Enumerative Problems in Intersection Theory

by

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Abstract

We develop and describe some of the basic tools of intersection theory in algebraic geometry. Some classical enumerative problems are then solved using these methods. In particular, we discuss the Fano variety of a cubic in surface in \( \mathbb{P}^3 \), determinantal varieties, and the number of conics tangent to five conics in \( \mathbb{P}^2 \).

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Chapter 1

Introduction

Enumerative problems in algebraic geometry were intensively studied in the nineteenth century. It is thus quite surprising to realize that some of the very well known and basic examples worked out in this paper have generalizations that are still unsolved. For example, consider the problem of counting lines on a hypersurface. In 1849, Salmon and Cayley showed that there are 27 lines on a cubic surface in $\mathbb{P}^3$. The general problem, however, is much more difficult. Even the dimension of the "Fano variety" of lines on a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d \leq n$ is still unknown, although both Johan de Jong and Olivier Debarre have conjectured it to be the expected value, $\dim F_1(X) = 2n - 3 - d$.

In this paper we construct and describe some of the basic tools of algebraic geometry and intersection theory. The topics we cover are selected to solve some of the classic enumerative geometry problems. For instance, we solve the problem of lines on a cubic surface, as mentioned above. Other, more intricate examples are mentioned below.

In the first chapter, we begin by considering the question of lines meeting four general lines in $\mathbb{P}^3$. Interestingly, this question can be rephrased as follows: Consider maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 3, modulo automorphisms of the target $\mathbb{P}^1$. These correspond to rational functions $\frac{P}{G}$, along with the equivalence relation $\frac{P}{G} \sim \frac{aP+bG}{cP+dG}$, with nonsingular coefficient matrix $\begin{bmatrix}a & b \\ c & d \end{bmatrix}$. Now, specify four points $p_1, \ldots, p_4$ in the domain $\mathbb{P}^1$. How many such maps have these four points as critical points? In general, we could ask how many degree $d$ maps $\mathbb{P}^1 \to \mathbb{P}^1$ have $2d - 2$ critical points.

The original question of lines meeting four lines leads us to the definition of the Grassmannian and its tangent space, and to the introduction of the Chow ring. We devote all of chapter 3 to the definition of the Chow ring and its properties. Another option would have been to define the algebraic Chow ring instead (where two $k$-cycles $Y$ and $Y'$ on $X$, a smooth projective variety, are equivalent if, for a smooth connected curve $C$ with $p, q \in C$, we have $Y \subset X \times C$, and $Y_p = Y'$ and $Y_q = Y'$). The algebraic Chow ring is easier to work with than the Chow ring. However, one of the reasons to use the Chow ring is that, in the case of curves, it coincides with the Picard group.

After the discussion of the Chow ring, we introduce Chern and Segre classes, which will turn out to generate the Chow ring in many of our examples. Although Segre classes were historically introduced first, we follow the modern custom of first defining Chern classes and then introducing Segre classes. In chapter 4, we give an original proof of the relationship between the total Chern class and the total Segre class of a vector bundle over a variety using Giambelli's formula.

Next, in chapter 6, we describe parameter spaces and introduce the Chow variety and the Hilbert scheme. In this chapter, we will see how important the appropriate choice of parameter space can be in solving an enumerative geometry question. In particular, we discuss two possible
solutions to Steiner's problem of finding the number of conics tangent to five general conics in $\mathbb{P}^2$. A solution to this problem was first attempted by Jacob Steiner in the mid-1800s using a synthetic construction of curves, which had correctly yielded the result that there is only one conic tangent to five general lines in $\mathbb{P}^2$.

Unfortunately, Steiner's construction for the problem of five conics gives the incorrect answer of 7776. The error in his calculation comes from Steiner's failure to realize that, in the $\mathbb{P}^5$ of plane conics, there is a Veronese surface $S$ of double lines and that $S$ is also contained in each set of conics tangent to one of the given five conics. There are three ways of correcting Steiner's oversight. We could blow-up the $\mathbb{P}^5$ of plane conics along $S$ and then carry out our calculations in $X = \text{Bl}_S \mathbb{P}^5$, or we could introduce the space of complete conics, the closure in $\mathbb{P}^5 \times \mathbb{P}^5^*$ of the set of smooth conics dual to each other. Both of these approaches are carried out in chapter 6. The third and final method is to apply the excess intersection formula. We discuss this formula in chapter 6.6.

Lastly, in chapter 7, we develop Porteous' formula, which leads us to a brief discussion of determinantal varieties. These varieties have equations that can be expressed as the minors of a matrix. Veronese and Segre varieties and rational normal scrolls are all in fact determinantal varieties. Porteous' formula, for example, allows us to calculate the number of quatriscant lines of a smooth, rational curve of degree $d$ in $\mathbb{P}^3$. 
Chapter 2

The Grassmannian

To motivate the construction of the Grassmannian and of the Chow ring, we will start by stating a classical enumerative problem. Its solution will be worked out in section 3.4.

**Question:** Given four general lines $L_1, L_2, L_3, L_4$ in $\mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ meet all four? For each $L_i$ consider the subset of $\mathbb{G}(1,3)$ given by

$$\Sigma_1(L_i) = \{\text{lines in } \mathbb{P}^3 \text{ meeting } L_i\}.$$ 

The answer to our question is simply given by the number of lines $L$ such that $L \in \Sigma_1(L_i)$ for all $i$.

First we need to show that $\Sigma_1(L_i)$ is a projective subvariety of $\mathbb{G}(1,3)$. Consider the incidence correspondence

$$\Gamma = \{(L,p) : p \in L \} \subset \mathbb{G}(1,3) \times \mathbb{P}^3,$$

then we have two projection maps:

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\beta} & \mathbb{P}^3 \\
\alpha \downarrow & & \\
\mathbb{G}(1,3) & &
\end{array}$$

Note that the fibers of $\alpha$ are isomorphic to $\mathbb{P}^1$'s, and that the fibers of $\beta$ are isomorphic to $\mathbb{P}^2$'s. Since $\mathbb{G}(1,3)$ is 4 dimensional and $\mathbb{P}^3$ is 3 dimensional, we see that $\Gamma$ must be 5 dimensional. Now $\beta$ is just a $\mathbb{P}^2$ bundle, thus $\beta^{-1}(L_i)$ is smooth and 3 dimensional. Furthermore, since $\Sigma_1(L_i) = \alpha(\beta^{-1}(L_i))$, $\Sigma_1(L_i)$ must be a 3 dimensional and smooth subvariety of $\mathbb{G}(1,3)$ except possibly at the point corresponding to $L_i$ itself.

Then, how do we count the lines in $\cap \Sigma_1(L_i)$? To answer this question we need to understand the Grassmannian and to be able to define and calculate its Chow ring, $A(\mathbb{G})$. Finally, we want to find the class of $\Sigma_1(L_i)$ in $A(\mathbb{G})$ and take its four-fold self-product.

This section closely follows [H], where further details may be found. The Grassmannian is defined to be the set of $k$ dimensional planes in $\mathbb{C}^n$, denoted by $G(k,n)$ or $G(k-1,n-1)$. We want to realize the Grassmannian as a projective variety. First we show that it is a subset of projective space. Start with a $k$ dimensional subset of $\mathbb{P}^n$, call it $W$, and suppose that it is spanned by the vectors $v_1, \ldots, v_k$. We can then associate to $W$ the multi-vector

$$\omega = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k(\mathbb{C}^n).$$
Clearly \( \omega \) is determined up to scalars by \( W \), if we chose a different basis, the corresponding vector would equal \( \omega \) times the determinant of the change of basis matrix. Thus, we have a well defined map of sets

\[
\varphi : G(k, n) \to \mathbb{P}(\bigwedge^k (\mathbb{C}^n)) \quad \text{given by} \quad W \mapsto [v_1 \wedge \cdots \wedge v_k].
\]

For any \( [\omega] = \varphi(W) \) in the image, the space of vectors \( v \in \mathbb{C}^n \) such that \( v \wedge \omega = 0 \in \bigwedge^{k+1}(\mathbb{C}^n) \) is simply \( W \). Thus, \( \varphi \) is an inclusion. Next, we show that \( G(k, n) \) is actually a subvariety of \( \bigwedge^k (\mathbb{C}^n) \). To do this, we need to characterize the subset of totally decomposable vectors \( \omega \in \bigwedge^k (\mathbb{C}^n) \). A multi-vector \( \omega \) will be totally decomposable if and only if the space of vectors \( v \) dividing it is \( k \) dimensional. To see this, note that a vector \( v \in \mathbb{C}^n \) will divide a vector \( \omega \in \bigwedge^k (\mathbb{C}^n) \) if and only if \( \omega \wedge \lambda = 0 \). Thus, \( [\omega] \) will lie in the image of the Grassmannian if and only if the rank of the map

\[
f(\omega) : \mathbb{C}^n \to \bigwedge^{k+1}(\mathbb{C}^n)
\]

\[
v \mapsto \omega \wedge v
\]

is \( n - k \). Note that the rank of \( f(\omega) \) is never strictly less than \( n - k \) and so \( [\omega] \) is in the image of the Grassmannian if and only if the rank(\( f(\omega) \)) \( \leq n - k \). Since the exterior product is multi-linear, the map \( \bigwedge^k (\mathbb{C}^n) \to \text{Hom}(\mathbb{C}^n, \bigwedge^{k+1}(\mathbb{C}^n)) \), sending \( \omega \) to \( f(\omega) \), is linear. This means that the entries of the matrix \( f(\omega) \in \text{Hom}(\mathbb{C}^n, \bigwedge^{k+1}(\mathbb{C}^n)) \) are homogeneous coordinates on \( \mathbb{P}(\bigwedge^k (\mathbb{C}^n)) \). Thus, we see that \( G(k, n) \subset \mathbb{P}(\bigwedge^k (\mathbb{C}^n)) \) is the subvariety defined by the vanishing of the \( (n - k + 1) \times (n - k + 1) \) minors of the matrix \( f(\omega) \).

The Grassmannian can also be viewed as complex manifold. Given a \( k \)-plane \( \Lambda \) in \( \mathbb{C}^n \), it can be represented by a \( k \times n \) matrix

\[
\begin{bmatrix}
v_{1,1} & \cdots & v_{1,n} \\
\vdots & & \vdots \\
v_{k,1} & \cdots & v_{k,n}
\end{bmatrix}
\]

of rank \( k \). Such a matrix determines \( \Lambda \) up to an element of \( GL_k \). Given any multi-index of cardinality \( k \), \( I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \), let \( V_I \subset \mathbb{C}^n \) be the \( (n - k) \)-plane in \( \mathbb{C}^n \) spanned by the vectors \( \{e_j : j \notin I\} \), and consider the set

\[
U_I = \{ \Lambda \in G(k, n) : \Lambda \cap V_I = \{0\} \}.
\]

\( U_I \) is the set of \( \Lambda \in G(k, n) \) such that the \( I \)-th \( (k \times k) \)-minor of any matrix representation of \( \Lambda \) is nonsingular, and \( \Lambda \in U_I \) can be represented uniquely by a matrix \( \Lambda^I \) of the form

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & * & \cdots & * \\
0 & 1 & \cdots & 0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & * & \cdots & *
\end{bmatrix}
\]

Conversely, any such \( k \times n \) matrix represents a \( k \)-plane \( \Lambda \in U_I \). Thus the \( k(n - k) \) entries of the \( I \)-th \( k \times (n - k) \) minor \( \Lambda^I_{k(n-k)} \) of \( \Lambda^I \) give a bijection of sets

\[
\varphi_I : U_I \to \mathbb{C}^{k(n-k)}
\]
for each $I$. Note that $\varphi_I(U_I \cap U_{I'})$ is open in $\mathbb{C}^{k(n-k)}$ for all $I, I'$. Furthermore, the map $\varphi \circ \varphi_{I'}^{-1}$ is holomorphic on this open set since if we choose some $\Lambda \in U_I \cap U_{I'}$ and let $\Lambda_{I'}^I$ be the $I'$-th $k \times k$ minor of $\Lambda^I$, then

$$\Lambda_{I'}^I = (\Lambda_{I'}^I)^{-1} \cdot \Lambda^I.$$ 

Since the entries of $(\Lambda_{I'}^I)^{-1}$ vary holomorphically with the entries of $\Lambda^I$, $\varphi \circ \varphi_{I'}^{-1}$ is holomorphic. Hence the maps $\varphi_I$ give $G(k, n)$ the structure of a complex manifold.

Moreover, the unitary group $U_n$ maps surjectively and continuously onto $G(k, n)$ by the map

$$g \mapsto g(V_k),$$

where $V_k = \{e_1, \ldots, e_k\} \subset \mathbb{C}^n$ (similarly $GL_n$ acts transitively on $G(k, n)$). Thus, with the above topology, $G(k, n)$ is compact and connected.

### 2.1 Schubert Cycles

Our next task is to give a cell decomposition of $G(k, n)$ to understand its additive cohomology. The results and discussion in this section closely follow [GH]. Begin by choosing a flag of linear subspaces of $\mathbb{C}^n$ as follows: let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{C}^n$, and set $V_i = \{e_1, \ldots, e_i\} \subset \mathbb{C}^n$. Then, for every $[\Lambda] \in G(k, n)$, consider the increasing sequence of subspaces

$$0 \subset \Lambda \cap V_1 \subset \Lambda \cap V_2 \subset \cdots \subset \Lambda \cap V_{n-1} \subset \Lambda \cap V_n = \Lambda.$$ 

Clearly, for a generic $\Lambda$, $\Lambda \cap V_i$ will be zero for $i \leq n - k$, and $(i + k - n)$-dimensional for $i > n - k$. Now, given any sequence of integers $a = a_1, \ldots, a_k$, let

$$W_{a_1, \ldots, a_k} = \{\Lambda \in G(k, n) \mid \dim(\Lambda \cap V_{n-k+i-a_i}) = i\}.$$ 

Since $\dim(\Lambda \cup V_{n-k+i-a_i}) = k + (n-k+i-a_i) - i = n - a_i$, $W_{a_1, \ldots, a_k}$ will be empty unless $a_i, \ldots, a_k$ is a non-increasing sequence of integers less than or equal to $n-k$. Furthermore, $\dim(\Lambda \cap V_{n-k+i-a_i}) = i$ if and only if the rank of the last $k \times (k+a_i-i)$ minor of the matrix representative for $\Lambda$ is exactly $k-i$. Thus, the closure

$$\overline{W_{a_1, \ldots, a_k}} = \{\Lambda \in G(k, n) \mid \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}$$

is a subvariety of $G(k, n)$.

Since the $W_a$ will be the cells we are looking for, we want to choose a special basis for any $k$-plane $\Lambda \in W_{a_1, \ldots, a_k}$. Let $v_1$ be a generator for the line $\Lambda \cap V_{n-k+1-a_1}$, normalized so that

$$\langle v_1, e_{n-k+1-a_1} \rangle = 1 \quad \text{or, in other words,} \quad v_1 = (*, *, \ldots, *, 1, 0, \ldots, 0).$$

Next, choose $v_2$ so that $\text{span}\{v_1, v_2\} = (\Lambda \cap V_{n-k+2-a_2})$, and normalized to give

$$\langle v_2, e_{n-k+1-a_1} \rangle = 0, \quad \text{and} \quad \langle v_2, e_{n-k+2-a_2} \rangle = 1.$$ 

We can continue to choose the $v_i$ such that $v_1, \ldots, v_i$ span $\Lambda \cap V_{n-k+i-a_i}$ and

$$\langle v_i, e_{n-k+j-a_j} \rangle = \begin{cases} 0 & \text{if } j < i, \\ 1 & \text{if } j = i. \end{cases}$$

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Since the choice of each $v_i$ is completely determined by the above conditions, we get a unique $k \times (n - k)$ matrix representative for the $k$-plane $\Lambda$

\[
\begin{bmatrix}
* * * & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
* * * & 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
* * * & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* * * & 0 & 0 & * * * & 0 & * * & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* * * & 0 & 0 & * * * & 0 & * * & \cdots & \cdots & \cdots 
\end{bmatrix},
\]

where, in the $i$-th column, the number of zeros after the one is $a_i - i$. Conversely, any such matrix determines a $k$-plane $\Lambda \in W_{a_1, \ldots, a_k}$. Now, $(k^2 + \sum a_i)$ entries of the matrix are fixed and the rest can vary freely. Thus, we have a homeomorphism

\[W_{a_1, \ldots, a_k} \cong \mathbb{C}^{(n-k)-\sum a_i},\]

and so the sets $W_{a_1, \ldots, a_k}$ give a cell decomposition of $G(k, n)$. Note that, we have cells only in even dimensions and therefore all the boundary maps are zero, hence

**Proposition 1 (Griffiths, Harris).** The integral homology of the Grassmannian $G(k, n)$ has no torsion and is freely generated by the cycles $\sigma_{a_1, \ldots, a_k} = [W_{a_1, \ldots, a_k}]$ in real codimension $2 \sum a_i$, where \( \{a_1, \ldots, a_k\} \) ranges over all non-increasing sequences of integers between 0 and $n - k$.

Since we can always find a continuous family of linear automorphisms of $\mathbb{C}^n$ taking any flag to another, the homology class of the subvariety $\sigma_\alpha$ is independent of the flag chosen. The subvarieties $\sigma_\alpha$ are called the Schubert cycles of the Grassmannian.

### 2.2 The Tangent Space to the Grassmannian

We have seen that the Grassmannian $G(k, n)$ is a smooth variety of dimension $(k + 1)(n - k)$. This follows from the explicit description of the covering of $G(k, n)$ by the open sets $U_{\Lambda} \cong \mathbb{A}^{(k+1)(n-k)}$. Thus the Zariski tangent spaces to $G$ are all vector spaces of dimension $(k + 1)(n - k)$.

**Claim 2.** The tangent space to the Grassmannian is given by

\[T_{[\Lambda]} G(k, n) \cong \text{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda).\]

We obtain this identification in terms of tangent vectors to arcs in $G(k, n)$. Suppose that $\{\Lambda_t\}$ is a holomorphic arc in $G(k, n)$, let $\Lambda_0 = \Lambda$, and let $v \in \Lambda$ be any vector. Choose a vector valued function $v(t)$ such that $v(t) \in \Lambda_t$ for all $t$ and, in particular, $v(0) = v$. Then we can associate to $v$ the vector

\[\varphi(v) = v'(0) = \frac{d}{dt} \bigg|_{t=0} v(t) \in \mathbb{C}^{n+1}/\Lambda.\]

This measures the movement of $\Lambda$ away from $v$, and if $\varphi(v)$ is nonzero, we cannot choose $v(t) \equiv v$. Now, we could have chosen a different vector valued function, say $w(t)$ with $w(t) \in \Lambda_t$ for all $t$. 

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Then, since \( w(0) = v(0) \), we can write

\[ w(t) - v(t) = tu(t) \]

where \( u(t) \in \Lambda_t \) for all \( t \). It follows that \( \varphi(v) \) is well defined as an element of \( \mathbb{C}^{n+1}/\Lambda \). Thus, the arc \( \{\Lambda_t\} \) determines a linear map \( \varphi : \Lambda \rightarrow \mathbb{C}^{n+1}/\Lambda \), which is a tangent vector of \( G(n, k) \) at \( \Lambda \). So the identification

\[ T_{\Lambda_t} G(k, n) \cong \text{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda) \]

holds.
Chapter 3

The Chow Ring

The most relevant for this section if [F 98], where the proofs of the following results may be found. Let $X$ be an algebraic variety. A $k$-cycle on $X$ is a formal sum

$$
\sum n_i[V_i]
$$

where the $V_i$ are $k$-dimensional subvarieties of $X$, and the $n_i$ are integers. Let $Z_n(X)$ be the set of $k$-cycles,

$$
Z_n(X) = \{ \sum n_i[V_i] : V_i \text{ $k$-dimensional subvariety of } X, n_i \in \mathbb{Z} \}.
$$

The first step in defining the Chow ring is to formulate the correct equivalence relation on $Z_n(X)$, called rational equivalence.

Let $R(X)$ be the field of rational functions on $X$, and let $R(X)^*$ denote the ring of nonzero elements of this field. Let $V$ be a subvariety of $X$ of codimension one, and thus the local ring $A = \mathcal{O}_{V,X}$ is a one-dimensional local domain. Let $r \in R(X)^*$, then we will define a homomorphism called the order of vanishing of $r$ along $V$, $\text{ord}_V(r)$. Note that, since any $r \in R(X)^*$ may be written as a ratio $r = a/b$, for $a, b \in A$, and since for a homomorphism,

$$
\text{ord}_V(rs) = \text{ord}_V(r) + \text{ord}_V(s)
$$

for any $r, s \in R(X)^*$, we only have to define $\text{ord}_V(r)$ for $r \in A$. It is clear what $\text{ord}_V(r)$ should be in two special cases. First, if $X$ is a curve over an algebraically closed field $K$,

$$
\text{ord}_V(r) = \dim_K(A/(r)).
$$

Next, if $X$ is nonsingular along $V$ of any dimension, $A$ is a DVR. Then $r = ut^m$, for $u \in A$ a unit, $t$ a generator of the maximal ideal and $m \in \mathbb{Z}$. Thus, it is natural to define

$$
\text{ord}_V(r) = m.
$$

In general, we define

$$
\text{ord}_V(r) = \text{length}(A/(r))
$$

as an $A$-module.

For any $(k+1)$-dimensional subvariety $W$ of $X$, and any $r \in R(X)^*$, define a a $k$-cycle on $X$ by

$$
[\text{div}(r)] = \sum \text{ord}_V(r)[V],
$$
where the sum is over all codimension 1 subvarieties $V$ of $W$.

**Definition 3.** A $k$-cycle $\beta$ is rationally equivalent to zero, $\beta \sim 0$, if there exists a finite number of $(k + 1)$-dimensional subvarieties $W_i$ of $X$ and $\tau_i \in R(W_i)^*$, such that $\beta = \sum [\text{div}(\tau_i)]$.

Since $[\text{div}(r^{-1})] = -[\text{div}(r)]$, the cycles rationally equivalent to zero form a subgroup $\text{Rat}_k(X)$ of $Z_k(X)$.

**Definition 4.** The group of $k$-cycles modulo rational equivalence on $X$ is the quotient group

$$A_k(X) = Z_k(X)/\text{Rat}_k(X).$$

Furthermore, we can define

$$Z_*(X) = \bigoplus_{k=0}^{n} Z_k(X) \quad A_*(X) = \bigoplus_{k=0}^{n} A_k(X),$$

where $n$ equals the dimension of $X$.

**Remark 1.** Two cycles are rationally equivalent if they are members of a family of cycles parametrized by $\mathbb{P}^1$.

Now, for an $n$-dimensional, nonsingular variety $X$, set

$$A^k(X) = A_{n-k}(X),$$

and let $1 \in A^0(X)$ correspond to the class $[X] \in A_n(X)$. By $A^*(X) = \oplus A^p(X)$.

### 3.1 Intersection Pairing and the Gysin Homomorphisms

We want to define a pairing: $A_k(X) \times A_l(X) \to A_{k+l-n}(X)$.

There are three different approaches to this problem.

1. If $V, W \subset X$ intersect generically transversely, set $[V] \cap [W] = [V \cap W]$, and then apply Chow's moving lemma, proven in [R].

2. If $V, W$ only intersect properly (i.e. in the correct dimension), we can try to define the intersection product as $[V] \cdot [W] = \sum_{Z \subset V \cap W} \text{mult}_Z(V \cdot W)[Z]$.

3. (Which we will follow). In general, given any two $V, W \subset X$, define $[V] \cdot [W] \in A_{k+l-n}(V \cap W) \to A_{k+l-n}(X)$. More precisely, if $X$ is a non-singular variety, the diagonal embedding $\delta : X \hookrightarrow X \times X$ is a regular embedding. Thus, for $[V], [W] \in A_*(X)$, we define the product $[V] \cdot [W] \in A_*(X)$ by the composition

$$A_k(X) \otimes A_l(X) \xrightarrow{\otimes} A_{k+l}(X \times X) \xrightarrow{\delta^*} A_{k+l-n}(X),$$

where $\delta^*$ is the Gysin homomorphism. We start by describing a special type of Gysin homomorphism, and then use it to obtain the general case.

**Definition 5.** Let $s = s_\mathcal{E}$ denote the zero section of a vector bundle $\pi : \mathcal{E} \to X$ of rank $r$. We define the $k$-th Gysin homomorphism

$$s^* : A_k\mathcal{E} \to A_{k-r}X,$$

by

$$s^*(\beta) = (\pi^*)^{-1}(\beta).$$
The following theorem shows that $s^*$ is a well defined homomorphism.

**Theorem 6 (Fulton).** Let $E$ be a vector bundle of rank $r$ on a scheme $X$, with projection $\pi : E \to X$. Then, the flat pull-back

$$\pi^* : A_{k-r}X \to A_kE$$

is an isomorphism for all $k$.

The Gysin homomorphism is an important intersection operation. Given any subvariety of $E$, or $k$-cycle $\beta$ on $E$, no matter how it meets the zero section, there is a well-defined cycle class $s^*(\beta)$ in $A_{k-r}X$. By the surjectivity of $\pi^*$, $s^*$ is determined by the facts that $s^*[\pi^{-1}(V)] = [V]$ for all $V \subset X$, and that $s^*$ preserves rational equivalence.

We actually need a more general form of the Gysin homomorphism to understand equation (3.1).

Let $i : X \to Y$ be a closed regular embedding of codimension $d$, and denote the normal bundle by $N_XY$. Let $V$ be a purely $k$-dimensional scheme, $f : V \to Y$ be a morphism, and let $f^{-1}(X) = W$, then form the fiber square

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ \downarrow{g} & & \downarrow{f} \\ X & \xrightarrow{i} & Y \end{array}$$

Let $N = g^*N_XY$, a bundle of rank $d$ on $W$, and let $\pi : N \to W$ be the projection map. Since the ideal sheaf $I$ of $X$ in $Y$ generates the ideal sheaf $\mathcal{J}$ of $W$ in $V$, there is a surjection

$$\bigoplus_n f^*(I^n/I^{n+1}) \to \bigoplus_n \mathcal{J}^n/\mathcal{J}^{n+1}.$$ 

This determines a closed embedding of the normal cone $C = C_WV$ (for the definition of the normal cone, see appendix A), as a subcone of the vector bundle $N$

$$\begin{array}{ccc} C & \subset & N \\ & & \downarrow{\pi} \\ & W \end{array}$$

Since $C$ is purely $k$-dimensional, it determines a $k$-cycle $[C]$ on $N$. Let $s$ be the zero section of the bundle $N$, then

**Definition 7.** The intersection product of $V$ by $X$ on $Y$, denoted $X \cdot V$, is the class on $W$ obtained by

$$X \cdot V = s^*[C] \in A_{k-d}(W),$$

where $s^* : A_k(N) \to A_{k-d}(W)$ is the Gysin homomorphism defined earlier.

Now we can define the refined Gysin homomorphism used in equation (3.1). Let $i : X \to Y$ be a regular embedding of codimension $d$ (i.e., every point in $X$ has an affine neighborhood $U$ in $Y$ such that, if $A$ is the coordinate ring of $U$, and $I$ the ideal of $A$ defining $X$, then $I$ is generated by
a regular sequence of length \( d \), and let \( f: Y' \to Y \) be a morphism. Form the fiber square

\[
\begin{array}{ccc}
X' & \to & Y' \\
\downarrow^g & & \downarrow^f \\
X & \to & Y,
\end{array}
\]

and define the homomorphisms

\[
i^! : Z_k Y' \to A_{k-d} X'
\]

by

\[
i^!(\sum n_j[V_j]) = \sum n_jX \cdot V_j,
\]

where \( X \cdot V_j \) is the intersection product defined above. \( i^! \) passes to rational equivalence, as shown in [F 98]. Assuming such result, we can finally make the following definition.

**Definition 8.** The induced homomorphisms

\[
i^! : A_k Y' \to A_{k-d} X'
\]

are called the **refined Gysin homomorphisms**.

If \( Y' = Y \) and \( f = id_Y \), then, the induced morphisms are called **Gysin homomorphisms** and denoted by

\[
i^* : A_k Y \to A_{k-d} X.
\]

Then, the intersection product can be rewritten as \([V] \otimes [W] \to [V] \cdot [W]\) and

\[
A^k(X) \otimes A^l(X) \to A^{k+l}(X),
\]

indexing by codimension.

**Definition 9.** With the above product \( A^*(X) = \oplus A^p(X) \) is a commutative, graded ring, with unit \([X]\). \( A^*(X) \) is called the **Chow ring** of \( X \).

### 3.2 The cycle map

If the base field is \( \mathbb{C} \) and \( V \subset X \) a subvariety of dimension \( k \), we can look at the class of \( V \), \([V] \in H^{2n-2k}(X, \mathbb{C})\), and the map: \( A_*(X) \to H^*(X, \mathbb{C}) \) sending the class of \( V \) in \( A_*(X) \) to the corresponding class in \( H^*(X, \mathbb{C}) \). In this case, we can define the intersection pairing to be the cup product in cohomology.

We want to find a class of spaces where

\[
A_* X \cong H_* X.
\]

We start with the following result.

**Lemma 10 (Fulton).** If \( X \) is an \( n \)-dimensional complex non-singular scheme, then \( H_i X = 0 \) for \( i > 2n \), and \( H_{2n} X \) is a free abelian group with one generator for each irreducible component of \( X \).
The proof of this lemma is simply given by the Poincaré isomorphism and the following property of homology:

$$H_iX = H_iX_1 \oplus \cdots \oplus H_iX_m,$$

where $X$ is the disjoint union of the $X_i$. The generator of $H_{2n}X$ corresponding to an $n$-dimensional irreducible component $X_i$ will be denoted $cl(X_i)$. In general, if $Y$ is a $k$-dimensional closed subvariety of a scheme $X$, we define the cycle class $cl_X(Y)$ to be

$$cl_X(Y) = i_*cl(Y) \in H_{2k}X,$$

where $i$ is the inclusion of $Y$ in $X$. Here are some properties of $cl_X$.

**Lemma 11 (Fulton).** Let $f : Y \to Z$ be a proper, surjective morphism of varieties. Then

$$f_*cl_X(Y) = \deg(Y/Z) \cdot cl(Z).$$

Furthermore, if a cycle $\alpha$ on a complex scheme $X$ is rationally equivalent to zero, then $cl(\alpha) = 0 \in H_*X$.

For a proof and a more general statement of the above lemma see [F 98]. It follows that the cycle map passes to rational equivalence, defining homomorphisms

$$cl : A_kX \to H_{2k}X,$$

which are covariant for proper morphisms.

The cycle map is the morphism we want since there are many interesting schemes $X$ for which

$$cl_X : A_*X \to H_*X$$

is an isomorphism. These varieties can have no odd-dimensional homology, and so the only curves in this class are $\mathbb{P}^1$ and $A^1$. On the other hand, if $X$ has a cellular decomposition (i.e., $X$ has a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$ by closed subschemes, with each $X_i - X_{i-1}$ a disjoint union of schemes $U_{ij}$ isomorphic to affine spaces $A^{m_{ij}}$), then $cl_X$ will be an isomorphism. The most relevant cases are those of $X$ a projective space, Grassmannian, or an arbitrary flag manifold, and then $cl_X$ will also be an isomorphism. More generally, if $X$ is a flag bundle over a scheme $Y$, then $cl_X$ is an isomorphism if and only if $cl_Y$ is an isomorphism.

### 3.3 Properties of the Chow Ring

Let $X$ be a nonsingular projective variety. Then its Chow ring has the following properties.

**C1** The cycles in codimension 1 are Weil divisors, and rational equivalence is the same as linear equivalence for them. Thus, since $X$ is nonsingular, we have that $A^1(X) \cong \text{Pic}X$.

**C2** The projection map $p : X \times A^m \to X$, $A^m$ any affine space, induces an isomorphism $p^* : A(X) \to A(X \times A^m)$.

**C3** If $Y$ is a nonsingular closed subvariety of $X$, $U = X - Y$, and $i : Y \to X$, $j : U \to X$ the inclusions, then there is an exact sequence

$$A(Y) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(U) \longrightarrow 0.$$
C4 Let $\mathcal{E}$ be a vector bundle of rank $k$ on $X$, let $\pi: \mathbb{P}\mathcal{E} \to X$ be the associated projective bundle, and let $\zeta \in A^1(\mathbb{P}\mathcal{E})$ be the class of the divisor whose restriction to the fiber of $\pi$ is the hyperplane class. Then $\pi^*$ makes $A(\mathbb{P}\mathcal{E})$ into a free $A(X)$-module generated by $1, \zeta, \zeta^2, \ldots, \zeta^{k-1}$.

3.4 Lines meeting four general lines

Recall our first question. Given four general lines $L_1, L_2, L_3, L_4$ in $\mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ meet all four? To answer it, we need to find the four-fold intersection product of the class of $\Sigma_1(L_i)$, where for each $L_i \Sigma_1(L_i)$ the subset of $\mathbb{G}(1, 3)$ given by

$$\Sigma_1(L_i) = \{\text{lines in } \mathbb{P}^3 \text{ meeting } L_i\}.$$ 

We start by calculating $A_*G$, where $G = \mathbb{G}(1, 3)$. The cohomology groups of $\mathbb{G}(1, 3)$ are as follows

$$A_4(G) = \mathbb{Z}\langle G \rangle = \mathbb{Z}\langle \sigma_0 \rangle$$

$$A_3(G) = \mathbb{Z}\langle \sigma_1 \rangle$$

$$A_2(G) = \mathbb{Z}\langle \sigma_{1,1}, \sigma_2 \rangle$$

$$A_1(G) = \mathbb{Z}\langle \sigma_{2,1} \rangle$$

$$A_0(G) = \mathbb{Z}\langle \sigma_{2,2} \rangle$$

Where the generators are

$$\sigma_0 = [\mathbb{G}(1, 3)],$$

$$\sigma_1 = [\Sigma_1(L) = \{\text{lines meeting } L \subset \mathbb{P}^3\}],$$

$$\sigma_{1,1} = [\Sigma_{1,1}(H) = \{\text{lines in the hyperplane } H\}],$$

$$\sigma_2 = [\Sigma_2(p) = \{\text{lines containing the point } p\}],$$

$$\sigma_{2,1} = [\Sigma_{2,1}(p, H) = \{\text{lines containing the point } p \text{ and contained in the hyperplane } H\}],$$

$$\sigma_{2,2} = [\text{point}].$$

Now, we need to calculate the ring structure, that is, we need to calculate the intersection pairings of the above Schubert cycles

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma_{1,1}$</th>
<th>$\sigma_{2,1}$</th>
<th>$\sigma_{2,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>$\sigma_2 + \sigma_{1,1}$</td>
<td>$\sigma_{2,1}$</td>
<td>$\sigma_{2,1}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$\sigma_{2,1}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{1,1}$</td>
<td>$\sigma_{2,1}$</td>
<td>0</td>
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<tr>
<td>$\sigma_{2,1}$</td>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>$\sigma_{2,2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then, the number of lines meeting four general lines in $\mathbb{P}^3$ is $\sigma_4^1 = 2$. \hfill ◊

**Note:** So far in our calculation we have been implicitly assuming that all of our intersections were transverse, that is
**Definition 12.** Suppose that $X, Y \subseteq \mathbb{G}(1,3)$ are two subvarieties and that their intersection has irreducible components $Z_i$. We say that $X$ and $Y$ intersect transversely at a point $p_i \in Z_i$, if they are smooth at $p_i$ with tangent spaces spanning $T_{p_i}(\mathbb{G}(1,3))$.

The following theorem, proved in [K], allows us to treat the intersections as transverse

**Theorem 13 (Kleiman).** Let $X$ be any variety on which an algebraic group $G$ acts transitively, $Y \subseteq X$ any smooth subvariety, and $f : Z \to X$ any map. For any $g \in G$ let $W_g = (g \circ f)^{-1}(Y) \subseteq Z$—that is, the inverse image of the translate $g(Y)$. Then for general $g \in G$,

$$(W_g)_{\text{sing}} = W_g \cap Z_{\text{sing}}.$$

Note that the theorem holds for Grassmannians. Furthermore, if $f$ is an inclusion, this says that a general translate of one smooth subvariety of a homogeneous space meets any other given smooth subvariety transversely.
Chapter 4

Chern and Segre Classes

We will start with a working definition of Chern Classes and then refine it to a more precise formulation. Let \( X \) be a \( n \)-dimensional variety and \( E \rightarrow X \) a vector bundle of rank \( k \). Given a section \( \sigma \) of \( E \), we would expect the zero locus of \( \sigma \) to be a cycle of dimension \( n - k \).

**Definition 14.** The top Chern class of \( E \) is defined to be

\[
c_k(E) = [(\sigma = 0)] \in A^k(X).
\]

More generally, given a collection of \( l \) sections \( \sigma_1, \ldots, \sigma_l \) of \( E \) we can look at the locus where they fail to be independent. We would expect this locus to be a cycle of codimension \( k - l + 1 \).

**Definition 15.** The \( r \)-th Chern class of \( E \) is defined to be

\[
c_r(E) = [(\sigma_1 \wedge \cdots \wedge \sigma_{k-r+1} = 0)] \in A^r(X).
\]

**Definition 16.** Let \( E \) be a vector bundle of rank \( r \) on a variety \( X \). The total Chern class \( c(E) \) is the sum

\[
c(E) = 1 + c_1(E) + \cdots + c_r(E).
\]

The Chern polynomial is:

\[
c_t(E) = \sum_{i=0}^{\infty} c_i(E)t^i = 1 + c_1(E)t + c_2(E)t^2 + \cdots .
\]

Although it is not clear at this point that Chern classes are well defined, we are going to use the above definitions to give a few examples of why Chern classes are useful.

First we need to introduce some ubiquitous vector bundles over the Grassmannian. Let \( G = G(k, n) \) and let \( V = \mathbb{C}^{n+1} \times G \) be the trivial bundle. We define the universal, or tautological, sub-bundle \( S \subset V \) to be the bundle with total space given by the pairs

\[
(k \text{ - plane } \Lambda \in \mathbb{C}^{n+1}, \text{ vector } v \in \Lambda),
\]

and with projection map \( \pi : V \rightarrow G \) given by \( \pi(\Lambda, v) = \Lambda \). Note that \( S_{[\Lambda]} = \Lambda \). There is another natural bundle on the Grassmannian arising from the two above. The universal quotient bundle \( Q \) is simply the quotient \( Q = V/S \) and each fiber is given by \( Q_{[\Lambda]} = \mathbb{C}^{n+1}/\Lambda \). This gives rise to the
following sequence of vector bundles

\[ 0 \longrightarrow S \longrightarrow V \longrightarrow Q \longrightarrow 0, \]

called the universal exact sequence over the Grassmannian. These vector bundles give us another way of describing the tangent space to the Grassmannian

\[ T_G = \text{Hom}(S, Q), \]

where the homomorphisms are to be taken as vector bundle maps.

**Example 1.** We want to calculate the Chern classes of the dual universal bundle \( S^* \) on \( G(1, 3) = G(2, 4) \), a rank 2 bundle. First note that a fiber of \( S^* \) over a point of the Grassmannian is simply \( S^*_{[\Lambda]} = \Lambda^* \). Now, given a homogeneous linear form \( \lambda : \mathbb{C}^4 \rightarrow \mathbb{C} \) we define a section \( \sigma \) of \( S^* \) by setting:

\[ \sigma(\Lambda) = \lambda|_{\Lambda}. \]

What is then \( c_2(S^*) \)? Using our working definition of the top Chern class we see that it simply the zero locus

\[ [\sigma = 0] = \{ \Lambda : \Lambda \subset H \} = \text{the Schubert cycle } \sigma_{1,1}, \]

where \( H = \ker(\lambda) \). Now for the next Chern class we need to define two sections of \( S^* \). Choose two homogeneous linear forms \( \alpha, \beta : \mathbb{C}^4 \rightarrow \mathbb{C} \). Then define the sections \( \omega, \tau \) to be \( \omega(\Lambda) = \alpha|_{\Lambda} \) and \( \tau(\Lambda) = \beta|_{\Lambda} \), both in \( \Lambda^* \). Now we can proceed to calculate \( c_1(S^*) = [(\omega \wedge \tau = 0)] \). Note that:

\[ \omega \wedge \tau = 0 \Leftrightarrow \ker \alpha|_{\Lambda} \cap \ker \beta|_{\Lambda} \neq 0 \Leftrightarrow \ker \alpha \cap \Lambda \cap \ker \beta \neq 0, \]

but the last statement simply says that \( c_1(S^*) \) is the class of lines in \( \mathbb{P}^3 \) meeting a given line, that is the Schubert cycle \( \sigma_1 \). \hfill \diamond

Related to the notion of Chern classes there is an inverse concept, the Segre classes. Recall that we defined the \( i \)-th Chern class to be

\[ c_i(E) = \text{[locus where } \sigma_1, \ldots, \sigma_{k-i+1} \text{ are linearly dependent]}. \]

On the other hand we could have considered the locus where \( \sigma_1, \ldots, \sigma_{k+i-1} \) fail to span. We are going to call this locus the \( i \)-th Segre class. More precisely,

**Definition 17.** Given a vector bundle \( E \) of rank \( k \) over a variety \( X \), we define its \( i \)-th Segre class to be:

\[ s_i(E) = (-1)^i \text{[locus where } \sigma_1, \ldots, \sigma_{k+i-1} \text{ fail to span}] \in A^i(X). \]

At this point we would like to see how these two notions are related. In the \( C^\infty \) category, every vector bundle \( E \rightarrow X \) of rank \( k \) is the pull-back of \( Q \), the universal quotient bundle over the Grassmannian, to \( X \) for some map \( X \rightarrow G(n-k, n) \) for some \( n \). Explicitly, find \( n \) generating sections for \( E \), call them \( \omega_1, \ldots, \omega_n \), and define the map \( X \rightarrow G(n-k, n) \) by
\[ \varphi : X \to G(n - k, n) \]
\[ x \mapsto \{(a_1, \ldots, a_n) : \Sigma a_i \omega_i(x) = 0 \in \mathcal{E}_x\}, \]

and hence we see that \( \mathcal{E} = \varphi^*Q \). Now we can make a second definition of the Chern and Segre classes.

**Definition 18.** Let \( \mathcal{E} \) be a rank \( k \) vector bundle over a variety \( X \), let \( \varphi : X \to G(n - k, n) \) be as above. Then we define the \( i \)-th Chern class of \( \mathcal{E} \) to be:

\[ c_i(\mathcal{E}) = \varphi^*\sigma_i. \]

Similarly we can define the \( i \)-th Segre class of \( \mathcal{E} \) to be:

\[ s_i(\mathcal{E}) = (-1)^i \varphi^*\sigma_{1,\ldots,1}. \]

Where \( \sigma_i \) and \( \sigma_{1,\ldots,1} \) are Schubert cycles over \( G(n - k, n) \).

**Proposition 19.** Let \( \mathcal{E} \) be a rank \( n \) bundle over a space \( X \), then

\[ s(\mathcal{E}) = \frac{1}{c(\mathcal{E})}. \]

**Proof.** By definition 18 we have that \( c_i(\mathcal{E}) = \varphi^*\sigma_i \), and thus

\[ c(\mathcal{E}) = 1 + \varphi^*\sigma_1 + \varphi^*\sigma_2 + \cdots + \varphi^*\sigma_n = \varphi^*(1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n). \]

Then we see that

\[
\frac{1}{1 + \sigma_1 + \cdots + \sigma_n} = 1 + \sum_{i=1}^{\infty} (-1)^i (\sigma_1 + \sigma_2 + \cdots + \sigma_n)^i
\]

\[
= \frac{1}{T_0} + \frac{(-\sigma_1)}{T_1} + \frac{(-\sigma_1^2 - \sigma_2)}{T_2} + \frac{(-\sigma_1^3 + 2\sigma_1\sigma_2 - \sigma_3)}{T_3} + \cdots,
\]

and thus the \( d \)-th term will simply be

\[ T_d = \sum_{\sum_{i=1}^k \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} = d} (-1)^k \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \]

and

\[ \frac{1}{1 + \sigma_1 + \cdots + \sigma_n} = 1 + \sum_{d=1}^{\infty} T_d. \]

Recall Giambelli's formula to express any Schubert cycle in terms of elementary ones

\[
\sigma_{a_1, \ldots, a_n} = \begin{vmatrix}
\sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+d-1} \\
\sigma_{a_2-1} & \sigma_{a_2} & \cdots & \cdots & \sigma_{a_2+d-2} \\
\sigma_{a_3-2} & \cdots & \cdots & \cdots & \sigma_{a_3} \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
\sigma_{a_d-d+1} & \cdots & \cdots & \cdots & \sigma_{a_d}
\end{vmatrix}. 
\]
By definition 18, \( s_i(E) = (-1)^i \varphi^* \sigma_1 \cdots \sigma_1 \), and so we only need a special case of the above determinant. Using the facts that \( \sigma_0 = 1 \) and \( \sigma_{r < 0} = 0 \), we see that

\[
\begin{vmatrix}
\sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_d \\
1 & \sigma_1 & \sigma_2 & \cdots & \sigma_{d-1} \\
0 & 1 & \sigma_1 & \cdots & \sigma_{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \sigma_1 & \sigma_1
\end{vmatrix}
\]

\[\text{d-times}\]

We will now calculate \( \sigma_1 \cdots 1 \) in terms of the \( \sigma_i \). Define \( P_d \) to be the above determinant and let

\[
Q(b_1, \cdots, b_d) = \begin{vmatrix}
b_1 & b_2 & b_3 & \cdots & b_d \\
1 & \sigma_1 & \sigma_2 & \cdots & \sigma_{d-1} \\
0 & 1 & \sigma_1 & \cdots & \sigma_{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \sigma_1 & \sigma_1
\end{vmatrix}
\]

so that \( Q(\sigma_1, \cdots, \sigma_d) = P_d \). By expanding the determinant along the first column, it is clear that

\[
Q(b_1, \cdots, b_d) = b_1 Q(\sigma_1, \cdots, \sigma_{d-1}) - Q(b_2, \cdots, b_d)
= b_1 P_{d-1} - Q(b_2, \cdots, b_d).
\]

Hence, we can write

\[
P_d = Q(\sigma_1, \cdots, \sigma_d) = \sigma_1 P_{d-1} - Q(\sigma_2, \cdots, \sigma_d)
= \sigma_1 P_{d-1} - (\sigma_2 P_{d-2} - Q(\sigma_3, \cdots, \sigma_d))
= \sigma_1 P_{d-1} - \sigma_2 P_{d-2} + (\sigma_3 P_{d-3} - Q(\sigma_4, \cdots, \sigma_d))
= \cdots
= \sigma_1 P_{d-1} - \sigma_2 P_{d-2} + \sigma_3 P_{d-3} - \cdots + (-1)^d \sigma_{d-1} P_1 + (-1)^{d+1} \sigma_d,
\]

that is

\[
\sigma_1 \cdots 1 = P_d = \sigma_1 P_{d-1} - \sigma_2 P_{d-2} + \sigma_3 P_{d-3} - \cdots + (-1)^d \sigma_{d-1} P_1 + (-1)^{d+1} \sigma_d. \tag{4.1}
\]

We claim that

\[
P_d = \begin{cases}
\sum_{i_j = -d} (-1)^{d-k} \sigma_{i_1} \cdots \sigma_{i_k} & \text{if } d > 0, \\
0 & \text{if } d \leq 0.
\end{cases}
\]

We will prove the claim by induction on (4.1). If \( d = 1 \), \( P_d = \sigma_1 \) and the claim is obviously true.

Then suppose that the claim holds for \( P_1, P_2, \cdots, P_{d-1} \), we want to prove that it holds for \( P_d \).
By (4.1) we have that
\[
P_d = \sigma_1 P_{d-1} - \sigma_2 P_{d-2} + \sigma_3 P_{d-3} - \cdots + (-1)^d \sigma_{d-1} P_1 + (-1)^{d+1} \sigma_d
\]
\[
= \sigma_1 \sum_{m=d-1} \sum_{\Sigma_i=m} (-1)^{d-1-m} \sigma_{i_1} \cdots \sigma_{i_m} - \sigma_2 \sum_{m=d-2} \sum_{\Sigma_i=m} (-1)^{d-2-m} \sigma_{i_1} \cdots \sigma_{i_m} + \cdots
\]
\[
+ (-1)^d \sigma_{d-1} \sigma_1 + (-1)^{d+1} \sigma_d
\]
\[
= \sum_r (-1)^{r-1} \sum_{m=d-r} (-1)^{d-r-m} \sigma_{i_1} \cdots \sigma_{i_m}
\]
\[
= \sum_r \sum_{m=d-r} (-1)^{d-(m+1)} \sigma_r \sigma_{i_1} \cdots \sigma_{i_m} = \sum_{m=d} (-1)^{d-(m+1)} \sigma_{i_1} \cdots \sigma_{i_{m+1}}.
\]

Let \( k = m + 1 \) and
\[
P_d = \sum_k (-1)^{d-k} \sigma_{i_1} \cdots \sigma_{i_k},
\]
as claimed. Finally recall that
\[
T_d = \sum_k (-1)^k \sigma_{i_1} \cdots \sigma_{i_k},
\]
and clearly \( T_d = (-1)^d P_d \), which proves that \( s(\mathcal{E}) = \frac{1}{c(\mathcal{E})} \). \( \square \)

### 4.1 Properties of Chern Classes

In general it very difficult to calculate Chern classes directly from the definition. In practice most calculations are carried out using properties of Chern classes rather than their definition. The following properties can be found in greater generality and with their proofs in [Hart].

The first four properties of Chern classes listed below can be used to show that, regardless of definition, there exists a unique theory of Chern classes. That is, for each vector bundle \( \mathcal{E} \) on \( X \) there is a way of assigning a class \( c_i(\mathcal{E}) \in A^iX \), satisfying C1, C2, C3 below. C4 is called the *splitting principle* and it is used to prove the uniqueness of such \( c_i(\mathcal{E}) \).

C1 If \( \mathcal{E} \cong O(D) \), then \( c_i(\mathcal{E}) = 1 + Dt \). In this case \( \mathbb{P}\mathcal{E} = X \) and the hyperplane class is just \( [D] \), that is, \( c_1(\mathcal{E}) = D \).

C2 The total Chern class \( c \) is a natural transformation of functors, \( c : \text{Vect}(X) \to A^*(X) \). Thus, if \( f : X' \to X \) is a morphism, and \( \mathcal{E} \) is a vector bundle on \( X \), then for each \( i \)
\[
c_i(f^*\mathcal{E}) = f^*c_i(\mathcal{E}).
\]

C3 *Whitney product* formula. If \( 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \) is an exact sequence of vector bundles on \( X \), then
\[
c_i(\mathcal{E}) = c_i(\mathcal{E}') \cdot c_i(\mathcal{E}'').
\]

C4 The *splitting principle*. A vector bundle \( \mathcal{E} \) is said to split if there exists a filtration \( \mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \cdots \supseteq \mathcal{E}_k = 0 \) whose successive quotients are all vector bundles. Now, suppose that \( \mathcal{E} \)
splits and, furthermore, that the filtration has line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_k \) as quotients, then
\[
c_t(\mathcal{E}) = \prod_{i=1}^{k} c_t(\mathcal{L}_i).
\]

The splitting principle is a very useful tool in calculating the Chern classes of tensor products, exterior products, and dual vector bundles. Let \( \mathcal{E} \) have rank \( k \), and let \( \mathcal{F} \) have rank \( s \). If we write
\[
c_t(\mathcal{E}) = \prod_{i=1}^{k} (1 + a_i t) \quad \text{and} \quad c_t(\mathcal{F}) = \prod_{i=1}^{s} (1 + b_i t),
\]
where the \( a_i \) and the \( b_i \) are just formal symbols, and then we have

\[C5\]
\[
c_t(\mathcal{E} \otimes \mathcal{F}) = \prod_{i,j} (1 + (a_i + b_j) t) \\
c_t(\wedge^p \mathcal{E}) = \prod_{1 \leq i_1 < \cdots < i_p \leq k} (1 + (a_{i_1} + \cdots + a_{i_p}) t) \\
c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E}).
\]

If we expand the expressions in \( C5 \), we see that the coefficients of each power of \( t \) are symmetric functions in the \( a_i \) and the \( b_j \). Thus, they can be expressed as polynomials in the elementary symmetric functions of the \( a_i \) and the \( b_j \), which are the Chern classes of \( \mathcal{E} \) and \( \mathcal{F} \).

### 4.2 Projective Bundles

In section 3.3 we mentioned the associated projective bundle, \( \mathbb{P}\mathcal{E} \), to a vector bundle \( \mathcal{E} \). In this section we will develop some its properties.

**Definition 20.** Consider a vector bundle \( \mathcal{E} \to X \) of rank \( k \), we define
\[
\mathbb{P}\mathcal{E} = \{(x,l) : l \subset \mathcal{E}_x \} \to X,
\]
called the *projective bundle* of \( \mathcal{E} \). It is a \( \mathbb{P}^{k-1} \) bundle over \( X \).

Now, is there a class \( \zeta \in H^2(\mathbb{P}\mathcal{E}) \) whose restriction to the fiber of \( \pi : \mathbb{P}\mathcal{E} \to X \) is the hyperplane class? Consider
\[
0 \to S \to \pi^* \mathcal{E} \to Q \to 0.
\]
Then, we claim that $\zeta = c_1(S^*)$. Furthermore,

$$H^*(\mathbb{P}\mathcal{E}, \mathbb{Z}) = \bigoplus_{i=0}^{k-1} \zeta^i H^*(X, \mathbb{Z}),$$

as a group.

We prove that $\zeta = c_1(S^*)$. Begin by considering the case where $X = \{x\}$, a single point. In this case $\mathbb{P}\mathcal{E} = \mathbb{P}^{k-1}$ and $\zeta = c_1(S^*)$, by definition. Now, for general $X$, consider the restriction of $\mathbb{P}\mathcal{E}$ to a point

$$\pi^{-1}(x) \xrightarrow{i} \mathbb{P}\mathcal{E}.$$ 

Then, we can apply property C2 of Chern classes to see that $c_1(i^*S^*) = i^*c_1(S^*)$, where $c_1(i^*S^*)$ is just the hyperplane class on a fiber and $i^*c_1(S^*)$ is the restriction of the class $c_1(S^*)$ to a fiber. Thus, we have that $\zeta = c_1(S^*)$, as claimed.

The statement that, as groups,

$$H^*(\mathbb{P}\mathcal{E}, \mathbb{Z}) \cong \bigoplus_{i=0}^{k-1} \zeta^i H^*(X, \mathbb{Z}),$$

is a direct consequence of the Leray-Hirsch theorem. For a proof of this theorem see [BT].

Now, consider the case where $\mathcal{E}$ is trivial. Then, as rings,

$$H^*(\mathbb{P}\mathcal{E}, \mathbb{Z}) = H^*(X, \mathbb{Z}) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{Z}) = H^*(X, \mathbb{Z})[\zeta]/(\zeta^k).$$

In general, $\zeta^k \neq 0$ and so it will be a measure of the twisting of $\mathcal{E}$. The following proposition gives $\zeta^k$ in terms of the Chern classes of $\mathcal{E}$.

**Proposition 21.** Let $\mathcal{E} \to X$ be a vector bundle of rank $k$, then

$$H^*(\mathbb{P}\mathcal{E}, \mathbb{Z}) = \frac{H^*(X, \mathbb{Z})[\zeta]}{(\zeta^k + \zeta^{k-1}c_1(\mathcal{E}) + \cdots + c_k(\mathcal{E}))}.$$ 

**Proof.** Consider the following sequence of vector bundles:

$$0 \to S \to \pi^*\mathcal{E} \to Q \to 0.$$ 

We want to think of $H^*(\mathbb{P}\mathcal{E})$ as an algebra over $H^*(X)$ and find the product relations in the powers of $\zeta$.

Since $\zeta = c_1(S^*)$, then $c(S) = 1 - \zeta$ and, by properties of Chern classes, we know that

$$\pi^*c(\mathcal{E}) = c(S) - c(Q)$$

and

$$c(Q) = \frac{c(\mathcal{E})}{1 - \zeta}$$

$$= (1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots)(1 + \zeta + \zeta^2 + \cdots).$$
However, note that \(\text{rank}(Q) = k - 1\), and thus \(c_k(Q) = 0\). Therefore, we have that

\[
\zeta^k + \zeta^{k-1} c_2(\mathcal{E}) + \zeta^{k-2} c_2(\mathcal{E}) + \cdots + c_k(\mathcal{E}) = 0.
\]

\(\square\)

**Remark 2.** Another way to define Chern classes is to start with the above result and to take the \(i\)th Chern class of the bundle \(\mathcal{E}\) to be the coefficient of \(\zeta^{k-i}\) in the expression for \(\zeta^n\).

More precisely,

**Definition 22.** Let \(\mathcal{E}\) be a vector bundle of rank \(k\) on a nonsingular projective variety \(X\). We define the \(i\)th Chern class \(c_i(\mathcal{E}) \in A^i(X)\) to be given by \(c_0 = 1\) and

\[
\sum_{i=0}^{k} \pi^* c_i(\mathcal{E}) \cdot \zeta^{k-i} = 0 \in A^k(\mathbb{P}\mathcal{E}).
\]

Where \(\pi : \mathbb{P}\mathcal{E} \to X\) is the projection.
Chapter 5

Examples of Chern Classes Calculations

Example 2. Consider a cubic surface $X \subset \mathbb{P}^3$ given by a polynomial $F = 0$. Define the first Fano variety of $X$ as

$$F_1(X) = \{\text{lines on } X\} \subset G(1,3).$$

What is the expected dimension and degree of $F_1(X)$?

First, we need to show that $F_1(X)$ is a variety. Consider the open subset $U \subset G(2,4)$ of 2-planes $\Lambda \subset \mathbb{C}^4$ complementary to a given 2-plane $\Lambda_0$. We are going to give explicit equations for $F_1(X) \cap U \subset U$. Start by choosing a basis $v_0(\Lambda), v_1(\Lambda)$ for each $\Lambda \in U$ by taking vectors $v_0, v_1 \in \mathbb{C}^4$ that, together with $\Lambda_0$, span all of $\mathbb{C}^4$. Then, set

$$v_i(\Lambda) = \Lambda \cap (\Lambda_0 + v_i).$$

The coordinates of these vectors are regular functions on $U$.

Now, the homogeneous polynomial $F$ is an element of $\text{Sym}^3(\mathbb{C}^4)^* \subset ((\mathbb{C}^4)^{\otimes 3})^*$, and we let

$$a_I(\Lambda) = F(v_{i_1}(\Lambda), v_{i_2}(\Lambda), v_{i_3}(\Lambda)),$$

where $I$ is a multi-index, $I = \{i_1, i_2, i_3\}$. The $a_I(\Lambda)$ then provide a system of polynomials defining $F_1(X)$ in $U$. In other words, the $a_I$ are the coefficients of the restriction of $F$ to $U$, written in terms of the basis for $\Lambda$ dual to the basis $\{v_0(\Lambda), v_1(\Lambda)\}$.

Next, we define a section $\sigma_F$ of $\text{Sym}^3 S^*$ by $\sigma_F|_\Lambda = F|_\Lambda$, where $\Lambda \in G(1,3)$. Then, $F_1(X) = [\sigma_F]$ and $\deg(F_1) = c_4(\text{Sym}^3 S^*)$. We know that

$$c_1(S^*) = \sigma_{1,1},$$

$$c_2(S^*) = \sigma_1,$$

and so we can write $c(S^*) = (1 + \alpha)(1 + \beta)$, where $\alpha + \beta = \sigma_1$ and $\alpha \cdot \beta = \sigma_{1,1}$. It follows that

$$c(\text{Sym}^3 S^*) = (1 + 3\alpha)(1 + 2\alpha + \beta)(1 + \alpha + 2\beta)(1 + 3\beta)$$
and thus
\[
c_4(\text{Sym}^3 S^*) = 9\alpha\beta(\alpha + 2\beta)(2\alpha + \beta) \\
= 9\alpha\beta(2\alpha^2 + 5\alpha\beta + 2\beta^2) \\
= 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) \\
= 9\sigma_{1,1}(3\sigma_{1,1}) \\
= 27.
\]

Hence there are 27 lines on a smooth cubic surface in \(\mathbb{P}^3\), counting multiplicities. In fact there are exactly 27 lines. To prove this, we begin by calculating the dimension of \(F_1(X)\).

Start by considering the space \(\mathbb{P}^N\) parametrizing all hypersurfaces of degree \(d\), with \(N = \binom{n+4}{d} - 1\). Now define the variety \(\Phi\) to be
\[
\Phi = \{(X, L) : L \subset X \} \subset \mathbb{P}^N \times \mathbb{G}(1, n).
\]
Then the fiber of \(\Phi\) over a point \(X \subset \mathbb{P}^n\) is the first Fano variety \(F_1(X)\). If the image of the projection \(\pi_1 : \Phi \to \mathbb{P}^N\) is of the maximal possible dimension, then we would expect the dimension of \(F_1(X)\) for general \(X\) to be the dimension of \(\Phi\) minus \(N\).

Now we need to calculate the dimension of \(\Phi\). Consider then the second projection \(\pi_2 : \Phi \to \mathbb{G}(1, n)\). The fiber of this map over any point \(L \in \mathbb{G}(1, n)\) is the space of hypersurfaces of degree \(d\) containing \(L\), since the restriction map
\[
\{\text{polynomials of degree } d \text{ on } \mathbb{P}^n\} \to \{\text{polynomials of degree } d \text{ on } L \cong \mathbb{P}^1\}
\]
is a surjective linear map, this is a subspace of codimension \(M = \binom{d+1}{d}\) in the space \(\mathbb{P}^N\). Thus \(\Phi\) is irreducible of dimension \(2(n - 1) + N - M\), and hence the expected dimension of the general fiber of \(\Phi\) over \(\mathbb{P}^N\) is
\[
\dim(F_1(X)) = 2(n - 1) - \binom{d+1}{d} = 2n - d - 3.
\]

In our case, \(\dim(F_1(X)) = 2n - d - 3 = 6 - 3 - 3 = 0\). Now we can calculate the multiplicities of the lines on \(X\). Let \(L \subset X\) be a line. We want to calculate \(N_{L/X}\). If \(H\) denotes the hyperplane class, then \(c_1(T_{\mathbb{P}^3}) = 4H\) and, by adjunction, \(c_1(T_X) = H\). Thus
\[
c_1(N_{L/X}) = c_1(T_X|_L) - c_1(T_L) = -1.
\]
\(N_{L/X}\) is a line bundle and, by the above, it must be \(O_L(-1)\). Thus it has no global sections. This tells us that, for any smooth cubic surface \(F_1(X)\) is reduced, zero dimensional, and \(\dim T_L F_1(X) = 0\). Thus, \(X\) has exactly 27 lines.

\textbf{Example 3.} Consider a pencil of degree \(d\) curves in the plane, \(C_\lambda = (\lambda_0 F + \lambda_1 G) \subset \mathbb{P}^2\). How many of them are singular?

Let
\[
V = H^0(\mathbb{P}^2, \mathcal{O}(d)) \quad \text{and} \quad V_p = H^0(\mathbb{P}^2, T_p^2(d)) \subset V,
\]
and so \(V_p\) is the codimension 3 subspace of curves with a singularity at \(p\). Define a vector bundle
$\mathcal{E}$ on $\mathbb{P}^2$ by $\mathcal{E}_p = V/V_p|_p$, or, more precisely, by

$$
\begin{array}{c}
0 \longrightarrow V_p \longrightarrow V \longrightarrow \mathcal{E} \longrightarrow 0.
\end{array}
$$

Then, $F$ and $G \in V$ induce sections $\sigma, \tau$ of $\mathcal{E} \to \mathbb{P}^2$. The number of singular curves in $C_\lambda$ will be given by the number of points $p \in \mathbb{P}^2$ such that $\sigma(p), \tau(p)$ are linearly dependent, i.e. $c_2(\mathcal{E})$. Introduce $W_p = H^0(\mathbb{P}^2, \mathcal{I}_p(d))$, and note that $V_p \subset W_p \subset V$. Thus, we have a short exact sequence

$$
\begin{array}{c}
0 \longrightarrow W_p/V_p \longrightarrow V/V_p \longrightarrow V/W_p \longrightarrow 0.
\end{array}
$$

Let $\omega$ denote the hyperplane class, then, from the short exact sequence

$$
0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n} \to 0,
$$

we see that $c(\mathcal{T}_{\mathbb{P}^n}) = (1+\omega)^{n+1}$. Apply property C5 of Chern classes to obtain

$$
c_2(\mathcal{T}_{\mathbb{P}^2}^*) = 1 - 3\omega + 3\omega^2 = (1+\alpha)(1+\beta),
$$

where $\alpha + \beta = -3\omega$ and $\alpha\beta = 3\omega^2$. Then,

$$
c(\mathcal{E}) = (1+\alpha + d\omega)(1+\beta + d\omega)(1+d\omega)
$$

$$
c_2(\mathcal{E}) = (\alpha + d\omega)(\beta + d\omega) + d\omega(\alpha + \beta + 2d\omega)
$$

$$
= \alpha\beta + 2(\alpha + \beta)d\omega + 3d^2\omega^2
$$

$$
= \omega^2(3 - 6d + 3d^2) = 3(d-1)^2\omega^2.
$$

Thus, we see that there will be $3(d-1)^2$ singular curves in a pencil of degree $d$ plane curves. \hfill \Box

**Example 4.** Consider the space of all plane cubics, canonically identified with $\mathbb{P}^9$. In this space we want to find the degree of the locus of cubics with a triple point, call it $\Sigma$. First note that $\Sigma$ is a codimension four subspace of $\mathbb{P}^9$. Now, a triple point $p$ locally looks like the intersection of three lines at $p$. Define $\Gamma \subset \mathbb{P}^9$ by

$$
\Gamma = \text{image of } \{(L_1, L_2, L_3) : \text{each } L_i \text{ a line in } \mathbb{P}^2\} = \mathbb{P}^{2*} \times \mathbb{P}^{2*} \times \mathbb{P}^{2*},
$$

under the 6 to 1 map

$$
(L_1, L_2, L_3) \mapsto L_1 \cup L_2 \cup L_3,
$$

$$
\mathbb{P}^{2*} \times \mathbb{P}^{2*} \times \mathbb{P}^{2*} \overset{\gamma}{\to} \mathbb{P}^9.
$$

Since $\Gamma$ is 6 dimensional, we want to find the number of points in the intersection of $\Gamma$ with a $\mathbb{P}^3 \subset \mathbb{P}^9$. If we denote by $\omega$ the hyperplane class in $H^2(\mathbb{P}^9, \mathbb{Z})$, then, since $\mathbb{P}^3 = H_1 \cap \cdots \cap H_6$, we
see that the class of $\mathbb{P}^3$ in $H^2(\mathbb{P}^9, \mathbb{Z})$ is simply $\omega^6$. Hence,

$$
\#(\Gamma \cap \mathbb{P}^3) = \frac{1}{6} \#(\gamma^{-1}(\mathbb{P}^3)) = \frac{1}{6} \#(\gamma^{-1}(H_1 \cap \cdots \cap H_6)) = \frac{1}{6} \#(\bigcap_{i=1}^{6} \gamma^{-1}(H_i)).
$$

If we let $\lambda$ denote the class $[\gamma^{-1}(\omega)]$ in $H^2(\mathbb{P}^2^* \times \mathbb{P}^2^* \times \mathbb{P}^2^*, \mathbb{Z})$, then the degree of $\Sigma$ is $\frac{\lambda^6}{6}$. Let's proceed with the calculation. Let $\alpha_i$ denote the pull-back of the hyperplane class of $\mathbb{P}^2^*$ under the projection $\pi_i : \mathbb{P}^2^* \times \mathbb{P}^2^* \times \mathbb{P}^2^* \rightarrow \mathbb{P}^2^*$, then

$$
H^*(\mathbb{P}^2^* \times \mathbb{P}^2^* \times \mathbb{P}^2^*, \mathbb{Z}) = \mathbb{Z}[\alpha_1, \alpha_2, \alpha_3]/(\alpha_i^3 = 0, \Pi \alpha_i^2 = 1).
$$

By symmetry $[\gamma^*(\omega)] = n(\alpha_1 + \alpha_2 + \alpha_3)$ for some $n \in \mathbb{Z}$. To find $n$, consider the class $\eta = \alpha_1 \alpha_2 \alpha_3^2$. Now, clearly $[\gamma^*(\omega)] \cdot \eta = 1$, and thus $(\alpha_1 + \alpha_2 + \alpha_3) \cdot \eta = 1$. Therefore, we see that $n = 1$ and $\lambda = \alpha_1 + \alpha_2 + \alpha_3$. Finally, by a straightforward calculation, we conclude that the degree of the locus of plane cubics with a triple point is $\frac{\lambda^6}{6} = \frac{90}{6} = 15$. \hfill \Diamond
Chapter 6

Parameter Spaces

So far, in solving enumerative problems we have always adopted the same underlying strategy; choose a good space to parametrize our objects, calculate its Chow or cohomology ring, and then find the class of the subvariety we are interested in. The scope of this section is to give a brief description of some common parameter spaces. We start by giving some definitions in these spaces as varieties. Once we understand this case, we will redefine a parameter space in scheme-theoretic language. In this section we follow the treatment in [H].

6.1 Parameter Spaces for Varieties and The Chow Variety

The basic situation we are interested in is that we are given a collection of subvarieties \( \{ X_\alpha \} \) of some projective space \( \mathbb{P}^n \). We would then like to give a natural bijection between the set \( \{ X_\alpha \} \) and the points of an algebraic variety \( \mathcal{H} \), and to require that the \( \{ X_\alpha \} \) form a reduced family with base \( \mathcal{H} \). We now need to explain what we mean by natural.

We start by noting that given a closed family \( \mathcal{V} \subset B \times \mathbb{P}^n \) of subvarieties of \( \mathbb{P}^n \) in the set \( \{ X_\alpha \} \), we have a set-theoretic map

\[
\varphi_\mathcal{V} : B \to \mathcal{H},
\]
given by sending each point \( b \in B \) to the point of \( \mathcal{H} \) corresponding to the fiber \( V_b = \pi^{-1}(b) \).

**Definition 23.** Let \( \{ X_\alpha \} \) be a collection of subvarieties of \( \mathbb{P}^n \). A **parameter space** for \( \{ X_\alpha \} \) is an algebraic variety \( \mathcal{H} \) together with a bijection between the points of \( \mathcal{H} \) and \( \{ X_\alpha \} \), such that, the \( \{ X_\alpha \} \) form a reduced family with base \( \mathcal{H} \). We require that, for any projective variety \( B \), the association to each family \( \mathcal{V} \subset B \times \mathbb{P}^n \) of the map \( \varphi_\mathcal{V} : B \to \mathcal{H} \) induces a bijection

\[
\left\{ \text{reduced closed families with base } B, \right\} \longleftrightarrow \left\{ \text{regular maps: } B \to \mathcal{H} \right\}.
\]

Similarly, a **cycle parameter space** is defined to be a space satisfying the above condition with respect to generically reduced families.

Suppose that \( X \in \mathbb{P}^n \) has pure dimension \( k \) and degree \( d \). Consider then the following incidence correspondence

\[
\Gamma = \{(p,H_1,\ldots,H_{k+1}) : p \in X \text{ and } p \in H_i \text{ for all } i \} \subset X \times \mathbb{P}^{n*} \times \cdots \times \mathbb{P}^{n*}.
\]

Where the \( H_i \) are hyperplanes in \( \mathbb{P}^n \). What is the dimension of \( \Gamma \)? The set of hyperplanes containing any \( p \in X \) is a hyperplane \( \mathbb{P}^{n-1} \subset \mathbb{P}^{n*} \), and so the set of \((k+1)\)-tuples of hyperplanes is irreducible.
of dimension \((k + 1)(n - 1)\). Therefore, \(\Gamma\) is of pure dimension \(k + (k + 1)(n - 1) = (k + 1)n - 1\), with an irreducible component for each irreducible component of \(X\). Given a general point \(p \in X\) and hyperplanes \(H_1, \ldots, H_{k+1}\), such that \(p \in H_i\) for all \(i\), the intersection of \(X\) and the \(H_i\) only contains the point \(p\). Thus, the projection \(\pi: \Gamma \to \mathbb{P}^{n*} \times \cdots \times \mathbb{P}^{n*}\) is birational. It follows that the image of \(\Gamma\) under the projection is a hypersurface in \(\mathbb{P}^{n*} \times \cdots \times \mathbb{P}^{n*}\), call it \(\Phi_X\).

Let \(F_X\) be the multi-degree \(d\), multi-homogeneous polynomial defining the hypersurface \(\Phi_X \subset \mathbb{P}^{n*} \times \cdots \times \mathbb{P}^{n*}\) and consider the vector space \(V\) of multi-homogeneous polynomials of multi-degree \((d, d, \cdots, d)\) in \(k+1\) sets of \(n+1\) variables. Then, we can associate to a variety \(X \subset \mathbb{P}^n\) of dimension \(k\) and degree \(d\) a well-defined element \([F_X] \in \mathbb{P}V\) and thus we have a set-theoretic map

\[
\begin{align*}
\{\text{varieties of pure dimension} & \}
\{\text{\(k\) and degree \(d\) in \(\mathbb{P}^n\)} & \xrightarrow{\xi} \mathbb{P}V.
\end{align*}
\]

The point \(\xi(X) = [F_X]\) in \(\mathbb{P}V\) is called the \(\text{Chow point}\) of \(X\).

The image of \(\xi\) is a quasi-projective variety, and it is called the \(\text{open Chow variety}\) of subvarieties of dimension \(k\) and degree \(d\) in \(\mathbb{P}^n\), denoted \(\mathcal{G}_{k,d} = \mathcal{G}_{k,d}(\mathbb{P}^n)\). Its closure is called the \(\text{Chow variety}\), \(\mathcal{C}_{k,d}\).

**Proposition 24.** \(\text{The open Chow variety is a cycle parameter space for the set of varieties of pure dimension \(k\) and degree \(d\) in \(\mathbb{P}^n\).}\)

### 6.2 The Hilbert Scheme

A good reference for this section is [EH] and we follow their treatment. We want to define a scheme, \(\mathcal{H}_P\), which will be a parameter space for subschemes of a projective space \(\mathbb{P}^n_K\) having a given Hilbert polynomial \(P\). How do we construct such a scheme? Note that if \(\mathcal{X} \subset \mathbb{P}^n_K \times B \rightarrow B\) is any flat family of subschemes of \(\mathbb{P}^n_K\) with Hilbert polynomial \(P\), we then get a map from the points of \(B\) with residue field \(K\) to the points of \(\mathcal{H}_P\) with residue field \(K\), by sending a point \(b \in B\) to the point of \(\mathcal{H}_P\) corresponding to the fiber \(X_b\) of \(\mathcal{H}_P\) over \(b\). Moreover, we want \(\mathcal{H}_P\) to have the property that, for any scheme \(\mathcal{B}\) over \(K\), the set of flat families of subschemes of \(\mathbb{P}^n_K\) with Hilbert polynomial \(P\) parametrized by \(B\) is naturally identified with the set of maps from \(B\) to \(\mathcal{H}_P\). Finally, we want to be able to define, for each \(P\) a single object \(\mathcal{H}_P\) over \(\text{Spec} \mathbb{Z}\) such that, for any field \(K\), the product \(\mathcal{H}_P \times \text{Spec} K\) parametrizes subschemes of \(\mathbb{P}^n_K\) with Hilbert polynomial \(P\).

**Definition 25.** The \(\text{Hilbert functor} \ h_P\) of flat families of schemes in \(\mathbb{P}^n_K\) with Hilbert polynomial \(P\), is the functor

\[
h_P: (\text{schemes}) \rightarrow (\text{sets})
\]

that associates to any \(B\) the set of subschemes \(\mathcal{X} \subset \mathbb{P}^n_B\) flat over \(B\) whose fibers over point so \(B\) have Hilbert polynomial \(P\).

We would then like for the Hilbert scheme \(\mathcal{H}_P\) to be the scheme that represents \(h_P\), that is, the scheme whose functor of points is \(h_P\). Then, by Yoneda's Lemma, this would determine the scheme \(\mathcal{H}_P\). To do this we need the following result

**Theorem 26 (Mumford).** \(h_P\) is representable. Define \(\mathcal{H}_P\), the Hilbert scheme, to be the scheme whose functor of points is \(h_P\).

For the proof of this theorem see [M].

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The statement of the above theorem has another useful interpretation. It tells us that there exists a universal family, that is, a scheme $\mathcal{H}$ and a subscheme $\mathcal{X} \subset \mathbb{P}_2^3 \times \mathcal{H}$ flat over $\mathcal{H}$ with Hilbert polynomial $P$, such that any subscheme $Y \subset \mathbb{P}_2^3 \times B$ flat over $B$ with Hilbert polynomial $P$ is equal to the fiber product $Y = \mathcal{X} \times_B B \subset \mathbb{P}_2^3 \times B$ for a unique morphism $B \rightarrow \mathcal{H}$. Since, if a universal family $\mathcal{X}$ exists, then $\mathcal{H}$ represents the functor $h_P$. Conversely, if a scheme $\mathcal{H}$ represents $h_P$, then the subscheme $\mathcal{X} \subset \mathbb{P}_2^3 \times \mathcal{H}$ associated to the identity map is universal in the above sense.

**Example 5.** Let $P(m) = m + 1$ be the Hilbert polynomial of a line. Then the Hilbert scheme of subschemes of $\mathbb{P}^3$ with Hilbert polynomial $P$ is the Grassmannian.

### 6.3 Examples of Parameter Spaces Calculations

**Example 6.** How many conics in $\mathbb{P}^3$ meet 8 general lines? Regard $\mathbb{P}^3^4$ as the Grassmannian of 2-planes in $\mathbb{P}^3$. With universal rank 3 sub-bundle $S$ and quotient bundle $Q$. The variety $\mathcal{H}$ of conics in $\mathbb{P}^3$ may be identified with the projective bundle of $\mathbb{P} \text{Sym}^2(S^*)$. Let $w$ denote the hyperplane class. Since $c(S^*) = 1 + w + w^2 + w^3$, by the splitting principle we see that $c(\text{Sym}^2(S^*)) = 1 + 4w + 10w^2 + 20w^3$. Then, if we let $z = c_1(O_{\mathbb{P}^3}(1))$, we see that

$$A_*(Y) = \frac{Z[w, z]}{(w^4, z^6 + 4wz^5 + 10w^2z^3 + 20w^3z^2)},$$

and

$$z^8 = -4, \quad z^7 \cdot w = 6, \quad z^6 \cdot w^2 = -4, \quad z^5 \cdot w^3 = 1.$$

Next, we express $\Sigma = \{\text{conics meeting a given line } L\}$ as a class in $A^*(\mathcal{H})$ and find its eight-fold product. It is easy to see that $[\Sigma] = z + 2w$, because the open subset of $\mathbb{P}^3^4$ consisting of planes not containing a give line has codimension 2, and any line in given plane will meet the conic contained in that plane.

Then,

$$(z + 2w)^8 = z^8 + 16z^7 \cdot w + 112z^6 \cdot w^2 + 448z^5 \cdot w^3$$

$$= -4 + 96 - 448 + 448 = 92$$

Hence, we have 92 conics in $\mathbb{P}^3$ meeting 8 general lines.

**Example 7.** Given 5 general conics $c_1, \ldots, c_5$ in $\mathbb{P}^2$, how many conics are there which are tangent to all 5 (i.e. having a point of intersection of multiplicity $\geq 2$ with each $c_i$)?

The first guess to the answer comes from a simple calculation. Let

$$\Sigma_c = \{\text{conics tangent to } c\} \subset Y = \{\text{conics in } \mathbb{P}^2\} = \mathbb{P}^5.$$  

Note that $\Sigma_c$, for $c$ smooth, is a degree 6 hypersurface in $\mathbb{P}^5$. To see this let $L \in Y$ be a generic line and $\{c_\lambda\}_{\lambda \in \mathbb{P}^1}$ the pencil of conics it represents. Then, the curves $\{c_\lambda\}$ cut out on $c$ a linear system of degree 4 without base points. The corresponding map expresses $c$ as a 4-sheeted cover of $\mathbb{P}^1$. Thus, by the Riemann-Hurwitz formula, the number of branch points of this map is

$$b = 2g(c) - 2 - 4(2g(\mathbb{P}^1) - 2) = 6.$$  

The pencil $\{c_\lambda\}$ therefore contains six conics tangent to $c$, and thus $\text{deg}(\Sigma_c) = 6$. We would then expect that there is a finite number of conics tangent to 5 distinct, general conics in $\mathbb{P}^2$. In particular
we see that
\[ \cap_{i=1}^5 \Sigma_{c_i} = [\Sigma_{c}]^5 = [6H]^5 = 6^5 = 7776. \]

Unfortunately, this guess is incorrect.

What is wrong with this calculation? The problem is that in the \( \mathbb{P}^5 \) of conics, there is a Veronese surface \( S \) of double lines and \( S \subseteq \Sigma_{c_i} \) for any \( c_i \). Thus, for 5 general conics
\[ \cap_{i=1}^5 \Sigma_{c_i} = S \cap \Gamma, \]
where \( \Gamma \) is the finite set corresponding to actual solutions, i.e. corresponding to smooth conics tangent to \( c_1, \ldots, c_5 \). How do we solve this problem? There are two approaches.

### 6.4 First Method

For this procedure to be valid, there must be some compactness assumption. One method is to construct a better (non-singular) compactification of \( Y \), where the intersections are proper and correspond only to non-degenerate solutions. In our case the space of complete conics is such compactification.

Introduce the space of complete conics,
\[ X = \{(c, c^*) : c, c^* \text{ smooth conics dual to each other} \} \subseteq \mathbb{P}^5 \times \mathbb{P}^5^*. \]

What does \( X \) look like? \( X \) is the graph of the rational map \( \mathbb{P}^5 \to \mathbb{P}^5^* \) sending a conic to its dual. In coordinates, this is the map which sends a symmetric \((3 \times 3)\) matrix to its inverse. A general point of \( X \) will simply be a smooth conic \( c \subseteq \mathbb{P}^5 \) and its dual, also a smooth conic \( c^* \subseteq \mathbb{P}^5^* \). A four dimensional subvariety of \( X \) is given by pairs of the type

![Diagram](attachment:image.png)

Where \( C_0 \) is obtained as the limit of a family of conics degenerating to two lines
\[ \text{Lim}_{t \to 0}(C_t)^* = 2 \cdot \{\text{line dual to } p\}. \]

Another four dimensional subvariety of \( X \) is given by pairs of the following type

![Diagram](attachment:image.png)

Where \( C_\infty \) is the limit of a one-parameter family degenerating to a double line.
Finally, there is a 3 dimensional subspace $S$ of $X$ given by double lines and their duals, which are simply double lines. $X \rightarrow \mathbb{P}^5$ is an isomorphism away from $\mathbb{P}^5 \setminus S$. For $c_0 \in S$, we have that
\[
\pi^{-1}(c_0) = \{(c_0, p^* \wedge q^*), p, q \text{ on } |c_0|\}.
\]
Thus, $\pi^{-1}(X) \rightarrow S$ is a $\mathbb{P}^2$ bundle. We'll see that $X$ is the correct space for our calculation.

The next step is to calculate $A_*(X)$. Let
\[ X \xrightarrow{\pi_1} \mathbb{P}^5 \xrightarrow{\pi_2} \mathbb{P}^{5*}, \]
and set $\omega = \pi^*_1(H_{\mathbb{P}^5})$, and $\nu = \pi^*(H_{\mathbb{P}^{5*}})$, the generators of $A^*(X)$. Note that, by symmetry, $\omega^i \nu^j = \omega^j \nu^i$, and so
\[
\begin{align*}
\omega^5 & = \text{(number of conics through 5 points)} = 1 \\
\omega^4 \cdot \nu & = \text{(number of conics through 4 points and tangent to 1 line)} = 2 \\
\omega^3 \cdot \nu^2 & = \text{(number of conics through 3 points and tangent to 2 lines)} = 4 \\
\omega^2 \cdot \nu^3 & = 4 \\
\omega \cdot \nu^4 & = 2 \\
\nu^5 & = 1 \\
\end{align*}
\]
dually.

Then, let
\[
\alpha = [(c, c^*) : c \text{ in a general pencil }] \\
\beta = [(c, c^*) : c^* \text{ in a general pencil }]
\]
and we see that
\[
\begin{array}{c|ccc}
. & \omega & \nu & \sigma \\
\hline
\alpha & 1 & 2 & 6 \\
\beta & 2 & 1 & 6 \\
\end{array}
\]
From this table we see that $\sigma = 2\omega + 2\nu$ and thus
\[
\sigma^5 = 2^5(\omega + \nu)^5 = 2^5(1 + 2 \cdot 5 + 10 \cdot 4 + 10 \cdot 4 + 5 \cdot 2 + 1) = 2^5 \cdot 102 = 3264.
\]

6.5 Second Method

Another possible solution to this problem is given by blowing up $Y$, the conics in $\mathbb{P}^2$, along $S$.

\[
\begin{array}{ccc}
E & \xrightarrow{\pi} & X = \text{Bl}_S \mathbb{P}^5 \\
\downarrow & & \downarrow \\
\mathbb{P}^2 \cong S & \xrightarrow{\pi} & Y = \mathbb{P}^5.
\end{array}
\]
For this calculation, we follow [GH]. Given any $c_i$, denote by $\Sigma_{c_i}$ the proper transform of $\Sigma_{c_i}$.

**Claim 27 (Griffiths, Harris).** For five generic smooth conics $c_1, \ldots, c_5$,

1. $\Sigma_{c_1}, \ldots, \Sigma_{c_5} \subset X$ meet transversally away from the subvariety of singular conics,
2. no conics consisting of two lines will be tangent to all five $c_1, \ldots, c_5$, and
3. the proper transforms $\tilde{\Sigma}_{c_1}, \ldots, \tilde{\Sigma}_{c_5} \subset X = \text{Bl}_S \mathbb{P}^5$ have no common points in the exceptional divisor $E$ of $X$.

**Proof.** To prove assertion 1, note that for $c$ smooth, the divisor $\Sigma_c$ is irreducible. To see this let

$$I' = \{(c_0, p) : c_0 \text{ is tangent to } c \text{ at } p \} \subset \Sigma_c \times c.$$

Since $c$ is irreducible and the fibers of the projection map $\pi_2 : I' \to c$ are linear subspaces of $Y$, $I'$ is irreducible. Thus $\Sigma_c$ is irreducible.

Now, let $U \subset Y$ be the open set of smooth conics,

$$I = \{(c_1, \ldots, c_5; c') : c' \subset \Sigma_{c_i} \text{ for all } i \},$$

and let $J \subset I$ be the closed subvariety of $I$ consisting of $(c_1, \ldots, c_5; c')$ such that $c'$ is a non-transverse point of intersection of $\Sigma_{c_1}, \ldots, \Sigma_{c_i}$. The fibers of the projection $\pi_2 : I \to U$ are isomorphic to $(\Sigma_{c_i})^5$, and thus irreducible. Furthermore, since the map $\pi_1 : I \to (Y)^5$ is generically finite to one we see that $J$ can map surjectively onto $(Y)^5$ only if $J = I$. However, if $J = I$, assertion 1 would be false, and thus, to finish its proof we need find a point in $I - J$. In other words, we need to find six conics $c_1, \ldots, c_5, c'$ such that $\Sigma_{c_1}, \ldots, \Sigma_{c_5}$ meet transversely at $c'$. If $c'$ is any smooth conic and $c_1, \ldots, c_5$ conics simply tangent to $c'$ at distinct points $p_1, \ldots, p_5$, then the tangent hyperplanes $T_{c_i}(\Sigma_{c_i}) = H_{p_i}$, the plane of conics containing $p_i$, are independent. Lastly, we verify the statement $T_{c_i}(\Sigma_{c_i}) = H_{p_i}$. Let $c''$ be a smooth point of $\Sigma_c$ which is simply tangent to $c$ at a single point $p \in c$. If $L \in Y$ is a generic line through $c''$ lying in the hyperplane $H_p \subset Y$ of conics containing $p$, the corresponding pencil $\{c'_i\}_{i \in \mathbb{P}^1}$ will cut out on $c$ a linear system of degree 4, with $p$ as a base point. The corresponding map then expresses $c$ as a 3-sheeted cover of $\mathbb{P}^1$, and so, by Riemann-Hurwitz, has only

$$b = 2(2g(c) - 2) - 3(2g(\mathbb{P}^1) - 2) = 4$$

branch points, i.e. the pencil $\{c''_i\}$ can contain at most four conics tangent to $c$ other than $c'$. Thus, $H_p$ is the tangent plane to $\Sigma_c$ at $c''$, and conversely, if $c''$ is simply tangent to $c$ at only one point $p$, then $c''$ is a smooth point of $\Sigma_c$.

To prove assertion 2, first note that in general, if $\{D_\lambda\}$ is any family of divisors without base points on an $n$-dimensional variety $V$, $n+1$ divisors $D_\lambda_1, \ldots, D_\lambda_{n+1}$ of the family have no common points, as long as their are selected generically. Then, since the locus $Y_1 \subset Y$ of conics of rank two has dimension 4, we only need to check that the family $\{\Sigma_{c_i}\}_{i \in \mathbb{P}^1}$ has no base points on this locus. This is clear, since we can obviously find a conic not tangent to any conic of rank 2.

Finally, we prove assertion 3. We need to show that the family $\{\Sigma_c\}$ has no base points in $E$. Now note that, given any point $2L \in S$ and a normal vector $v$ to $S$ at $2L$ represented by a line $\{c_\lambda\}$ in $Y$, the proper transform $\tilde{\Sigma_c}$ will contain the point of $E$ corresponding to $v$ if and only if the line $\{c_\lambda\}$ has intersection multiplicity 3 or more with $\Sigma_c$ at $2L$. Thus, we only need to show that, for any point $2L \in S$ and any line $\{c_\lambda\}$ through $2L$ but not tangent to $S$ at $2L$, there exists a conic $c$ such that

$$\text{mult}_{2L}(\Sigma_c, \{c_\lambda\}) = 2.$$
But, any pencil of conics containing two double lines, $2L$ and $2L'$, has a base point of order 4, and hence it must consist entirely of singular conics. We see then that, in the limiting case, any pencil tangent to $S$ at $2L$ has only singular conics. Thus, the tangent plane $T_{2L}(S)$ is equal to the 2-plane

$$\{ L + L' \}_{L \in (\mathbb{P}^2) \times Y}.$$

Now, any pencil $\{c_\lambda\}$ through $2L$, but not in the tangent space $T_{2L}(S)$, has only finitely many base points. Then, we choose the conic $c$ to miss these base points, and we see that $\{c_\lambda\}$ meets $\Sigma_c$ with multiplicity 2 at $2L$.

We can then solve our problem. We begin by computing the multiplicity of the locus $S$ of double lines in the generic divisor $\Sigma_c$. Consider a conic $c$, a generic double line $2L$, and a generic pencil of conics containing $2L$ as an element, $\{c_\lambda\}$. Then, $\{c_\lambda\}$ cuts out on $c$ a base point free pencil of degree 4. The corresponding map, as we have seen, has 6 branch points, but 2 of these are just the points of intersection of $L$ with $c$. Thus, $\{c_\lambda\}$ has 4 points of intersection with $\Sigma_c$ other than $2L$. It follows that

$$\text{mult}_{2L}(\{c_\lambda\}, \Sigma_c) = 2,$$

and so, for generic $c$,

$$\text{mult}_S(\Sigma_c) = 2.$$

Then, we have

$$\sigma = \tilde{\Sigma}_c \equiv 6\tilde{\omega} - 2e \in H^2(X, \mathbb{Z}),$$

where $\tilde{\omega} = \pi^*\omega$ is the pullback of the hyperplane class $\omega$ in $Y$, and $e$ is the class of the exceptional divisor $E$.

Now, to finish our calculation we need to compute the five-fold self-intersection of $\sigma$. Recall that $S$ is the Veronese surface $\nu(\mathbb{P}^2)$. Thus, let $l$ and $p = l^2$ denote the classes of a line and a point in $S \cong \mathbb{P}^2$, and let $\tilde{p} = \pi^*p$ and $\tilde{l} = \pi^*l$ be the pullback classes in $E \subset X$. Then,

$$\omega|_S = 2l,$$

and

$$\omega^2|_S = (2l)^2 = 4p.$$

We also know that

$$c(T(Y)|_S) = (1 + 6\omega + 15\omega^2)|_S = 1 + 12l + 60p,$$

and

$$c(T(S)) = 1 + 3l + 3p.$$

$$T(Y)_S = T(S) \otimes N_{S/Y}$$ implies $c(N_{S/Y}) = c(T(Y)) / c(T(S))$, and so we see that

$$c(N_{S/Y}) = 1 + 9l + 30p.$$

Thus, if $\zeta \in H^2(E, \mathbb{Z})$ denotes the Chern class of the tautological bundle on $E \cong \mathbb{P}(N_{S/Y})$, proposition 21 tells us that

$$\zeta^3 - 9l \cdot \zeta^2 + 30\tilde{p} \cdot \zeta = 0. \quad (6.1)$$
On each fiber $E_x$ of $E \to S$, the tautological bundle restricts to the universal bundle $[-H]$, and so
\[
\zeta^2 \cdot \bar{\nu} = c_1(T|_{E_x})^2 = 1.
\]
Multiplying relation (6.1) by $\bar{\nu}$ we obtain
\[
\bar{\nu} \cdot \zeta^3 - 9\bar{\nu} \cdot \zeta^2 = 0,
\]
and hence
\[
\bar{\nu} \cdot \zeta^3 = 9.
\]
Finally, multiplying equation (6.1) by $\zeta$, we see that
\[
\zeta^4 - 9\bar{\nu} \cdot \zeta^3 + 30\bar{\nu} \cdot \zeta^2 = 0,
\]
and thus
\[
\zeta^4 = 9\bar{\nu} \zeta^3 - 30\bar{\nu} \zeta^2 = 51.
\]

Now, we can calculate $\sigma^5$. Since the class $\omega$ restricts to the class $2\nu$ on $S$, we see that $\bar{\omega}|_E = 2\bar{\nu}$. The tautological bundle $T = N_{E/P^2}$, and
\[
e|_E = c_1(T) = \zeta.
\]
Furthermore,
\[
(\bar{\omega}^5)_X = (\omega^5)_Y = 1.
\]
\[
\bar{\omega}^5 = 1,
\]
\[
\bar{\omega}^4 \cdot e = ((2\bar{\nu})^4)_E = 0,
\]
\[
\bar{\omega}^3 \cdot e^2 = ((2\bar{\nu})^3 \cdot \zeta)_E = 0,
\]
\[
\bar{\omega}^2 \cdot e^3 = ((2\bar{\nu})^2 \cdot \zeta^2)_E = 4(\bar{\nu} \cdot \zeta^4)_E = 4,
\]
\[
\bar{\omega} \cdot e^4 = (2\bar{\nu} \cdot \zeta^3)_E = 18,
\]
\[
e^5 = \zeta^4 = 51.
\]

Finally we see that
\[
(6\bar{\omega} - 2e)^5 = 6^5 \bar{\omega} - 5 \cdot 6^4 \cdot 2 \cdot \bar{\omega}^4 \cdot e + 10 \cdot 6^3 \cdot 2^2 \cdot \bar{\omega}^3 \cdot e^2
\]
\[
- 10 \cdot 6^2 \cdot 2^3 \cdot \bar{\omega}^2 \cdot e^3 + 5 \cdot 6 \cdot 2^4 \cdot \bar{\omega} \cdot e^4 - 2^5 \cdot e^5
\]
\[
= 6^5 - 10 \cdot 6^2 \cdot 2^3 \cdot 4 + 5 \cdot 6 \cdot 2^4 \cdot 18 - 2^5 \cdot 51
\]
\[
= 7776 - 11520 + 8640 + 1632 = 7776 - 4512
\]
\[
= 3264.
\]

Thus, we confirm that there are 3264 conics tangent to 5 general conics in $P^2$.  
\[\Diamond\]
6.6 Excess Intersection Formula

We would like to end this paper by talking about some interesting formulas in intersection theory and algebraic geometry in general. We will not give the proofs in detail or otherwise, but rather we will try to justify why such formulas hold and give examples of how they may be useful.

Given a projective $n$-dimensional variety $X$ and $l$ subvarieties of $X$, $Z_1, \ldots, Z_l \subset X$, each of codimension $c_i$, we would expect $\cap Z_i$ to have codimension $c = \sum c_i$. Now suppose that, instead, $\cap Z_i = \cup W_\alpha$, where some of the $W_\alpha$ have lower codimension than $c$. We want to obtain a formula which assigns to each $W_\alpha$ a class

$$\gamma_\alpha \in A_{n-c}(W_\alpha) \to A_{n-c}(X) = A_c(X),$$

such that

$$\sum \gamma_\alpha = \pi[Z_i] \in A_c(X).$$

Before deriving the general formula, we give two easier examples.

**Example 8.** Suppose that $X = \mathbb{P}^3$, and that we are given $S, T, U \in \mathbb{P}^3$ hypersurfaces of degrees $d, e, f$ respectively. Furthermore assume that $S$ is smooth and $S, T, U$ are all irreducible. Then, their intersection is simply

$$S \cup T \cup U = L \cap \Sigma,$$

where $L$ is a line and $\Sigma$ a finite set. What is the degree of $\Sigma$?

Note that $S \cap T = L \cup C$ and $S \cap U = L \cup C'$. Their cohomology classes are $[C] = e \cdot h - l$ and $[C'] = f \cdot h - l$ in $H^2(S)$, for $h = [H]$, the hyperplane section, and $l = [L]$, the class of the line. Then, $\Sigma = C \cap C'$, and the degree of $\Sigma$ is given by

$$\deg \Sigma = (e \cdot h - l)(f \cdot h - l)$$

$$= ef(h \cdot h) - (e + f)(h \cdot l) + l^2$$

$$= def - e - f + l^2.$$

Now, we only have to calculate the product $l \cdot l \in A^*(S)$. If $D \subset X$ is a smooth divisor of a smooth variety then, by adjunction, we see that $K_D = (K_X + D)|_D$. In the case of $S \subset \mathbb{P}^3$, $K_S = (d-4) \cdot H$. Furthermore, for $L \subset S$, we have that

$$K_L = ((d-4) \cdot H + l)|_L$$

$$-2 = (d-4)H \cdot l + l^2,$$

and thus $l^2 = 2 - d$. We see then that $\deg \Sigma = def - e - f - d + 2$. ⊿

**Example 9.** Consider two smooth surfaces in $\mathbb{P}^4$, $S, T$, intersecting along a smooth curve $C$ and a finite number of points $\Sigma$, that is, $S \cap T = C \cup \Sigma$. We want to find a formula which assigns to $C$ a zero-dimensional class.

We start by perturbing $T$ until $C$ breaks into points, but $\Sigma$ is still a finite collection of points. Once we do this, we can calculate the class of the resulting collection of points. Recall that first order deformations of $T$ correspond to sections $\sigma$ of $N_{T/\mathbb{P}^4}$. Let $\{ T_\lambda \}$ be the family obtained by deforming $T$, so that $T_0 = T$. Then, a point $p$ is in $T$ if and only if, under the same deformation, $p_\lambda \in T_\lambda$. Denote by $\overline{p}$ the tangent vector at $p = p_0$ of the arc $\{ p_\lambda \}$. Suppose that $p \in C$, then $p_\lambda \in S \cap T_\lambda$ only if $\overline{p}$ is tangent to $S$.  

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Now, consider the restriction of $N_{T/P^4|C}$. It maps onto the line bundle $L = \frac{T_{P^4|C}}{(T_{S} + T_{C})}$ and then, the zero locus of the section $\sigma|_C$ is exactly the points which remain in the intersection of $S \cap T_{\lambda}$, as $\lambda$ varies. Thus, the answer to this problem is given by the first Chern class of $L$.

Given the following short exact sequence

$$0 \to T_{C} \to T_{T/C} \otimes T_{S/C} \to T_{T/C} + T_{S/C} \to 0,$$

it follows that

$$c_1(L) = c_1(T_{S/C}) - c_1(T_{T/C}) + c_1(T_{C})$$

$$= c_1(T_{S/C}) - (c_1(T_{S/C}) - c_1(N_{S/C})) + c_1(T_{C}) + (c_1(P^4|C) - c_1(N_{C/P^4}))$$

$$= c_1(N_{S/C}) + c_1(N_{T/C}) - c_1(N_{C/P^4})$$

$$= c_1(N_{S/C}) + c_1(N_{T/P^4|C}) - c_1(N_{C/P^4}),$$

where $c_1(L) = c_1(N_{S/C}) + c_1(N_{T/P^4|C}) - c_1(N_{C/P^4})$ is the class we have been looking for. ⊗

6.7 The General Case

Instead of deriving the excess intersection formula in the utmost generality, we will motivate its existence in a somewhat restricted case. The proof of a more general incarnation of this formula can be found in [F 98].

Let $X$ be a smooth $n$-dimensional projective variety, and let $Z_1, \cdots, Z_l$ be smooth codimension $c_1$ subvarieties of $X$, such that $\sum c_i = n$. Assume that

$$\cap Z_i = \prod_{X} W_{\alpha},$$

where each $W_{\alpha}$ is smooth of dimension $d_{\alpha}$, and let $W_{\alpha} \xrightarrow{i_{\alpha}} X$ be the inclusion maps. We want the excess intersection formula to assign to each $W_{\alpha}$ a class $\gamma_{\alpha} \in A_0(W_{\alpha})$, such that

$$\prod [Z_i] = \sum i_{\alpha} \cdot \gamma_{\alpha} \in A_0(X).$$

To obtain the desired excess intersection formula, we proceed as in example 9 above. Suppose that we can perturb the surfaces $Z_i$, i.e. that there exist $Z_1, \cdots, Z_l$ in $X \times \Delta$, for $\Delta$ a disk, flat over $\Delta$, such that

$$Z_i(t) = Z_i \cap (X \times \{t\})$$

and

$$Z_i(0) = Z_i.$$

Note that this assumption is not overly restrictive. Deformations always exist in the $C^\infty$ category, and from this we could deduce that they exist in under our assumptions. Furthermore, assume that, for $t \neq 0$, the $Z_i(t)$ intersect transversely, that is

$$\cap Z_i = \cup W_{\alpha} \times \{0\} \cup \Phi,$$

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where $\Phi$ is flat over $\Delta$, consists of points of intersection of transverse fibers, and $\deg \Delta = \prod [Z_i]$.

Now, for each $\alpha$, we want to describe $(W_\alpha \times \{0\}) \cap \Phi$. Choose a particular $\alpha$ and let $W := W_\alpha$. Suppose that $W$ is $k$-dimensional, then we have a map

$$N(W \times \{0\}/\{X \times \Delta\}) \to \oplus N(Z_i/\{X \times \Delta\})|_{W \times \{0\}}.$$ 

We want to find the points where the above map fails to be an inclusion. Note that

$$\oplus N(Z_i/\{X \times \Delta\})|_{W \times \{0\}} = \oplus N(Z_i/X)|_W,$$

and

$$N(W \times \{0\}/\{X \times \Delta\})^{rk 1}_{W \times \{0\}} = \mathcal{O},$$

$$N(W \times \{0\}/\{X \times \Delta\})^{rk n-k+1}_{W \times \{0\}} \to \oplus N(Z_i/X)^{rk n}_W,$$

$$N(W/X)^{rk n-k} \to 0$$

is short exact. Then we obtain a map

$$\mathcal{O}^{rk 1} \to (\oplus N(Z_i/X))/(N(W/X))^{rk k},$$

and therefore the class of $(W_\alpha \times \{0\}) \cap \Phi$ is

$$c_k(\oplus N(Z_i/X))/(N(W/X)),$$

the excess intersection formula that we wanted.
Chapter 7

Porteous Formula

In this section we follow chapter III of [ACGH] for most of the material presented. Consider a vector bundle map \( \varphi : \mathcal{E} \to \mathcal{F} \), where \( \mathcal{E} \) has rank \( m \), \( \mathcal{F} \) has rank \( n \), and they are both bundles over a smooth projective variety \( X \). If we choose local trivializations of \( \mathcal{E} \) and \( \mathcal{F} \) on an affine open \( U \), the homomorphism \( \varphi \) is represented by an \((m \times n)\) matrix \( A \). This corresponds to a morphism

\[ f : U \to \mathcal{M} = \mathcal{M}(m, n), \]

where \( \mathcal{M}(m, n) \) is the variety of complex \((m \times n)\) matrices, and we denote the rank at most \( k \) such matrices by \( \mathcal{M}_k \). Let \( U_k \) be the pre-image of \( \mathcal{M}_k \) under this map, i.e. the ideal of \( U_k \) is generated by the \((k + 1) \times (k + 1)\) minors of \( A \). Since \( U_k \) does not depend on the choice of trivialization, there is a well defined subvariety \( X_k(\varphi) \subset X \), such that \( X_k(\varphi) \cap U = U_k \) for any \( U \).

**Definition 28.** The variety \( X_k(\varphi) \) is called the \( k \)-th determinantal variety associated to \( \varphi \), and it supported on the set

\[ \{ p \in X : \operatorname{rank}(\varphi_p) \leq k \}. \]

It is clear from the definition that when \( X_k(\varphi) \) is non-empty, it has codimension at most \((m - k)(n - k)\).

**Proposition 29 (ACGH).** Let \( X \) be a complex manifold, and let

\[ \varphi : \mathcal{E} \to \mathcal{F} \]

be a homomorphism of holomorphic vector bundles of ranks \( n \) and \( m \), respectively. If \( X_k(\varphi) \) has codimension \((m - k)(n - k)\), then it is Cohen-Macaulay.

We now proceed to derive Porteous formula, which will give us the class of \( X_k = X_k(\varphi) \) in the Chow ring when \( X_k \) has the expected codimension. Clearly, the class of \( X_k \) will depend on \( \mathcal{E} \) and \( \mathcal{F} \), but we will see that we only need to know the total Chern classes of \( \mathcal{E} \) and \( \mathcal{F} \).

Let \( \mathcal{E} \) and \( \mathcal{F} \) be vector bundles of respective ranks \( n \) and \( m \) over a smooth projective variety \( X \), and let \( \varphi : \mathcal{E} \to \mathcal{F} \), be as above. Recall that, from our first definition of Chern classes, the Chern classes of a vector bundle \( \mathcal{E} \to X \) describe the loci where one or more sections of \( X \) will be linearly dependent, in other words, the locus \( X_0 \) is the zero locus of \( \varphi \), considered as a section of \( \operatorname{Hom}(\mathcal{E}, \mathcal{F}) \), and so

\[ [X_0] = c_{m,n}(\mathcal{E}^* \otimes \mathcal{F}) \in A^*(X). \]
In the case that \( m = n \), we see that \( X_{m-1} \) is the zero locus of \( \Lambda^m \varphi \), and thus

\[
[X_{m-1}] = c_1(\Lambda^m \mathcal{E}^* \otimes \Lambda^m \mathcal{F}) = c_1(\mathcal{F}) - c_1(\mathcal{E}).
\]

In general, we can always write

\[
X_k = \{ \Lambda^{k+1} \varphi = 0 \},
\]

but as a section of \( \Lambda^{k+1} \mathcal{E}^* \otimes \Lambda^{k+1} \mathcal{F}, \Lambda^{k+1} \varphi \) vanishes in the wrong codimension. To solve this problem, start by considering the Grassman bundle \( \pi : G(n - k, \mathcal{E}) \to X \), and denote by \( S \) and \( Q \), the universal sub-bundle and quotient bundle on \( G(n - k, \mathcal{E}) \) respectively,

\[
0 \to S \to \pi^* \mathcal{E} \to Q \to 0.
\]

Composing the lifted homomorphism \( \pi^* (\varphi) : \pi^* \mathcal{E} \to \pi^* \mathcal{F} \) with the inclusion of \( S \) in \( \pi^* \mathcal{E} \) gives a homomorphism \( \tilde{\varphi} : S \to \pi^* \mathcal{F} \). Denote by \( \tilde{X}_k \) the subvariety of \( G(n - k, \mathcal{E}) \) defined by the vanishing of \( \tilde{\varphi} \). The support of \( \tilde{X}_k \) is the set of couples \( (x, W) \), where \( x \) is point of \( X_k \) and \( W \) is an \( (n-k) \)-plane contained in the kernel of \( \varphi_k \). Now \( \tilde{X}_k \) can be viewed as a section of \( \text{Hom}(S, \pi^* \mathcal{F}) \). Therefore,

\[
[X] = c_{m(n-k)}(S^* \otimes \pi^* \mathcal{F}),
\]

and we may compute the class of \( X_k \) as

\[
[X_k] = \pi_*([\tilde{X}]),
\]

where \( \pi_* : A^*(G(n - k, \mathcal{E})) \to A^*(X) \) is the Gysin homomorphism.

At this point we only have to evaluate the top Chern class of the tensor product of two vector bundles, apply this formula to the bundles \( S^* \) and \( \pi^* \mathcal{F} \) on \( G(n - k, \mathcal{E}) \), and finally evaluate the image of the class we get under the Gysin map \( \pi_* \). The first step is easily given by \( \mathbf{C}5 \) from section 4.1. Recall that

\[
c_t(\mathcal{E} \otimes \mathcal{F}) = \prod_{i,j}(1 + (a_i + b_j)t),
\]

where \( a_i, b_j \) are just formal symbols derived from the splitting of \( \mathcal{E} \) and \( \mathcal{F} \) respectively. In other words, if we assume that \( \mathcal{E} \) and \( \mathcal{F} \) split into direct sums of line bundles

\[
\mathcal{E} = \oplus L_i \quad c_1(L_i) = a_i, \\
\mathcal{F} = \oplus M_i \quad c_1(M_i) = a_i,
\]

so that

\[
c_t(\mathcal{E}) = \prod (1 + a_it), \quad c_t(\mathcal{F}) = \prod (1 + b_it).
\]

Then, we can write \( \mathcal{E}^* \otimes \mathcal{F} \) as

\[
\mathcal{E}^* \otimes \mathcal{F} = \bigoplus_{i,j} L_i^* \otimes M_j,
\]

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and thus
\[ c_{mn}(\mathcal{E}^* \otimes \mathcal{F}) = \prod_{i,j} c_1(L_i^* \otimes M_j) \]
\[ = \prod_{i,j} (b_j - a_i). \]

Thus, if we define for any formal series \( p(t) = \sum d_i t^i \) the determinant
\[ \Delta_{i,j}(p) = \det \begin{pmatrix} p_i & \cdots & p_{i+j-1} \\ \vdots & & \vdots \\ p_{i-j+1} & \cdots & p_1 \end{pmatrix}, \]
the \( mn \)-th Chern class of the product above is
\[ c_{mn}(\mathcal{E}^* \otimes \mathcal{F}) = \prod_{i,j} (b_j - a_i) \]
\[ = \Delta_{m,n}(c_t(\mathcal{F})/c_t(\mathcal{E})). \]

Now, recall that \( [X_k] = \pi_* (c_{m(n-k)}(S^* \otimes \pi^* \mathcal{F})) \), and so we see that
\[ [X_k] = \pi_* \Delta_{m,n-k}(\pi^* c_t(\mathcal{F})/c_t(S)). \]

For any monomial \( \prod c_i(Q)^{\alpha_i} \) we have
\[ \pi_* \left( \prod_{i=0}^{k} c_i(Q)^{\alpha_i} \right) = 0, \quad \text{if} \quad \sum i \alpha_i < k(n - k). \]

The above direct image is a multiple of the fundamental class of \( X \) when \( \sum i \alpha_i = k(n - k) \). Since \( c(S)c(Q) = \pi^* c(E) \), we have that
\[ [X_k] = \pi_* \Delta_{m,n-k}(\pi^* c_t(\mathcal{F})/c_t(S)) \]
\[ = \pi_* \Delta_{m,n-k}\left( \pi^* \left( \frac{c_t(\mathcal{F})}{c_t(E)} \right) \cdot c_t(Q) \right). \]

Also note that, since the entries of the \( (n-k) \times (n-k) \) determinant \( \Delta \) are all linear combinations of \( c_0(Q), \ldots, c_k(Q) \) with coefficients in \( A^*(X) \), by the above remark, only those terms in the entries of the determinant \( \Delta \) which contain the factor \( c_k(Q) \) will contribute to its Gysin image \( \pi_* \Delta \). Thus, by the push-pull formula, we see that
\[ [X_k] = \Delta_{m-k,n-k}(c_t(\mathcal{F})/c_t(E)) \cdot (\pi_* c_k(Q)^{n-k}). \]

Since our goal is to express \([X_k] \) in terms of the Chern classes of \( E \) and \( F \) only, we want to show that \( \pi_*(c_k(Q)^{n-k}) = 1_X \). Since \( Q \) restricts to the universal bundle on each fiber of \( \pi \), this will follow from

**Lemma 30 (ACGH).** If \( Q \) is the universal quotient bundle on the Grassmannian \( G(n - k, n) \),
then
\[ \int_{G(n-k,n)} c_k(Q)^{n-k} = 1. \]

Proof. The integral is just the number of zeros common to \( n - k \) general sections of \( Q \). Any vector \( v \in \mathbb{C}^n \) determines a section of \( Q \), which vanishes precisely at those \((n-k)\)-planes \( W \) which contain \( v \). If we choose \( n - k \) linearly independent vector \( v_1, \ldots, v_{n-k} \in \mathbb{C}^n \), the corresponding sections of \( Q \) vanish simultaneously only at the span of \( v_1, \ldots, v_{n-k} \). \( \square \)

Thus we have proved Porteous' formula. If \( X(\varphi) \) is either empty or has the expected codimension \( (n-k)(m-k) \), then
\[ [X_k] = \Delta_{m-k,n-k}(c_t(\mathcal{F} - \mathcal{E})). \]

**Example 10.** We want to evaluate the rank of the variety \( M_k(m,n) \in \mathbb{P}^N \), where \( N = mn - 1 \). Denote by \( x_{ij} \), \( i = 1, \ldots, m \), \( j = 1, \ldots, n \), the homogeneous coordinates in \( \mathbb{P}^N \). Then, the matrix \( x_{ij} \) can be viewed as a matrix with entries in \( H^0(\mathbb{P}^N, \mathcal{O}(1)) \), and hence it gives a vector bundles homomorphism
\[ \varphi : \mathcal{O}^m_{\mathbb{P}^N} \to \mathcal{O}^m_{\mathbb{P}^N}(1). \]

We will use Porteous' formula to calculate the fundamental class of \( M_k \). Replacing \( \varphi \) by its transpose, if necessary, we assume that \( m \leq n \). If \( \omega \) denotes the class of a hyperplane, we see that
\[ c_t(\mathcal{O}^m(1)) = c_t(\mathcal{O}(1))^m = (1 + \omega t)^m = \sum_{i=0}^{m} \binom{m}{i} \omega^i t^i. \]

Therefore, the fundamental class, \( \Delta_{m-k,n-k}(c_t(\mathcal{O}^m(1))) \), of \( M_k \) is
\[ \det \begin{pmatrix} \binom{m}{m-k} \omega^{m-k} & \cdots & \binom{m}{m+2-2k-1} \omega^{mn+n-2k-1} \\ \vdots & & \vdots \\ \binom{m}{m-n+1} \omega^{m-n+1} & \cdots & \binom{m}{m-k} \omega^{m-k} \end{pmatrix}, \]

and thus the degree of \( M_k \) is given by
\[ \det \begin{pmatrix} \binom{m}{m-k} & \binom{m}{m-k+1} & \cdots & \binom{m}{m+n-2k-1} \\ \binom{m}{m-k-1} & & & \\ \vdots & & & \\ \binom{m}{m-n+1} & & & \binom{m}{m-k} \end{pmatrix}. \]

Now, we only have to evaluate this determinant. We replace the second column with the sum of the first two columns, the third column with the sum of the original second and third columns, and so on. In the resulting matrix we replace the third column with the sum of the second and third columns, the fourth column with the sum of the third and fourth columns, and so on. The
resulting matrix tells us that

\[
\begin{vmatrix}
\frac{m}{m-k} & \frac{m+1}{m-k+1} & \cdots & \frac{m+n-k-1}{m+n-2k-1} \\
\frac{m}{m-k} & \frac{m+1}{m-k+1} & \cdots & \frac{m+n-k-1}{m+n-2k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{m}{m-n+1} & \cdots & \cdots & \frac{m+n-k-1}{m-k}
\end{vmatrix}
= \prod_{i=0}^{n-k-1} \frac{(m+i)!}{(k+i)!} \frac{1}{(m-n-1)!} \frac{1}{(m-k)!} \cdots \frac{1}{(m-n)!} \frac{1}{(m-k)!}
= \prod_{i=0}^{n-k-1} \frac{(m+i)!}{(k+i)!(m-k+i)!} \begin{vmatrix}
1 & 1 & \cdots \\
0 & m-k & m-k+1 & \cdots \\
(m-k)^2 & (m-k)^2+1 & \cdots \\
\vdots & \vdots & \ddots
\end{vmatrix}
\]

The last determinant above is simply the Vandermonde determinant

\[
\begin{vmatrix}
1 & 1 & \cdots \\
m-k & m-k+1 & \cdots \\
(m-k)^2 & (m-k+1)^2 & \cdots \\
\vdots & \vdots & \ddots
\end{vmatrix}
\]

and thus it equals

\[
\prod_{n-k-1 \geq i \geq j \geq 0} ((m-k+i)-(m-k+j)) = \prod_{i=0}^{n-k-1} i!.
\]

Finally, we see that the degree of \(M_k(m,n)\) is

\[
\deg(M_k(m,n)) = \prod_{i=0}^{n-k-1} \frac{(m+i)!}{(k+i)!(m-k+i)!}.
\]

To see how the formula works, we’ll compute a simple case. If \(k = 1\) then, according to our formula,

\[
\deg(M_1) = \binom{m+n-2}{n-1}.
\] (7.1)

Now, recall that

\[
M_1 \cong \left( \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \right) \text{Segre} \mathbb{P}^{mn-1}.
\]

If we let \(\omega_1\) and \(\omega_2\) denote the hyperplane class of \(\mathbb{P}^{n-1}\) and \(\mathbb{P}^{m-1}\) respectively, then

\[
\deg(M_1) = (\omega_1 + \omega_2)^{m+n-2}.
\]

Which, by equation (7.1), is simply

\[
\binom{m+n-2}{n-1} = \frac{(m+n-2)!}{(n-1)!(m-1)!}.
\]
Example 11. Let $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth rational curve of degree $d$. How many quadririscant lines does $C$ have?

The natural parameter space to consider is $\text{Sym}^4 C \cong \mathbb{P}^4$, which parametrizes subschemes $D \subset C$ of degree 4 and dimension 0. Let $E = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_C$, and construct a fiber bundle $F$, with fiber at a point $D$ of $\text{Sym}^4 C$ given by

$$F_D = H^0(D, \mathcal{O}_{\mathbb{P}^3}(1) / \mathcal{O}_{\mathbb{P}^3}(1)(-D)).$$

To actually construct $F$, look at the universal divisor

$$\Delta = \{(D, p) : p \in D\} \subset (\text{Sym}^4 C) \times C,$$

where $(\text{Sym}^4 C) \times C$ has projection maps $\pi_1$ and $\pi_2$ to $\text{Sym}^4 C$ and $C$, respectively. Then $F = \pi_1^*(\pi_2^* \mathcal{O}_{\mathbb{P}^3}(1) |_{\Delta})$. If we consider the evaluation map $\varphi : \mathcal{E} \to \mathcal{F}$, the degree 4 points contained in a line are exactly the points where $\varphi$ has rank 2. Note that we expect a finite number of such points, since this locus should have codimension $(4 - 2)(4 - 2) = 4$.

In order to apply Porteous' formula we need to first calculate the Chern classes of $E$ and $F$. $E$ is easy, since it is a trivial bundle. Given that

$$A^*(\text{Sym}^4 C) = A^*(\mathbb{P}^4) = \mathbb{Z}[\omega]/(\omega^5),$$

we write

$$c(F) = 1 + \alpha \omega + \beta \omega^2 + \gamma \omega^3 + \delta \omega^4.$$  

To calculate $\alpha, \beta, \gamma$, and $\delta$, we will solve other, easier enumerative problems that we can express in terms of $\alpha, \beta, \gamma$, and $\delta$.

Choose one section $\sigma_1$ of $E$. We want to find where its image in $F$ is zero. Such section corresponds to a hyperplane, which cuts out $d$ points on $C$. Then any of the $\binom{d}{4}$ degree 4 divisors contained in the intersection will be a vanishing point of $\sigma_1$. Thus $\delta = \binom{d}{4}$.

Next, choose two sections of $E$, $\sigma_1$ and $\sigma_2$. $\gamma$ is the locus where $\sigma_1$ and $\sigma_2$ are linearly dependent. To find $\gamma$, we have to find the degree of this locus. Intersecting this locus with a hyperplane in $\text{Sym}^4 C$ gives a divisor vanishing at a specified point of $C$. Thus, the desired degree is the number of the number of 4-tuples of points in a pencil on $C$ containing a specified point $p$. Then, $\gamma = \binom{d-1}{3}$.

Similarly, to find $\beta$, we take a net of sections and we count how many have a 4-tuple of points containing two specified points $p$ and $q$. Thus, $\beta = \binom{d-2}{2}$.

Finally, to find $\alpha$, we consider a 3 dimensional family of sections and count how many of them have a 4-tuple of points containing three specified points. Which tells us that $\alpha = \binom{d-3}{1}$.

Therefore,

$$c(F) = 1 + \omega \binom{d-3}{1} + \omega^2 \binom{d-2}{2} + \omega^3 \binom{d-1}{3} + \omega^4 \binom{d}{4}$$

$$= (1 - \omega)^{d-3}.$$ 

Now, we can use Porteous' formula to find the number of quadririscant lines to $C$:

$$\Delta_{2,2}(c(F)) = \begin{vmatrix} c_2(F) & c_1(F) \\ c_3(F) & c_2(F) \end{vmatrix} = \left(\binom{d-2}{2}\right)^2 - \binom{d-3}{1} \binom{d-1}{3}.$$  

\diamondsuit
Appendix A

Normal Cones

We start by defining the normal cone in the case of subvarieties of an affine variety, as in [F 84].

**Definition 31.** If $W$ is a subscheme of an affine variety $V$, defined by an ideal $I$ in the coordinate ring $A$ of $V$, the normal cone $C = C_W V$ to $W$ in $V$ is

$$C = \text{Spec}(\oplus I^t/I^{t+1}).$$

The isomorphism of the coordinate ring of $W$ with $A/I$ determines a morphism $p_C : C \to W$ called the *projection*, and a closed embedding $s_C : W \to C$, called the *zero section*, with $p_C \circ s_C = \text{id}_W$. If $I$ is generated by $f_1, \ldots, f_d$, the canonical surjection $A/I[x_1, \ldots, x_d]$ onto $\oplus I^t/I^{t+1}$ determines a closed embedding in $C$ in $W \times \mathbb{C}^d$

$$C \leftarrow p_C \longrightarrow W \times \mathbb{C}^d.$$

The following lemma contains the main facts from commutative algebra that we will need. For proofs, see [E].

**Lemma 32 (Eisenbud).** (a) If $A$ is Cohen-Macaulay, a sequence $f_1, \ldots, f_d$ of elements in the maximal ideal of $A$ is a regular sequence if and only if

$$\dim(A/(f_1, \ldots, f_d)) = \dim(A) - d.$$

(b) Let $f_1, \ldots, f_d$ be a regular sequence in a local ring $A$, and let $I = (f_1, \ldots, f_d)$. Then the canonical homomorphism

$$A/I[x_1, \ldots, x_d] \to \bigoplus_{t=0}^{\infty} I^t/I^{t+1},$$

which takes the $x_i$ to the image of $f_i$ in $I/I^2$, is an isomorphism. Moreover, the kernel of the canonical surjection

$$A/I[x_1, \ldots, x_d] \to \bigoplus_{t=0}^{\infty} I^t$$

is generated by the elements $f_ix_j - f_jx_i$, for $1 \leq i < j \leq d$.

By this lemma it follows that, if $f_1, \ldots, f_d$ is a regular sequence, then $C = W \times \mathbb{C}^d$. In general, since $C$ is defined by a homogeneous ideal, it is a subcone of $W \times \mathbb{C}^d$, i.e. $C$ is invariant under
multiplication by \( \mathbb{C}^* \) on the fibers of \( \mathbb{C}^d \).

**Definition 33.** The projective normal cone \( \mathbb{P}(C) = \mathbb{P}(C_W V) \) is defined by

\[
\mathbb{P}(C) = \text{Proj}(\oplus I^t/I^{t+1}).
\]

If generators for \( I \) are chosen as above, \( \mathbb{P}(C) \) is the subscheme of \( W \times \mathbb{P}^{d-1} \) defined by the same equations that define \( C \) in \( X \times \mathbb{C}^d \).

If the embedding of \( W \) in \( V \) is a regular embedding, that is, local equations for the ideal of \( W \) in \( V \) form a regular sequence in local rings of \( V \), then \( C_W V \) is the normal bundle to \( W \) in \( V \). If \( V \) and \( W \) are nonsingular, this agrees with the definition of the normal bundle as the quotient of tangent bundles,

\[
0 \to T_W \to T_V |_W \to N_W V \to 0.
\]

Now, consider the blow-up \( \tilde{V} = \text{Bl}_W V \) of a variety \( V \) along a subscheme \( W \), and let \( E \) be the exceptional divisor, \( \tilde{V} = \text{Proj} \left( \bigoplus_{t=0}^{\infty} I^t \right) \), and \( E = \pi^{-1}(W) \).

Then, the induced mapping \( (\tilde{V} - E) \to (V - W) \) is an isomorphism, and \( E \) is isomorphic to \( \mathbb{P}(C_W V) \). The identification of the exceptional divisor \( E \) with \( \mathbb{P}(C) \) follows from the canonical isomorphism

\[
(\oplus I^t) \otimes_A A/I = \oplus(I^t/I^{t+1}).
\]

Then, the mapping from \( E \) to \( W \), induced by \( \pi : \tilde{V} \to V \), is the projection from \( \mathbb{P}(C) \) to \( W \)

When \( D \) is an effective Cartier divisor on a variety \( X \), \( N_D X \) is the restriction to \( D \) of the associated line bundle \( \mathcal{O}_X(D) \) on \( X \). If \( E = \mathbb{P}(C) \) is the exceptional divisor on the blow-up \( \tilde{V} \) of a variety \( V \) along a subscheme \( W \), then

\[
N_E \tilde{V} = \mathcal{O}_{\tilde{V}}(E)|_E = \mathcal{O}_C(-1)
\]

is also the dual line bundle to the canonical line bundle \( \mathcal{O}_C(1) \) on \( \mathbb{P}(C) \).

We will now look at an example of a normal cone which is not a vector bundle. If \( W \) is the intersection of two curves which have common components in the plane \( V \), then \( C_W V \) will have irreducible components which lie over each irreducible component of \( W \). If the curves are written in the from \( D_1 + E \) and \( D_2 + E \), where \( D_1 \) and \( D_2 \) meet properly, then \( C_W V \) has components over each component of \( E \) and over each point in \( D_1 \cap D_2 \), including those points which are in \( E \). To see this, let \( d_1, d_2, e \) be the polynomials in \( R = \mathbb{C}[x, y] \) defining \( D_1, D_2, E \), and set \( I = (d_1 e, d_2 e) \), the ideal of \( W \). It is easy to see that the kernel of the homomorphism \( A/I[u_1, u_2] \to \oplus I^n/I^{n+1} \), which takes \( u_i \) to \( d_i e \mod I^2 \), is generated by \( d_2 u_1 - d_1 u_2 \). Thus, \( C_W V \) is the subscheme of \( W \times \mathbb{C}^2 \) defined by \( d_2 u_1 - d_1 u_2 \).

We can now translate the above definitions into scheme-theoretic language. The references for the following discussion are [EGA] II.8 and [EGA] IV. 16.9, 17, 19.1.

**Definition 34.** Let \( X \) be a closed subscheme of a subscheme \( Y \), defined by the ideal sheaf \( \mathcal{I} \). The normal cone \( C_X Y \) to \( X \) in \( X \) is the cone over \( X \) defined by the spectrum of the graded sheaf of
$\mathcal{O}_X$ algebras $\oplus (\mathcal{I}_n/\mathcal{I}^{n+1})$, that is

$$C_X Y = \text{Spec}( \oplus_{n \geq 0} \mathcal{I}_n/\mathcal{I}^{n+1}).$$

A closed immersion $i : X \to Y$ as schemes is a regular embedding of codimension $d$ if every point in $X$ has an affine neighborhood $U$ in $Y$, such that, if $A$ is coordinate ring of $U$ and $I$ the ideal in $A$ defining $X$, then $I$ is generated by a regular sequence of length $d$. If $\mathcal{I}$ is the ideal sheaf of $X$ in $Y$, is follow that the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank $d$ on $X$. The normal sheaf to $X$ in $Y$, denoted $\mathcal{N}_X Y$, is the vector bundle on $X$ whose sheaf of sections is dual to $\mathcal{I}/\mathcal{I}^2$.

If $f : Y' \to Y$ is a morphism, $X' = f^{-1}(X)$, and $g$ is the induced morphism from $X'$ to $X$, then there is a canonical closed embedding

$$C'_X Y' \subset g^* C_X Y = C_X Y \times_X X'.$$

Indeed, there is a canonical surjection of $f^* \mathcal{I}$ onto the ideal sheaf $\mathcal{I}'$ of $X'$ in $Y'$, which gives a surjection of $\oplus g^*(\mathcal{I}_n/\mathcal{I}^{n+1})$ onto $\oplus (\mathcal{I}'_n/\mathcal{I}'^{n+1})$.

If the embedding of $X$ in $Y$ is a regular embedding of codimension $d$, then $C_X Y$ is a vector bundle of rank $d$ on $X$, and it is isomorphic to $\mathcal{N}_X Y$, the sheaf of sections of $\mathcal{N}_X Y$ is $(\mathcal{I}/\mathcal{I}^2)^*$. In particular, if $i : X \to Y$ embeds $X$ as a Cartier divisor on $Y$, then

$$C_X Y = \mathcal{N}_X Y = i^* \mathcal{O}_Y (X).$$

A related concept to the normal cone of a subscheme is the blow-up of a scheme along a certain subscheme.

**Definition 35.** The blow-up of $Y$ along $X$, denoted $\text{Bl}_X Y$ is the projective cone over $Y$ of the sheaf of $\mathcal{O}_Y$-algebras $\oplus \mathcal{I}_n$

$$\text{Bl}_X Y = \text{Proj}( \oplus \mathcal{I}_n).$$

Let $\tilde{Y} = \text{Bl}_X Y$, and let $\pi$ denote the projection from $\tilde{Y}$ to $Y$. The canonical invertible sheaf $\mathcal{O}(1)$ on the projective cone $\tilde{Y}$ is the ideal sheaf of $\pi^{-1}(X)$, which is therefore a Cartier divisor on $\tilde{Y}$, called the exceptional divisor. Let $E = \pi^{-1}(X)$. By construction $E$ is the projective cone of $(\oplus \mathcal{I}_n) \otimes_{\mathcal{O}_Y} \mathcal{O}_X = \oplus (\mathcal{I}_n/\mathcal{I}_n^{n+1})$, so

$$E = P(C_X Y)$$

is the projective normal cone to $X$ in $Y$. From this description we see that

$$N_{E \tilde{Y}} = \mathcal{O}_{\tilde{Y}}(E)|_E = \mathcal{O}_C(-1),$$

where $C = C_X Y$. Let $\eta$ be the projection from $E$ to $X$. If the embedding of $X$ in $Y$ is regular, then the canonical embedding of normal cones $N_{E \tilde{Y}} \subset \eta^* \mathcal{N}_X Y$ is the embedding of the universal line bundle $\mathcal{O}_N(-1)$ in $\eta^* \mathcal{N}_X Y$. In general, $\pi$ induces an isomorphism from $\tilde{Y} - E$ onto $Y - X$.

Now, let $S^* = S^0 \oplus S^1 \oplus \ldots$ be a graded sheaf of $\mathcal{O}_X$-algebras on a scheme $X$, such that the canonical map from $\mathcal{O}_x$ to $S^0$ is an isomorphism, and that $S^*$ is locally generated as an $\mathcal{O}_X$-algebra by $S^1$. To $S^*$ we associate two schemes over $X$.

**Definition 36.** Let $S^*$ be as above. Then the cone of $S^*$ is

$$C = \text{Spec}(S^*), \quad \text{with map} \quad \pi : C \to X;$$

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and the \textit{projective cone} of $S^*$, Proj($S^*$), with projection $p$ to $X$, a proper morphism. The latter is also called the projective cone of $C$, denoted $\mathbb{P}(C)$. The projective cone has a canonical line bundle, $\mathcal{O}_C(1)$.

If $S^* \to S'^*$ is a surjective, graded homomorphism of such graded sheaves of $\mathcal{O}_X$-algebras, and $C = \text{Spec}(S^*)$, $C' = \text{Spec}(S'^*)$, then there are closed embeddings $C' \hookrightarrow C$, and $\mathbb{P}(C') \hookrightarrow \mathbb{P}(C)$, such that $\mathcal{O}_C(1)$ restricts to $\mathcal{O}_{C'}(1)$.

Now, let $t$ be a variable, $S^*[t]$ the graded algebra whose $n$-th graded piece is

$$S^n \oplus S^{n-1}t \oplus \ldots \oplus S^1t^{n-1} \oplus St^n.$$ 

The corresponding cone is denoted $C \oplus 1$.

\textbf{Definition 37.} The projective cone $\mathbb{P}(C \oplus 1)$ of $C \oplus 1$ is called the \textit{projective completion} of $C$.

For proofs of the above statements and for more information on cones and projective cones see [Hart] II.7.

\textbf{A.1 Segre Classes of Cones}

As in the previous section, we will first define Segre classes over varieties and then over schemes. The references for the section are [F 84] and [F 98]. Let $W$ be a subvariety of a variety $V$. If $V$ and $W$ are nonsingular or if the embedding of $W$ in $V$ is regular, then the normal bundle $N_W V$ is defined, and we may construct invariants of the embedding by using Chern and Segre classes

$$c_i(N_W V) \cap [W] \quad \text{and} \quad s_i(N_W V) \cap [W].$$

However, in general, the normal bundle is not defined and we have only the normal cone to work with, $C_W V$. Unfortunately, Chern classes cannot be defined over normal cones, but there is a useful notion of Segre classes. We now see how this construction works. Let $W$ be any closed subscheme of a variety $V$, such that $W \neq V$. Let $\tilde{V}$ denote the blow-up of $V$ along $W$, $E$ be the exceptional divisor, and $\eta: E \to W$ be the projection. The $i$-fold self-intersections $E^i$ of the divisor $E$ are well defined classes in $A_{k-i}(E)$, $k = \dim(V) = \dim(\tilde{V})$.

\textbf{Definition 38.} We define the \textit{total Segre class} $s(W, V) \in A_* W$ to be equal to the class $[W]$ if $V = W$, otherwise

$$s(W, V) = \sum_{i \geq 1} (-1)^{i-1} \eta_*(E^i).$$

Identifying $E$ with $\mathbb{P}(C)$, the restriction of $\mathcal{O}_E(E)$ to $E$ is the dual of the universal line bundle $\mathcal{O}_C(1)$ on $\mathbb{P}(C)$. It follows that $E^i = (-1)^{i-1} c_i(\mathcal{O}_C(1))^{i-1} \cap [\mathbb{P}(C)]$, and hence

$$s(W, V) = \eta_*(c(\mathcal{O}_C(1))^i \cap [\mathbb{P}(C)]).$$

Then, we can make the following definition.

\textbf{Definition 39.} The \textit{Segre class of a normal cone} $C = C_W V$, $s(C)$, is given by

$$s(C) = \eta_*(c(\mathcal{O}_C(-1))^{-1} \cap [\mathbb{P}(C)]).$$
Now, we are ready to generalize the above constructions. Let $C$ be a cone over a scheme $X$, i.e. $C = \text{Spec}(S^*)$, where $S^*$ is a sheaf of graded $\mathcal{O}_X$-algebras. Assume that $\mathcal{O}_X \to S^0$ is surjective, $S^1$ is coherent, and $S^*$ is generated by $S^1$. Let

$$\mathbb{P}(C \oplus 1) = \text{Proj}(S^*[t])$$

be the projective completion of $C$, with projection $g : \mathbb{P}(C \oplus 1) \to X$, and let $\mathcal{O}(1)$ be the canonical line bundle on $\mathbb{P}(C \oplus 1)$. Then,

**Definition 40.** The Segre class of $C$, denoted $s(C)$, is the class in $A_* X$ defined by the formula

$$s(C) = g_* \left( \sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus 1)] \right).$$

We then need the following proposition from [F 98].

**Proposition 41 (Fulton).** (a) If $\mathcal{E}$ is a vector bundle on $X$, then

$$s(\mathcal{E}) = c(\mathcal{E})^{-1} \cap [X].$$

(b) Let $C_1, \ldots, C_s$ be the irreducible components of $C$, and $m_i$ the geometric multiplicity of $C_i$ in $C$. Then

$$s(C) = \sum_{i=1}^s m_i s(C_i).$$

Furthermore, note that, for any cone $C$, $s(C \oplus 1) = s(C)$.

Now we can define the Segre class of any closed subscheme $X$ of a scheme $Y$.

**Definition 42.** Let $C$ denote the normal cone of $X$ in $Y$. The Segre Class of $X$ in $Y$, denoted $s(X, Y)$, is defined to be the Segre class of the normal cone $C$,

$$s(X, Y) = s(C_X Y) \in A_* X.$$

When $X$ is regularly embedded in $Y$, $C = N$ is a vector bundle, and

$$s(X, Y) = c(N)^{-1} \cap [X].$$

The interesting result, which can be found in [F 98], is that these Segre classes have fundamental birational invariance. If $f : Y' \to Y$ is a birational proper morphism, and $X' = f^{-1}(X)$, then $s(X', Y')$ pushes forward to $s(X, Y)$. More precisely,

**Corollary 43 (Fulton).** Let $f : Y' \to Y$ be a morphism of pure dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ regularly embedded in $Y'$, with normal bundle $N'$, and $g : X' \to X$ the induced morphism. If $f$ is proper, $Y$ irreducible, and $f$ maps each irreducible component of $X$ onto $Y$, then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)(c(N)^{-1} s(X, Y)).$$

If $X \subset Y$ is also regularly embedded, with normal bundle $N$, then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)(c(N)^{-1} \cap [X]).$$

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Bibliography


