Graetz problem for slip flow in a channel in the presence of axial conduction, viscous dissipation and flow work

by

Lowell L. Baker

Submitted to the Mechanical Engineering Department in partial fulfillment of the requirements for the degree of Bachelor of Science in Mechanical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 2002

© Lowell L. Baker, MMII. All rights reserved.

The author hereby grants to MIT permission to reproduce and distribute publicly paper and electronic copies of this thesis document in whole or in part.

Author .......................................................... Lowell L. Baker

Mechanical Engineering Department

May 10, 2002

Certified by .................................................. Nicholas Hadjiconstantinou

Assistant Professor of Mechanical Engineering

Thesis Supervisor

Accepted by .................................................. Ernest H. Cravalho

Professor Ernest Cravalho

Chairman of the Undergraduate Thesis Committee
Graetz problem for slip flow in a channel in the presence of axial conduction, viscous dissipation and flow work

by

Lowell L. Baker

Submitted to the Mechanical Engineering Department on May 10, 2002, in partial fulfillment of the requirements for the degree of Bachelor of Science in Mechanical Engineering

Abstract

This thesis investigates developing heat transfer in laminar, hydrodynamically fully developed slip flow in a channel with a constant wall temperature. Slip flow is a good approximation when the Knudsen number, the ratio of the molecular mean free path to the channel width, is less than approximately 0.1. Our analysis includes the effects of viscous dissipation, flow work, and axial heat conduction. A series solution is used to solve for the temperature distribution and provide predictions for the Nusselt number. It is found that inclusion of axial conduction increases both the non-dimensional temperature and the Nusselt number at a given location. The effect of slip flow is to increase the non-dimensional temperature and decrease the Nusselt number at a given location. The effects of viscous dissipation and flow work are more complex, and can either increase or decrease the non-dimensional temperature and the Nusselt number at various points along the length of the channel. It is found that a non-zero Eckert number will result in an asymptotic Nusselt number (based on energy transfer to the wall) of zero. The joint effects of slip flow, axial heat conduction, and viscous dissipation and flow work, are also illustrated through exploration of the relevant parameter space.

Thesis Supervisor: Nicolas Hadjiconstantinou
Title: Assistant Professor of Mechanical Engineering
Acknowledgments

I would like to thank my advisor, Nicolas Hadjiconstantinou, for the opportunity to work on this project and for his help in completing it. I would also like to thank my friends and family for their help and support over the years.
3.3 Combined effect of slip flow and axial conduction on Nusselt number . . . . . 38

4 Conclusions 45

A Wall temperature jump 47
A.1 Thermally fully developed flow ........................................ 47
A.2 Energy equation ............................................................... 48
A.3 Solution of the energy equation ......................................... 48
A.4 Boundary conditions ......................................................... 49
A.5 Solution ................................................................. 50

B Representing a function with Legendre Polynomials 51

C Numerical solution method 55
C.1 Maple script ................................................................. 55
C.2 Root finding program .......................................................... 59
List of Figures

3-1 Spatial variation of $\theta$ for $Re = 1, Pr = 0.7, Ec = 1, Kn = 0.05, \gamma = 1.4$. .................. 26
3-2 Developing $\theta$ profile at various axial locations for $Re = 100, Pr = 0.7, \gamma =
1.4, Kn = 0$. ...................................................... 26
3-3 $\theta$ profile at $\xi = \frac{1}{10RePr}$ for $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. ................. 28
3-4 $\theta$ profile at $\xi = \frac{1}{2RePr}$ for $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. ................. 28
3-5 $\theta_{bulk}$ at $x = \frac{\xi}{RePr}$ with $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. .................. 29
3-6 $\theta_{bulk}$ at $x = \frac{\xi}{RePr}$ with $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = -0.5$. ................. 30
3-7 $\theta_{bulk}$ at $x = \frac{\xi}{RePr}$ with $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0.5$. ................. 30
3-8 Nusselt number $Nu_f$ vs $x = \frac{\xi}{RePr}$ with $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. ... 31
3-9 Nusselt number in thermal entrance region for $x = \frac{\xi}{RePr}$ with with $Pr =
0.7, \gamma = 1.4, Kn = 0, Ec = -0.5$. .............................................. 32
3-10 Nusselt number in thermal entrance region for $x = \frac{\xi}{RePr}$ with with $Pr =
0.7, \gamma = 1.4, Kn = 0, Ec = 0.5$. .............................................. 32
3-11 Effect of Kn on temperature profile at $\xi = 0$ for $Re = 1000, Pr = 0.7, \gamma =
1.4, Kn = 0, Ec = 0$. .............................................. 34
3-12 Effect of Kn on temperature profile at $\xi = 20$ with $Re = 1000, Pr = 0.7, \gamma =
1.4, Kn = 0, Ec = 0$. .............................................. 34
3-13 Effect of Kn on temperature profile at $\xi = 100$ with $Re = 1000, Pr = 0.7, \gamma =
1.4, Kn = 0, Ec = 0$. .............................................. 35
3-14 Effect of Kn on $\theta_{bulk}$ with $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. ........ 36
3-15 $\theta_{bulk}$ variation dependent on Kn with $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn =
0, Ec = 0$. .............................................. 36
3-16 The effect of Kn on $Nu_f$ with $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. .... 37
3-17  The difference between $\text{Nu}_f$ and $\text{Nu}_w$ with $\text{Re} = 1000$, $\text{Pr} = 0.7$, $\gamma = 1.4$, $\text{Kn} = 0$, $\text{Ec} = 0.5$. ................................................................. 38

3-18  The difference between $\text{Nu}_f$ and $\text{Nu}_w$ with $\text{Re} = 1000$, $\text{Pr} = 0.7$, $\gamma = 1.4$, $\text{Kn} = 0$, $\text{Ec} = -0.5$. ................................................................. 40

3-19  Axial variation of $\text{Nu}_w$ with $\text{Re} = 0.1$, $\text{Pr} = 0.7$, $\text{Kn} = 0$. ................................. 41

3-20  Axial variation of $\text{Nu}_w$ with $\text{Re} = 0.1$, $\text{Pr} = 0.7$, $\text{Kn} = 0.05$. ................................. 41

3-21  Axial variation of $\text{Nu}_w$ with $\text{Re} = 10$, $\text{Pr} = 0.7$, $\text{Kn} = 0$. ................................. 42

3-22  Axial variation of $\text{Nu}_w$ with $\text{Re} = 10$, $\text{Pr} = 0.7$, $\text{Kn} = 0.05$. ................................. 42

3-23  Axial variation of $\text{Nu}_w$ with $\text{Re} = 1000$, $\text{Pr} = 0.7$, $\text{Kn} = 0$. ................................. 43

3-24  Axial variation of $\text{Nu}_w$ with $\text{Re} = 1000$, $\text{Pr} = 0.7$, $\text{Kn} = 0.05$. ................................. 43
List of Tables

3.1 Fully developed Nusselt number $\text{Nu}_f = \text{Nu}_w$ for $\text{Ec} = \text{Kn} = 0$.

3.2 Variation of asymptotic Nusselt number $\text{Nu}_f (= \text{Nu}_w)$ with Re and Kn for $\text{Pr} = 0.7, \gamma = 1.4, \text{Ec} = 0$.

3.3 Variation of asymptotic Nusselt number $\text{Nu}_f (= \text{Nu}_w)$ with Re and Kn for $\text{Pr} = 2/3, \gamma = 5/3, \text{Ec} = 0$.

3.4 Variation of asymptotic Nusselt number $\text{Nu}_f$ with Kn for $\text{Pr} = 0.7, \gamma = 1.4, \text{Ec} \neq 0, \text{Re} > 0$. (Note that $\text{Nu}_w \to 0$ asymptotically for $\text{Ec} \neq 0$.)

3.5 Variation of asymptotic Nusselt number $\text{Nu}_f$ with Kn for $\text{Pr} = 2/3, \gamma = 5/3, \text{Ec} \neq 0, \text{Re} > 0$. (Note that $\text{Nu}_w \to 0$ asymptotically for $\text{Ec} \neq 0$.)
Chapter 1

Introduction

Recent interest in small scale engineering systems has resulted in the need for better understanding of physical processes at the microscale. In particular, this thesis focuses on convective heat transfer in two-dimensional channels which are sufficiently small for slip effects to be important. Slip effects appear when the characteristic system dimension approaches the mean distance traveled by molecules between collisions, known as the mean free path. More precisely, the continuum approximation with no-slip boundary conditions is found to be reasonable when Kn ≤ 0.001, where Kn = λ/(2L) is the Knudsen number, λ is the molecular mean free path and 2L is the channel width. As the Knudsen number increases, the continuum approximation breaks down. For 0.001 < Kn ≤ 0.1, the Navier-Stokes equations hold in the interior of the flow field, but the boundary conditions on the temperature and velocity must be modified: Discontinuities such as a velocity slip and a temperature jump appear at the walls. Beyond the slip flow regime lies the transition flow regime (0.1 < Kn ≤ 10) and the free molecular flow regime (Kn > 10). Our work will focus on the slip flow regime, though the results obtained will also be applicable to the continuum regime.

Due to the small transverse dimensions of the channels associated with slip flow, we expect the Reynolds numbers of interest to be small, and thus for axial heat conduction to be important. In addition, viscous dissipation and flow work may have a significant effect on the heat transfer. In this thesis, the effect of axial heat conduction, flow work and viscous dissipation on the temperature profile, bulk temperature, and Nusselt number are investigated.
A solution to the Gratz problem is presented (in the presence of axial heat conduction, viscous dissipation and flow work), in which a gas with a known initial temperature profile and a fully developed velocity profile enters a channel having constant wall temperature. In Chapter 2, this entrance temperature profile is assumed to be symmetric. In the results section, Chapter 3, the fluid is assumed to be at a constant temperature at the entrance. The extension of this solution to more general boundary conditions is straightforward. In Appendix B, we show how the method of Chapter 2 can be applied to the case of hydrodynamically fully developed flow in an infinite channel with a wall temperature jump.

The Graetz problem was solved by finding an infinite series expansion of the temperature that satisfies the energy equation and the associated boundary conditions. This temperature field was then used to find other quantities of interest. The solutions found were compared with previous results for limiting cases when possible, and were found to be in agreement with the latter.

Our results were compared to those of Hadjiconstantinou and Simek [1] who considered slip flow with axial conduction but neglected flow work and viscous dissipation, the results of Ou and Cheng [3] who investigated no-slip flow with viscous dissipation and flow work but without axial conduction, and finally, the results of Rodriguez [4] who considered slip flow without axial heat conduction, viscous dissipation or flow work.

For no-slip flow with $Ec = 0$, it was observed that the asymptotic Nusselt numbers ranged between $Nu = 7.54$ for the case $RePr \to \infty$ (corresponding to no axial heat conduction), and $Nu = 8.117$ for the case $RePr \to 0$ in agreement with the results of [1]. Our results for $Ec = 0$, $Kn \neq 0$ (corresponding to no viscous dissipation or flow work) were also in agreement with the results of [1].

A positive Knudsen number results in larger values of the non-dimensional temperature, but lower values of the Nusselt number based on thermal energy transfer from the fluid. However, the non-zero fluid velocity at the wall results in work being done on the wall. This work will result in an additional energy transfer to the wall, but no change to the temperature profile within the fluid. The slip work done on the wall serves to reduce the effect of slip flow on the Nusselt number. Despite this, a non-zero value of the Knudsen number still has the effect of reducing the amount of energy transferred to the wall.

Ou and Cheng show in [3], that any non-zero value of the Brinkman number, $Br = PrEc$, results in an asymptotic value of zero for the Nusselt number. This result is a reflection
of the fact that sufficiently far from the origin, axial temperature gradients have gone to zero, thus the energy exchange with the wall is dominated by the competition between flow work and viscous dissipation. As shown by Shah and London [5], these contributions balance out for an ideal gas if integrated over the channel cross section, thus leading to zero net energy exchange between the wall and the gas and hence a zero Nusselt number. In a similar fashion, our results suggest that the asymptotic Nusselt number, based on the total energy transfer to the wall, will asymptotically approach 0. (The Nusselt number based only on thermal energy transfer from the fluid does not approach zero for a positive Knudsen number.) Although the asymptotic result \( (x \to \infty) \) corresponds to the limiting case in a strict sense, the typical quantity of interest is the Nusselt number due to the "convective" temperature difference, that is due to the fact that the fluid enters the channel at a different temperature from the temperature of the wall or the wall temperature changes suddenly. Therefore the relevant "asymptotic" Nusselt number is not \( \text{Nu}(x \to \infty) \) but the Nusselt number at an axial distance where entrance effects have died out but viscous heat generation and flow work do not yet dominate, in the sense that the temperature difference because of "convective" effects (as defined above) is larger than the contribution of flow work and viscous heat dissipation to the temperature profile. As will be seen later, for sufficiently small values of the Eckert number, \( \text{Nu}(\text{Ec} = 0) \) is a reasonable approximation for the actual Nusselt number for some region of axial distances.

Our analysis neglects several factors which could be important in practical applications. Potentially the most important assumptions are those of constant transport properties and incompressible flow. The latter assumption in particular warrants further investigation, as density changes are expected to arise because of both temperature gradients and the large pressure gradients required for flow in small channels. If these results are applied to a given problem, care must be taken that these assumptions are valid. The method of solution is also not analytical, thus more caution than usual must be used in interpreting the results. Due to time constraints, no effort was made to quantify the effect of rounding errors, series truncation and other factors on the solution. In the interest of speed, a relatively simple algorithm was used to find the eigenvalues. All of these factors could potentially lead to an incorrect solution, however the agreement between this solution and previous works suggests that the present solution is reasonably accurate.
Chapter 2

Problem solution

2.1 Problem statement

We investigate the heat transfer characteristics of hydrodynamically fully developed flow in an infinite two-dimensional channel. The flow enters the region of interest with a known, symmetric\(^1\), temperature profile. The walls of the channel at \(y = \pm L\) are maintained at a constant temperature. All of the fluid properties are assumed to be constant, and natural convection effects and compressibility are neglected; both axial and lateral conduction are included, as well as viscous dissipation and flow work.

2.2 Governing equations

2.2.1 Slip flow between flat plates

For \(Kn \leq 0.1\) the velocity slip is given by

\[
\left. u \right|_{y = \pm L} = \mp a^* \lambda \left. \frac{du}{dy} \right|_{y = \pm L} \tag{2.1}
\]

where \(u\) is the fluid velocity, \(y\) is the coordinate perpendicular to the flow and \(a^* = \frac{2-a}{a}\)

where \(a\) is the momentum accommodation coefficient.

The velocity profile is expected to be symmetric about the channel centerline so the

\(^1\)It is straightforward to generalize to an arbitrary temperature profile
velocity gradient in the $y$ direction will be zero there

$$\frac{du}{dy}\bigg|_{y=0} = 0 \quad (2.2)$$

For hydrodynamically fully developed flow of a fluid with constant properties in a channel, the Navier-Stokes equations reduce to

$$\frac{dp}{dx} = \mu \frac{d^2 u}{dy^2} \quad (2.3)$$

Solving (2.3) with the boundary conditions (2.1) and (2.2) we obtain

$$u = -\frac{L^2 dp}{2\mu dx} \left( 1 + 4a^* Kn - \left( \frac{y}{L} \right)^2 \right) \quad (2.4)$$

The mean velocity is defined as

$$u_m = \frac{1}{2L} \int_{-L}^{L} u \, dy = -\frac{L^2 dp}{3\mu dx} \left( 1 + 6a^* Kn \right) \quad (2.5)$$

and the ratio of the velocity to the mean velocity is given by

$$\frac{u}{u_m} = \frac{3 \left( 1 + 4a^* Kn - \left( \frac{y}{L} \right)^2 \right)}{2 \left( 1 + 6a^* Kn \right)} \quad (2.6)$$

### 2.2.2 Energy equation

For the present problem, the equation for energy conservation is

$$\rho c u \frac{\partial T}{\partial x} = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + u \frac{dp}{dx} + \mu \left( \frac{du}{dy} \right)^2 \quad (2.7)$$

This equation will be solved to find the temperature field.

### 2.2.3 Non-dimensionalization

It is convenient to change the variables to a non-dimensional form. Dimensionless variables can be defined as

$$\eta \equiv \frac{y}{L} \quad (2.8)$$

$$\xi \equiv \frac{x}{L} \quad (2.9)$$
\[ \theta = \frac{T - T_w}{T_0 - T_w} \quad (2.10) \]
\[ U \equiv \frac{u}{u_m} \quad (2.11) \]
\[ \text{Re} \equiv \frac{4\rho u_m L}{\mu} \quad \text{Reynolds number} \quad (2.12) \]
\[ \text{Pr} \equiv \frac{\mu c}{k} \quad \text{Prandtl Number} \quad (2.13) \]
\[ \text{Ec} \equiv \frac{u_m^2}{c(T_0 - T_w)} \quad \text{Eckert number} \quad (2.14) \]

The derivatives appearing in the equations can be non-dimensionalized as follows.

\[ \frac{\partial T}{\partial x} = \frac{T_0 - T_w}{L} \frac{\partial \theta}{\partial \xi} \quad (2.15) \]
\[ \frac{\partial^2 T}{\partial x^2} = \frac{T_0 - T_w}{L^2} \frac{\partial^2 \theta}{\partial \xi^2} \quad (2.16) \]
\[ \frac{\partial^2 T}{\partial y^2} = \frac{T_0 - T_w}{L^2} \frac{\partial^2 \theta}{\partial \eta^2} \quad (2.17) \]
\[ \frac{du}{dy} = \frac{u_m}{L} \frac{dU}{d\eta} \quad (2.18) \]

Substituting the above expressions for the derivatives into the energy equation and using (2.5) to eliminate \( dp/dx \) gives the non-dimensional form of the energy equation\(^2\)

\[ U \frac{\partial \theta}{\partial \xi} = \frac{4}{\text{RePr}} \left( \frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2} \right) + \frac{\text{Ec}}{\text{Re}} \left( -\frac{12}{1 + 6a^*\text{Kn}} + 4 \left( \frac{dU}{d\eta} \right)^2 \right) \quad (2.19) \]

where \( U = U(\eta) \) is given by 2.6.

For \( \text{Kn} \leq 0.1 \), the boundary conditions on \( \theta \) are

\[ \theta|_{\xi=0} = \theta_0 \quad (2.20) \]
\[ \frac{\partial \theta}{\partial \eta}|_{\eta=0} = 0 \quad (2.21) \]
\[ \theta|_{\eta=1} = -4F^* \gamma^* \frac{\text{Kn}}{\text{Pr}} \frac{\partial \theta}{\partial \eta}|_{\eta=1} \quad (2.22) \]

where \( F^* \equiv \frac{2-F}{F} \) with \( F \) the energy accommodation coefficient, and \( \gamma^* \equiv \frac{7}{\gamma+1} \) where \( \gamma \) is the ratio of specific heats.

\(^2\)This expression assumes \( (u_m \rho c(T_0 - T_w))/L \neq 0 \).
2.3 Solution for temperature field

To find the temperature field, we must find a solution to (2.19) subject to the boundary conditions (2.20)-(2.22).

2.3.1 Change of variables

We define a new variable $\phi$ as

$$\phi \equiv \theta - f(\eta)$$  \hspace{1cm} (2.23)

where $f(\eta)$ satisfies

$$\frac{d^2 f}{d\eta^2} = -\frac{EcPr}{4} \left( -\frac{U}{1 + 6a^*Kn} + 4 \left( \frac{dU}{d\eta} \right)^2 \right)$$  \hspace{1cm} (2.24)

For a $\phi$ satisfying this relationship, (2.19) takes on a simpler form when written in terms of $\phi$. Because (2.24) only determines $f$ up to the addition of a multiple of $\eta$ and a constant, $f$ can be chosen so that the boundary conditions on $\phi$ are similar to those on $\theta$. More specifically, we choose\(^3\)

$$f = \frac{9}{8} \frac{EcPr}{(1 + 6a^*Kn)^2} \left[ -\eta^4 + (2 + 8a^*Kn) \eta^2 - 1 - 8a^*Kn - 64F^*a^*\gamma^* \frac{Kn^2}{Pr} \right]$$  \hspace{1cm} (2.25)

With $\phi$ defined by (2.23), equation (2.19) can be written as

$$U \frac{\partial \phi}{\partial \xi} = \frac{4}{RePr} \left( \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} \right)$$  \hspace{1cm} (2.26)

The boundary conditions on $\phi$ are found by subtracting $f$ from the boundary conditions (2.20)-(2.22) on $\theta$.

$$\phi|_{\eta=1} = \left. -4F^*a^*\gamma^* \frac{Kn}{Pr} \frac{\partial \phi}{\partial \eta} \right|_{\eta=1}$$  \hspace{1cm} (2.27)

$$\phi|_{\xi=0} = \theta_0 - \frac{9}{8} \frac{EcPr}{(1 + 6a^*Kn)^2} \times$$

$$\left[ -\eta^4 + (2 + 8a^*Kn) \eta^2 - 1 - 8a^*Kn - 64F^*a^*\gamma^* \frac{Kn^2}{Pr} \right]$$  \hspace{1cm} (2.28)

$$\left. \frac{\partial \phi}{\partial \eta} \right|_{\eta=0} = 0$$  \hspace{1cm} (2.29)

\(^3\)This expression for $f(\eta)$ turns out to be the fully developed temperature profile; see (A.2)
2.4 Series solution

A solution for the temperature profile can be found by assuming $\phi$ to have the following form

$$\phi = \sum_{n=1}^{\infty} \left( C_n \exp \left( - \frac{\beta_n^2}{\text{RePr}} \xi \right) \sum_{j=0}^{\infty} a_{2j} \eta^{2j} \right)$$  \hspace{1cm} (2.30)

This form for $\phi$ satisfies the boundary condition (2.29).

Assuming that (2.30) can be twice differentiated term-by-term, we obtain

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{\text{RePr}} \sum_{n=1}^{\infty} \left( - C_n \beta_n^2 \exp \left( - \frac{\beta_n^2}{\text{RePr}} \xi \right) \sum_{j=0}^{\infty} a_{2j} \eta^{2j} \right)$$  \hspace{1cm} (2.31)

$$\frac{\partial^2 \phi}{\partial \xi^2} = \frac{1}{\text{RePr}^2} \sum_{n=1}^{\infty} \left( C_n \beta_n^4 \exp \left( - \frac{\beta_n^2}{\text{RePr}} \xi \right) \sum_{j=0}^{\infty} a_{2j} \eta^{2j} \right)$$  \hspace{1cm} (2.32)

$$\frac{\partial \phi}{\partial \eta} = \sum_{n=1}^{\infty} \left( C_n \exp \left( - \frac{\beta_n^2}{\text{RePr}} \xi \right) \sum_{j=0}^{\infty} (2j) a_{2j} \eta^{2j-1} \right)$$  \hspace{1cm} (2.33)

$$\frac{\partial^2 \phi}{\partial \eta^2} = \sum_{n=1}^{\infty} \left( C_n \exp \left( - \frac{\beta_n^2}{\text{RePr}} \xi \right) \sum_{j=0}^{\infty} (2j)(2j-1) a_{2j} \eta^{2j-2} \right)$$  \hspace{1cm} (2.34)

By substituting the expression for $\phi$ in (2.30) into the energy equation (2.26) and equating like coefficients of $\eta$, a relationship can be found among the coefficients $a_{2j}$. Taking $a_0 = 1$, we obtain

$$a_2 = - \left( \frac{3}{16} + 4a^* Kn \beta_n^2 + \frac{1}{2} \left( \frac{1}{\text{RePr}} \right) \beta_n^4 \right)$$  \hspace{1cm} (2.35)

$$a_{2j} = \frac{3}{8(2j)(2j-1)} \times \left[ - \left( \frac{1 + 4a^* Kn \beta_n^2}{1 + 6a^* Kn \beta_n^2} \right) + \frac{8}{3} \left( \frac{1}{\text{RePr}} \right) \beta_n^4 a_{2j-2} + \frac{1}{1 + 6a^* Kn \beta_n^2} a_{2j-4} \right]$$  \hspace{1cm} (2.36)

By substituting (2.30) for $\phi$ in (2.27), we find that the temperature jump boundary condition at $\eta = 1$ is satisfied if

$$1 + \sum_{j=1}^{\infty} a_{2j} \left( 1 + 8j F^* \gamma_n^* \frac{Kn}{Pr} \right) = 0$$  \hspace{1cm} (2.37)

Because the $a_{2j}$ are known functions of the eigenvalues, $\beta_n$, the above can be solved for the set of eigenvalues. Find the $C_n$ coefficients by first expressing the boundary condition.
(2.28) as a sum of polynomials. One method for doing this is discussed in Appendix B. Once such an expression for the boundary condition is found, we require that \( \phi|_{\zeta=0} \) satisfies the boundary condition (2.29). This gives

\[
\sum_{n=1}^{\infty} \left( C_n \sum_{n=0}^{\infty} a_{2n} \eta^{2n} \right) = \sum_{m=0}^{\infty} L_n P_n \left( \frac{1}{2} \eta \right)
\]  

(2.38)

where \( L_n \) and \( P_n \) are discussed in Appendix B. This can be solved by equating coefficients of like powers of \( \eta \). This is simplified by recognizing that the coefficient of \( \eta^{2j} \) on the left side of (2.38) is

\[
\sum_{n=1}^{\infty} C_n a_{2j} (\beta_n)
\]  

(2.39)

2.5 Bulk temperature and Nusselt number

After finding the temperature profile, the latter can be used to obtain other useful information, such as the bulk temperature and the Nusselt number.

2.5.1 Bulk temperature

The bulk temperature, \( T_b \) is defined as

\[
T_b = \frac{\int_L^T T \, dy}{\int_{-L}^T u \, dy} = \frac{\int_L^T T \, dy}{2u_m L}
\]  

(2.40)

\( T_b \) can be expressed in terms of dimensionless variables

\[
T_b = \int_{-1}^{1} ((T_0 - T_w) \theta + T_w) \frac{U}{2} \, d\eta
\]  

(2.41)

The dimensionless bulk temperature is

\[
\theta_b = \frac{T_b - T_w}{T_0 - T_w} = \frac{1}{2} \int_{-1}^{1} \theta U \, d\eta
\]  

(2.42)

We have an expression for \( U \) and for \( \theta = \phi + f(\eta) \), so this equation can be simplified to

\[
\theta_b = \frac{1}{2} \int_{-1}^{1} U f \, d\eta + \frac{3}{4} \frac{1}{1 + 6a^* Kn} \times
\]

22
\[
\int_{-1}^{1} \sum_{n=1}^{\infty} \left( C_n \exp \left( -\frac{\beta_n^2}{\text{RePr} \, \xi} \right) \sum_{j=0}^{\infty} a_{2j} \left( \left( 1 + 4a^* \text{Kn} \right) \eta^{2j} - \eta^{2j+2} \right) \right) d\eta
\]

(2.43)

\[
= -\frac{\text{EcPr}}{(1 + 6a^* \text{Kn})^3} \left[ \frac{27}{35} + \frac{54}{5} a^* \text{Kn} + 36 \left( a^* \right)^2 + \frac{2}{\text{Pr}} F^* a^* \gamma^* \right] \text{Kn}^2 + \frac{432}{\text{Pr}} F^* (a^* \gamma^*)^2 \text{Kn}^3
\]

\[
+ \frac{3}{2} \frac{1}{1 + 6a^* \text{Kn}} \sum_{n=1}^{\infty} \left( C_n \exp \left( -\frac{\beta_n^2}{\text{RePr} \, \xi} \right) \sum_{j=0}^{\infty} a_{2j} \left( \frac{1 + 4a^* \text{Kn}}{2j + 1} - \frac{1}{2j + 3} \right) \right)
\]

(2.44)

2.5.2 Nusselt number

The energy exchange between the wall and the gas can be divided into two parts. The first part is the exchange of thermal energy due to a temperature gradient at the wall. The second part is the work done by the slipping flow at the wall against the wall shear stress. This dissipation mechanism scales with the Eckert number (see below) and is only present in slip flow.

Considering the thermal energy transfer from the fluid, we can write

\[
h_f = \frac{k}{T_w - T_b} \left. \frac{\partial T}{\partial y} \right|_{y=L} = -\left. \frac{k}{L} \frac{\partial \theta}{\partial \eta} \right|_{\eta=1}
\]

(2.45)

where the subscript \( f \) is used to denote the thermal energy transfer.

The Nusselt number (due to thermal energy transfer) is thus defined as

\[
\text{Nu}_f \equiv \frac{4L h_f}{k} = -\frac{4}{\theta_b} \left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=1}
\]

(2.46)

Evaluating the derivative \( \frac{\partial \theta}{\partial \eta} = \frac{\partial \theta}{\partial \eta} + \frac{\partial \eta}{\partial \eta} \) at \( \eta = 1 \), we obtain

\[
\text{Nu}_f = -\frac{4}{\theta_b} \left[ \sum_{n=1}^{\infty} \left( C_n \exp \left( -\frac{\beta_n^2}{\text{RePr} \, \xi} \right) \sum_{j=0}^{\infty} 2ja_{2j} \right) + 18a^* \frac{\text{EcPr} \text{Kn}}{(1 + 6a^* \text{Kn})^2} \right]
\]

(2.47)

where \( \theta_b \) is as given above.

The energy transfer to the wall due to the shear work does not depend on the temperature, although the effective heat transfer coefficient will vary with the changing bulk temperature. We can similarly define a shear work Nusselt number that accounts for the energy dissipated at the wall,

\[
\text{Nu}_s = \frac{4L \dot{w}_s}{k (T_b - T_w)}
\]

(2.48)
Using

\[ w_s = -\mu u_{wall} \left. \frac{\partial u}{\partial y} \right|_{wall} \]  \hspace{1cm} (2.49)

and changing to non-dimensional variables gives

\[ \text{Nu}_s = - \frac{4 \text{PrEc} U_{\eta=1} \left. \frac{\partial U}{\partial \eta} \right|_{\eta=1}}{\theta_b} \]
\[ = \frac{72 \text{PrEc}}{\theta_b} \frac{a^* \text{Kn}}{(1 + 6a^* \text{Kn})^2} \]  \hspace{1cm} (2.50, 2.51)

The Nusselt number for the total energy transfer to the wall is

\[ \text{Nu}_w = \text{Nu}_s + \text{Nu}_f \]  \hspace{1cm} (2.52)
Chapter 3

Results and discussion

Previous studies have considered special cases of the analysis presented in this thesis. This section will emphasize the results in which the assumptions of the previous analyses fail. To show the validity of the procedure used in this thesis, our results will be compared to the previous results in the appropriate limiting cases.

Results are given for a constant entering temperature profile, and the momentum and energy accommodation coefficients are both assumed to be unity.

A typical 3d plot of the $\theta$ field is given as figure 3-1. The oscillations near $\xi = 0$ are a result of the series approximation used for the boundary condition, and are discussed in greater detail in Appendix B. It can be observed that the effects of these oscillations die out relatively quickly. An illustration of how the oscillations decay is shown in Fig. 3-2.

This figure also shows how the temperature profile evolves with the axial distance $\xi$.

3.1 Effect of axial heat conduction

By considering the ratio of axial heat conduction to convection, one can see that axial heat conduction is negligible for flows when $\mathrm{RePr} \gg 1$. For typical engineering applications, neglecting axial heat conduction has little effect on the result. However, for flow in microchannels the Reynolds number will be small, thus one might expect axial conduction to play a larger role. The effect of axial conduction is shown below.
Figure 3-1: Spatial variation of $\theta$ for $Re = 1, Pr = 0.7, Ec = 1, Kn = 0.05, \gamma = 1.4$.

Figure 3-2: Developing $\theta$ profile at various axial locations for $Re = 100, Pr = 0.7, \gamma = 1.4, Kn = 0$. 
3.1.1 Temperature profile

As shown in [3], if axial heat transfer is neglected for $Kn = 0$, the axial variation of both the temperature and Nusselt number can be parameterized by the product $\xi RePr$. This is not the case when axial heat conduction is included; in this case the axial profile depends jointly on $\xi$ and $RePr$. The case where $RePr \to \infty$ (here approximated by $Re = 1000^1$) corresponds to negligible axial heat conduction. Thus, by plotting the actual value of the temperature profile, Nusselt number, and bulk temperature against $RePr\xi$ and comparing to the case $Re = 1000$, we can isolate the effect of axial heat transfer.

Figures 3-3 and 3-4 show the effect of changes in the Reynolds number on the temperature profile. Increasing the value of $Re$ while holding all other parameters constant has the effect of decreasing the relative effect of axial heat transfer.

The value for $\theta$ increases as the value of $Re$ decreases. This indicates that including axial conduction makes the temperature at a given point closer to the entrance temperature than it would be if axial conduction were neglected. The reason for this is that the temperature gradient (and thus the magnitude of axial conduction) is a decreasing function of $\xi$. Therefore, the fluid at a given point in the flow field will be more strongly affected by the fluid behind it, so the temperature at a given point will be closer to the entrance temperature than in the case where axial conduction is neglected.

As $RePr \to 0$, the deviation from the case where axial heat transfer is neglected becomes more apparent. For $RePr = 1000$ the temperature profiles are observed to be virtually identical to the limiting case $RePr \to \infty$. Even when $RePr = 10$, the deviation from the case where axial conduction is neglected is moderate. This gives some idea over the range which axial conduction is important, although the magnitude of the change will depend on the flow parameters used.

3.1.2 Bulk temperature variation

As shown in the last section, including axial heat transfer increases the temperature at a given point as compared to the case where axial heat conduction is neglected. It is thus little surprise that the bulk temperature increases also. Figures 3-5 through 3-7 show the effect of axial conduction on the bulk temperature. As in the temperature profile graphs,

$^1$For typical small scale flow, $Re = 1000$ (or even 10) is not reasonable, but this value is used as an example of the limiting case of negligible axial heat conduction.
Figure 3-3: $\theta$ profile at $\xi = \frac{1}{10 \text{RePr}}$ for $\text{Pr} = 0.7, \gamma = 1.4, \text{Kn} = 0, \text{Ec} = 0.$

Figure 3-4: $\theta$ profile at $\xi = \frac{1}{2 \text{ReT}}$ for $\text{Pr} = 0.7, \gamma = 1.4, \text{Kn} = 0, \text{Ec} = 0.$
in the case where axial heat conduction is negligible all graphs should collapse onto the line for \( \text{RePr} = 1000 \) (which closely approximates the case \( \text{RePr} \to \infty \)).

Figures 3-6 and 3-7 show the effect of viscous dissipation and flow work. These effects lead to a non-zero asymptotic bulk temperature.

The graphs of bulk temperature for different values of \( \text{Re} \) have a similar appearance, except for being translated in \( \xi \). These graphs also show that axial heat conduction has no effect on the final temperature (which is expected because as \( \xi \to \infty \) the axial temperature gradient goes to 0).

![Graph showing bulk temperature for different values of Re](image)

Figure 3-5: \( \theta_{\text{bulk}} \) at \( x = \frac{\xi}{\text{RePr}} \) with \( \text{Pr} = 0.7, \gamma = 1.4, \text{Kn} = 0, \text{Ec} = 0 \).

### 3.1.3 Developing Nusselt number

As shown in (2.46), the Nusselt number is inversely proportional to the bulk temperature, which increases when axial heat conduction is considered. The Nusselt number is also proportional to the lateral temperature gradient at the wall, which is expected to increase when axial conduction is included, thus the effect of axial conduction on the Nusselt number is ambiguous without further calculation. Evaluating the Nusselt number shows that the axial conduction leads to a net increase in the Nusselt number in the thermal entrance
Figure 3-6: $\theta_{bulk}$ at $x = \frac{\xi}{RePr}$ with Pr = 0.7, $\gamma = 1.4$, Kn = 0, Ec = 0.5.

Figure 3-7: $\theta_{bulk}$ at $x = \frac{\xi}{RePr}$ with Pr = 0.7, $\gamma = 1.4$, Kn = 0, Ec = 0.5.
Figure 3-8: Nusselt number $Nu_f$ vs $x = \frac{e}{Re Pr}$ with $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$.

region. The Nusselt number in the thermal entrance region is plotted in figures 3-8 through 3-10, again for the three cases $Ec = -0.5, Ec = 0, Ec = 0.5$

The Nusselt number is higher when axial conduction is considered, and increases rapidly as one moves toward the entrance. Figure 3-8 shows how the final value of the Nusselt number varies with the Reynolds number for zero Eckert number. For $Ec > 0$, the bulk temperature $\theta_{bulk}$ will pass through zero eventually; this location is indicated by the vertical asymptotes in figure 3-10. Notice that as the Reynolds number is decreased, the point where this temperature inversion occurs moves downstream because of axial conduction. This was previously shown in Figure 3-7.

3.1.4 Asymptotic Nusselt number

The asymptotic Nusselt number is seen to be affected by axial heat conduction for $Ec = 0$. However, similarly to as was shown by [3] this effect is transient. For any non-zero value of $Ec$ with $Kn = 0$, the Nusselt number will eventually go to zero. Values for the asymptotic Nusselt number with $Ec = Kn = 0$ are tabulated in table 3.1. While these values will not be strictly accurate for any real fluid, they do serve as a useful approximation as moderate
Figure 3.9: Nusselt number in thermal entrance region for $x = \frac{\xi}{Re_{ft}}$ with $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = -0.5$.

Figure 3.10: Nusselt number in thermal entrance region for $x = \frac{\xi}{Re_{Pr}}$ with $Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0.5$.  

32
<table>
<thead>
<tr>
<th>RePr</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>8.117</td>
</tr>
<tr>
<td>0.01</td>
<td>8.116</td>
</tr>
<tr>
<td>0.1</td>
<td>8.105</td>
</tr>
<tr>
<td>1</td>
<td>8.005</td>
</tr>
<tr>
<td>10</td>
<td>7.631</td>
</tr>
<tr>
<td>100</td>
<td>7.542</td>
</tr>
<tr>
<td>1000</td>
<td>7.541</td>
</tr>
</tbody>
</table>

Table 3.1: Fully developed Nusselt number $\text{Nu}_f = \text{Nu}_{w}$ for $Ec = Kn = 0$.

axial distances for $Ec$ small.

### 3.2 Effect of non-zero Knudsen number

When the Knudsen number is nonzero, there is a temperature and velocity jump at the wall. The temperature jump affects the temperature profile, bulk temperature and Nusselt number. The Nusselt number is additionally affected by the shear work at the wall.

#### 3.2.1 Temperature profile

A nonzero Knudsen number results in a temperature jump at the wall that is proportional to the temperature gradient at the wall as given in (2.22). However, the entrance temperature is also prescribed in this model as $\theta_{\xi=0} = 1$, which is inconsistent with the above boundary condition. To partially remedy this, I used a polynomial series that approximates the boundary condition (2.22), but also satisfies the boundary condition at $\xi = 0$. Because of the approximations inherent in this method, the results are not necessarily very accurate near $\xi = 0$.

Figure 3-11 shows the difficulty in satisfying the boundary conditions at $\xi = 0$. However, as shown in Figs. 3-12 and 3-13 these oscillations die out and converge to a common value, although we do not prove this is the correct value. Further discussion can be found in Appendix B.

Figures 3-11 through 3-13 show the effect of increasing the Knudsen number. (To show the effect of the Knudsen number in the absence of axial conduction an unrealistically large Reynolds number is used.) At higher values of the Knudsen number, there is a larger temperature jump at the wall. This corresponds to a smaller temperature gradient at the
Figure 3-11: Effect of Kn on temperature profile at $\xi = 0$ for $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$.

Figure 3-12: Effect of Kn on temperature profile at $\xi = 20$ with $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. 
wall, which results in lower heat transfer at the wall, which results in a higher value of $\theta$. Notice that at $\xi = 20$ there is much less difference in the centerline temperature than at $\xi = 100$. The effect of the different boundary conditions at the wall takes time to propagate inward.

![Figure 3-13: Effect of Kn on temperature profile at $\xi = 100$ with $Re = 1000$, $Pr = 0.7$, $\gamma = 1.4$, $Kn = 0$, $Ec = 0$.](image)

3.2.2 Bulk temperature

A nonzero Knudsen number also affects the bulk temperature in the same manner as the temperature profile. A larger value for Kn leads to a higher value of $\theta_{bulk}$, however for $Ec = 0$ the final value of the bulk temperature does not depend on the Knudsen number. With no dissipation or flow work, the temperature profile eventually goes to the wall temperature with zero lateral temperature gradients. In figure 3-15, one can see another measure of the error resulting from the approximation of the boundary condition. The value for $\theta_{bulk}|\xi = 0$ should be 1 in all cases. The effect of this error is unknown.
Figure 3-14: Effect of Kn on $\theta_{bulk}$ with $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$.

Figure 3-15: $\theta_{bulk}$ variation dependent on Kn with $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = 0$. 
Figure 3-16: The effect of Kn on $\text{Nu}_f$ with $\text{Re} = 1000, \text{Pr} = 0.7, \gamma = 1.4, \text{Kn} = 0, \text{Ec} = 0$.

3.2.3 Entrance Nusselt number

Figure 3-16 shows the effect of Kn on the Nusselt number based on the heat transfer from the fluid, $\text{Nu}_f$. A positive value of Kn results in a lower asymptotic $\text{Nu}_f$, and seems to result in a lower $\text{Nu}_f$.

Figures 3-17 and 3-18 illustrate the difference between the Nusselt numbers based on the heat flow from the fluid, $\text{Nu}_f$, and that based on the total energy transfer to the wall, $\text{Nu}_w$. In Fig. 3-17, both $\text{Nu}_f$ and $\text{Nu}_w$ are lower than in the case Kn = 0, but $\text{Nu}_w$ is initially higher. This is because $\theta_b$ is positive at first, so the work done at the wall is added to the heat transfer. Notice that the asymptote where $\theta_b = 0$ is the same for both cases Kn = 0.1, and is after the asymptote for Kn = 0. After the asymptote, $\theta_b < 0$ because of flow work on the fluid. However, the energy transfer due to slip work is still to the wall while the heat transfer is from the wall. Thus $\text{Nu}_w$ is less than $\text{Nu}_f$.

Figure 3-18 is similar, except Ec < 0 so the bulk temperature is always positive. Because Ec < 0 the slip work serves to decrease the Nusselt number. It was observed that $\text{Nu}_w$ approaches 0 when Ec ≠ 0. This is expected; as remarked in the introduction, the flow work and the sum of the viscous dissipation and shear work at the wall balance for an ideal
gas. This will result in no net energy exchange with the wall asymptotically far from the entrance.

![Graph showing the difference between \( \text{Nu}_f \) and \( \text{Nu}_w \) with \( \text{Re} = 1000, \text{Pr} = 0.7, \gamma = 1.4, \text{Kn} = 0, \text{Ec} = 0.5 \).]

Figure 3-17: The difference between \( \text{Nu}_f \) and \( \text{Nu}_w \) with \( \text{Re} = 1000, \text{Pr} = 0.7, \gamma = 1.4, \text{Kn} = 0, \text{Ec} = 0.5 \).

**Fully developed Nusselt number**

The values for the asymptotic Nusselt number \( \text{Nu}_f \) for a monatomic and diatomic gas are given in tables 3.2-3.5. The variation in the Nusselt number is found to depend on both \( \text{Re} \) and \( \text{Kn} \) for \( \text{Ec} = 0 \), but only on \( \text{Kn} \) for \( \text{Ec} \neq 0 \). Note, however, that the asymptotic Nusselt number for \( \text{Ec} \neq 0 \) is in fact dominated by the viscous heat generation and flow work, as discussed in the introduction.

### 3.3 Combined effect of slip flow and axial conduction on Nusselt number

We include various figures of the Nusselt number \( \text{Nu}_w \) as a function of axial distance and Eckert number for several different values of the Reynolds number and Knudsen number. The general conclusions about the effect of slip flow and axial heat conduction still hold;
<table>
<thead>
<tr>
<th>Kn = 0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re = 0.01</td>
<td>8.117</td>
<td>7.459</td>
<td>6.835</td>
</tr>
<tr>
<td>0.1</td>
<td>8.109</td>
<td>7.452</td>
<td>6.828</td>
</tr>
<tr>
<td>1</td>
<td>8.036</td>
<td>7.384</td>
<td>6.768</td>
</tr>
<tr>
<td>10</td>
<td>7.685</td>
<td>7.056</td>
<td>6.483</td>
</tr>
<tr>
<td>100</td>
<td>7.543</td>
<td>6.930</td>
<td>6.376</td>
</tr>
<tr>
<td>1000</td>
<td>7.541</td>
<td>6.926</td>
<td>6.374</td>
</tr>
</tbody>
</table>

Table 3.2: Variation of asymptotic Nusselt number $\text{Nu}_f(= \text{Nu}_w)$ with Re and Kn for $\text{Pr} = 0.7, \gamma = 1.4, \text{Ec} = 0$.

<table>
<thead>
<tr>
<th>Kn = 0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re = 0.01</td>
<td>8.117</td>
<td>7.365</td>
<td>6.670</td>
</tr>
<tr>
<td>0.1</td>
<td>8.109</td>
<td>7.358</td>
<td>6.664</td>
</tr>
<tr>
<td>1</td>
<td>8.040</td>
<td>7.293</td>
<td>6.607</td>
</tr>
<tr>
<td>10</td>
<td>7.693</td>
<td>6.971</td>
<td>6.332</td>
</tr>
<tr>
<td>100</td>
<td>7.544</td>
<td>6.837</td>
<td>6.222</td>
</tr>
<tr>
<td>1000</td>
<td>7.541</td>
<td>6.835</td>
<td>6.220</td>
</tr>
</tbody>
</table>

Table 3.3: Variation of asymptotic Nusselt number $\text{Nu}_f(= \text{Nu}_w)$ with Re and Kn for $\text{Pr} = 2/3, \gamma = 5/3, \text{Ec} = 0$.

<table>
<thead>
<tr>
<th>Kn = 0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.568</td>
<td>2.588</td>
<td>3.632</td>
</tr>
</tbody>
</table>

Table 3.4: Variation of asymptotic Nusselt number $\text{Nu}_f$ with Kn for $\text{Pr} = 0.7, \gamma = 1.4, \text{Ec} \neq 0, \text{Re} > 0$. (Note that $\text{Nu}_w \to 0$ asymptotically for $\text{Ec} \neq 0$.)

<table>
<thead>
<tr>
<th>Kn = 0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.563</td>
<td>2.560</td>
<td>3.500</td>
</tr>
</tbody>
</table>

Table 3.5: Variation of asymptotic Nusselt number $\text{Nu}_f$ with Kn for $\text{Pr} = 2/3, \gamma = 5/3, \text{Ec} \neq 0, \text{Re} > 0$. (Note that $\text{Nu}_w \to 0$ asymptotically for $\text{Ec} \neq 0$.)
Figure 3-18: The difference between $\text{Nu}_f$ and $\text{Nu}_w$ with $Re = 1000, Pr = 0.7, \gamma = 1.4, Kn = 0, Ec = -0.5$.

decreasing the Reynolds number and decreasing the Knudsen number both increase the Nusselt number. However, the plots can help explain what happens when the effects are combined.
Figure 3-19: Axial variation of $\textit{Nu}_w$ with $\textit{Re} = 0.1, \textit{Pr} = 0.7, \textit{Kn} = 0$.

Figure 3-20: Axial variation of $\textit{Nu}_w$ with $\textit{Re} = 0.1, \textit{Pr} = 0.7, \textit{Kn} = 0.05$. 

41
Figure 3-21: Axial variation of $\text{Nu}_\text{w}$ with $\text{Re} = 10, \text{Pr} = 0.7, \text{Kn} = 0$.

Figure 3-22: Axial variation of $\text{Nu}_\text{w}$ with $\text{Re} = 10, \text{Pr} = 0.7, \text{Kn} = 0.05$. 
Figure 3-23: Axial variation of $Nu_w$ with $Re = 1000, Pr = 0.7, Kn = 0$.

Figure 3-24: Axial variation of $Nu_w$ with $Re = 1000, Pr = 0.7, Kn = 0.05$. 

43
Chapter 4

Conclusions

This thesis has presented a solution for the Graetz problem in the slip flow regime. The effects of viscous dissipation, flow work and axial heat conduction were included. The temperature profile was evaluated numerically as a series, and the results were used to show the effect of slip flow, axial heat conduction, viscous dissipation, and flow work on the temperature profile and Nusselt number.

The results agree with previous analyses where such analyses were available. This lends credibility to the model, however in interpreting the results one must be cautious. The temperature profile and the boundary condition on the temperature at the entrance were approximated by truncating an infinite series. All calculations were done with finite-precision arithmetic, and the root solver used to find the eigenvalues uses a simple algorithm that could give erroneous results under certain circumstances.

Despite these limitations, the results obtained appear reasonable. They show the effect of including axial heat conduction, and show the circumstances when axial heat conduction is most important, and gives an indication of when axial heat conduction can be neglected.

The results also show the effect that slip flow has on the heat transfer. Slip flow generally decreases the effectiveness of heat transfer. Including slip flow results in a non-zero asymptotic Nusselt number based on heat transfer from the fluid, however the Nusselt number based on energy transfer to the wall asymptotically approaches zero.

While the solution presented here might be helpful for design or analysis work, it is still limited in the degree of understanding it yields. As it is fundamentally a numerical solution, it can not give the degree of physical understanding that a purely analytical solution would
could. Discovering an analytical solution to this problem, though not necessarily possible, would be a great aid in our understanding of heat transfer at the microscale.
Appendix A

Wall temperature jump

This problem deals with the case where thermally and hydrodynamically fully developed flow in a channel with constant wall temperature encounters a sudden jump in wall temperature at $\xi = 0$. In the absence of axial heat conduction, information can not flow backward in the flow, so for $\xi < 0$ the temperature will be given by the fully developed temperature profile, and the temperature profile for $\xi > 0$ can be found from the solution given before for the semi-infinite region with the temperature at $\xi = 0$ the fully developed temperature by considering a semi-infinite region with a known temperature profile at $\xi = 0$.

Inclusion of axial heat conduction makes the problem more difficult, but it can be solved in a manner analogous to the semi-infinite problem discussed earlier. The problem can be divided into two regions; $\xi < 0$ and $\xi > 0$ with the requirement that the temperature and the temperature gradient are continuous across the interface.

A.1 Thermally fully developed flow

The temperature profile for thermally fully developed flow can easily be solved analytically. All derivatives in the $x$ direction vanish. In this case, the energy equation (2.19) reduces to

$$\frac{d^2 \theta}{d\eta^2} = -\frac{EcPr}{4} \left(-12\frac{U}{1 + 6a^*Kn} + 4 \left(\frac{dU}{d\eta}\right)^2\right)$$ \hspace{1cm} (A.1)

with boundary conditions corresponding to (2.20)-(2.22).
Solving this equation for fully developed flow gives

\[
\theta = \frac{9}{8} \frac{EcPr}{(1 + 6Kn)^2} \left( -\eta^4 + (2 + 8Kn)\eta^2 - 1 - 8Kn - 64 \frac{2 - F}{F} \frac{\gamma}{\gamma + 1} \frac{Kn^2}{Pr} \right) \tag{A.2}
\]

Notice that this exactly matches the expression for \( f(\eta) \) in (2.24).

**A.2 Energy equation**

Denote the region \( \xi < 0 \) as region 1 and the region \( \xi > 0 \) as region 2. All dimensionless parameters are to be bases on the wall temperature in their region and the obvious subscripts to distinguish the variables in one region 1 from 2.

In this case, the energy equation (2.19) for \( \theta_1 \) and \( \theta_2 \) is the same as it was for the semi-infinite case (except for the addition of subscripts), and the same substitution can be made to obtain the energy equation (2.26) in terms of \( \phi \).

The boundary conditions on the \( \phi_m \) in the \( \eta \) direction are again given by (2.27)-(2.29). The boundary condition at \( \xi = 0 \) is different. The temperatures in the two regions must be adjusted so that the both the temperatures and temperature gradients in the \( \xi \) direction match at \( \xi = 0 \). This condition is given in (A.8) and (A.15).

**A.3 Solution of the energy equation**

Following the method of the last chapter, let

\[
\phi_1 = \sum_{n=1}^{\infty} \left( A_n \exp \left( \frac{\lambda_n^2}{RePr} \xi \right) \sum_{j=0}^{\infty} b_{2j} \eta^{2j} \right) \tag{A.3}
\]

and

\[
\phi_2 = \sum_{n=1}^{\infty} \left( C_n \exp \left( -\frac{\beta_n^2}{RePr} \xi \right) \sum_{j=0}^{\infty} a_{2j} \eta^{2j} \right) \tag{A.4}
\]

As previously, this substitution reduces the energy equations for regions 1 and 2 to the simpler form (2.26). For \( \phi_2 \) the eigenvalues, \( \beta_n \) and the relationship between the \( a_{2j} \) coefficients are the same as in the semi-infinite case, as given in (2.35)-(2.36) and (2.37). For \( \phi_1 \) the relationships are similar, except for signs. Taking \( b_0 = 1 \) the eigenvalues for
region 1 are related by

\[
b_2 = \frac{3}{16} \left( 1 + 4Kn \lambda_n^2 \right) - \frac{1}{2} \left( \frac{1}{RePr} \right)^2 \lambda_n^4
\]

(A.5)

\[
b_{2j} = \frac{3}{8 (2j) (2j - 1)} \times \left[ \left( 1 + 4Kn \lambda_n^2 \right) - \frac{8}{3} \left( \frac{1}{RePr} \right)^2 \lambda_n^4 \right] b_{2j-2} - \frac{1}{1 + 6Kn} \lambda_n^2 b_{2j-4}
\]

(A.6)

Substituting the definition for \( \phi_1 \) into the boundary condition at \( \eta = 1 \) gives the relationship between the coefficients \( b_{2j} \).

\[
1 + \sum_{j=1}^{\infty} b_{2j} \left( 1 + 8j \frac{2 - F}{2} \frac{\gamma Kn}{1 + \gamma Pr} \right) = 0
\]

(A.7)

### A.4 Boundary conditions

The above relationships are sufficient to find the eigenvalues and all of the coefficients of \( \eta \) for both regions. All that remains is to find the sets of coefficients \( A_n \) and \( C_n \). These coefficients are found by requiring that the temperature be continuous and have a continuous \( \xi \) derivative at \( \xi = 0 \).

Requiring the temperatures to be equal at \( \xi = 0 \) gives the condition

\[
\frac{T_0 - T_w,1}{T_w,2 - T_w,1} \theta_1|_{\xi=0} = \frac{T_0 - T_w,2}{T_w,2 - T_w,1} \theta_2|_{\xi=0} + 1
\]

(A.8)

In terms of \( \phi_m \) A.8 is

\[
\frac{T_0 - T_w,1}{T_w,2 - T_w,1} \left( \phi_1|_{\xi=0} + f_1 \right) = 1 + \frac{T_0 - T_w,2}{T_w,2 - T_w,1} \left( \phi_2|_{\xi=0} + f_2 \right)
\]

(A.9)

This expression is simplified by the fact that the only difference in \( f_1 \) and \( f_2 \) is the temperature difference in the denominator of the Ec number. Since the \( f \) terms are multiplied by this temperature difference, they cancel out, leaving

\[
\frac{T_0 - T_w,1}{T_w,2 - T_w,1} \phi_1|_{\xi=0} = 1 + \frac{T_0 - T_w,2}{T_w,2 - T_w,1} \phi_2|_{\xi=0}
\]

(A.10)
We can choose $T_0$ to be defined as

$$T_0 = \frac{T_{w,1} + T_{w,2}}{2}$$  \hspace{1cm} (A.11)

In this case the condition for equality of temperature at the interface reduces to

$$\frac{1}{2} \phi_1 |_{\xi=0} = 1 - \frac{1}{2} \phi_2 |_{\xi=0}$$  \hspace{1cm} (A.12)

By balancing the coefficients of powers of $\eta$ on either side of this equation, we obtain the result

$$\sum_{n=1}^{\infty} C_n a_0 (\beta_n) = 2 - \sum_{n=1}^{\infty} A_n b_0 (\lambda_n)$$  \hspace{1cm} (A.13)

$$\sum_{n=1}^{\infty} C_n a_{2j} (\beta_n) = \sum_{n=1}^{\infty} A_n b_{2j} (\lambda_n)$$  \hspace{1cm} (A.14)

Requiring the derivative of the temperature to be the same at the interface gives the condition

$$\frac{\partial \theta_1}{\partial \xi} \bigg|_{\xi=0} = \frac{T_0 - T_{w,2}}{T_0 - T_{w,1}} \frac{\partial \theta_2}{\partial \xi} \bigg|_{\xi=0}$$  \hspace{1cm} (A.15)

Equation (A.15) is equivalent to

$$\frac{\partial \phi_1}{\partial \xi} \bigg|_{\xi=0} = -\frac{\partial \phi_2}{\partial \xi} \bigg|_{\xi=0}$$  \hspace{1cm} (A.16)

Balancing the coefficients of $\eta$ across this equation gives

$$\sum_{n=1}^{\infty} C_n \beta_n^2 a_{2j} (\beta_n) = \sum_{n=1}^{\infty} A_n \lambda_n^2 b_{2j} (\lambda_n)$$  \hspace{1cm} (A.17)

These equations are sufficient to determine the values of the $A_n$ and $C_n$ and thus to solve for the temperature profile.

### A.5 Solution

Once the temperature profile is known the bulk temperature, heat transfer coefficients, and Nusselt numbers can all be found from the equivalent expressions in section 2.5.
Appendix B

Representing a function with Legendre Polynomials

It is necessary to express the boundary condition (2.28) as a sum of polynomials. This can be done by writing the boundary condition as a sum of Legendre polynomials\(^1\).

The Legendre polynomials are given by Rodrigues’ formula

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( x^2 - 1 \right)^n \quad (B.1)
\]

where \( P_n(x) \) is the \( n^{th} \) Legendre polynomial. The first several Legendre polynomials are

\[
\begin{align*}
P_0(x) & = 1 \quad (B.2) \\
P_1(x) & = x \quad (B.3) \\
P_2(x) & = \frac{1}{2} \left( 3x^2 - 1 \right) \quad (B.4) \\
P_3(x) & = \frac{1}{2} \left( 5x^3 - 3x \right) \quad (B.5) \\
P_4(x) & = \frac{1}{8} \left( 35x^4 - 30x^2 + 3 \right) \quad (B.6)
\end{align*}
\]

The Legendre polynomials form an orthogonal set on \((-1, 1)\), thus any piecewise continuous function can be represented as a series of Legendre polynomials on this interval. The Legendre polynomials are not the only choice for this though. In (2.30), polynomials were somewhat arbitrarily chosen as they are easier to work with. Perhaps a series of sines

\(^1\)Properties of Legendre polynomials and orthogonal functions in this section come from [2]
would have been more natural, as sines are the eigenfunction of the conduction equation when convection is neglected.

In this case, orthogonality on the interval $(-1,1)$ is not adequate. The polynomials $P_n \left( \frac{3}{2} \right)$ are orthogonal on the region $(-2,2)$, thus a generalized Fourier series (Fourier-Legendre series) expansion for a function on $(-2,2)$ can be constructed

$$ g(x) = \sum_{0}^{\infty} L_n P_n \left( \frac{x}{2} \right) $$

where

$$ L_n = \left[ \int_{-2}^{2} \left( P_n \left( \frac{x}{2} \right) \right)^2 dx \right]^{-1} \int_{-2}^{2} g(x) P_n \left( \frac{x}{2} \right) dx $$

$$ = \frac{2n + 1}{4} \int_{-2}^{2} g(x) P_n \left( \frac{x}{2} \right) dx $$

(B.8)  

(B.9)

This Fourier expansion of a given piecewise continuous function $g(x)$ on this region will converge to $g(x)$ at points where $g(x)$ is continuous, and to $(g(x^+) + g(x^-))/2$ at points of discontinuity. The boundary condition (2.27) requires that $\phi|_{\eta=1} \propto \frac{d\phi}{dy}|_{\eta=1}$, however it is easier to consider the case where $\phi|_{\eta=1} = 0$ first.

This condition is met if $g$ in (B.7) is defined by

$$ g|_{\eta=1^+} = - g|_{\eta=1^-} $$

(B.10)

One way to satisfy this condition, while maintaining $g(x) = \phi(x)$ on the interval $(-1,1)$ is to take

$$ g(\eta) = \begin{cases} 
\phi & \text{if } |x| < 1 \\
-\phi & \text{if } |x| > 1
\end{cases} $$

(B.11)

With this definition of $g$, one can use (B.7) to find a Legendre series that converges to a given function $\phi$ on the range $(-1,1)$ and converges to zero at the points $\pm 1$.

To satisfy the boundary condition (2.27), one can substitute $\eta = k x$. This will change the value of the function at $\eta = 1$ so that the boundary condition can be satisfied at $\eta = 1$.

The parameter $k$ can be found by solving

$$ g|_{kx=1} = -4 F* \gamma* \frac{Kn}{Pr} \frac{dg}{dx} \bigg|_{kx=1} $$

(B.12)
for a value of $k$ nearest 1. Because we are dealing with polynomials, this can be simplified to

$$g\bigg|_k = -4F^\gamma \frac{Kn}{Pr} \frac{1}{k} \frac{dg}{dx}\bigg|_k$$

(B.13)

This method will yield a series $g(\eta)$ that converges to $\phi(\eta)$ on the interval $(-1, 1)$, but also satisfies the appropriate boundary condition at $\eta = \pm 1$. As with all series methods, $g$ is not a perfect representation of $\phi$. In particular, any finite series approximation for $g$ will exhibit oscillations. The magnitude of some of these oscillations will decrease as more terms are kept in the series. Regardless of the number of terms kept, the Gibbs phenomenon will be present. As a result of the discontinuity in the function $g$ at $\pm 1$, there will always be a finite overshoot in the series representation.

This oscillation and overshoot is observed to die out relatively quickly in $\xi$, so it does not seem to present a problem except for very near the entrance region. In the entrance region, this approximation will result in errors in the calculated temperature, errors in the bulk temperature (although some of the oscillations should serve to cancel each other out), and errors in the Nusselt number. In particular, the series solution will have a finite slope at the entrance, resulting in an upper limit to the Nusselt number. The magnitude of these effects has not been quantified. It it thought that their effect is minor outside of the immediate entrance region, however there is no guarantee of this.
Appendix C

Numerical solution method

The majority of the programming for this thesis was done in Maple. One of the most time-consuming steps was the determination of the zeros of a polynomial. As the polynomials encountered were typically well-behaved, a simple-minded approach to finding the zeros was implemented in C. This program is also included.

C.1 Maple script

For the most part this is a direct implementation of the equations found in the text of this thesis.

```maple
> with(StringTools):

> p_roots := proc(poly, variable, step, max)
> local str, i, roots, numroots, A;
> roots := {};
> str := cat("C/pzero4.exe ", convert(step,string)," ", convert(max,string));
> for i from 0 to degree(simplify(poly)) do
> str := cat(str," ", convert(coef(poly, variable, i),string));
> end do;
> A := Split(system(str)[2]);
```
numroots := nops(A) - 1;  
for i from 1 to numroots do
  roots := roots union {parse(A[i])};
end do;

roots := sort(convert(roots, list));
return roots;
end proc:

a := proc(twoj, RePr, Kn)
option remember;
if (twoj = 0) then return 1;
elif (twoj = 2) then return -(3./16*(1+4.*Kn)/(1+6.*Kn)*
beta2+0.5/RePr^2*beta2^2);
elif (twoj >= 4) then return collect(3./8/twoj/(twoj-1)*
(-((1+4.*Kn)/(1+6.*Kn)*beta2+8./3./RePr^2*beta2^2)*
a(twoj-2, RePr, Kn) + 1/(1+6*Kn)*beta2*a(twoj-4, RePr, Kn)), beta2);
else return false;
end if;
end proc:
eigenvalue_condition := proc(terms, Ren, Pr, gamm, Kn)
return collect(1+add(a(2*j, Ren*Pr, Kn)*(1+8*j*gamm/(gamm+1)*Kn/Pr),
j = 1..terms), beta2);
eend proc:
eigenvalues := proc(Ren, Pr, gamm, Kn, terms, step, max)
option remember;
local poly, roots;
poly := collect(eigenvalue_condition(terms, Ren, Pr, gamm, Kn), beta2);
roots := p_roots(poly, beta2, step, max);
return roots;
eend proc:
with(orthopoly):
with(LinearAlgebra):
LegendreCoeff := proc(f,n)
  option inline;
  (2*n+1)/4*int(eval(f,eta=x)*P(n,1/2*x),x=-1..1)-(2*n+1)/4*int(eval(f,eta=x)*P(n,1/2*x),x=1..2)-(2*n+1)/4*int(eval(f,eta=x)*P(n,1/2*x),x=-2..-1);
end proc:
C_coeff := proc(eigs, Ec, Ren, Pr, Kn, gamm, phizero)
  local neigs, Leg, A, i, j, B, C, k, con;
  neigs := nops(eigs);
  Leg := collect(add(eval(LegendreCoeff(phizero,2*n)*P(2*n,0.5*eta)),
    n=0..neigs-1), eta);
  con := -4*gamm/(gamm+1)*Kn/Pr;
  # and its derivative at the wall boundary condition
  k := -1*fsolve(simplify(eval(Leg, eta=K)-con/K*eval(diff(Leg, eta), eta=K)),
    K=-1..0,maxsols=1);
  # desired boundary condition
  A := Matrix(neigs, neigs);
  for i from 0 to neigs-1 do
    for j from 1 to neigs do
      A[i+1,j] := eval(a(2*i, Ren*Pr, Kn), beta2 = eigs[j]);
    end do;
  end do;
end proc:
B := Matrix(neigs, 1);
for i from 0 to neigs-1 do
  B[i+1,1] := coeff(Leg, eta, 2*i);
> end do;
>
> C:=LinearSolve(A,B);
> return C;
> end proc:

> theta:=proc(Ren, Pr, Ec, gamm, Kn, terms, etamult, step, max)
> option remember, system;
> local eigs, coeffs, theta, neigs, phi, f, U, thetab, Nu,
> ret, t, A2j, Asums, i, j, A;
> eigs:=eigenvalues(Ren, Pr, gamm, Kn, terms-10, step, max);
> neigs:=nops(eigs);
> printf("Eigenvalues are %a\n", eigs);

> f:=9./8*Ec*Pr/(1+6*Kn)^2*(-eta^4+(2+8*Kn)*eta^2-1-8*Kn-
> 64*gamm/(gamm+1)*Kn^2/Pr);
> coeffs:=Transpose(C_matrix coeffs, Ec, Ren, Pr, Kn, gamm, 1-f));
> coeffs:=convert(coeffs, listlist)[1]; ## convert to a list
> printf("Coefficients are %a\n", coeffs);
> t:=time();
> A2j:=Matrix(terms+1, neigs);
> for i from 0 to terms do ##most time consuming loop....
> #if (i mod 10 = 0) then printf("On step %a out of %a\n",i,terms); end if;
> A:=convert(a(2*i,Ren*Pr,Kn),horner);
> for j from 1 to neigs do
> A2j[i+1,j]:=eval(A,beta2=eigs[j])*eta^(2*i);
> end do;
> end do;
> Asums:=Matrix(neigs, 1);
> for j from 1 to neigs do
> Asums[j,1]:=add(A2j[i,j],i=1..terms+1);
> end do;
> phi:=add(coeffs[n]*exp(-1*eigs[n]/Ren/Pr*xi)*Asums[n,1], n=1..neigs);
> phi:=collect(phi, xi);
> printf("It took %a seconds to calculate phi", time()-t);
> theta:=phi+f;
> return theta;
> end proc:
> theta_bulk:=proc(Ren, Pr, Ec, gamm, Kn, terms, etamult, step, max)
> local U, t;
> U:=1.5*(1+4*Kn-eta^2)/(1+6*Kn);
> t:=theta(Ren, Pr, Ec, gamm, Kn, terms, etamult, step, max);
> return simplify(0.5*int(t*U, eta=-1..1));
> end proc:
> Nusselt:=proc(Ren, Pr, Ec, gamm, Kn, terms, etamult, step, max)
> local tb, t;
> t:=theta(Ren, Pr, Ec, gamm, Kn, terms, etamult, step, max);
> tb:=theta_bulk(Ren, Pr, Ec, gamm, Kn, terms, etamult, step, max);
> return simplify(-4/tb*eval(diff(t, eta), eta=1));
> end proc:

C.2 Root finding program

This C program employs a crude method for finding the positive zeros of a polynomial, which is used to find the eigenvalues $\beta_n$. The function is simply evaluated at regular intervals until a change of sign is detected. Then the program refines the location of the zero several times to ensure accuracy. Because the polynomial coefficients are typically very small, this program was run on a machine with 128 bit floating point numbers. As a result of this, and to optimize the program for successive calculation of increasing powers of a variable some standard functions were rewritten. These are also included.

The program takes command-line input. The first parameter is the step size between successive iterations, the second is the maximum value to consider. After that come the coefficients of the polynomial with zeroth order first.
// Lowell Baker

// A program to find the zeros of a polynomial with large coefficients
// takes command line arguments
// first argument after the program name is the step size
// second is the maximum zero value to look for
// after that they are the coefficients of the polynomial, in order from
// zeroth order coefficient to highest order.

#include <iostream.h>
#include <ctype.h>

#define Length 1024 // the maximum number of coefficients that can be entered

long double eval (long double coeffs[Length], long double x, int terms);
int sgn(long double x);
long double powld(long double a, int b);
long double atolld(char* a);

int main(int argc, char* argv[]) {  
  long double coeffs[Length]; // coeffs of the poly, zeroth first
  long double zeros[Length]; // the zeros, to be returned
  long double step=0.01; // step size between checks
  long double max=500; // the last (MOL) value to check
  long double x=0; // the last (MOL) value to check
  long double y=0; // the value currently being tested
  int lastsign=0; // the sign of the last number evaluated
  int currentsign=0; // the sign of the poly at the current location
  int numzeros=0; // the number of zeros found
  int i=0; // generic counter;

const long double refine=128;  //the reduction in step size when refining
long double revisionstep; //the size of steps to be taken while revising
const int numrevisions=6; //the number of revisions to do to the zeros

if (argc<3)
{
    cout <<"Error, not enough arguments.\n";
    exit(1);
}
if(argc-2>Length)
{
    cout<<"Error, too many coefficients entered, max is "<<Length<<',\n';
    exit(1);
}

step=atold(argv[1]); //the first argument after the name
max=atold(argv[2]); //the second argument after the name

for(i=0;i<Length;i++) // zero (all) the coefficients
{
    coeffs[i]=0;
}
for(i=3;i<argc && i<Length+2;i++) //read coefficients in from command line
{
    coeffs[i-3]=atold(argv[i]);
    // cout << coeffs[i-3]<<',';
}

lastsign=sgn(eval(coeffs,0,argc-3)); //the sign of the polynomial at zero

for(x=0;x<=max+step;x+=step) //look for changes in sign of the polynomial
{
currentsign = sgn(eval(coeffs, x, argc - 3));

if (currentsign != lastsign || currentsign == 0) // if a zero found
{
    revisionstep = step;
    y = x;
    for (i = 0; i < numrevisions; i++)
    {
        y = y - revisionstep; // move back to before where the zero was
        revisionstep = revisionstep / refine; // change the step size
        while (sgn(eval(coeffs, y, argc - 3)) == lastsign) // move forward
            y += revisionstep;
    }
    zeros[numzeros++] = y - revisionstep / 2;
}
lastsign = currentsign;
}

// print the results

cout.precision(20);
for (i = 0; i < numzeros - 1; i++)
    cout << zeros[i] << " ";
if (numzeros >= 1)
    cout << zeros[numzeros - 1];
// cout << \\
return 0;
}
long double eval(long double *coeffs, long double x, int terms)
{
    int i=0;  //generic counter
    long double result=0;  //the value of the polynomial
    for (i=0;i<terms;i++)
    {
        result+=coeffs[i]*powld(x,i);
        //cout <<result<< \n;
    }
    return result;
}

int sgn(long double x)
{
    int ret;
    if (x==0)
        ret=0;
    else
        ret=(x>0)?1:-1;
    return ret;
}
long double powld(long double x, int n) //returns x^i for i an int
{
    int i=0;
    long double result=1;
    static long double lastx=1;
    static long double lastresult=1;
    static int lastn=1;

    if(lastn==n-1 && lastx==x)
    {
        result=x*lastresult;
    }
    else if (n>0)
    {
        if(n%2==0)
            result=powld(x*x,n/2);
        else
            result=x*powld(x*x,(n-1)/2);
    }
    else if (n<0)
    {
        for(i=0;i>n;i--)
        {
            result/=x;
        }
    }

    lastresult=result;
    lastx=x;
lastn=n;
return result;
}

long double atold (char str[]) //convert text to a long double
    //correct range, but will lose some precision
{
    long double x=0;
    long double decpart=0;
    char decimal[30];
    char exponent[30];
    int i=0;
    int j=0;

    for(i=0;i<30;i++)
    {
        decimal[i]=0;
        exponent[i]=0;
    }

    i=0;
    while(isdigit(str[i]) || str[i]=='.' || str[i]=='-')
    {
        decimal[i]=str[i];
        i++;  
    }
    decimal[i]=NULL;
if (str[i] == 'e' || str[i] == 'E')
{
    i+=1;

    while(isdigit(str[i]) || str[i] == '-')
    {
        exponent[j]=str[i];
        i++;  
        j++;  
    }
}
exponent[j]=NULL;

} else
{
    exponent[0]=0;
    exponent[1]=NULL;
}

return (long double)atof(decimal)*powld(10., atoi(exponent));
}
Bibliography


