Delta_1, n\Delta_2)$ is computed as

$$\hat{z}(m\Delta_1, n\Delta_2) = \begin{cases} \hat{z}_0(0) & \text{if } m = 0 \\ \sum_{|k|\leq K} \hat{z}_k(m\Delta_1)\, e^{jkn\Delta_2} & \text{otherwise,} \end{cases} \quad (5.5)$$

where

$$\hat{z}_k(m\Delta_1) = \begin{cases} E[z_0(m\Delta_1)|y_0(l\Delta_1) \ : 0 \leq l \leq M] & k = 0 \\ E[z_k(m\Delta_1)|y_k(l\Delta_1) \ : 1 \leq l \leq M] & k \neq 0, \end{cases} \quad (5.6)$$

and where K is some number suitably chosen (see the previous section).

6 ASYMPTOTIC BEHAVIOR OF THE DIFFERENTIAL MODELS AT INFINITY

The Fourier coefficient processes $x_k(r)$ have a finite variance for all $r \in \mathbf{R}$ since by definition $x(\vec{r})$ has finite variance over the whole plane (see Section 2.) Hence, the optimal estimator for the Fourier coefficient process $x_k(r)$ written in integral form, must have a well-behaved kernel for all r. However, as we have already discussed, there is an issue to be addressed in examining the estimator as r tends to zero, as all Fourier coefficients except the zeroth-order one approach zero. There is also a similar issue to be faced for large values of r. In particular, as we will discuss, our earlier analysis shows that the variance of any individual Fourier coefficient decreases as r^{-1} as r tends to infinity. This has two implications. The first is that in order to obtain an accurate estimate for large r, one must retain a large number of terms in (3.10). The second is that one must develop a method to implement our Fourier coefficient estimators that is well-behaved as r tends to infinity. As one might expect, this will involve a scaling of the Fourier coefficients to obtain a well-posed estimation problem for large r.

The need to do scaling can be seen immediately in (3.11)-(3.13) and (4.5)-(4.6) which are *not* well-behaved as r tends to zero or infinity. This ill-behavior is due to the singularity of $K_k(Ar)$ and $I_k(Ar)$ as r tends to zero and infinity respectively [1], [4]. In Section 5, we discussed a strategy for dealing with the singularity of the models (3.11)-(3.13) and (4.5)-(4.6) as r tends to zero. Here, we introduce differential models for the Fourier coefficient processes that are well-behaved as r tends to infinity. These models show that we can interpret the Fourier coefficient process $y_k(r)$ as being the output of a cascaded system which is driven by non-singular noise processes. The cascaded system consists of a system which is well-behaved as r tends to infinity followed by a gain stage with a gain of $r^{-1/2}$, as shown in Fig. 1. For large values of r, we use these models to obtain smoothed estimates of the Fourier coefficients $x_k(r)$ by feeding the observations $y_k(r)$ into an input gain stage with a gain of $r^{1/2}$ followed by the smoothing filter associated to the models that we develop and an output gain stage with a gain of $r^{-1/2}$, as is shown in Fig. 2. In the sequel, we show that the smoothing filter corresponding to the center block in Fig. 2 can be implemented by using asymptotically stable Kalman filters. Furthermore, the asymptotic forms of these Kalman filters are *identical* for *all* of the Fourier coefficients and also lead to an important spectral factorization result.

A Models

The models that we develop are obtained by applying the state transformation

$$\chi_k(r) = T_k(r) \begin{bmatrix} \xi_k(r) \\ \eta_k(r) \end{bmatrix} \tag{6.1}$$

$$T_k(r) = \begin{bmatrix} K_k(Ar) & I_k(Ar) \\ -K_{k+1}(Ar) & I_{k+1}(Ar) \end{bmatrix} \tag{6.2}$$

to models (3.11)-(3.13) and (4.5)-(4.6), followed by a normalization of all the processes. The normalization consists in multiplying all processes by $r^{1/2}$, which forces the intensity of the noise processes to be a constant.

Note that by using (3.12) we can identify

$$\chi_k(r) = \begin{bmatrix} x_k(r) \\ A^{-1}(\frac{d}{dr}x_k(r) - \frac{k}{r}x_k(r)) \end{bmatrix}. \tag{6.3}$$

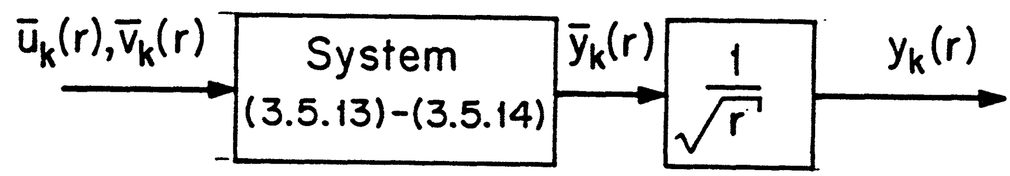

Figure 1: A model for $y_k(r)$ for large values of r.

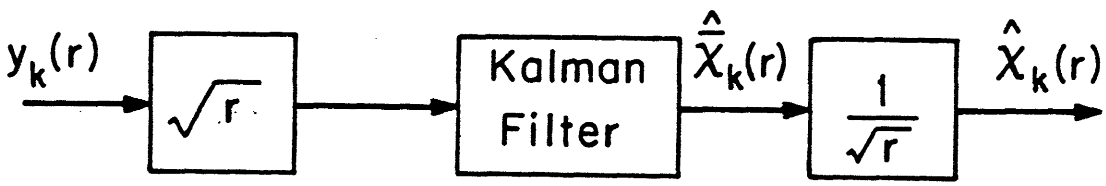

Figure 2: Filtering procedure for large values of r.

Note also that the transformation $T_k(r)$ has the properties that

$$\frac{d}{dr}T_k(r) = \begin{bmatrix} \frac{k}{r}I & A \\ A & -\frac{(k+1)}{r}I \end{bmatrix} T_k(r) \tag{6.4}$$

$$T_k^{-1}(r) = r\begin{bmatrix} AI_{k+1}(Ar) & -AI_k(Ar) \\ AK_{k+1}(Ar) & AK_k(Ar) \end{bmatrix}. \tag{6.5}$$

Identities (6.4)-(6.5) can be derived by using the recurrence relations for modified Bessel functions [1], [4] and the Wronskian identity [1], [4]

$$I_{k+1}(Ar)K_k(Ar) + I_k(Ar)K_{k+1}(Ar) = A^{-1}r^{-1}. \tag{6.6}$$

If we apply the state transformation $T_k(r)$ to the model (3.11)-(3.13) and if we introduce the *normalized processes*

$$\bar{\alpha}_k(r) = \sqrt{r}\alpha_k(r) \tag{6.7}$$

where $\alpha_k(r)$ stands for $\chi_k(r)$, $u_k(r)$, $y_k(r)$ or $v_k(r)$, we obtain

$$\frac{d}{dr}\bar{\chi}_k(r) = (A_k(r) + \frac{I}{2r})\bar{\chi}_k(r) + \bar{B}\bar{u}_k(r) \tag{6.8}$$

$$\bar{y}_k(r) = \bar{C}\bar{\chi}_k(r) + \bar{v}_k(r) \tag{6.9}$$

where

$$A_k(r) = \begin{bmatrix} \frac{k}{r}I & A \\ A & -\frac{(k+1)}{r}I \end{bmatrix} \tag{6.10}$$

$$\bar{B} = \begin{bmatrix} 0 \\ A^{-1}B \end{bmatrix} \tag{6.11}$$

$$\bar{C} = [C \quad 0], \tag{6.12}$$

and where we have used (6.4)-(6.5). In (6.8)-(6.9) $\bar{u}_k(r)$ and $\bar{v}_k(r)$ are two uncorrelated zero-mean Gaussian noise processes with intensities $I/2\pi$ and $V/2\pi$ respectively. Hence, (6.8)-(6.9) does not lead to a singular estimation problem.

Similarly, by using the state transformation $T_k(r)$ and normalizing all processes we find that model (4.5)-(4.6) is transformed to

$$\frac{d}{dr}\bar{\chi}_k(r) = (A_k'(r) + \frac{I}{2r})\bar{\chi}_k(r) + \bar{B}\bar{\tilde{u}}_k(r) \tag{6.13}$$

$$\bar{y}_k(r) = \bar{C}\bar{\chi}_k(r) + \bar{v}_k(r) \tag{6.14}$$

where

$$A'_k(r) = \begin{bmatrix} \frac{k}{r}I & A \\ A + D_k(r) & -\frac{(k+1)}{r}I + E_k(r) \end{bmatrix} \quad (6.15)$$

$$D_k(r) = -\frac{r}{2\pi}A^{-1}BB^T K_k^T(Ar)\Pi_{\eta_k}^{-1}(r)K_{k+1}(Ar)A \quad (6.16)$$

$$E_k(r) = -\frac{r}{2\pi}A^{-1}BB^T K_k^T(Ar)\Pi_{\eta_k}^{-1}(r)K_k(Ar)A, \quad (6.17)$$

and where $\bar{\bar{u}}_k(r)$ and $\bar{v}_k(r)$ are two uncorrelated zero-mean Gaussian noise processes with intensities $I/2\pi$ and $V/2\pi$ respectively.

Let us now make two comments. First observe that the transformation $T_k(r)$, its inverse $T_k^{-1}(r)$ and the normalization gain $r^{1/2}$ blow up as r tends to infinity. (The transformation $T_k(r)$ and its inverse $T_k^{-1}(r)$ blow up as r tends to infinity because of the singularity of the matrix functions $I_k(Ar)$ as r tends to infinity.) However, the normalized processes that appear in (6.8)-(6.9) and (6.13)-(6.14) are well-behaved and have a finite non-zero variance as r tends to infinity. In fact, by using the asymptotic forms of $K_k(Ar)$ and $I_k(Ar)$ as r tends to infinity (cf. Appendix A) and using equation (6.1), it can be shown that the process $\chi_k(r)$ has a variance that tends to zero as r^{-1} as r tends to infinity. Furthermore, recall that the intensity of the noise processes $u_k(r)$ and $v_k(r)$ is also proportional to r^{-1}. Hence, the variance of all the Fourier coefficient processes tends to zero as r^{-1} as r tends to infinity. This is precisely the reason why we have to keep a very large number of terms in (3.10) to obtain meaningful results as r tends to infinity. Note that this also implies that all the normalized processes are well-behaved with variances and noise intensities that tend to a finite constant as r tends to infinity. Second, note that as mentioned earlier, the models (6.8)-(6.9) and (6.13)-(6.14) can be used to obtain smoothed estimates of the Fourier coefficient processes $x_k(r)$ for large values of r by feeding the observations $y_k(r)$ into an input gain stage with a gain of $r^{1/2}$ followed by the smoothing filter associated to (6.8)-(6.9) and (6.13)-(6.14) and an output gain stage with a gain of $r^{-1/2}$.

B Asymptotic forms

We now turn to the examination of the asymptotic behavior of models (6.8)-(6.9) and (6.13)-(6.14) as r tends to infinity. Not only will this provide us with the basis for a stability result for our Kalman filter implementation but it also yields a spectral factorization result for the random field. To begin, we note that as r tends to infinity the modified Bessel functions $K_k(Ar)$ and $I_k(Ar)$ have the asymptotic forms [1]

$$\begin{aligned} I_k(Ar) &\sim (2\pi Ar)^{-\frac{1}{2}} e^{Ar} && (6.18)\\ K_k(Ar) &\sim (\frac{2Ar}{\pi})^{-\frac{1}{2}} e^{-Ar}. && (6.19)\end{aligned}$$

Hence, if we assume that the pair (A, B) is controllable we obtain

$$\begin{aligned} \lim_{r\to\infty} D_k(r) &= \lim_{r\to\infty} -A^{-1}BB^Te^{-A^Tr}(\int_r^\infty e^{-As}BB^Te^{-A^Ts}\,ds)^{-1}e^{-Ar}A \\ &= -A^{-1}BB^TQ^{-1}A \\ &= D, && (6.20)\end{aligned}$$

where Q is the matrix

$$Q = \int_0^\infty e^{-As}BB^Te^{-A^Ts}\,ds. \quad (6.21)$$

Note that since $-A$ is a stable matrix and since the pair (A, B) is controllable then Q is the unique positive definite solution of the matrix equation [5]

$$-AQ - QA^T + BB^T = 0. \quad (6.22)$$

Similarly, we have

$$\lim_{r\to\infty} E_k(r) = D. \quad (6.23)$$

Thus, as r tends to infinity the TPBV model (6.8)-(6.9) takes the form

$$\begin{aligned} \frac{d}{dr}\bar{\chi}_k(r) &= \bar{A}\bar{\chi}_k(r) + \bar{B}\bar{u}_k(r) && (6.24)\\ \bar{y}_k(r) &= \bar{C}\bar{\chi}_k(r) + \bar{v}_k(r) && (6.25)\end{aligned}$$

where

$$\bar{A} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}, \quad (6.26)$$

whereas the Markovian model (6.13)-(6.14) takes the form

$$\frac{d}{dr}\bar{\chi}_k(r) = \bar{A}'\bar{\chi}_k(r) + \bar{B}\bar{\tilde{u}}_k(r) \tag{6.27}$$

$$\bar{y}_k(r) = \bar{C}\bar{\chi}_k(r) + \bar{v}_k(r) \tag{6.28}$$

where

$$\bar{A}' = \begin{bmatrix} 0 & A \\ A+D & D \end{bmatrix}. \tag{6.29}$$

Note that the asymptotic models (6.24)-(6.25) and (6.27)-(6.28) imply that the models (6.8)-(6.9) and (6.13)-(6.14) are well-behaved as r tends to infinity. Note also that the asymptotic models (6.24)-(6.25) and (6.27)-(6.28) are *space invariant* models that do *not* depend on the order k of the Fourier coefficient process under consideration. This reflects the fact that as r tends to infinity all the Fourier coefficient processes have an equal importance in the sense that we would have to retain a very large number of terms in (3.10) to obtain meaningful results, as was already observed in Section 5. This also implies that for large values of r, we can use the *same* filter to obtain smoothed estimates of all the Fourier coefficients.

C Stability analysis

We now show that the stability of the matrix $\bar{A}'$ implies that the Kalman filter associated with (6.13)-(6.14) is stable. To do this we will need the following lemma which is an adaptation of a result of Coddington and Levinson ([7], p. 314).

Lemma 6.1 *Let*

$$\dot{x} = Ax + f(t,x) \tag{6.30}$$

where A is a real constant matrix with eigenvalues all having negative real parts. Furthermore, let f be real, continuous for small $|x|$ and $t \geq 0$, and such that

$$f(t,x) = o(|x|) \quad \text{as} \quad |x| \to 0 \tag{6.31}$$

uniformly in t, $t \geq 0$. Then, the system (6.30) is exponentially stable in a neighborhood of $x = 0$.

Proof

Let $\phi(t)$ be a solution of (6.30). So long as $\phi(t)$ exists, it follows from (6.30) that

$$\phi(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)} f(s, \phi(s))\, ds. \tag{6.32}$$

Because the real parts of the eigenvalues of A are negative, there exists positive constants K and σ such that

$$|e^{At}| \leq Ke^{-\sigma t} \quad \text{for} \quad t \geq 0. \tag{6.33}$$

Hence, we have

$$|\phi(t)| \leq K|\phi(0)|e^{-\sigma t} + K\int_0^t e^{-\sigma(t-s)}|f(s, \phi(s))|\, ds. \tag{6.34}$$

Given $\epsilon > 0$, there exists by assumption a $\delta > 0$ such that $|f(t, x)| \leq \epsilon|x|/K$ for $|x| \leq \delta$. Thus, as long as $|\phi(t)| \leq \delta$, it follows that

$$e^{\sigma t}|\phi(t)| \leq K|\phi(0)| + \epsilon\int_0^t e^{\sigma s}|\phi(s)|\, ds. \tag{6.35}$$

This inequality yields

$$e^{\sigma t}|\phi(t)| \leq K|\phi(0)|e^{\epsilon t}, \tag{6.36}$$

or

$$|\phi(t)| \leq K|\phi(0)|e^{-(\sigma-\epsilon)t} \quad \text{for} \quad t \geq 0. \tag{6.37}$$

The above discussion now implies that if initially $|\phi(0)| \leq \delta/K$, then $|\phi(t)|$ will decay exponentially to zero.

□□

Lemma 6.1 can now be used to prove the following result.

Theorem 6.1 *The system defined by equations (6.13)-(6.14) is exponentially stable.*

Proof

The proof follows by writing

$$A'_k(r) = \bar{A}' + \bar{A}'_k(r) \tag{6.38}$$

where $\bar{A}'$ is defined in (6.29). By taking $f(r,x)$ in Lemma 6.1 as

$$f(r, \bar{\chi}_k) = \bar{A}'_k(r)\bar{\chi}_k(r), \tag{6.39}$$

and noting that

$$\lim_{r\to\infty} \bar{A}'_k(r) = 0, \tag{6.40}$$

we obtain the desired result by invoking Lemma 2.

□□

By using Theorem 6.1 we can state and prove the main result of this section.

Theorem 6.2 *The Kalman filter associated with the model (6.13)-(6.14) is asymptotically stable. Furthermore, the error covariance associated with the normalized process $\bar{\chi}_k(r)$ converges to a non-negative definite matrix $\bar{P}$ as r tends to infinity, where $\bar{P}$ is the solution of the algebraic Riccati equation*

$$0 = \bar{A}'\bar{P} + \bar{P}\bar{A}'^T + \bar{B}\bar{B}^T - \bar{P}\bar{C}^T V^{-1}\bar{C}\bar{P}, \tag{6.41}$$

where the matrix $\bar{A}'$ is defined in (6.29).

Proof

The result follows by direct application of Theorem 4.11 of [14].

□□

D Stable spectral factorizations

Model (6.27)-(6.28) also provides a *stable spectral factorization* of $S_x(\lambda)$. In particular, observe that the transfer function associated with equation (6.27) is

$$\begin{aligned} W_f(s) &= A(sI+A)^{-1}(sI-A+A^{-1}BB^TQ^{-1}A)^{-1}A^{-1}B \\ &= (sI+A)^{-1}(sI-A+BB^TQ^{-1})^{-1}B. \end{aligned} \tag{6.42}$$

The formula

$$-A+BB^TQ^{-1}=QA^TQ^{-1} \tag{6.43}$$

(which is easily derived from (6.22)) now shows that $-A+BB^TQ^{-1}$ and A have the same eigenvalues. Therefore, $W_f(s)$ will have its poles in the left half-plane since all the eigenvalues of A have a positive real part by assumption. Note that this also implies that the matrix $\bar{A}'$ is a stable matrix. Furthermore, observe that

$$W_f(s)U(s)=W_b(s) \tag{6.44}$$

where

$$\begin{aligned} W_b(s) &= (sI+A)^{-1}(sI-A)^{-1}B \\ &= (s^2I-A^2)^{-1}B \\ &= (s^2I-M)^{-1}B \end{aligned} \tag{6.45}$$

and where

$$U(s)=I+B^TQ^{-1}(sI-A)^{-1}B. \tag{6.46}$$

It is easy to verify that $U(s)$ is a paraunitary or allpass transfer function in the sense that

$$U(s)U^T(-s)=U^T(-s)U(s)=I. \tag{6.47}$$

Hence, we have

$$\begin{aligned} W_f(s)W_f^T(-s) &= W_b(s)W_b^T(-s) \\ &= (sI+A)^{-1}(sI-A)^{-1}BB^T(-sI-A^T)^{-1}(-sI+A^T)^{-1} \\ &= S_x(\lambda)|_{\lambda=-js}, \end{aligned} \tag{6.48}$$

which proves that the asymptotic model (6.27)-(6.28) does lead to a stable spectral factorization of $S_x(\lambda)$. Finally, observe that the results of [24] imply that $(sI - A + BB^TQ^{-1})^{-1}B$ is the transfer function of a stable *forward* Markovian model corresponding to the stable *backwards* Markovian model with transfer function $(sI - A)^{-1}B$.

7 CONCLUSION

In this paper we have obtained efficient recursive estimation techniques for isotropic random fields described by non-causal internal differential realizations. By exploiting the properties of isotropic random fields, we showed that the problem of estimating an isotropic random field given noisy observations over a finite disk of radius R is equivalent to a countably infinite set of decoupled one-dimensional two-point boundary value system (TPBV) estimation problems for the Fourier coefficient processes of the random field. We then solved the 1-D TPBV estimation problems using either the method of Adams et al. [2] or by using a Markovianization approach followed by standard 1-D smoothing techniques. We have also studied the asymptotic behavior of the Markovian models that we developed as the radius R of the disk of observation tends to infinity, and we have shown that the 1-D Kalman filters associated to these models are asymptotically stable. The smoothing schemes that we have developed involve recursive structures in which the data is processed outwards or inwards with respect to a disk of observation as shown in Fig. 3. Observe that this concept of causality follows naturally form the special geometrical structure of isotropic random fields.

Note that the approach that we have used in this paper carries over to the case where the source term $u(\cdot)$ appearing on the right hand side of (2.1) is not spatially white but has a covariance function which is invariant under rotations only. In particular, it applies to the case where the field $u(\cdot)$ has a covariance function of

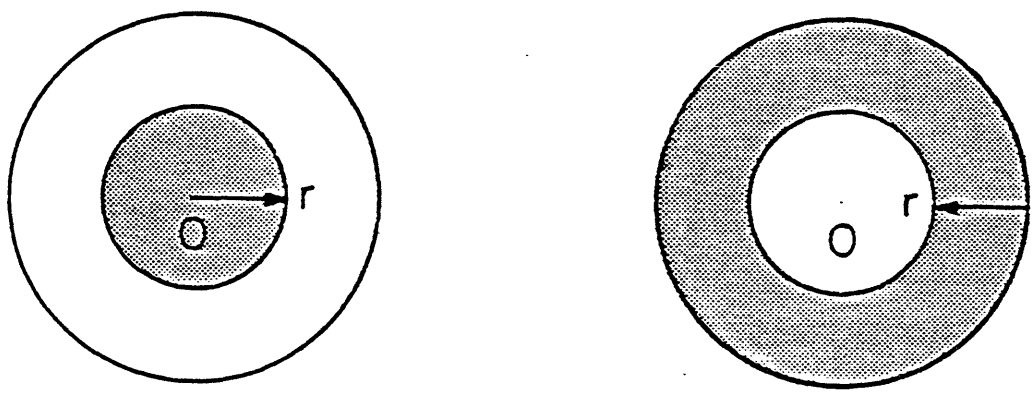

Figure 3: Outgoing and incoming radial recursions.

the form

$$E[u(\vec{r})u^T(\vec{s})] = K_1(r,s)K_2(\theta-\phi) = K_1(r,s)\sum_k a_k e^{jk(\theta-\phi)}, \tag{7.1}$$

where $K_1(r,s)$ is a positive definite function of the variables r and s which is assumed to have a *finite* dimensional state-space realization. In such a case the noise process $u(\cdot)$ has a Fourier series expansion with uncorrelated Fourier coefficients. By substituting the Fourier series expansion of $u(\cdot)$ into (2.1) we find that the field $z(\cdot)$ has also a Fourier series expansion with uncorrelated coefficients. Hence, the 2-D estimation problem for the field $z(\cdot)$ can be reduced to a countably infinite number of 1-D estimation problems for its Fourier coefficient processes. Note however that in such a case the TPBV model (3.11)-(3.13) describing the 1-D Fourier coefficient processes has to be properly augmented to account for the fact that the processes $u_k(r)$ have covariance functions $a_k K_1(r,s)$ and must be realized as the output of a 1-D dynamical system driven by white noise.

Several interesting extensions of our results suggest themselves. One of these is the development of alternate "recursive " structures. In particular, one can imagine developing algorithms that process data recursively in the angular direction as shown in Fig. 4, rather than radially (see [23] for such an algorithm in a different setting). Also, note that an important problem of great practical interest which is special to applications in several dimensions, is the problem of estimating random *vector* fields governed, for example , by Maxwell's equations, the heat equation or the gravitational field equations. This problem has not yet been considered in the literature and constitutes a natural generalization of the estimation problem for scalar fields that we have examined here. A preliminary investigation of 2-D estimation problems for isotropic random vector fields is currently under way and is based on the ideas introduced in [10], [29], [22].

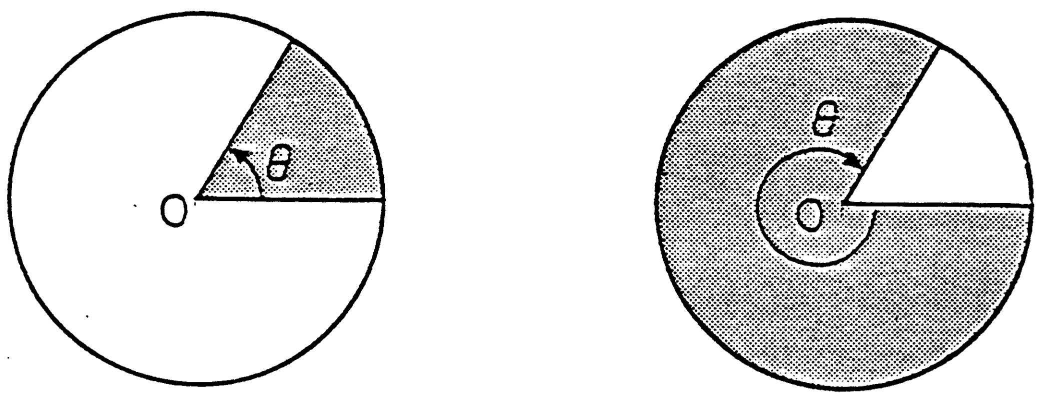

Figure 4: Clockwise and anticlockwise angular recursions.

APPENDIX A

In this paper, we make frequent use of the matrix modified Bessel functions of the first and second kinds, $I_k(Ar)$ and $K_k(Ar)$. These functions are a generalization of the corresponding scalar modified Bessel functions, and they satisfy the matrix differential equation

$$(I_n(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k^2}{r^2}) - A^2)F(r) = 0 \tag{A.1}$$

with the limiting forms

$$I_k(Ar) \sim (k!)^{-1}(\frac{Ar}{2})^k \tag{A.2}$$

$$K_0(Ar) \sim \ln(Ar) \tag{A.3}$$

$$K_k(Ar) \sim \frac{(k-1)!}{2}(\frac{Ar}{2})^{-k}, \quad k \geq 1 \tag{A.4}$$

as r tends to zero, and with the asymptotic forms

$$I_k(Ar) \sim (2\pi Ar)^{-\frac{1}{2}}e^{Ar} \tag{A.5}$$

$$K_k(Ar) \sim (\frac{2Ar}{\pi})^{-\frac{1}{2}}e^{-Ar} \tag{A.6}$$

as r tends to infinity. Thus $I_k(Ar)$ and $K_k(Ar)$ are regular at $r = 0$, and as r tends to infinity, respectively.

Bessel functions have a number of useful properties which are listed in [4].

APPENDIX B

In this appendix we show that the boundary condition for the process $x(\vec{r})$ given in Theorem (3.3) is independent of the noise inside the disk D_R, where $D_R = \{\vec{r} : \quad r \leq R\}$, and that this boundary condition leads to a well-posed problem. From Green's identity, we have

$$\int_{D_R} G(\vec{r},\vec{s})((I_n\nabla^2 - A^2)x(\vec{s})) - ((I_n\nabla^2 - A^2)G(\vec{r},\vec{s}))x(\vec{s})\,d\vec{s} = \int_\Gamma [G(\vec{r},\vec{s})\frac{\partial}{\partial n}x(\vec{s}) - (\frac{\partial}{\partial n}G(\vec{r},\vec{s}))x(\vec{s})]\,dl \tag{B.1}$$

where $\vec{r} \in D_R$, $G(\vec{r},\vec{s}) = \frac{1}{2\pi}K_0(A|\vec{r}-\vec{s}|)$, and where $\frac{\partial}{\partial n}$ denotes the normal derivative with respect to the curve $\Gamma = \{\vec{r} : \quad r = R\}$. Here, dl is an element of arc length along Γ. Equation (B.1) implies that

$$x(\vec{r}) = -\int_{D_R} G(\vec{r},\vec{s})Bu(\vec{s})\,d\vec{s} + \int_\Gamma [G(\vec{r},\vec{s})\frac{\partial}{\partial n}x(\vec{s}) - (\frac{\partial}{\partial n}G(\vec{r},\vec{s}))x(\vec{s})]\,dl. \tag{B.2}$$

However, from (2.5) $x(\vec{r})$ can be expressed as

$$x(\vec{r}) = -\int_{D_R} G(\vec{r},\vec{s})Bu(\vec{s})\,d\vec{s} - \int_{D_R^c} G(\vec{r},\vec{s})Bu(\vec{s})\,d\vec{s} \tag{B.3}$$

where D_R^c denotes the complement of D_R in $\mathbf{R}^2$. Hence, we conclude from (B.2) and (B.3) that

$$\Phi_R(\vec{r}) = \int_\Gamma [G(\vec{r},\vec{s})\frac{\partial}{\partial n}x(\vec{s}) - (\frac{\partial}{\partial n}G(\vec{r},\vec{s}))x(\vec{s})]\,dl \tag{B.4}$$

$$= \int_{D_R^c} G(\vec{r},\vec{s})Bu(\vec{s})\,d\vec{s} \tag{B.5}$$

and the above identity can be used to specify a boundary condition for $x(\vec{r})$ which is independent of the noise inside the disk D_R. Specifically, $\Phi_R(R,\theta)$ depends only on the noise $u(\vec{r})$ outside the disk D_R, and is therefore independent of the noise inside D_R. Let

$$\beta(R,\theta) = \Phi_R(R,\theta). \tag{B.6}$$

By taking the expression (B.5) into account, and using the expansion [4]

$$K_0(A|\vec{r}-\vec{s}|) = \sum_k I_k(Ar_<)K_k(Ar_>)\cos(k(\theta-\phi)) \tag{B.7}$$

where $\vec{r}=(r,\theta)$, $\vec{s}=(s,\phi)$, $r_< = \min(r,s)$ and $r_> = \max(r,s)$, we obtain

$$\beta(R,\theta) = -\sum_{k=-\infty}^{\infty} I_k(Ar)\int_R^{\infty} K_k(As)Bu_k(s)s\,ds\,e^{jk\theta}, \tag{B.8}$$

where

$$u_k(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)e^{-jk\theta}\,d\theta. \tag{B.9}$$

Since the random variables $u_k(s)$ and $u_l(s)$ with $k \neq l$, are independent zero-mean white Gaussian noise processes of intensity $I/2\pi s$, it follows that

$$E[\beta(R,\theta)] = 0, \tag{B.10}$$

$$\begin{aligned} E[\beta(R,\theta)\beta^T(R,\phi)] &= \Pi_\beta(R;\theta-\phi) \\ &= \sum_{k=-\infty}^{\infty} I_k(AR)\Pi_{\eta_k}(R)I_k^T(AR)e^{jk(\theta-\phi)}, \end{aligned} \tag{B.11}$$

where $\Pi_{\eta_k}(R)$ is given by (2.35). Then, as indicated in Theorem (3.3), equation (B.5) together with (B.10) and (B.11) can be used to specify a boundary condition for the 2-D field $x(\vec{r})$ in terms of the boundary process $\beta(R,\theta)$.

To show that the boundary condition (2.32) leads to a well-posed problem, note that (B.4) implies that $\Phi_R(\vec{r})$ satisfies

$$(I_n\nabla^2 - A^2)\Phi_R(\vec{r}) = 0, \quad \text{for } \vec{r}\in D_R \tag{B.12}$$

$$\Phi_R(R,\theta) = \beta(R,\theta). \tag{B.13}$$

Let $G_R(\vec{r},\vec{s})$ denote that Green's function corresponding to the system (B.12)-(B.13). Then $G_R(\vec{r},\vec{s})$ obeys the equation

$$(I_n\nabla^2 - A^2)G_R(\vec{r},\vec{s}) = -I_n\delta(\vec{r}-\vec{s}), \tag{B.14}$$

for $\vec{r},\vec{s}\in D_R$, with the boundary condition

$$G_R(\vec{R},\vec{s}) = 0 \text{ for } \vec{R}\in\Gamma. \tag{B.15}$$

Now using Green's identity, we obtain

$$\Phi_R(\vec{r}) = -\int_\Gamma \left(\frac{\partial}{\partial n} G_R(\vec{r}, \vec{s})\right)\beta(\vec{R})\, dl. \tag{B.16}$$

Then, combining relations (B.3), (B.4) and (B.16), $x(\vec{r})$ can be expressed as

$$x(\vec{r}) = -\int_{D_R} G(\vec{r}, \vec{s}) Bu(\vec{s})\, d\vec{s} - \int_\Gamma \left(\frac{\partial}{\partial n} G_R(\vec{r}, \vec{s})\right)\beta(\vec{R})\, dl, \tag{B.17}$$

which shows that the boundary value problem for $x(\vec{r})$ is well-posed.

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