String Field Theory and Tachyon Condensation

by

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Abstract

In this thesis I discuss various aspects of Witten's cubic string field theory. After a brief review of the basics of string field theory we begin by showing how string field theory can be used to check certain conjectures about the tachyon vacuum. We then discuss the problem of trying to globally gauge fix string field theory. We end with a discussion of various results in the quantization of the theory.

Thesis Supervisor: Washington Taylor
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Chapter 1

Introduction

Since the mid-1990’s, string theory has been widely viewed as the most promising candidate for a theory unifying all the fundamental forces. Perhaps the greatest virtue of the theory is that it appears to be more or less unique. All of the supersymmetric critical string theories written down to date have been related to each other by various dualities and are thought to be just different vacua of one parent theory, known tentatively as M-theory. Moreover, the theory has no free parameters, but only depends on one dimensionful parameter which may be taken to be the Planck length.

Unfortunately, the uniqueness just described is somewhat misleading. While there seems to be a well-defined sense in which there is only one string theory, it is clear that the theory has an enormous number of vacua. If one studies the low energy theories that these various vacua describe one can find standard-model-like physics, but, one can also find just about anything else. This problem of vacuum selection is perhaps the largest obstacle in attempting to relate string theory to the real world, since even if one can find a string vacuum which precisely reproduces standard model physics at low energies, it will be only one of a dizzyingly large array of other vacua which bear no resemblance to the standard model.

This problem is made worse by the fact that, currently, there is no single formulation of string theory which can describe the complete set of vacua of the theory. Instead, one has a large collection of incomplete formulations which describe the physics on small patches in the space of vacua, but whose regime of validity cannot be extended. This is the problem of background independence and it stems from the fact that we still have no fundamental
principle which defines string theory.

In the early days of string theory, it was thought that one of the best ways to formulate a string theory with background independence was to use the language of string field theory, or SFT. The basic idea of an SFT is to try to construct a Lagrangian whose Feynman rules reproduce string perturbation theory. To do this, it was found that one should second quantize the theory, using the wave function of the free string as a classical field.

The advantage of this framework is that considering another background is equivalent to just shifting the classical field. Moreover, as we will see later, if one considers backgrounds that are solutions to the equations of motion, the action takes a canonical form. This situation is very different from standard string perturbation theory, where a shift in the background can only be accomplished by changing the world-sheet sigma model.

The first string field theories to be written down came from the light-cone quantization of the string action [1, 2, 3, 4]. While light-cone quantized SFTs have a number of advantages, especially for the superstring [5, 6, 7], and are the only available SFTs in backgrounds such as the pp-wave [8, 9], the lack of covariance can complicate simple problems such as finding the minimum of the classical potential. While a covariantized light-cone SFT was constructed in [10], this action contained a number of subtleties that were never resolved. In 1984, however, Witten constructed a simple covariant cubic SFT which captured the complete physics of bosonic open string theory [11].

While Witten's open string field theory (OSFT) was studied by a number of authors in the late 1980's, interest in the subject eventually waned and string field theory went into a long dormant period. There were a number of reasons for this, which one has to keep in mind when studying OSFT. The main reason was that the subject is enormously complicated. While the action takes a simple cubic form, the fields are multiplied together with a complicated star-product which mixes all of the infinite number of fields with each other. A second reason was that the supersymmetric version of the theory proposed in [12, 13, 14], was found to have various problems from colliding picture-changing operators. This meant that OSFT was restricted to the bosonic theory which suffers from various potentially fatal instabilities.

The final and perhaps most important reason that OSFT was abandoned was that it wasn't obvious what one could calculate in OSFT. Indeed it was often stated that OSFT had never lead to any new results in string theory [15].
This situation changed in 1999, when it was realized by Sen [16] that OSFT could be used to address questions in the decay of unstable D-branes. As is true for all brane configurations in bosonic string theory, and for some configurations in superstring theory, the open string spectrum contains a tachyon. This tachyon indicates that the system is unstable and will decay into some lower-energy state.

Sen made three conjectures about the tachyon potential: Given an unstable brane in string theory with a tachyonic mode $T$,

1. The effective tachyon potential $V(T)$ has a minimum at $T = T_*$ and the energy difference $V(0) - V(T_*)$ equals the tension of the brane.

2. The minimum of the tachyon potential corresponds to a state where the brane has decayed. Thus, there are no open string excitations. Since in SFT we start with a theory of open strings, we expect that there are no perturbative excitations of any kind.

3. There should exist lump solutions of the tachyon potential which correspond to lower dimensional branes.

As it turns out, OSFT is an excellent laboratory for examining each of these conjectures. In fact, conjecture 1 had already been examined in 1987 by Kostelecky and Samuel in [17], where they had observed that the tachyon potential had a minimum. Their results, however, were not understood at the time since it was not appreciated that OSFT was the theory of a space-filling brane.

As well as being ahead of its time, Kostelecky and Samuel’s paper is also important because it introduced the notion of level truncation. At present, it is not possible to solve the equations of OSFT exactly. Instead, following Kostelecky and Samuel, one truncates the OSFT action by throwing out all fields which have a mass greater than some cutoff referred to as the level. After doing this, there are only a finite number of fields and solving the equations of motion becomes straightforward. One then studies the action as a function of level. In general, results such as the energy of a solution to the equations of motion converge reasonably rapidly as the level is increased. While there is no clear understanding of why this procedure works, there is ample numerical evidence that it is a sensible approximation.
scheme.

Following Sen’s conjectures, the calculation of Kostelecky and Samuel was extended to very high levels in [18, 19, 20], confirming conjecture 1 with high precision. Similarly, conjecture 3, that there should be lump solutions of the tachyon potential corresponding to lower dimensional branes was verified in [21, 22, 23].

Sen’s second conjecture, that there are no open string excitation around the tachyon vacuum took longer to verify. We examined this problem in two different complimentary ways in [24, 25] which we discuss in detail in chapters 2 and 3.

Our first method, which we discuss in chapter 2, is very direct. To examine the open string spectrum around the tachyon vacuum, we take the known value of the vacuum itself and then linearize the OSFT action about it. This procedure yields a quadratic action of the form

$$S_{\text{linearized}} = -\frac{1}{2} \int dp \sum_{ij} \psi^i(p)\tilde{Q}^{ij}(p)\psi^j(p),$$

where the $\psi^i$ are the various string fields and $\tilde{Q}^{ij}$ is just a matrix of numbers which depends on the momentum $p^\mu$. As we will see later $\tilde{Q}^{ij}$ is not diagonal and the various fluctuations around the vacuum mix with each other. At each momentum $p$, we can diagonalize $\tilde{Q}^{ij}$ and study its eigenvalues. Whenever we find a vanishing eigenvalue, the corresponding eigenvector is an on-shell state.

One might hope, then, that if there were no states around the tachyon vacuum, that there would be no null eigenvalues of the matrix $\tilde{Q}^{ij}$. Unfortunately this is not quite right, as one must also worry about the gauge invariance of the theory. Around the tachyon minimum, the matrix $\tilde{Q}^{ij}$ corresponds to a BRST operator which generates gauge transformations via

$$\Psi \rightarrow \Psi + \tilde{Q} \Lambda$$

To find the physical spectrum around the tachyon minimum one must find null eigenvectors of $\tilde{Q}$ which are not also images of $\tilde{Q}$. In other words we must find the cohomology of $\tilde{Q}$. States which are images of $\tilde{Q}$ are referred to as $\tilde{Q}$-exact states.

Hence, to show that there are no states around the tachyon vacuum we must show that, whenever we find a zero of $\det(\tilde{Q}^{ij})$, the corresponding null eigenvector is $\tilde{Q}$-exact to a high
degree of numerical precision. Unfortunately, to do this calculation, we must work in the level-truncation scheme. As such, we can only check that there are no states below some mass which we take to be less than the mass scale of the truncation. However, within this limitation we are able to show that there are no states.

Our second method for trying to show that the cohomology of \( \tilde{Q} \) vanishes is somewhat more abstract. As we just mentioned, at the bottom of the hill we have a BRST operator \( \tilde{Q} \) whose cohomology gives the spectrum of physical states. As it turns out, whenever the cohomology of an operator vanishes, one can find another operator \( R \) such that \( \{ Q, R \} = 1 \). While the proof of this is given in chapter 3, if we assume that such an operator exists, it is easy to show that the cohomology vanishes. Suppose that a state \( \Psi \) satisfies \( Q \Psi = 0 \). Then we have

\[
Q(R \Psi) = \{ Q, R \} \Psi = \Psi
\]

So that \( \Psi \) is also in the image of \( Q \).

Finding such an operator \( R \) is in general non-trivial, however OSFT provides an extra ingredient which makes things somewhat simpler. As we will discuss later, OSFT defines an associative product between string fields called the star product which is denoted \( \Psi_1 \star \Psi_2 \). The BRST operator acts as a graded derivation on this algebra

\[
\tilde{Q}(\Psi_1 \star \Psi_2) = (\tilde{Q} \Psi_1) \star \Psi_2 + (-1)^{\Psi_1} \Psi_1 \star (\tilde{Q} \Psi_2)
\]

where by \((-1)^{\Psi_1}\) we mean \(-1\) if \( \Psi_1 \) has odd ghostnumber and \(1\) otherwise. The star product also has an identity \( \mathcal{I} \) which satisfies \( \mathcal{I} \star \Psi = \Psi \star \mathcal{I} = \Psi \). Since \( \tilde{Q} \) is a derivation of the star-algebra, it must annihilate \( \mathcal{I} \) for consistency. If in fact, as conjectured, \( \tilde{Q} \) has no cohomology, we must have also that \( \mathcal{I} = \tilde{Q} A \) for some state \( A \).

The converse is also true. If there exists a state \( A \) such that \( \tilde{Q} A = \mathcal{I} \), then we may take the left star multiplication of \( A \) to be the operator \( R \) defined above. Explicitly, if \( \tilde{Q} \Psi = 0 \) then we have

\[
\tilde{Q}(A \star \Psi) = (\tilde{Q} A) \star \Psi = \mathcal{I} \star \Psi = \Psi.
\]

Hence, the statement that \( \tilde{Q} \) has trivial cohomology is equivalent to the existence of a state \( A \) satisfying \( \tilde{Q} A = \mathcal{I} \). Thus, we proceed to check that such an \( A \) can be found, at least
approximately, in the level truncation procedure. While we discuss the extent that our numerical calculation succeeds in chapter 3, the upshot is that to within the approximation of the level truncation scheme one can in fact find such a state.

While this argument seems very convincing, there are a few subtleties which must be addressed. As mentioned above, any derivation of the star algebra must annihilate the identity state for consistency. It is known, however, that $c_0$ is a derivation of the star algebra which does not annihilate $\mathcal{I}$ [28]. The reason for this anomaly is that the fields in the identity do not fall off quickly at high level. This makes the star product $\mathcal{I} \star \Psi$ somewhat poorly defined. In spite of this potential issue, we argue in chapter 3 that the anomalous behavior of $\mathcal{I}$ does not affect our arguments about the cohomology of $\hat{Q}$.

Having addressed Sen's second conjecture in two ways, we turn to other interesting issues in OSFT. One of the most basic assumptions in OSFT is that the gauge can be fixed using Feynman-Siegel (FS) gauge given by

$$b_0 \Psi = 0.$$  

This gauge has been shown work for all perturbative questions around the top of the tachyon potential and numerical calculations around the bottom of the potential suggest that the gauge works there as well. It is important, however, to ask whether or not this gauge works globally.

The basic way that FS-gauge can break down is that given a field $\Psi$ in FS-gauge, the gauge variation $\Psi \to \Psi + \delta \Psi$ satisfies $b_0(\delta \Psi) = 0$. Working in the sector of scalar fields at zero momentum, we check at various points in field space whether or not such a gauge transformation exists which preserves FS-gauge. Surprisingly, we find that only a short distance away from the top and bottom of the tachyon potential FS-gauge breaks down.

This breakdown of gauge invariance explains one of the mysterious properties of the tachyon potential. As was noticed in [17, 19], if one calculates the effective tachyon potential in level truncation, one finds branch cuts in the potential to the left of the perturbative vacuum and to the right of the tachyon vacuum. Using our knowledge of the validity of FS-gauge, we check that these points are precisely the places where FS-gauge breaks down.

In chapter 5, we turn to various issues in the quantization of OSFT. Having examined the classical physics of tachyon vacuum, our goal was to consider how quantum corrections
might effect our results. Unfortunately, quantizing OSFT around the tachyon vacuum is not possible with existing techniques, so we consider instead asking how quantum corrections affect the physics around the perturbative vacuum. To this end we study what is probably the simplest loop diagram in the theory, the one-loop one-point function or tadpole diagram.

This diagram is interesting because it represents the first quantum correction to the closed string background as seen by the open strings. This may be understood as follows. Since OSFT is the theory of open strings on a D-brane, we may ask: How does the presence of the D-brane affect the closed string background? In general, one expects that a massive object in spacetime will source closed string fields such as the graviton and the dilaton. While open string theory cannot see these closed strings directly, if one of the closed strings emitted by the brane is reabsorbed as an open string, it will make a contribution to the open string one-point function. The one-loop tadpole diagram is the simplest example of this process.

While we leave the details of how one actually goes about evaluating this diagram to chapter 5, we mention some of the most interesting results here. First, we find that provided one considers an OSFT defined on a brane with sufficiently many transverse directions, the diagram is finite and may be interpreted as giving the first correction to the propagation of the open strings in the closed string background generated by the brane.

Second, we find that when there are less than three transverse directions, the diagram suffers from an infrared divergence due to the long distance propagation of massless closed string modes. These divergences correspond to the fact that, when there are too few transverse directions, the closed string fields don’t fall off at infinity.

When OSFT is defined on a space-filling brane, we find that in addition to diverging, the diagram suffers from a BRST anomaly. This anomaly is essentially the same anomaly one finds in the analogous calculation in ordinary string perturbation theory where it is removed using the so called Fischler-Susskind mechanism [29, 30, 31, 32, 33, 34, 35, 36].

Unfortunately, in OSFT there is no analogue of the Fischler-Susskind mechanism and perturbation theory breaks down. While this is the only pathology we find at one-loop, at two-loops we argue that there will be a BRST anomaly for any brane background.

Having found various flaws with the bosonic theory, it is interesting to imagine what we would expect to find in a SUSY OSFT. There we expect that as long as we have three transverse directions that the theory should be completely well defined quantum mechanically.
Chapter 2

The open string spectrum around the tachyon vacuum

In this chapter, following our work with W. Taylor in [24], we discuss our first approach to studying the spectrum of Witten’s open string field theory (OSFT) around the tachyon vacuum. We verify for a limited set of states that in fact there are no open string excitations, confirming Sen’s second conjecture.

2.1 Introduction

The 26-dimensional open bosonic string has a tachyon in its spectrum with $M^2 = -1/\alpha'$. The presence of this tachyon indicates that the perturbative vacuum of the theory is unstable. While some early work [17] indicated the possible existence of a more stable vacuum at lower energy (see also [37, 38]), until fairly recently the significance of this other vacuum was not understood, and the tachyon was taken to be an indication of fundamental problems with the open bosonic string.

In 1999, Sen suggested that the open bosonic string should be interpreted as ending on an unstable space-filling D25-brane [16]. Sen argued that the condensation of the tachyon should correspond to the decay of the D25-brane, and that it should be possible to give an analytic description of this condensation process using the language of Witten’s cubic open string field theory [11]. In particular, Sen made three concrete conjectures:
1. The difference in the action between the unstable vacuum and the perturbatively stable vacuum should be $\Delta E = VT_{25}$, where $V$ is the volume of space-time and $T_{25}$ is the tension of the D25-brane.

2. The perturbatively stable vacuum should correspond to the closed string vacuum. In particular, there should be no physical open string excitations around this vacuum.

3. Lower-dimensional Dp-branes should be realized as soliton configurations of the tachyon and other string fields.

Conjecture (1) has been verified to a high degree of precision in level-truncated cubic open string field theory [18, 19, 20]. Similarly, conjecture (3) has been verified for a wide range of single and multiple Dp-brane configurations [21, 22, 23].

Until our work in [24], conjecture (2) had received relatively little attention. See, however, [17, 39] for an early discussion. In this chapter we review our work with W. Taylor [24], where we demonstrate, for a specific set of states, that there are no on-shell degrees of freedom. To do this, we explicitly compute the scalar open string spectrum in the cubic open string field theory expanded around the perturbatively stable vacuum, using the level-truncation approximation. The results of this computation give direct evidence that there are no physical open string states in this vacuum, and that the open strings are removed from the spectrum by purely classical effects in the string field theory.

2.2 String Field Theory in the Stable Vacuum

We begin with a brief summary of Witten's cubic formulation of open bosonic string field theory [11] (see [15, 40, 41, 42, 43, 44] for reviews). The string field $\Phi$ contains an infinite family of space-time fields, one field being associated with each state in the open string Fock space. Physical fields are associated with states in the Hilbert space of ghost number one. The string field may be formally written as

$$\Phi = \phi(p)|\hat{0}; p\rangle + A_\mu(p)\alpha^\mu_{-1}|\hat{0}; p\rangle + \cdots$$  (2.1)
where $|0\rangle$ is the ghost number one vacuum related to the $SL(2,R)$-invariant vacuum $|0\rangle$ through $|0\rangle = c_1|0\rangle$. Witten's cubic string field theory action is

$$S = -\frac{1}{2} \int \Phi \ast Q\Phi - \frac{g}{3} \int \Phi \ast \Phi \ast \Phi \quad (2.2)$$

where $Q$ is the BRST operator of the string theory and the “star product” $\ast$ is defined by dividing each string evenly into two halves and “gluing” the right side of one string to the left side of the other through a delta function interaction. The action (2.2) is invariant under the stringy gauge transformations

$$\delta \Phi = Q \Lambda + g (\Phi \ast \Lambda - \Lambda \ast \Phi) \quad (2.3)$$

where $\Lambda$ is a ghost number zero string field.

While there are an infinite number of component fields in the string field $\Phi$, for any particular component fields the quadratic and cubic interactions in (2.2) and the related terms in the gauge transformations (2.3) can be computed in a straightforward fashion using a Fock space representation of the BRST operator $Q$ and the star product [45, 46, 47]. It has been found [17, 48] that truncating the theory by including only fields up to a fixed level $L$ is an effective approximation technique for many questions relevant to the tachyon condensation problem. (By convention the tachyon is taken to have level zero). At fixed level $L$, the theory can be further simplified by only considering interactions between fields whose levels total to some number $I < 3L$. Empirical evidence [17, 19] indicates that truncating at level $(L, I) = (L, 2L)$ is the most effective cutoff to maximize accuracy for a fixed number of computations and that calculations at truncation level $(L, 2L)$ give similar results to calculations at truncation level $(L, 3L)$.

Sen’s conjecture states that there is a Lorentz invariant solution $\Phi_0$ of the full string field theory equations of motion

$$Q\Phi_0 = -g\Phi_0 \ast \Phi_0 \quad (2.4)$$

corresponding to the closed string vacuum without a D25-brane. The existence of a nontrivial solution to (2.4) has been analyzed in the level-truncated theory [17, 18, 19, 20].

As a simple example, consider the level $(0, 0)$ truncation. This truncation reduces the
string field $\Phi$ to just the tachyon field $\phi|\hat{0}; p)$. The tachyon potential is the given by $-\frac{1}{2}\phi^2 + g\kappa\phi^3$, where $\kappa$ is a numerical constant. This potential gives a locally stable vacuum at $\phi_0 = 1/(3g\kappa)$. Evaluating the potential at this point gives 68% of Sen's conjectured value $T_{25}$ for the energy gap between the unstable vacuum and the perturbatively stable vacuum. When other scalar fields are included in the string field by raising the level at which the theory is truncated, many of these fields couple to the tachyon $\phi$ and take expectation values when $\phi$ becomes nonzero, but similar solutions to the level-truncated string field theory equations of motion continue to exist. In the level (4, 8) truncation, the energy gap between the two vacua becomes 98.6% of the predicted value [18], and in the level (10, 20) truncation the energy gap becomes 99.91% of the predicted value [19]. These results provide us with a close approximation to a field $\Phi_0$ satisfying (2.4), which we can use to study the perturbatively stable vacuum in the level truncated theory$^1$.

We can describe the physics around a nontrivial vacuum $\langle \Phi \rangle = \Phi_0$ satisfying the equation (2.4) by shifting the string field

$$\Phi = \Phi_0 + \tilde{\Phi}.$$  \hfill (2.5)

In terms of the new field $\tilde{\Phi}$, the action becomes

$$S = S_0 - \frac{1}{2} \int \tilde{\Phi} \star \tilde{Q} \tilde{\Phi} - \frac{g}{3} \int \tilde{\Phi} \star \tilde{\Phi} \star \tilde{\Phi}$$  \hfill (2.6)

where the new BRST operator $\tilde{Q}$ acts on a string field $\Psi$ of ghost number $n$ through

$$\tilde{Q}\Psi = Q\Psi + g (\Phi_0 \star \Psi - (-1)^n \Psi \star \Phi_0).$$  \hfill (2.7)

The identity

$$\tilde{Q}^2 = 0$$  \hfill (2.8)

for the new BRST operator follows from (2.4). The BRST invariance of the level-truncated approximation to the vacuum $\Phi_0$ was studied in [49].

We are interested in studying the physics of the new string field theory defined through (2.6,2.7). According to Sen, the vacuum $\Phi_0$ should be the closed string vacuum, and should

$^1$Since the work [24] appeared, the level truncation analysis has been extended to level (18,54) in [20], however we only use the results up to level (10,20).
not admit open string excitations. To study the spectrum of excitations of the theory, we need to explicitly calculate the quadratic terms in the action, or equivalently to compute the action of the new BRST operator (2.7) on a general string field. This requires us to compute all cubic couplings in the original string field theory of the form

$$t_{ijk}(p) \phi_i(0)\phi_j(-p)\phi_k(p).$$  \hspace{1cm} (2.9)

For simplicity, we restrict attention to scalar excitations, so we need to compute all terms of the form (2.9) where $\phi_i, \phi_j$ and $\phi_k$ are scalar fields. Because $\phi_j, \phi_k$ are momentum-dependent, we must include among these fields longitudinal polarizations of all higher-spin tensor fields as well as the zero-momentum scalars $\phi_i(0)$ which can take nonzero vacuum expectation values in $\Phi_0$. We also restrict attention to scalars at even levels, which decouple from odd-level scalars at quadratic order due to the twist symmetry of the theory [17, 18].

With the assistance of the symbolic manipulation program Mathematica we have computed all 58,481 scalar interactions of the form (2.9) in the level (6, 12) truncation of the theory. There are 160 scalar fields $\phi_j(p)$ at even levels $\leq 6$, including longitudinal polarizations of tensor fields, and 31 momentum-independent scalar fields $\phi_i(0)$, each of which takes a nonzero value in the vacuum $\Phi_0$. (Actually, the vacuum lies in a subspace $\mathcal{H}_1$ of the full scalar field space [16], but we do not use this decomposition in our analysis). As a check on our calculations we have also computed all the coefficients associated with gauge transformations (2.3) where one of the three fields involved has vanishing momentum. We have verified that our terms of the form (2.9) give rise to an action (in the perturbative vacuum) which is invariant at order $g^1$ under arbitrary momentum-independent gauge transformations and a random sampling of momentum-dependent gauge transformations.

Using the complete set of terms of the form (2.9) in the level (6, 12) truncation, we have calculated all the quadratic terms for even-level scalars in the action (2.6) around the perturbatively stable vacuum. In the remainder of this chapter we summarize the results of using this quadratic action to study the spectrum of open string states in the theory (2.6). Earlier attempts to study the spectrum of physical states in a subset of the level (2, 6) truncation appeared in [17, 39].
2.3 BRST Cohomology

The spectrum of physical states in the theory (2.6) is given by the BRST cohomology

\[ \text{Ker} \tilde{Q}_1 / \text{Im} \tilde{Q}_0, \]

where \( \tilde{Q}_n \) describes the action of the BRST operator \( \tilde{Q} \) on a string field of ghost number \( n \). From (2.8), it follows that \( \tilde{Q}_1 \tilde{Q}_0 = 0 \) in the full string field theory. The states associated with vanishing eigenvalues of the kinetic operator \( \tilde{Q}_1 \) at a fixed value of \( p^2 \) are the \( \tilde{Q} \)-closed states in the theory. Two \( \tilde{Q} \)-closed states are physically equivalent if they differ by a \( \tilde{Q} \)-exact state \( \tilde{Q}_0 \Lambda \) where \( \Lambda \) is a string field of ghost number 0.

Level truncation of the open string field theory breaks the general gauge invariance (2.3) at order \( g^2 \), although gauge invariance is preserved at order \( g^0 \) and \( g^1 \). This breaking of gauge invariance means that the level-truncated BRST operator no longer squares to zero. In other words,

\[ \tilde{Q}_1^{(L,I)} \tilde{Q}_0^{(L,I)} \neq 0 \]

where \( \tilde{Q}_n^{(L,I)} \) is the level \( (L, I) \) truncated approximation to \( \tilde{Q}_n \). The inequality (2.11) means that \( \tilde{Q} \)-closed states which are also \( \tilde{Q} \)-exact in the full string field theory (2.6) will be approximated in the level truncated theory by \( \tilde{Q} \)-closed states which are not precisely \( \tilde{Q} \)-exact. This fact makes the identification of physical states in the theory only possible in an approximate sense.

We have systematically computed \( \tilde{Q} \)-closed states in the level-truncated theory by finding values of \( M^2 = -p^2 \) where

\[ \text{det} \tilde{Q}_1^{(L,I)} = 0 \]

and then computing the eigenvectors associated with the vanishing eigenvalues.

As an example of this computation, consider the level \( (0, 0) \) truncation of the theory, which includes only the tachyon field \( \phi \). The quadratic term for the tachyon field in the nontrivial vacuum is

\[ \phi(-p) \left[ \frac{p^2 - 1}{2} + g\kappa \left( \frac{16}{27} \right)^2 \cdot 3\langle \phi \rangle \right] \phi(p). \]

The determinant of \( \tilde{Q}_1^{(0,0)} \) is simply the quantity in square brackets. This quantity does not
vanish for any real value of $p^2$, so there are no $\tilde{Q}$-closed states in the spectrum at this level [17].

In the level $(2, 6)$ truncation there are seven scalar fields to be considered, associated with the Fock space states

\[
|0; p\rangle, \quad (\alpha_{-1} \cdot \alpha_{-1}) |0; p\rangle, \\
(\alpha_0 \cdot \alpha_{-2}) |0; p\rangle, \quad (\alpha_0 \cdot \alpha_{-1})^2 |0; p\rangle, \\
b_{-1}c_{-1}|0; p\rangle, \quad (\alpha_0 \cdot \alpha_{-1}) b_{-1}c_0|0; p\rangle, \\
b_{-2}c_{-1}|0; p\rangle
\] (2.14)

At this level of truncation, using the vacuum expectation values determined in [19] with the level $(10, 20)$ truncation, we found five values of $p^2$ where $\det \tilde{Q}^{(2,6)} = 0$, associated with states having

\[
M^2 = 0.9067, \quad 2.0032, \quad 12.8566, \quad 13.5478, \quad 16.5998 \tag{2.15}
\]

in units where $M^2 = -1$ for the tachyon. At level $(4, 12)$ we found 18 $\tilde{Q}$-closed states with $M^2 < 20$, of which the lightest has $M^2 = 0.58817$. At level $(6, 12)$ we found 33 $\tilde{Q}$-closed states with $M^2 < 20$, of which the lightest has $M^2 = 0.85562$. The complete set of $\tilde{Q}$-closed states we found\(^3\) is graphed in Figure 2-1.

To test the $\tilde{Q}$-exactness of a given $\tilde{Q}$-closed state at level $(L, I)$, we computed $\tilde{Q}_0^{(L,I)} \Lambda_i$ for each ghost number zero field $\Lambda_i$ with level $\leq L$. The span of the fields $\tilde{Q}_0^{(L,I)} \Lambda_i$ gives an approximation to the subspace of $\tilde{Q}$-exact states at each level. Suppose that $\{e_i\}$ is an orthonormal basis for this subspace and $s$ is one of the $\tilde{Q}$-closed states we found. We can then measure the extent to which a state is $\tilde{Q}$-exact by the norm squared of its projection.

---

\(^2\)Note: a similar calculation was done at level $(2, 6)$ in Feynman-Siegel gauge by Kostelecky and Samuel [17]. They did not check, however, whether their spectrum was associated with physical or exact states. Our calculation can be restricted to this gauge, where it agrees for the most part (but not exactly) with their calculation.

\(^3\)Our algorithm for locating momenta associated with $\tilde{Q}$-closed states proceeded by calculating the determinant of $\tilde{Q}_1^{(L,I)}$ at equally spaced values of $p$ (with $\Delta p = 0.0001$) and looking for changes of sign in the determinant. The spacing of our $p$ values was significantly less than the smallest distance we observed between $\tilde{Q}$-closed states (0.0039), so we believe that we have found all the $\tilde{Q}$-closed states at $M^2 < 20$. Some possibility remains that we have missed pairs of $\tilde{Q}$-closed states which are very close in momentum. It is remotely possible that physical states are hiding in such closely spaced pairs of $\tilde{Q}$-closed states.
Figure 2-1: Spectrum of $\tilde{Q}$-closed states in level truncations $(0, 0), (2, 6), (4, 12)$ and $(6, 12)$. States below the cutoff $M^2 = L - 1$ lie mostly in the exact subspace, confirming Sen's conjecture.
onto the \(\tilde{Q}\)-exact subspace. Explicitly,
\[
\text{fraction in exact subspace} = \frac{\sum_i (s \cdot e_i)^2}{s \cdot s}.
\] (2.16)

There is no natural positive definite inner product defined on the single string Hilbert space \(\mathcal{H}\), so to compute (2.16) we had to make an ad hoc choice of such an inner product. We did the calculation using two choices for this inner product, and found similar results in both cases. The first choice, which seems most natural, is to take the inner product \(\langle s|s\rangle\) with \(p \rightarrow |p|\) in the matter sector and a Kronecker delta function in the ghost sector. The second inner product we tried was simply defined by a Kronecker delta function on a basis of states spanned by all possible scalar products of matter and ghost operators (such as (2.14) at level (2, 6), giving a unit normalization to each of these states). Using these two definitions of the inner product, we find for example that the \(\tilde{Q}\)-closed state at \(M^2 = 0.9067\) found in the (2, 6) truncation lies 97.90\% in the exact subspace using the first inner product, and 95.24\% in the exact subspace using the second inner product. In the remainder of this chapter all calculations use the first definition of the inner product.

In the full string field theory, there are continuous families of \(\tilde{Q}\)-closed states which are also \(\tilde{Q}\)-exact at all \(p\), given by states of the form \(\tilde{Q}_0|s;p\rangle\). In the level truncation approximation we expect these continuous families to be replaced by a discrete spectrum of almost-exact states, approaching a continuous distribution as the level of truncation is increased. The extent to which we see a continuous distribution of \(\tilde{Q}\)-exact states arising in the level-truncation approximation to the theory around the vacuum \(\Phi_0\) is a measure of how well level truncation works in the new vacuum, and how close the level-truncation approximation comes to giving a BRST operator satisfying \(\tilde{Q}_1\tilde{Q}_0 = 0\). A complete list of \(\tilde{Q}\)-closed states at \(M^2 < 20\) and the exactness of these states is given in Table 2.1; qualitative results for the exactness of all \(\tilde{Q}\)-closed states are depicted in Figure 2-1. As we would hope, as the level of truncation is increased we see a discrete distribution of almost-exact states which become both more exact and more closely spaced as the level of truncation is lifted. We interpret these almost-exact states as the remnant in the level-truncated theory of the continuous families of \(\tilde{Q}\)-exact states in the full theory.

Physical states in the theory correspond to \(\tilde{Q}\)-closed states satisfying \(\tilde{Q}_1|s;p\rangle = 0\) which
are not $\hat{Q}$-exact. Because states in the cohomology of $\hat{Q}$ will not be removed by a generic small perturbation, all physical states in the theory should appear in the level truncation as $\hat{Q}$-closed states with $-p^2$ approaching some fixed value $M^2$ as the level of truncation is increased. To verify Sen's conjecture, we would hope to find that the states lying below the cutoff $M^2 = L - 1$ are all approximately $\hat{Q}$-exact, to the precision allowed by the level-truncation approximation, so that no physical states appear in the limiting theory. Indeed, we find that beyond the level (2, 6) truncation all states below the cutoff lie more than 99% in the exact subspace, using either choice of inner product described above. For example, the lowest lying states mentioned above in the level truncations (2, 6), (4, 12) and (6, 12) lie 97.90%, 99.900% and 99.997% in the exact subspace. We would expect physical states in the theory to appear consistently in each level truncation as states with significant components outside the exact subspace, since the average state in the level-truncated space lies less than 35% in the exact space. We see no sign in our data of such physical open string states at low levels, even up to values of $M^2$ several times larger than the cutoff. We interpret this result as strong evidence for Sen's conjecture that there are no physical open string excitations around the vacuum $\Phi_0$, and that this string field configuration should be identified with the closed string vacuum.

It may seem surprising that we expect to see the physical states in the cohomology of $\hat{Q}$, which form a set of measure zero in the full space of $\hat{Q}$-closed states, through this approach. The difference between the behavior of physical and exact states under level truncation of $\hat{Q}$ can be understood by considering the behavior of the zeros of the functions $f(x) = x - 1$ and $g(x) = 0$ under a small perturbation by a noise function $\eta(x)$. In the first case, generically $f(x) + \eta(x)$ will have a single zero near $x = 1$. In the second case, $g(x) + \eta(x)$ will develop a discrete spectrum of randomly spaced zeros. The physical states in the cohomology of $\hat{Q}$ are controlled by functions like $f(x)$, while the continuous families of exact states are controlled by functions like $g(x)$. While this argument suggests that physical states should indeed continue to be present in the level-truncation approximation, as a check on our methodology we have used the same method we used to compute the approximate cohomology of $\hat{Q}$ to compute the approximate cohomology of the BRST operator $Q$ in the perturbative vacuum, after adding a small random perturbation $\hat{Q} = Q + \eta$. We find that unless the perturbation $\eta$ is large enough to dominate the system (e.g., by pushing the exactness of a generic $\hat{Q}$-closed
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Table 2.1: Masses and exactness of all $\tilde{Q}$-closed states found in level truncations (2, 6), (4, 12), and (6, 12) with $M^2 < 20$.

state below 90%), the physical states at $M^2 = -1$ and $M^2 = 3$ are easily distinguishable in a level (4, 12) truncation of the theory.

### 2.4 Summary

We have explicitly calculated the quadratic terms in the open string field theory action around the nonperturbative vacuum $\Phi_0$ in the (6, 12) level truncation. We computed the BRST cohomology by computing all closed states under the truncated BRST operator $\tilde{Q}_1^{(L,I)}$, and comparing with the subspace of exact states formed by the operator $\tilde{Q}_0^{(L,I)}$. We found evidence that all $\tilde{Q}$-closed states in the theory become $\tilde{Q}$-exact in the limit when fields of all levels are included.
There are several directions in which it would be interesting to proceed, given the results in this letter. For one thing, it would be very nice to have a better conceptual understanding of the decoupling of open string states in the perturbatively stable vacuum. While some interesting perspectives on this phenomenon have been given [50, 51, 52, 53, 54, 55, 56, 57], a convincing picture which explains the classical decoupling of open strings in the cubic string field theory picture has yet to be given.
Chapter 3

Spectrum of open strings around the tachyon vacuum II

In this chapter, following our work with B. Feng, Y. He, and N. Moeller [25], we take another approach to the problem of showing that there are no physical fluctuations around the tachyon vacuum in open string field theory. We do this by constructing an operator $A$ which acts as an inverse of the BRST operator when acting on states which are BRST closed.

3.1 Introduction

In the previous chapter we examined the spectrum of open strings around the tachyon vacuum in a rather direct manner. In this chapter we take a more abstract approach in an attempt to make a more general claim about the cohomology of the BRST operator.

Recall that as it is most simply stated, Sen’s second conjecture says just that there are no open string excitations around the tachyon vacuum. However, as proposed in [58, 59] there is a slightly stronger version for the second conjecture: Not only are there no physical excitations around the tachyon vacuum, but the complete cohomology of $\tilde{Q}$ is empty at every ghostnumber. Here we try to show this more general claim.

Our discussion relies heavily upon the existence of a string field $\mathcal{I}$ of ghost number 0
which is the identity of the $\ast$-product. It satisfies

$$I \ast \psi = \psi \ast I = \psi$$

for any state\(^1\) \(\psi\). The state \(I\) was first constructed in the oscillator basis in [45]. More recently, the work [28] gave a recursive way of constructing the identity in the (background independent) total-Virasoro basis which shows its universal property in string field theory. As a by-product of our analysis, we have found a new and elegant analytic expression for \(I\) without recourse to the complicated recursions.

Ignoring anomalies, the fact that \(\tilde{Q}\) is a derivation of the $\ast$-algebra implies that \(I\) is \(\tilde{Q}\) closed and the problem is to determine whether it is also \(\tilde{Q}\)-exact, i.e., if there exists a ghost number \(-1\) field \(A\), such that \(I = \tilde{Q}A\). If so, then for an arbitrary \(\tilde{Q}\)-closed state \(\Phi\) we would have

$$\tilde{Q}(A \ast \Phi) = (\tilde{Q}A) \ast \Phi - A \ast (\tilde{Q}\Phi)$$

$$= I \ast \Phi$$

$$= \Phi,$$

(3.1)

where in the second step, we used the fact that \(\Phi\) is \(\tilde{Q}\)-closed, and in the last step, that \(I\) acts as the identity on \(\Phi\). This means that any \(\tilde{Q}\)-closed field \(\Phi\) is also \(\tilde{Q}\)-exact, in other words, the entire cohomology of \(\tilde{Q}\) is trivial.

Therefore we have translated the problem of the triviality of the cohomology of \(\tilde{Q}\) into the issue of the exactness of the identity \(I\). In this paper, we will use the level truncation method to show that the state \(A\) indeed exists for the tachyon vacuum \(\Phi_0\) up to an accuracy of 3.2%.

This chapter is structured as follows. In Section 2, we explain the above idea of the exactness of \(I\) in detail. In Section 3, we use two different methods to find the state \(A\): one without gauge fixing and the other, in the Feynman-Siegel gauge. They give the results

\(^1\)There are some mysteries regarding of the identity. For example, in [28] the authors showed that this identity string field is subject to anomalies, with consequences that \(I\) may be the identity of the $\ast$-algebra only on a subspace of the whole Hilbert space. In the following, we will first assume that \(I\) behaves well on the whole Hilbert space, and postpone some discussions thereupon to Section 4.
up to an accuracy of 2.4% and 3.2% respectively. In Section 4, we discuss the behaviour of \( \mathcal{I} \) under level truncation and perform a few consistency checks on our approximations. Finally, in Section 5 we make some concluding remarks and address some further problems and directions.

A few words on nomenclature before we proceed. By \( |0\rangle \) we mean the \( SL(2, \mathbb{R}) \)-invariant vacuum and \( |\Omega\rangle := c_1 |0\rangle \). We consider \( |\Omega\rangle \) to be level 0 and hence \( |0\rangle \) is level 1. Furthermore, in this paper we expand our fields in the universal basis (matter Virasoro and ghost oscillator modes).

## 3.2 The Proposal

We begin with a simple proposition: Given a \( Q \) which has vanishing cohomology we can always construct an \( A \) such that \( \{A, Q\} = I \). Suppose that we denote the string Hilbert space at ghost level \( g \) by \( V_g \). Define the subspace \( V_g^C \) as the set of all closed elements of \( V_g \).

We can then pick a complement, \( V_g^N \), to this subspace satisfying \( V_g = V_g^C \oplus V_g^N \). Note that it consist of vectors which are not killed by \( Q \). This subspace \( V_g^N \), is not gauge invariant but any specific choice will do. The important point is that because we have assumed that \( Q \) has no cohomology, the restriction of \( Q \) to \( V_g^N \) given by

\[
Q\big|_{V_g^N} : V_g^N \to V_{g+1}^C,
\]

has no kernel and is surjective on \( V_{g+1}^C \). Thus it has an inverse which we denote \( A \)

\[
A\big|_{V_g^C} \equiv Q^{-1}\big|_{V_g^C} : V_{g+1}^C \to V_g^N.
\]

This insures that on the space \( V_g^C \), \( \{A, Q\} = I \) holds since if \( \Phi \) is \( Q \)-closed,

\[
\{A, Q\} \Phi = AQ\Phi + QA\Phi = QQ^{-1}\Phi = \Phi.
\]

The above discussion only defines the action of \( A \) on \( V_g^C \), what remains is to define its action on the complement \( V_g^N \). Here there is quite a bit of freedom since one can choose any
map that takes $V^N_g$ into $V^C_{g-1}$. Assuming this, we have that for $\Phi \in V^N_g$,

$$\{A, Q\} \Phi = A Q \Phi + Q A \Phi = Q^{-1} Q \Phi + Q^2 \chi = \Phi,$$

where by assumption $A \Phi$ is $Q$-closed (because it is in $V^C_{g-1}$) and thus equals $Q \chi$ for some $\chi \in V^N_{g-2}$. In general one can insist that $A$ satisfies more properties. For example if we set $A|_{V^N_g} = 0$ we get that $A^2 = 0$. We summarize the above discussion as

**PROPOSITION 3.2.1** The cohomology of $Q$ is trivial iff there exists an operator $A$ such that $\{A, Q\} = I$.

The basic hypothesis of this paper is that not only does such an operator $A$ exist for $\tilde{Q}$, but that, the action of $A$ can be expressed as the left multiplication by the ghost number $-1$ string field which we denote as $A \ast$. Thus we are now interested in satisfying the equation $\{A \ast, \tilde{Q}\} = I$. Writing this out explicitly we have

$$\{A \ast, \tilde{Q}\} \Phi = A \ast (\tilde{Q} \Phi) + \tilde{Q} (A \ast \Phi)$$

$$= A \ast \tilde{Q} (\Phi) + (\tilde{Q} A) \ast \Phi - A \ast (\tilde{Q} \Phi)$$

$$= (\tilde{Q} A) \ast \Phi.$$

In order for the last line to equal $\Phi$ for all $\Phi$ we need that

$$\tilde{Q} A = I,$$  \hspace{1cm} \text{(3.2)}

where $I$ is the identity of the $\ast$-algebra.

For the case of interest, we wish to study the physics around the minimum of the tachyon potential. We recall that for a state $\Psi$, the new BRST operator around the solution $\Phi$ of the EOM is given by

$$Q_\Phi \Psi = Q_B(\Psi) + \Phi \ast (\Psi) - (-)^\Psi(\Phi) \ast \Psi.$$  \hspace{1cm} \text{(3.3)}

Using this expression for the BRST operator we can rewrite the basic equation (3.2) as $Q_\Phi A = Q_B(A) + \Phi \ast (A) + (A) \ast \Phi = I$. For general vacua $\Phi$, such a string field $A$ will
not exist. For example in the perturbative vacuum, $\Phi = 0$, $Q_{\Phi}$ is simply $Q_B$. It is easy to show here that there is no solution for $A$ because the $Q_B$ action preserves levels while $\mathcal{I}$ has a component at level one (namely $|0\rangle$), but the minimum level of a ghost number $-1$ state $A$ is 3. Indeed, for a more general solution $\Phi \neq 0$ (such as the tachyon vacuum), the star product will not preserve the level and so it may be possible to find $A$. Our endeavor will be to use the level truncation scheme to find $A$ for the tachyon vacuum $\Phi_0$, i.e., to find a solution $A$ to the equation

$$\tilde{Q}A = \mathcal{I}. \quad (3.4)$$

Note that this equation is invariant under

$$A \rightarrow A + \tilde{Q}B$$

for some $B$ of ghost number $-2$, thereby giving $A$ a gauge freedom. This is an important property to which we shall turn in the next section.

Having expounded upon the properties of $A$, our next task is clear. In the following section, we show that for the tachyon vacuum $\Phi_0$, we can find the state $A$ satisfying (3.4) in the approximation of the level truncation scheme.

### 3.3 Finding The State $A$

We now solve (3.4) by level truncation. We proceed in two ways. Recall from the previous section that $A$ is well-defined up to the gauge transformation $A \rightarrow A + \tilde{Q}B$ where $B$ is a state of ghost number $-2$. Because in the level truncation scheme, this gauge invariance is broken, we first try to find the best fit results without fixing the gauge of $A$. The fitting procedure is analogous to that used in [60] and we shall not delve too much into the details. We shall see below that at level 9, the result is accurate to 2.4%. However, when we check the behavior of the numerical coefficients of $A$ as we increase the accuracy from level 3 to 9, we found that they do not seem to converge. We shall explain this phenomenon as the consequence of the gauge freedom in the definition of $A$; we shall then redo the fitting in the Feynman-Siegel gauge. With this second method, we shall find that the coefficients do converge and the best fit at level 9 is to 3.2% accuracy. These results support strongly the
existence of a state \( A \) in (3.4) and hence the statement that the cohomology around the tachyon vacuum is indeed trivial. In the following subsections let us present our methods and results in detail.

### 3.3.1 The Fitting without Gauge Fixing \( A \)

To solve the condition (3.4), we first need an explicit expression of the identity \( I \). Such an expression has been presented in [45] and [28], differing by a mere normalization factor \(-4i\). In this paper, we will follow the conventions of [28] which has\(^2\)

\[
|I\rangle = e^{L_{-2} - \frac{1}{2} L_{-4} + \frac{1}{4} L_{-6} - \frac{1}{12} L_{-8} + \frac{1}{3} L_{-10} + \ldots} |0\rangle
\]

\[
= |0\rangle + L_{-2} |0\rangle + \frac{1}{2} (L_{-2}^2 - L_{-4}) |0\rangle + \left( \frac{1}{6} L_{-2}^3 - \frac{1}{4} L_{-2} L_{-4} - \frac{1}{4} L_{-4} L_{-2} + \frac{1}{2} L_{-6} \right) |0\rangle + \left( \frac{1}{24} L_{-2}^4 + \frac{1}{4} (L_{-2} L_{-6} + L_{-6} L_{-2}) + \frac{1}{8} L_{-4}^2 - \frac{7}{12} L_{-8} \right) - \frac{1}{12} (L_{-2}^2 L_{-4} + L_{-2} L_{-4} L_{-2} + L_{-4} L_{-2}^2) |0\rangle
\]

where \( L_n = L_n^m + L_n^g \), the sum of the ghost (\( L_n^g \)) and matter (\( L_n^m \)) parts, is the total Virasoro operator. For later usage we have expanded the exponential up to level 9. Furthermore, we split \( L_n \) into matter and ghost parts and expand the latter into \( b_n, c_n \) operators as

\[
L_n^g := \sum_{n=-\infty}^{\infty} (2m-n) : b_n c_{m-n} : - \delta_{m,0}. \]

In other words, we write the states in the so-called “Universal Basis” [28].

As a by-product, we have found an elegant expression for \( I \) which avoids the recursions\(^3\) needed to generate the coefficients in the exponent. In fact, one can show that only \( L_{-m} \) for \( m \) being a power of 2 survive in the final expression, thus significantly reducing the

\(^2\)With the normalization \( \langle c_1, c_1, c_1 \rangle = 3 \) that we are using, we should scale this expression by a factor of \( K^3/3 \), where \( K = 3\sqrt{3}/4 \). However, as the normalization of the identity will not change our analysis, we will use this right normalization only in Section 4, where we are dealing with expressions like \( I \ast \Phi \).

\(^3\)Indeed the expression given in [45] has no recursion either, however their oscillator expansion is not normal-ordered due to ghost insertions at the string mid-point.
complexity of the computation of level-truncation for $I$:

$$
|I| = \left( \prod_{n=2}^{\infty} \exp \left\{ -\frac{2}{2^n} L_{-2n} \right\} \right) e^{L_{-2}} |0\rangle \\
= \ldots \exp \left( -\frac{2}{2^3} L_{-2^3} \right) \exp \left( -\frac{2}{2^2} L_{-2^2} \right) \exp (L_{-2}) |0\rangle ,
$$

(3.7)

where we emphasize that the Virasoro’s of higher index stack to the left ad infinitum. We leave the proof of this fact to appendix C.

It is worth noticing that in the expansion of $I$ only odd levels have nonzero coefficients. This means that we can constrain the solution $A$ of (3.4), if it exists, to have only odd levels in its expansion. The reason for this is as follows. Equation (3.4) states that $Q_B A + \Phi_0 \ast A + A \ast \Phi_0 = I$, moreover we recall that (cf. e.g. Appendix A.4 of [60]) the coefficient $k_{\ell,t}$ in the expansion of the star product $x \ast y = \sum_{\ell,t} k_{\ell,t} \psi_{\ell,t}$ is $k_{\ell,t} = \langle \psi_{\ell,t}, x, y \rangle$ for the orthogonal basis $\psi$ to $\psi$. Now the triple correlator has the symmetry $\langle x, y, z \rangle = (-)^{1+g(x)g(y)+\ell(x)+\ell(y)+\ell(z)} \langle x, z, y \rangle$, where $g(x)$ and $\ell(x)$ are the ghost number and level of the field $x$ respectively. Thus, one can see that the even levels of $\Phi_0 \ast A + A \ast \Phi_0$ will be zero because the tachyon vacuum $\Phi_0$ has only even levels and $A$ is constrained to odd levels. Furthermore, $Q_B = \sum_n c_n L_{-n} + \frac{1}{2} (m-n) : c_n c_n b_{m-n} : -c_0$ preserves level. Therefore, in order that both the left and right hand sides of (3.4) have only odd levels, $A$ must also have only odd level fields.

Now the procedure is clear. We expand $A$ into odd levels of ghost number $-1$ with coefficients as parameters and calculate $\hat{Q}A$. Indeed as with [60], all the states will be written as Euclidean vectors whose basis is prescribed by the fields at a given level; the components of the vectors are thus the expansion coefficients in each level. Then we compare $\hat{Q}A$ with $I$ up to the same level and determine the coefficients of $A$ by minimizing the quantity

$$
\epsilon = \frac{|\hat{Q}A - I|}{|I|},
$$

which we of course wish to be as close to zero as possible. We refer to this as the “fitting of the coefficients”. The norm $|.|$ is the Euclidean norm (for our basis, see the Appendix). As discussed in chapter 2, different normalizations do not significantly change the values from the fitting procedure, so for simplicity we use the Euclidean norm to define the above
measure of proximity $\epsilon$. The minimum level of the ghost number $-1$ field $A$ is 3, so we
start our fitting from this level and continue to up to level 9 (higher levels will become
computationally prohibitive).

First we list the number of components of odd levels for the fields $A$ and $I$ up to given
levels:

<table>
<thead>
<tr>
<th></th>
<th>level 3</th>
<th>level 5</th>
<th>level 7</th>
<th>level 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Components of $A$</td>
<td>1</td>
<td>4</td>
<td>14</td>
<td>43</td>
</tr>
<tr>
<td>Number of Components of $I$</td>
<td>4</td>
<td>14</td>
<td>43</td>
<td>118</td>
</tr>
</tbody>
</table>

From this table, we see that at level 3, we have only one parameter to fit 4 components. At
level 5, we have 4 parameters to fit 14 components. As the level is increased the number of
components to be fitted increases faster that the number of free parameters. Therefore it is
not a trivial fitting.

**$A$ up to level 3**

At level 3 the identity is:

$$I_3 = |0\rangle + L_{-2} |0\rangle$$

$$= |0\rangle - b_{-3} c_1 |0\rangle - 2 b_{-2} c_0 |0\rangle + L_{-2}^m |0\rangle$$

and we find the best fit of $A$ (recall that at level 3 we have only 1 degree of freedom) to be

$$A_3 = 1.12237 b_{-2} |0\rangle ,$$

with an $\epsilon$ of 17.1%.

**$A$ up to level 5**

Continuing to level 5, we have

$$I_5 = |0\rangle + L_{-2} |0\rangle + \frac{1}{2} (L_{-2}^2 - L_{-4}) |0\rangle$$

$$= |0\rangle - b_{-3} c_1 |0\rangle - 2 b_{-2} c_0 |0\rangle + L_{-2}^m |0\rangle + b_{-5} c_1 |0\rangle - b_{-2} c_2 |0\rangle$$

$$+ b_{-3} c_{-1} |0\rangle + 2 b_{-3} b_{-2} c_0 c_1 |0\rangle + 2 b_{-4} c_0 |0\rangle - \frac{1}{2} L_{-4}^m |0\rangle$$

$$- b_{-3} c_1 L_{-2}^m |0\rangle - 2 b_{-2} c_0 L_{-2}^m |0\rangle + \frac{1}{2} L_{-2}^m L_{-2}^m |0\rangle .$$
To this level we have determined the best-fit $A$ to be

$$A_5 = 1.01893 \ b_{-2} | 0 \rangle + 0.50921 \ b_{-3} b_{-2} c_1 | 0 \rangle - 0.518516 \ b_{-4} | 0 \rangle + 0.504193 \ b_{-2} L_{-2}^m | 0 \rangle,$$

with an $\epsilon$ of 11.8%.

The detailed data of the field $A$ to levels 7 and 9 are given in table B.1 of the Appendix. Here we just summarize the results of the best-fit measure $\epsilon$:

| $\epsilon = \frac{|\tilde{Q} A - I|}{|I|}$ | level 3 | level 5 | level 7 | level 9 |
|-------------------------------------|---------|---------|---------|---------|
| 0.171484                           | 0.117676 | 0.0453748 | 0.0243515 |

This indicates that up to an accuracy of 2.4% at level 9, there exists an $A$ that satisfies (3.4); moreover the accuracy clearly gets better with increasing levels.

3.3.2 The Stability of Fitting

There is, however, a problem with our fit. Looking carefully at the coefficients of $A$ given in the table B.1, especially the fitting coefficients between levels 7 and 9, we see that these two groups of data have a large difference. Naively it means that our solution for $A$ does not converge as we increase level. How do we solve this puzzle?

We recall that $A$ is well-defined only up to the gauge freedom

$$A \rightarrow A + \tilde{Q}B.$$

It means that the solutions of (3.4) should consist of a family of gauge equivalent $A$. However, because $\tilde{Q}^2 \neq 0$ under the level truncation approximation, the family (or the moduli space) is broken into isolated pieces. This is similar to what we found in chapter 2, where the momentum-dependent closed states were given by points instead of a continuous family. Using this fact, our explanation is that the fitting of levels 7 and 9 are related by $\tilde{Q}B$ for some field $B$ of ghost number $-2$. To show this, we solve a new $\tilde{A}$ up to level 9 that minimizes

$$\frac{|(\tilde{A})_7 - A_7|}{|A_7|} + \frac{|\tilde{Q}\tilde{A} - I_9|}{|I_9|}$$

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where $A_7$ is the known fitting data at level seven, $I_9$ is the identity up to level nine and $(\tilde{A})_7$ refers to the first 14 components (i.e., the components up to level seven) of the level 9 expansion of $\tilde{A}$. By minimizing this above quantity, we balance the stability of fitting from level 7 to 9. The data is given in the last column of B.1. Though having gained stability, the fitting for level 9 is a little worse, with $\epsilon$ increasing from 2.44\% to 3.56\%.

The next thing is to check whether $\tilde{A} - A_9$ is an exact state $\check{Q}B$. We find that this is indeed true and we find a state $B$ such that

$$\frac{|(\tilde{A} - A_9) - \check{Q}B|}{|\tilde{A} - A_9|} = 0.28\%.$$

### 3.3.3 Fitting $A$ in the Feynman-Siegel Gauge

Alternatively, by gauge-fixing, we can also avoid the instability of the fit. If we require the state $A$ to be in the Feynman-Siegel gauge, $A$ will not have the gauge freedom anymore and the fitting result should converge as we do not have isolated points in the gauge moduli space to jump to. We have done so and do find much greater stability of the coefficients.

Notice that in the Feynman-Siegel gauge, $A$ has the same field bases in levels 3 and 5, so the fitting at these two levels is the same as in Subsections 3.1.1 and 3.1.2. However, in this gauge it has one parameter less at level 7 and 5 less in level 9. Performing the fit with these parameters we have reached an accuracy of $\epsilon = 4.8\%$ at level 7 and $\epsilon = 3.2\%$ at level 9, which is still a good result. The details are presented in Table B.2 in the Appendix.

### 3.4 Some Subtleties of the Identity

As pointed out in the Introduction, there are some mysterious and anomalous features of the identity $I$. For example, $I$ is not a normalizable state [61], moreover, $c_0$, contrary to expectation, does not annihilate $I$ even though it is a derivation [28]. We shall show in the following that with a slight modification of the level truncation scheme, this unnormalizability does not effect the results and furthermore that in our approximation $\check{Q}I$ indeed vanishes as it must for consistency.

Let us first show how problems may arise in a naive attempt at level truncation. Consider
the quantity $\mathcal{I}_\ell \star |\Omega\rangle - |\Omega\rangle$, where $\mathcal{I}_\ell$ denotes the identity truncated to level $\ell$ and $|\Omega\rangle := c_1 |0\rangle$. We of course expect this to approach 0 as we increase $\ell$. Using the methods of the previous section, we shall define the measure of proximity

$$\eta = \frac{|\mathcal{I}_\ell \star |\Omega\rangle - |\Omega\rangle|}{|\Omega\rangle} = |\mathcal{I}_\ell \star |\Omega\rangle - |\Omega\rangle|,$$

where $|\cdot|$ is our usual norm. We list $\eta$ to levels 3, 5, 7, and 9 in the following Table:

<table>
<thead>
<tr>
<th>level $\ell$</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta =</td>
<td>\mathcal{I}_\ell \star</td>
<td>\Omega\rangle -</td>
<td>\Omega\rangle</td>
<td></td>
</tr>
</tbody>
</table>

Our $\eta$ obviously does not converge to zero, hence star products involving $\mathcal{I}$ do not converge in the usual sense of level truncation. It is however not yet necessary to despair, as weak convergence will come to our rescue\(^4\).

Indeed, instead of truncating the result to level $\ell$, let us use a slightly different scheme. We truncate $\mathcal{I}_\ell \star |\Omega\rangle$ to a fixed level $m < \ell$ and observe how the coefficients of the fields up to level $m$ converge as we increase $\ell$. In the following table we list the values of the coefficients $\text{coeff}(x)$ of the basis for $m = 2$ (i.e., fields $x$ of level 0, 1 and 2) for the expression $\mathcal{I}_\ell \star |\Omega\rangle$.

| $\mathcal{I}_\ell \star |\Omega\rangle$ | coeff($|\Omega\rangle$) | coeff($b_{-1}c_0 |\Omega\rangle$) | coeff($b_{-1}c_{-1} |\Omega\rangle$) | coeff($b_{-2}c_0 |\Omega\rangle$) | coeff($L_{-2}^m |\Omega\rangle$) |
|-------------|------------------|------------------|------------------|------------------|------------------|
| $\ell = 3$  | 0.6875           | 0.505181         | -0.905093        | -0.930556        | 0.465278         |
| $\ell = 5$  | 1.16898          | -0.278874        | 0.38846          | 0.520748         | -0.260374        |
| $\ell = 7$  | 0.911094         | 0.16252          | -0.197833        | -0.296607        | 0.148304         |
| $\ell = 9$  | 1.05767          | -0.0971502       | 0.0902728        | 0.163579         | -0.0817895       |

We see that the $|\Omega\rangle$ component converges to 1 while the others converge to 0, as was hoped. We note however that this (oscillating) convergence is rather slow and we thus expect slow weak convergence for other calculations involving the identity.

Having shown that as $\ell \to \infty$ we get a weak convergence $\mathcal{I}_\ell \star |\Omega\rangle \to |\Omega\rangle$, we now consider $\tilde{Q}\mathcal{I}_\ell$ as $\ell \to \infty$, which should tend to zero. Since $Q_B$ preserves level and $Q_B \mathcal{I} = 0$, we have that $Q_B \mathcal{I} = 0$ in the level expansion; thus $\tilde{Q}\mathcal{I} = \Phi_0 \star \mathcal{I} - \mathcal{I} \star \Phi_0$, which should converge to

\(^4\)We thank B. Zwiebach for this suggestion.
zero.

Figure 3-1: A plot of $q_{0,1}(\ell)$ (solid curve), $q_{2,1}(\ell)$ (dotted curve) and $q_{2,3}(\ell)$ (dashed curve) as functions of the level $\ell$ of the identity. $\ell$ goes from 3 to 17.

As the expression $\tilde{Q}\mathcal{I}$ is linear in every component of $\Phi_0$, that $\mathcal{I}$ is $\tilde{Q}$-closed will be established if we can show that for each component $\phi$ in $\Phi_0$, $\phi \star \mathcal{I} - \mathcal{I} \star \phi \equiv [\phi \star, \mathcal{I}]$ converges to zero as the level of $\mathcal{I}$ is increased. We plot in Fig.3-1, the absolute values of the coefficient of $c_0 |0\rangle$ in the expressions $[(c_1 |0\rangle)\star, \mathcal{I}_\ell]$, $[(c_{-1} |0\rangle)\star, \mathcal{I}_\ell]$ and $[(L_{-2}^m c_1) |0\rangle \star, \mathcal{I}_\ell]$, which we denote by $q_{0,1}(\ell)$, $q_{2,1}(\ell)$ and $q_{2,3}(\ell)$ respectively. It seems clear that the coefficients do converge to zero.

The weak convergence we have shown above can be interpreted in a more abstract setting. Let us examine the quantity $|\mathcal{I}_\ell \star \Phi - \Phi|$. It was shown in [62] that the $\star$-algebra of the open bosonic string field theory is a $C^*$-algebra. A well-known theorem dictates that any $C^*$-algebra $M$ (with or without unit) has a so-called approximate identity which is a set of operators $\{I_i\}$ in $M$ indexed by $i$ satisfying (i) $\|I_i\| \leq 1$ for every $i$ and (ii) $\|I_i x - x\| \to 0$ and $\|x I_i - x\| \to 0$ for all $x \in M$ with respect to the (Banach) norm $\|\cdot\|$ of $M$ (cf. e.g. [63]).

The level $\ell$ in our level truncation scheme is suggestive of an index for $\mathcal{I}$. Furthermore the weak convergence we have found in this section is analogous to property (ii) of the theorem (being of course a little cavalier about the distinction of the Banach norm of the $C^*$-algebra with the Euclidean norm used here). Barring this subtlety, it is highly suggestive that our $\mathcal{I}_\ell$ is an approximate identity of the $\star$-algebra indexed by level $\ell$. 

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3.5 Summary

According to a strong version of Sen's Second Conjecture, there should be an absence of any open string states around the perturbatively stable tachyon vacuum $\Phi_0$. This disappearance of all states, not merely the physical ones of ghost number 1, means that the cohomology of the new BRST operator $\tilde{Q}$ should be completely trivial near the vacuum. It is the key observation of this paper that this statement of triviality is implied by the existence of a ghost number $-1$ field $A$ satisfying

$$\tilde{Q}A = Q_A + \Phi_0 A + A \Phi_0 = \mathcal{I}.$$ 

That is to say that if the identity of the $\star$-algebra $\mathcal{I}$ is a $\tilde{Q}$ exact state, then the cohomology of $\tilde{Q}$ would be trivial.

The level truncation scheme was subsequently applied to check our proposal. We have found that such a state $A$ exists up to an accuracy of 3.2% at level 9. Although these numerical results give a strong support to the proposal for the existence of $A$ and hence the triviality of $\tilde{Q}$-cohomology near the vacuum, an analytic expression for $A$ would be most welcome. However, to obtain such an analytic form of $A$, it seems that we would require the analytic expression for the vacuum $\Phi_0$, bringing us back to an old problem.

Finally, an interesting question is about the identity $\mathcal{I}$. In this paper we have given an elegant analytic expression for $\mathcal{I}$ which avoids the usage of complicated recursion relations. Furthermore, we have suggested that though the $\star$-algebra of OSFT may be a non-unital $C^*$-algebra, $\mathcal{I}$ still may serve as a so-called approximate identity. However, as we discussed before, anomalies related to the identity in the String Field Theory make the calculation in level truncation converge very slowly. It will be useful to understand more about $\mathcal{I}$. 

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Chapter 4

Gauge fixing in string field theory

In this chapter we discuss our work with W. Taylor [26] on the gauge fixing procedure in OSFT. In particular, we find that Feynman-Siegel gauge, the gauge used in almost all OSFT calculations, is not a good gauge globally. As a consequence of this, we show that singularities in the tachyon effective potential which appear in Feynman-Siegel gauge are gauge artifacts associated with the boundary of the region in field space where Feynman-Siegel gauge is valid. We also briefly discuss the problem of solving the tachyon condensation equations in gauges other than Feynman-Siegel and without any gauge fixing.

4.1 Gauge Symmetry in OSFT

We now describe the gauge symmetry of the cubic string field theory. Recall that the OSFT action, given by

\[ S = -\frac{1}{2} \int \Phi \star Q \Phi - \frac{g}{3} \int \Phi \star \Phi \star \Phi \]  \hspace{1cm} (4.1)

has the gauge invariance

\[ \delta \Phi = Q \Lambda + g (\Phi \star \Lambda - \Lambda \star \Phi) \] \hspace{1cm} (4.2)

The standard procedure for fixing the gauge is take Feynman-Siegel (FS) gauge which restricts the space of fields to those satisfying

\[ b_0 \Phi = 0. \]
We now want to study whether or not this gauge choice is good globally. For simplicity, we will restrict attention to scalar fields at zero momentum, which are all that is relevant for calculations of a Lorentz-invariant vacuum. The zero-momentum scalar string field can be expanded as

$$\Phi = \sum_i \phi^i |s_i^{(1)}\rangle$$

(4.3)

in terms of ghost number one scalar states in the Fock space. The action for the component fields $\phi^i$ is a simple cubic polynomial

$$S = \sum_{i,j} d_{ij} \phi^i \phi^j + g\kappa \sum_{i,j,k} t_{ijk} \phi^i \phi^j \phi^k$$

(4.4)

with constant coefficients $d_{ij}, t_{ijk}$, where the constant $\kappa$ has been chosen so that $t_{111} = 1$. The scalar gauge parameters $\mu^a$ are the components of a ghost number zero string field

$$\Lambda = \sum_a \mu^a |s_0^{(0)}\rangle.$$  

(4.5)

The variation of the fields $\phi^i$ under gauge transformations generated by the $\mu^a$ can be written as

$$\delta \phi^i = D_{ia} \mu^a + g\kappa T_{ija} \phi^j \mu^a$$

(4.6)

with constant coefficients $D_{ia}, T_{ija}$.

We have computed all the coefficients in (4.4) up to level (10, 20), and the coefficients in (4.6) up to level (8, 16). We have checked that the action is invariant under the gauge transformations up to order $g^1$. At order $g^2$ the gauge invariance is broken by level truncation, but we have checked that the gauge invariance is approximately satisfied at this order and improves as the level of truncation is increased. Figure 4.1 lists the numbers of fields and gauge parameters at each level.

As a simple example of the level-truncated gauge transformations, let us consider the level (2, 6) truncation. At this level, the scalar string field has an expansion

$$\Phi = \phi|\bar{0}\rangle + B(\alpha_{-1} \cdot \alpha_{-1})|\bar{0}\rangle + 3b_{-1}c_{-1}|\bar{0}\rangle + \eta b_{-2} c_0 |\bar{0}\rangle$$

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\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Level & Total fields & Gauge DOF \\
\hline
2 & 4 & 1 \\
4 & 15 & 5 \\
6 & 50 & 19 \\
8 & 152 & 61 \\
10 & 431 & 179 \\
\hline
\end{tabular}
\end{center}

Figure 4-1: The number of fields and gauge parameters at various levels.

The gauge parameter at level two is

$$\Lambda = \mu b_{-2} |\tilde{0}\rangle$$

(4.7)

and the gauge transformations are (in units where $S = -\phi^2/2 + g\kappa \phi^3 + \cdots$)

\begin{align*}
\delta \phi &= g\kappa \mu \left( -\frac{16}{9} \phi + \frac{2080}{243} B - \frac{464}{243} \beta + \frac{128}{81} \eta \right) \\
\delta B &= \frac{\mu}{2} + g\kappa \mu \left( \frac{40}{243} \phi - \frac{9296}{6561} B + \frac{1160}{6561} \beta - \frac{320}{2187} \eta \right) \\
\delta \beta &= -3\mu + g\kappa \mu \left( -\frac{176}{243} \phi + \frac{22880}{6561} B - \frac{11248}{6561} \beta - \frac{6016}{6561} \eta \right) \\
\delta \eta &= -\mu + g\kappa \mu \left( -\frac{224}{81} \phi + \frac{29120}{2187} B + \frac{992}{6561} \beta + \frac{1792}{729} \eta \right)
\end{align*}

(4.8)

At this level, the action (4.4) was computed in [28]. It can be checked that the action given there, which agrees with our results, is invariant under this set of gauge transformations.

### 4.2 Region of validity of Feynman-Siegel gauge

We can now ask the question of when Feynman-Siegel is a valid gauge choice. Near the origin $\phi^i = 0$, the gauge transformations (4.6) are constant vector fields

$$\delta \phi^i = D_{ia} \mu^a$$

(4.9)

Feynman-Siegel gauge sets to zero the fields $\phi^a = 0$ associated with scalar states in the Fock space annihilated by $c_0$. It is easy to see that this gauge choice is good near the origin, in
that all scalar gauge parameters $\mu^b$ associated with scalar states $|s_b\rangle$ annihilated by $b_0$ give rise to a variation $\delta\Phi$ consisting of a part proportional to $c_0|s_b\rangle$ plus a part annihilated by $b_0$. The statement that Feynman-Siegel gauge is valid locally is equivalent to the condition $\det D_{q^b} \neq 0$, where $q$ ranges over fields associated with ghost number one states annihilated by $c_0$, and $b$ ranges over fields associated with ghost number zero states annihilated by $b_0$. Note that we are using Feynman-Siegel gauge to fix the set of allowable gauge transformations as well as to determine which fields are set to vanish. From this point of view, a breakdown of Feynman-Siegel gauge can occur either because a valid infinitesimal gauge transformation produces a vector tangent to the space of fields in the kernel of $b_0$ or because the gauge transformation parameters annihilated by $b_0$ no longer span the space of effective gauge transformations.

At a general point in field space, the gauge transformations are given by

$$\delta\phi^i = M_{ia} \mu^a$$  \hspace{1cm} (4.10)

where

$$M_{ia} = D_{ia} + g\kappa T_{ija} \langle \phi^j \rangle.$$  \hspace{1cm} (4.11)

The Feynman-Siegel gauge choice breaks down when

$$\det M_{qb} = 0$$  \hspace{1cm} (4.12)

where $q, b$ are restricted as above to the spaces of states of ghost number 1/0 annihilated by $c_0/b_0$. Such a breakdown of gauge fixing is familiar from nonabelian gauge theories, where Gribov ambiguities are a well-studied phenomenon. In string field theory the locus of points where (4.12) is satisfied defines a codimension one boundary dividing field space into regions where Feynman-Siegel gauge is locally valid. The existence of such a boundary means that not only might several apparently distinct field configurations in Feynman-Siegel gauge actually be physically equivalent, but there may also be physical field configurations which have no representative in Feynman-Siegel gauge.

In the level-truncated theory, $M_{qb}$ is a finite size matrix. Therefore, we can solve (4.12) at a fixed level of truncation, and we can consider the stability of the locus of points where this
determinant vanishes as we increase the level of truncation. We have done this up to level (8, 16) and we find that, at least near the origin, the boundary of the region of Feynman-Siegel gauge validity determined by (4.12) seems to be nicely convergent under level truncation.

Before discussing the higher level truncations of (4.12), let us consider a simple illustrative example involving only two fields. A very simple truncation of the full string field theory can be defined by dropping all fields except the level 0 and level 2 fields $\phi$ and $\eta$ defined in (4.7). Restricted to these two fields, the gauge transformations are

\[
\delta \phi = g\kappa \mu \left( -\frac{16}{9} \phi + \frac{128}{81} \eta \right), \\
\delta \eta = -\mu + g\kappa \mu \left( -\frac{224}{81} \phi + \frac{1792}{729} \eta \right).
\]

Feynman-Siegel gauge sets $\eta = 0$. This gauge choice goes bad when

\[
\eta = \delta \eta = 0 \rightarrow \phi = \phi_c = -\frac{1}{g\kappa 224}. 
\]

For $\phi < \phi_c$, the Feynman-Siegel gauge configurations with $\eta = 0$ are equivalent under the restricted gauge transformations (4.13) to configurations with $\eta = 0, \phi > \phi_c$. This is analogous to the appearance of Gribov copies in nonabelian gauge theories. Furthermore, there is a range of field configurations $(\phi, \eta)$ which have no representatives in Feynman-Siegel gauge. These are the points with $\phi, \eta < 0$ which lie below the gauge orbit passing through the point $(\phi, \eta) = (\phi_c, 0)$. While the full theory is of course much more complicated after all other fields are included, this simple two-field example illustrates clearly the problems which arise when the boundary of Feynman-Siegel gauge validity is reached.

From the level two gauge transformations (4.8), we can see that the level (2, 6) approximation to the determinant condition (4.12) is

\[
g\kappa \left( -\frac{224}{81} \phi + \frac{29120}{2187} B + \frac{992}{6561} \beta \right) = 1
\]

This equation defines a plane in the three-dimensional space spanned by $\phi, B, \beta$. This plane divides the $\phi - B - \beta$ space into two regions, in each of which Feynman-Siegel gauge would be locally valid if the level two gauge transformations were exact. In Figure 4-2 we have graphed
Figure 4-2: Successive approximations to boundary of region of Feynman-Siegel gauge validity near origin, in units of $1/3g\kappa$

the vanishing locus of the determinant in the $\phi - B$ plane in level 2, 4, 6 and 8 truncations. It is clear from the figure that the part of this curve near the origin defines a boundary for the region of Feynman-Siegel gauge validity which converges fairly well as the level of truncation is increased. It seems that the vanishing locus of the determinant becomes less exactly described by low level truncations as one moves away from the origin. The sign of the determinant (4.12) in the $\phi - B$ plane is shown in Figure 4-3 for $-50/3g\kappa \leq \phi, B \leq 50/3g\kappa$ for level 4, 6 and 8 truncations. The black elliptical region containing the origin is the region of Feynman-Siegel gauge validity. The boundary of this region converges fairly well near the origin. Outside this region, the structure of the sign of the determinant becomes significantly more complicated as the level of truncation is increased, indicating a very complex structure for the gauge orbits of the theory.

The results we have summarized here give good evidence that the level truncation method gives a good systematic approximation scheme for the location of the boundary of the region of Feynman-Siegel gauge validity near the origin in field space. In the remainder of this chapter we discuss briefly a few applications of these results to the problem of tachyon condensation.
4.3 Applications to tachyon condensation

4.3.1 Branch points in tachyon effective potential

As discussed in Section 1, the calculations in [19] of the tachyon effective potential using the level-truncation method indicated the presence of two branch points in the effective potential, near $\phi \approx -0.25/g$ and $\phi \approx 1.5/g$. These calculations were performed using Feynman-Siegel gauge. The tachyon effective potential is computed by choosing a value of $\phi$, solving the equations of motion for all the other scalar fields, and computing the energy using these values for the fields. By following the trajectory in field space associated with this potential, we have found that, as the tachyon value approaches the points associated with branch points of the effective potential, the field configuration given by solving the equations of motion for the other fields approaches the boundary of the region of Feynman-Siegel gauge validity. It is
Figure 4-4: $\det M_{qb}$ in level (4, 8) truncation along tachyon potential.

difficult to numerically follow the field trajectory close to the branch points since numerical methods become unstable in this region. By following the trajectory to a point near the branch point, and extrapolating the trajectory further along a quadratic curve matching the first and second derivatives of the computed curve, however, we find that at each level in level truncation, the value of $\phi$ where the trajectory crosses the Feynman-Siegel validity region comes within a few percent of the value where the effective potential encounters a branch point. An example of such a calculation is shown in Figure 4-4, where the determinant (4.12) is graphed as a function of $\phi$ along the trajectory of fields giving the effective potential at level (4, 12). The first branch point at this level is near $\phi \approx -0.286/g$, and the boundary of Feynman-Siegel gauge validity is encountered near this point. Calculating the determinant along the line of the effective potential to within $0.001/g$ of the branch point and continuing on a quadratic trajectory, we find that the boundary of the Feynman-Siegel validity region is encountered at $\phi \approx -0.290$, within 2% of the branch point location. Similar results are found for the second branch point at level (4, 12) and for both branch points at other levels of truncation.

This analysis gives strong evidence that the branch points in the tachyon effective potential encountered in [19] are gauge artifacts. Thus, cubic string field theory seems completely consistent with background independent string field theory, where the tachyon potential is unbounded below for sufficiently negative values of the tachyon field. We expect that if a
method could be found for using cubic string field theory to compute the tachyon effective potential below the branch point $\phi \approx -0.25/g$ a similar result would be found.

4.3.2 Extra constraints on Feynman-Siegel gauge solution

Another interesting result which can be obtained from the level truncation of the full (non-gauge-fixed) SFT action is an infinite family of new constraints on the FS gauge nontrivial vacuum state. It is now generally believed that there is a well-defined state $\Phi_0$ in the Fock space which is annihilated by $b_0$ and which satisfies the SFT equation of motion $Q\Phi_0 + g\Phi_0 \star \Phi_0 = 0$. The coefficients of the low-level fields in this state were determined to a high degree of numerical accuracy in [17, 18, 19] by solving the equations of motion $\partial S/\partial \phi^b = 0$ for all the fields associated with states annihilated by $b_0$. It should also be the case, however, that the equations of gauge invariance $\partial S/\partial \phi^q = 0$ associated with states annihilated by $c_0$ should be satisfied in the nontrivial vacuum. This gives an infinite family of additional conditions which should be satisfied by $\Phi_0$. At any finite level of truncation, these additional equations of motion will only be satisfied approximately. For example, the equation of motion for the level two field $\eta$ in (4.7) vanishes in the stable vacuum to 92% when only level two fields are considered, 98% when level four fields are considered, and 99% when level six fields are included. Related gauge invariance conditions for the vacuum were previously considered in [49, 64]; other algebraic constraints on the vacuum were found in [65, 66]. It is interesting to ask whether all these constraints can be combined to give a more efficient method for determining the stable vacuum.

4.3.3 Finding the vacuum without gauge fixing or in other gauges

Since we have found that the Feynman-Siegel gauge choice is only valid locally, it is interesting to ask whether we can find the true vacuum and/or the tachyon effective potential without gauge fixing. While in the complete theory there is an infinite-dimensional gauge orbit of equivalent locally stable vacua, the breakdown of gauge invariance caused by level truncation means that even without gauge fixing, the equations of motion of the level-truncated theory have only a discrete set of solutions. In the level (2, 6) truncation, for example, there are two solutions in the vicinity of the stable vacuum, with energy densities $E/VT_{25} = -0.880$
and -1.078 (vs. -0.959 in FS gauge). (The first of these solutions was also found in [28]). At level (4, 12), there are at least three solutions, with \( E/VT_{25} = -0.927, -0.963, \) and -1.075 (vs. -0.988 in FS gauge). As the level of truncation is increased, the multiplicity of the candidate solutions continues to grow. While some solutions approach the correct value, others do not, so without some further criterion for selecting branches, it does not seem possible to isolate a good candidate for the vacuum in the level-truncated, non-gauge-fixed theory. A unique branch of the effective potential at each level has the property that it can be determined by a power-series expansion of the equations of motion around the unstable vacuum, as was done in [19] for the FS gauge tachyon potential. Above level 4, however, this branch encounters a branch point before reaching the stable vacuum. This difficulty in solving the theory without gauge fixing clearly arises from the presence of a continuous family of equivalent vacua in the full theory.

While solving the theory numerically without gauge fixing does not seem practical, it is natural to ask whether other gauges may work as well or better than the Feynman-Siegel gauge for determining the stable vacuum or the effective tachyon potential. One simple way of modifying the Feynman-Siegel gauge choice, for example, is to choose a different set of fields to vanish at each level from those dictated by the Feynman-Siegel gauge choice. As long as these fields are associated with states in the image of \( Q \) in the perturbative vacuum, this will locally be a valid gauge choice. As a simple example of such a different gauge choice, \( \text{At level two we could choose to fix } B = 0 \text{ or } \beta = 0 \text{ instead of } \eta = 0. \) More generally, we could choose to set any linear combination of these fields to vanish which is not invariant under the linear terms in (4.8). We can then take the usual Feynman-Siegel gauge choice for all the higher-level fields. Each of these gauge choices defines a new gauge in which we can perform level truncated calculations to arbitrary level. We have tried a variety of gauges of this type. We find that in general, these gauges behave rather similarly to Feynman-Siegel gauge, although the vacuum energy in a generic gauge seems to converge somewhat more slowly than in Feynman-Siegel gauge. For example, using the gauge fixing \( B = 0 \) for level two fields and FS gauge for all higher level fields, we find that \( E/VT_{25} \) is given by \(-0.901, -0.960, -0.979 \) at levels (2, 6), (4, 12) and (6, 18) (compared to -0.959, -0.988, -0.995 in FS gauge). By choosing different fields to vanish at various low levels, we can choose a wide variety of gauges of this type. Tuning the coefficients of the linear combinations that are
fixed to zero, we can produce a vacuum approximation at any particular level of truncation which has an energy density which is arbitrarily close to the desired value of $-VT_{25}$. In general, however, the approach to the vacuum energy is not monotonic, so this is not a particularly useful way to choose a gauge—even if the energy is exact at one level, including the next highest level of fields moves the energy away from the desired value by some small quantity.

The upshot of this investigation is that while various other gauges can be chosen which have similar behavior to Feynman-Siegel gauge, none seem to be particularly better than FS gauge, and generic other gauge choices seem to overshoot the vacuum energy more quickly than FS gauge which doesn’t overshoot until level (14, 24) [20].

It would be interesting to investigate other more general gauge choices, such as restricting to fields annihilated by some particular ghost operator other than $c_0$. This problem is left to future work.

4.4 Summary

We have investigated the range of validity of Feynman-Siegel gauge in Witten’s cubic string field theory. We found that this gauge choice breaks down outside a fairly small region in field space, and that the boundary of the region containing the origin in which Feynman-Siegel gauge is a good gauge choice can be stably computed using the level truncation method. We found that branch points appearing in earlier calculations of the tachyon effective potential are gauge artifacts arising when the field configuration along the effective potential leaves the region of validity of FS gauge. We investigated the possibility of determining the locally stable vacuum and/or the tachyon effective potential either without gauge fixing or by choosing a different gauge than Feynman-Siegel gauge, but found no approach which was substantially better than the Feynman-Siegel gauge-fixed approach.
Chapter 5

Issues in quantizing string field theory

In this chapter, we discuss our work with J. Shelton and W. Taylor [27]. Having discussed the classical theory of OSFT, we discuss various interesting aspects of its quantization. In particular, we discuss the open string one-point function at one loop. We show that this diagram incorporates the lowest order correction to the closed string background and we discuss the different ways in which the diagram can diverge.

5.1 Introduction

Much recent interest in Witten’s open string field theory (OSFT) [11] has been centered around the discovery that this theory can describe D-branes as classical solitons, so that distinct open string backgrounds not related through marginal deformations can appear as solutions of a single set of equations of motion. Most of the work in this area has focused on classical aspects of OSFT (although, for some recent papers which address quantum features of the theory, see [67]).

In order for string field theory to have a real chance at addressing any of the deep unsolved problems in string theory/quantum gravity, it is clearly necessary that the theory should be well defined quantum mechanically. In an earlier phase of work, some progress was made in understanding the quantum structure of OSFT. This work is summarized and described in the language of BV quantization by Thorn in his review [15]. In this chapter we extend this earlier work by carrying out a systematic analysis of the one-loop open string tadpole
diagram in Witten's bosonic OSFT. We analyze the divergence structure of this diagram and the role which closed strings play in the structure of the tadpole, and we describe the implications of this analysis for the quantum theory.

An important aspect of quantum open string field theory is the role which closed strings play in the theory. As has been known since their first discovery [68, 69], closed strings appear as poles in nonplanar one-loop amplitudes of open strings. An analysis of these poles in the one-loop nonplanar two-point function of OSFT was given in [70]. Because of the existence of these intermediate closed string states, any unitary quantum open string field theory must include some class of composite asymptotic states which can be identified with closed strings. These asymptotic states have not yet been explicitly identified in OSFT, although related open string states which can be used to compute amplitudes including closed strings in OSFT are described in [71, 72, 73, 74, 75, 76]; other approaches to understanding how closed strings appear in OSFT were pursued in [77]. We consider the appearance of closed strings in OSFT from a different point of view than has been taken in previous work on the subject. We show that an important part of the structure of the open string tadpole comes from the closed string tadpole, which, in the presence of a D-brane, describes the linearized gravitational fields around the brane. This demonstrates that not only do the closed strings appear as poles in the open string theory, but that they also take expectation values in response to D-brane sources within the context of OSFT; this provides a new perspective on the role of closed strings in OSFT.

The relationship between open and closed strings is central to the concept of holography and the AdS/CFT correspondence [78, 79, 80]. In the AdS/CFT correspondence, a decoupling limit is taken where open strings on a brane are described by a conformal field theory; this theory has a dual description as a near-horizon limit of the closed string (gravity) theory around the D-brane. A complete quantum open string field theory would generalize this picture; if OSFT can be shown to be unitary without explicitly including the closed strings as additional dynamical degrees of freedom, we would have a more general holographic theory in which the open string field theory on a D-brane would encode the gravitational physics in the full D-brane geometry in a precise fashion. While we do not directly address these ideas in this chapter, some further discussion in this direction is included in the summary.

Until recently, only a few diagrams had been explicitly evaluated in OSFT: the Veneziano
amplitude [81, 82, 83], and the non-planar two-point function [70]. These diagrams were computed by explicitly mapping the Witten parameterization of string field theory to a parameterization more natural for conformal field theory, and then computing the diagram explicitly in CFT. There has recently been some renewed interest in studying perturbative aspects of OSFT by developing new techniques for calculating diagrams in the theory [84, 85]. Using these methods it is possible to compute any OSFT diagram to a high degree of accuracy using the level truncation method on oscillators. This method provides an alternative to the CFT method, and gives some information about a wider range of diagrams while lacking the analytic control of the CFT method. In this chapter we use both methods, finding that each gives useful information.

The one-loop tadpole diagram is perhaps the simplest of the one-loop diagrams in OSFT. While a preliminary study of this diagram was done in [15, 84], we expand on the analysis presented in [84] and augment it by using the conformal field theory method to give an alternate expression for the diagram. We also generalize the discussion by computing the diagram for OSFT defined on a Dp-brane background for any p.

The one-loop tadpole diagram has divergences of several kinds. As the modular parameter $T$ describing the length of the internal open string propagator becomes large, there is a divergence from the open string tachyon. This divergence is easy to understand, and can be removed by analytic continuation in the oscillator approach to OSFT. In addition to the large $T$ divergence of the diagram, there are divergences as $T \to 0$. In the conformal frame natural to OSFT this limit corresponds to a pinching off of the world-sheet. In an alternate conformal frame, however, the small open string loop gives rise to a long closed string tube. These two conformal frames are displayed in figure 5-1.
Since the propagation over long distances of massive fields is suppressed, only the tachyon and the massless sector of the closed strings contribute to the $T \to 0$ divergences of the tadpole. Using both the conformal field theory and oscillator approaches, we isolate the $T \to 0$ divergences in the one-loop tadpole diagram. By extracting the leading divergences of the tadpole, we can separate the divergence arising from the tachyon from any divergences associated with the graviton/dilaton in the massless sector.

The divergence from the closed string tachyon arises because of the usual problem that the Euclidean theory has a real exponent in the Schwinger parameterization of the propagator, and diverges for tachyonic modes. This problem is usually dealt with by a simple analytic continuation. In this case, however, the analytic continuation is rather subtle, as the closed string degrees of freedom which are causing the divergence are not fundamental degrees of freedom in the theory. In the one-loop diagram we study in this chapter, the analytic continuation can be done by hand in the CFT approach by explicitly using our understanding of the closed string physics underlying the divergence. Even for this relatively simple diagram, however, there are a number of subtleties in this analytic continuation, and to ensure that we completely remove all the tachyonic divergences we are forced to consider lower-dimensional brane backgrounds. In a more general context, such as for higher-loop diagrams, it would be difficult to systematically treat this type of divergence using open string field theory.

Assuming that the divergence from the closed string tachyon is dealt with by a form of analytic continuation, we are left with possible divergences from the massless closed string states. Such divergences appear only when considering the open string theory on a D$p$-brane with $p \geq 23$. This is essentially because the open string tadpole is generated directly from the closed string tadpole, and the closed string tadpole arises from the solution of the linearized gravitational equations with a D$p$-brane source [32, 33]. For a related discussion of divergences in brane scattering, see also [34, 35, 36]. Since a brane of codimension 2 has a long-range potential which goes as $\ln r$, while a brane of codimension 3 has a potential going as $1/r$, we need at least three codimensions to remove the divergences from the massless sector. In general, whether the tadpole is finite or divergent, we find that the structure of the linearized closed string fields in the D-brane background is encoded in the open string tadpole.
After analyzing the divergence structure of the one-loop tadpole, we also consider briefly what one might expect of OSFT at two or more loops. We consider in particular the two loop non-planar diagram which represents a a torus with a hole in it. This diagram contains as a subdiagram the closed string one point function which suffers from a BRST anomaly [29, 30, 31, 32] in world-sheet perturbation theory. We conjecture that in OSFT this diagram will lead to a divergence and BRST anomaly. Since this divergence is purely closed-string in nature, it should occur for the theory on a Dp-brane for any p, and poses a serious problem for any attempt to make sense of the bosonic open string field theory as a complete quantum theory.

In Section 3 we compute the one-loop tadpole using conformal field theory methods. In section 4 we compute the same diagram using oscillator methods. Section 5 contains a discussion of the one-loop open string tadpole in Zwiebach’s open-closed string field theory, where the structure of the diagram is somewhat more transparent. Section 6 synthesizes the results of the preceding sections, and contains a general discussion of the divergences of the tadpole and the role of closed strings in the tadpole. Section 7 contains some concluding remarks. In two appendices we include some technical points. Appendix A contains a discussion of the BRST anomaly in the D25-brane theory. Appendix B contains some comments on the infinite-level limit of the level truncation method used in Section 4.

5.2 Perturbation theory in open string field theory

We begin with a summary of perturbation theory in OSFT [11]. For a general review of these ideas see [15].

Recall that the classical action is given by

\[ S(\Psi) = \frac{1}{2} \int \Psi \star Q_B \Psi + \frac{g}{3} \int \Psi \star \Psi \star \Psi. \]  

(5.1)

The definitions for the \( \star \)-product and string integration are given in [40, 86, 87, 46, 47, 88, 89] in terms of both oscillator expressions and conformal field theory correlators and are often defined in terms of the two-string and three-string vertices, \( \langle V_2 \rangle \) and \( \langle V_3 \rangle \), which are given
by

\begin{align}
\langle V_2 | \Psi_1 \rangle | \Psi_2 \rangle &= \int \Psi_1 \star \Psi_2, \\
\langle V_3 | \Psi_1 \rangle | \Psi_2 \rangle | \Psi_3 \rangle &= \int \Psi_1 \star \Psi_2 \star \Psi_3,
\end{align}

and are elements of the two-string and three-string Fock spaces respectively.

The theory has a large gauge group. Infinitesimally the gauge transformations are given by

\[ \Psi \rightarrow \Psi + Q_B \Lambda + g(\Psi \star \Lambda - \Lambda \star \Psi), \]

where \( \Lambda \) is any ghost number 0 field.

To quantize the theory we must fix a gauge. The standard choice for gauge fixing is Feynman-Siegel (FS) gauge fixing which imposes the condition \( b_0 \Psi = 0 \). As discussed in chapter 3, this gauge appears to be good near the perturbative vacuum. It is straightforward to perform tree-level calculations in this gauge, but some care is required when trying to impose this condition on path integrals. Roughly speaking, it turns out that if one tries to introduce Fadeev-Popov ghosts to fix \( b_0 \Psi = 0 \), the ghosts themselves suffer from a gauge redundancy similar to the gauge redundancy of the original action. To fix this new gauge redundancy one must introduce ghosts for ghosts. As the new ghosts have their own redundancy, this process proceeds forever, creating an infinite tower of ghost fields. Happily, at the end of the day this entire procedure can be summarized as follows [15]:

1. The field \( \Psi \) is fixed by \( b_0 \Psi = 0 \).
2. The ghost number of \( \Psi \) is allowed to range over all ghost numbers, not just ghost number 1. The fields of ghost numbers other than one are all ghost fields.
3. \( \Psi \) is a grassmann odd field. To define what this means, suppose the states \( \{|s\}\} \) form a basis for the open string Fock space such that each \( |s\) \) has definite ghost number. Then if we write \( \Psi \) in a Fock space expansion as \( \Psi = \sum_s |s\rangle \psi_s \), then \( \psi_s \) has the opposite grassmannality of \( |s\rangle \).

The form of the action remains the same as in equation (5.1). Using the FS gauge condition
of $\Psi$ we can simplify the kinetic term:

$$S_{FS}(\Psi) = \frac{1}{2} \int \Psi \star c_0 L_0 \Psi + \frac{g}{3} \int \Psi \star \Psi \star \Psi.$$  \hspace{1cm} (5.5)

Given the gauge fixed action we can now develop the Feynman rules for perturbation theory. We can do this in two ways, which we will refer to as the conformal field theory method and the oscillator method.

In the conformal field theory method, the Feynman rules are given in terms of rules for sewing strips of world-sheet together. Amplitudes may be evaluated by conformally mapping the resulting diagrams to the upper half plane for genus 0 or the cylinder for genus 1.

In the oscillator method the Feynman rules are calculated directly from the action using the usual methods from field theory but summing over the infinite number of fields. For any amplitude this gives rise to correlators which can be evaluated using squeezed state methods.

In the next two sections we consider each of these two methods in turn.

### 5.3 Evaluation of the tadpole using conformal field theory

In this section we calculate the one-point function using conformal field theory methods. We begin the calculation with the assumption that we are working with the theory on a D25-brane. In section 5.3.1 we describe how the diagram can be computed by constructing a map from the original Witten diagram to the cylinder. In section 5.3.2, we specialize to the limit where the internal loop of the diagram is small and study the divergences in this limit. In section 5.3.3 we discuss the origin of these divergences from the propagation of closed string modes over long distances (these divergences are discussed further in Section 6). Finally, in section 5.3.4 we discuss how the calculation differs for the theory on a Dp-brane with $p \neq 25$.

#### 5.3.1 Mapping the Witten diagram to the cylinder

We begin with a brief discussion of the world-sheet interpretation of FS gauge fixed OSFT. The derivation of this interpretation is given in [90, 91, 15, 92]. The Feynman rules consist
of one propagator and one vertex.

The propagator is given by an integral over world-sheet strips of fixed width. By convention the strips are of width $\pi$, and length $T$, where $T$ is integrated from 0 to $\infty$. To ensure the right measure on moduli space $b(\sigma)$ is integrated across the strip [90]. The only vertex in the theory is a prescription for gluing three strips together. The right half of the first strip is glued to the left half of the second and similarly for the second and third strips and the third and first strips.

Using these rules we can construct the one-point amplitude. We start with an external state $|A\rangle$ which propagates along a strip of length $T_A$. We then take a second strip of length $T$ and glue both ends of it and the end of the first propagator together using the vertex. The resulting diagram is pictured in figure 5-2.

We wish to study this diagram using conformal field theory methods. To do this we use the methods of [70] to map the diagram to a cylinder. For another approach to this conformal mapping problem, see [93]. Taking the limit $T_A \to \infty$, we can map the external state, $|A\rangle$, to a puncture at the boundary of the cylinder.

It is convenient to flatten the diagram by cutting along the folded edge of the external propagator in figure 5-2 and cutting the internal propagator in half. The resulting diagram is displayed in figure 5-3 a. We will let $\rho$ be the coordinate on the Witten diagram and $u$ be the coordinate on the cylinder. To enforce Neumann boundary conditions along the boundaries of the diagram, we use the doubling trick. Since the double of the cylinder is a torus, we may use the theory of elliptic functions to determine $\rho(u)$.

Consider the image of the Witten diagram under $\rho \to u$ shown in figure 5-3 b. The top
and bottom of the diagram are identified as well as the left and right edges. The external state, |A⟩, is mapped to a puncture at point A which we choose to be at u = 0. The midpoint is mapped to point β. By the symmetry in the diagram, we can set the real part of β to zero. Since the vertex in the original Witten diagram had an angle of 3π, the function ρ(u) must behave as ρ(u) − ρ(β) ∼ (u − β)3/2 near β. This implies that dρ/du has a branch cut. The height of the torus is given by a purely imaginary parameter τ to be determined later. The integral around the puncture A of dρ/du corresponds1, in the original Witten diagram, to the total width of the external propagator plus its double, which is fixed to be 2πi. This implies that dρ/du has a simple pole at A of residue one.

We now define the quadratic differential φ(u) = (dρ/du)^2(u). From the form of dρ/du near β we see that φ(u) has no branch cut, just a simple zero at β. In order to preserve Neumann boundary conditions we must also include a zero at the image of β under the doubling. Since we have put the top of the diagram along the real axis, we just get a second zero at β*. The only other piece of analytic structure we need is that since dρ/du had a simple pole at A, φ(u) has a double pole at A.

Now since φ(u) is a meromorphic function on a torus we may determine it using ϑ-

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1One has to be careful that the contour does not cross the branch cut so that, in the ρ coordinates, the contour is continuous.
functions:

$$\phi(u) = C \frac{\vartheta_1(u - \beta, q) \vartheta_1(u - \beta^*, q)}{\vartheta_1^2(u, q)},$$

(5.6)

where $q = e^{i\pi \tau}$. The constant $C$ is determined from the condition that $\sqrt{\phi(u)}$ has a pole of residue one at $A$:

$$C = \frac{(\vartheta_1'(0))^2}{\vartheta_1(-\beta) \vartheta_1(-\beta^*)}.$$

(5.7)

The two constants $\beta$ and $\tau$ can be determined from the height and width of the diagram by integrating $d\rho/du$ along the curves $\gamma_1$ and $\gamma_2$. We have

$$\int_{\gamma_1} du \sqrt{\phi(u)} = 2\pi i,$$

$$\int_{\gamma_2} du \sqrt{\phi(u)} = T.$$

(5.8)

In general these relations cannot be solved analytically, but it is straightforward to solve them numerically and thus to determine $\tau$ and $\beta$ as functions of $T$.

At this point one could, in principle, evaluate the diagram for any given $A$. If we suppose that the state $A$ is defined by a vertex operator inserted on a half-disk with coordinates $v$, we can easily compute the map $\rho(v)$ from the half-disk to the tadpole diagram. The diagram at a fixed modular parameter is then computed by evaluating

$$\langle (u(\rho) \circ \rho(v) \circ A)(u(\rho) \circ \frac{1}{2\pi i} \int d\rho b(\rho)) \rangle_{\text{torus}}$$

(5.9)

where the contour of integration runs across the internal propagator. This correlator implicitly defines a fock space state, $|T(T)\rangle$, given by

$$\langle A|T(T)\rangle \equiv \langle (u(\rho) \circ \rho(v) \circ A)(u(\rho) \circ \frac{1}{2\pi i} \int d\rho b(\rho)) \rangle_{\text{torus}}.$$

(5.10)

Note that the state $|T(T)\rangle$ is a function of the modular parameter $T$. The full tadpole diagram is given by integrating over this modular parameter. We thus define the full tadpole state $|T\rangle$ by

$$|T\rangle = \int_0^\infty dT \ |T(T)\rangle.$$

(5.11)

The expression (5.9) can only be evaluated numerically, since we do not know $\rho(u)$ explicitly.
(only its derivative) and we cannot analytically solve the constraints (5.8). Furthermore, there are several types of divergences in the integral over $T$ in (5.11). The integrand (5.10) diverges as $T \to \infty$ due to the open string tachyon. While this divergence is difficult to treat in the CFT approach, its physical origin is clear and is quite transparent in the oscillator approach, where this divergence can be treated by a suitable analytic continuation. We discuss the open string tachyon divergence further in Sections 4 and 6. In addition to the the divergence as $T \to \infty$, there are further divergences as $T \to 0$ arising from the closed string. These divergences are much more subtle, as closed strings are not explicitly included among the degrees of freedom in OSFT, but arise as composite states of highly excited open strings. We thus seek to evaluate the tadpole diagram in an expansion around $T = 0$, where much of the interesting physics in the diagram is hidden.

5.3.2 The $T \to 0$ limit

We now focus on the region of moduli space near $T = 0$. Unfortunately the map $\sqrt{\phi(u)}$ cannot easily be expanded around this limit. To get around this we use a trick. It turns out that the conformal map greatly simplifies if, instead of fixing the integral along $\gamma_1$ to be $2\pi i$, we set it equal to some parameter $H$ and take $H \to i\infty$ holding $T$ fixed. This is equivalent to gluing a semi-infinite cylinder to the bottom of the tadpole. Later we will see how to replace this long cylinder with a boundary state to reduce back to the finite length cylinder case, but for the moment we just consider the conformal map in this limit.

By solving the constraints (5.8) numerically, one can verify that as $H \to \infty$ with $T$ fixed, $\beta$ limits to a constant $\beta_0$, while $\tau \to i\infty$. Recalling that $q = e^{i\pi \tau}$, we see that since $\tau$ is pure imaginary, $q \to 0$ as $H \to \infty$. Thus we may set $q = 0$ in our map to get

$$\frac{d\rho}{du} \bigg|_{q=0} = \sqrt{\phi(u)} \bigg|_{q=0} = \sqrt{\cot^2(u) - \cot^2(\beta_0)}. \quad (5.12)$$

We can now solve for $T$ in terms of $\beta_0$ by performing the integral along $\gamma_2$:

$$T = \oint_{\gamma_2} du \sqrt{\cot^2(u) - \cot^2(\beta_0)} = -\frac{i\pi}{\sin(\beta_0)}. \quad (5.13)$$
Using this relation, we can eliminate $\beta_0$ from the definition of our conformal map:

$$\lim_{H \to \infty} \left( \frac{d\rho}{du} \right) = \sqrt{1 + \left( \frac{T}{\pi} \right)^2 + \cot^2(u)}. \quad (5.14)$$

This function may even be integrated analytically although the resulting expression is cumbersome. For notational simplicity we now consider $d\rho/du$ only in the limit of $H \to \infty$ and we will assume that the tadpole diagram has an infinitely long tube extending from the bottom.

Before we consider the effect of replacing the long tube at the bottom of the diagram with a boundary state, we consider the effect of the map $\rho(u)$ on the external state $A$ as $T \to 0$. Note that if we take the limit that $T \to 0$, $d\rho/du$ simplifies even further.

$$\lim_{T \to 0} \left( \frac{d\rho}{du} \right) = \sqrt{1 + \cot^2(u)} = \frac{1}{\sin(u)}, \quad (5.15)$$

where one must be careful about the interpretation of the branch cuts. Integrating this function yields

$$\int \frac{du}{\sin(u)} = \log \left( -\tan \left( \frac{u}{2} \right) \right). \quad (5.16)$$

While this may not seem like a familiar map, it is actually a representation of the identity state. Putting

$$h(z) = \frac{1 + iz}{1 - iz}, \quad (5.17)$$

we consider the circle of conformal maps pictured in figure 5-4. One can verify that traversing the diagram counterclockwise (starting from the vertical strip at the bottom and proceeding to the representation of the identity in the upper left), gives the same map as $\rho(u)$ when $T \to 0$. This implies that the tadpole diagram with a long tube attached to the bottom is conformally equivalent to the identity state with an operator inserted in the corner in the limit $T \to 0$. Such states are known as Shapiro-Thorn states [71, 72].

We now must deal with the fact that the original diagram had a tube of finite length extending from the bottom. We thus consider the effect of replacing the infinite tube at the bottom of the tadpole with the closed string boundary state with Neumann boundary conditions. The boundary state in disk coordinates for Neumann boundary conditions can
Figure 5-4: A circle of conformal maps showing the equivalence of the two prescriptions for the identity state. In the limit that $T \to 0$ traversing the diagram clockwise from the surface in the upper left to the tadpole in the bottom left is equivalent to the trivial map $f(z) = z$. 
be written
\[ \langle B \rangle = \langle 0 | (\tilde{c}_{-1} + c_1)(\tilde{c}_0 + c_0)(\tilde{c}_1 + c_{-1}) \exp \left( \sum_{m > 1} b_m \tilde{c}_m + \tilde{b}_m c_m \right) \exp \left( \sum_{n \geq 1} \frac{1}{n} \alpha_n \cdot \tilde{\alpha}_n \right) \] (5.18)

where the oscillators are the usual closed string oscillators and \( \langle 0 | \) is the closed string \( SL(2, \mathbb{R}) \) vacuum. When we map to cylinder coordinates, the disk becomes a semi-infinite tube with the boundary state being propagated in from infinity. By rescaling the size of the cylinder we can make it the same circumference as the long tube at the bottom of the tadpole.

Consider taking the long tube that extends from the bottom of the diagram and cutting it off a distance \( \pi / 2 \) below the vertex. We can then attach the boundary state, in cylinder coordinates, onto the bottom of the diagram. Before adding the boundary state, the topology of the tadpole diagram is a annulus with one vertex operator (representing the external state) inserted on the outer boundary. Attaching the boundary state plugs the hole in the annulus, changing the topology to a disk. The cost of doing this is that there is now an additional vertex operator, representing the boundary state, inserted in the interior of the disk. For general values of the modulus \( T \), the complexity of the boundary state makes this replacement impractical, but, for small \( T \), only a few terms in the boundary state become relevant.

After this modification, we can map the tadpole diagram to the unit disk with the image of the boundary state at \( z = 0 \) and the image of the state \( |A\rangle \) at \( z = 1 \), where \( z \) is the coordinate on the disk. We call the coordinates on the boundary state disk \( w \). Since the boundary state is not conformally invariant, it will be mapped to \( z(w) \circ B \). As before, we denote by \( v \) the coordinates on the half disk where the vertex operator for the external state \( A \) is defined and we let \( z(v) \) be the map that takes the external state \( A \) into the disk coordinates. We can then write down a formal expression for the tadpole diagram
\[ \langle (z(v) \circ A)(z(\rho) \circ \frac{1}{2\pi i} \int d\rho \, b(\rho))(z(w) \circ B) \rangle_{\text{disk}}. \] (5.19)

Mapping everything to the upper-half plane using the map \( h(z) \) defined in (5.17), we can write this as an inner product between \( A \) and the ket \( |T(T)\rangle \) defined in (5.10). If \( |A\rangle \) is the
Figure 5-5: The map $w(z)$ is shown as a series of maps. The shaded regions show the images of the boundary state.

The external state in half-disk coordinates, the diagram at fixed modular parameter is given by

$$\langle A | U_{h(z)oz(v)}^\dagger (h(z) \circ z(\rho) \circ \frac{1}{2\pi i} \int d\rho b(\rho)) (h(z) \circ z(w) \circ \mathcal{B}) | 0 \rangle$$

which implies that

$$| T(T) \rangle = U_{h(z)oz(v)}^\dagger (h(z) \circ z(\rho) \circ \frac{1}{2\pi i} \int d\rho b(\rho)) (h(z) \circ z(w) \circ \mathcal{B}) | 0 \rangle,$$

where the operator $U_f$ is defined by its action on local operators

$$U_f \mathcal{O} U_f^{-1} = f \circ \mathcal{O}.$$  

and $U_f^\dagger$ is the BPZ dual of $U_f$. Such operators have been considered in [94, 28, 95, 96].

Since, as we’ve already discussed, the map $z(v)$ just limits to a representation of the identity state as $T \to 0$, the operator $U_{h(z)oz(v)}^\dagger$ has a well behaved limit as $T \to 0$. Thus, when we analyze the small $T$ limit, we only need to focus on the behavior of $z(w) \circ \mathcal{B}$ and $z(\rho) \circ \int d\rho b(\rho)$. As it turns out, it is computationally easier to calculate $w(z)$ rather than $z(w)$. Pictorially, $w(z)$ is given in figure 5-5.

The map from the disk to the vertical strip is given by

$$u(z) = \frac{i}{2} \log(z).$$

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Since we already know the derivative of the map from the vertical strip to the Witten diagram, $d\rho/du$, we can easily compute the derivative of the map from the disk to the Witten diagram. To eliminate excess factors of $\pi$, we put $W = T/\pi$. We then have

$$\frac{d\rho}{dz} = \frac{d\rho}{du}(u(z)) \frac{du}{dz} = \frac{i}{2} \frac{\sqrt{W^2(z - 1)^2 - 4z}}{z(z - 1)}. \quad (5.24)$$

Integrating this function gives

$$\rho(z) = -\frac{i}{2} \left( 2 \tan^{-1}(\frac{1 + z}{\sqrt{W^2(z - 1)^2 - 4z}}) - W \log(z) \right. \left. + W \log(-2z + (1 + z)W \sqrt{W^2(z - 1)^2 - 4z + W^2(1 + z^2)}) \right) + g(W), \quad (5.25)$$

where $g(W)$ is the constant of integration. By fixing the image of the midpoint, we can determine $g(W)$ to be

$$g(W) = \frac{i}{2} \left( \pi + 2 \cot^{-1}(W) + i \log\left(\frac{W + i}{W - i}\right) + W \log(2 + 2W^2) \right). \quad (5.26)$$

Finally, we map the bottom of the Witten diagram to the disk with the boundary state in the center. This map is given by

$$w(\rho) = \exp\left(\frac{\pi - 2i\rho}{W}\right). \quad (5.27)$$

We can thus calculate $w(\rho(z)) = w(z)$. Since the full expression is quite complicated we only display the series expansion:

$$w(z) = e^{\pi/W + i\log\left(\frac{W + i}{W - i}\right)/W} \left[ \frac{1 + W^2}{W^2} z + 2 \frac{1 + W^2}{W^4} z^2 + \frac{2W^4 + W^2 - 1}{W^6} z^3 + \mathcal{O}(z^4) \right]. \quad (5.28)$$

We can now find $z(w)$ as a power series in $w$. Putting

$$k(W) = e^{-\pi/W - i\log\left(\frac{W + i}{W - i}\right)/W}, \quad (5.29)$$

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we have

\[
z(w) = -k \frac{W^2}{1 + W^2} w + 2k^2 \frac{W^2}{(1 + W^2)^2} w^2 - k^3 \frac{W^2(7 + 2W^2)}{(1 + W^2)^3} w^3 + \mathcal{O}(w^4).
\] (5.30)

As one might expect, the unit disk is mapped to smaller and smaller regions as \( W \to 0 \). This suggests that the boundary state might be mapped to some local operator. In fact, all the positive weight parts of \(|B\rangle\) will be suppressed in the small \( W \) limit. Thus the only relevant terms from the boundary state are the weight zero and weight \((-1, -1)\) fields. These are given by

\[
|B\rangle = c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle \\
+ \alpha_{-1} \cdot \tilde{\alpha}_{-1} c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle - (c_1c_{-1} + \tilde{c}_{-1}\tilde{c}_1)(c_0 + \tilde{c}_0)|0\rangle \\
+ \text{higher weight states.}
\] (5.31)

Since the term \( c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle \) is a weight \((-1, -1)\) primary it picks up the coefficient

\[
\left( \frac{1 + W^2}{kW^2} \right)^2
\] (5.32)

under the \( z(w) \) map. Similarly, the term \( \alpha_{-1} \cdot \tilde{\alpha}_{-1} c_1(c_0 + \tilde{c}_0)c_1|0\rangle \) is a weight \((0, 0)\) primary and thus is unchanged by the \( z(w) \) map.

Unfortunately, the term, \(- (c_1c_{-1} + \tilde{c}_{-1}\tilde{c}_1)(c_0 + \tilde{c}_0)|0\rangle\), is not a primary and is mapped to

\[
-(c_1c_{-1} + \tilde{c}_{-1}\tilde{c}_1)(c_0 + \tilde{c}_0)|0\rangle - \frac{8}{W^4} c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle \\
+ \frac{4}{W^2} c_1(c_{-1} + \tilde{c}_{-1})\tilde{c}_1|0\rangle + \frac{2}{W^2} (c_1 - \tilde{c}_1)c_0\tilde{c}_0.
\] (5.33)

As can be seen in the second term, this state mixes with the weight \((-1, -1)\) state \( c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle \) under conformal maps. This mixing will play a role later.

To fully account for the behavior of the tadpole near \( W \to 0 \), we must take care of the insertion of the \( b_0 \). Since the conformal map from the disk to the tadpole has already been found, the transformation of \( b_0 \) to the disk coordinates is straightforward. Calling the
resulting operator $B$, we get

$$B = \frac{1}{\pi} \int dz \frac{z(z-1)}{\sqrt{W^2(z-1)^2 - 4z}} b(z) - \text{c.c.},$$

(5.34)

where the contour runs along the real axis from $-1$ to $z(-1)$. Note that as $W \to 0$, $z(-1) \to 0$ also. Thus we can get collisions between $b(z)$ and the various $c$’s in the boundary state. We can expand $B$ in terms of the modes found in equation (5.31), keeping only the most divergent terms for each mode $b_n$:

$$B \sim \ldots - \frac{4}{3\pi} b_{-1} + \frac{1}{W^2} b_0 - \frac{1}{\pi W^3} b_1 + \ldots - \text{c.c.}$$

(5.35)

We can now let $B$ act on the conformally transformed boundary state. We keep only the most divergent terms from each state and drop all finite terms:

$$B \left( z(w) \circ |B\rangle \right) = -\frac{2}{k^2 W^6} c_1 \tilde{c}_1 |0\rangle - \frac{1}{k^2 W^4} \left( c_0 + \tilde{c}_0 \right) \tilde{c}_1 B : |0\rangle - \frac{4}{3\pi k^2 W^4} (c_0 + \tilde{c}_0)(\tilde{c}_1 - c_1) |0\rangle + \frac{2}{W^2} \alpha_{-1} \cdot \tilde{\alpha}_{-1} \tilde{c}_1 c_1 |0\rangle - \frac{2}{W^2} (c_1 c_{-1} + \tilde{c}_{-1} \tilde{c}_1) |0\rangle - \frac{16}{3\pi W^2} (c_{-1} + \tilde{c}_{-1})(\tilde{c}_1 - c_1) |0\rangle - \frac{8}{3\pi W^2} c_0 \tilde{c}_0 |0\rangle + \ldots$$

(5.36)

To see directly how these terms cause the tadpole to diverge, consider for example the term $-\frac{2}{k^2 W^6} c_1 \tilde{c}_1 |0\rangle$, which is the most divergent term in the expansion (5.36). Plugging this term into the expression for $|T(T)\rangle$ given in equation (5.21) gives

$$|T(T)\rangle \sim -\frac{2}{k^2 W^6} U_{h(z) \circ z(v)}^\dagger (h(z) \circ c(0) \tilde{c}(0)) |0\rangle \propto \frac{e^{2\pi^2 / T}}{T^6} U_{h(z) \circ z(v)}^\dagger c(i) \tilde{c}(i) |0\rangle.$$  

(5.37)

As discussed above, as $T \to 0$ the map $h(z) \circ z(v)$ limits to the map representing the identity
state. Thus if we are only interested in $|\mathcal{T}(T)\rangle$ near $T = 0$ we can replace

$$h(z) \circ z(v) = f(v) = \frac{2v}{1 - v^2},$$

where $f(z)$ is the usual expression for the identity state in the upper-half plane geometry. Thus we can write, for small $T$,

$$|\mathcal{T}(T)\rangle \sim \frac{e^{2\pi^2/T}}{T^6} U_{f(v)}^\dagger c(i) \bar{c}(i)|0\rangle.$$  

(5.39)

The full tadpole state is then given by the integral of this state over $T$. Since the state $U_{f(v)}^\dagger c(i) \bar{c}(i)|0\rangle$ does not depend on $T$, the small $T$ region of the integral is determined by the integral over the function $e^{2\pi^2/T}/T$, which diverges. One can also consider what we would get if we take, for example, the term $\frac{2}{W^2} \alpha_{-1} \cdot \bar{\alpha}_{-1} \bar{c}_1 c_1 |0\rangle$ from the expansion (5.36). A similar calculation yields

$$|\mathcal{T}(T)\rangle_{\alpha_{-1} \cdot \bar{\alpha}_{-1} \bar{c}_1 c_1 |0\rangle} \sim \frac{1}{T^2} \left( \frac{2}{W^2} \right) \alpha_{-1} \cdot \bar{\alpha}_{-1} c_1 \bar{c}_1 |0\rangle,$$

(5.40)

where the subscript indicates that this is only the behavior of the terms in $|\mathcal{T}(T)\rangle$ arising from term $\alpha_{-1} \cdot \bar{\alpha}_{-1} \bar{c}_1 c_1 |0\rangle$ in the expansion (5.36). Since one cannot integrate $1/T^2$ near zero, this also leads to a divergence in $|\mathcal{T}\rangle$.

For comparisons with the results in section 5.4, it is useful to note that the expression for the leading divergence of the $|\mathcal{T}(T)\rangle$, given in (5.39), can be rewritten as

$$|\mathcal{T}(T)\rangle \sim \frac{e^{2\pi^2/T}}{T^6} \exp \left( -\frac{1}{2} a_m^\dagger C_{mn} a_n^\dagger - c_m^\dagger C_{mn} b_n^\dagger \right) c_0 c_1 |0\rangle,$$

(5.41)

where $a_m^\dagger$, $c_m^\dagger$ and $b_m^\dagger$ are the usual open string oscillators and $C_{mn} = (-1)^m \delta_{mn}$. Equation (5.41) may be verified using the methods of [40].

### 5.3.3 Interpretation of the divergences

The divergences in (5.36) may be interpreted as arising from the propagation of tachyonic and massless closed string modes down a long cylinder. The tube at the bottom of the tadpole is a tube of length $\frac{5}{2}$ and circumference $T$. Rescaling this tube by a factor of $\frac{2\pi}{T}$
gives a tube of length $\frac{\pi^2}{T}$ and constant circumference $2\pi$. These are the standard lengths for closed string theory. If we think of the boundary state at the end of the tube as a closed string state, we may propagate it along the length of the tube using the operator

$$\exp \left(-\frac{\pi^2}{T} (L_0 + \tilde{L}_0) \right).$$

For the term in the boundary state given by $c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle$, which we may think of as coupling to the closed string tachyon, this gives a prefactor of

$$e^{2\pi^2/T} \sim \frac{1}{k^2}.$$  \hspace{1cm} (5.43)

For the weight zero terms from the boundary state there is no term picked up from the propagation. However we must account for the measure on the moduli. Taking the modulus to be the length of the cylinder $s = \frac{\pi^2}{T}$, the usual measure on a cylinder is just $ds = -\frac{\pi^2 \, dt}{T^2}$.

Of course the full tadpole diagram is not just a cylinder. Complicated conformal factors can and do arise from the specific manner in which the cylinder at the bottom of the diagram is attached to the external open string state. Furthermore since the boundary state is not a conformal primary, conformal transformations can mix the behavior of the various terms. This is seen in equation (5.36). The first three terms diverge as $\sim \frac{1}{k^2}$ suggesting that these divergences come from the propagation of the closed string tachyon over the long cylinder. The next four terms diverge as $\sim \frac{1}{T^2}$ suggesting that these terms arise from the massless closed string sector fields. The term $\alpha_{-1} \cdot \tilde{\alpha}_{-1}\tilde{c}_1c_1|0\rangle$ is of the right form to correspond to the graviton/dilaton. Since this field does not mix with any of the other fields under the conformal map $z(w)$, there is no ambiguity that this divergence arises purely from the massless sector. The other fields diverging as $1/W^2$, correspond to auxiliary fields. Because these states are not conformal primaries, they mix with the states coupling to the tachyon field. Thus, these divergences may be due in part to the closed string tachyon.

The tachyon divergence may be partially treated by an analytic continuation. If we take the weight of the state $c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle$ to be $(h_1, h_1)$, we can try to perform the integrals over the modular parameter with the assumption that $h_1 > 0$. Since the term, $-(c_1\tilde{c}_{-1} + \tilde{c}_{-1}\tilde{c}_{-1})(c_0 + \tilde{c}_0)|0\rangle$, mixes with $c_1(c_0 + \tilde{c}_0)\tilde{c}_1|0\rangle$ under conformal maps we must also take this

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state to have some arbitrary weight \((h_2, h_2)\). We can then substitute \(h_1 = -1\) and \(h_2 = 0\) at the end of the calculation. This prescription works for all the terms containing powers of \(1/k\). Unfortunately there are subleading terms which contain no factors of \(1/k\), but still diverge as badly as \(1/W^5\).

These divergences, which mix with the massless divergences, seem to be an unfortunate consequence of the geometry of the Witten diagram. In other versions of string field theory, such as the one discussed in section 5.5, these divergences do not arise. It is possible that a more sophisticated method for treating these divergences would eliminate them and that we are merely limited by our inability to correctly identify the tachyonic degrees of freedom, since they are encoded in a highly nontrivial fashion in terms of the fundamental open string degrees of freedom. It is also possible, since we are dealing with off-shell physics, that spurious divergences are arising. Such unexpected off-shell physics was found in [70], where extra poles were found in an internal closed string propagator. Fortunately, however, we can eliminate these \(1/W^n\) divergences by considering lower dimensional Dp-brane backgrounds, which is the subject of the next section.

### 5.3.4 Lower dimensional Dp-branes

Having examined the case of the D25-brane, we turn to the general case of the Dp-brane for \(p < 25\). The only change in the analysis we need to make is to replace the boundary state, \(|\mathcal{B}\rangle\), with the correct boundary state for a Dp-brane. This state is given by

\[
|\mathcal{B}_p\rangle = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot \alpha_{-n} - b_{-m} \tilde{c}_{-m} - \tilde{b}_{-m} c_{-m} \right) \int d^{25-p} q_\perp e^{-i q_\perp \cdot y_\perp} |q_\perp, q_{\parallel} = 0\rangle, \tag{5.44}
\]

where \(y_\perp\) is the location of the brane, \(q_\perp\) is the momentum transverse to the brane, and \(q_{\parallel}\) is the momentum parallel to the brane. We will set \(y_\perp^\mu = 0\) for convenience. The minus sign in front of the \(\alpha'\)s is chosen for Neumann boundary conditions and the plus sign for Dirichlet.

We can now study the divergence structure of the tadpole diagram by studying \(B(z(w) \circ |\mathcal{B}_p\rangle)\). Since the sign changes in (5.44) do not affect the divergence structure we need only consider the effect of the momentum dependence. Because it is simpler, we begin with the
massless sector. Consider the contribution from the graviton/dilaton.

\[ B(z(w) \circ \mathcal{B}_p)) = \ldots - \int d^{25-p}q_\perp \left( \frac{kW_2}{1 + W_2} \right)^{\frac{q_\perp^2}{2W^2}} \alpha_{-1} \cdot \tilde{\alpha}_{-1} c_1 \tilde{c}_1 |q_\perp| + \ldots \] (5.45)

Note that the factor of \( \left( \frac{kW_2}{1 + W_2} \right)^{\frac{q_\perp^2}{2W^2}} \) kills the \( W \to 0 \) divergence if \( q_\perp^2 > 0 \). To see whether the point \( q_\perp = 0 \) contributes a divergence in the integral, we map this state to a local operator at \( i \) in the upper-half plane using the map \( h^{-1}(z) \). This gives the operator

\[ \int d^{25-p}q_\perp (2)^{\frac{q_\perp^2}{2W^2}} \frac{kW^2_2}{1 + W^2_2} \frac{2}{W^2} c_\partial X(i) \tilde{c}_{\partial X(i)} e^{iq_\perp \cdot X(i)}. \] (5.46)

In the upper-half plane geometry, the external state is mapped to some local operator at the origin. By Lorentz invariance, this operator cannot have any momentum dependence.

We can now evaluate the tadpole in this geometry using the Green’s function relevant to the Dirichlet/Neumann boundary conditions. The term \( e^{iq_\perp \cdot X(i)} \) produces a factor of \( (2)^{-q_\perp^2} \) when it contracts with itself. There can also be additional momentum dependent factors when \( e^{iq_\perp \cdot X(i)} \) contracts with other \( X \)'s in the boundary state and in the external state. These will produce additional factors of \( q_\perp^2 \). As we will see below, these factors will only make things more convergent, so we can ignore them. If we take just the momentum-dependent and \( W \)-dependent terms we get

\[ \int d^{25-p}q_\perp \left( \frac{kW^2_2}{1 + W^2_2} \right)^{\frac{q_\perp^2}{2W^2}} = \text{Const} \times \int_0^\infty dr r^{25-p-1} \left( \frac{kW^2_2}{1 + W^2_2} \right)^{\frac{1}{2}} \frac{2}{W^2}. \] (5.47)

Dropping the constant from the angular integral, this gives

\[ \frac{2}{W^2} \left[- \log \left( \frac{kW^2}{1 + W^2} \right) \right]^{\frac{1}{2}(p-25)}. \] (5.48)

Expanding this function around \( W = 0 \) gives

\[ \left( \frac{W}{\pi} \right)^{\frac{25-p}{2}} \frac{1}{W^2} + \mathcal{O}(W^{\frac{25-p-2}{2}}). \] (5.49)

We are interested in when this can be integrated near \( W = 0 \). Additional factors of \( W \)}
could arise from the external state map but these will only aid the convergence. From the expansion (5.49) we see that we can integrate (5.47) with respect to \( W \) near \( W = 0 \) if

\[
p \leq 22. \tag{5.50}
\]

Thus, for sufficiently many transverse dimensions, there are no divergences from the massless sector. As we discuss in Section 6, this constraint has a natural interpretation in terms of the long range behavior of the massless fields around the brane.

Now let’s look at the tachyon sector. From the calculation in the massless sector, we can see that the only effect of momentum dependence is to give an extra power of \( W^{(25-p)/2} \). Unfortunately, this is not enough to suppress the factor of \( 1/k^2 \) so one must still resort to some form of analytic continuation as we did in the D25-brane theory. Unlike the D25-brane case, however, this analytic continuation can now be used to make the diagram completely finite provided \( p \) is small enough. Recall that the terms which still diverged after analytic continuation were at worst of the form \( 1/W^5 \). Thus to make the tadpole integrable around \( W = 0 \), we must choose \( p \leq 16 \). While we have a natural interpretation for the constraint (5.50), this additional restriction of the dimensionality of the branes seems to be an artifact of the interplay between our somewhat ad hoc choice of analytic continuation and the details of the Witten OSFT. We do not believe that there is any universal significance to this constraint; indeed, it is not evident in the alternative string field theory discussed in section 5.5.

As in the D25-brane case, it is instructive to compare these results with the propagation of the boundary state along a closed string tube of length \( \frac{\pi}{W} \) and circumference \( 2\pi \). As before, the propagation of the boundary state is represented by

\[
\exp \left( -\frac{\pi}{W} (L_0 + \tilde{L}_0) \right) |B_p\rangle. \tag{5.51}
\]

Consider decomposing the boundary state into a sum over states of definite weight, \((h, h)\), and momentum, \( q_{\perp} \),

\[
|B_p\rangle = \sum_{h=-1}^{\infty} \int q^{25-p} q_{\perp} |B_p(q_{\perp}, h)\rangle. \tag{5.52}
\]
We can then write (5.51) as

\[ \sum_{h=-1}^{\infty} \int d^{25-\rho} q_\perp \exp \left( -\frac{\pi}{W} (2h + k^2_\perp) \right) |\mathcal{B}_p(q_\perp, h)|. \quad (5.53) \]

Now consider the different terms in the sum over \( h \). For \( h > 0 \), the term in the exponent \( 2h + k^2_\perp \) is always greater than zero. Thus the limit as \( W \to 0 \) is well defined. For \( h = 0 \) the limit \( W \to 0 \) is well defined for the region of integration where \( q^2_\perp > 0 \). As we saw above, the region of the integral around \( q^2_\perp = 0 \) is also defined for sufficiently small \( p \).

For the tachyon, however, we have \( h = -1 \). Now, whenever \( q^2_\perp < 2 \), the limit as \( W \to 0 \) is divergent. Thus the added momentum dependence does nothing to help the tachyon divergence and we must resort to analytic continuation.

### 5.4 Evaluation of the tadpole using oscillator methods

Having examined the one-point function using conformal field theory methods, we now evaluate it using the oscillator approach of [84]. In this section we primarily specialize to the D25-brane. We will comment on the lower dimensional branes at the end of the section. In section 5.4.1 we review the oscillator form of the two- and three-string vertices and use squeezed state methods to compute the one-loop tadpole in terms of infinite matrices of Neumann coefficients. In sections 5.4.2 and 5.4.3, we analyze the results using numerical and analytical methods and compare with our results from section 5.3.

#### 5.4.1 Oscillator description of the one-loop tadpole

We begin by writing an oscillator description for the tadpole diagram, following [15, 84]. The oscillator expressions for the the two- and three-string vertices \( |V_2\rangle \) and \( |V_3\rangle \) are squeezed states in the two-fold and three-fold tensor product of the string Fock space with itself [40, 86, 87, 46, 47, 88, 89]. Explicit formulae for these vertices are given below. In Feynman-Siegel gauge, \( b_0 |\Psi\rangle = 0 \), and the propagator is given by \( b_0 / L_0 \), which we can represent using a Schwinger parameter

\[ \frac{b_0}{L_0} = b_0 \int_0^\infty dT e^{-TL_0}. \quad (5.54) \]
We can use the vertices and the propagator to write the tadpole as

\[ |T\rangle = -g \int_0^\infty dT\, T^{1,2} V_2 \left| b_0^2 e^{-\frac{1}{2} T (T_0^{(1)} + T_0^{(2)})} \right| V_3 \rangle_{1,2,3}. \]  

(5.55)

The three-string vertex is given by

\[
|V_3\rangle = \int d^2p_1 d^2p_2 \left( \prod_{n} \left( -\frac{1}{2} c_n^{iV_{nm} a_n^{jV_{nm} a_n^{jV_{nm} a_n^{jV_{nm}}}}} - \frac{1}{2} b_0^{V_{nm} b_0^{V_{nm} b_0^{V_{nm} b_0^{V_{nm}}}}} \right) \right) \times \exp \left( -c_n^{iV_{nm} b_0^{V_{nm} b_0^{V_{nm} b_0^{V_{nm}}}}} \right) c_0^{iV_{nm} b_0^{V_{nm} b_0^{V_{nm} b_0^{V_{nm}}}}} |0; p_1\rangle_1 c_0^{iV_{nm} b_0^{V_{nm} b_0^{V_{nm} b_0^{V_{nm}}}}} |0; p_2\rangle_2 c_0^{iV_{nm} b_0^{V_{nm} b_0^{V_{nm} b_0^{V_{nm}}}}} |0; p_3 = -p_1 - p_2\rangle_3,
\]

(5.56)

where sums on all repeated indices are understood. The vacuum \( |0\rangle = c_1 |0\rangle \), where as before \( |0\rangle \) is the \( SL(2, \mathbb{R}) \)-invariant vacuum. The Neumann coefficients \( V_{nm}^{ij} \) and \( X_{nm}^{ij} \) are given by standard formulae \([86, 87, 89]\) and are tabulated in \([97]\) among other places. The two-string vertex \( |B_2\rangle \) is related to the usual two-string vertex \( |V_2\rangle \) through

\[ |\tilde{V}_2\rangle = (-1)^{N_f^{(1)}} |V_2\rangle \]

(5.57)

and has the following oscillator representation

\[
|\tilde{V}_2\rangle = \int d^2p \left( c_0^1 + c_0^2 \right) \exp \left( -a_n^{iC_{nm} a_n^{jC_{nm}} - c_n^{iC_{nm} b_n^{jC_{nm}} - c_n^{iC_{nm} b_n^{jC_{nm}}}}} \right) |0; p\rangle_1 |0; -p\rangle_2,
\]

(5.58)

where

\[ C_{nm} = \delta_{nm} (-1)^n. \]

(5.59)

The extra sign in (5.57) arises from the fact that the \( b_0 \) in the propagator anticommutes with Grassmann-odd states in \( |V_2\rangle \) \([15]\). Note that in (5.55) we have used the property

\[ e^{-T L_0^{(1)}} |\tilde{V}_2\rangle = e^{-\frac{T}{2} (L_0^{(1)} + L_0^{(2)})} |\tilde{V}_2\rangle \]

(5.60)

to make the expression more symmetric.

We may represent the action of the propagator on the vertex by multiplying each term in the vertex by the appropriate function of its conformal weight. That is, since \( [L_0^{(k)}, a_n^{iV_{nm} a_n^{jV_{nm}}} ] = \)
(n\delta_{ik} + m\delta_{jk})a_n^{i\dagger}V_{mn}a_m^{j\dagger}$, we have
\begin{equation}
e^{-Tl_0^{(k)}} \exp \left( -\frac{1}{2}a_{n}^{i\dagger}V_{mn}a_{m}^{j\dagger} \right) \left| 0; 0 \right>_{1} c_{0}^{1} \left| 0; 0 \right>_{2} c_{0}^{2} \left| 0; 0 \right>_{3} = e^{T} \exp \left( -\frac{1}{2}a_{n}^{i\dagger}V_{mn}a_{m}^{j\dagger} e^{-T(n\delta_{ik} + m\delta_{jk})} \right) \left| 0; 0 \right>_{1} c_{0}^{1} \left| 0; 0 \right>_{2} c_{0}^{2} \left| 0; 0 \right>_{3} , \tag{5.61}\end{equation}

with an analogous result in the ghost sector. As in [84], we denote a vertex which has absorbed a propagator in this fashion by a hat, so that
\begin{equation}\hat{V}_{nm}^{i\times j}(T_{k}, T_{l}) \equiv e^{-nT_{k}/2} V_{nm}^{i\times j} e^{-mT_{l}/2}. \tag{5.62}\end{equation}

Using the explicit representations (5.54, 5.56, 5.58) of the vertices and propagators, the evaluation of $|T\rangle$ at fixed modular parameter $T$ reduces to computing the inner product of two squeezed states. The formula for squeezed state expectation values is given by [98]
\begin{equation}\left< 0 \right| e^{-\frac{1}{2}aSa - \mu a^{\dagger} - \frac{1}{2}V a^{\dagger}} |0\rangle = \det(1 - SV)^{-1/2} e^{-\frac{1}{2}\mu(1 - SV)^{-1}} \delta_{\mu}, \tag{5.63}\end{equation}

where $a$ and $a^{\dagger}$ are understood to be vectors, and matrix multiplication is implicit. We also need the corresponding fermionic formula
\begin{equation}\left< 0 \right| e^{SB} e^{-\lambda_{\alpha} b^{\dagger} - \epsilon^{\alpha} \lambda_{\alpha} - \epsilon^{\dagger} X b^{\dagger}} |0\rangle = \det(1 - SX) e^{-\lambda_{\alpha} (1 - SX)^{-1}} \delta_{\lambda_{\alpha}} \tag{5.64}\end{equation}

where we have suppressed the ghost zero mode dependence.

We now apply these formulas to the tadpole. Since the diagram factorizes into separate matter and ghost portions, we will discuss the matter and ghost parts in turn. The matter portion of the expectation value in the integrand of (5.55) is given by
\begin{equation}\int d^{26}q e^{-T(q^{2} - 1) - q^{2}V_{00}^{11}} \left< 1, 2 \left| e^{-\frac{1}{2}aSa - \frac{1}{2}a^{\dagger}Va^{\dagger} - \mu a^{\dagger} - \frac{1}{2}V a^{\dagger}} |0\rangle_{1, 2, 3} \right. \tag{5.65}\end{equation}
where we have defined
\[ S = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \]  \( (5.66) \)
\[ \tilde{V} = \begin{pmatrix} \tilde{V}^{11}(T,T) & \tilde{V}^{12}(T,T) \\ \tilde{V}^{21}(T,T) & \tilde{V}^{11}(T,T) \end{pmatrix}, \]  \(5.67\)
\[ \mu = \begin{pmatrix} \tilde{V}^{21}(T,0) \\ \tilde{V}^{12}(T,0) \end{pmatrix} a^{3\dagger} + \begin{pmatrix} \tilde{V}^{11}_{n0}(T,0) - \tilde{V}^{12}_{n0}(T,0) \\ \tilde{V}^{21}_{n0}(T,0) - \tilde{V}^{11}_{n0}(T,0) \end{pmatrix} q. \]  \(5.68\)

Using the squeezed state formula (5.63) and carrying out the resulting Gaussian integral over momentum (with the appropriate analytic continuation), equation (5.65) becomes
\[ e^{T} \left( \frac{2\pi}{Q \det(1 - \tilde{S}\tilde{V})} \right)^{13} \exp \left( -\frac{1}{2} a_{n}^{\dagger} M_{nm} a_{m} \right) |0\rangle, \]  \(5.69\)

where
\[ M_{nm} = Q_{nm} - \frac{Q_{n}Q_{m}}{Q}, \]

and
\[ Q_{nm} = V_{nm}^{11} + \begin{pmatrix} \tilde{V}^{11}_{n}(0,T) \\ \tilde{V}^{21}_{n}(0,T) \end{pmatrix}^{T} \frac{1}{1 - \tilde{S}\tilde{V}} S \begin{pmatrix} \tilde{V}^{21}_{m}(T,0) \\ \tilde{V}^{12}_{m}(T,0) \end{pmatrix}, \]  \(5.70\)
\[ Q_{n} = V_{0n}^{11} - V_{0n}^{23} + \begin{pmatrix} \tilde{V}^{11}_{0}(0,T) - \tilde{V}^{21}_{0}(0,T) \\ \tilde{V}^{12}_{0}(0,T) - \tilde{V}^{11}_{0}(0,T) \end{pmatrix}^{T} \frac{1}{1 - \tilde{S}\tilde{V}} S \begin{pmatrix} \tilde{V}^{21}_{n}(T,0) \\ \tilde{V}^{12}_{n}(T,0) \end{pmatrix}, \]  \(5.70\)
\[ Q = 2V_{00}^{11} + 2T + \begin{pmatrix} \tilde{V}^{11}_{0}(0,T) - \tilde{V}^{21}_{0}(0,T) \\ \tilde{V}^{12}_{0}(0,T) - \tilde{V}^{11}_{0}(0,T) \end{pmatrix}^{T} \frac{1}{1 - \tilde{S}\tilde{V}} S \begin{pmatrix} \tilde{V}^{21}_{0}(T,0) - \tilde{V}^{12}_{0}(T,0) \\ \tilde{V}^{11}_{0}(T,0) - \tilde{V}^{12}_{0}(T,0) \end{pmatrix}. \]  \(5.70\)

Meanwhile, the ghost portion of the expression (5.55) is
\[ _{1,2} \langle 0 | e^{cS_{b}e^{-c\tilde{F}}b\lambda - c^{1}b\lambda^{1} - c^{2}b\lambda^{2} - c^{3}X^{11}b^{3\dagger}} |0\rangle_{1,2,3}, \]  \(5.71\)
with

\[ \tilde{X} = \begin{pmatrix} \hat{X}^{11}(T, T) & \hat{X}^{12}(T, T) \\ \hat{X}^{21}(T, T) & \hat{X}^{11}(T, T) \end{pmatrix}, \]

and

\[ \lambda_c = c^t \begin{pmatrix} \hat{X}^{12}(0, T) & \hat{X}^{21}(0, T) \end{pmatrix}, \]
\[ \lambda_b = \begin{pmatrix} \hat{X}^{21}(T, 0) \\ \hat{X}^{12}(T, 0) \end{pmatrix} b^t + \begin{pmatrix} \hat{X}^{21}_{n0}(T, 0) \\ \hat{X}^{12}_{n0}(T, 0) \end{pmatrix} b^0. \tag{5.72} \]

Using the squeezed state formula (5.64), we find

\[ \det(1 - S \tilde{X}) \exp \left( -c^t \bar{R}_{nm} b^t_m \right) c_0 |\bar{0}\rangle, \tag{5.73} \]

where we have defined

\[ R = X^{11} + \begin{pmatrix} \hat{X}^{12}(0, T) & \hat{X}^{21}(0, T) \end{pmatrix} \frac{1}{1 - S \tilde{X}} S \begin{pmatrix} \hat{X}^{21}(T, 0) \\ \hat{X}^{12}(T, 0) \end{pmatrix}. \tag{5.74} \]

Up to an overall constant, then, \(|\mathcal{T}\rangle\) is given by,

\[ |\mathcal{T}\rangle = \int_0^\infty dT e^{T} \frac{\det(1 - S \tilde{X})}{(Q \det(1 - SV))^{13}} e^{-\frac{1}{2} a^t M a - c^t R b c_0} |\bar{0}\rangle. \tag{5.75} \]

\subsection{5.4.2 Divergences in the tadpole}

We are now interested in evaluating the tadpole integral (5.75). At any finite value of \(T\), the integrand is a state in the string Fock space with finite coefficients for each zero-momentum state. The matrices \(X^{rs}_{nm}\) and \(V^{rs}_{nm}\) are infinite-dimensional, and while we have expressions for each matrix element, we cannot analytically compute the integrand of (5.75). Nonetheless, by truncating the matrices \(X\) and \(V\) at finite oscillator level, we can numerically estimate the value of the integrand. Empirical evidence indicates that the integrand converges exponentially quickly as the level of truncation is increased, where the rate of convergence depends on the modular parameter \(T\). For large \(T\), we can also consider expanding (5.75)
Figure 5-6: $\log S_0$ vs. $s = 1/T$. The straight line plotted has a slope of $2\pi^2$, which is the analytical prediction from section 3.

as a function of $e^{-T}$. The contribution at order $e^{(1-k)T}$ represents the portion of the tadpole arising from the propagation of open string states at level $k$ around the loop.

It is immediately clear from (5.75) that the integral for the tadpole diverges as $T \to \infty$ due to the term of order $e^T$. This term arises from the propagation of the open string tachyon around the loop. In an expansion in terms of the level of the field propagating around the loop, it is easy to see how this divergence can be removed by analytic continuation. Unlike the divergence from the closed string tachyon considered in the CFT calculation, the field causing this divergence is explicitly considered as one of the fundamental degrees of freedom in OSFT, and thus the associated tachyon divergence can easily be removed by analytic continuation. This analytic continuation is most transparent when we do the calculation level by level in fields. The method of level truncation by fields has previously been used in many OSFT calculations, and is probably the most effective way of dealing with the open string tachyon; because we are more interested in closed string physics here, however, we will not pursue this approach further and we will instead approximate (5.75) by truncation in oscillator level, which leads to a simpler calculation of the integrand at small $T$.

From the CFT analysis, we expect to see a divergence in the integrand of (5.75) as $T \to 0$. Indeed, we see numerical evidence for such a divergence. Let us first consider the
scalar portion of the state (5.75),

\[ S_0(T) = e^T \frac{\det(1 - S \tilde{X})}{\det(1 - S V)^{13} Q^{13}}. \]  \hspace{1cm} (5.76)

Our numerical computations of $S_0$ are plotted in figure 5.4.2. We find that as $T \to 0$, $S_0$ first increases exponentially, and then falls off, taking a finite value at $T = 0$. Both aspects of this behavior have simple explanations. The exponential increase is the divergence arising from the negative mass-squared of the closed string tachyon, and takes the form $S_0 \propto e^{B/T}$.

We can compute the coefficient $B$ numerically, as we discuss below. The fall-off occurs because as $T \to 0$ we consider smaller and smaller world-sheet distances, which require higher and higher oscillator modes to resolve. Level truncation essentially acts as a UV cutoff, rendering the divergence finite as $T \to 0$.

From the conformal field theory analysis, we expect that if we could compute $S_0(T)$ to infinite level we would find

\[ S_0(T) \sim e^{2\pi^2/T}. \]  \hspace{1cm} (5.77)

In figure 5.4.2 we plot the quantity $\log S_0$ versus $s = 1/T$ at successively higher levels. The straight line with slope $2\pi^2$ is the CFT prediction for the infinite level behavior in the region $s \gg 1$. As is evident from the plot, our numerical data is in good accord with this prediction. As expected, as the level increases the region of linear behavior becomes larger, while the fall-off becomes less rapid, so that successively better estimations of $B$ are obtained.

Our data, considered as a function of level, converges exponentially quickly; that is, to a good approximation, we may write the amplitude obtained at finite level as

\[ \log S_0^{(L)}(T) = \log S_0^{(exact)}(T) - A e^{-r(T)L}, \]  \hspace{1cm} (5.78)

where $L$ is the level. The rate of convergence $r(T)$ is approximately given by

\[ r(T) \approx 0.002 + 0.949 T. \]  \hspace{1cm} (5.79)

This approximation to the function $r(T)$ is obtained by numerically determining the rate of convergence for several points at fixed values of $T$ using the ansatz $r(T) = A + B T$. Since the
first terms dropped in the level $L$ truncation are of order $e^{-LT}$, we indeed expect $r(T) = T$, in close agreement with our empirical data. These results are consistent with the general empirical observation that, while integrated amplitudes converge polynomially in $1/L$ (when finite), the integrand itself converges exponentially in $L$ with a rate that depends on the modular parameter [84].

Looking at the region $T > .007$, we find the best fit given by equations (5.78) and (5.79) gives

$$B \approx .9911 \times 2\pi^2.$$  \hspace{1cm} (5.80)

This is within less than 1% of the value $2\pi^2$ found analytically in section 3, so we have a very strong agreement between our two methods of calculation for the leading rate of divergence.

5.4.3 The matrices $M$ and $R$

In this subsection, we consider the form of the matrices $M$ and $R$ in the exponential part of (5.75). We show that in the limit $T \to 0$, these matrices reduce to $C_{nm}$, so that just as in the CFT calculation, the leading divergence is associated with a Shapiro-Thorn state. Furthermore, we compute the first subleading terms in $M$ and $R$. We show that these subleading terms agree with what is expected from the CFT calculation.

In the small $T$ limit, the loop of the tadpole diagram reduces to an identification of the left and right halves of the incoming string. Therefore, the matrices $M_{nm} = Q_{nm} + Q_n Q_m / Q$ and $R_{nm}$ should both limit to $C_{nm}$ in order to describe the identification of the sides of the string. For any finite level, we can demonstrate analytically that this is indeed the case for the matrix $M$ of (5.71) and $R$ of (5.74). Consider the identity

$$\begin{pmatrix} J_2 & J_3 \\ J_3 & J_2 \end{pmatrix} \left( \begin{array}{cc} 1 - CJ_3 & -CJ_1 \\ -CJ_1 & 1 - CJ_2 \end{array} \right)^{-1} = \left( \begin{array}{cc} C & C \end{array} \right)$$  \hspace{1cm} (5.81)

which holds for any matrices $J_i$ that satisfy $J_1 + J_2 + J_3 = C$.

In level truncation, we can simply apply the identity (5.81) at $T = 0$, with $J_i$ equal to
\( X^{11} \) and \( V^{11} \) in the ghost and matter sectors respectively. This gives us the result

\[
\begin{align*}
R_{nm}(T = 0) &= C_{nm} \\
M_{nm}(T = 0) &= C_{nm}.
\end{align*}
\tag{5.82}
\]

Since the sum condition needed to demonstrate this result is linear, we find that \( M(T = 0) \) and \( R(T = 0) \) are equal to \( C \) level by level when we calculate them in level truncation. This reproduces the formula we found in equation (5.41). Unfortunately, one can show that there is no analogue of equation (5.81) without level truncation. This makes the verification of (5.82) without truncating the matrices quite difficult. We discuss some of the subtleties of comparing the level truncated analysis and the infinite dimensional matrix analysis in appendix E.

We can also look at the first order corrections to \( M_{nm} \) and \( R_{nm} \). Doing a linear fit near \( T = 0 \) to the various coefficients gives for the first five diagonal elements at level 25,

\[
\begin{align*}
M_{11} &\approx -0.999999 + 1.0000031 \, (T) \\
M_{22} &\approx 0.9999994 - 1.0000095 \, (2T) \\
M_{33} &\approx -0.9999982 + 1.0000265 \, (3T) \\
M_{44} &\approx 0.9999953 - 1.0000349 \, (4T) \\
M_{55} &\approx -0.9999912 + 1.0000694 \, (5T).
\end{align*}
\tag{5.83}
\]

The off diagonal elements are consistent with being zero to \( \mathcal{O}(T) \). We find similar behavior in the ghost sector. At level 150 we find the following coefficients for a linear fit near \( T = 0 \)

\[
\begin{align*}
R_{11} &\approx -0.999995 + 0.999775 \, (T) \\
R_{22} &\approx 0.999989 - 0.999642 \, (2T) \\
R_{33} &\approx -0.999985 + 0.999761 \, (3T) \\
R_{44} &\approx 0.999979 - 0.999627 \, (4T) \\
R_{55} &\approx -0.999975 + 0.999741 \, (5T). \tag{5.84}
\end{align*}
\]

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As with the matter sector, the off-diagonal elements are approximately zero to $\mathcal{O}(T)$.

These coefficients suggest that the first order corrections to $M_{nm}$ and $R_{nm}$ are given by

$$
M_{nm} = C_{nm} - m C_{nm} T + \mathcal{O}(T^2)
$$

$$
R_{nm} = C_{nm} - m C_{nm} T + \mathcal{O}(T^2).
$$

(5.85)

It might seem that this formula should be easy to derive by just Taylor expanding the expressions for $M_{nm}$ and $R_{nm}$ to first order in $T$. In level truncation, this expansion is straightforward and yields the result

$$
M^L_{nm} = C_{nm} - 2m C_{nm} T + \mathcal{O}(T^2)
$$

$$
R^L_{nm} = C_{nm} - 2m C_{nm} T + \mathcal{O}(T^2),
$$

(5.86)

where we have put the superscript $L$ in the matrices to emphasize that this expression is only valid in level truncation. Note that equations (5.85) and (5.86) are off by a factor of two! This disagreement stems from the fact that one must take the level to infinity and then expand around $T = 0$. Expanding around $T = 0$ and then taking the level to infinity gives incorrect results. For a more complete discussion of this issue see appendix E.

We can now compare (5.85) with our results from the conformal field theory method. As it turns out the linear correction comes entirely from the map that acts on the external state. The corrections that arise from the map that acts on the boundary state only enter at $\mathcal{O}(T^2)$. Using the notation of section 5.3.2 we can write the external state map as $f(z) = h(z) \circ z(v)$. At $T \to 0$, $f(z)$ is just the map corresponding to the identity state

$$
f(z) \bigg|_{T=0} = \frac{2z}{1 - z^2}.
$$

(5.87)

As it turns out, the first order correction in $T$ takes a simple form

$$
f(z) = f(z) \bigg|_{T=0} \circ \left( \frac{z}{1 + T/2} \right) + \mathcal{O}(T^2).
$$

(5.88)
Thus to $\mathcal{O}(T)$ we can write

$$f(z) \circ |A\rangle = f(z)\bigg|_{T=0} \circ \left(\frac{z}{1+T/2}\right) \circ |A\rangle.$$  

(5.89)

Since we also have

$$\left(\frac{z}{1+T/2}\right) \circ |A\rangle = (1+T/2)^{-L_0}|A\rangle,$$  

(5.90)

the change in the tadpole state from this correction to the external state map can be accounted for by just taking

$$|T\rangle\bigg|_{T=0} \rightarrow (1+T/2)^{-L_0} \left(|T\rangle\bigg|_{T=0}\right) + \mathcal{O}(T^2),$$  

(5.91)

where we are only setting $T = 0$ in the map from the external state to the geometry. Acting with $(1+T/2)^{-L_0}$ on an arbitrary state is straightforward since one just takes, for example,

$$a_m^\dagger \rightarrow (1+T/2)^{-m}a_m^\dagger,$$  

(5.92)

and similarly for the ghosts. Thus the terms in the exponent become

$$-\frac{1}{2}a_m^\dagger C_{mn}a_n^\dagger - c_m C_{mn}b_n \rightarrow -\frac{1}{2}(1+T/2)^{-2m}a_m^\dagger C_{mn}a_n^\dagger - (1+T/2)^{-2m}c_m^\dagger C_{mn}b_n^\dagger$$

$$= -\frac{1}{2}a_m^\dagger (C_{mn} - mT C_{mn})a_n^\dagger - c_m^\dagger (C_{mn} - mT C_{mn})b_n^\dagger + \mathcal{O}(T^2),$$

which reproduces what we found in equations (5.85).

### 5.4.4 Summary of oscillator calculation

We have given an analytic expression for the tadpole in terms of infinite-dimensional matrices of Neumann coefficients. A divergence associated with the open string tachyon arises for large Schwinger parameter $T$, and can be dealt with by straightforward analytic continuation when the amplitude is expanded in the level of the open string field propagating in the loop. We have numerically analyzed the behavior of the tadpole integrand as $T \rightarrow 0$. Our numerical approximations have reproduced to a high degree of accuracy the leading divergence in this
limit. Near $T = 0$ the tadpole takes the form

$$|\mathcal{T}\rangle \sim \int_0 dT e^{2\pi^2/T} e^{-\frac{1}{2} a^\dagger a - c^\dagger c b^\dagger} |0\rangle + \ldots$$

(5.93)

This leading term is the zero-momentum Shapiro-Thorn state $|\Phi\rangle = e^{-\frac{1}{2} a^\dagger a - c^\dagger c b^\dagger} |0\rangle$ that describes the closed-string tachyon [71, 72]. The oscillator calculation thus agrees with the conformal calculation of section 3.

We have also identified the leading corrections (in $T$) to the limit $M, R = C$. These corrections are linear in $T$ and do not represent Shapiro-Thorn states for massless closed string fields. In terms of the conformal calculation discussed in the previous section, these corrections may be understood as coming from the conformal transformation of the incoming string.

Ideally we would like to be able to see the Shapiro-Thorn states for the graviton and dilaton. These could arise from terms in $M$ that vanish as $e^{-2\pi^2/T}$ as $T \to 0$; such terms can give a finite contribution to the amplitude because of the exponential divergence in the scalar portion of the amplitude $S_0$. However, such exponentially dying corrections are not only many orders of magnitude smaller than the leading corrections, but also many orders of magnitude smaller than the error introduced by level truncation. Without analytic control over the matrices $\tilde{V}$ and $\tilde{X}$ or some way of explicitly removing the divergence due to the closed string tachyon, we cannot examine the subleading terms directly in numerical experiments.

Finally, we note that it would be easy to redo the calculations for the lower-dimensional branes. We would simply introduce a few minus signs for the Dirichlet coordinates and eliminate the momentum integrals for the transverse directions. Without some new approach to the calculation, however, all we would find is the same divergence from the closed string tachyon. We would not be able to observe that the massless sector was no longer contributing to the divergence.
Figure 5-7: The relevant vertices in OCSFT. a) The open string propagator is the same as in OSFT. b) The closed string propagator is a tube of circumference $2\pi$. Its length, $s$, is integrated from 0 to $\infty$. c) The open string vertex is similar to the OSFT vertex but world-sheet stubs of length $2\pi$ are added on each side. d) The open-closed string transition vertex represents an open string which turns into a closed string. e) The vertex representing a closed string being absorbed by the brane.

5.5 The open string tadpole in open-closed string field theory

To understand the physics hidden in the OSFT tadpole, it is useful to study the same diagram in a version of string field theory which includes closed strings explicitly. This open-closed string field theory (OCSFT), due to Zwiebach [99, 100], has the nice property that it divides up moduli space so that there are never any pinched-off world-sheets. This is convenient for us, since it implies that in this theory the tube at the bottom of the OSFT tadpole will be explicitly written as a closed string propagator. For a related discussion in a different version of string field theory, see [101].

OCSFT is complicated by the fact that its action is non-polynomial but, since we are working only to order $g$, we will only need to consider a few terms. In fact since we are only interested in the divergent part of the tadpole, we truncate the theory to just the terms in the action relevant to the $T \to 0$ part of the tadpole moduli space. These terms may be summarized as follows

1. Start with the usual OSFT action but modify the cubic term by adding to the vertex strips of length $\pi$ to each of the three sides. The resulting vertex is shown in figure 5-7 c. The stubs added to the Witten vertex eliminate the region of the tadpole near $T \to 0$. 

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2. Add to the theory a set of closed string fields $\Phi$ with kinetic term $\langle \Phi | (c_0 - \bar{c}_0)(Q_B + \bar{Q}_B) | \Phi \rangle$. The field $\Phi$ is assumed to satisfy $(b_0 - \bar{b}_0)\Phi = 0$. We also impose the analogue of FS gauge $(b_0 + \bar{b}_0)\Phi = 0$. The propagator of the theory is given by integrating over closed string tubes of length $s$ with insertions of $b_0\bar{b}_0$.

3. Add the open-closed string vertex shown in figure 5-7 d. We will denote this vertex $\langle V_{OC} | \Psi \rangle | \Phi \rangle$.

4. Add a vertex in which a closed string is absorbed by the brane. This is given by $\langle \Phi | (c_0 - \bar{c}_0)e^{-\pi(\hat{L}_0 + \hat{\bar{L}}_0)} | \mathcal{B} \rangle$. Pictorially this vertex is shown in figure 5-7 e.

We can now consider the OCSFT version of the open string tadpole. Using the new open string vertex in figure 5-7 c, and the open string propagator one can construct a tadpole diagram that looks just like the one we considered on OSFT. The only difference is that because of the stubs, the loop cannot have length $T < 2\pi$. The part of the tadpole moduli space near $T \to 0$ is covered by a new diagram formed by gluing an open string propagator to an open-closed vertex, then attaching a closed string propagator and finally capping the diagram with the vertex in figure 5-7 e. The resulting diagram is pictured in figure 5-8.

Note that the dependence on $s$ is quite simple since it is just the propagation of the boundary state a distance $s$. Thus everything to the right of the open-closed vertex can be represented as

$$
(b_0 + \bar{b}_0) \int_{\pi}^{\infty} ds' e^{-s'(\hat{L}_0 + \hat{\bar{L}}_0)} | \mathcal{B} \rangle.
$$

(5.94)

As in the OSFT version of the tadpole, this integral diverges because of the weight $(-1, -1)$ field in $| \mathcal{B} \rangle$. As in OSFT we can treat this divergence by analytic continuation in the weight
of the tachyon field. This analytic continuation is equivalent to the replacement

\[
(b_0 + \tilde{b}_0) \int_\pi^\infty ds' e^{-s'(L_0 + \tilde{L}_0)} |B\rangle = \frac{b_0 + \tilde{b}_0}{L_0 + \tilde{L}_0} e^{-\pi(L_0 + \tilde{L}_0)} |B\rangle
\]  

(5.95)

where the operator \((L_0 + \tilde{L}_0)^{-1}\) is defined on all states that are not weight \((0, 0)\). This analytic continuation is closer in spirit to the analytic continuation used in Section 4 to remove the open string tachyon divergence than the analogous analytic continuation used in Section 3 to deal with the closed string tachyon, since the closed string tachyon is explicitly included among the fundamental degrees of freedom in OCSFT. We can also have divergences from the massless sector, but as we showed for the OSFT tadpole, these are only relevant for the D25, D24 and D23-branes and we postpone discussion of these cases until the end of the section.

Now consider our truncated action (we drop the open string three-vertex since it is not relevant here)

\[
S[\Psi, \Phi] = \frac{1}{2} \int \Psi \ast Q_B \Psi + \frac{1}{2} \langle \Phi | (c_0 - \tilde{c}_0) (Q_B + \tilde{Q}_B) | \Phi \rangle + g \langle \Phi | (c_0 - \tilde{c}_0) e^{-\pi(L_0 + \tilde{L}_0)} | B \rangle.
\]

(5.96)

Note that the closed string tadpole appears even at the classical level. We can try to shift the closed string field by \(\Phi \to \Phi + \delta \Phi\) to cancel this tadpole. This gives

\[
\delta S[\Psi, \Phi] = g \langle \Phi || \Psi \Phi \rangle + \langle \Phi | (c_0 - \tilde{c}_0) (Q_B + \tilde{Q}_B) | \delta \Phi \rangle + \text{const.}
\]

(5.97)

Notice that if we can find a \(\delta \Phi\) such that \((Q_B + \tilde{Q}_B) | \delta \Phi \rangle = -e^{-\pi(L_0 + \tilde{L}_0)} | B \rangle\), then the closed string tadpole will be canceled. We then get the new action

\[
S[\Psi, \Phi + \delta \Phi] = \frac{1}{2} \int \Psi \ast Q_B \Psi + \frac{1}{2} \langle \Phi | (c_0 - \tilde{c}_0) (Q_B + \tilde{Q}_B) | \Phi \rangle + g \langle \Phi || \Psi \rangle + g \langle \Phi || \Psi \delta \Phi \rangle.
\]

(5.98)

Notice that the closed string tadpole is now eliminated, but there is now a new contribution to the open string tadpole.
Let’s recalculate the open string tadpole in the shifted theory. The original diagram that we calculated before is now gone because the closed string tadpole (which made up the right half of the diagram) has been canceled. However, there is a new diagram coming from the term \( g(V_{OC}|\Psi)\delta\Phi \). This diagram may be thought of as the original open string tadpole, but with the closed string propagator chopped off and the state \(-|\delta\Phi\rangle\) stuck onto the end. Note that since this diagram has no closed string modulus to integrate, it is finite as long as \(|\delta\Phi\rangle\) is finite. This fact will be useful for us when we discuss the cases where we have divergences from the massless sector in the original tadpole.

Now, as discussed in [32], the equation \( (Q_B + \tilde{Q}_B)|\Phi\rangle = -e^{\pi(L_0 + \tilde{L}_0)}|B\rangle \) is equivalent to the linearized Einstein’s equations in the background of the brane. Thus the new open string tadpole represents a coupling between the closed string background and the the open strings.

For OSFT this may not seem important since there is no closed string background to shift. However, what makes this discussion relevant for OSFT is that the open string tadpole after the shift in the closed string background is actually equal to the open string tadpole before we made the shift. This is seen by solving the equation for \(|\Phi\rangle\) to get

\[
|\Phi\rangle = -\frac{b_0 + \tilde{b}_0}{L_0 + \tilde{L}_0} e^{\pi(L_0 + \tilde{L}_0)}|B\rangle.
\]

Thus, replacing the closed string propagator with \(-|\Phi\rangle\) is equivalent to doing nothing. The implication for the open string tadpole in OSFT is that as long as the diagram is finite it naturally incorporates the linearized shift in the closed string background.

This leads to the question: what can we say about the tadpole diagrams which diverge because of the massless sector? For the finite diagrams the closed string propagator essentially represents the inverse of the BRST operator. For the divergent diagrams this representation is not defined when acting on the boundary state. To cure this problem one must invert the BRST operator by hand. To see how this is done it is useful to note how BRST invariance is maintained in the shifted and unshifted theory. Recall that in string theory BRST invariance for scattering diagrams reduces to the fact that exact states should decouple from on-shell states. For the tadpole diagram this simply implies that the tadpole should be annihilated by the BRST operator.

In the unshifted theory, when \( Q_B + \tilde{Q}_B \) is pulled through the OCSFT tadpole diagram,
it picks a contribution from the closed string propagator given by

$$\int_\pi^\infty ds' \left\{ Q_B + \tilde{Q}_B, b_0 + \tilde{b}_0 \right\} e^{-s'(L_0 + \tilde{L}_0)} |B\rangle = \int_\pi^\infty ds' (L_0 + \tilde{L}_0) e^{-s'(L_0 + \tilde{L}_0)} |B\rangle$$

$$= - \int_\pi^\infty ds' \frac{\partial}{\partial T} e^{-s'(L_0 + \tilde{L}_0)} |B\rangle$$

$$= - e^{-s'(L_0 + \tilde{L}_0)} \big| s=\pi \big| B\rangle.$$

(5.100)

As with the open string, we only pick up contributions at the endpoints of integration. The contribution at $s = \pi$ cancels with a surface term from a part of the tadpole moduli space that we haven’t included.

In the shifted theory, we have replaced the closed string propagator with a surface term. Acting on the surface term with $Q_B + \tilde{Q}_B$ we get

$$- (Q_B + \tilde{Q}_B)|\delta \Phi\rangle = e^{-\pi(L_0 + \tilde{L}_0)} |B\rangle,$$

(5.101)

which is the same surface term at $s = \pi$ that we found in (5.100). Note, though, that (5.101) is the same equation that we used to find $|\Phi\rangle$ in the first place. Thus the condition for BRST invariance essentially determines the surface term.

This fact is quite useful for studying the tadpole for the cases where it diverges. If we take one of the OSFT tadpoles which diverges, we can remove the small $T$ region of integration and replace it with a surface term. By the above discussion, this surface term is mostly determined by BRST invariance. We will employ this fact in section 5.6.3 where we study the physics hidden in the divergent diagrams.

### 5.6 Divergences and closed strings

In this section we discuss the various divergences which arise in the one-loop tadpole as $T \to 0$, and the role which closed strings play in the structure of the tadpole. In subsection 5.6.1 we discuss the leading divergence in the tadpole and the closed string tachyon which is responsible for this divergence. In subsection 5.6.2 we discuss the massless closed string modes and the piece of the tadpole arising from them. In subsection 5.6.3 we study the physics hidden in the divergent diagrams using BRST invariance. Finally, in subsection
5.6.4 we discuss some further problems which arise at two loops in OSFT.

5.6.1 The leading divergence and the closed string tachyon

In both the conformal field theory and oscillator calculations, we found a leading divergence in the open string tadpole which arises from the region of the modular integration near \( T = 0 \). In terms of the dual parameter \( s = \pi^2/T \), the divergence arises from an integral of the form

\[
\int_{\infty}^{s} ds \left[ e^{2s} \exp \left( -\frac{1}{2} a^{\dagger} \cdot C \cdot a \cdot a^{\dagger} - c^{\dagger} \cdot C \cdot b^{\dagger} \right) |0\rangle + \text{subleading terms} \right]. \tag{5.102}
\]

This divergent type of integral is a standard problem when we have a Schwinger parameter associated with a tachyonic state. In principle, we would like to simply analytically continue the integral, using

\[
\int_{\Lambda}^{\infty} e^{\alpha s} ds \rightarrow -\frac{1}{a} e^{\alpha \lambda}. \tag{5.103}
\]

Since the closed string channel associated with the divergent integral is not included explicitly in OSFT, however, it is rather subtle to carry out an analytic continuation of this type. In the CFT calculation, we can do this explicitly once we have expanded around \( T = 0 \) as in (5.36). As suggested in Section 3, the terms in the small \( T \) expansion associated with the tachyon component of the boundary state can be separately analytically continued. Even here, however, we run into difficulties, and have to resort to considering lower dimensional branes to ensure a completely finite diagram.

Moreover, the expansion around \( T = 0 \) is a difficult starting point for any exact or approximate calculation of the complete tadpole diagram. In order to get a numerical approximation to the tadpole amplitude including the analytic continuation for the closed string tachyon, it is necessary to break the modular integral into several parts. The integral for \( T > T_0 \) is finite (after the open string tachyon is analytically continued as discussed in Section 4), and can in principle be computed approximately using level truncation on fields in the oscillator formalism. The integral for \( T < T_0 \) can be approximately computed when \( T_0 \) is small using the small \( T \) expansion and an explicit analytic continuation of the tachyon term. While this approach allows us to deal with the divergence from the closed
string tachyon "by hand" in the particular diagram considered here, we should emphasize that this method is not general, and for higher-loop diagrams it is much more difficult to see how analogous divergences can be controlled.

In the oscillator approach, it is even less clear how the divergence from the closed string tachyon can be treated. Since level truncation regulates the divergence, the region of the modular integral near $T = 0$ is softened, and the divergent piece cannot be precisely isolated. It seems that in the level-truncated theory, unlike in the CFT picture, there is no way to implement by hand an analytic continuation to deal with the closed string divergence. For this reason, we cannot use the oscillator approach to study other parts of the tadpole, such as the finite part and the part depending on massless closed strings. While this is unfortunate, this problem is an artifact of the closed string tachyon. It seems likely that in superstring field theory, this problem would not occur, so that the oscillator method would be a much more useful approach for analyzing detailed features of loop amplitudes.

It is also worth pointing out at this point that a small modification of the usual formulation of OSFT might make the closed string tachyon divergence much more tractable. If we were to explicitly formulate OSFT in terms of a Lorentzian world-sheet, the real Schwinger parameterization used here could be replaced by an integral with an imaginary exponent. Unlike the divergent integral $\int e^{as}$, the complex integral $\int e^{ias}$ is oscillatory. With such an oscillatory integrand, level truncation should simply suppress the integrand at large values of $s$, effectively regulating the theory and giving a finite result for integrals which diverge in the Euclidean formulation. Unlike the ad hoc approach used to implement analytic continuation in the CFT calculation of the tadpole, this approach would immediately generalize to all diagrams, and would in one stroke deal with the open string tachyon as well as the closed string tachyon. We will not pursue this approach further here, but it is an interesting possible avenue for further investigations.

5.6.2 Tadpole contributions from massless closed strings

Let us now turn our attention to the massless closed string modes. As mentioned in the previous subsection, the terms in the tadpole arising from these modes cannot be seen in the oscillator calculation without a new formulation of the theory. These terms do appear
explicitly, however, in the small $T$ expansion of the tadpole (5.40).

To understand the structure of the terms associated with massless closed strings it is helpful to consider the schematic form of the OCSFT calculation from Section 5. In OCSFT, it is clear that the open string tadpole arises directly from the closed string tadpole. The closed string tadpole in turn encodes the structure of the D-brane as a source for the closed string fields. The D-brane boundary state $|\mathcal{B}\rangle$ only couples to the closed string fields. If we forget about the open strings and solve the equations of motion for the closed string, the linearized equation of motion corresponds to the linearized gravity equations in the presence of the brane. For a $Dp$-brane source, the linearized gravitational equations take the schematic form

$$\partial^2 \phi(x) = \delta^{25-p}(x_\perp).$$  \hfill (5.104)

(We ignore here the details of the tensor structure of the full gravity multiplet in order to elucidate the underlying physics; the exact equations for the gravitational fields are written in the following subsection.) In momentum space, the solution of (5.104) is

$$\phi(k) = \frac{1}{k_\perp^2} \delta(k_\parallel).$$  \hfill (5.105)

In position space, the solution is just the usual

$$\phi(x) \propto r^{p-23}, \quad p \neq 23; \quad \phi(x) \propto \ln r, \quad p = 23.$$  \hfill (5.106)

As discussed in Section 5, the open string tadpole in OCSFT is unchanged if we explicitly shift the closed string background to cancel the tadpole. This corresponds to turning on a linearized gravity background of the form (5.105). In OCSFT, the open string tadpole then arises by acting on this closed string background with the open-closed interaction vertex. Thus, in OCSFT, we can naturally think of the open string tadpole as coming directly from the closed string background arising from the D-brane.

In OSFT, the analysis in Section 3 of the small $T$ expansion of the one-loop tadpole reveals a highly parallel structure to that just described. The terms in (5.36) associated with massless fields are those with a $T$ dependence of the form $1/T^2$. The integration measure $dT/T^2$ is proportional to the measure $ds$ for the dual (closed string) modular parameter. Unlike the
closed string tadpole, the open string tadpole only has support at vanishing momentum $p = 0$. As discussed in Section 3, the integral over $q_\perp$ in the closed string boundary state gives extra powers of $T$ in the modular integral, so that when $p \leq 22$ the modular integral is convergent. This is analogous to smoothing out the solution (5.105) by integrating with the measure $\int d^{25-p}k_\perp \sim \int k_\perp^{24-p}dk_\perp$ in the vicinity of the singular point $k_\perp$. For $p \leq 22$ the resulting integral is convergent, while for $p \geq 23$ the integral diverges.

This somewhat schematic discussion indicates that part of the open string tadpole can be seen as arising from the closed string background associated with the D$p$-brane. For $p \leq 22$, this piece of the tadpole is finite. In the following subsection, we consider D-branes for which this part of the tadpole diverges, and we develop the line of reasoning just described in more detail.

### 5.6.3 Divergent diagrams and BRST invariance

Having analyzed the case where the tadpole diagram is finite, we now turn to the cases where the diagram diverges\textsuperscript{2}. While we have no satisfactory way of regulating these diagrams, we can still extract some interesting physics from them. What we attempt to show is that BRST invariance mostly determines the physics hidden in the divergent part of the modular integral.

Recall from section 5.5 that, for the tadpole diagram, BRST invariance amounts to checking that $Q_B$ annihilates the tadpole state, $|\mathcal{T}\rangle$. Since $Q_B$ is a total derivative on moduli space, we will only pick up contributions at the boundaries of moduli space. In other words, we can only pick up surface terms at $T = 0$ and $T = \infty$. Such surface terms can, in fact, arise because of the divergences in the tadpole and are discussed in appendix D.

Since the $T \to 0$ limit of the tadpole diagram is divergent, we must introduce some sort of regulator. A very simple choice is to just cut off the modular integral at some minimum value $T_0$. In other words, we replace $|\mathcal{T}\rangle$ with the regulated state

$$|\mathcal{T}\rangle_{T_0} = \int_{T_0}^{\infty} |\mathcal{T}(T)\rangle.$$

\textsuperscript{2}We would like to thank Eva Silverstein for very useful discussions on the issues in this section and in Appendix A.1. A world-sheet discussion of issues related to the divergences we describe in this section will be given in [103]
where we have included the subscript \( T_0 \) to explicitly indicate that this is a regulated form of the tadpole. Since the subscript \( T_0 \) is cumbersome, we drop it in subsequent equations, but, throughout this section, we will assume that \(|\mathcal{T}\rangle\) is regulated in this way. Introducing the cutoff, \( T_0 \), explicitly breaks the BRST invariance of the diagram. Acting on \(|\mathcal{T}\rangle\) with \( Q_B \) gives a surface term at \( T_0 \). This is seen in the following calculation

\[
\langle \mathcal{T} | Q_B \rangle = \int_{T_0}^{\infty} dT \langle V_3 | b_0^{(2)} e^{-T L_0} | \tilde{V}_2 \rangle Q_B \\
= \int_{T_0}^{\infty} dT \frac{d}{dT} \langle V_3 | e^{-T L_0} | \tilde{V}_2 \rangle \\
= \langle V_3 | e^{-T_0 L_0} | \tilde{V}_2 \rangle.
\] (5.107)

Note that the last line of (5.107) is similar to the original tadpole diagram except that we have fixed the modular parameter at \( T_0 \) and dropped the insertion of \( b_0 \). Thus in the conformal field theory method, we can represent the surface term by just taking \( T \to T_0 \) and dropping the integral of \( b(z) \) across the world-sheet.

For the tadpole diagram to be BRST invariant, we would require that \( Q_B \) acting on the region of integration \( T < T_0 \) should cancel this surface term. Unfortunately, for the divergent diagrams, we can not define this region of integration. Thus we consider replacing the \( T < T_0 \) region of integration with a new diagram. We then try to understand what restrictions the condition of BRST invariance imposes. This condition will not fix the new diagram completely but will tell us something about the physics of the \( T < T_0 \) region.

To construct the new diagram, we start with the surface term we found from acting on \(|\mathcal{T}\rangle\) with \( Q_B \) and modify it. As we discussed for the tadpole diagram, we may replace the boundary at the bottom of the surface term with a boundary state without changing the diagram. Instead of doing this, however, we replace the bottom of the diagram with some arbitrary closed string field \( \Phi \). Call this new diagram \(|\mathcal{T}_\Phi\rangle\). The surface term we found from acting on \(|\mathcal{T}\rangle\) with \( Q_B \) can then be written as \(|\mathcal{T}_B\rangle\).

We now propose replacing the region of integration \( T < T_0 \) with the surface term \(|\mathcal{T}_\Phi\rangle\). To ensure that this choice of surface term is valid, one would have to show that any open string state can be written in the form \(|\mathcal{T}_\Phi\rangle\) for some closed string state \( \Phi \). While, from the conformal field theory expression for \(|\mathcal{T}_\Phi\rangle\), this seems likely, we will content ourselves with
the fact that this choice of surface term is general enough for our purposes. With this caveat in mind, we can then ask: what condition does BRST invariance impose on $\Phi$? To check this we need to consider the action of $Q_B$ on $|\mathcal{T}_\Phi\rangle$.

We now show that $Q_B|\mathcal{T}_\Phi\rangle = |\mathcal{T}_{(Q_B + \tilde{Q}_B)\Phi}\rangle$. This can be shown by a contour pulling argument. We start with the BRST current contour running across the external leg of the diagram and pull it to the right. Since the Witten vertex is BRST invariant, we can freely slide the contour over it. If we slide the two ends of the contour all the way to the right of the diagram, they eventually meet each other and we can join them together to form a closed contour. This contour can then be pulled down the tube at the bottom of the diagram to act on $|\Phi\rangle$. This process is pictured in figure 5-9.

Note that in the notation we are using we can write, $Q_B|\mathcal{T}\rangle = |\mathcal{T}_B\rangle$. Thus, if we modify the tadpole diagram by $|\mathcal{T}\rangle \rightarrow |\mathcal{T}\rangle + |\mathcal{T}_\Phi\rangle$, the condition for BRST invariance becomes

$$0 = Q_B(|\mathcal{T}\rangle + |\mathcal{T}_\Phi\rangle) = |\mathcal{T}_B\rangle + |\mathcal{T}_{(Q_B + \tilde{Q}_B)\Phi}\rangle = |\mathcal{T}_{B+(Q_B + \tilde{Q}_B)\Phi}\rangle.$$  \hspace{1cm} (5.108)

So to cancel the BRST anomaly we should choose

$$(Q_B + \tilde{Q}_B)|\Phi\rangle = -|\mathcal{B}\rangle.$$ \hspace{1cm} (5.109)

Notice that this is essentially the same closed string field we used when we shifted the OCSFT action.

We now try to solve for $|\Phi\rangle$. We restrict discussion to the D25-brane case since it is the simplest. Equation (5.109) was already solved in [32] and we follow the discussion there.
First, note that equation (5.109) requires that the boundary state is BRST-closed and BRST-exact. While it is true that \((Q_B + \tilde{Q}_B)|\mathcal{B}\rangle = 0\), it is not true that \(|\mathcal{B}\rangle\) is exact. This can be seen by looking at the expansion of \(|\mathcal{B}\rangle\) in equation (5.31). We can resolve this issue by noting that the cohomology of \(Q_B\) is defined using states with well-behaved momentum dependence. For example one can check that

\[
Q_B x_0^\mu|0\rangle = c_1 \alpha_{-1}^\mu|0\rangle,
\]

contrary to the fact that \(c_1 \alpha_{-1}^\mu|0\rangle\) is in the cohomology of \(Q_B\). Thus, we proceed by just taking a general state of weight \((0, 0)\) and \((-1, -1)\) fields, acting on it with \((Q_B + \tilde{Q}_B)\) and comparing it with \(|\mathcal{B}\rangle\). We use the same parametrization for \(|\Phi\rangle\) as [32]

\[
|\Phi\rangle = \left( A(p)c_1 \tilde{c}_1 - \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu h_{\mu\nu}(p)c_1 \tilde{c}_1 + (c_1 c_{-1} - \tilde{c}_1 \tilde{c}_{-1})(\Phi(p) - h_{\mu}(p)/2) + (c_0 + \tilde{c}_0)(c_1 \alpha_{-1}^\mu - \tilde{c}_1 \tilde{\alpha}_{-1}^\mu)i\zeta_{\mu}(p) \right)|0\rangle.
\]  

We then compute

\[
(Q_B + \tilde{Q}_B)|\Phi\rangle = \left( \frac{1}{2} p^2 - 1 \right) A(p)c_1 (c_0 + \tilde{c}_0)c_1|0\rangle
\]

\[
+ \left[ -\frac{1}{2} p^2 h_{\mu\nu}(p) + ip_{\mu} \zeta_{\nu}(p) + ip_{\nu} \zeta_{\mu}(p) \right] \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu (c_0 + \tilde{c}_0)c_1 \tilde{c}_1|0\rangle
\]

\[
+ \left[ \frac{1}{2} p^2 (\Phi(p) - h_{\mu}(p)/2) + ip_{\mu} \zeta_{\mu}(p) \right] (c_0 + \tilde{c}_0)(c_1 c_{-1} - \tilde{c}_1 \tilde{c}_{-1})|0\rangle
\]

\[
+ \left[ -p^\mu h_{\mu\nu}(p) - p^\nu (\Phi(p) - h_{\mu}(p)/2) + 2i\zeta_{\mu}(p) \right] (\alpha_{-1} c_{-1} + \alpha_{-1} \tilde{c}_{-1})c_1 \tilde{c}_1|0\rangle.
\]

Substituting equation (5.112) into \((Q_B + \tilde{Q}_B)|\Phi\rangle = -|\mathcal{B}\rangle\) gives

\[
\left( \frac{1}{2} \partial^2 + 1 \right) A(x) = 1
\]

\[
\frac{1}{2} \partial^2 h_{\mu\nu}(x) - \partial_\mu \zeta_\nu(x) - \partial_\nu \zeta_\mu(x) = \eta_{\mu\nu}
\]

\[
\frac{1}{2} \partial^2 \Phi(x) - \frac{1}{4} \partial^2 h_{\mu}(x) + \partial^\mu \zeta_\mu(x) = 1
\]

\[
\frac{1}{2} \partial^\mu h_{\mu\nu}(x) + \frac{1}{2} \partial_\mu \Phi(x) - \frac{1}{4} \partial_\mu h_{\nu}(x) = \zeta_\mu(x).
\]

If we eliminate \(\zeta_\mu(x)\), these equations are equivalent to the linearized gravity equations in the background of the brane [32]. Note that the closed string tachyon shift is simply given
by \( A(p) = 1 \).

To generalize to arbitrary \( p \) one must replace the R.H.S. of equation (5.113) with the appropriate source terms from the lower dimensional boundary states. These source terms will have \( \delta \)-functions for the transverse dimensions. Solving these equations one can check that the long range behavior of the fields described in section 5.6.2 is reproduced.

Since the terms in \( |B \rangle \) of higher weight are BRST-exact we can act on them with

\[
- \frac{b_0 + \tilde{b}_0}{L_0 + \tilde{L}_0}
\]

(5.114)

to solve for the rest of \( |\Phi \rangle \).

We have thus shown that BRST invariance of the tadpole implies that the region of integration \( T < T_0 \) should, for consistency, represent a closed string background. The fact that this region of integration actually diverges, even when the solutions to the equations (5.113) are finite, arises from the specific prescription which OSFT uses to define the inverse of the BRST operator in terms of a closed string propagator.

One might ask why we cannot simply adopt this construction as a way of defining the tadpole. These are a number of problems with this idea. First, there is an ambiguity in the choice of \( \Phi \) under the shift \( \Phi \rightarrow \Phi + Q_B \delta \Phi \). This implies that if we were to use this scheme for one of the finite diagrams, we would, in general, get a different answer than the answer we would get from using analytic continuation. There is also the problem that we have introduced an unphysical parameter \( T_0 \). One can check that this is not much of a problem since a shift in \( T_0 \) can be accommodated by a corresponding BRST exact shift of \( \Phi \). Perhaps the most serious problem with this somewhat arbitrary division of moduli space is that it would lead to a breakdown in unitarity since for \( T < T_0 \) one can no longer consider the tadpole to be made up of an open string loop.

Finding a sensible way of systematically dealing with the divergent massless closed string tadpoles encountered in this section seems to be an important problem for string field theory. Although these divergences are only associated with Dp-branes having codimension 3 or less, such branes are important to understand in string theory. While it seems difficult to make sense of the theory in, for example, a space-filling brane background, we know from the recent work on the Sen conjectures that string field theory in such a background contains
Figure 5-10: The two-loop non-planar tadpole diagram contributing to a breakdown in BRST invariance.

within it other backgrounds corresponding to lower-dimensional D-branes and empty spacetime. Since the theory is well defined in these other backgrounds, it seems that the technical problems encountered in the presence of high-dimensional D-branes should have some robust technical solution which does not require a significant modification of the theory. On the other hand, it may be possible that the theory does not have a well-defined perturbation series in all backgrounds. This question is particularly pressing for superstring field theory, where we might expect the theory to be completely well-defined with finite tadpoles for Dp-branes with $p < 7$, while Dp-branes with $p \geq 7$ would have similar divergences to those discussed in this section.

### 5.6.4 Beyond one loop

Having examined the divergences in the tadpole diagram we discuss the situation for higher-loop diagrams. Although we have no explicit computations, we predict that there will be additional divergences of a new kind. For example at the two-loop level one can consider a torus diagram with a hole. If we put an open string state on the edge of the hole we have a contribution to the two-loop tadpole diagram. In the conformal frame natural to OSFT this diagram is pictured in figure 5-10. There is a region of moduli space in which the hole is separated from the rest of the diagram by a long tube. In this limit the diagram may be viewed as an open string which turns into a closed string which propagates over a long tube and then ends in a torus. This long tube leads to a divergence and associated BRST anomaly in world-sheet string theory and represents a shift in the cosmological constant [32].
The divergence occurs both because of the closed string tachyon and the massless fields and thus cannot be treated by a simple analytic continuation.

We expect a similar divergence and BRST anomaly in OSFT. Just as the one-loop open string tadpole could be thought of as a coupling between the open strings and the closed string tadpole, here the two-loop open string tadpole can be thought of as a coupling between the open strings and the one-loop closed string tadpole. The closed string tadpole we studied in the context of the one-loop open string tadpole arose because the brane was a source for the closed string fields. The one-loop closed string tadpole occurs even in pure closed string theory in the absence of the brane. Thus we expect that the divergence we find will be independent of the number of transverse dimensions.

At higher loops still more diagrams with divergent long tubes appear. For every divergent closed string tadpole we will find a corresponding set of divergences in OSFT. Without some general framework, each of these divergences must be treated individually. To some extent, this same problem arises in ordinary world-sheet string theory, but one might have hoped that in OSFT the problem would be more tractable.
5.7 Summary

In this chapter we have explicitly calculated the one-loop open string field theory tadpole using both conformal field theory and oscillator methods and found that the calculations agree. There is a divergence in the loop diagram due to the propagation of an open string tachyon around the loop; this divergence is easily dealt with by analytic continuation in an expansion of the diagram in the level of the open string field in the loop using the oscillator approach to OSFT. We also find, however, that the one-loop tadpole diagram diverges due to the propagation of the closed string tachyon over a long closed string tube. This divergence arises only because we are working in bosonic string theory, and can also in principle be treated by analytic continuation. In practice, we can perform this analytic continuation for the one-loop tadpole by explicitly using our knowledge of the role of closed strings in this diagram, but this is much more subtle than in the case of the open string tachyon, as the closed strings are not explicitly included as degrees of freedom in OSFT. The analytic continuation we use does not deal with the closed string tachyon divergence successfully for low-codimension branes. This procedure is not easily generalized to more complicated diagrams, and we do not have any systematic way of dealing with such divergences other than order-by-order in perturbation theory.

Once we have treated the leading divergences in the one-loop tadpole by hand, the tadpole becomes finite as long as we work in a D-brane background with sufficiently many codimensions. For D-branes of dimension \( p \geq 23 \), we find divergences from the propagation of massless modes over long distances. For the D25-brane theory, this divergence also contributes to a BRST anomaly. It does not seem to be possible to treat these divergences without stepping outside the framework of OSFT. Indeed, recent work indicates that even the Fischler-Susskind mechanism for dealing with such divergences in the world-sheet formalism may have previously unexpected subtleties \([102, 103]\). We describe, however, a procedure for using BRST invariance that reveals much of the physics hidden in these divergent diagrams. Achieving a better understanding of these divergences seems to be an important unsolved problem for string field theory.

We have found that the open string tadpole essentially arises from the coupling between the closed string background and the open strings. The closed string has a tadpole in the
presence of a D-brane, expressing the linearized massless fields produced by the D-brane source. The fact that this shift in the closed string background is seen in the one-loop tadpole diagram considered here demonstrates that OSFT actually captures the running of the background geometry, and thus contains a highly nontrivial aspect of closed string physics.

At two loops, OSFT runs into further difficulties with divergences and probably BRST anomalies as well. We believe that these problems, which are already well-known from perturbative string theory, doom any attempt to construct a complete quantum theory from bosonic open string field theory, without the introduction of some fundamental new idea. It is, of course, possible that some as-yet unknown mechanism stabilizes the closed string tachyon and turns bosonic string theory into a consistent theory. But if this is the case, this mechanism is no more apparent from the point of view of OSFT than from the traditional perturbative approach to the theory. In this sense, our study of quantum effects in bosonic OSFT has not led to any surprising new insights.

The positive results we have found here are, however, firstly that OSFT does not have any new problems or complications in the definition of the quantum theory which could not be predicted based on known issues in the perturbative theory, second that the one-loop tadpole in bosonic OSFT can be understood in a physically sensible fashion, and third that the OSFT tadpole naturally contains information about the shift in the closed string background due to the D-brane with respect to which the open string theory is originally defined.

While bosonic string theory has been a nice toy model with which to explore a variety of phenomena in string field theory, such as tachyon condensation, it remains unclear whether this theory is connected in any fundamental way with supersymmetric string theory and M-theory, and even whether this theory can be defined in a complete and consistent fashion quantum mechanically. In order to make further progress in understanding how far string field theory can take us in probing the fundamental nature of string theory, when we begin to consider seriously the construction of a nonperturbative quantum theory, it is probably necessary to work with a supersymmetric string field theory. The results of this chapter seem to give positive support to the hypothesis that supersymmetric open string field theory, if it exists classically, may naturally extend to a sensible quantum theory. Two possible
candidates for SUSY OSFT are the Berkovits theory [104] and the Witten theory [12, 13, 14]. If one of these theories can be shown to be completely consistent and to correctly incorporate the Ramond as well as the NS sector of open strings, it seems likely that an analogous calculation to the one in this chapter will, for Dp-branes with $p < 7$, give rise to a finite one-loop tadpole which encodes the linearized gravitational fields from the D-brane source. Such a tadpole should not suffer from any of the divergences encountered in this paper. In the supersymmetric theory, there seems to be no reason why higher loop calculations should not continue to incorporate the higher-order gravitational effects of the Dp-brane, so that the full theory should completely encode the Dp-brane geometry seen by closed strings.

There is clearly a significant amount of work remaining to be done to substantiate this story, but if this picture can be realized explicitly, it could open several exciting new directions for progress. This would give a new nonperturbative quantum definition of string theory. In this theory, if it is unitary, closed strings should arise as composite states. This would give a new more general open string model exhibiting open-closed string duality, from which the CFT description of strings in AdS space would arise as a special case in the usual decoupling limit. Because OSFT can be defined nonperturbatively, it is possible to imagine using level truncation to compute numerical approximations to finite nonperturbative quantities, thus potentially accessing new nonperturbative features of string theory. Because the closed string background is encoded in the quantum open string diagrams, it is possible that changes in the closed string background might be studied completely in terms of OSFT, given a full definition of the quantum theory exists.

At a more technical level, we have seen that the two different approaches to computing OSFT diagrams, the oscillator approach and the CFT approach, give somewhat orthogonal information about the structure of the theory. The CFT approach has the advantage of giving analytic expressions for amplitudes. From these analytic expressions, it is possible to take the limit where open string modular parameters become small, which in some situations corresponds to the limit where closed string physics plays an important role. In our analysis, as in [70], it was possible to analytically study the contribution from closed strings by expanding around this limit. On the other hand, the CFT approach is only tractable for simple diagrams. For any diagram at genus $g > 1$, the CFT approach is not easily applicable. Furthermore, even for computing finite diagrams at genus $g \leq 1$, the CFT approach gives
a complicated integral expression in terms of implicitly defined functions, which can only be numerically approximated. The oscillator approach, unlike the CFT approach, can be applied to arbitrary diagrams of OSFT. It is even possible to imagine truncating OSFT at finite level and including a momentum cutoff so that even nonperturbative quantities in OSFT can be approximated by finite-dimensional integrals. While closed strings are not included exactly when OSFT is truncated at finite oscillator level, we have found that the effects of closed strings can be clearly seen when the oscillator cutoff is sufficiently high. Although we found here that the closed string tachyon divergence made it difficult to extract other physical effects in the oscillator calculation, in a theory without closed string tachyons, like the superstring, it seems that the oscillator method should be a viable approach for approximating any loop diagram. Even for the bosonic theory, a general method for analytically continuing or otherwise taming the tachyon divergence—such as a Lorentzian world-sheet formulation—would allow us to extract useful physics from the oscillator method. Summarizing these observations, it seems likely that in further developments, such as in a systematic formulation of a quantum supersymmetric open string field theory, both these approaches to computations will play useful roles.
Chapter 6

Conclusions and future directions

In this chapter, we conclude with a few remarks about what we’ve learned from string field theory and what we might hope to learn in the future.

As we saw in the first few chapters, OSFT is a powerful tool for studying non-perturbative questions. Conceptually, the problem of finding new vacua in OSFT takes a very simple form: one simply simply looks for minima of the classical potential. It is one of the elegant properties of OSFT that the action around a new vacuum only differs from the usual action by the choice of BRST operator. This simple change, however, is enough to encode the difference between a theory of open strings living on a brane and a theory with no open string excitations at all.

It is clear that the downside of the SFT approach is that the string field is really an infinite number of ordinary fields, so that even the classical action is enormously complex. Thus, we are reduced to using the level truncation scheme, which makes any results we find approximate. However, in spite of this limitation, we have found that by using level truncation we can make definite statements about interesting questions in OSFT. Indeed, we found strong evidence that the cohomology of the BRST operator vanishes around the tachyon vacuum and we were able to study the global validity of the gauge fixing procedure.

Still, in spite of these successes, it is hard not wish for some analytical results. Unfortunately, we still have no way of writing down even one exact solution of the string field theory equations of motion. This makes any attempt to get analytic control over any of the more complicated questions like the BRST cohomology seem completely out of reach.
Even if we accept the fact that our computations will be only approximate, it is not clear that level truncation can be applied to general questions in OSFT. For example, a basic challenge in OSFT is to try and find a solution of the equations of motion describing two coincident D-branes. In spite of a fair amount of effort to find such a solution in level truncation [105], no such solution has ever been found.

This problem may also be related to the fact that, as we discussed in chapter 4, FS-gauge breaks down only a small distance away from the known vacua. While for an exact solution of OSFT, fixing the gauge is not necessary, in level truncation the stability of the solutions and, thus, the numerical accuracy of the results depends on a good gauge choice.

It is not clear how best to address these problems within the OSFT framework. It may be that the best approach is to consider the closely related vacuum string field theory, first proposed in [58, 59], where one can find analytic solutions for arbitrary numbers of branes as projectors of the star algebra [108, 107, 106, 109, 75, 110]. Unfortunately, it has never been clear how one can generate solutions of OSFT from solutions of vacuum string field theory. Hence, it seems like an important problem to try to understand the relationship between OSFT and vacuum string field theory more completely.

In the quantum theory, our results have only just scratched the surface of what is still a somewhat mysterious subject. As was first found in the non-planar “turret” diagram [71, 70], OSFT naturally incorporates closed string physics at the quantum level. In our calculation, we computed the tadpole diagram and found that it represented the first correction to the open string propagator from the non-trivial closed string background around the brane. Our calculation had many subtleties, which arose mostly because we were considering the bosonic string. If we imagine, however, that we had performed the same calculation in the superstring we could ask: how does the corrected open string propagator differ from the uncorrected propagator? It might be that one simply finds that the masses of the usual open string modes are shifted, but one might also find that there are new physical states, perhaps corresponding to closed strings propagating in the near horizon geometry of the brane. At this time it is hard to say, but it seems like this is an important direction to pursue.

While our results suggest an interesting way in which closed string physics might appear in OSFT, they also suffer from what is perhaps OSFT’s greatest flaw. Working with the
theory on a space-filling brane, we found a BRST anomaly. While this anomaly is just an artifact of our poor choice of background, the correct closed string background, de Sitter space, seems completely unreachable in OSFT. This seems to be a general problem with the theory. Whenever OSFT is able to capture some piece of closed string physics in terms of open string degrees of freedom, it appears to be a complete theory, however, whenever one wants to treat a major shift in the closed string background, one must switch to a new OSFT tailored to that background. Hence, from the point of view of closed string physics, OSFT is not a background independent theory.

This lack of background independence is only really a flaw, however, if one is trying to think of string field theory as the fundamental theory which should define string theory. Perhaps a better perspective is to think of OSFT as being analogous to SYM side of the Ads/CFT correspondence. While no one claims that SYM can describe string theory in an arbitrary background, the very fact that it captures all the physics in one background with a simple classical action is itself remarkable. Similarly, if we believe that OSFT captures all the closed string physics away from the brane, it seems like an important problem to try to understand precisely how this occurs and what it may imply about string theory as a fundamental theory.
Appendix A

The Perturbatively Stable Vacuum Solution at Level $(M, 3M)$

We tabulate the coefficient of the expansion of the stable vacuum solution $\Psi_0$ at various levels and interaction [19].

<table>
<thead>
<tr>
<th>gh = 1 field basis</th>
<th>level (2, 6)</th>
<th>level (4, 12)</th>
<th>level (6, 18)</th>
<th>level (8, 24)</th>
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<td>0.400720030749924</td>
<td>0.400379075568671</td>
<td>0.400379075568671</td>
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</tr>
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<td>$b_{-1}c_{-1}$ [0]</td>
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<td>$-0.400379075568671$</td>
<td>$-0.400379075568671$</td>
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<td>0.401589452148973691</td>
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<tr>
<td>$b_{-2}c_{-2}$ [0]</td>
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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
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gh = 1 field basis & \text{level (2, 6)} & \text{level (4, 12)} & \text{level (6, 18)} & \text{level (8, 24)} \\
\hline
\hline
\( b_{-1}c_{-1} \) & 0.014312021693536028 & & & \\
\hline
\( b_{-2}c_{-1} \) & 0.00915805058940239 & & & \\
\hline
\( b_{-3}c_{-1} \) & 0.0056742859934370511 & & & \\
\hline
\( b_{-4}c_{-1} \) & 0.004839557768226669 & & & \\
\hline
\( b_{-5}c_{-1} \) & 0.0034045613681820345 & & & \\
\hline
\( b_{-6}c_{-1} \) & 0.003052735328646744 & & & \\
\hline
\( b_{-7}c_{-1} \) & 0.002044574527640116 & & & \\
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\( b_{-8}c_{-1} \) & 0.00375275501998804 & & & \\
\hline
\( b_{-9}c_{-1} \) & 0.000430248004449016 & & & \\
\hline
\( b_{-10}c_{-1} \) & 0.0011807519406179202 & & & \\
\hline
\( b_{-11}c_{-1} \) & 0.0018763777509999383 & & & \\
\hline
\( L_{m}^{m} \) & 0.00001901038699211334 & & & \\
\hline
\hline
\( b_{-1}c_{-2} \) & 0.00029329813765591303 & & & \\
\hline
\( b_{-2}c_{-2} \) & 0.2614897317373461 x 10^{-6} & & & \\
\hline
\( b_{-3}c_{-2} \) & 3.140744865868727 x 10^{-6} & & & \\
\hline
\( b_{-4}c_{-2} \) & 0.00005028127313894 & & & \\
\hline
\( b_{-5}c_{-2} \) & 0.0000379159554779373 & & & \\
\hline
\( b_{-6}c_{-2} \) & 0.00016684272410463048 & & & \\
\hline
\( L_{m}^{m} \) & 0.0000219997029451806 & & & \\
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\( b_{-1}c_{-3} \) & 0.000349619452657495 & & & \\
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\( b_{-2}c_{-3} \) & -3.7535611693688668 x 10^{-6} & & & \\
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\( b_{-3}c_{-3} \) & -2.1835707446245427 x 10^{-6} & & & \\
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\( b_{-4}c_{-3} \) & 8.74037636163817 x 10^{-6} & & & \\
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\( b_{-2}c_{-4} \) & 1.259443290617533 x 10^{-6} & & & \\
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\( b_{-3}c_{-4} \) & 0.00153453432927412 & & & \\
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\( b_{-4}c_{-4} \) & 0.001356579245221895 & & & \\
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\( b_{-6}c_{-4} \) & 0.000677654626100039 & & & \\
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\end{tabular}
\end{table}

(A.1)
Appendix B

Fitting of the Parameters of $A$

B.1 $A$ up to Level 9 without Gauge Fixing

As $A$ is of ghost number $-1$ and has only odd levels, we here tabulate such field basis at levels 3, 5, 7 and 9. The best-fit numbers are the coefficients of $A$ obtained by best-fit via minimizing $\epsilon = \frac{|Q_{\phi}A - \bar{T}|}{|\bar{T}|}$. The stable fit at level 9 is constructed so as to control the convergence behaviour of the coefficients.
<table>
<thead>
<tr>
<th>Field Basis</th>
<th>level 3 fit</th>
<th>level 5 fit</th>
<th>level 7 fit</th>
<th>level 9 fit</th>
<th>stable level 9 fit</th>
</tr>
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<td>0.400769</td>
<td>0.406562</td>
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<th>level 5 fit</th>
<th>level 7 fit</th>
<th>level 9 fit</th>
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<td>0.163428</td>
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<th>level 7 fit</th>
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\[ \epsilon = \frac{|Q_{fe} A - Z|}{|Z|} \]

\[ 0.171484 \quad 0.117676 \quad 0.0453748 \quad 0.0243515 \quad 0.0356226 \]
B.2 Fitting $A$ in the Feynman-Siegel gauge

As $A$ enjoys the gauge freedom $A \rightarrow A + Q_{\Psi_0}B$, we can fix it to be in the Feynman-Siegel gauge. This is another way to control the convergence behaviour of the coefficients.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{fields} & \text{level 3 fit} & \text{level 5 fit} & \text{level 7 fit} & \text{level 9 fit} \\
\hline
b_{-3}b_{-2}c_{1} & 0.50921 & 0.467 & 0.500266 \\
b_{-4} & -0.518516 & -0.503772 & -0.53228 \\
b_{-2}L_{-2}^{m} & 0.504193 & 0.476325 & 0.504269 \\
\hline
-b_{-3}b_{-2}c_{1} & 0.333428 & 0.326986 \\
b_{-3}b_{-2}c_{1} & -0.330557 & -0.328381 \\
b_{-6} & 0.346811 & 0.331188 \\
-b_{-4}b_{-2}c_{1} & 0.325862 & 0.327997 \\
b_{-4}b_{-2}c_{0} & 0 & 0 \\
b_{-2}L_{-2}^{m} & -0.166799 & -0.164306 \\
b_{-2}L_{-2}^{m} & 0.00133026 & 0.000334022 \\
b_{-3}b_{-2}c_{1}L_{-2}^{m} & 0.341592 & 0.328637 \\
b_{-3}b_{-2}L_{-2}^{m} & -0.332864 & -0.327326 \\
b_{-2}L_{-2}^{m}L_{-2}^{m} & 0.1686 & 0.165931 \\
\hline
b_{-3}b_{-3}c_{1} & 0.245499 \\
b_{-4}b_{-3}c_{1} & -0.253014 \\
b_{-7}b_{-2}c_{1} & 0.250149 \\
b_{-8} & -0.257672 \\
b_{-3}b_{-3}b_{-2}c_{2} & 0.249999 \\
b_{-4}b_{-3}b_{-2}c_{2} & -0.256812 \\
b_{-4}b_{-3}b_{-2}c_{1} & 0.246226 \\
b_{-2}b_{-3}b_{-2}c_{1} & -0.25213 \\
b_{-4}b_{-3}b_{-2}c_{0}c_{1} & 0 \\
b_{-3}b_{-3}c_{3} & 0 \\
b_{-4}b_{-3}c_{3} & 0 \\
b_{-2}L_{-2}^{m} & 0.00104113 \\
b_{-3}L_{-2}^{m} & 0.0000151443 \\
b_{-3}b_{-2}c_{1}L_{-2}^{m} & -0.126025 \\
b_{-3}L_{-2}^{m} & 0.12448 \\
b_{-4}b_{-3}c_{3}L_{-2}^{m} & -0.0004548 \\
b_{-3}L_{-2}^{m} & -0.000819122 \\
b_{-3}b_{-3}b_{-2}c_{0}L_{-2}^{m} & 0 \\
b_{-2}L_{-2}^{m}L_{-2}^{m} & 0.0000903030 \\
b_{-3}b_{-2}c_{1}L_{-2}^{m} & 0.250728 \\
b_{-3}b_{-2}c_{1}L_{-2}^{m} & -0.251499 \\
b_{-3}L_{-2}^{m} & 0.250865 \\
b_{-3}b_{-2}c_{1}L_{-2}^{m} & 0.249179 \\
b_{-4}b_{-3}b_{-2}c_{0}L_{-2}^{m} & 0 \\
b_{-3}L_{-2}^{m}L_{-2}^{m} & -0.123363 \\
b_{-3}b_{-3}c_{2}L_{-2}^{m} & 0.000457948 \\
b_{-3}b_{-3}c_{2}L_{-2}^{m} & 0.126358 \\
b_{-2}L_{-2}^{m}L_{-2}^{m} & -0.152248 \\
b_{-2}L_{-2}^{m}L_{-2}^{m} & 0.0406385 \\
\hline
\end{array}
\]

$\epsilon = \{Q_{\Psi_0} A - T / I \} = 0.171484 \quad 0.117676 \quad 0.0480685 \quad 0.0320384$
B.3 Expansion of $I$ up to level 9

Immediately below the field basis at ghost number 0 and levels 1, 3, 5, 7 and 9 is given the coefficient of the expansion of $I$. 

\[ \begin{array}{|c|c|c|c|c|c|}
\hline
(0) & b_{-3c1} & 0 & -b_{-2c2} & 0 & L_{-x}^2 & b_{-4c3} & 0 \\
\hline
1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
-b_{-3c-3} & 0 & -b_{-3c-1} & 0 & b_{-3c_{-2c2}} & 0 & -b_{-4c-2} & 0 \\
\hline
1 & -1 & 2 & -2 & 2 & -2 & -1 & 1 \\
\hline
b_{-2c_{-1}} L_{m}^2 & 0 & b_{-3c1} L_{m}^2 & 0 & -b_{-2c2} L_{m}^2 & 0 & L_{m}^2 L_{m}^2 & b_{-4c3} & 0 \\
\hline
0 & -1 & 2 & 2 & -2 & -1 & 1 & 1 \\
\hline
-b_{-3c-4} & 0 & -b_{-3c-3} & 0 & b_{-3c_{-2c2}} & 0 & -b_{-4c-2} & 0 \\
\hline
1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} & 0 & b_{-3c_{-2c2}} & 0 & b_{-3c_{-2c2}} & 0 & -b_{-4c-2} & 0 \\
\hline
1 & 2 & -2 & 2 & -2 & 2 & 0 & 0 \\
\hline
L_{m}^2 & 0 & b_{-3c1} L_{m}^2 & 0 & -b_{-2c2} L_{m}^2 & 0 & b_{-4c3} & 0 \\
\hline
0 & 0 & 1/2 & -1 & 1 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & L_{m}^2 L_{m}^2 & b_{-4c3} L_{m}^2 & -b_{-3c-2} L_{m}^2 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\hline
-b_{-3c-1} & 0 & b_{-3c_{-2c2}} & 0 & L_{m}^2 L_{m}^2 & b_{-4c3} L_{m}^2 & -b_{-3c-2} L_{m}^2 & 0 \\
\hline
1 & -1 & 2 & -2 & -1/2 & 0 & 0 & 0 \\
\hline
b_{-3c1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & L_{m}^2 L_{m}^2 & b_{-4c3} L_{m}^2 & -b_{-3c-2} L_{m}^2 & 0 \\
\hline
1 & 2 & -2 & -2 & -1/2 & 0 & 0 & 0 \\
\hline
-b_{-3c-5} & 0 & b_{-3c_{-2c2}} & 0 & -b_{-3c-4} & 0 & b_{-3c_{-2c2}} & 0 \\
\hline
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
b_{-4b_{-3c-2c1}} & 0 & b_{-3b_{-2c2}} & 0 & -b_{-3c-2} & 0 & b_{-3b_{-2c2}} & 0 \\
\hline
0 & -1 & 2 & -2 & 2 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} & 0 & b_{-3b_{-2c2}} & 0 & b_{-3b_{-2c2}} & 0 & -b_{-3c-2} & 0 \\
\hline
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-0} & 0 & b_{-3b_{-2c2}} & 0 & b_{-3b_{-2c2}} & 0 & -b_{-3c-2} & 0 \\
\hline
-2 & -2 & 2 & -2 & 2 & 0 & 0 & 0 \\
\hline
L_{m}^2 & 0 & b_{-2c1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & b_{-3c1} L_{m}^2 & 0 & -b_{-3c-2} L_{m}^2 & 0 \\
\hline
0 & 0 & 1/2 & -1 & 1/2 & -1/2 & 1/2 & 1/2 \\
\hline
b_{-3b_{-2c0} c1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & L_{m}^2 L_{m}^2 & b_{-3c1} L_{m}^2 & -b_{-3c-2} L_{m}^2 & 0 \\
\hline
-1 & -1 & 1/4 & 1/8 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-2} L_{m}^2 & 0 & b_{-3b_{-2c2}} & 0 & L_{m}^2 L_{m}^2 & b_{-3c1} L_{m}^2 & -b_{-3c-2} L_{m}^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
L_{m}^2 L_{m}^2 & 0 & b_{-2c1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & -b_{-3c-2} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
L_{m}^2 & 0 & b_{-2c1} L_{m}^2 & 0 & -b_{-3c0} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-4} & 0 & -b_{-3c0} L_{m}^2 & 0 & -b_{-3c-2} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} L_{m}^2 & 0 & b_{-3b_{-2c2}} & 0 & -b_{-3c-2} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
1 & 2 & -2 & -2 & 2 & 0 & 0 & 0 \\
\hline
L_{m}^2 & 0 & b_{-2c1} L_{m}^2 & 0 & b_{-3c1} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} & 0 & b_{-3b_{-2c2}} & 0 & b_{-3b_{-2c2}} & 0 & -b_{-3c-2} & 0 \\
\hline
1 & 2 & -2 & 2 & -2 & 2 & 0 & 0 \\
\hline
L_{m}^2 & 0 & b_{-2c1} L_{m}^2 & 0 & b_{-3c1} L_{m}^2 & 0 & b_{-4c3} L_{m}^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} L_{m}^2 & 0 & b_{-3b_{-2c2}} & 0 & b_{-3b_{-2c2}} & 0 & -b_{-3c-2} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-b_{-3c-1} (L_{m}^2)^2 & 0 & -b_{-3b_{-2c2}} (L_{m}^2)^2 & 0 & -b_{-3c-1} (L_{m}^2)^2 & 0 & b_{-4c3} (L_{m}^2)^2 & 0 \\
\hline
-1/2 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\
\hline
b_{-3c1} (L_{m}^2)^2 & 0 & -b_{-3c0} (L_{m}^2)^2 & 0 & (L_{m}^2)^2 & 0 & b_{-4c3} (L_{m}^2)^2 & 0 \\
\hline
-1/6 & 1/3 & 1/24 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array} \]
Appendix C

The Proof for the Simplified Expression for the Identity

In this appendix we present the proof for the analytic expression for the identity as given in (3.7). We remind the reader of the expression:

$$|\mathcal{I}\rangle = \left( \prod_{n=2}^{\infty} \exp \left\{ -\frac{2}{2^n} L_{-2^n} \right\} \right) e^{L_{-2}} |0\rangle$$

$$= \ldots \exp \left( -\frac{2}{2^3} L_{-2^3} \right) \exp \left( -\frac{2}{2^2} L_{-2^2} \right) \exp (L_{-2}) |0\rangle,$$

(C.1)

or its BPZ conjugate form\(^1\)

$$\langle \mathcal{I} | = \langle 0 | U_h U_{f_2} U_{f_3} U_{f_4} \ldots ,$$

(C.2)

where \( U_{f_n} = e^{-\frac{2}{2^n} L_{2^n}} \) for \( n \geq 2 \) and \( U_h = e^{L_2} \). In [28], the identity is given by \( \langle \mathcal{I} | = \langle 0 | U_{f_2} \) where \( U_{f_2} \) is the operator corresponding to the function

$$f_\mathcal{I}(z) = \frac{z}{1 - z^2}.$$ 

Using the composition law \( U_{g_1} U_{g_2} = U_{g_1 \circ g_2} \), what we need is to prove

$$U_h U_{f_2} U_{f_3} U_{f_4} \ldots = U_{h \circ f_2 \circ f_3 \circ \ldots} = U_{f_2}$$

\(^1\)Please notice that, besides the replacement \( L_n \to (-)^n L_{-n} \), the orders under BPZ-conjugation are also reversed. This is because we use \( L_n \) instead of the oscillators \( \alpha_m \), whose orders do not get reversed under BPZ.
which is equivalent to proving

$$\lim_{k \to \infty} h \circ f_2 \circ \cdots \circ f_k(z) = f_1(z) = \frac{z}{1 - z^2}. \quad (C.3)$$

For the operator $U_f = e^{aL_n}$, the corresponding function $f$ is given by [40]

$$f(z) = \exp \left\{ az^{n+1} \partial_z \right\} z = \frac{z}{(1 - az^n)^{1/n}},$$

so we have

$$h(z) = \frac{z}{(1 - 2z^2)^{1/2}},$$

$$f_n(z) = \frac{z}{(1 + 2z^2)^{1/2}}.$$

A useful property of the $f_n$ is that $f_n(z) = (g(z^{2^n}))^{1/2^n}$ where

$$g(z) := \frac{z}{1 + 2z} = \frac{1}{2 + 1/z}.$$

Before writing down the general form, first let us do an example:

$$f_2 \circ f_3 \circ f_4(z) = f_2 \circ f_3(\left( g(z^{2^4}) \right)^{1/2^4})$$

$$= f_2(\left( g(\left( g(z^{2^4}) \right)^{1/2^3}) \right)^{1/2^3})$$

$$= f_2(\left( g(\left( g(z^{1/2}[z^{2^4}]) \right)^{1/2^3}) \right)^{1/2^3})$$

$$= (g(\left( g(\left( g(z^{1/2}[z^{2^4}]) \right)^{1/2^3}) \right)^{1/2^2})^{1/2^2})$$

$$= (g^{1/2}(g^{1/2}[g^{1/2}[z^{2^4}]]))^{1/2}.$$

Now it is easy to see that the general form is

$$h \circ f_2 \circ f_3 \circ \cdots \circ f_{k+1}(z) = h \circ (g^{\frac{1}{2}} \circ \cdots \circ g^{\frac{1}{2}}(z^{2^{k+1}}))^{\frac{1}{2}}.$$

Thus equation (C.3) is equivalent to showing that

$$\lim_{k \to \infty} g^{\frac{1}{2}} \circ \cdots \circ g^{\frac{1}{2}}(z^{2^{k+1}}) = (h^{-1}(f(z)))^2 = \frac{z^2}{1 + z^4}.$$
The left hand side can be written as

\[
(2 + (2 + \cdots + (2 + \frac{1}{z^{2k+1}})\frac{1}{2} \cdots)\frac{1}{2} )^{-1} = z^2((2z^{2^2} + (2z^{2^3} + \cdots (2z^{2^{k+1}} + 1)\frac{1}{2} \cdots)\frac{1}{2})\frac{1}{2})^{-1}.
\]

Thus (C.3) reduces to the verification of the equation

\[
\lim_{k \to \infty} (2z^{2^2} + (2z^{2^3} + \cdots (2z^{2^{k+1}} + 1)\frac{1}{2} \cdots)\frac{1}{2})\frac{1}{2} = 1 + z^{2^2}.
\]

This can be done as follows. Consider first squaring both sides of the above equation and canceling \(2z^{2^2}\) from the two sides, we get

\[
\lim_{k \to \infty} (2z^{2^3} + \cdots (2z^{2^{k+1}} + 1)\frac{1}{2} \cdots)\frac{1}{2} = 1 + z^{2^3}.
\]

Repeating the above operation \(k\) times, the left hand side gives 1 while the right hand side gives \(1 + z^{2k+2}\). Thus as long as \(z < 1\), we get that the left and right hand sides do converge to each other as \(k \to \infty\).
Appendix D

The BRST anomaly in the D25-brane theory

We now study the BRST invariance of the tadpole in the D25-brane theory. In any BRST-quantized string theory, a basic condition for gauge invariance is that BRST-exact states decouple from on-shell diagrams. For the tadpole diagram, this condition is simply that $Q_B$ should annihilate the tadpole.

In sections D.0.1 and D.0.2 we use conformal field theory and oscillator methods to compute the action of the BRST operator on the tadpole and find that the tadpole is not BRST closed. This breakdown of gauge invariance is familiar from ordinary string perturbation theory in the context of the Fischler-Susskind mechanism [29, 30, 31, 32, 101]. Our results differ from previous work on BRST invariance in OSFT. In [111] it was argued that $Q_B |\mathcal{T}\rangle$ should vanish, while in [15] it was suggested that it might be possible to avoid a breakdown in BRST invariance in Feynman-Siegel gauge. Our results indicate that in fact $Q_B |\mathcal{T}\rangle \neq 0$, even in Feynman-Siegel gauge.

D.0.1 The BRST anomaly in the conformal field theory method

We now study the BRST invariance of the OSFT tadpole for the D25-brane theory. On the grounds of gauge invariance, we expect that $Q_B |\mathcal{T}\rangle = 0$. To check this identity we first need to render $|\mathcal{T}\rangle$ a finite state by imposing a cutoff.

We choose to simply cut off the integrals over the modular parameter $T$ so that it never
gets smaller than some minimum value \( T_0 \). We can then evaluate \( Q_B |T\) using equation (5.107). This gives

\[
\langle T | Q_B = \langle V_2 | e^{-T_0 L_0} | V_2 \rangle.
\]

We can then examine the behavior as \( T_0 \to 0 \). Conveniently, we have already done all the work for this calculation in section 5.3. Notice that the only difference between equation (D.1) and equation (5.55), is that \( T \) is evaluated at a specific point, \( T_0 \), and that the \( b_0 \) is missing. In the conformal field theory language we can accommodate this change by simply fixing the length of the internal propagator to be \( T_0 \) and dropping the insertion of \( b_0 \). This gives

\[
Q_B | T \rangle = U_{h(z) \circ z(w)}^{\dagger} \bigg|_{T=T_0} (h(z) \circ z(w)) \bigg|_{T=T_0} \circ B \rangle.
\]

To map the resulting diagram to the disk, we can just use the same map \( z(w) \) given in equation (5.30). Since there is no \( b_0 \) to worry about, the divergence structure is actually much simpler. Since \( U_{h(z) \circ z(w)}^{\dagger} \) is well behaved as \( T_0 \to 0 \) and \( h(z) \) is independent of \( T_0 \), we only need to consider \( z(w) \circ B \). Using equation (5.31) we get that the boundary state is mapped to

\[
z \circ \langle B \rangle \rightarrow \left( \frac{1 + W_0^2}{k_0 W_0^2} \right)^2 c_1 (c_0 + \bar{c}_0) \bar{c}_1 \langle 0 | - \alpha_{-1} \cdot \bar{\alpha}_{-1} \bar{c}_1 (c_0 + \bar{c}_0) c_1 | 0 \rangle - 2 (c_1 c_{-1} + \bar{c}_{-1} \bar{c}_1) (c_0 + \bar{c}_0) | 0 \rangle + \text{terms that vanish as } W_0 \to 0,
\]

where \( W_0 \) is \( T_0/\pi \) and \( k_0 \) is \( k(W_0) \). Clearly as \( T_0 \to 0 \) this expression is non-vanishing.

For theories on \( Dp \) branes with \( p < 25 \) we will still find an anomaly for the terms coming from the closed string tachyon but we no longer find an anomaly in the massless sector. For example, the graviton/dilaton term in (D.3) in the \( Dp \)-brane theory becomes

\[
\int d^{25-p} q_\perp \left( \frac{1 + W_0^2}{k_0 W_0^2} \right)^{2} \alpha_{-1} \cdot \bar{\alpha}_{-1} \bar{c}_1 (c_0 + \bar{c}_0) c_1 | q_\perp \rangle.
\]
Mapping to the upper-half plane this becomes

$$\int d^{25-p}q_\perp (2)^{q_\perp} \left( \frac{1 + W_0^2}{k_0 W_0^2} \right)^{q_\perp} : c \partial X(i) \cdot \bar{c} \partial X(i) e^{i q_\perp \cdot X(i)} :. $$  \tag{D.5}

Now consider evaluating the correlation between this operator and the external state. When $e^{i q_\perp X(i)}$ contracts with itself we pick up a factor of $(2)^{q_\perp}$. The operator $e^{i q_\perp X(i)}$ can also contract with other $X$’s but these just pick up factors of $q_\perp^2$. Such terms will go to zero as $W \to 0$. Thus the largest contribution comes from the momentum integral

$$\int d^{25-p}q_\perp \left( \frac{1 + W_0^2}{k_0 W_0^2} \right)^{q_\perp} \propto \left[ - \log \left( \frac{1 + W_0^2}{k_0 W_0^2} \right) \right]^{\frac{1}{2}(p-25)}, $$  \tag{D.6}

which vanishes as $W_0 \to 0$.

### D.0.2 The BRST anomaly in level truncation

It is also possible to demonstrate the BRST anomaly in level truncation. It is easiest to frame this discussion in terms of truncation on field level rather than oscillator level. Let us begin by considering a level expansion of the regulated equation (5.107)

$$Q_B \int_{T_0}^{\infty} d\tau \langle \tilde{V}_2 | b_0 e^{-T_0 L_0} | V_3 \rangle = \langle \tilde{V}_2 | e^{-T_0 L_0} | V_3 \rangle = \sum_n c_n(T_0) e^{(1-n)T_0}, $$  \tag{D.7}

where the term associated with the coefficient $c_n$ arises from fields in the loop of level $n$. We can deal with the open string tachyon divergence through analytic continuation. The coefficients $c_n(T_0)$ are finite for all values of $n > 1, T_0 > 0$, and can be directly calculated by computing the contribution to either the first or the second expression in (D.7) at each level. Direct computation at low levels confirms that both ways of computing $c_n(T_0)$ give the same result. Truncating the theory at finite field level reduces the summation in (D.7) to a finite sum, from which we can take the $T_0 \to 0$ limit in the level-truncated theory. Thus, we see that (D.1) holds in level truncation, even as $T_0 \to 0$, so that level truncation naturally regulates the divergences in this equation arising from closed strings.

To show that (D.7) does not vanish in the limit $T_0 \to 0$, let us examine the action of $Q_B$ on the level 2 sector of $|T \rangle$. This calculation is straightforward, as $Q_B$ conserves level. The
fields appearing in $|\mathcal{T}\rangle$ at level 2 are

$$|\mathcal{T}^{(2)}\rangle = \left( \beta^\mu_{\mu} a_1^\dagger a_1^\dagger + \gamma c_2^\dagger b_0 + \delta c_1^\dagger b_1^\dagger \right) c_0 |\bar{0}\rangle.$$  \hfill \text{(D.8)}

Upon acting with $Q_B$ we find

$$Q_B |\mathcal{T}^{(2)}\rangle = (\beta^\mu_{\mu} - \gamma - 3\delta) c_2^\dagger c_0 |\bar{0}\rangle.$$  \hfill \text{(D.9)}

The condition that $|\mathcal{T}^{(2)}\rangle$ be $Q_B$-closed thus reduces to the condition

$$\beta^\mu_{\mu} - \gamma - 3\delta = 0$$  \hfill \text{(D.10)}

on the coefficients of the component states.

The coefficients $\beta^\mu_{\mu}$, $\gamma$ and $\delta$ are given by

$$\beta^\mu_{\mu} = \eta_{\mu\nu} \int_0^\infty dT \, S_0(T) \left( -\frac{1}{2} M_{11}(T) \right),$$

$$\gamma = \int_0^\infty dT \, S_0(T) (-R_{20}(T)),$$

$$\delta = \int_0^\infty dT \, S_0(T) (-R_{11}(T)).$$  \hfill \text{(D.11)}

As discussed above, level truncation serves to render the integrals in equation (D.11) finite, so that they are calculable numerically. The integrands become very sharply peaked near $T = 0$, so numerical analysis becomes difficult for small $T$. Nevertheless, it is clear that effects near $T = 0$ give rise to a violation of (D.10). At successively higher levels, the weight factor $S_0$ becomes more and more sharply peaked at $T = 0$, and the integral is well approximated by its endpoint value $-\frac{1}{2} M_{11}(0) S_0(0)$. As we have demonstrated above, $M_{11}(0) = R_{11}(0) = -1$, while $R_{20}(0) = 0$; these values manifestly fail to satisfy (D.10). Moreover, as we remove the regularization by sending the level to infinity, the violation becomes infinite.

We have thus seen in the operator formalism the presence of a breakdown in BRST invariance which can be directly ascribed to the presence of short-distance divergences, interpretable in terms of the closed string tachyon.
Appendix E

The infinite level limit

In this section we look at some of the subtleties of the infinite level limit of the matrices in the oscillator calculations. In particular, we are interested in comparing how level-truncated calculations near $T = 0$ compare with what we expect if we take the level to infinity.

For definiteness, we work in the ghost sector. The calculation in the matter sector is completely analogous. We begin by expanding the expression for $R_{nm}$, given in (5.74), around $T = 0$ in level truncation. The only tricky part of the calculation is the small $T$ expansion of

$$\frac{1}{1 - S^{\bar{X}}}. \quad (E.1)$$

It turns out, however, that it is sufficient to consider the expansion of (E.1) acting on $S \left( \begin{array}{c} X^{21} \\ X^{12} \end{array} \right)$, which gives

$$\frac{1}{1 - S^{\bar{X}} S \left( \begin{array}{c} X^{21} \\ X^{12} \end{array} \right)} = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) - T \left( \frac{1}{1 - S^{\bar{X}}} \bigg|_{T=0} \right) \left( \begin{array}{c} \{G, CX^{21} + CX^{11}\} \\ \{G, CX^{12} + CX^{11}\} \end{array} \right) + \mathcal{O}(T^2), \quad (E.2)$$

where we have defined the matrix

$$G_{mn} = \frac{m}{2} \delta_{mn}. \quad (E.3)$$

Using these formulas, one can work out the expansion

$$R_{mn} = C_{mn} - 2T \{G, C\}_{mn} + \mathcal{O}(T^2) = C_{mn} - 2mT C_{mn} + \mathcal{O}(T^2). \quad (E.4)$$

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Figure E-1: Plot of $-\frac{d}{dT} R_{11}(T)$ at various levels, $L$. We see that near $T = 0$, $-\frac{d}{dT} R_{11}(T)$ always approaches $-2$, but at high levels comes arbitrarily close to hitting $-1$.

In fact since the only identity needed to prove this relation is the sum rule $X^{11} + X^{12} + X^{21} = C$, equation (E.4) holds exactly at every fixed level.

We now make the following claim: In spite of the fact that the leading order correction is the same at every finite level, if we did not truncate the matrices at all, we would get instead

$$R_{mn} = C_{mn} - mT C_{mn} + \mathcal{O}(T^2).$$

To see how this happens, consider a single matrix element $R_{11}$. We expect from level-truncated analysis that $R_{11}(T) = -1 + 2T + \mathcal{O}(T^2)$. In figure E-1, we plot $-\frac{d}{dT} R_{11}(T)$ near zero at various levels. We see that if we go extremely close to zero, $-\frac{d}{dT} R_{11}(T)$ always eventually approaches $-2$ as predicted by (E.4).

As the level is increased, however, the region where $-\frac{d}{dT} R_{11}(T)$ is close to $-2$ becomes arbitrarily small. This falloff near $T = 0$ is analogous to the falloff we found when studying the tachyon divergence in section 5.4.2. If we imagine taking the level to infinity and then
taking \( T \to 0 \), we can ignore the falloff region and just take

\[
\lim_{T \to 0} \left( -\frac{d}{dT} R_{11}(T) \right) = -1, 
\]

as predicted by the conformal field theory method.

We can now ask: what went wrong with the calculation of equation (E.4)? To analyze this question, it is convenient to work with the matrices \( M^{ij} = C X^{ij} \). In level truncation these matrices satisfy one identity

\[
M^{11} + M^{12} + M^{21} = 1. 
\]

(E.7)

If we do not truncate the matrices, they also satisfy \([112]\)

\[
(M^{11})^2 + (M^{12})^2 + (M^{21})^2 = 1, 
\]

(E.8)

and commute. These identities imply that at \( T = 0 \), the matrix \( 1 - S \tilde{X} \) has a non-vanishing kernel. Thus the matrix

\[
\left. \frac{1}{1 - S \tilde{X}} \right|_{T=0}
\]

is ill-defined. This invalidates the expansion (E.2). Even worse, the expression (5.81) that we used to see that \( R_{nm} \to C \) as \( T \to 0 \) is not defined for the untruncated matrices. In fact one can check that while it is true that

\[
\lim_{\text{Level} \to \infty} \left\{ \lim_{T \to 0} \left( \hat{X}^{12} \hat{X}^{21} \right) \frac{1}{1 - S \tilde{X}} \right\} = \begin{pmatrix} C & C \end{pmatrix},
\]

(E.10)

one finds that

\[
\lim_{T \to 0} \left\{ \lim_{\text{Level} \to \infty} \left( \hat{X}^{12} \hat{X}^{21} \right) \frac{1}{1 - S \tilde{X}} \right\} \neq \begin{pmatrix} C & C \end{pmatrix}
\]

(E.11)

in a similar manner to what we found for \( -\frac{d}{dT} R_{11}(T) \).

In spite of this, we find that \( R_{nm} \to C \) in the small \( T \) limit unambiguously in numerical tests. At this time, we do not have enough control over the matrices \( \tilde{M} \) and \( \tilde{X} \) to give an infinite level proof of the identity E.5.
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